CS543 Final Project Report

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1 Introduction

Testing whether an unknown discrete distribution is uniform is a fundamental problem in theoretical computer science and statistics, with broad applications in randomness testing, data validation, and property testing. Given a distribution over a finite domain of size m, the task is to distinguish whether the distribution is uniform or ϵ-far from uniform in total variation distance, using as few independent samples as possible.

Two classical approaches to this problem are the collision-based uniformity test (the one we covered in class) [1] and singleton-based uniformity test (as introduced by Paninski) [3]. The collision-based test relies on counting the number of collisions— pairs of identical samples—observed in the sample set. This test is motivated by the fact that the collision probability is minimized under the uniform distribution. On the other hand, Paninski's singleton-based test uses the number of elements that appear exactly once in the sample set (singletons), capitalizing on the idea that the frequency of such rare events varies significantly between uniform and non-uniform distributions.

While both tests are conceptually simple and practically appealing, they differ in their theoretical guarantees. Early analyses suggested that the collision-based test required $O(\frac{\sqrt{n}}{\epsilon^4})$ samples to reliably detect non-uniformity, whereas Paninski's singleton-based test achieved an improved sample complexity of $O(\frac{\sqrt{n}}{\epsilon^2})$, aligning with known lower bounds for uniformity testing. This led to a common belief that collision-based testers were suboptimal.

However, recent results challenge this view. In particular, the paper "Collision-based Testers are Optimal for Uniformity and Closeness" [2] demonstrates that with careful tuning and a more refined analysis, the collision-based test can in fact achieve the optimal sample complexity of $O(\frac{\sqrt{n}}{\epsilon^2})$. This finding bridges a gap between the practical simplicity of the collision method and its theoretical performance, and invites a re-evaluation of its utility compared to the singleton-based approach.

In this project, I aim to explore both the theoretical derivations and empirical behavior of these two uniformity tests. Specifically, I will:

- Analyze the mathematical foundations that lead to the sample complexity bounds for both testers.
- Investigate the assumptions, variance bounds, and concentration inequalities that underpin these results.
- Empirically compare the sample efficiency of both tests across a variety of distributional shifts
- Evaluate whether the theoretical improvements suggested by recent work on collision-based testers manifest in practice.

2 Theoretical Derivations

Note: The derivations below are simplified versions of the results mentioned in the papers.

High-Level Idea

All tests aim to solve the uniformity testing problem:
Given i.i.d. samples from an unknown distribution D over a domain of size n, distinguish:

- Null Hypothesis (H_0): $D = U_{[n]}$ (the uniform distribution)
- Alternative Hypothesis (H_1): $||D - U_{[n]}||_1 \geq \epsilon$

Each test uses a statistic computed from the sample, and decides YES (uniform) or NO (non-uniform) depending on whether the statistic deviates too much from what we'd expect under uniformity.

2.1 Collision-Based Uniformity Test

I used the derivations showed in lecture. [1]

Let D be the unknown distribution that we are testing its uniformity.
Let $U_{[n]}$ be the uniform distribution over $[n] = \{1, 2, ..., n\}$.

Then, $||D||_2^2$ would be the probability of collision for the unknown distribution and $||U_{[n]}||_2^2$ would be the collision probability of the uniform distribution.

It is known that the uniform distribution's collision probability is:

$$||U_{[n]}||_2^2 = \frac{1}{n}$$

In class we found that,

$$||D||_2^2 \geq \frac{4\epsilon^2+1}{n}$$

Now, what we want is to find $||D||_2^2$ so that we can argue whether D looks more like uniform distribution or not.

We can argue that if $||D||_2^2$ is close to $\frac{1}{n}$ then it is a uniform distribution. And, we can argue the opposite if it is far from $\frac{1}{n}$.

To write this mathematically, we can find a threshold by using the two lower bounds of probability of collision in the two cases. The threshold is the middle of the two lower bounds:

$$t = \frac{1}{2}\left(\frac{1}{n} + \frac{4\epsilon^2+1}{n}\right) = \frac{1+2\epsilon^2}{n}$$

.

