Geometric invariant theory (GIT)

Talk 1: Introduction

Setup: G an algebraic group acting on an algebraic variety $G \curvearrowright X$. (e.g. $\mathbb{G}_m \curvearrowright K$)

GIT asks the following:

dim 0 scheme?

What is a sensible notion of a quotient "space" X/G? What properties? ($K/\mathbb{G}_m = \underline{\underline{\{0,1\}}}$?)

Why are we interested? It is a very useful tool for classifying (algebro-)geometric objects up to isomorphism, and find "moduli spaces" of

Riemann 19th century

1. smooth projective algebraic curves (of genus g) $\mathcal{M}_g$
2. Riemann surfaces (of genus g) $\mathcal{M}_g$

Mumford Seshadri Narasimhan 1965

3. Vector bundles on X (of rank r, degree d) $\mathrm{Bun}_{r,d}(X)$
4. principal-G-bundles on X $\mathrm{PrincBun}_G(X)$

Donaldson 1983

(G reductive linear algebraic group)

5. (k-dimensional) subspaces of V, an n-dimensional K-vector space $\mathrm{Gr}_k(V)$

$\mathbb{P}^1_K = \mathrm{Gr}_1(K^2)$

2

(1) was studied indirectly by Riemann in 1857 (Theorie der Abel'schen Funktionen).

Idea: * View curves X, genus g as simply branched covers $X \to \mathbb{P}^1$ of degree d.

May have ramification points

Riemann-Hurwitz

* Use properties of such covers to calculate the number of "parameters"

$$\dim(\mathcal{M}_g) = 3g-3$$

Issues: It's only a "local deformation space", Riemann's argument fixes a curve X, and understands how $\mathcal{M}_g$ varies locally around X.

To understand $\mathcal{M}_g$ globally, we need to find a geometric structure on $\mathcal{M}_g$ (scheme/Manifold/variety)

$$\mathcal{M}_g \stackrel{?}{=} X/PGL \text{ using GIT?}$$

What is a fine/coarse moduli space?

Such a space $\mathcal{M}$ is in bijection w/ geometric objects up to isomorphism, & respects how they "vary" in "families"

What do we mean by family?

Curves: $\mathcal{C} \xrightarrow{\pi} S$

ensures dim. of fibers are locally constant (sensible "variations")

