Homework #4
Automata and Computation Theory
Fall 2018

Written by Eric Rothman

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Problem 1

Prove the following languages are not regular:

a

$A = \{0^n 1^m 0^{n+m} | m, n \geq 0\}$

Proof:

FSOC assume A is regular. Then $\exists$ a number p such that the pumping lemma holds.

Let $S = 0^p 1^p 0^{2p}$. This string has the length of $4p > p$, so the pumping lemma holds here.

So $S = xyz$.

Since $|xy| \leq p$ and the first p values of S are all 0, y contains only 0's.

So WLOG lets say y is of length $b > 0$, by the second condition.

So y is a string of b0's.

Now lets look at xy^2z.

xy^2z would be a string of the structure $0^{p+b}1^p0^{2p}$.

Since $b > 0$, $2p \neq p + b + p = 2p + b$.

So the sum of the lengths of the first two segmants, the first 0 segment and the 1 segmant, is not equal to the length of the third segmant.

Since xy^2z is not of the form $0^n 1^m 0^{n+m}$, $xy^2z \notin A$.

This is a contradiction with the pumping lemma.

Since there is a contradiction, the assumption must be false.
So A is not regular.

b

$B = \{w | w \in \{0,1\}^*$ is not a palindrome $\}$
Proof:

FSOC let B be regular
Since B is a regular language and the set of regular languages is closed under complement, $\overline{B}$ is a regular language.
Since $\overline{B}$ is a regular language, $\{w | w \in \{0,1\}^*$ is a palindrome $\}$ is a regular language.
This contradicts what we proved in class.

To cover my bases though, I will prove again that $\overline{B} = \{w | w \in \{0,1\}^*$ is a palindrome $\}$ is not a regular language.
FSOC let let $\overline{B}$ be a regular language. So $\exists p$ such that the pumping lemma holds.

Let $S = 0^p 1 0^p$, $S \in \overline{B}$ $|S| = 2p + 1 > p$.
So $S = xyz$, since $|xy| \leq p$, $y = o^t$ for some $t > 0$.
So $xy^2z = 0^{p+t}10^p \notin \overline{B}$
That contradicts with the pumping lemma, so $\overline{B}$ is not regular.

Since there is a contradiction, the assumption must be false.
So B is not a regular language.

Problem 2

a

Let $B = \{1^k y | y \in \{0,1\}^* \text{ and } y \text{ contains at least } k \text{ 1s, for any } k \geq 1\}$. Is B a regular language? Prove your answer.

B is not a regular language.
Proof:

FSOC let B be a regular language. Than $\exists p$ such that the pumping lemma hold.
Let $S = 1^p 0 1^p$.
Since $p \geq 1$, and $y = 01^p$ contains at least p ones, $S \in B$.
$|S| = 2p + 1 > p$, so the pumping lemma holds for S.
So $S = xyz$.
Since $|xy| \leq p$ by the third condition, and the first p spots in S are 1, and $y > 1$ by the first condition $y = 1^t$ where $t \geq 1$.
Now examine $S_2 = xy^2 z$.

S_2 is of the form $1^{p+t}01^p$ since xy is within the first subset of 1's.
Since S_2 is of that form, it can be divided into two sections, 1^k where $k = p + t \geq 1$ and $y' = 01^p$.
The number of 1's in y' is $p < (p + t)$ since $t > 1$.
So since y' does not have at least $(p + t)$1's in it, $S_2 \notin B$.
This contradicts the pumping lemma, so the assumption must be false.
So B is not a regular language.

b

Let $C = \{1^k y | y \in \{0,1\}^* \text{ and } y \text{ contains at most } k \text{ 1s, for any } k \geq 1\}$. Is C a regular language? Prove your answer.

C is not a regular language.
Proof:

FSOC let C be a regular language So $\exists p$ such that the pumping lemma holds.
Let $S = 1^p 0 1^p$
Since $p \geq 1$, and $y = 01^p$ contains at most p ones, $S \in B$.
$|S| = 2p + 1 > p$, so the pumping lemma holds for S.
o $S = xyz$.

Since $|xy| \leq p$ by the third condition, and the first p spots in S are 1, and $y > 1$ by the first condition $y = 1^t$ where $t \geq 1$ and $t \leq p$.

Now examine $S_2 = xy^0z$.

S_2 is of the form $1^{p-t}01^p$ since xy is within the first subset of 1's.

Since S_2 is of that form, it can be divided into two sections, 1^k where $k = p - t \geq 0$ and $y' = 01^p$.

The number of 1's in y' is $p > (p - t)$ since $t > 1$.

So since y' has more than $(p - t)$1's in it, $S_2 \notin C$.

This contradicts the pumping lemma, so the assumption must be false.

So C is not a regular language.

Problem 3

Consider the language $F = \left\{a^i b^j c^k | i, j, k \geq 0 \text{ and if } i = 1 \text{ then } j = k\right\}$, where $\Sigma = \{a, b, c\}$.

a

Show that F is not regular.