So the algorithm is:
IF $||D||_2^2 \geq \frac{1+2\epsilon^2}{n}$, OUTPUT NO (D is not uniform)
ELSE, OUTPUT YES (D is uniform)

But, we need to find $||D||_2^2$ in order to perform this test. The Straightforward way is to keep drawing pairs and see the number of collisions you get in them. But, this way requires $\Omega(n)$ samples.

The better way that is proposed in class is: Draw S samples, $X_1, ..., X_S$ and use the number of collisions on all pairs:

$$||D||_2^2 => \frac{number\ of\ collisions\ in\ S\ samples}{\binom{S}{2}}$$

So, our test statistics is

$$number\ of\ collisions\ in\ S\ samples = Y = \sum_{i<j} Y_{i,j}$$

$$where\ Y_{i,j} = \begin{cases} 1, & if\ X_i = X_j \\ 0, & otherwise \end{cases}$$

So the algorithm becomes:
Draw S samples from D
Set $t = \binom{S}{2}\frac{1+2\epsilon^2}{n}$
Find Y

IF $Y \geq t$, OUTPUT NO (D is not uniform)
ELSE, OUTPUT YES (D is uniform)

Now, we need to find S- the number of samples required.

Finding S:
Let, $\hat{c} = \frac{Y}{\binom{S}{2}}$ be the estimator for $||D||_2^2$.

We want our estimator $\hat{c}$ to be close to $||D||_2^2$ with high constant probability. Given that

$$E(\hat{c}) = ||D||_2^2$$

We can use the Chebyshev's Inequality to find a bound for S:

$$Pr(|\hat{c} - E(\hat{c})| \geq \alpha) \leq \frac{Var(\hat{c})}{\alpha^2}$$

We don't want the error to be bigger than the gap $\triangle = \frac{1+4\epsilon^2}{n} - \frac{1}{n} = \frac{4\epsilon^2}{n}$

Setting $\alpha = \frac{\triangle}{2} = \frac{2\epsilon^2}{n}$

The Chebyshev's Inequality becomes:

$$Pr(|\hat{c} - E(\hat{c})| \geq \frac{2\epsilon^2}{n}) \leq \frac{4Var(\hat{c})}{\frac{16\epsilon^4}{n^2}} = 4Var(\hat{c})\frac{n^2}{16\epsilon^4}$$

Under uniformity the upper bound becomes,

$$O(\frac{1}{S}\sqrt{||D||_2^2}) = O(\frac{1}{S\sqrt{n}})$$

Under Non-Uniformity, we infer that,

$$S \geq \frac{C\sqrt{n}}{\epsilon^4}$$

So, it suffices to get $S = \left\lceil \frac{C\sqrt{n}}{\epsilon^4} \right\rceil$
We were able to have this conclusion by using the following lemma,

$$Var(Y) \leq 7(\binom{S}{2}||D||_2^2)^{3/2}$$

2.2 Optimal Collision-Based Uniformity Test [2]

Note: the variables used in this paper is not the same with the ones we used in class. So, I am converting the variables in the paper to keep things consistent.

The test analyzed in the paper is the same: collision-based uniformity test. But, they differ in the analysis of the test statistic, threshold, and variance bound, which directly affects the sample complexity.

The estimate for $||D||_2^2$ used is the same: $\hat{c} = \frac{Y}{\binom{S}{2}}$. So, the test statistic is Y

Recall, that we used the the bounds of $||D||_2^2$ under completeness (near-uniform) and soundness ($\epsilon - far$).

In this paper they used a slightly different bounds for $||D||_2^2$:

Under completeness:

$$||D||_2^2 \leq \frac{1 + \epsilon^2/2}{n}$$

Under soundness:

$$||D||_2^2 \geq \frac{1 + \epsilon^2}{n}$$

As we did before, to set the threshold we will get the middle point:

$$t = \frac{1}{2}(\frac{1+\epsilon^2}{n}) = \frac{1+\epsilon^2/2}{n} = \frac{1+3\epsilon^2/4}{n}$$

.