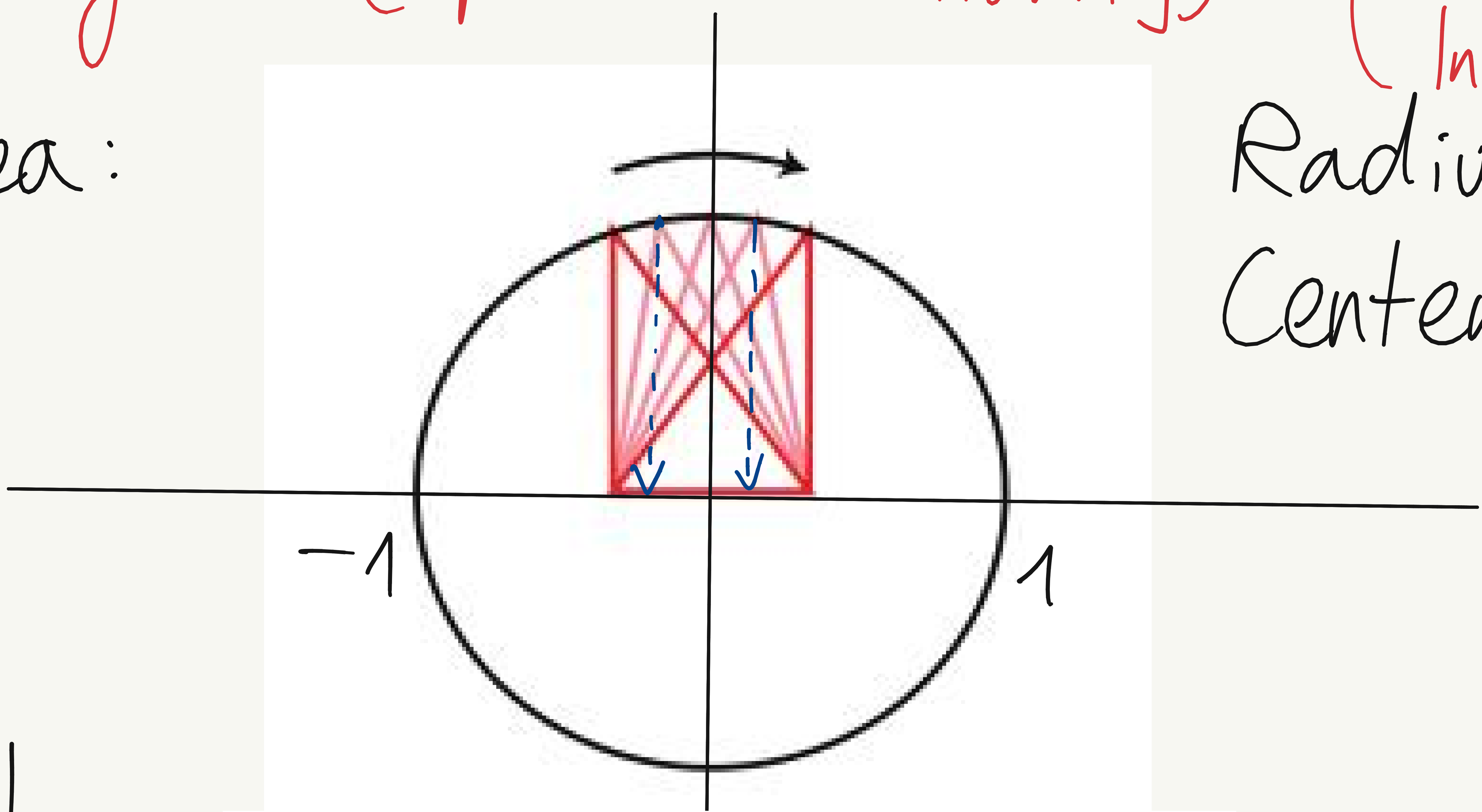
Flat morphism of schemes s.t. $\forall s \in S$: $\pi^{-1}(s)$ is a smooth curve/subspace of V

