Homework #1
Automata and Computation Theory
Fall 2018

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1 Problem 1

Show that for any three sets A, B, C, we have that

$$(A \cap B) \cup C = (A \cup C) \cap (B \cup C).$$

PROOF: There are two parts to the proof.

Part 1

Prove $(A \cap B) \cup C \subset (A \cup C) \cap (B \cup C)$
Let $a \in (A \cap B) \cup C$
So $a \in (A \cap B)$ or $a \in C$
Case 1: $a \in (A \cap B)$
So $a \in A$ and $a \in B$
Since $a \in A$, $a \in (A \cup C)$
and since $a \in B$, $a \in (B \cup C)$
Since $a \in (B \cup C)$ and $a \in (A \cup C)$, $a \in (A \cup C) \cap (B \cup C)$
Case 2: $a \in C$
So $a \in (A \cup C)$ and $a \in (B \cup C)$
So $a \in (A \cup C) \cap (B \cup C)$
So in either case if $a \in (A \cap B) \cup C$, then $a \in (A \cup C) \cap (B \cup C)$
So $(A \cap B) \cup C \subset (A \cup C) \cap (B \cup C)$

Part 2

Prove $(A \cup C) \cap (B \cup C) \subset (A \cap B) \cup C$

Let $b \in (A \cup C) \cap (B \cup C)$

So $b \in (A \cup C)$ and $b \in (B \cup C)$.

There are two cases

Case 1: WLOG let $b \in C$

Then $b \in (A \cap B) \cup C$ because $b \in C$.

Case 2: $b \in A$ and $b \in B$

Then $b \in (A \cap B)$

So $b \in (A \cap B) \cup C$

So in either case if $b \in (A \cup C) \cap (B \cup C)$, then $b \in (A \cap B) \cup C$

So $(a \cup C) \cap (B \cup C) \subset (A \cap B) \cup C$.

Since $(A \cup C) \cap (B \cup C) \subset (A \cap B) \cup C$ and $(A \cap B) \cup C \subset (A \cup C) \cap (B \cup C)$, $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$

2 Problem 2

In the textbook it is shown that $\sqrt{2}$ is an irrational number. Use this fact to show that the following statement is true: there exist two irrational numbers p and q, such that q^p is a rational number.

ANSWER:

We know that $\sqrt{2}$ is an irrational number.

Lets look at $\sqrt{2}^{\sqrt{2}}$

There are two cases, $\sqrt{2}^{\sqrt{2}}$ is rational and it is irrational

Case 1: Rational

If $\sqrt{2}^{\sqrt{2}}$ is a rational number then the proof ends there since $\sqrt{2}$ is an irrational number.

So there exists p and q in the irrational numbers such that q^p is a rational number.

Case 2: Irrational

Let $\sqrt{2}^{\sqrt{2}}$ be irrational.

then let $p = \sqrt{2}^{\sqrt{2}}$ and let $q = \sqrt{2}$.

So we have $(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}}$

By properties of exponential, this equals $\sqrt{2}^{\sqrt{2}*\sqrt{2}} = \sqrt{2}^2 = 2$

2 is an integer and as such is a rational number.

Since 2 is a rational number, $\exists 2$ irrational numbers p and q such that q^p is a rational number.

3 Problem 3

Show that every undirected graph with 2 or more nodes contains two nodes with the same degree.

PROOF:
Lets construct an arbitrary undirected graph with $n \geq 2$ nodes.

Lets also contruct it so that it maximizes the number of nodes with unique degrees.

Finally let D be the set of all possible degrees and V be the set of all n nodes.

There are two options of how to construct the graph based on whether or not the graph is connected.

Case 1: Graph is disconnected

Since the graph is disconnected, $\exists$ a node v with a degree of 0.

Since v has a degree of 0, there are no edges connecting v to any other node.

So it is impossible for a node to have a degree of $n-1$ since that requires that node having an edge connecting it to every other node including v.

So the maximum degree a node can have is $n-2$ which occurs when that node has an edge connecting it to every other node except v.

So $D = \{x | x \in \mathbb{Z}$ and $0 \leq x \leq n-2\}$

So $|D| = n-1$.

Since $|D| = n-1$ and $|V| = n$ and $n > n-1$, $|V| > |D|$

Since $|V| > |D|$, by the pigeon-hole principal there is no function from set V to set D that is one-to-one.

Since there is no function that is one-to-one, any function from V to D, such as putting degree counts to nodes, will always have atleast 2 elements in V that map to the same element in D

So for any arbitrary disconnected graph with nodes $n \geq 2$, there will always be atleast 2 nodes that share a degree.

Case 2: The graph is connected

Since the graph is connected, every node has atleast one edge connecting it to another node.

So the minimum degree a node can have is 1.

The maximum degree a node can have is $n-1$ which is when it has an edge connecting it with every other node in the graph.

So $D = \{x | x \in \mathbb{Z}$ and $1 \leq x \leq n-1\}$

So $|D| = n - 1$.

Since $|D| = n - 1$ and $|V| = n$ and $n > n - 1$, $|V| > |D|$

Since $|V| > |D|$, by the pigeon-hole principal there is no function from set V to set D that is one-to-one.

Since there is no function that is one-to-one, any function from V to D, such as putting degree counts to nodes, will always have atleast 2 elements in V that map to the same element in D

So for any arbitrary connected graph with nodes $n \geq 2$, there will always be atleast 2 nodes that share a degree.

Since in any connected or in any disconnected undirected graph with $n \geq 2$ nodes there will always be atleast 2 nodes that share a degree it holds true that:

Any undirected graph with 2 or more nodes contains two nodes with the same degree.