Finally,

$$t = \binom{S}{2}\frac{1+3\epsilon^2/4}{n}$$

In an attempt to find a bound for S, we are going to use the Chebyshev's Inequality:

$$Pr(|Y - E(Y)| \geq k\sigma) \leq \frac{1}{k^2}$$

where σ is $\sqrt{Var(Y)}$

The paper says that we want S to be closer to its expected value then threshold is to its expected value. This implies:

$$|Y - E(Y)| \leq |t - E(Y)|$$

So, the error happens if:

$$|Y - E(Y)| \geq |t - E(Y)|$$

Using Chebyshev's Inequality:

$$Pr(||Y - E(Y)| \geq |t - E(Y)|) \leq \frac{Var(Y)}{(t - E(Y))^2}$$

The authors said that we want this error probability to be at most $\frac{1}{4}$

$$\frac{Var(Y)}{(t - E(Y))^2} \leq \frac{1}{4}$$

$$\sqrt{4Var(Y)} \leq \sqrt{(t - E(Y))^2}$$

$$|t - E(Y)| \geq 2\sigma$$

Using lemma 2: $E(Y) = \binom{S}{2}||D||_2^2$

$$|t - E(Y)| = |E(Y) - t| = |\binom{S}{2}||D||_2^2 - (1 + 3\epsilon^2/4)/4| = \binom{S}{2}|\alpha - 3\epsilon^2/4|/n$$

$$\binom{S}{2}|\alpha - 3\epsilon^2/4|/n \geq 2\sigma$$

It suffices for the number of samples S to satisfy the slightly stronger condition that

$$\sigma \leq S^2 \frac{|\alpha - 3\epsilon^2/4|}{5n}$$

After isolating S,

$$S \geq \sqrt{\frac{5\sigma n}{|\alpha - 3\epsilon^2/4|}}$$

In the paper this inequality is denoted as lemma 4.

But, in order to find the exact lower bound for S, we need to find what σ is. As in the previous case, the bounds for σ change depending on the uniformity of D.

Let's take a look at the lemma 3, to find the upper bound of $Var(Y)$:

$$Var(Y) \leq S^2||D||_2^2 + S^3(||D||_3^3 - ||D||_2^4)$$

This new upper bound opens the path to find the optimal solution.

In the completeness case, $||D||_2^2 = \frac{1}{n}$ and $||D||_3^3 = ||D||_2^4 = \frac{1}{n^2}$
By plugging these into lemma 3, we get:

$$\sigma \leq \frac{S}{\sqrt{n}}$$

We also know that $\alpha = 0$, when D is uniform. Substituting these two facts into lemma 4, we get:

$$S \leq \frac{6\sqrt{n}}{\epsilon^2}$$

Now, let's take a look at the soundness case. In this case, since we don't know what D is, we don't know which term dominates the the upper bound of the $Var(Y)$. Recall that we have a quadratic and a cubic term of S in the upper bound:

$$Var(Y) \leq S^2||D||_2^2 + S^3(||D||_3^3 - ||D||_2^4)$$

Let's see what happens when the quadratic term $(S^2||D||_2^2)$ dominates the upper bound. For the sake of brewity of the report, I will skip the proof and just mention the result. Lemma 6 takes care of this case:

We know that $||D||_2^2 = \frac{1+\alpha}{n}$, for $\alpha \geq \epsilon^2$. If $S^2||D||_2^2$ dominates the upper bound, it implies $S^2||D||_2^2 \geq S^3(||D||_3^3 - ||D||_2^4))$. Substituting these two facts into lemma 4, we get:

$$S \leq \frac{48\sqrt{n}}{\epsilon^2}$$

Now, let's see what happens when the cubic term $(S^3(||D||_3^3 - ||D||_2^4))$ dominates the upper bound. For the sake of brewity of the report, I will skip the proof and just mention the result. Lemma 7 takes care of this case:

