# Four estimators for shifted death densities

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## 1 The model

Consider two subgroups of the population, which for convenience we will call "men" and "women" and denote via the subscripts $m$ and $w$. Assume we observe the counts of deaths for these two subgroups between ages $\alpha$ and $\beta$, but we do not know the relative sizes of these two subgroups. Let $a$ denote age, and let $f_m(a)$ and $f_w(a)$ denote the true but unobserved density of deaths for our two subgroups.

Our model: we assume that

$$f_m(a) = f_w(a + S)$$

where $S$ is some unknown constant. It follows from this assumption that the mean-ages-at-death for these two groups differ by the unknown $S$. Our goal is to recover $S$ from the observed counts of deaths.

## 2 Notation

Let $v_m(a)$ and $v_w(a)$ be the conditional densities for men and women on the observed interval of ages $\alpha$ to $\beta$, with observed means $\mu_m$ and $\mu_w$. $v'_m(a)$ and $v'_w(a)$ denote the derivatives of these functions. These derivatives, and some of quantities below, may be obtained by interpolating the conditional densities.

- $\Delta$ represents successive approximations of $S$. As a first approximation, set $\Delta = \mu_w - \mu_m$, where $\mu_w > \mu_m$. All of the following quantities depend on $\Delta$. In order to use the estimation formulae iteratively, these quantities should be coded as functions of $\Delta$, so that with each iteration an updated value for $\Delta$ is obtained.

- $\hat{\mu}_w = \int_{\alpha+\Delta}^{\beta} a v_w(a) da$ and $\hat{\mu}_m = \int_{\alpha}^{\beta-\Delta} a v_m(a) da$
- $\hat{\sigma}_w^2 = \int_{\alpha+\Delta}^{\beta} (a - \hat{\mu}_w)^2 v_w(a) da$ and $\hat{\sigma}_m^2 = \int_{\alpha}^{\beta-\Delta} (a - \hat{\mu}_m)^2 v_m(a) da$
- $Q = \int_{\alpha+\Delta}^{\beta} v_w(a) da$ and $\tilde{Q} = \int_{\alpha}^{\beta-\Delta} v_m(a) da$
- $R = \hat{\mu}_m + \Delta$ and $\tilde{R} = \hat{\mu}_w - \Delta$
- $N = ((\beta - R)^2 - \frac{\hat{\sigma}_m^2}{\tilde{Q}}) v_w(\beta) - ((\alpha - \hat{\mu}_m)^2 - \frac{\hat{\sigma}_m^2}{\tilde{Q}}) v_w(\alpha + \Delta) - 2\hat{\mu}_w + 2RQ$
- $D = ((\beta - R)^2 - \frac{\hat{\sigma}_m^2}{\tilde{Q}}) v'_w(\beta) - ((\alpha - \hat{\mu}_m)^2 - \frac{\hat{\sigma}_m^2}{\tilde{Q}}) v'_w(\alpha + \Delta) - 2(\beta - R) v_w(\beta) + 2(\alpha - \hat{\mu}_m) v_w(\alpha + \Delta) + 2Q$
- $\tilde{N} = ((\beta - \hat{\mu}_w)^2 - \frac{\hat{\sigma}_w^2}{Q}) v_m(\beta - \Delta) - ((\alpha - \tilde{R})^2 - \frac{\hat{\sigma}_w^2}{Q}) v_m(\alpha) - 2\hat{\mu}_m + 2\tilde{R}\tilde{Q}$
- $\tilde{D} = ((\beta - \hat{\mu}_w)^2 - \frac{\hat{\sigma}_w^2}{Q}) v'_m(\beta - \Delta) - ((\alpha - \tilde{R})^2 - \frac{\hat{\sigma}_w^2}{Q}) v'_m(\alpha) - 2(\beta - \hat{\mu}_w) v_m(\beta - \Delta) + 2(\alpha - \tilde{R}) v_m(\alpha) + 2\tilde{Q}$

# 3 The Formulae

## 3.1 Estimator 1

We approximate $S$ by

$$\Delta - \frac{N}{D} - \frac{\sqrt{(N^2 - 2D(\hat{\sigma}_w^2 + 2\hat{\mu}_w^2 - Q\hat{\mu}_w^2 - 2R\hat{\mu}_w + QR^2 - \frac{Q}{\tilde{Q}}\hat{\sigma}_m^2)}}{D}$$

## 3.2 Estimator 2

We approximate $S$ by

$$\Delta + \frac{\tilde{N}}{\tilde{D}} + \frac{\sqrt{\tilde{N}^2 - 2\tilde{D}(\hat{\sigma}_m^2 + 2\hat{\mu}_m^2 - \tilde{Q}\hat{\mu}_m^2 - 2\tilde{R}\hat{\mu}_m + \tilde{Q}\tilde{R}^2 - \frac{\tilde{Q}}{Q}\hat{\sigma}_w^2)}}{\tilde{D}}$$

### 3.3 Estimator 3

We approximate $S$ by

$$\Delta - \frac{\hat{\sigma}_w^2 + 2\hat{\mu}_w^2 - Q\hat{\mu}_w^2 - 2R\hat{\mu}_w + QR^2 - \frac{Q}{\tilde{Q}}\hat{\sigma}_m^2}{N}.$$

### 3.4 Estimator 4

We approximate $S$ by

$$\Delta + \frac{\hat{\sigma}_m^2 + 2\hat{\mu}_m^2 - \tilde{Q}\hat{\mu}_m^2 - 2\tilde{R}\hat{\mu}_m + \tilde{Q}\tilde{R}^2 - \frac{\tilde{Q}}{Q}\hat{\sigma}_w^2}{\tilde{N}}.$$

## 4 Implementation Issues

These formulae are decorated with subscripts, tildes, hats, and other easily overlooked marks. I found typos hard to avoid when coding the formulae. (And I hope there are no typos in this write-up!)

If the data does not fit the model, estimators 1 or 2 have the potential to return non-real values. All four estimators exhibit some instability with noisy data. It's possible for them to return unrealistically large or unrealistically negative values, and at that point I don't believe iteration will return the estimators to plausible values. Some smoothing of the data might reduce this instability.