Bundles: $\mathcal{E} \to X \times S$

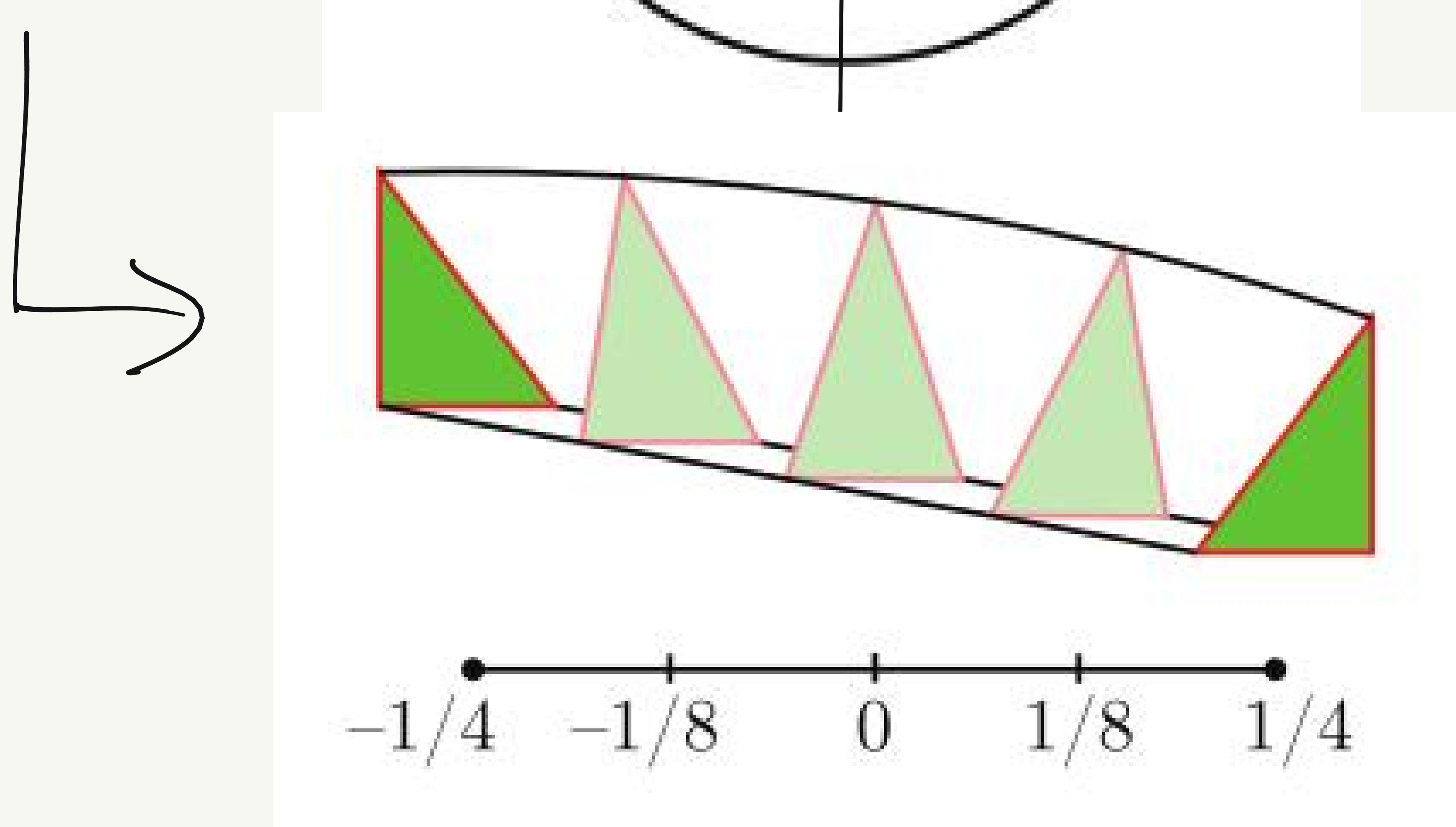
Flat morphism of schemes s.t. $\forall s \in S$ closed: $\mathcal{E}_s = \mathcal{E}|_{X \times s}$ is a vector principal bundle

Triangles (up to similarity): (Images from Behrend Intro to algebraic stacks)

Idea:



Radius 1 circle Centered at (0, 0)



Family (Beware, we don't respect orientation)

What is a fine moduli space?

There is a family $\mathcal{V}$ over $\mathcal{M}$, such that

called universal family

$\forall$ families $\begin{array}{c} \mathcal{E} \\ \downarrow \\ S \end{array}$, $\exists!$ morphism $S \to \mathcal{M}$

such that:

$$\begin{tikzcd} \mathcal{E} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathcal{V} \arrow[d] \\ S \arrow[r] & \mathcal{M} \end{tikzcd}$$

is cartesian

in the appropriate Category of spaces

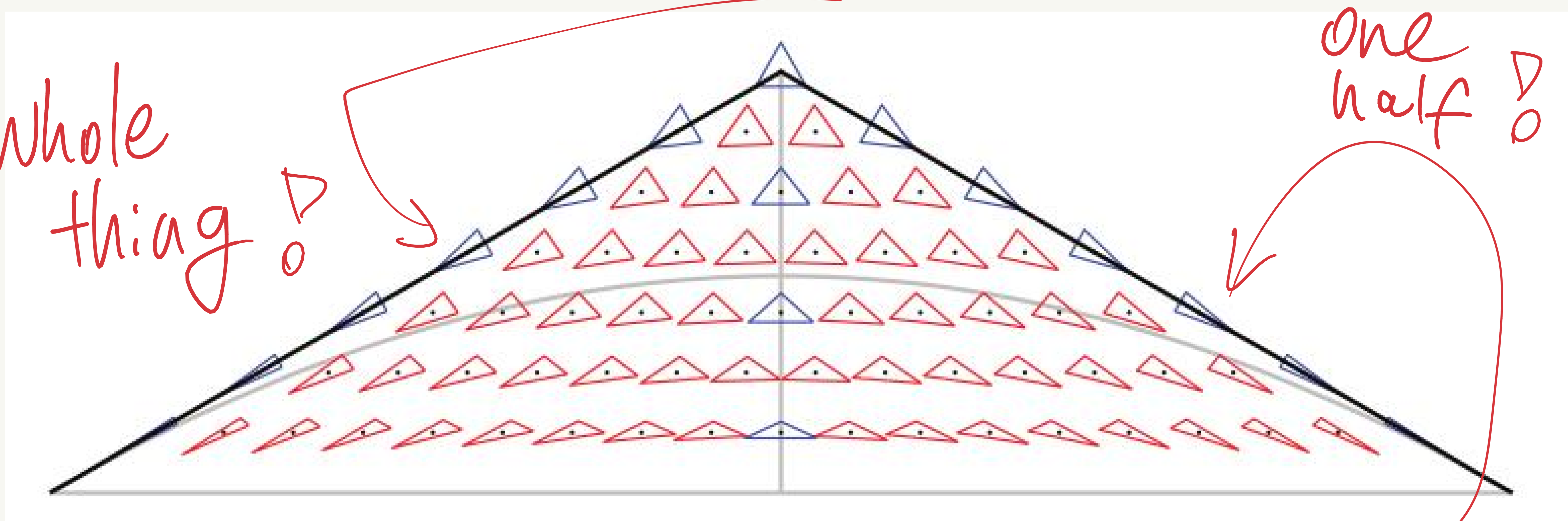
(Varieties, Schemes, Algebraic Spaces, Stacks)

