Notes on PhD Research

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Contents

1 Note on parahoric groups

1.1 Motivation and Narasimhan-Seshadri correspondence

We fix a compact Riemann surface X of genus $g > 1$. The *Narasimhan-Seshadri correspondence* from [**NS**] is a correspondence between the moduli space of degree 0, rank r, slope-stable holomorphic vector bundles $\mathcal{N}_0$, and irreducible unitary representations of the fundamental group $\pi_1(X)$:

$$\mathcal{N}_0 \leftrightarrow \mathrm{Hom}(\pi_1(X), \mathbf{U}(r))/\mathbf{U}(r). \tag{1.1}$$

This correspondence is an isomorphism of complex analytic manifolds.

We follow [**WG08**, Appendix, Section 4] to generalize this to the degree d case of $\mathcal{N}_d$, we need universal central extensions of $\pi_1(X)$. The fundamental group $\pi_1(X)$ has $2g$ generators $A_1, \dots, A_g, B_1, \dots, B_g$ with the relation:

$$\prod_{i=1}^{g} [A_i, B_i] = 0. \tag{1.2}$$

A universal central extension is a short exact sequence of groups:

$$0 \to \mathbb{Z} \to \Gamma \to \pi_1(X) \to 0, \tag{1.3}$$

where $1 \in \mathbb{Z}$ maps to a central element $J \in \Gamma$. The group Γ has the generators with relations:

$$\prod_{i=1}^{g} [A_i, B_i] = J. \tag{1.4}$$

Using (1.3), any representation $\tilde{\rho} : \pi_1(X) \to \mathbf{PGL}(r, \mathbb{C})$ lifts to a representation $\rho : \Gamma \to \mathbf{GL}(r, \mathbb{C})$. From the principal-$\mathbf{PGL}(r, \mathbb{C})$-bundle $\xi = (\tilde{X} \times_{\tilde{\rho}} \mathbf{P}(\mathbb{C}^r))/\mathbf{PGL}(r, \mathbb{C})$ on X, we find a vector bundle E_ρ on X whose extension from $\mathbf{GL}(r, \mathbb{C})$ to $\mathbf{PGL}(r, \mathbb{C})$ is ξ. There exists a projectively flat connection Θ on E_ρ such that at any fiber $x \in X$, $\tilde{\rho} : \pi_1(X, x) \to \mathbf{PGL}((E_\rho)_x)$ is the holonomy of Θ at x. From this, we can define a subgroup of $\mathrm{Hom}(\Gamma, \mathbf{GL}(r, \mathbb{C}))$:

$$\mathrm{Hom}_d(\Gamma, \mathbf{GL}(r, \mathbb{C})) = \{\rho \in \mathrm{Hom}(\Gamma, \mathbf{GL}(r, \mathbb{C})) | c_1(E_\rho) = d\}, \tag{1.5}$$

and so we get a more general *Narasimhan-Seshadri correspondence* for degree d:

$$\mathcal{N}_d \leftrightarrow \mathrm{Hom}_d(\Gamma, \mathbf{U}(r))/\mathbf{U}(r). \tag{1.6}$$

Question: What if $X' \subset X$ had finitely many punctures, and we looked at bundles over X'? Or more algebraically, what if X' was a quasi-projective smooth algebraic curve over $\mathbb{C}$? Would there still be a version of Narasimhan-Seshadri correspondence over X'?

The introduction of punctures allows for vector bundles on X' to exhibit local behavior around them that was previously impossible only on X. One could record this extra information as a flag on the puncture points $X \setminus X'$, with weights, leading to the notion of *parabolic vector bundles.* We follow [**MS**] in the first section of this note.

One can then ask how this would generalize to principal-G-bundles for a reductive group G. It would make sense that principal bundles on X' could have a reduction to a parabolic H on the puncture points $X \setminus X'$. How this works will be explained in the second section, following [**PR**].