Again, we know that $||D||_2^2 = \frac{1+\alpha}{n}$, for $\alpha \geq \epsilon^2$. If $S^3(||D||_3^3 - ||D||_2^4))$ dominates the upper bound, it implies $S^2||D||_2^2 \leq S^3(||D||_3^3 - ||D||_2^4))$. Substituting these two facts into lemma 4, this time we get:

$$S \leq \frac{3200\sqrt{n}}{\epsilon^2}$$

Conclusion: As can be seen in both cases, completeness and soundness, S is upper bounded similarly:

Let's take a look at them at the same time. In the completeness case:

$$S \leq \frac{6\sqrt{n}}{\epsilon^2}$$

In the soundness case, we have two different bounds depending of the domination of different terms:

$$S \leq \frac{48\sqrt{n}}{\epsilon^2}$$

and

$$S \leq \frac{3200\sqrt{n}}{\epsilon^2}$$

Looking at these three upper bounds, we can safely infer that

$$S = O(\frac{\sqrt{n}}{\epsilon^2})$$

Since we all know how to set S and t and we already know how to find Y ($\sum_{i<j} Y_{i,j}$) we can construct the algorithm:

Algorithm:
Draw S samples from D
Set $t = \binom{S}{2}\frac{1+3\epsilon^2/4}{n}$
Find Y
IF $Y \geq t$, OUTPUT NO (D is not uniform)
ELSE, OUTPUT YES (D is uniform)

2.3 Paninski's Singleton-Based Uniformity Test [3]

Before talking about the details of Paninski's singleton-based uniformity test, I want to highlight two main different approaches that Paninski accomplished compared to collision-based uniformity test:

Firstly, he used a different type of test statistic. Instead of relying on second-moment statistics (like $||D||_2^2$ from collisions), he used the number of elements seen exactly once. These elements are called as singletons.

Secondly, he analyzed the test statistic via Poisson approximation. He showed that under sparse sampling (i.e., $S << n$), the counts X_i can be approximated as Poisson random variables. This simplifies the variance analysis of the singleton count S.

I will show the details of these in this section.

In his paper, Paninski started by mentioning the relation that as we observe more collisions, the distribution, D, more tends to be non-uniform. Based on this fact, he defines

$$K_1 = number\ of\ X_i's\ that\ are\ observed\ exactly\ once$$

or, equivalently,

$$K_1 = number\ of\ singletons$$

In contrast, as we observe more collisions, the value of K_1 tends to drop.

He defines a test statistic based on the following difference under the sparse regime ($S << n$):

$$T = E_U(K_1) - K_1$$

where $E_U(K_1) = expected\, number\, of\, singletons\, under\, uniform\, distribution$

The key idea is that under null hypothesis K_1 is expected to be high. But, under the alternative hypothesis, K_1 is expected to be low. s

Now, let's talk about how we set the threshold, t.

Looking at lemma 1,

$$E_U(K_1) - E_D(K_1) \geq \frac{S^2\epsilon^2}{n}(1 + O(\frac{S}{n}))$$

where $E_U(K_1) = S(\frac{n-1}{n})^{S-1}$ and $E_D(K_1) = \sum_{i=1} \binom{S}{1} D_i(1 - D_i)^{S-1}$

In order to find the upper bound of the variance let's take a look the lemma 2:

$$Var_D(K_1) \leq E_U(K_1) - E_D(K_1) + O(\frac{S^2}{n})$$

due to Efron-Stein Inequality

To find the threshold, we can find the middle point of the gap (difference between $E_U(K_1)$ *and* $E_D(K_1)$), defined in lemma 1.