In this case, we say $\mathcal{M}$ is a fine moduli space.

Examples Grassmanians, Similar oriented triangles

Whole thing!

one half!



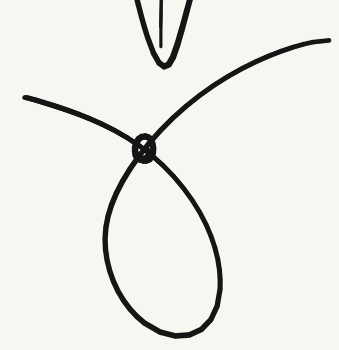
not $\mathcal{M}_g$, $\mathrm{Bun}_{r,d}(X)$, Similar triangles

Is a fine moduli space feasible? (Generally no...)

As soon as the objects we study have nontrivial automorphisms, naturality is impossible (i.e. no <u>fine</u> moduli space).

e.g. compare

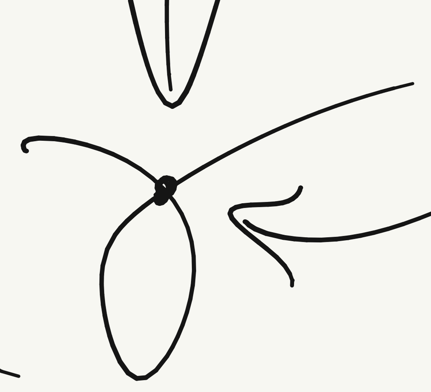
$$\left(\mathcal{C} \xrightarrow{\pi} \right)$$



Constant family
glue through identity
$\mathcal{C}_s \cong \mathcal{C}_{s'} \ \forall s, s'$