1.2 Parabolic vector bundles

As a reminder, a compact Riemann surface X of genus $g > 1$ has a universal covering of the upper half plane $\mathbb{H}$ (see: uniformization theorem), for which a discrete subgroup Γ of $\mathbf{PSL}(2, \mathbb{R})$ acts (freely, properly discontinuously). X can be recovered as $\mathbb{H}/\Gamma$, so that $\Gamma = \pi_1(X)$. *This is not the same* Γ *as in the last section!*

Now instead, we look at $X' = \mathbb{H}/\Gamma$ with only finite measure, so Γ may not act properly discontinuously and X' may have punctures. In [**Mat**], parabolic cusps of Γ in $\mathbb{H}$ are defined, and we write $\mathbb{H}^+ = \mathbb{H} \sqcup \{\textit{Parabolic cusps}\}$. By giving $\mathbb{H}^+$ the appropriate holomorphic structure and by lifting the action of Γ from $\mathbb{H}$ to $\mathbb{H}^+$, $X = \mathbb{H}^+/\Gamma$ can be viewed as a compact unpunctured Riemann surface containing $X' = \mathbb{H}/\Gamma$. Note that the holomorphic structure of X restricted to X', and that of X' itself, may not be the same.

For an element $P \in X \setminus X'$, which WLOG corresponds to a parabolic cusp $P = \infty \in \mathbb{H}$, one can find a suitable neighborhood of P in X appearing like U/Γ_∞, where U is an open neighborhood of ∞ in $\mathbb{H}^+$, and Γ_∞ is generated as a subgroup of Γ by one element $(z \mapsto z + 1)$ of Γ.

For a representation $\rho : \Gamma \to \mathbf{U}(r)$, we can induce a holomorphic vector bundle $E'_\rho = (\mathbb{H} \times_\rho \mathbb{C}^r)/\Gamma$ on X' that extends to a holomorphic vector bundle $E_\rho = (\mathbb{H}^+ \times_\rho \mathbb{C}^r)/\Gamma$ on X. In [**MS**, Section 1], by choosing a basis of E_ρ at P (which WLOG corresponds to the cusp at ∞), the representation ρ at P, acting on the isotropy group $\mathbf{GL}((E_\rho)_P)$, can be represented by a matrix that is invariant under the action of Γ_∞, or more explicitly the generator $(z \mapsto z + 1)$ of Γ_∞. The entries of this matrix can be used to find the *weights* of E'_ρ at P.

The subsequent propositions and corollaries of [**MS**, Section 1] show that E_ρ, equipped with the extra information of these weights at punctures in $X \setminus X'$, recovers E'_ρ completely. This is called a *parabolic structure.*

Definition 1.1. For a compact Riemann surface X with marked points $D_1, \ldots, D_n$ and a vector bundle E on X, a *parabolic structure* on E is given by the data on each marked point D_i:

(a) A flag $0 = V_0 \subset \ldots \subset V_k = E_{D_i}$.

(b) Weights $0 \leq \alpha_1 < \ldots < \alpha_{k-1} \leq 1$.

The dimensions of the quotients of the flag V_{l+1}/V_l are called the multiplicity k_l of α_l.

One can define morphisms of parabolic vector bundles, which preserve the flag (only when an inequality on the weights are fulfilled). Other constructions include parabolic subbundles, quotients, etc. that are similar to that of vector bundles, but respecting the parabolic structure.

Much like for vector bundles, we define a parabolic version of degree. We define it such that a parabolic structure on E comes from a holomorphic bundle E'_ρ on $X' \subset X$ if and only if the parabolic degree is 0.

Definition 1.2. Let E be a holomorphic vector bundle on X with a parabolic structure.

(a) The *parabolic degree* of E is:

$$\operatorname{pardeg}(E) = \deg(E) + \sum_P \sum_i k_i \alpha_i. \tag{1.7}$$

(b) E is parabolic-(semi)-stable if for all subbundles F of E, inheriting its parabolic structure, fulfills the inequality:

$$\frac{\operatorname{pardeg}(F)}{\operatorname{rk}(F)}(\leq)\frac{\operatorname{pardeg}(E)}{\operatorname{rk}(E)} \tag{1.8}$$

For more on parabolic vector bundles, see [**LM**, Section 2.1]. For more on moduli spaces of parabolic vector bundles, see [**MS**, Section 4], which is constructed analogously to the moduli space of stable vector bundles.

1.3 Parahoric groups

1.4 Parahoric $\mathcal{G}$-torsors

2 Note on the Hitchin map

We describe roughly what Hitchin did in his paper [**Hita**], observing the cotangent bundle $T^*\mathcal{N} \to \mathcal{N}$ of the moduli space $\mathcal{N}$ of stable holomorphic vector bundles over a Riemann surface X (of rank r and degree d), and proved that it is an *algebraically completely integrable system* (ACIS).

