Real Term Premia in Consumption-Based Models*

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Abstract

Can consumption-based mechanisms generate positive and time-varying real term premia as we see in the data? I show that only models with time-varying risk aversion or models with high consumption risk can produce these patterns. The latter explanation has not been analysed before with respect to real term premia, and it relies on a small group of investors exposed to high consumption risk. Additionally, it can give rise to a "consumption-based arbitrageur" story of term premia. In relation to preferences, I consider models with both time-separable and recursive utility functions. Specifically for recursive utility, I introduce a novel perturbation solution method in terms of the intertemporal elasticity of substitution. This approach has not been used before in such models, it is easy to implement, and it allows a wide range of values for the parameter of intertemporal elasticity of substitution.

JEL: C65, E43, G12

Keywords: term premia, consumption-based models, habit, long-run risk, limited arbitrage, high consumption volatility, recursive utility, solution methods

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1 Introduction

Risk-free bonds hold a central position in financial theory, and in practice government bonds hold a central position in financial markets. Yet we do not fully understand the associated risk premia, and how they are connected to the consumption of households. Understanding this connection is crucial for consumption-based asset pricing and also important for other fields. For instance, it would facilitate households' investment decisions, and it would lead to a better understanding of the effects of monetary policy which are associated with changes in prices of real bonds.

[Figure 1]

The focus of this paper is explaining real term premia, i.e., the risk premia of inflation-adjusted bonds over a specific holding period.[1] Thus, throughout this paper, terms like yields, returns, term premia etc. should be understood as referring to their real counterparts, unless otherwise specified. Term premia in the data are mostly positive and significantly time-varying (Abrahams, Adrian, Crump, Moench and Yu 2016; d 'Amico, Kim and Wei 2018; Pflueger and Viceira 2016). Estimates from d 'Amico *et al.* (2018) are shown in Figure 1. Given these main features, the question is whether they can be explained by economic theory. In this paper, I answer this question within consumption-based models, and to the best of my knowledge, I contribute to the literature in the following ways. Firstly, I provide explicit values of term premia as a function of the state of the economy for a large range of model variations. This is useful, because consumption-based models in the literature often focus on nominal term premia, and even when they focus on real term premia, explicit state-dependent term premia are rarely displayed. In addition, I am providing the code to reproduce the calculations, which can easily be adapted for other specifications or calibrations. Secondly, I am the first to analyse the explanation, according to which a small group of investors assuming large consumption risk is driving the behaviour of term premia. This is the first explanation of term premia based on the consumption process, and it can also be adapted to tell a " consumption based arbitrageur" story of term premia. The only alternative within the standard consumption-based framework

[1] Term premia reflect the expected difference in log return from holding long-term bonds compared to short-term bonds over the same time period. On the contrary, risk premia usually refer to the same difference in expected returns taken over a single period (or instantaneously in continuous time). The exact definition of term premia along with the exact definition of all terms in this paper can be found in Appendix A. Actual bonds may also have liquidity premia, which are deviations in the price of bonds due to their liquidity in the market. Liquidity premia are separate from term premia, or, in other words, my definition of term premia assumes that bonds are perfectly liquid.

involves models with time-varying risk aversion (for instance Wachter 2006), and I discuss the properties of both kinds of explanations, without advocating for one of them. Finally, apart from time-separable utility (TSU), my analysis also includes models with recursive utility (RU), and I contribute a novel perturbation method to easily and robustly solve such models. My perturbation method builds on the approach of Tsai and Wachter (2018). While they used an approximation to the value function that is constant in terms of the intertemporal elasticity of substitution (IES), and analytically correct only for IES equal to 1, I consider the full perturbation series in terms of the IES. This provides a global approximation in terms of the state variable of the economy that allows the easy solution of the model for most values of the IES that are economically interesting. It is also the first perturbation method in terms of the IES within recursive utility models.

Overall, the challenge of explaining positive and time-varying nominal term premia based on consumption-based models is referred to as the bond premium puzzle (Backus, Gregory and Zin 1989).[2] The source of the puzzle is that consumption-based mechanisms typically generate small, negative and often constant term premia, namely the exact opposite of what we see in the data. This is due to bond prices typically being counter-cyclical in these models, while consumption risk is relatively small and varies little with the business cycle. In addition, contrary to nominal term premia, it is not possible to explain real term premia by relying on an inflation premium, which arises due to risk associated with the inflation process.

In my analysis, I employ standard consumption-based asset pricing and I assume a single state variable following a stationary autoregressive process.[3] In such a setup, I show that only two kinds of models can independently generate mostly positive and time-varying term premia, without generating a short-term rate that is excessively volatile.[4] The first employs time-varying risk aversion, and notable examples include models with an external habit in the utility function. The second relies on high

[2]Nominal term premia have the same definition as real term premia with the underlying bonds not being inflation-adjusted.

[3]Thus, my analysis does not include models that are driven by higher order beliefs as in Angeletos, Collard and Dellas (2018). In addition, there is literature addressing the fact that macro processes could be more elaborate than steady state reverting autoregressive processes. For example, Bauer and Rudebusch (2020) decompose the nominal yield curve by taking into account long-run macroeconomic trends. Also, there is a long literature investigating the time-series properties of interest rates (a survey is provided by Neely, Rapach *et al.* 2008). It would be interesting for further research to expand my analysis, in order to include more elaborate processes for the state variable.

[4]In recent years the term premium has also moved to negative territory. In this paper, while one of my goals is the generation of a positive term premium, most of the emphasis is placed on generating a substantially time-varying term premium.

consumption risk, and to my knowledge the current paper is the first to show that such models generate positive, sizeable, and time-varying term premia similar to the data. Apart from high consumption risk, the mechanism also requires a negative correlation between consumption and the short-term interest rate, which implies that long-term bonds do not act as a hedge and that term premia are positive. While my paper focuses more on the properties of this latter mechanism, I do not argue that it is superior in explaining term premia compared to a model with time-varying risk aversion. On the contrary, the focus of the paper is to analyse both mechanisms and show that within this framework these are the only available mechanisms, able to produce the features of term premia.

Given that aggregate consumption is not risky enough, this high consumption risk explanation requires that the investors are a small part of the overall economy. Thus, the approach follows heterogeneous agent models, with a small group of investors having a different consumption process than the average in the economy. However, I do not examine a full heterogeneous agent model, as I restrict my analysis to marginal investors in the bond market. Furthermore, my approach does not follow the intermediary asset-pricing paradigm (He and Krishnamurthy 2013). In particular, despite the fact that a small group of investors is driving asset prices in the bond market, these investors do not act as intermediaries for the households, and the results do not stem from any intermediation constraints.

The high consumption risk approach can also be adapted to tell a "consumption-based arbitrageur story" of the term structure of interest rates related to Vayanos and Vila (2021). They explained the term structure of interest rates by a preferred habitat model, in which so-called arbitrageurs integrate the yield curve by taking advantage of differences in expected return between different maturities of bonds. While Vayanos and Vila (2021) associate their arbitrageurs with banks and/or hedge funds, they do not take a position whether natural persons could correspond to arbitrageurs.[5] The current paper shows that arbitrageurs can be modelled as consumers, who drive positive and time-varying term premia, as long as consumption risk is high.

The rest of the paper is organised as follows: In section 2, I provide more information regarding the literature on the bond premium puzzle. In section 3, I discuss interest rates in the data. In section 4, I present the setup that will allow me to price bonds in the context of TSU and RU. In section 5, I show and comment on the results. Finally, section 6 concludes.

[5]This could also be the case if arbitrageurs are investing on behalf of natural persons without significant intermediation distortions.

2 Literature on the Bond Premium Puzzle

While I analyse real term premia, the bond premium puzzle originally referred to nominal term premia.[6] One of the first papers to address this was Backus *et al.* (1989). Utilising a consumption-based asset-pricing model of an endowment economy, they discovered the model's inability to yield significant positive term premia. Subsequent studies by Donaldson, Johnsen and Mehra (1990) and Den Haan (1995) further indicated that standard real business cycle models also could not resolve the puzzle. Rudebusch and Swanson (2008) incorporated an external habit into DSGE models but found that the bond premium puzzle remains. Specifically, including a habit with non-flexible working hours can generate positive term premia, but at the cost of inducing volatile wages, prices and short-term interest rates. Duffee (2013) showed that basic properties of nominal yields cannot be explained macroeconomically, at least according to standard asset-pricing models. Also in a more generic contribution, Duffee (2002) shed light on the challenges of fitting both interest rate and term premium dynamics within affine models.

Next, a series of papers provided explanations that focused on nominal term premia, and not on real term premia. Notably, Piazzesi and Schneider (2006) showed that parameter uncertainty in a model where inflation brings bad news about future consumption growth can produce positive nominal term premia.[7] Gabaix (2012) and Tsai (2015), following Rietz (1988) and Barro (2006), showed that positive nominal term premia can be explained, if inflation is on average high during consumption disasters. Bansal and Shaliastovich (2013), following Bansal and Yaron (2004), demonstrated that the risk premium of a nominal bond can be positive in a model with long-run risk, as long as inflation is correlated with consumption trend. Rudebusch and Swanson (2012) used a similar model within a DSGE framework, which has real and nominal long-term risks, and they show that positive nominal term premia are generated; nevertheless real term premia are again negative in this model. Gomez-Cram and Yaron (2021) also used a model following Bansal and Yaron (2004), but they focused on explaining nominal term premia, using an inflation channel, while claiming that the apparent under-performance of their model with respect to real term premia should be expected due to liquidity premia in the TIPS market.

Alternatively, some articles also consider real term premia. For instance, Katagiri

[6] Rudebusch and Swanson (2008) also offers a good summary of this extensive literature.

[7] Collin-Dufresne, Johannes and Lochstoer (2016) introduces a model with bayesian learning of parameters. However, this model does not emphasise bond term premia and it generates *negative* term premia.

(2022) explored a model with monetary policy, in which consumption changes can be negatively correlated with consumption trend, and risk aversion is very high. As a result term premia can be positive, but the premia time variability is not examined. Ellison and Tischbirek (2021) went beyond standard rational expectations models by using a beauty contest mechanism as introduced by Angeletos *et al.* (2018), in which agents anticipate the expectations of other agents; their model generates positive term premia.

Using a similar approach to the current paper, some articles tackle the problem by deviating from the representative agent model. Vayanos and Vila (2021) suggested that term premia are generated by arbitrageurs interacting with so-called preferred habitat investors, namely investors that have a tendency to hold specific maturities of bonds. Kekre, Lenel and Mainardi (2022) built on Vayanos and Vila (2021), and showed that the characteristics of the arbitrageur portfolio can have important implications for the sign of term premia. Jappelli, Subrahmanyam and Pelizzon (2023) also built on Vayanos and Vila (2021) by integrating the repo market in their analysis. Schneider (2022) showed that positive term premia can arise in models with heterogeneous agents exhibiting different attitudes towards risk and different preferences to substituting consumption through time. Finally, returning to models with a representative agent, Wachter (2006) showed that term premia can be positive and time-varying, within a model with an external habit following Campbell and Cochrane (1999). Kliem and Meyer-Gohde (2022) used the same mechanism within a DSGE model, and they found positive term premia. Hsu, Li and Palomino (2021) also used this mechanism within a DSGE model, and they verified that a habit element is key in generating positive and time-varying term premia. Campbell, Pflueger and Viceira (2020) also used a habit model to explain the time-variability of term premia. More generally, a model with external habit can be classified as a model with time-varying effective risk aversion, and within this class of models, Lettau and Wachter (2011) showed that positive and time-varying term premia can be obtained, and Bekaert, Engstrom and Grenadier (2010) showed that time-varying term premia can be obtained. These papers all use time-varying risk aversion, which is to my knowledge the only mechanism in the literature that achieves positive and time-varying term premia within a rational representative agent model.[8]

[8]Yet, a utility with a time-varying degree of risk aversion may not be considered the most standard rational utility function.

3 Real Rates in the Data

3.1 TIPS as real rates

The first challenge regarding real rates is that they are not directly observable from standard bonds. The real interest rate is the yield of a nominal bond whose payoff is adjusted for inflation. So deducing real interest rates from nominal bonds requires at least the calculation of expected inflation, which is not trivial. The closest thing that we have in the data for real interest rates is inflation-adjusted government bonds. Such data are available for the UK and the US. In the UK, inflation-adjusted government bonds (inflation-adjusted GILTs) have been available since the 1980s. In the US, the corresponding securities are called TIPS (Treasury Inflation-Protected Securities) and corresponding price data are available for roughly twenty years (Gürkaynak, Sack and Wright 2010).[9] A severe limitation of TIPS is that they are not as liquid as normal US treasuries. For this reason, I focus on term premia measures produced by d 'Amico *et al.* (2018) who computed risk-neutral yields and term premia, after taking account of the liquidity premia of TIPS over normal US treasuries.[10] As can be seen in Figure 2, in some periods liquidity premia of TIPS are considerable. Nevertheless, as has been discussed already and shown in Figure 1, term premia are still significantly time-varying.

3.2 Real rates as a component of nominal rates

Figure 2 shows real yields at the top and nominal yields at the bottom. The graph reveals several key conclusions. Firstly, both nominal and real interest rates are time-varying. In addition, different maturities have different yields and the term structure seems to be upward-sloping in both cases. In other words, longer maturities are associated with higher yields. The slope of the term structure is also not constant, as the spread between yields of different maturities varies. Secondly, it is clear from Figure 2 that nominal rates are highly correlated with real rates. Considering the

[9] Gürkaynak *et al.* (2010) provides data starting from 1999. However, the full set of maturities is provided starting in 2002.

[10] Apart from liquidity issues related to TIPS, there is also a small concern (mostly with recently issued TIPS) that negative inflation is not correctly accounted for. This is because TIPS are guaranteed to pay investors at least the original principal value of the bond, even if the rate of inflation is negative. This makes inflation adjustment somewhat skewed. However, the effect will probably be small for securities that were issued several years prior, given that likely some inflation has already taken place and the probability that negative inflation will overcome it is small. Lastly, the accuracy of inflation adjustment can be debated, as the consumer price index might not capture the specific inflation concerns of investors.