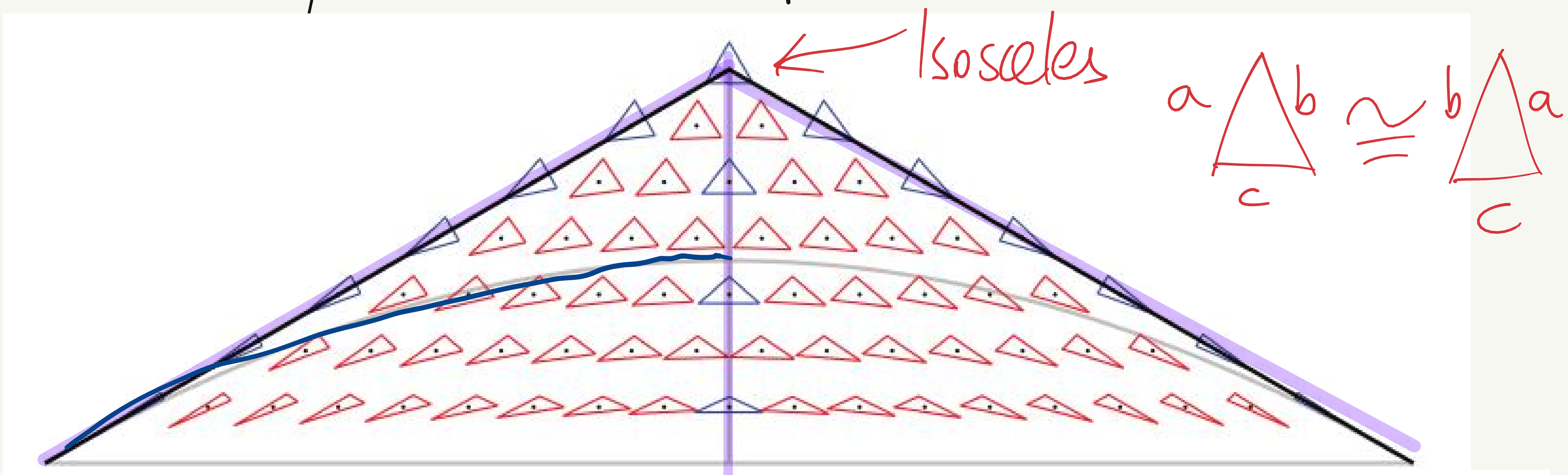
with

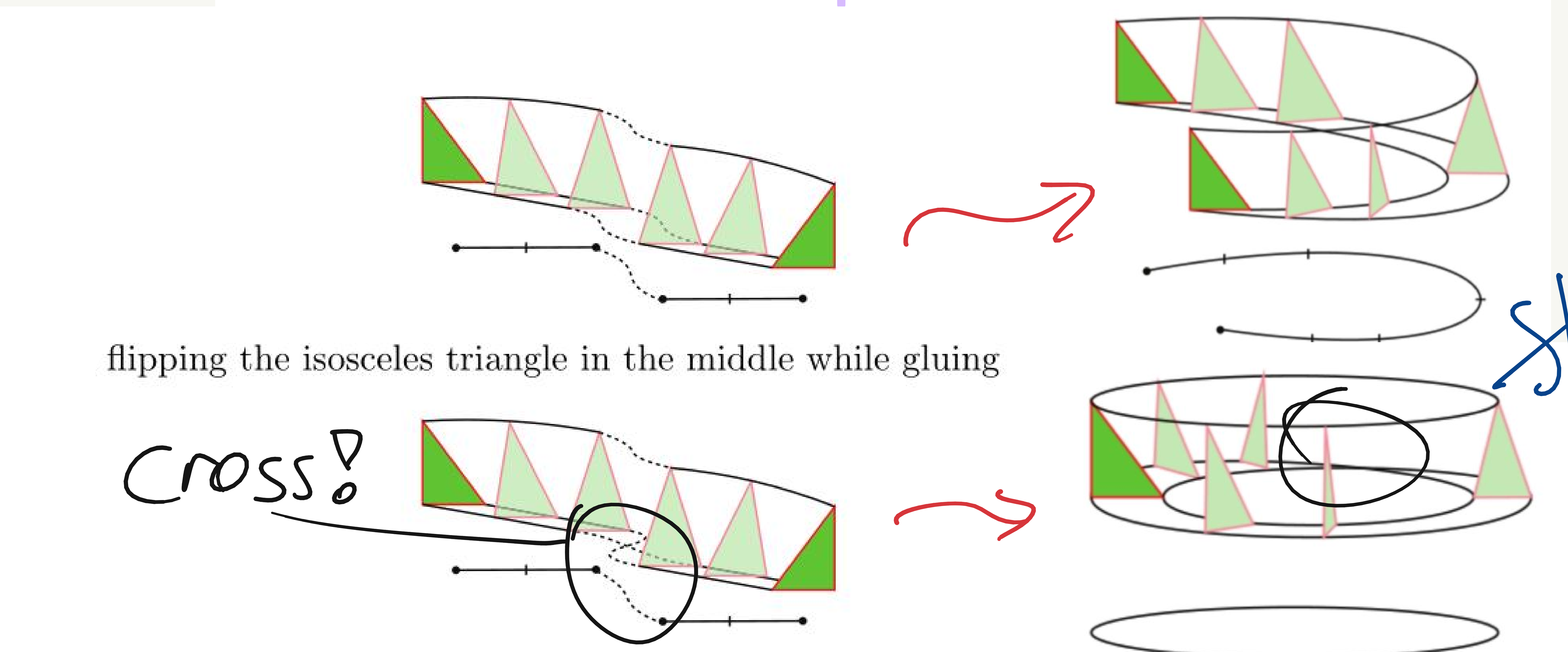
$$\left(\tilde{\mathcal{C}} \longrightarrow \right)$$