If $n >> S$, then $E_U(K_1) - E_D(K_1) \sim \frac{S^2\epsilon^2}{n}$

So,

$$t = \frac{1}{2}\frac{S^2\epsilon^2}{n} = \frac{S^2\epsilon^2}{2n}$$

As we found the threshold, now we can construct a test for the null hypothesis:

If,

$$T \equiv E_U(K_1) - K_1 = S(\frac{n-1}{n})^{S-1} - K_1 > t$$

then reject H_0

Let's define the error: $|T - E_U(T)| \geq t$. Then, by Chebyshev's Inequality,

$$Pr_U(|T - E_U(T)| \geq t) \leq \frac{Var(T)}{t^2}$$

where $E_U(T) = 0$ and $Var_U(T) = O(\frac{S^2}{n})$ by lemma 2.
Equivalently,

$$Pr_U(T \geq t) \leq \frac{Var(T)}{t^2}$$

If $n >> S$,

$$\frac{Var(T)}{t^2} \sim \frac{S^2/n}{(S^2\epsilon^2/n)^2} = \frac{n}{S^2\epsilon^4}$$

This is small if $S^2\epsilon^4 >> n$. In other words, $\frac{n}{S^2\epsilon^4} \to 0$ if $S^2\epsilon^4 >> n$.

Thus,

$$S^2\epsilon^4 >> n$$

$$\sqrt{S^2} >> \sqrt{\frac{n}{\epsilon^4}}$$

$$S >> \frac{\sqrt{n}}{\epsilon^2}$$

To reliably test uniformity vs. $\epsilon - far$ alternatives, it suffices to set

$$S = \frac{C\sqrt{n}}{\epsilon^2}$$

where C is a constant

Since we all know how to set S and t and we already know how to find K_1, we can construct the algorithm:

$$t = \frac{S^2\epsilon^2}{2n}$$

Algorithm:
Draw S samples from D
Set $t = \frac{S^2\epsilon^2}{2n}$
Compute $E_U(K_1) = S(\frac{n-1}{n})^{S-1}$
Compute test statistic: $T = E_U(K_1) - K_1$
Find Y

IF $Y \geq t$, OUTPUT NO (D is not uniform)
ELSE, OUTPUT YES (D is uniform)

3 Empirical Bounds

To complement the theoretical analysis, I conducted a series of experiments aimed at estimating empirical sample complexity bounds for both the collision-based and singleton-based uniformity tests. My goal was to identify the minimum number of samples required to reliably distinguish between uniform and non-uniform distributions across varying domain sizes and distances ϵ. In doing so, I implemented both testing procedures and systematically measured their empirical success rates, enabling a direct comparison between observed and theoretical sample complexities.

3.1 Methodology

In class, we saw that by setting the number of samples $S = \frac{C\sqrt{n}}{\epsilon^4}$ and choosing the threshold $t = \binom{S}{2}\frac{1+2\epsilon^2}{n}$, the collision-based tester can distinguish between uniform and non-uniform distributions with high constant probability, for some sufficiently large constant C. However, this approach requires a large number of samples, even when ϵ is not very small. Later work showed that the same test works with only $S = \frac{800\sqrt{n}}{\epsilon^2}$ samples (with a different threshold) to achieve 3/4 success probability, but this still involves a large constant, making the sample requirement impractical in many cases—especially when higher confidence is needed.

Given that the sample complexities derived from existing theoretical analyses are often too large for practical use, my goal is to use empirical methods to identify more realistic bounds. Specifically, I designed experiments to determine how many samples are needed to reliably distinguish a uniform distribution from a carefully constructed ϵ-far distribution that minimizes the expected number of collisions. These experiments aim to find practical thresholds that still maintain strong discriminatory power with fewer samples. [4]

In order to accomplish this, I started creating three different types of subroutines.

Subroutine 1: A subroutine generating samples from the uniform distribution on [n].

Subroutine 2: A subroutine generating samples from some distribution on [n] that is ϵ–far from uniform in total variation distance and minimizes the expected number of collisions. Let's denote this distribution D_{far1} [4]

Here is the construction for D_{far1}:

Partition [n] into two groups: A set of size ϵn. The complement [n]/S of size $(1-\epsilon)n$.