Fisher equation:

$$y_t^{nom,m} \approx y_t^{real,m} + E[\pi_{t,t+m}] \tag{1}$$

where m denotes the maturity of the underlying bond. The nominal rate can be thought of as a composite rate that includes two separate components, the real rate and expected inflation.[11] Figure 2 also shows that real interest rates are a significant and non-trivial component of nominal interest rates. Namely, real rates are moving substantially and mostly in parallel to nominal rates. This means that in order to fully understand the movements in nominal rates, it is important to also consider the movements in real rates. In addition, models that seek to explain nominal rates solely or primarily based on processes related to inflation are not able to provide a comprehensive understanding of interest rates. This underscores the importance of finding models that can accurately describe real rates. In Appendix B, I statistically verify that the information contained in the movements of in real rates explains a large proportion of the variation of nominal rates.

[Figure 2]

3.3 Empirical evidence regarding term premia

Empirical research has predominantly focused on nominal bonds in relation to term premia, *return predictability*, the *expectations hypothesis* (EH), and *excess volatility*. Specifically, predictability in nominal rates has been found by Fama and Bliss (1987) and Singleton (1980), who showed that yield spreads can partially predict excess returns of bonds over extended periods. This implies both the existence of term premia, and that they are time-varying. In addition, this is equivalent to a violation of the EH, which has been verified by Cochrane and Piazzesi (2005) among others. The existence of excess volatility (Shiller 1979) also indicates time-varying term premia, because the excess volatility is evidence that changing economic conditions affect the value of long-term bonds beyond what can be explained by movements in expected short rates. Even though the literature has focused less on real term premia, relatively recent studies have concluded that real term premia are also positive and time-varying, after accounting for liquidity premia. In particular, Abrahams *et al.* (2016) estimated the five-to-ten year real forward term premium and find that it has ranged roughly from 0% to 4% between 2000 and 2014 (Figure 14 in Appendix E).[12] d 'Amico *et al.* (2018) find that the five-to-ten year forward term premium has ranged roughly from

[11] The Fisher equation can be made into an equality by adding an inflation risk premium.

[12] Shown in Figure 5 of Abrahams *et al.* (2016).

-0.5% to 4% between 1980 and 2022 (Figure 1).[13] Pflueger and Viceira (2016) have demonstrated the existence of predictability of real excess returns, also implying the existence of time-varying real term premia. The conclusion that real term premia are substantial and time-varying is significant, because it implies that the variability in nominal term premia is not exclusively (or even primarily) driven by inflation. Therefore, models cannot rely only on inflation to explain nominal term premia.

4 The Consumption-Based Framework

I adopt a consumption-based framework in continuous time, which can accommodate a range of model variations. The framework is built upon three main components from which everything else is derived: 1) an exogenous consumption process; 2) a utility specification; and 3) a process for the state variable. The state variable determines the state of the economy, and it is either connected with some component of the consumption process or with some components of the utility function. Specifically, in the variations in this paper, the state variable is either connected to consumption trend (otherwise referred to as consumption drift, CD), or connected to consumption volatility (CV), or connected to the external habit of the utility function. These three options in combination with different calibrations and utility specifications give rise to a long list of variations and interpretations. To keep things simple, I only use one state variable for each model variation. Utility will either be time-separable (TSU), or recursive (RU) following Duffie and Epstein (1992).

4.1 Naming the variations

As mentioned already, I analyse several model variations in the main text of this paper, and many more in Appendix F. While I explain the models variations in detail in Sections 4 and 5, for convenience, I also provide abbreviations for the model variations in the following table. I use these abbreviations throughout the paper:

[Table 1]

[13] For the period between 2000 and 2014, the results of d 'Amico *et al.* (2018) imply a somewhat smaller variability of term premia compared to the results of Abrahams *et al.* (2016).

4.2 Consumption process

Although consumption is often considered a fundamental choice variable for economic agents, it is assumed to be exogenous in this paper.[14] This approach is consistent with consumption having been decided at some earlier stage that is not explicitly modelled, and it significantly simplifies the analysis. In the most general form, consumption (C_t) follows the stochastic process expressed below:[15]

$$\mathrm{d}\log(C_t) = \mathrm{d}c_t = \mu_{ct}\mathrm{d}t + \sigma_{ct}\mathrm{d}W_{ct} \tag{2}$$

μ_{ct} denotes the CD at time t and σ_{ct} is the volatility coefficient of consumption growth at time t, which is multiplying the stochastic component $\mathrm{d}W_{ct}$.[16] In the remainder of the paper, CV refers to σ_{ct}.

4.3 Utility

Lifetime utility at time 0 takes the following form depending on the utility specification:

$$\underbrace{U_0 = E_0 \int_0^\infty e^{-\rho t} u(C_t, S_t)\mathrm{d}t,}_{\text{TSU}} \qquad \underbrace{V_0 = E_0 \int_0^\infty f(C_t, V_t)\mathrm{d}t}_{\text{RU}} \tag{3}$$

In both cases there is an infinite horizon, with ρ representing the time preference parameter. In the case of TSU, flow utility u depends on the consumption flow and potentially on the surplus consumption ratio S_t, which is connected to the external habit.[17] In the variations without habit, S_t is taken to be equal to 1. In the case of RU, the aggregator function f depends on the consumption flow and on the current lifetime utility V_t which in the context of RU is referred to as the value function. u

[14] This is a standard choice in this literature. See for example Campbell and Cochrane (1999) and Bansal and Yaron (2004).

[15] It should be noted that I use the same parameter symbols for all model variations, and they should be distinguished by context. For example, in TSU-CD μ_{ct} is time-varying and a function of the state variable, while in TSU-CV μ_{ct} is a constant. The same applies to the symbols: σ_{ct} and σ_{xt}.

[16] W_{ct} is a standard Wiener Process associated with consumption such that $W_{ct} - W_{cs} \sim \text{Normal}(0, s-t)$.

[17] In the habit model of Campbell and Cochrane (1999), which is followed here, this variable is actually equal to $(C_t^a - X_t)/C_t^a$, where X_t is the level of habit and C^a is aggregate consumption.

and f take the following form:

$$\underbrace{u(C,S)=\frac{(CS)^{1-\gamma}-1}{1-\gamma}}_{\text{TSU}},\qquad \underbrace{f(C,V)=\frac{\rho\,(1-\gamma)\,V}{1-1/\psi}\left(\left(\frac{C}{((1-\gamma)\,V)^{-1/(1-\gamma)}}\right)^{1-1/\psi}-1\right)}_{\text{RU}} \tag{4}$$

γ is the risk aversion parameter, and in the standard TSU case it is equal to relative risk aversion, which also equals the inverse of IES. ψ is the IES parameter in the RU case.[18]

4.4 State variable process

At time t, the state variable x_t follows the process:

$$\mathrm{d}x_t = -\log(\phi)(\mu_{x0} - x_t)\mathrm{d}t + \sigma_{xt}\mathrm{d}W_{xt} \tag{5}$$

This expression describes an autoregressive stochastic process that reverts to the steady state μ_{x0}.[19] The rate of reversion to the steady state is governed by ϕ, which is constrained to be between 0 and 1. Thus, $\log(\phi)$ is non-positive and it implies that when $x_t > \mu_x$ ($x_t < \mu_x$) the drift is downward-sloping (upward-sloping), always towards the steady state. $\mathrm{d}W_{xt}$ is also a standard Wiener process, and σ_{xt} is the volatility coefficient of the state variable and it is either a constant or it also depends on x_t. $\mathrm{d}W_{xt}$ can be correlated with $\mathrm{d}W_{ct}$, and the value of the correlation is captured by ρ_{cx}. In economic terms, the state variable plays a different role for each model variation. The full dependence of the model variations on the state variable is shown in Table 2:

[Table 2]

In some variations the steady state is at $x_t = 0$, while in others it is at $x_t = 1$, and x_t is positive with probability 1. This specification is used for the variations in which CV σ_{ct} is proportional to the state variable, to ensure that σ_{ct} is positive.

[18] f has the form of a normalised aggregator as in Duffie and Epstein (1992).

[19] The steady state $x_t = \mu_{x0}$ does not necessarily coincide with the ergodic mean or median of the process when the diffusion of the process is not symmetric around the steady state value.

4.5 Stochastic discount factor

4.5.1 Time-separable utility Case

In the TSU case, the stochastic discount factor is the derivative of the flow utility with respect to consumption. In the general case the formula is the following:

$$\Lambda = e^{-\rho t}(C_t S_t)^{-\gamma} \tag{6}$$

where S_t is only relevant in the habit model. Using the above expression, along with the consumption process (Equation 2) and the state variable process (Equation 5), Ito's Lemma can be implemented to get the stochastic differential equation (SDE) of the SDF:

$$\begin{aligned} \frac{\mathrm{d}\Lambda}{\Lambda} =& \Big(-\rho - \gamma\mu_{ct} + \frac{\gamma^2}{2}\sigma_{ct}^2\Big)\mathrm{d}t - \gamma\sigma_{ct}\mathrm{d}W_{ct} \\ & \underbrace{+\big(-\gamma\log(\phi)x_t + 2\rho_{cx}\sigma_{ct}\sigma_{xt} + \sigma_{xt}^2\big)\mathrm{d}t - \gamma\sigma_{xt}\mathrm{d}W_{xt}}_{\text{habit model only}} \end{aligned} \tag{7}$$

For the details of the derivation, see Appendix H.1.

4.5.2 Recursive utility case

In the case of RU, the stochastic process of the SDF is derived from the expressions for the value function and the aggregator function. The latter is given in Equation 4, and the value function turns out to have the following form:

$$V_t = \frac{C_t^{1-\gamma}e^{(1-\gamma)K(x_t)}}{1-\gamma} \tag{8}$$

V_t increases with K, which is a specific function of x_t that captures the full dependence of the value function on the state variable. At the end of this section, I show how the expression above is justified, and I compute a novel perturbation approximation that provides a formula for K. Given the expression for the value function, Ito's Lemma can be implemented to get to the SDE of the SDF. The calculation here follows Chen, Cosimano, Himonas and Kelly (2009). In particular, the fundamental relationship is:

$$\frac{\mathrm{d}\Lambda}{\Lambda} = f_V(C,V)\mathrm{d}t + \frac{\mathrm{d}f_C(C,V)}{f_C(C,V)} \tag{9}$$

f_C and f_V denote partial derivatives of f with respect to consumption and the value function respectively. The first term on the right hand side is the derivative of the flow utility with respect to the value function. The second term can be computed by applying Ito's lemma on the derivative of flow utility with respect to consumption.[20] The result is the following:

$$\begin{aligned}\frac{\mathrm{d}\Lambda}{\Lambda} =&\Bigg(\frac{\rho\left(-(1-\gamma\psi)e^{\frac{(1-\psi 1)K[x_t]}{\psi}}-\gamma\psi+\psi\right)}{1-\psi}-\gamma\mu_{ct}+\frac{\gamma^2\sigma_{ct}^2}{2}+\frac{\gamma(\gamma\psi-1)\rho_{cx}\sigma_{xt}\sigma_{ct}K'\left(x_t\right)}{\psi}\\ &+\frac{(\gamma\psi-1)\left(2\psi(\mu_{x0}-x_t)\log(\phi)K'\left(x_t\right)+\sigma_{xt}^2\left((\gamma\psi-1)K'\left(x_t\right)^2-\psi K''\left(x_t\right)\right)\right)}{2\psi^2}\Bigg)\mathrm{d}t\\ &-\frac{(\gamma\psi-1)\sigma_{xt}K'(x_t)}{\psi}\mathrm{d}W_{xt}-\gamma\sigma_{ct}\mathrm{d}W_{ct}\end{aligned} \tag{10}$$

The details of the derivation can be found in Appendix H.2. It is notable that the special case of $\gamma = 1/\psi$, is time-separable, and the equation above simplifies to the formula in Equation 7 from the standard TSU case. Also, the stochastic component relating to consumption ($-\gamma\sigma_{ct}\mathrm{d}W_{ct}$), is exactly the same as in TSU, and there is an extra component, namely $-\frac{(\gamma\psi-1)\sigma_{xt}K'(x_t)}{\psi}\mathrm{d}W_{xt}$, due to the direct dependence of the SDF on the state variable.

4.6 Instantaneous rate

From the SDF the short-term rate is derived as follows:

$$\text{TSU: } r(x_t)\mathrm{d}t = -\mathrm{E}_t\left[\frac{\mathrm{d}\Lambda}{\Lambda}\right] = \left(\rho+\gamma\mu_{ct}-\frac{\gamma^2}{2}\sigma_{ct}^2\right)\mathrm{d}t\underbrace{+\left(\gamma\log(\phi)x_t-2\rho_{cx}\sigma_{ct}\sigma_{xt}-\sigma_{xt}^2\right)\mathrm{d}t}_{\text{habit model only}}$$

$$\begin{aligned}\text{RU: } r(x_t)\mathrm{d}t = -\mathrm{E}_t\left[\frac{\mathrm{d}\Lambda}{\Lambda}\right] =&\\ &\frac{\rho\left((1-\gamma\psi)e^{\frac{(1-\psi)K[x_t]}{\psi}}+\gamma\psi-\psi\right)}{1-\psi}+\gamma\mu_{ct}-\frac{\gamma^2\sigma_{ct}^2}{2}-\frac{\gamma(\gamma\psi-1)\rho_{cx}\sigma_{xt}\sigma_{ct}K'\left(x_t\right)}{\psi}\\ &-\frac{(\gamma\psi-1)\left(2\psi(\mu_{x0}-x_t)\log(\phi)K'\left(x_t\right)+\sigma_{xt}^2\left((\gamma\psi-1)K'\left(x_t\right)^2-\psi K''\left(x_t\right)\right)\right)}{2\psi^2}\end{aligned} \tag{11}$$

[20] This operation is performed by substituting the value function using Equation (19) and applying Ito's lemma based on consumption and the state variable as independent variables.