Noncostant family:
glue through nontrivial isomorphism
$\mathcal{C}_s \cong \mathcal{C}_{s'}$

Here $\mathcal{C} \not\cong \tilde{\mathcal{C}}$, so pullback property falters





flipping the isosceles triangle in the middle while gluing

Not $\ncong$ isomorphic!

Possible solutions:

1. Wish for less (a <u>coarse</u> moduli space)
2. Incorporate more data into the objects we study, killing automorphisms
3. Enlarge the category $\mathcal{M}$ resides in, ($\mathcal{M}$ is only an algebraic stack rather than a scheme)

(for another seminar ☺)

(With 2 we have a fine moduli space X for which we could quotient out by automorphisms $X /\!/ G$ using GIT)

What is a coarse moduli space?

(☆) $\forall$ families $\begin{matrix} \mathcal{C} \\ \downarrow \\ S \end{matrix}$, the induced map $S \to \mathcal{M}$

is in the relevant category (of varieties, schemes, etc.)

For any other space $\mathcal{M}'$ in bijection with our geometric objects fulfilling (☆), we have a map $\mathcal{M} \to \mathcal{M}'$ ($\mathcal{M}$ is a "refinement" of $\mathcal{M}'$)

Examples $\mathcal{M}_g^{s}$, $\mathrm{Bun}^{ss}_{r,d}(X)$,

"stable" curves

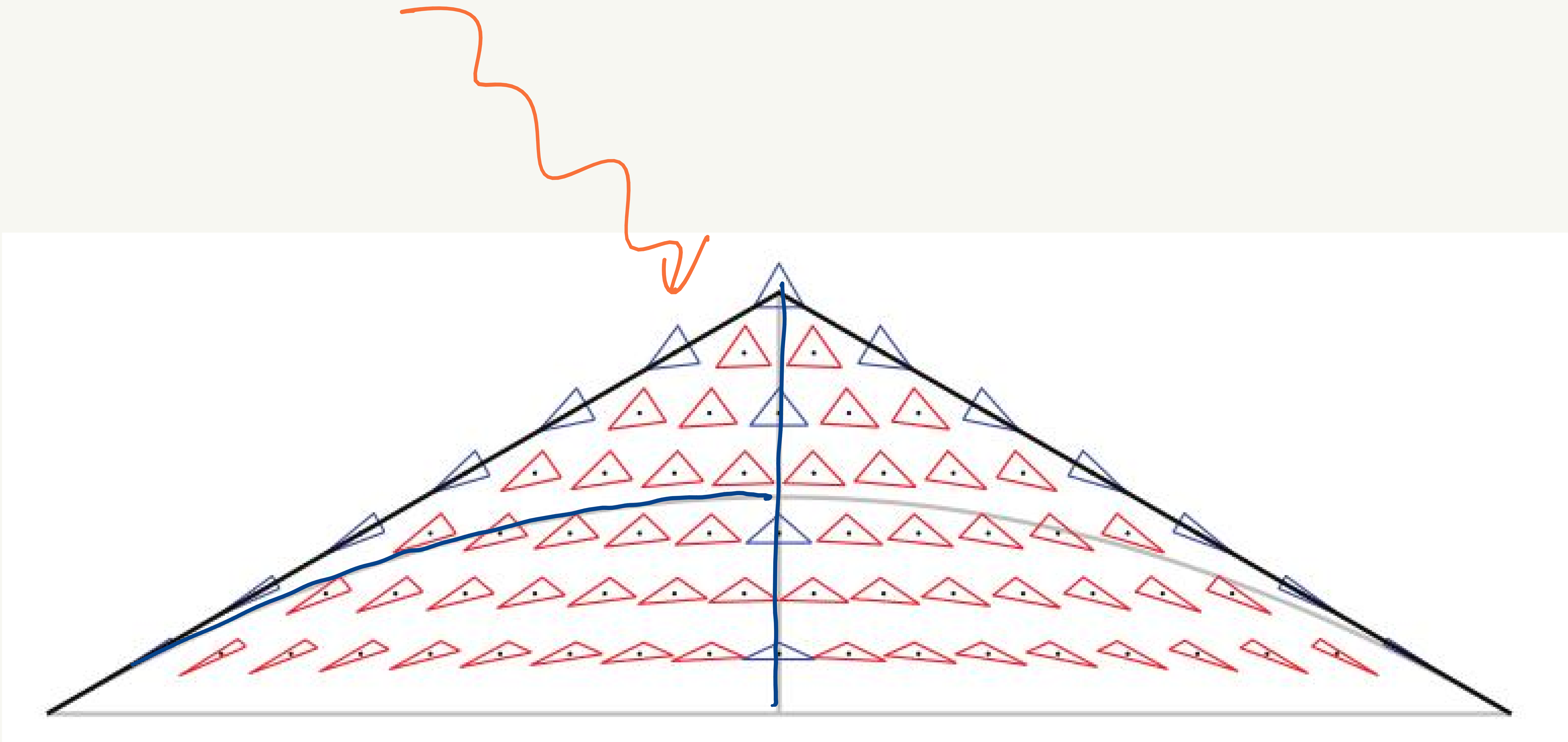
Mumford - "semistable" vec. bun.