And then I defined D_{far1} as:

$$p_i = \begin{cases} \frac{1}{n} + \frac{2\epsilon}{n}, & if\ i \in S \\ \frac{1}{n} - \frac{2\epsilon}{n(1-\epsilon)}, & otherwise \end{cases}$$

Subroutine 3:

The ϵ-perturbed Bernoulli-hypercube distribution: Assume that n is even. We choose q randomly according to the following distribution $\mu(q)$: choose n/2 independent Bernoulli RVs $z_j \in -1, 1$ (i.e., z samples uniformly from the corners of the n/2-dimensional hypercube). Let's denote this distribution D_{far2} [2]

Given z_j set:

$$p_i = \begin{cases} \frac{(1+\epsilon z_{i/2})}{n}, & if\ i\ even \\ \frac{(1-\epsilon z_{(i+1)/2})}{n}, & otherwise \end{cases}$$

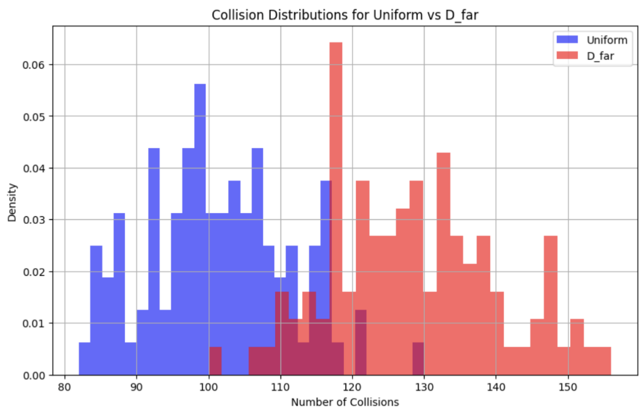
Procedure:

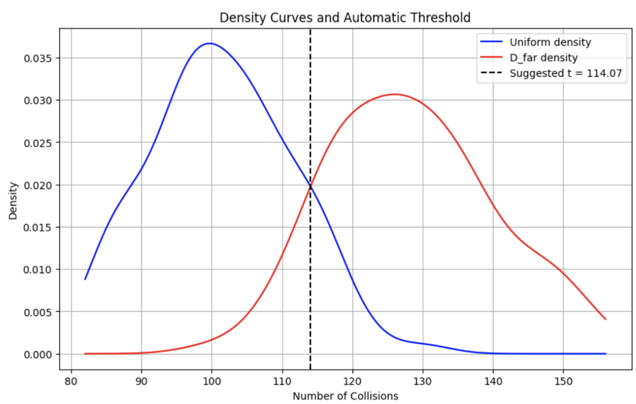
To find a suitable number of samples S and a threshold t that distinguish the uniform distribution from one that is ϵ-far in total variation distance, I used an empirical procedure built on carefully constructed mixtures of my subroutines. Specifically, I combined samples from Subroutine 1 (uniform distribution) with samples from either Subroutine 2 or Subroutine 3 to produce distributions that are guaranteed to be ϵ-far from uniform in l_1-distance. For each candidate sample size S, I repeatedly drew samples from both the uniform distribution and the constructed far-from-uniform distribution. By running many trials (well beyond $1/\delta$) and recording the number of observed collisions, I estimated the empirical distributions of the collision statistic under both cases. This allowed me to determine whether a given S could reliably distinguish the two distributions with high confidence.

Next, I search for a threshold t such that at least a 1-δ fraction of the uniform samples produce fewer than t collisions, and at least a 1-δ fraction of the non-uniform samples produce more than or equal to t collisions. If such a t exists, the current S is accepted; otherwise, I increase S slightly and repeat. This procedure yields empirical bounds on the number of samples needed to reliably distinguish the two distributions with confidence $\geq 1 - \delta$. (Please see the function $find\_s\_and\_t()$ $Uniformity_testing.ipynb$ and $Paninski\_Uniformity\_Test.ipynb$ for implementation.)

To support the empirical search for a distinguishing threshold, I implemented a kernel density estimation (KDE)–based method that automatically identifies a threshold t where the distribution of collision counts under the ϵ-far distribution begins to dominate that of the uniform distribution. This method smooths the empirical collision histograms, computes the point of intersection between the two density curves, and uses this crossover as a candidate threshold. The resulting plots visually highlight the separation between the two distributions and help verify whether a given sample size S is sufficient for reliable testing.