In the standard TSU case, the short rate depends on three components. The first is the time preference parameter ρ. The second is $\gamma\mu_{ct}$ and it relates to the consumption smoothing motive. As CD increases, agents try to borrow to increase current consumption, and in equilibrium the short rate increases. The third is $-\gamma^2\sigma_{ct}^2/2$ and it relates to the precautionary saving motive. As consumption becomes more risky, agents try to save, and in equilibrium the short rate decreases. In TSU-Habit, there are extra components that relate both to the consumption smoothing motive and the precautionary saving motive, and they are due to the state variable being part of the utility function. Thus, as the surplus consumption ratio falls, marginal consumption increases even more than in standard TSU. So, the agent has an even higher motive to smooth consumption. However, in the same state of the world, the surplus consumption ratio is also much more volatile and the agent also has a higher precautionary saving motive. In Campbell and Cochrane (1999) these two opposite effects on the short-term rate are regulated by a parameter denoted b. If $b = 0$, then the short rate becomes a constant. If $b > 0$ ($b < 0$), then the short rate is decreasing (increasing) in the surplus consumption ratio.

In the RU case, the short rate becomes more complicated. However, for the main calibrations the dominating additional effect comes from the fact that the marginal utility of consumption is expected to change as the state variable changes. The effect of this is that short rates are affected less by the consumption smoothing effect and the precautionary savings effect, and short rates under RU are less sensitive to the state variable than short rates under TSU.

4.7 Long-term bond

4.7.1 Bond pricing equation

Next, given the process of the SDF, the price of the long-term bond Q can be computed in the same way for both TSU and RU cases. The bond price is a function of the state variable x_t and its remaining maturity m. Thus, by using Ito's Lemma the stochastic process follows:[21]

$$dQ(x,m) = \big(-\log(\phi)(\mu_{x0} - x_t)Q_x - Q_m + \frac{1}{2}\sigma_{xt}^2 Q_{xx}\big)\mathrm{d}t + \sigma_{xt}Q_x \mathrm{d}W_{xt} \tag{12}$$

[21] Given the flow utility function, investors' decisions are not affected by the level of consumption. This implies that the long-term bond is not going to be a function of consumption itself (see for example Tsai and Wachter 2018).

In the equation above, subscripts $\cdot_x$ and $\cdot_m$, denote partial derivatives with respect to the corresponding variable. The next step is to derive the partial differential equation (PDE) that Q obeys in these models. Thus, I use the pricing equation following the approach in Cochrane (2009) and Chen, Cosimano and Himonas (2010):

$$\mathrm{E}\big[d(\Lambda Q)\big] = 0 \to \mathrm{E}\left[\frac{\mathrm{d}\Lambda}{\Lambda}Q + \mathrm{d}Q + \frac{\mathrm{d}\Lambda}{\Lambda}\mathrm{d}Q\right] = 0 \tag{13}$$

Substituting the expressions for Λ, $\mathrm{E}[\mathrm{d}\Lambda/\Lambda]$ and $\mathrm{d}Q$ from Equations (7), (11) and (12) respectively, gives rise to the PDE obeyed by Q:[22]

$$\begin{aligned} -Q_m - r(x_t)Q + \big(-\log(\phi)(\mu_{x0} - x_t) + A(x_t)\big)Q_x + \frac{\sigma_{xt}^2}{2}Q_{xx} = 0 \\ \text{where: } A(x_t)\mathrm{d}t = \frac{\mathrm{d}\Lambda}{\Lambda}\mathrm{d}Q \end{aligned} \tag{14}$$

The expression comprises five terms. The first is the derivative with respect to maturity Q_m. The second is the short rate term $r(x_t)Q$, which differs depending on the variation, as shown in Equation (11). The third is the expectation term $-\log(\phi)(\mu_{x0}-x_t)$, that captures the information that short rates may be expected to change in the future. The fourth is what I call the A term, and it is responsible for term premia, as it captures consumption-based risk. [23] The fifth is the diffusion term $\frac{\sigma_{xt}^2}{2}Q_{xx}$.[24] The solution of this equation is discussed next, while Appendix C shows in more detail how these five terms affect the term structure of interest rates and its dynamics.

4.7.2 Solution of the pricing equation

Equation (14) is a PDE, and I solve it by making use of the Feynman-Kac formula, which re-expresses the solution of a PDE as an expectation of a stochastic process. In particular, the solution of Equation (14) is:

$$Q(m, x_t) = \mathrm{E}_t\left[\exp\left\{\int_m^0 r(\tilde{x}_{t+s})ds\right\}\right] = \mathrm{E}_t\left[\exp\left\{-\int_0^m r(\tilde{x}_{t+s})dt\right\}\right] \tag{15}$$

[22] This equation is similar to a Black-Scholes equation.

[23] Risk is understood in the context of consumption-based asset pricing. Therefore, if the the price of the bond does not co-vary with the SDF, then the A term is 0. The A term being 0 does not mean that the price of the bond is deterministic.

[24] This term is connected with the idea of *convexity* in finance.

where $\tilde{x}_0 = x_t$ and $\tilde{x}_t$ follows the modified stochastic process compared to the state variable:[25]

$$\mathrm{d}\tilde{x}_t = \Big(- \log(\phi)(\mu_{x0} - \tilde{x}_t) + A(\tilde{x}_t)\Big)\mathrm{d}t + \sigma_{xt}(\tilde{x}_t)\mathrm{d}W_{xt} \tag{16}$$

The expectation is computed using Monte Carlo simulations.

4.7.3 Risk-neutral yield and term premium

Instead of using a modified process, the original process of the state variable can also be used in the Feynman-Kac formula:

$$H(m, x_t) = \mathrm{E}_t\left[\exp\left\{\int_m^0 r(x_{t+s})ds\right\}\right] = \mathrm{E}_t\left[\exp\left\{-\int_0^m r(x_{t+s})dt\right\}\right] \tag{17}$$

This is by definition the expected gross return from rolling over the short-term rate. Thus, $-\log(H(m, x_t))/m$ is by definition the risk-neutral yield, and the argument above shows that it corresponds to the solution of Equation (14), after setting $A(x_t) = 0$ for all x_t. In other words, the risk-neutral yield can be thought of as deriving from a bond priced by a risk-neutral investor. This also provides a natural way for computing term premia, which is:

$$TP(x_t, m) = \frac{-\log Q(x_t, m) - (-\log H(x_t, m))}{m} \tag{18}$$

Namely, the term premium is the difference between the yield of the bond and the risk-neutral yield. Unfortunately, there is no analytic expression for term premia, given that Q and H are computed numerically. However, there is an analytic expression for function A in Equation 14, and it can serve as a diagnostic of term premia, as it is the component that distinguishes Q from H. Especially when the short-term rate is linear in the state variable, the sign of A determines the sign of term premia,[26] the time variability of A determines the time variability of term premia, and the size of A determines the size of term premia.[27] In addition, the size of A can easily be

[25] Here I show the dependence of σ_{xt} on $\tilde{x}_t$, in order to clarify that it is the same function as before, but it takes the modified variable as the argument.

[26] In particular term premia have the sign of the product of A with the derivative of $-Q$ with respect to the state variable x_t, which usually has the same sign as the derivative of short-rete with respect ot the state variable x_t.

[27] To be precise, term premia depend on the entire pricing Equation (14). However, if the short-term rate is linear and the effect of the diffusion term is small, then the bulk of the time-varying behaviour of term premia is determined by A. In the explanation provided here, I assume that the diffusion term and non-linearities have a small effect on the yield spread. A detailed analysis is conducted in Appendix C.

judged by comparison to the size of the expectation term $-\log(\phi)(\mu_{x0} - x_t)$, which also multiplies Q_x in the PDE. The expectation term and the A term are the two main drivers of the yield spread. Therefore, if the typical values of the expectation term are much larger than the typical values of the A term, this implies that the yield spread is due to expected changes of short-term rates in the future. On the other hand, if the values of the two terms are comparable in size, then the yield spread likely contains a component due to the term premium as large as a component due to the expectation term. This comparison is illustrated in practice in Section 5.

4.8 Preview of mechanisms

Here, I provide a preview of the mechanisms that are further analysed in Section 5. As shown in the previous subsection function A determines the characteristics of term premia. In order to generate substantial, positive and time-varying term premia, A needs to be large, opposite in sign to the slope of the short-term rate with respect to the state variable, and time-varying respectively. As we shall see later, models with time-varying risk aversion like the habit model analysed in Wachter (2006) achieve the main features of term premia, by introducing a high time-varying state volatility and a short rate that co-varies negatively with consumption. High time-varying state volatility makes A large in absolute value and time-varying, while the negative correlation between the stochastic changes of consumption and the short-term rate generates positive instead of negative term premia. Alternatively, in the mechanism that is first introduced in this paper, A is large and time-varying, because CV is large and time-varying, while A has the opposite sign to the slope of the short-term rate, because the stochastic changes of CV and consumption are positively correlated.[28] This assumption can be justified in this version of the model, because increased CV is associated with a lower short-term rate and higher level of wealth for bondholders, who are also marginal investors.

4.9 Perturbation approximation for K function in the recursive utility case

As mentioned in Subsection 4.5.2, given the process of the SDF it is possible to compute the price of bonds in the RU case in the same way as in the TSU case. This in turn requires an expression for the value function. This subsection is dedicated to

[28] This is the case for the TSU-HCV and the Arb-DP variations, which both produce positive and substantially time-varying term premia. As I discuss later Arb-IN

explaining a novel perturbation method to approximate function K that was used in the RU value function.

As shown by Tsai and Wachter (2018), the value function follows:[29]

$$J(C,x) = V = \frac{C^{1-\gamma}e^{(1-\gamma)K(x)}}{1-\gamma} \tag{19}$$

where $K(x)$ solves the following ordinary differential equation (ODE):

$$-\frac{1}{2}\gamma\sigma_{ct}^2+\frac{1}{2}\sigma_{xt}^2\left(K''(x)-(\gamma-1)K'(x)^2\right)-\log(\phi)(\mu_{x0}-x_t)K'(x)+\frac{\beta\left(e^{-\epsilon K(x)}-1\right)}{\epsilon}+\mu_{ct}=0 \tag{20}$$

where the substitution $\psi = \frac{1}{1-\epsilon}$ has been made. For $\psi = 1$ ($\epsilon = 0$) Equation (20) has an analytic solution. This can then be used to create a global perturbation solution in terms of the state variable, which I express in terms of ϵ.[30] In particular, Equation (20) can be expanded to:

$$\begin{aligned}-\frac{1}{2}\gamma\sigma_{ct}^2+\frac{1}{2}\sigma_{xt}^2(-(\gamma-1)\left(\epsilon^2K_2'(x)+\epsilon K_1'(x)+K_0'(x)\right)^2+\epsilon^2K_2''(x)+\epsilon K_1''(x)+K_0''(x))+\mu_{ct}\\ +\frac{\beta\left(e^{-\epsilon\left(\epsilon^2K_2(x)+\epsilon K_1(x)+K_0(x)\right)}-1\right)}{\epsilon}-\log(\phi)(\mu_{x0}-x_t)\left(\epsilon^2K_2'(x)+\epsilon K_1'(x)+K_0'(x)\right)\approx 0\end{aligned} \tag{21}$$

Here function K has been expanded up to second order, but it could also be expanded further. Given this expansion the equation admits a solution of the form:

$$\begin{aligned}K_0(x) &= a_{0,0}+a_{0,1}x\\ K_1(x) &= a_{1,0}+a_{1,1}x+a_{1,2}x^2\\ K_2(x) &= a_{2,0}+a_{2,1}x+a_{2,2}x^2+a_{2,3}x^3\\ &\dots\end{aligned} \tag{22}$$

This solution can by plugged into the ODE (21), and for each m,n, $a_{m,n}$ can be

[29] A similar setup is used in Benzoni, Collin-Dufresne and Goldstein (2011) building upon earlier literature such as (Duffie and Epstein 1992). The setup here is simpler than Tsai and Wachter (2018), because there is only one state variable and there are no Poisson jumps in the consumption process and the state variable process.

[30] The approximation is global in terms of the state variable x, as the perturbation is done with respect to parameter ϵ. Nevertheless, it is not valid for all values of x. In particular, the approximation takes such a form, so that its validity depend on different regions of the state variable. In the region that it converges, the quality of the approximation is high for all values of x, but outside this region the series diverge.

derived by setting each factor of $x^m \epsilon^n$ equal to zero. This leads to a linear equation for each coefficient.[31] Conveniently, these equations can be solved successively so that for each equation there is only one unknown. Unfortunately, as can be seen from equation (22), the full solution is a sum of polynomials in terms of x. For each successive order of ϵ, the order of the polynomial increases by one. While it is possible to compute many orders of approximation, eventually the computation becomes expensive, as each order of ϵ requires the solution of more linear equations, and each equation has an increasing complexity.

The solution in Tsai and Wachter (2018) only derived $K_0(\cdot)$ which is the first term in formula (22) and it is the "zeroth" order approximation in terms of ϵ or equivalently ψ. My approximation is useful, because it allows a much larger range of values for ψ, and it provides an analytic expression that is easy to include in the Monte Carlo simulations, that solve the pricing equation. My method is described in detail in Melissinos (2023).[32] On the contrary, given that the solution provided by Tsai and Wachter (2018) was analytically correct only for $\epsilon = 0$ ($\psi = 1$), implementing the method for other values of ψ is not easily justifiable, even if in practice it would generate qualitatively or even quantitatively similar results. It should also be noted that the full perturbation series, that I provide, is the exact solution to the ODE. Namely it is the unique perturbation series that represents the solution. Thus, it is highly likely that with some extra mathematical analysis, it can be re-expressed in terms of known special functions, and we can get an exact answer that is practically trivial to compute for arbitrary order. Nevertheless, the method in its current form allows the researcher to easily approximate the value function, while also practically checking its convergence. The value function can then be used in the pricing equation to directly get the price of assets, while being robust to a large range of parameters for the IES. Thus, this method can be implemented widely in RU models.

The details and the properties of the method are described further in Melissinos (2023), while the perturbation can be expanded in further research to more general setups including more than one state variables. Given that the full perturbation series represents the exact solution to the problem it can likely also give rise to an exact solution in terms of special mathematical functions. Such a solution would be

[31] Apart from coefficient $a_{0,1}$ which may require the solution of a second order equation.