2.1 General setup

We fix a compact Riemann surface X of genus $g > 1$, and fix a smooth vector bundle $\mathbb{E}$ on X of complex rank r and degree d. A holomorphic vector bundle E of rank r and degree d is determined by a *holomorphic structure* d_A on $\mathbb{E}$, which is an operator:

$$d_A : \Omega^0(X, \mathbb{E}) \to \Omega^{0,1}(X, \mathbb{E}), \tag{2.1}$$

with the Leibniz rule:

$$d_A(fs) = \overline{\partial} f \otimes s + f d_A(s), \tag{2.2}$$

where $\Omega^0(X, \mathbb{E})$ is the sheaf of smooth sections of $\mathbb{E}$, and $\Omega^{0,1}(X, \mathbb{E}) = \Omega^{0,1}(X) \otimes \Omega^0(X, \mathbb{E})$.

By subtracting two holomorphic structures, we get $C^\infty(X)$-linearity:

$$(d_A - d_{A'})(fs) = f(d_A - d_{A'})(s), \tag{2.3}$$

ensuring $d_A - d_{A'} \in \Omega^{0,1}(X, \operatorname{End}(\mathbb{E}))$ (the maps induced on the fibers of $\mathbb{E}$ are linear). Thus, the holomorphic structures form an infinite dimensional complex affine space $\mathcal{A}$ based on $\Omega^{0,1}(X, \operatorname{End}(\mathbb{E}))$.

Let $\mathcal{G} = \operatorname{Aut}(\mathbb{E})$ be the Lie group of smooth vector bundle automorphisms on $\mathbb{E}$, acting on $\mathcal{A}$ through conjugation:

$$g \in \mathcal{G} : \quad d_A \mapsto g^{-1} d_A g, \tag{2.4}$$

which are affine transformations on $\mathcal{A}$. Through $\mathcal{G}$ acting on $\mathcal{A}$, we can obtain a quotient $\mathcal{N} = \mathcal{A}^{\text{st}}/\mathcal{G}$ of slope-stable holomorphic vector bundles on X, and write $E = [d_A]$ in $\mathcal{N}$ (this is known as the de Rham moduli space).

Through symplectic reduction, it can be shown that $\mathcal{N}$ is a complex submanifold of a compact complex manifold, there is also an algebraic construction through projective

GIT, using Quot schemes, that is equivalent due to Kempf-Ness. Through a Riemann-Roch calculation, we have that:

$$\dim_{\mathbb{C}} \mathcal{N} = r^2(g-1)+1, \qquad (=m). \tag{2.5}$$

In both situations, we can talk about the cotangent bundle $T^*\mathcal{N} \to \mathcal{N}$.

Hitchin claims in [**Hita**] that with the canonical symplectic structure ω, there exists m smooth functions $a_1, \ldots, a_m : T^*\mathcal{N} \to \mathbb{C}$, such that

(a) The induced Hamiltonian fields X_{a_i} are linearly independent everywhere (equivalently $da_1 \wedge \ldots \wedge da_m$ is generically nonzero).

(b) We have $\{a_i, a_j\} = 0$ (they Poisson-commute).

In this case, $T^*\mathcal{N}$ is called a *completely integrable Hamiltonian system.* Hitchin claims further that it is *algebraically* completely integrable, so we also have

(c) A generic fiber of $h = (a_1, \ldots, a_m) : T^*\mathcal{N} \to \mathbb{C}^m$ embeds as an open subset of an abelian variety.

These extra conditions give the functions "compatibility" with the algebraic structure of $\mathcal{N}$, and are fulfilled precisely by the Hitchin map h constructed in [**Hita**].

2.2 The Hitchin Map

How can an element in $T^*\mathcal{N}$ be actually written down? Fixing $E \in \mathcal{N}$, deformation theory tells us that $T_E\mathcal{N} \cong H^1(X, \mathrm{End}(E))$. Then by using Serre duality, we have $T_E\mathcal{N}^* \cong H^0(X, \mathrm{End}(E) \otimes K)$, for the canonical bundle K of X. Giving an element $T^*\mathcal{N}$ is the same as giving a pair (E, φ), where $\varphi : E \to E \otimes K$ is called a *Higgs field.*

Hitchin's idea is to define a smooth map

$$h : T^*\mathcal{N} \to \mathbb{H} = \bigoplus_{i=1}^{r} \Omega^0(X, K^{\otimes i}), \tag{2.6}$$

where (E, φ) is mapped to its characteristic polynomial $\chi_\varphi(t) = \det(t - \varphi)$. More precisely, on a local trivialization of E, $\chi_\varphi(t)$ is the determinant of an $r \times r$-matrix valued in local sections of K varying in t. On the determinant bundle $\det(\mathrm{End}(E) \otimes K)$, we get a decomposition

$$\chi_\varphi(t) = t^r + a_1 t^{r-1} + \ldots + a_{r-1} t + a_r, \tag{2.7}$$

with coefficients $a_i \in \Omega^0(X, K^{\otimes i})$.