$\mathrm{PrincBun}^{ss}_G(X)$ ← Ramanathan - "semistable" - principal bundles

$\mathrm{Bun}^{\mathrm{Higgs},ss}(X)$ ← Semistable Higgs bundles

(Bundles w/ Higgs fields)

Similar triangles



Back to GIT In $G \curvearrowright X$, a quotient space "$f: X \to X/G$" has, ideally, the following properties:

1. X/G is a variety/scheme (or whatever other geometric structure)
2. X/G is the orbit space (as a set) $= \{ G \cdot x \mid x \in X \}$ (Geometric quotient)
3. f is G-invariant, such that $(f_* \mathcal{O}_X)^G = \mathcal{O}_Y$ (Good quotient) (Hoskins, Def 3.27)
4. $f: X \to X/G$ is a categorical quotient
   $\forall h: X \to Y$ G-invariant (i.e. $\forall g \in G \ \forall x \in X: h(gx) = h(x)$)
   $\exists! j: X/G \to Y$ s.t.
$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "f"'] & Y \\ X/G \arrow[ur, "j"'] & \end{tikzcd}$$

Often all 4 is impossible to obtain...

In GIT, we first aim for 1 and 3

From this, 4 comes for free ⚬

↑ Hoskins, proposition 3.30

Passing to "GIT-stable points" - get 2

The payment is that G must be reductive, and we must restrict X to GIT-semistable points

Idea of const. of an affine GIT quotient $X/\!/G$

Def: G is lin. reductive if its a smooth conn. lin. alg. group, $G \subset GL(r, k)$, such that every smooth unipotent alg. subgr. is trivial

If $X = \operatorname{Spec}(A)$, $G \curvearrowright X$ acts rationally i.e. all $f \in A$ is contained in a finite-dim G-inv subspace of A, then

$$X = \operatorname{Spec}(A) \longrightarrow X/\!/G = \operatorname{Spec}(A^G)$$

finite type $\xRightarrow{\text{Nagata}}$ finite type

Nagata $\Rightarrow$ this is a good quotient

Def (a) $x \in X$ is GIT-stable if $G \cdot x$ is closed in X and $\dim G_x = 0$ (or orbit is of max dimension equal to $\dim(G)$)

(b) In projective GIT, $x \in X$ is GIT-semistable if $\exists$ homogeneous polynomial f, G-inv. such that $f(X) \neq 0$.

Example $\mathbb{G}_m \curvearrowright \mathbb{G}_a$ over $k = \bar{k}$, all points are GIT-unstable $\left(f(rx) = r^d f(x) \neq f(x)\right)$