[32] One limitation of the method, that my contribution does not overcome, is that the parameters of the processes should be at most linear functions of the state variable. In particular, σ^2_{xt} and σ^2_{ct} are linear in the state variable. Therefore, unlike in the TSU case where I set $\sigma_{ct} \propto x_t$, here I set $\sigma_{ct} \propto \sqrt{x_t}$. This is investigated in more detail in Melissinos (2023), but the main implication is that CV is relatively restricted in its variability.

significant, and it would likely help with the proof of existence and uniqueness of the solution, which has long been an open problem in the literature (Tsai and Wachter 2018).

4.10 Calibration

[Table 3]

Given that the goal of the paper is to identify consumption-based mechanisms that are consistent with the patterns of term premia in the data, the emphasis is not on providing a perfect calibration. Instead, the focus is on finding the combination of the utility specification and the consumption process that generate the observed patterns in term premia. Thus, several parameter choices are explored, and the calibration of each model variation is reported in the corresponding figure showing the results. Nevertheless, there are shared parameter choices across model variations. These are reported in Table 3 and they follow Wachter (2013), while the risk aversion parameter γ is set equal to 2 following Wachter and Zhu (2019).

5 Results

5.1 How variations are evaluated

In evaluating the features of term premia, I also require that the model variations generate an empirically plausible short rate volatility.[33] Higher short rate volatility can give rise to higher term premia. Thus, I assume a relatively large short rate volatility in order to give these variations the best chance of success. Their performance is compared, by plotting model-implied term premia as a function of the state variable next to the time series of estimated term premia (these were already shown in Figure 1). In making the comparison, the focus is more on the variability than the level.[34] If the models generate roughly the same pattern of variability, then they are

[33] The calibration of the state variable is described in Appendix D.

[34] For example, in the data, especially recently, term premia seem to also become negative. However, the successful models that I present here all have exclusively positive term premia. I do not consider this a large setback, as my focus is on the variability of term premia and the models investigated here only have one state variable. In a full explanation of term premia and interest rates more generally, at least two variables would be necessary, given that a principal component analysis of the yields and spreads requires at least two principal components to explain the bulk of the variation (this is shown in Appendix B)

considered a success. In most cases it is obvious, when the models fail to generate the patterns of term premia in the data.

Table 4 shows information for function A for six separate variations with moderate CV, and Figures 3 and 5 show the corresponding five and ten-year term premia, which are discussed in Subsections 5.2.1 and 5.2.2 respectively. Further variations and calibrations are shown in Appendix F. With these results, apart from illustrating the standard consumption-based mechanisms, I aim to provide a helpful reference, as in the literature state-dependent term premia are rarely provided. Furthermore, Table 5 and Figure 6 show variations with high CV, whose consequences have not been analysed before with respect to term premia. I am the first to show that these variations can generate the features of term premia in the data. These latter results are discussed in Subsection 5.3. While for my main results I use estimated premia from d 'Amico *et al.* (2018), Abrahams *et al.* (2016) also estimate the term premium, and they provide a decomposition of the five-to-ten year forward. Estimations from both papers are shown in Appendix E,[35] and the five-to-ten forward term premium generated by the variations analysed in the current paper are shown in Appendix F.

5.2 Moderate consumption volatility

5.2.1 Time-separable utility

[Table 4]

[Figure 3]

As mentioned earlier, the three main mechanisms analysed are time-varying CD, time-varying CV, and time-varying surplus consumption ratio (in TSU-Habit).[36] The effect of these mechanisms on term premia can first be understood by looking at function A for each of the variations. Table 4 shows the functional form of A, and which components are time-varying. It also shows the typical range for the size of the A term and the expectation term. As mentioned earlier, the sign of A, in conjunction with the slope of the short-term rate, determines the sign of term premia, while the size and variability of A also determine the size and variability of term premia. In TSU-CD the short rate is increasing with CD, due to the consumption smoothing

[35] The two measures are similar, with the term premium in Abrahams *et al.* (2016) reaching relatively higher values. In addition, the risk-neutral yield has much less variability in the Abrahams *et al.* (2016) estimation.

[36] The first two mechanisms can also be found in the long-run risk models introduced by Bansal and Yaron (2004), who used a recursive utility.

motive.[37] As a result, in conjunction with $\rho_{cx} > 0$ the term premia are negative and constant in the state variable. The intuition for negative term premia is that the short-term rate goes up and bond prices go down when CD rises, which is also the time that consumption tends to increase (due to $\rho_{cx} > 0$). This means that long-term bonds act as a hedge, and they command a negative term premium. Apart from the negativity of term premia, A typically takes much smaller values in absolute value compared to the expectation term, implying that term premia should be very small. Thus, instead of positive, time-varying and sizeable, term premia are negative, constant and small. Alternatively, for TSU-CV, the short rate is decreasing in CV, due to the precautionary savings motive, and I assume $\rho_{cx} < 0$. Therefore, the A term is positive and time-varying in the state variable, as it includes CV σ_{ct} (in this specification σ_{xt} is also time-varying). As a result, the term premia are again negative (they have the same sign as the slope of the short-term rate), but in this case they are time-varying. However, the A term is much smaller in absolute value compared to the expectation term, so term premia apart from negative are again very small. Figure 3 shows the term premia for these two variations in comparison to the time series of term premia in the data.[38] It is evident from the figure that as the state of the economy changes, term premia would hardly move away from 0, and they would not be able to generate the variation estimated in the time-series. From the functional form of A it also follows that assuming a different sign for ρ_{cx}, would imply term premia of the opposite sign in both cases. However, for a representative consumption process it is reasonable that an increase of CD is associated with an increase in consumption itself, while an increase in CV is associated with a decrease in consumption.[39] In Appendix F the results above are verified for several different calibrations.

The mechanisms discussed above use the power utility setup. Here, I discuss the effect of including external habit in the utility function as in Campbell and Cochrane

[37]This means that the stochastic component of consumption is positively correlated with the stochastic component of the state variable, which is associated with CD. To avoid this long description, I will mostly use ρ_{cx}.

[38]In TSU-CV the short rate is also insensitive to CV.

[39]This is intuitive if the consumption process is thought of as a relatively independent consumption process that determines the short-term rate. However, if the short-term rate is the independent variable, and the consumption process is reacting, then it makes sense that as the short-term rate decreases, borrowing becomes cheaper and consumption temporarily increases. This can can either imply that CD decreases, as consumption comes back to its normal level, or that CV increases as the agent has less savings. In both cases, the sign of ρ_{cx} is the opposite compared to the first scenario. My conjecture is that this should not happen in a large economy with a short-term rate determined by the behaviour of a representative agent. However, it could also be argued that the short-term rate is the independent force in the economy, due to the actions of the monetary authority.

(1999). As shown by Wachter (2006), TSU-Habit can generate the basic patterns of term premia that we see in the data. As mentioned previously, models with time-varying risk aversion, like the habit model, belong to one of only two kinds of models that can explain the patterns of term premia in a consumption-based setup with a single stationary autoregressive process. Thus, I analyse this model within my setup, in order to comprehensively describe consumption-based explanations to real term premia, and delineate its main differences compared to the alternative explanation that I introduce in the next subsection. Table 4 shows that the habit model has an extra term in the functional form of A. It turns out that this second term is dominant because the state variable volatility is in most states much larger compared to CV ($\sigma_{xt} \gg \sigma_{ct}$).[40] As a result, the sign of A does not depend on ρ_{cx} (which in the canonical habit model is equal to 1 anyway, as consumption completely determines the habit variable.), and the sign of term premia is determined exclusively by the slope of the short-term rate as a function of the surplus consumption ratio. As discussed in Subsection 4.6, this relationship in TSU-Habit depends on parameter b, which is chosen positive so that the short-term rate is decreasing and the term premia are positive.[41] Furthermore, term premia are large, as the value of A is large compared to the expectation term. Lastly, term premia are time-varying, given that A includes σ_{xt}^2, which is time-varying. Namely, variability of term premia is due to the heteroskedasticity of the state variable, which is amplified because A includes the square of the volatility.[42] Therefore, term premia are positive, time-varying, and large. This is explicitly shown in Figure 3, and the typical amount of variability, captured between the dashed lines, matches closely the variability in the estimated term premia.

[Figure 4]

5.2.2 Recursive utility

[Figure 5]

In this subsection, the results are extended to RU. This case is arguably of higher interest, as it separates risk aversion and IES. Moreover, Bansal and Yaron (2004)

[40] The size of the two terms is shown in the right plot of Figure 4

[41] This was also the choice of Wachter (2006), while Campbell and Cochrane (1999) set $b = 0$ in the final version of their paper (in an earlier version they also investigated $b > 0$). In Appendix F, I also derive the results of a variation in which $b < 0$. In this case the short rate is increasing in the surplus consumption ratio, and term premia are negative, time-varying and large in absolute value.

[42] In Appendix F, I impose homoskedasticity, and this leads to constant term premia. Admittedly, this is contrary to the spirit of the habit model.

were able to use this feature in conjunction with time-varying CD and CV in long-run risk models, to explain the equity premium puzzle. Indeed, similar to TSU-Habit, as is shown in Table 4, the RU variations have an extra term in function A. This term disappears for $\gamma = 1/\psi$ which coincides with the special case in which utility becomes time-separable. Similar to TSU-Habit, this term dominates the A function. Thus, the sign of term premia does not depend on the sign of ρ_{cx}, but on the slope of K, which turns out to match the slope of the short-term rate both in RU-CD and in RU-CV. This means that negative term premia are now a more robust prediction compared to TSU-CD and TSU-CV. However, in the case of RU-CD A is significantly larger compared to TSU-CD. Therefore, term premia are negative, constant, but can be somewhat sizeable in absolute value. In contrast, RU-CV shows the same patterns as TSU-CV. Given that the short-rate hardly exhibits variability in RU-CV, I also compute RU-Mixed which includes both time-varying CD and CV, governed by the same state variable. However, A in this variation is also quite small, and the term premia are small in absolute value. The term premia for RU-CD, RU-CV and RU-Mixed are shown in Figure 5, and it is clear that they cannot generate the variability in the estimated term premia. Appendix F has further variations with different calibrations verifying these results.

Intuitively, RU models might be considered as good candidates for explaining term premia due to their flexibility in separating risk aversion and IES. However, term premia are constant in RU-CD and very small in RU-CV, while they are in both cases negative. This result is consistent with the literature. Specifically, Bansal and Shaliastovich (2013) study term premia in RU models, but they investigate the variability in *nominal* term premia and their mechanism involves inflation. Gomez-Cram and Yaron (2021) provide a similar explanation for nominal term premia using RU that also relies on inflation. Hence, the real term premia that they generate are not substantially time-varying. Van Binsbergen, Fernández-Villaverde, Koijen and Rubio-Ramírez (2012) also consider a RU setup with inflation, and they find that nominal term premia can be positive, for very high risk aversion values. However, they also find that real term premia are negative.

5.3 High consumption volatility

[Table 5]

[Figure 6]

In general, agents should be independently adjusting their investment and consumption. Thus, given the same asset-pricing processes, if optimising agents are heterogeneous in their utility function, they will have different consumption processes. Given a utility function the consistence of term premia with the consumption process can be checked independently for each consumer. Previously, I have shown that representative consumer explanations of term premia require time-varying risk aversion. This raises the question whether there is *any* consumer group whose consumption process is consistent with term premia, without assuming time-varying risk aversion. Given the negativity and the small size of term premia found in the previous subsection, it is reasonable to assume that the answer is again no. However, it turns out that there are other explanations that rely on the dynamics of the consumption process within TSU. Table 3 shows information on function A for these cases, while Figure 6 shows the corresponding state-dependent term premia. As has been shown previously, time-varying CV implies time-varying term premia. Thus, starting from time-varying CV, the way forward is in principle simple based on the expression for A. By changing the sign of ρ_{cx} and increasing the steady state level of σ_{ct}, term premia become positive and large. Indeed, this works in generating the amount of variability in the estimated term premia (Figure 6). This is noteworthy given the difficulty encountered previously in generating any amount of significant time-varying term premia. However, are these two necessary modifications economically sensible?

As shown in Figure 6 the typical variability of the state variable (area between dashed lines) ranges from 5% to more than 20% CV per year. Even accounting for potential mismeasurement of aggregate consumption, this range is excessively large. Therefore, this approach is not consistent with a representative consumer whose consumption coincides with aggregate consumption. However, this does not mean that the consumption process is too extreme for *any* consumer. Firstly, if financial markets are incomplete, and risk sharing is not possible, then idiosyncratic CV is relevant for asset prices (Constantinides and Duffie 1996). This means that aggregate CV could already be underestimating the CV that should be used in the models. Next, while 12% steady state CV is large compared to aggregate CV, it is not large compared to asset price volatility in financial markets. For people whose wealth lies in the financial sector, 11% wealth volatility is entirely plausible, and according to standard consumption-based portfolio theory, CV should follow wealth volatility. Lastly, there is also direct evidence that CV is much higher for some groups of consumers. While I do not take a position whether these investors are rich or poor, Ait-Sahalia, Parker and Yogo (2004) showed that the CV of rich individuals could be much higher compared to aggregate CV. In particular, while they reported that the annual standard deviation of non-durables and services was 2.3% according to the standard NIPA data, they measure an annual standard deviation of 19.6% for luxury retail sales and 20.4% for charitable contributions of wealthy individuals.[43] These values are both significantly larger than the steady state CV of the model variations in this subsection, which is equal to 12%.[44] Based on these results, Ait-Sahalia *et al.* (2004) also argued that the equity premium puzzle is less of a puzzle when considering the consumption process of rich consumers. In Appendix G, I also show that TSU-HCV implies a sizeable equity premium. Lastly, Malloy, Moskowitz and Vissing-Jørgensen (2009) provided evidence that wealthy stockholders' consumption volatility is roughly three times higher compared to non-stockholders, while also showing that bond returns can be predicted by the covariance of wealthy stockholders' consumption growth with returns. This evidence is consistent with the idea that a small group of investors with high CV are driving term premia.