By calculating the complex dimension of $\mathbb{H}$ to be m using Riemann-Roch, as in [**Hita**, Section 4], when we fix an isomorphism $\mathbb{H} \cong \mathbb{C}^m$, we have $h = (a_1, \ldots, a_m)$. Hitchin then shows that these smooth functions $a_i : T^*\mathcal{N} \to \mathbb{C}$ Poisson-commute in [**Hita**, Proposition 4.5], using Hamiltonian reduction from $\mathcal{A}$ to $\mathcal{A}/\mathcal{G}$.

We have that $da_1 \wedge \ldots \wedge da_r$ is generically nonzero, since the coefficients a_i all being nonzero is a generic property of $T^*\mathcal{N}$. Thus, we have a completely integrable Hamiltonian system!

2.3 Spectral Curves (away from Ramification)

For the total space $|K|$ of K, and the canonical morphism $\pi : |K| \to X$, observe the pullback diagram of vector bundles,

$$\begin{tikzcd} \pi^*K \arrow[r] \arrow[d] & K \arrow[d] \\ |K| \arrow[r, "\pi"] \arrow[u, bend left, "\lambda"] & X \end{tikzcd}, \tag{2.8}$$

with the tautological section $\lambda = (\lambda_1, \lambda_2) : p \mapsto (p, p)$. We then define the *spectral curve at* φ

$$X_\varphi = \{p \in |K| : \chi_\varphi(\lambda_2(p)) = 0\}, \tag{2.9}$$

by which $\chi_\varphi(t)$ can take in values at K and return a value in $\det(\mathrm{End}(E) \otimes K)$. Using linear systems of divisors of X_φ for all (E, φ), Bertini's theorem states that X_φ is smooth for generic (E, φ) in $T^*\mathcal{N}$.

When X_φ is smooth at (E, φ), φ is diagonalizable for all $x \in X$, and $\pi_\varphi : X_\varphi \to X$ appears as a cover of degree r. Elements of $\pi_\varphi^{-1}(x)$ correspond to eigenvalues $p_x \in K$ of φ at x. The eigenspaces of E at these eigenvalues p are isomorphic to line subbundles $L_p^E \in \mathrm{Jac}(X_\varphi)$ of $\pi_\varphi^*\mathbb{E}$, as the eigenvalues p are distinct when X_φ is smooth.

For a characteristic polynomial $\chi \in \mathbb{H}$, the fiber $h^{-1}(\chi)$ consists of the families of pairs (E, φ) where $\chi_\varphi = \chi$, and the spectral curve X_φ is fixed. What differentiates between elements in $h^{-1}(\chi)$ are the line bundle eigenspaces L_p^E in X_φ. Precisely, we get an injective morphism of varieties

$$h^{-1}(\chi) \to \mathrm{Jac}(X_\varphi), \qquad (E, \varphi) \mapsto (L_p^E). \tag{2.10}$$

A possible inverse could be

$$\mathrm{Jac}(X_\varphi) \to h^{-1}(\chi), \qquad L \mapsto \pi_{\varphi *}(L) \cong \bigsqcup_{x \in X} \bigoplus_{p_x \in \pi^{-1}(x)} L_{p_x}. \tag{2.11}$$

However, not every line bundle on $\mathrm{Jac}(X_\varphi)$ induces a stable vector bundle of degree d (by construction rank r is true), hence $h^{-1}(\chi)$ will be embedded as an open subset of the abelian variety $\mathrm{Jac}(X_\varphi)$, corresponding to the stable locus of vector bundles with Higgs field φ, and degree d. Otherwise, the pushforward $\pi_{\varphi *}(L)$ is how we recover the pair (E, φ) from a line bundle of X_φ.

2.4 Spectral Curves (at Ramification)

At ramification the situation is more complicated, (E, φ) induces a ramified cover $\pi_\varphi : X_\varphi \to X$ of degree r, with finitely many branch points in X.