4 Problem 4

Show that there exist no integers x, y, z such that $x^2 + y^2 = 3z^2$, except $x = y = z = 0$.

PROOF:
NOTE: Since $\forall a \in \mathbb{Z}\ a^2 = (-a)^2$, if $a, b, c \in \mathbb{Z}$ is a solution to $x^2 + y^2 = 3z^2$, then $a, b, -c$ is a solution and so is any version of -1 times one or multiple of the answers.

So WLOG lets look at the solution a, b, c in which $a, b, c > 0$.

FSOC lets say $\exists x, y, z$ such that $x, y, z \in \mathbb{Z}$, $x^2 + y^2 = 3z^2$, $x, y, z > 0$, and z is the smallest possible positive value that satisfies the equation on the right hand side.

Since z is an integer, z^2 is an integer.

Since $x^2 + y^2 = 3z^2$, $z^2 = \dfrac{x^2 + y^2}{3}$

Since z^2 is an integer and $z^2 = \dfrac{x^2 + y^2}{3}$, $x^2 + y^2$ must be divisible by 3.

Since $x^2 + y^2$ is the sum of squares of two integers and is divisible by 3, x, y must both be divisible by 3.

Since x and y are both divisible by 3, x^2 and y^2 are both divisible by 9.

So $x^2 + y^2 = 3z^2$ is the same as $9(x')^2 + 9(y')^2 = 3z^2$ where $x = 3x'$, $y = 3y'$ and $x', y' \in \mathbb{Z}$

So $(x')^2 + (y')^2 = \dfrac{3z^2}{9}$

So $(x')^2 + (y')^2 = \dfrac{z^2}{3}$.

Since x' and y' are integers, so are $(x')^2$ and $(y')^2$

Since $(x')^2$ and $(y')^2$ are integers, $(x')^2 + (y')^2$ is the sum of two integers so it is also an integer.

Since $(x')^2 + (y')^2$ is an integer and $(x')^2 + (y')^2 = \dfrac{z^2}{3}$, $\dfrac{z^2}{3}$ is also an integer.

Since $\dfrac{z^2}{3}$ is an integer, z^2 is divisible by 3.

Since z^2 is the square of an integer and is divisible by 3, z must also be divisible by 3.

So $\exists z' \in \mathbb{Z}$ such that $z = 3z'$

So $z^2 = 9(z')^2$

So $(x')^2 + (y')^2 = \dfrac{9(z')^2}{3}$

So $(x')^2 + (y')^2 = 3(z')^2$.

Since z is a positive integers and $z = 3z'$, $z > z'$ and $z' > 0$.

Since $x', y', z' \in \mathbb{Z}$ and $x', y', z' \neq 0$ and $(x')^2 + (y')^2 = 3(z')^2$, x', y', z' are valid solutions to the equation.

This is a contradiction since $z > z'$, both z and z' are part of a solution and z was defined to be the smallest possible positive value that satisfies the equation for the right hand side.

Since there is a contradiction when it is assumed that the hypothesis is false, by proof by contradiction there must not exist integers x, y, and z such that $x^2 + y^2 = 3z^2$ that is not $x = y = z = 0$.

5 Problem 5

Let r be a number such that $r + \frac{1}{r}$ is an integer. Use induction to show that for every positive integer n, $r^n + \frac{1}{r^n}$ is an integer.

PROOF:

Basic Cases

$n = 0$: $r^0 + \frac{1}{r^0} = 1 + \frac{1}{1} = 1 + 1 = 2$ and 2 is an integer.

$n = 1$: It is given to us that $r^1 + \frac{1}{r^1} = r + \frac{1}{r}$ is an integer.

Induction Hypothesis

$\exists kn \in \mathbb{N}$ such that $\forall k \leq n$, $r^k + \frac{1}{r^k}$ is an integer.

Inductive Step

Lets look at $n + 1$: so we have $r^{n+1} + \frac{1}{r^{n+1}}$

First observe that $(r + \frac{1}{r}) * (r^n + \frac{1}{r^n}) = r^{n+1} + \frac{1}{r^{n-1}} + r^{n-1} + \frac{1}{r^{n+1}}$.

So $r^{n+1} + \frac{1}{r^{n+1}} = (r + \frac{1}{r}) * (r^n + \frac{1}{r^n}) - (dfrac1r^{n-1} + r^{n-1})$.

By the induction hypothesis we see that $(r^n + \frac{1}{r^n})$ is an integer

We also are give that $r + \frac{1}{r}$ is an integer.

So $(r + \frac{1}{r}) * (r^n + \frac{1}{r^n})$ is the product of two integers, so it is also an integer.

Also by the induction hypothesis we get that $(dfrac1r^{n-1} + r^{n-1})$ is an integer

So $(r + \frac{1}{r}) * (r^n + \frac{1}{r^n}) - (dfrac1r^{n-1} + r^{n-1})$ is the difference between two integers, so it is also an integer.

Since $(r + \frac{1}{r}) * (r^n + \frac{1}{r^n}) - (dfrac1r^{n-1} + r^{n-1})$ is an integer and since $r^{n+1} + \frac{1}{r^{n+1}} = (r + \frac{1}{r}) * (r^n + \frac{1}{r^n}) - (dfrac1r^{n-1} + r^{n-1})$, $r^{n+1} + \frac{1}{r^{n+1}}$ is an integer.

So by the law of strong induction, $\forall x \in \mathbb{N}$, $r^x + \frac{1}{r^x}$ is an integer.

Since the set of positive integers is a subset of the natural numbers, $\forall n \in \mathbb{N}$ such that $n > 0$, $r^n + \frac{1}{r^n}$ is an integer.