The second required assumption for the mechanism is that ρ_{cx} is positive. Previously, I have argued that this is not plausible for a representative consumer, because an increase in consumption risk should induce consumers to consume less and save more. However, the consumer-investors in TSU-HCV could be a small part of the overall population, and in this case $\rho_{cx} > 0$ can be justified. As CV increases, the short-term rate goes down, and this leads to an increase in bond prices. Thus, bondholders would then increase their consumption, given that their wealth also increases. An alternative intuition is that, as the short-term rate decreases, consumption increases due to borrowing, which in turn increases CV.

While I have shown the effect of high CV on term premia, I have only done so for $\gamma = 2$. Apart from further variations in Appendix F, Figure 7 shows the different levels of term premia on the same scale for various values of γ and for various values of steady state CV. The results are interesting in several ways. Firstly, it stands out that different values of γ lead to huge changes in term premia, when consumption volatility is high. This means that term premia in TSU-HCV are highly sensitive to risk aversion levels. On the other hand, term premia are so small when CV is low, that moderate increases in risk aversion are not able to generate the required variability. Thus, even if $\gamma = 4$, CV needs to be able to reach at least 10%, so that

[43] NIPA refers to the national income and product accounts produced by the Bureau of Economic Analysis of the US Department of Commerce. Ait-Sahalia *et al.* (2004) also include other measurements on the sales of luxury retail products.

[44] The standard deviation of consumption growth calculated from simulations also takes values similar to the CV of the model.

time-variability in term premia is generated.

[Figure 7]

This subsection shows how some consumers could have consumption processes that are consistent with the main features of term premia. By restricting my attention to these investors, and not introducing a full heterogeneous agent model, I can examine many different variations. Nevertheless, it is important to also consider the potential behaviours of the remaining agents in the economy. For instance, they could be investing in the bond market, but their behaviour could be explained by more complicated or alternative models. It could also be the case that other investors in the bond market are entities, such as hedge funds and pension pension funds that are not appropriately modelled as consumers. The only requirement for the remaining investors is that they do not trade in such a way, that induces extensive risk sharing with high CV investors. If they did, then this would lead to a decrease in the CV of the high CV investors. Alternatively, many consumers may not be participating in the bond market at all.[45] In both cases the other agents can have moderate consumption processes, and be primarily responsible for aggregate consumption dynamics.[46]

TSU-HCV has been the simplest consumption-based variation that is able to generate large term premia. However, given a high CV, slightly more complicated variations can be examined, in which CD and CV are simultaneously changing. I refer to these as "arbitrageur" variations in relation to Vayanos and Vila (2021), who suggested that the term structure of interest rates is driven by "arbitrageurs", who take advantage of investment opportunities in the bond market. As these opportunities can be risky, arbitrageurs are not able to fully equate rates and eliminate the effect of the demand of idiosyncratic investors or "preferred habitat investors", as they are called in the article.[47] Here, I abstract from these latter investors and restrict my attention to arbitrageurs. They are marginal investors in the bond market. Their consumption process should be consistent with the observed term structure of interest rates, including term premia. I argue that the consumption process of the arbitrageurs has two main features. Firstly, their CV is high (similar to TSU-HCV). Secondly, as the investment opportunity increases, both CV and CD rise. This occurs because the

[45] Or they may not be marginal investors due to short-selling constraints. For instance, an investor who is constrained from shorting one end of the term structure could be holding some long-term bonds, but this does not make her a marginal investor of bonds in general.

[46] A fuller analysis would provide a full heterogeneous agent model explaining to what extent idiosyncratic consumption risk can be insured through financial markets.

[47] Given that there is risk, these investment opportunities fall under the category of "limited arbitrage".

higher investment opportunity offers higher expected returns, which implies a higher CD. At the same time, the higher investment opportunity brings more risk, and CV also rises. This setup can give rise to four separate variations depending on the behaviour of the short-term rate and the sign of ρ_{cx}. These are shown in Table 6. The movements in CD and CV have opposite effects on the short-term rate. Depending on the dominating component, the short-term rate can either be increasing or decreasing in the magnitude of the investment opportunity. In addition, the sign of function A is fully determined by ρ_{cx}, which in turn depends on the portfolio composition of arbitrageurs, and how its value fluctuates given the changing state of the economy.[48] These two binary choices give rise to the four possibilities shown in Table 6.

[Table 6]

While each of these possibilities seems plausible, I focus on the two that generate positive term premia. In Arb-IN, term premia are positive and increasing with CV, as is the short-term rate. As shown in Figure 6, Arb-IN generates positive, time-varying and sizeable term premia. However, the size of the term premia is not as high as in TSU-Habit, TSU-HCV and Arb-DP. $\rho_{cx} < 0$ could be justified in Arb-IN, because an increase in the short-term rate could be inducing the arbitrageurs to invest more in the bond market and decrease their consumption. In addition, despite CD rising, an increase in the short-term rate can also imply a decrease in their wealth, if the arbitrageurs are bondholders.

In Arb-DP, $\rho_{cx} > 0$ can also be justified because it makes sense for consumption to increase when CD goes up. In addition, if the arbitrageurs are bond holders, then their wealth increases, as CV increases and the short rate goes down. This is also the variation that is most akin to the intuition provided in Vayanos and Vila (2021). As the short-term rate decreases, long-term bond yields underreact, and this leads to an increase in term premia. The arbitrageurs in Vayanos and Vila (2021) optimise between the mean and variance of their wealth, and consumption is not part of the analysis. To the best of my knowledge, I am the first to show that this behaviour can be rationalised within a consumption-based setup. Arb-DP also provides the characteristics of the consumption process that are consistent with term premia, and it shows that a low CV would not generate substantial variability in the arbitrageur setup. The mechanism driving term premia is basically the same as in TSU-HCV. Thus, explaining the main features of term premia requires high CV. Reaching a final conclusion whether actual bondholders' consumption process exhibits such volatility is

[48]This is true to the extent that arbitrageurs do not have income external to their portfolio.

not possible within this paper. However, my paper provides the theoretical prediction that can be evaluated and tested empirically. If such CV is judged to be too high, then arbitrageurs are likely not acting as consumers or on behalf of consumers. This would be evidence for the existence of frictions, such as the ones in the intermediary asset-pricing literature. Alternatively, if it is found that some bondholders have high CV as the model predicts, then it would be interesting to further research the reasons that distinguish these investors, and why they are not able to share their risk with the remaining population.

Apart from asset-pricing implications, the variations presented in this subsection are also significant for monetary policy, to the extent that monetary policy affects term premia (Beechey and Wright 2009). In particular, according to Arb-DP, central banks decreasing (increasing) interest rates is equivalent to increasing (decreasing) the CV of the marginal investors. An increasing CV implies higher term premia, and this mechanism hinges on stochastic consumption changes being positively correlated with CV. In addition, the effect on CV is very strong, as it can roughly range from 5% to 20%. On the contrary, the effect of monetary policy on the consumption process of non-investors might be muted, if they are indeed disconnected from the effects of bond markets. A full understanding of the effects of monetary policy on all agents in the economy would benefit from a full heterogeneous agent model that explains the investment behaviour of all households.[49]

Furthermore, the high CV and the arbitrageur variations have implications for household finance. In particular, the participants in these markets are assuming large consumption risks. Therefore, a usual household whose CV is low and whose utility function is similar to the utility function of the marginal investors, could benefit from investing in long-term bonds, when term premia are high. This is valid, as long as CV of the household does not become too volatile due to this investment. However, the benefit is conditional on the state of the economy, and it is not clear if the state of the economy is transparent to most households, as the current CV of marginal investors is not directly observable.[50] This advice would not be valid in the context of the habit model. In that case high term premia reflect states in which households

[49] Schneider (2022) provides such a model, in which the state variable captures "aggregate conditions in the credit market". Similar models would be interesting, in which the state variable captures CD and CV.

[50] One could argue that the state of the economy is directly observable by the level of the short-term rate. However, here I have focused on explaining term premia, and I am using a single state variable. In a full explanation of the dynamics of interest rates, at least two state variables would be needed. Hence, the level of the short-term rate would most likely not directly imply the level of term premia.

have a high risk aversion, and investing in risky securities would not be appropriate.

6 Conclusion

In conclusion, consumption-based models encounter three key challenges in explaining the features of term premia. Firstly, they typically generate long-term bonds that provide a hedge against risk, which leads to negative instead of positive term premia. Specifically, for a representative consumer, it is reasonable that a rise in CD is associated with a stochastic consumption increase. Therefore, bond prices increase when CD decreases, and vice versa. Therefore, bonds are extra valuable, because they provide insurance against macroeconomic risk, and the associated term premia will be negative. Similarly, for an aggregate representative consumer it is reasonable that increased CV is associated with a stochastic consumption decrease. A similar argument implies that term premia are again negative. Secondly, time-varying CD generates constant instead of variable term premia. The paper shows that this turns out to be the case even in RU models. In contrast, time-varying CV always produces time-varying term premia, because by definition the state variable affects consumption uncertainty and, hence, risk. Thirdly, in calibrations according to an aggregate consumption process, term premia are typically very small in absolute value. The intuition for this is that consumption processes that are relatively stable give rise to term premia that are small. For term premia to be large it means that consumers are assuming large risks. Thus, given that aggregate consumption is relatively stable, the corresponding models imply low term premia. With the exception of the third shortcoming, these issues arise both in the TSU case and in the RU case.

However, I have identified model variations that do yield positive and significantly time-varying term premia. Firstly, a model with external habit, as in Campbell and Cochrane (1999) and Wachter (2006), produces better results. This occurs because a) the short rate is counter-cyclical, b) the state variable is relatively large and directly affects the utility function, and c) the state variable is heteroskedastic. These three factors respectively imply that the term premia are a) positive, b) large in absolute value and c) time-varying. In this variation the time-variability of term premia is directly related to the heteroskedasticity of the state variable, while a potential drawback is that effective risk aversion is highly volatile and takes extreme values.[51] Nevertheless, my analysis shows that the habit model (or models with time-varying risk aversion more generally) is the best model we have that is able to explain term

[51] This was also the main criticism of the habit model by Mehra *et al.* (2007).

premia while using a representative consumption process.

I also demonstrate how large term premia can be explained by model variations that deviate from standard representative consumption processes. In particular, model variations for which a) CV is high, ranging for example from 5 to 20% per year, and for which b) stochastic consumption changes are positively correlated with CV, can generate positive and highly variable term premia. The first component contributes to term premia being high in absolute value, and the second component implies that term premia are positive. Apart from time-varying risk aversion this is the only available consumption-based mechanism to generate positive and substantially time-varying term premia. An important implication of this model is the high CV for many states of the economy. However, it is not ludicrously high. If the consumption-based setup were completely wrong and disconnected from the actual mechanisms generating term premia, then it could imply almost any value of CV for term premia to become highly variable. Moreover, the CV levels in these variations mirror return volatility levels in certain financial markets, and there is literature measuring a high CV in products consumed by rich households. An interesting empirical question would then be to ask, what the CV is for marginal optimising investors of the term structure of interest rates. Another important aspect is that a large part of the population does not actively participate in the bond market. Thus, maybe the consumption process of these households is not so relevant regarding the levels of term premia. In a separate variation, which also performs quite well, I combine high levels of CV with a time varying CD. In this variation there is a tradeoff between CV and CD, and I claim that this interpretation is similar to the arbitrageur story in Vayanos and Vila (2021), which to the best of my knowledge has not been implemented in a consumption-based framework. While I describe the consumption process of arbitrageurs, I do not take a stance whether its volatility is too high or not, and the final answer to this question probably requires further empirical research in the consumption process of direct and indirect bond holders. Whatever the answer is, further interesting questions emerge. If the CV implied by term premia is implausibly high, then arbitrageurs likely do not correspond to actual households, and they do not invest according to households' wishes, at least based on this high consumption risk explanation.[52] This could indicate the existence of intermediation constraints. Alternatively, if CV of actual bondholders is indeed high, then the question is why

[52] The intuition for this statement comes from consumption-based portfolio selection theory, according to which the portfolio weights of risky assets should agree whether households are investing directly or through funds.

these bondholders do not engage in risk sharing with the rest of the agents in the economy.

Finally, given that a couple of different mechanisms can generate the basic features of term premia, the question arises which one is the correct explanation. The answer requires further research. Nevertheless, one approach is to combine some of the explanations provided here within a full heterogeneous model that also accounts for households not participating in financial markets. Non-participation can be rationalised given the high volatility in financial markets and the existence of some friction. As a result, there would be reduced risk sharing, justifying CV being large. This setup is likely to jointly explain term premia, stock market non-participation, reduced risk sharing in the economy and the equity premium puzzle. Therefore, I consider it a promising direction for further research.

Tables and Figures

Tables

Model Variation Description	**Abbreviation**
Time-varying consumption drift with time-separable utility.	TSU-CD
Time-varying consumption volatility with time-separable utility.	TSU-CV
Time-varying habit with time-separable utility.	TSU-Habit
Time-varying consumption drift with recursive utility.	RU-CD
Time-varying consumption volatility with recursive utility.	RU-CV
Time-varying consumption drift and consumption volatility with recursive utility.	RU-Mixed
High time-varying consumption volatility with positive correlation $\rho_{cx} > 0$, and time-separable utility.	TSU-HCV
Arbitrageur case with short-term rate decreasing in the investment opportunity and positive correlation $\rho_{cx} > 0$, with time-separable utility.	Arb-DP
Arbitrageur case with short-term rate decreasing in the investment opportunity and negative correlation $\rho_{cx} > 0$, with time-separable utility.	Arb-IN

Table 1: Names of main model variations. The models are explained in Section 5.