At a branch point $x \in X$, we have generalized eigenvalues of E at x (appearing more than once on the Jordan normal form), where the ramification index of π_φ at x is equal to the sum of dimensions of these generalized eigenspaces at x.

A pair (E, φ) in $h^{-1}(\chi)$ still embeds into $\mathrm{Jac}(X_\varphi)$ as follows, for a branch point $x \in X$ caused by generalized eigenvalues in $\pi^{-1}(x)$, with ramification index $e(p_x)$, we search for $L \in \mathrm{Jac}(X_\varphi)$ (torsion free sheaf of rank 1 as in BNR?) by looking at its *jet* $J_{e(p_x)}(L)_{p_x}$. The jet is the fiber at $p_x \in \pi^{-1}(x)$ of the vector bundle $J_{e(p_x)}(L)$ over

X_φ, where for sections $\sigma : X_\varphi \to L$ with local coordinates around $p_x \in X_\varphi$ (in a local trivialization of L)

$$b_0 + b_1 y + b_2 y^2 + \ldots + b_{e(p_x)} y^{e(p_x)} + \ldots \tag{2.12}$$

the stalk of $J_{e(p_x)}(L)$ at p_x consists of the germs of these sections that agree up to degree $y^{e(p_x)}$. This should be independent of choice of trivialization, and Vakil shows in [**Vak**], that in general

$$J_n(L) \cong L \otimes (\mathcal{O}(X_\varphi) \oplus K \oplus \ldots \oplus \operatorname{Sym}^n(K)). \tag{2.13}$$

The stalk of $J_{e(p_x)}(L)_{p_x}$ should be isomorphic to germs of sections of E at x, which evaluate in the generalized eigenspaces of p_x of E at x (that caused ramification). To find E in terms of L in $\operatorname{Jac}(X_\varphi)$, Hitchin in [**Hita**] looks at the short exact sequence of vector spaces

$$0 \to \bigoplus_{p_x \in \pi^{-1}(x)} L^*_{p_x} \to (\pi_* L)^*_x \to \bigoplus_{p_x \in \pi^{-1}(x)} J^*_{e(p_x)}(L)/L^*_x \to 0. \tag{2.14}$$

In the unramified case, the first two spaces are equal. When passing to sheaves of $\mathcal{O}(X)$-modules, we obtain

$$0 \to W \to (\pi_* L)^* \to \mathcal{S} \to 0, \tag{2.15}$$

where $\mathcal{S}$ is a finite sum of skyscraper sheaves (at the branch points), and W is a locally free sheaf of rank r such that $W = E^*$.

Like in [**LM**, Proposition 2.2], we can also restrict from X to a Zariski-open subset $U \subset X$ containing x, such that L is trivial on $\pi|_\varphi^{-1}(U)$, and K is trivial over U. In this case, $\pi_* L$ restricted to U is given as an $\mathcal{O}(U)$-module by $\mathcal{O}(U)[t]/(\chi(t))$. Here, the zeros of $\chi(t)$ with higher multiplicity introduce torsion elements at their stalks of $\mathcal{O}(U)[t]/(\chi(t))$, corresponding to $\mathcal{S}$ in (2.15).

2.5 Conclusion

For a generic fiber $h^{-1}(\chi)$, we can embed $h^{-1}(\chi)$ as an open subset of Jacobian variety. Thus, the Hitchin map h is an algebraic completely integrable Hamiltonian system.

It would be nice to have that $h^{-1}(\chi)$ is Lagrangian (as a leaf in a Lagrangian foliation of the distribution induced by $X_{a_1}, \ldots, X_{a_m}$), as it would agree with Liouville's theorem for completely integrable systems (over $\mathbb{R}$). For this, we need to pass to the moduli space of stable Higgs bundles $\mathcal{M}$, where $T^*\mathcal{N}$ is contained as the complement of a submanifold of at least codimension g. Hitchin showed explicitly that these fibers form a Lagrangian foliation vector bundles of rank 2 in [**Hitb**, Section 7].

3 Note on Higgs bundles

3.1 Strongly parabolic Higgs bundles

3.2 Weakly parabolic Higgs bundles

4 Note on Fano Varieties and K-Stability

Definition 4.1. A *Fano variety* is a variety whose anticanonical line bundle is ample.

5 Note on Springer Theory

6 Note on ADHM Spaces and Instantons

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