As proven in HW 2, regular languages are closed under reverse, so if and only if F is regular, $F^R = \left\{c^k b^j a^i | i, j, k \geq 0 \text{ and if } i = 1 \text{ then } j = k\right\}$ must also be regular.

Now I'll prove F^R is not regular.

FSOC let assume F^R is regular, so $\exists p$ such that the pumping lemma holds.

let $S = c^p b^p a$. Since $i = 1$ and $j = p$ and $k = p$, so $j = k$, $S \in F^R$.

So $S = xyz$.

Since $|xy| < p$ by the third condition and the first p values of S is c, $y = c^t$ where $t \geq 1$ by the second condition.

Lets look at $S_2 = xy^2 z$

S_2 is of the form $c^{p+t} b^p a$.

Since $t > 1$, $p + t > p$, so $p + t \neq p$.

Since in S_2, $i = 1$, and $j \neq k$, $S_2 \notin F^R$.

This contradicts the pumping lemma, so the assumption must be false.

So F^R is not regular.

Since F^R is not regular and the set of regular languages is closed under reverse, $(F^R)^R = F$ is not regular.

So F is not regular.

b

Show that F acts like a regular language in the pumping lemma.

Let the pumping length $p = 2$.

For all $S \in F$ such that length of $S \geq p$, there are four cases.

Case 1

$i = 0$.

So $a \notin S$ at any spot. So $a \notin y$ at any spot of the subset. So no matter the value of $k \geq 1$, there are 0 occurances of a in $xy^k z$. Since $i = 0$ for every value of k, $xy^k z \in F$ always.

NOTE: If $j = 1$, then y could be b or c, and no matter what the power it is raised to, the string will still be of the form $b^x c^z$ where $x, z \geq 0$. If $j \neq 1$ and $i = 0$, then no matter the y, as long as it matches the conditions of $|y| > 0$ and $|xy| \leq 2$ it will be in F since no matter the k, y will either contain only b's or c's, but never a mix.

There will always be a b or c to choose to be F since the length of S is greater or equal to 2 and there are 0 a's in S.

Case 2

$i = 1$

Then let $y = a$ which is the first spot in the string. $|y| > 1$ since it is not the empty string, and $|xy| \leq 2$ since $|x| = 0$ and $|y| = 1$. So $|xy| = 1 < 2$. Since $S \in F$, and $i = 1$ $z = b^n c^n$ where $n \geq 0$.

if $k = 1$, then $a \neq 1$ and $xy^k z = x^i y^j z^k$ where $i, j, k \geq 0$, so $xy^k z \in F$

if $k = 1$, then we have xyz which is assumed to be in F.

So no matter the value of $k \geq 0$, $xy^k z \in F$.

Case 3

$i = 2$

Let $y = aa$, which is the first two spots in S and the only a's in S. $|y| = 2 > 1$, $|xy| = 2 \leq 2$ since $|x| = 0$ and $y = aa$.

Since y is the first two spots in S and the only a's in S, no matter what $k \geq 0$ is $y^k \neq a$. So no matter the value of k, $i \neq 1$.

Since $i \neq 1$, and $xy^k z = a^i b^j c^k$ where $i, j, k \geq 0$, $xy^k z \in F$

Case 4

$i > 2$

Let $y = a$, which is the first spot in S and the first of more than 2 a's.

$|y| = 1$, and $|xy| = 1 < 2$ since $|x| = 0$ and $y = a$.

Then if $k = 0$, $i \geq 2$ since $i \geq 2$ when $k = 1$ and since $|y| = 1$.

If $k > 0$, $i > 2$ because the number of a's is not getting smaller.

So no matter the value of k, $i > 1$, so $i \neq 1$.

So $xy^k z = a^i b^j c^k$ where $i, j, k \geq 0$ and $i \neq 1$, so $xy^k z \in F$ since it is of the proper form and matches the condition of the statement.

So no matter the case, $\forall S \in F$ such that $|S| \geq 2$, all three conditions are met.

So F acts like the pumping lemma.

c

Explain why parts (a) and (b) do not contradict the pumping lemma.

They do not contradict the pumping lemma because the pumping lemma is defined only one way.

It is defined as if A then B.

But since A, that F is a regular language, is not fulfiled as of part a, the lemma makes no assumptions about whether or not b is true.

So despite the fact that A is false and B is true, that does not violate the binary logic of the if then statements.

If the pumping lemma was if and only if, then parts (a) and (b) would violate the pumping lemma, but that is not defined as such.

Problem 4

Give a context-free grammar that generates the following language, where the alphabet Σ is $\{0, 1\}$: $\{w|w \text{ is not empty and starts and ends with the same symbol}\}$.

G:

$S --> 0A0|1A1|0|1$

$A --> 0A|1A|\epsilon$

NOTE: start variable is S.