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Model variation			
TSU-CD:	$\mu_{ct} = \mu_{c0} + x_t$	$\sigma_{xt} = \sigma_{x0}$	$\mu_{x0} = 0$
TSU-CV:	$\sigma_{ct} = \sigma_{c1} x_t$	$\sigma_{xt} = \sigma_{x1}\sqrt{x_t}$	$\mu_{x0} = 1$
TSU-Habit:*	$S_t = S_0 e^{x_t}$	$\sigma_{xt} = \sigma_{ct}\lambda(x_t)$	$\mu_{x0} = 0$
RU-CD:	$\mu_{ct} = \mu_{c0} + x_t$	$\sigma_{xt} = \sigma_{x0}$	$\mu_{x0} = 0$
RU-CV:	$\sigma_{ct} = \sigma_{c1}\sqrt{x_t}$	$\sigma_{xt} = \sigma_{x1}\sqrt{x_t}$	$\mu_{x0} = 1$
TSU-HCV:	$\sigma_{ct} = \sigma_{c1} x_t$	$\sigma_{xt} = \sigma_{x1}\sqrt{x_t}$	$\mu_{x0} = 1$
Arb-IN:	$\mu_{ct} = \mu_{c1} x_t^{1/4}, \sigma_{ct} = \sigma_{c1}\sqrt{x_t}$	$\sigma_{xt} = \sigma_{x1}\sqrt{x_t}$	$\mu_{x0} = 1$
Arb-DP:	$\mu_{ct} = \mu_{c1} x_t^{3/2}, \sigma_{ct} = \sigma_{c1} x_t^{2/3}$	$\sigma_{xt} = \sigma_{x1}\sqrt{x_t}$	$\mu_{x0} = 1$

Table 2: Dependence on the state variable for each model variation

* Following Campbell and Cochrane (1999) and Wachter (2006), the exact form of $\lambda(\cdot)$ is:

$$\sigma_{xt} = \sigma_{ct}\lambda(x_t) = \begin{cases} \sigma_{ct}\left(\frac{\sqrt{1-2x_t}}{S_0} - 1\right) & \text{if } x_t < \frac{1-S_0^2}{2} \\ 0 & \text{if } x \geq \frac{1-S_0^2}{2} \end{cases}, \quad S_0 = \sqrt{\frac{\gamma}{-\log(\phi) - b/\gamma}} \tag{23}$$

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General paramaters	
Relative risk aversion γ	2.0
Rate of time preference ρ	0.012/yr
Steady state CD μ_{c0}	0.0252 /yr
Steady state CV σ_{c0}	0.02/yr
Steady state reversion $\log\phi$	log(0.92)/yr

Table 3: Calibration of common parameters

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Model variation	$A(x_t)$	ρ_{cx}	Range of A term*	Range of expectation term*
TSU-CD	$-\gamma\rho_{cx}\sigma_c\sigma_x$	$+$	(-0.0002,-0.0002)	(0.0023, -0.0022)
TSU-CV	$-\gamma\rho_{cx}\sigma_{ct}\sigma_{xt}$	$-$	(0.0012, 0.0086)	(0.048, -0.052)
TSU-Habit	$-\gamma\rho_{cx}\sigma_c\sigma_{xt}$ $-\gamma\sigma_{xt}^2$	$+$	(-0.083, -0.011)	(0.034, -0.030)
RU-CD	$-\gamma\rho_{cx}\sigma_c\sigma_x$ $-\frac{(\gamma\psi-1)\sigma_x^2K'(x_t)}{\psi}$	$+$	(-0.001250, -0.001253)	(0.0023, -0.0022)
RU-CV	$-\gamma\rho_{cx}\sigma_{ct}\sigma_{xt}$ $-\frac{(\gamma\psi-1)\sigma_{xt}^2K'(x_t)}{\psi}$	$-$	(0.0018, 0.0040)	(0.030, -0.034)
RU-Mixed	$-\gamma\rho_{cx}\sigma_{ct}\sigma_{xt}$ $-\frac{(\gamma\psi-1)\sigma_{xt}^2K'(x_t)}{\psi}$	$+$	(-0.0045, -0.010)	(0.033, -0.031)

Table 4: Information on function A from Equation (14) in different model variations with moderate CV. The t-subscript has been dropped from the quantities that are not time-varying according to the variation.
* This range covers the typical values of the state variable. The values correspond to the dashed vertical lines in Figures 3 and 5.

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Model variation	$A(x_t)$	ρ_{cx}	Range of A term*	Range of steady state reversion term*
High CV	$-\gamma\rho_{cx}\sigma_{ct}\sigma_{xt}$	$+$	(-0.0079,-0.0057)	(0.0047, -0.0053)
Arb-IP	$-\gamma\rho_{cx}\sigma_{ct}\sigma_{xt}$	$-$	(-0.0069, -0.052)	(0.048, -0.052)
Arb-DN	$-\gamma\rho_{cx}\sigma_c\sigma_{xt}$	$+$	(0.0096, 0.045)	(0.047, -0.054)

Table 5: Information on function A from Equation (14) in different model variations with high CV.
* This range covers the typical values of the state variable. The values correspond to the dashed vertical lines in Figure 6.

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Short-term rate	Positive ρ_{cx}	Negative ρ_{cx}
Short-term rate Decreasing with CV (CV dominates)	Arb-DP, **positive** term premia	Arb-DN, **negative** term premia
Short-term rate Increasing with CV (CD dominates)	Arb-IP, **negative** term premia	Arb-IN, **positive** term premia

Table 6: Term Premia Sign in Basic Arbitrageur Variations

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Figures

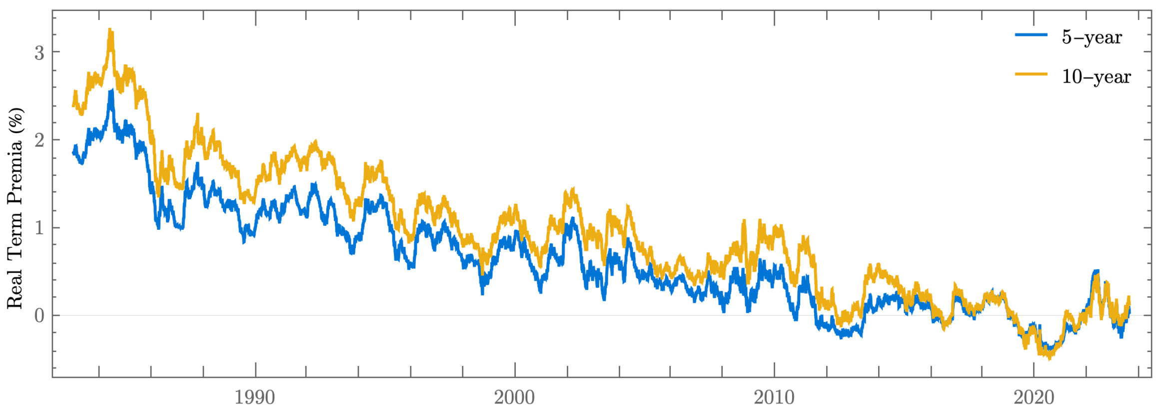

Figure 1: Time series of real term premia for the US
Data is taken from d 'Amico *et al.* (2018), who decomposed nominal yields into risk-neutral real yields (expected short-term rates averaged over the corresponding period), real term premia, expected inflation, inflation premia and liquidity premia.
Data Source: `https://www.federalreserve.gov/econres/notes/feds-notes/tips-from-tips-update-and-discussions-20190521.html`

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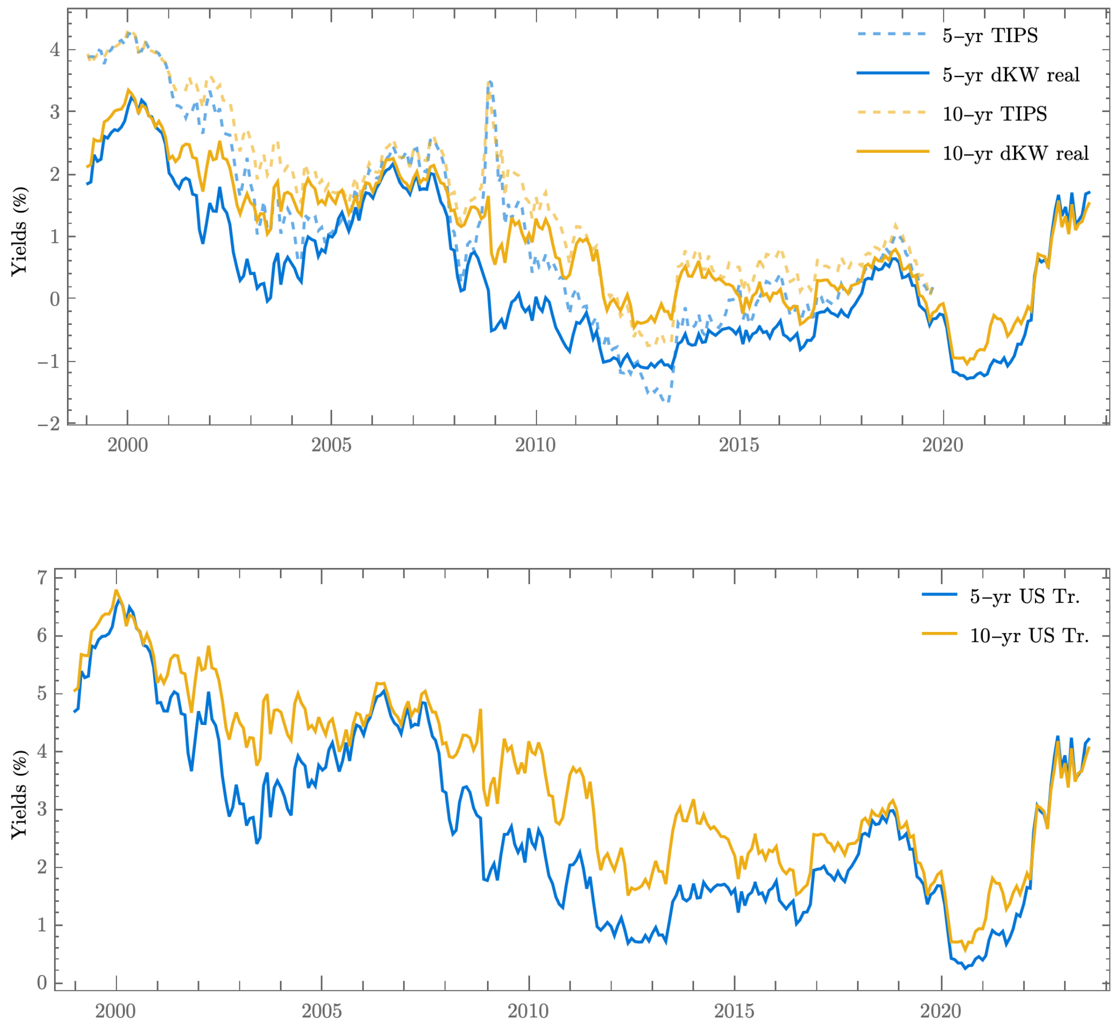

Figure 2: Yields of US Treasuries
TIPS data is taken from Gürkaynak *et al.* (2010), normal US treasury yields data is taken from Gürkaynak *et al.* (2007), and dKW real yields are taken from d 'Amico *et al.* (2018). Real yields are the sum of risk-neutral yields and the real term premia. The difference between the dashed and solid lines are the liquidity premia of the TIPS over the normal treasuries. Thus, this assumes that normal treasuries are perfectly liquid.

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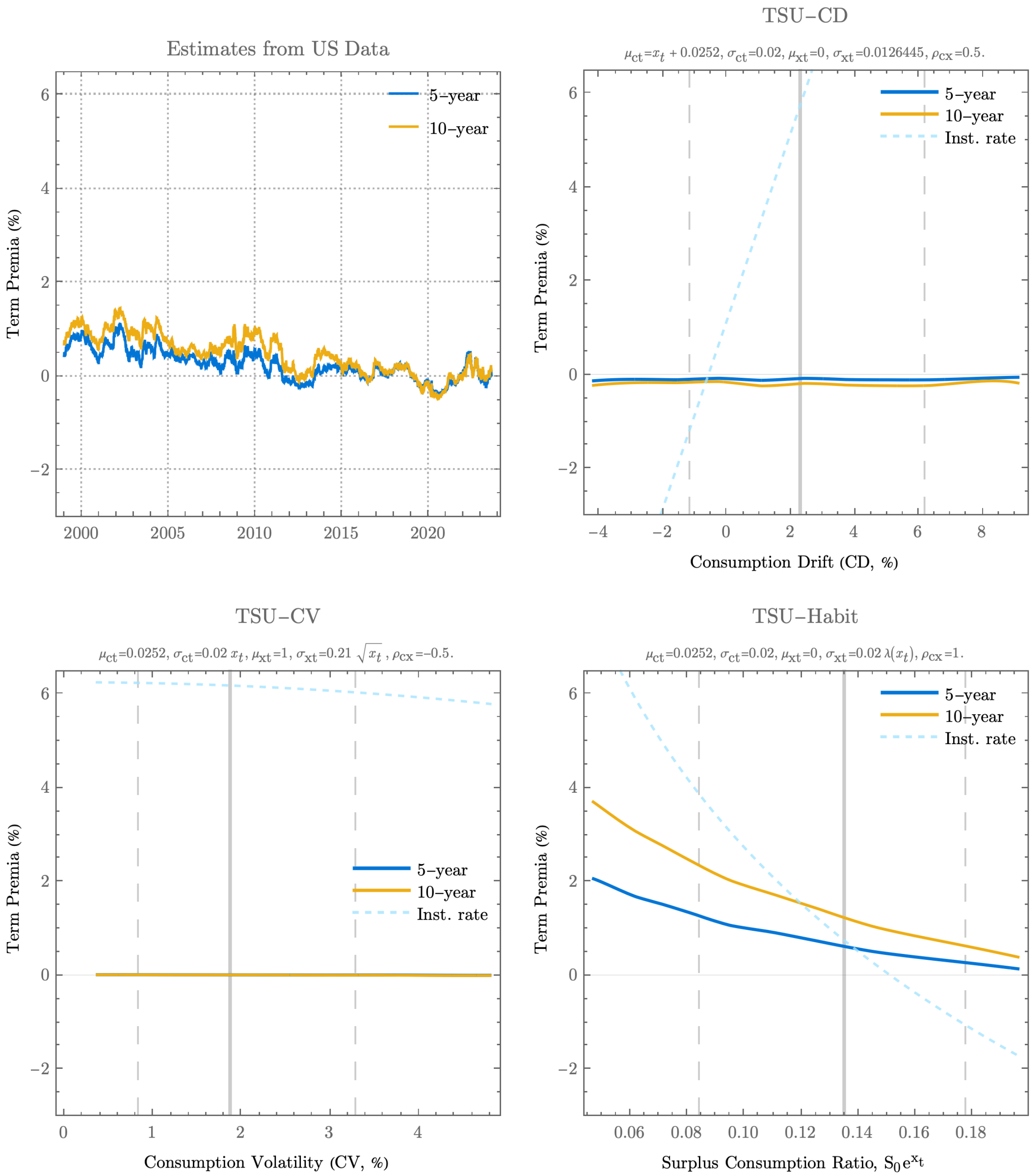

Figure 3: Term Premia in standard models with Time-Separable Utility
The top left plot shows estimates of term premia according to d 'Amico *et al.* (2018). The remaining plots show state-dependent term premia for three standard variations, namely variations with a) time-varying CD, b) time-varying CV, and c) an external habit in the utility function respectively. The dashed line shows the short-term rate. The vertical dashed lines correspond to the typical values of the state variable based on simulations. The full range of the x-axis includes extreme values of the state variable, which are still possible (see Appendix D or Figure 16 for the exact definition of the ranges).