Here is an example:



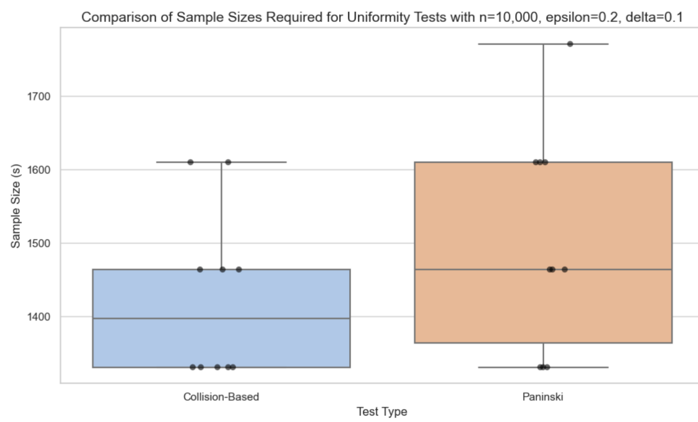


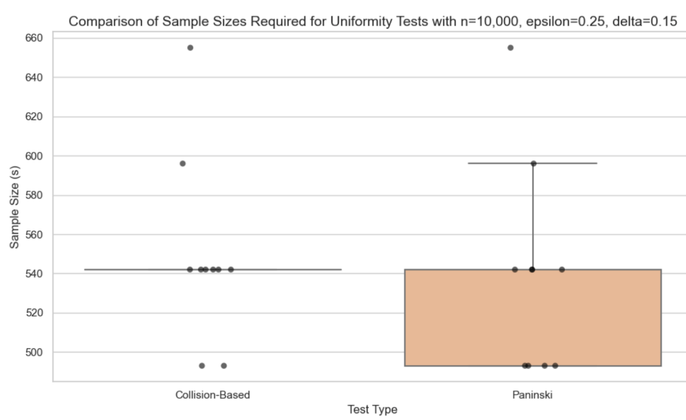
3.2 Results

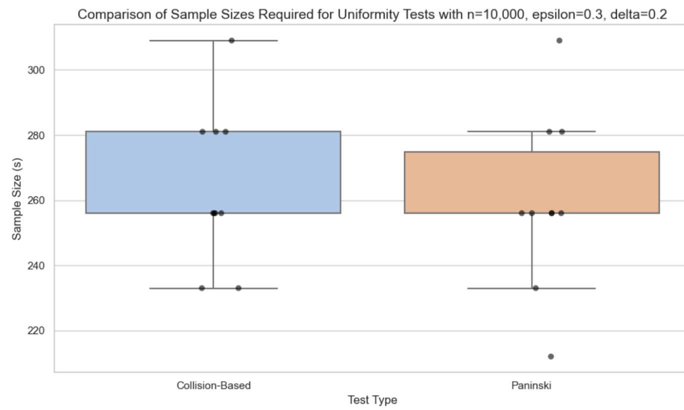
Results from Combination of Subroutines 1 and 2:

To evaluate and compare the empirical performance of the collision-based uniformity test and Paninski's singleton-based test, I conducted a series of experiments using the distribution constructed from the combination of Subroutines 1 and 2, which guarantees a total variation distance of at least ϵ from the uniform distribution. For each setting of n, ϵ, and δ, I applied both testing methods and recorded the resulting values of the required sample size s that achieved the desired confidence level 1-δ. These experiments were repeated across multiple ϵ and δ configurations to observe how the sample requirements of each test varied under different conditions.