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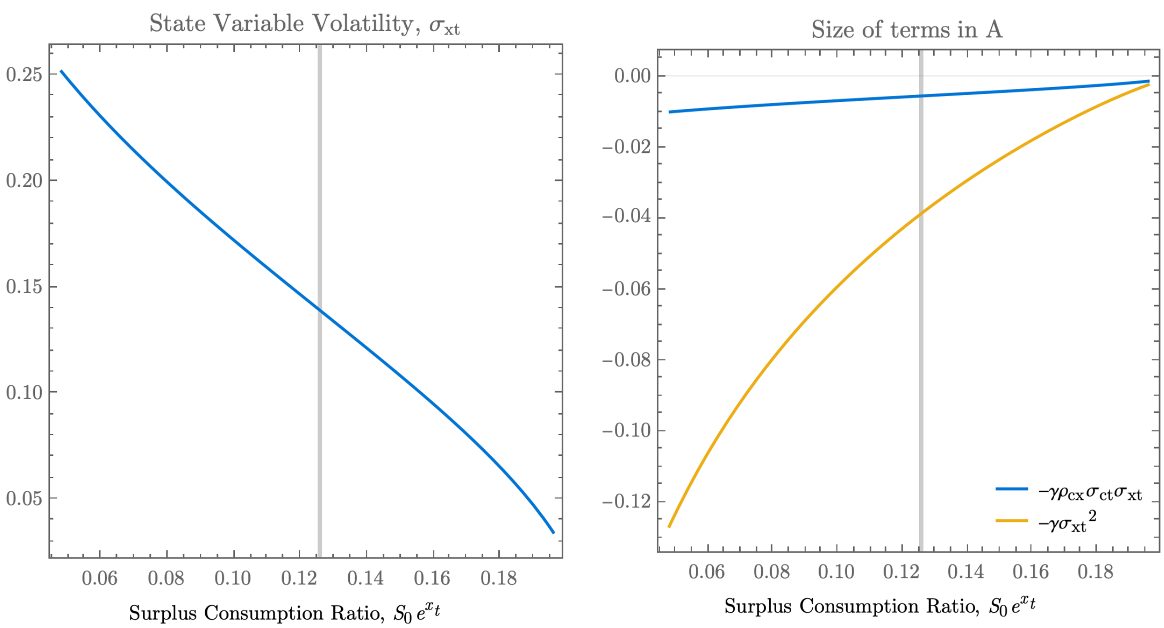

Figure 4: Terms related to TSU-Habit
The left plot shows the value of the volatility coefficient of the state variable in TSU-Habit. The right plot shows the magnitude of the two terms in the A function in TSU-Habit.

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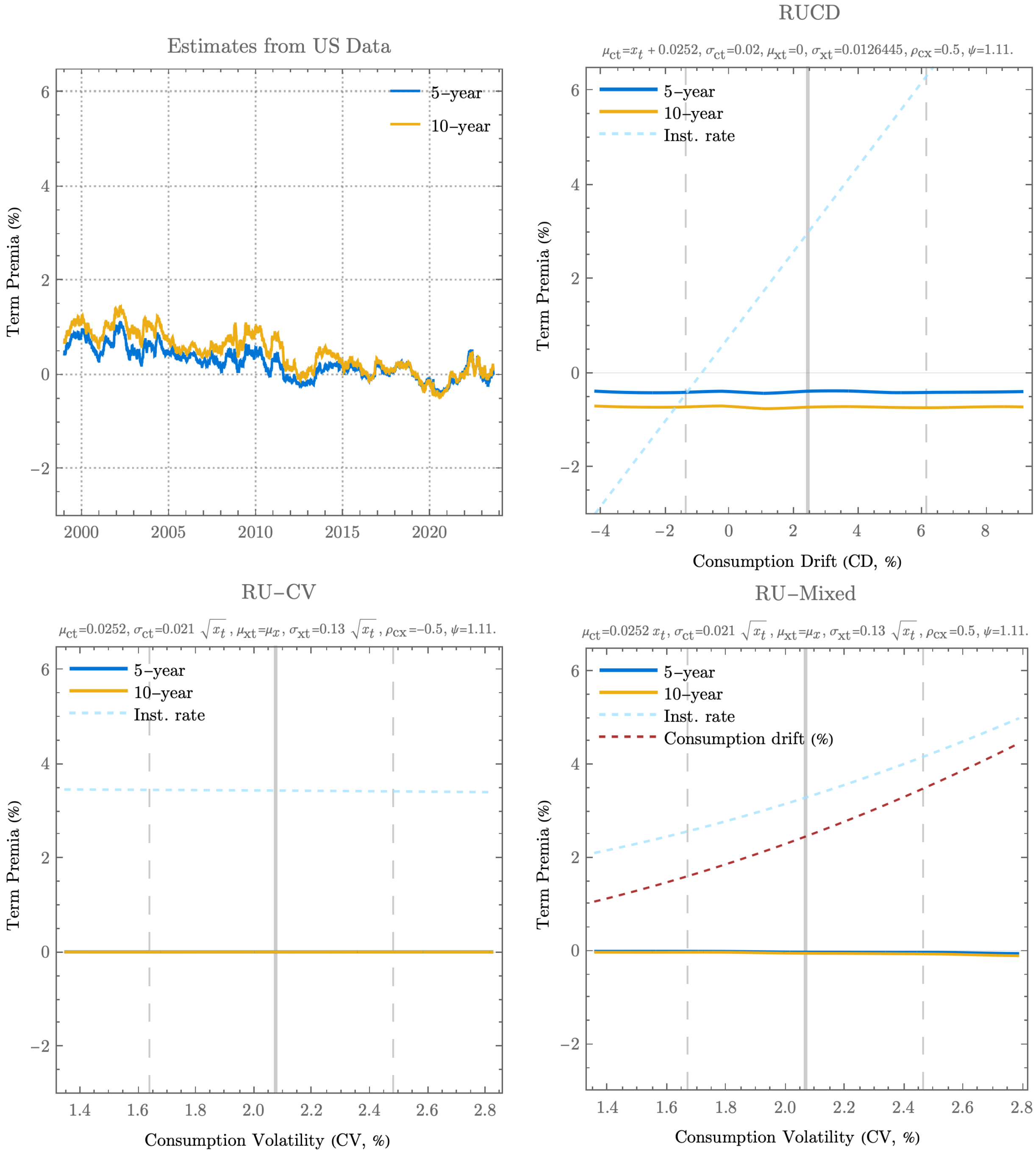

Figure 5: Term Premia in standard models with Recursive Utility
See Figure 3 for details.

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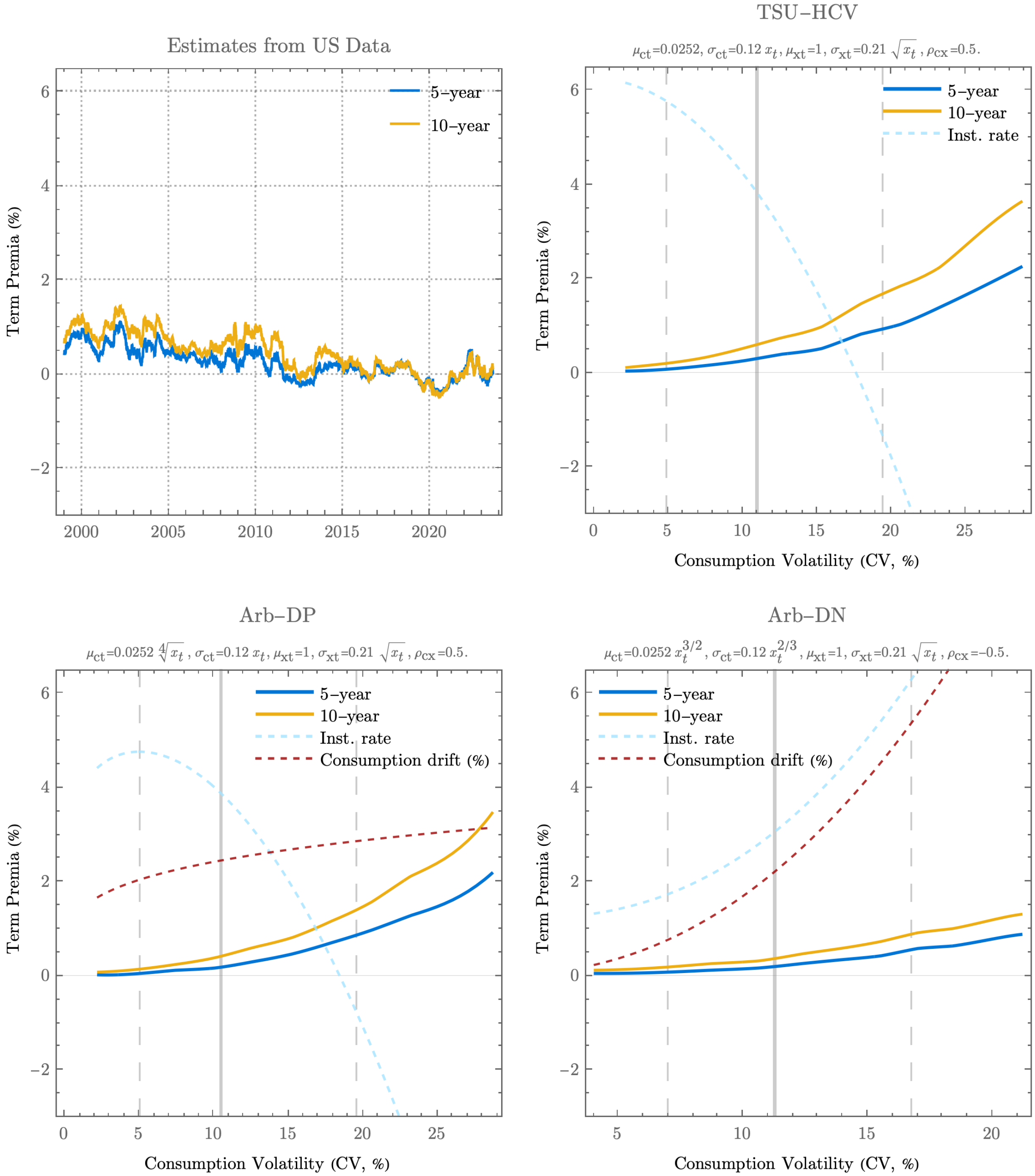

Figure 6: Term Premia in models with Time-Separable Utility and High consumption volatility
See Figure 3 for details.

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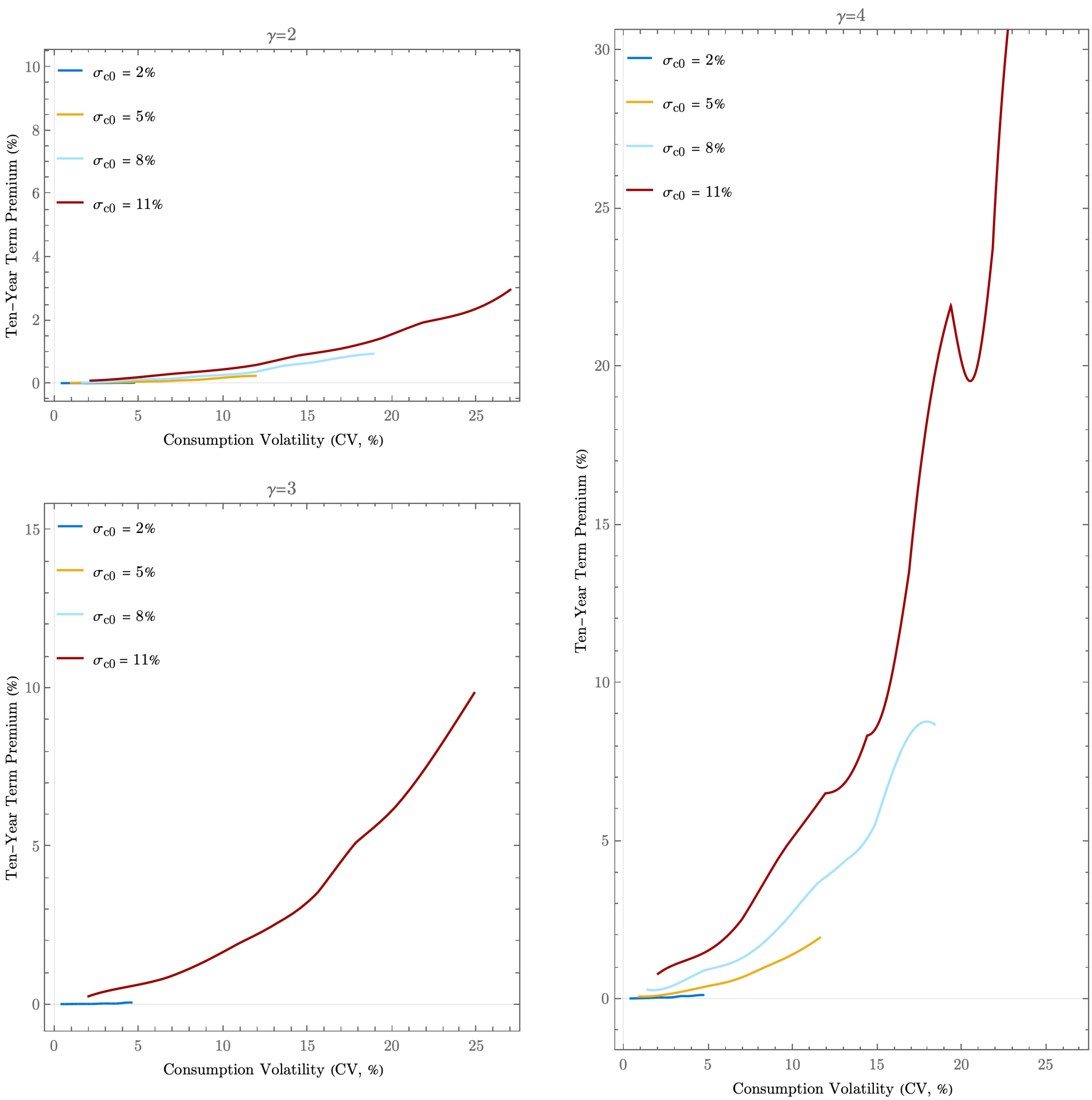

Figure 7: Ten-Year Term Premium in the TSU-HCV Variation for Different Steady State CV σ_{c0}, Levels and Risk Aversion Levels

The plots show the ten-year term premium for different variations, and they are drawn with the same scale. Each plot corresponds to a different value of the risk aversion paramter γ, and each line corresponds to a different value for the steady state value of CV. The range of CV over which the lines are drawn correspond to the values of CV that can reasonably be acquired (these are the same ranges as in the previous figures).