(See results in the next page)







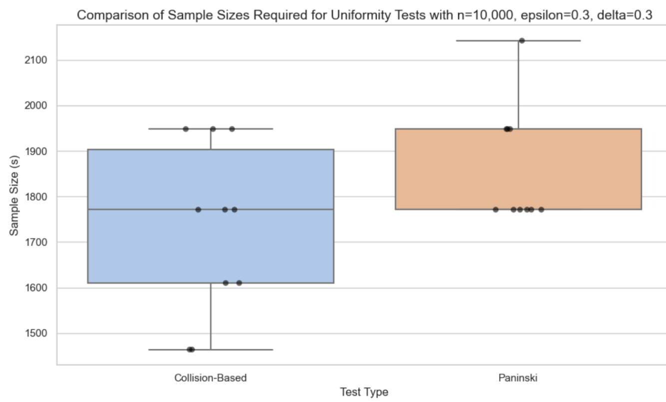
My Comments on the Results

Across the different settings of ϵ and δ, the results show that both the collision-based and Paninski's singleton-based tests generally yield comparable sample size requirements, though their fluctuations can differ depending on the distribution used. In scenarios with lower ϵ (e.g.,$\epsilon = 0.2$), both tests tend to require larger sample sizes, as expected, due to the increased difficulty in distinguishing distributions that are closer to uniform. As ϵ increases, the required sample sizes drop significantly for both methods. While the collision-based test occasionally produces more consistent values across trials, Paninski's test sometimes shows greater variability, especially when the number of singletons becomes sensitive to minor distributional shifts.

Results from Combination of Subroutines 1 and 3:

Same steps for this combination:

To further evaluate the robustness of both uniformity testing methods, I performed additional experiments using the combination of Subroutines 1 and 3, which generates a distribution that is ϵ-far from uniform via an ϵ-perturbed Bernoulli-hypercube construction. This structured deviation from uniformity introduces controlled randomness through sign-flipping in a high-dimensional hypercube, ensuring a guaranteed total variation distance while introducing a rich mixture of support variation. As before, I recorded the number of samples S required by each test to meet the confidence threshold $1 - \delta$ under fixed parameters n=10,000, ϵ= 0.3 and $\delta = 0.3$.



My Comments on the Results

In this setting, both the collision-based and singleton-based tests required similar sample sizes to achieve the desired confidence level. This aligns with earlier findings and reinforces the empirical observation that both tests, despite relying on different statistics, can perform comparably in practice. The sample counts for both tests mostly ranged between 1464 and 2142, reflecting the added complexity introduced by the hypercube-based distribution's structure. Notably, the Paninski test results showed less variability and tended to cluster around the higher end of the sample size spectrum, while the collision test exhibited slightly more spread. This suggests that the perturbation-based distribution D_{far2} may reduce the contrast in singleton frequency relative to uniform, making it harder for Paninski's method to detect divergence efficiently. In contrast, the collision-based test remains comparably stable, reinforcing its reliability across structurally different alternatives to uniformity.

4 Conclusion

This project explored both the theoretical foundations and empirical behavior of two classical uniformity testing methods: the collision-based test and Paninski's singleton-based test. Initially, I was under the impression—based on earlier analyses and classroom discussions—that the collision-based test was suboptimal compared to Paninski's method, requiring a larger number of samples due to its $O(\frac{\sqrt{n}}{\epsilon^4})$ complexity. Before preparing my presentation, I was not aware that this test could be made optimal. However, after the presentation, I read the paper "Collision-based Testers are Optimal for Uniformity and Closeness", which completely changed my understanding. The paper shows that with a refined variance analysis and improved threshold setting, the collision-based test can indeed achieve the optimal sample complexity of $O(\frac{\sqrt{n}}{\epsilon^2})$, matching that of Paninski's test.

This new perspective motivated a deeper empirical investigation. Initially, I had tested a single pair of ϵ and δ values for both subroutine combinations and found that Paninski's test appeared to require slightly fewer samples than the collision-based test. This led to a misleading impression that Paninski's test was empirically more efficient. However, after expanding the experiments across a broader set of ϵ and δ values, I observed that this was not consistently the case. In fact, the sample sizes required by both methods were generally similar, and their performance was closely aligned regardless of the underlying far-from-uniform distribution used.

5 References

Here are the lecture notes and papers I got reference from:

[1]: https://onak.pl/teaching/download/2024-spring-ds563/lecture19.pdf
[2]: https://arxiv.org/pdf/1611.03579
[3]: https://ieeexplore.ieee.org/document/4626074
[4]: https://onak.pl/teaching/download/2024-spring-ds563/hw6.pdf