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References

Abrahams, M., Adrian, T., Crump, R. K., Moench, E. and Yu, R. (2016). Decomposing real and nominal yield curves. *Journal of Monetary Economics*, **84**, 182–200.

Ait-Sahalia, Y., Parker, J. A. and Yogo, M. (2004). Luxury goods and the equity premium. *The Journal of Finance*, **59** (6), 2959–3004.

Angeletos, G.-M., Collard, F. and Dellas, H. (2018). Quantifying confidence. *Econometrica*, **86** (5), 1689–1726.

Backus, D. K., Gregory, A. W. and Zin, S. E. (1989). Risk premiums in the term structure: Evidence from artificial economies. *Journal of Monetary Economics*, **24** (3), 371–399.

Bansal, R. and Shaliastovich, I. (2013). A Long-Run Risks Explanation of Predictability Puzzles in Bond and Currency Markets. *The Review of Financial Studies*, **26** (1), 1–33.

— and Yaron, A. (2004). Risks for the long run: A potential resolution of asset pricing puzzles. *The Journal of Finance*, **59** (4), 1481–1509.

Barro, R. J. (2006). Rare disasters and asset markets in the twentieth century. *The Quarterly Journal of Economics*, **121** (3), 823–866.

Bauer, M. D. and Rudebusch, G. D. (2020). Interest rates under falling stars. *American Economic Review*, **110** (5), 1316–1354.

Beechey, M. J. and Wright, J. H. (2009). The high-frequency impact of news on long-term yields and forward rates: Is it real? *Journal of Monetary Economics*, **56** (4), 535–544.

Bekaert, G., Engstrom, E. and Grenadier, S. R. (2010). Stock and bond returns with moody investors. *Journal of Empirical Finance*, **17** (5), 867–894.

Benzoni, L., Collin-Dufresne, P. and Goldstein, R. S. (2011). Explaining asset pricing puzzles associated with the 1987 market crash. *Journal of Financial Economics*, **101** (3), 552–573.

Campbell, J. Y. and Cochrane, J. H. (1999). By force of habit: A consumption-based explanation of aggregate stock market behavior. *Journal of Political Economy*, **107** (2), 205–251.

—, Pflueger, C. and Viceira, L. M. (2020). Macroeconomic drivers of bond and equity risks. *Journal of Political Economy*, **128** (8), 3148–3185.

Chen, Y., Cosimano, T. F. and Himonas, A. A. (2010). Continuous time one-dimensional asset-pricing models with analytic price–dividend functions. *Economic theory*, **42** (3), 461–503.

—, —, — and Kelly, P. (2009). Asset pricing with long run risk and stochastic differential utility: An analytic approach. *Available at SSRN 1502968*.

Cochrane, J. H. (2009). *Asset pricing: Revised edition*. Princeton University Press.

— and Piazzesi, M. (2005). Bond risk premia. *American Economic Review*, **95** (1), 138–160.

— and — (2009). Decomposing the yield curve. In *AFA 2010 Atlanta Meetings Paper*.

Collin-Dufresne, P., Johannes, M. and Lochstoer, L. A. (2016). Parameter learning in general equilibrium: The asset pricing implications. *American Economic Review*, **106** (3), 664–698.

Constantinides, G. M. and Duffie, D. (1996). Asset pricing with heterogeneous consumers. *Journal of Political Economy*, **104** (2), 219–240.

d 'Amico, S., Kim, D. H. and Wei, M. (2018). TIPS from TIPS: the informational content of treasury inflation-protected security prices. *Journal of Financial and Quantitative Analysis*, **53** (1), 395–436.

Den Haan, W. J. (1995). The term structure of interest rates in real and monetary economies. *Journal of Economic Dynamics and Control*, **19** (5-7), 909–940.

Donaldson, J. B., Johnsen, T. and Mehra, R. (1990). On the term structure of interest rates. *Journal of Economic Dynamics and Control*, **14** (3-4), 571–596.

Duffee, G. R. (2002). Term premia and interest rate forecasts in affine models. *The Journal of Finance*, **57** (1), 405–443.

— (2013). Bond pricing and the macroeconomy. *Handbook of the Economics of Finance*, **2**, 907–967.

Duffie, D. and Epstein, L. G. (1992). Stochastic differential utility. *Econometrica: Journal of the Econometric Society*, pp. 353–394.

Ellison, M. and Tischbirek, A. (2021). Beauty contests and the term structure. *Journal of the European Economic Association*, **19** (4), 2234–2282.

Fama, E. F. and Bliss, R. R. (1987). The information in long-maturity forward rates. *The American Economic Review*, pp. 680–692.

Gabaix, X. (2012). Variable rare disasters: An exactly solved framework for ten puzzles in macro-finance. *The Quarterly Journal of Economics*, **127** (2), 645–700.

Gomez-Cram, R. and Yaron, A. (2021). How important are inflation expectations for the nominal yield curve? *The Review of Financial Studies*, **34** (2), 985–1045.

Gürkaynak, R. S., Sack, B. and Wright, J. H. (2007). The US treasury yield curve: 1961 to the present. *Journal of Monetary Economics*, **54** (8), 2291–2304.

—, — and Wright, J. H. (2010). The TIPS yield curve and inflation compensation. *American Economic Journal: Macroeconomics*, **2** (1), 70–92.

He, Z. and Krishnamurthy, A. (2013). Intermediary asset pricing. *American Economic Review*, **103** (2), 732–770.

Hsu, A., Li, E. X. and Palomino, F. (2021). Real and nominal equilibrium yield curves. *Management Science*, **67** (2), 1138–1158.

Jappelli, R., Subrahmanyam, M. G. and Pelizzon, L. (2023). Quantitative easing, the repo market, and the term structure of interest rates.

Katagiri, M. (2022). Equilibrium yield curve, the phillips curve, and monetary policy. *Journal of Money, Credit and Banking*, **54** (8), 2235–2272.

Kekre, R., Lenel, M. and Mainardi, F. (2022). Monetary policy, segmentation, and the term structure.

Kliem, M. and Meyer-Gohde, A. (2022). (un) expected monetary policy shocks and term premia. *Journal of Applied Econometrics*, **37** (3), 477–499.

Lettau, M. and Wachter, J. A. (2011). The term structures of equity and interest rates. *Journal of Financial Economics*, **101** (1), 90–113.

Malloy, C. J., Moskowitz, T. J. and Vissing-Jørgensen, A. (2009). Long-run stockholder consumption risk and asset returns. *The Journal of Finance*, **64** (6), 2427–2479.

Mehra, R. *et al.* (2007). The equity premium puzzle: A review. *Foundations and Trends® in Finance*, **2** (1), 1–81.

Melissinos, E. (2023). A perturbation solution for the pricing of long-term bonds under recursive utility.

Neely, C. J., Rapach, D. E. *et al.* (2008). Real interest rate persistence: evidence and implications. *Federal Reserve Bank of St. Louis Review*, **90** (6), 609–641.

Pflueger, C. E. and Viceira, L. M. (2016). Return predictability in the treasury market: real rates, inflation, and liquidity. *Handbook of Fixed-Income Securities*, pp. 191–209.

Piazzesi, M. and Schneider, M. (2006). Equilibrium yield curves. *NBER Macroeconomics Annual*, **21**, 389–442.

Rietz, T. A. (1988). The equity risk premium a solution. *Journal of Monetary Economics*, **22** (1), 117–131.

Rudebusch, G. D. and Swanson, E. T. (2008). Examining the bond premium puzzle with a DSGE model. *Journal of Monetary Economics*, **55**, S111–S126.

— and — (2012). The bond premium in a DSGE model with long-run real and nominal risks. *American Economic Journal: Macroeconomics*, **4** (1), 105–43.

Schneider, A. (2022). Risk-sharing and the term structure of interest rates. *The Journal of Finance*, **77** (4), 2331–2374.

Shiller, R. J. (1979). The volatility of long-term interest rates and expectations models of the term structure. *Journal of Political Economy*, **87** (6), 1190–1219.

Singleton, K. J. (1980). Expectations models of the term structure and implied variance bounds. *Journal of Political Economy*, **88** (6), 1159–1176.

Tsai, J. (2015). Rare disasters and the term structure of interest rates. *Available at SSRN 2669992*.

— and Wachter, J. A. (2018). Pricing long-lived securities in dynamic endowment economies. *Journal of Economic Theory*, **177**, 848–878.

Van Binsbergen, J. H., Fernández-Villaverde, J., Koijen, R. S. and Rubio-Ramírez, J. (2012). The term structure of interest rates in a DSGE model with recursive preferences. *Journal of Monetary Economics*, **59** (7), 634–648.

Vayanos, D. and Vila, J.-L. (2021). A preferred-habitat model of the term structure of interest rates. *Econometrica*, **89** (1), 77–112.

Wachter, J. A. (2006). A consumption-based model of the term structure of interest rates. *Journal of Financial Economics*, **79** (2), 365–399.

— (2013). Can time-varying risk of rare disasters explain aggregate stock market volatility? *The Journal of Finance*, **68** (3), 987–1035.

— and Zhu, Y. (2019). Learning with rare disasters. *Available at SSRN 3407397*.

The full paper can be found at: https://errikos-melissinos.com.

Appendix

A Definitions

In the following I provide a complete set of definitions.[53]

- Throughout the paper, terms like yields, returns, term premia etc. should be understood as referring to their real counterparts, unless otherwise specified. The distinction is still made explicit when necessary to avoid confusion.

- A **nominal zero-coupon bond** with maturity m is a security paying one unit of currency after m years.[54]

- A **real bond** with maturity m is a security paying one unit of currency times an adjustment, that corrects for the elapsed inflation from the time it was issued until its maturity. The payment occurs after m years. Equivalently, a real bond is a security that pays the value of some basket of goods[55] when it matures.[56]

- Q_t^m is the price of the bond with maturity m at time t.

- **Real (or nominal) yield** at time t of a real or (nominal) bond with maturity m years where Q_t^m is the price of the corresponding bond, which is perfectly liquid:[57]
$$y_t^m = \frac{-\log(Q_t^m)}{m}, \quad m > 0$$

- **Yield spread** at time t between maturity m and n, where typically $m > n$:
$$y_t^m - y_t^n$$

- **The yield curve or the term structure of interest rates** refers to yields as a function of maturity. The yield curve is sloping upward/downward (or

[53] Including for some concepts to which I make reference in the main paper, without ever using in expressions.

[54] In the paper bonds always refer to zero-coupon bonds.

[55] Here there is an implicit assumption that individuals primarily care about this specific basket of goods. This basket of goods is also relevant for the calculation of inflation. Without this assumption the study of real interest rates would be significantly hindered.

[56] A real bond of maturity $m+1$ one year ago is also equivalent to a real bond with maturity m today up to a renormalisation so that the principals match.

[57] Actual bonds' prices may deviate from Q_t^m due to liquidity considerations.

the slope of the yield curve is positive/negative) when yields are an increasing/decreasing function with respect to maturity. It is also possible that the slope is positive for some maturities and flat or negative for other maturities.

- **Annualised Gross Return** of a bond with maturity m from time t to $t+s$:

$$R_{t,t+s}^{m} = \sqrt[s]{Q_{t+s}^{m-s}/Q_{t}^{m}}$$

- **Log return or just return**[58] of a bond with maturity m from time t to $t+s$:

$$r_{t,t+s}^{m} = \log\left(R_{t,t+s}^{m}\right) = \frac{\log\left(Q_{t+s}^{m-s}\right) - \log\left(Q_{t}^{m}\right)}{s}$$

- **Instantaneous return** of a bond with maturity m at time t:

$$r_{t}^{m} = \lim_{s\to 0} r_{t,t+s}^{m}$$

- **Instantaneous short rate or just short rate** at time t:

$$r_{t} = \lim_{m\to 0} r_{t}^{m} = \lim_{m\to 0} y_{t}^{m}$$

- In the main paper yields are also referred to as *long-term interest rates*, whereas *interest rates* in general also include the short rate.

- **m-to-n year forward** at time t:

$$f_{t}^{m,n} = \frac{\log(Q_{t}^{m}) - \log(Q_{t}^{n})}{n-m}$$

- **Instantaneous m-year forward** is:

$$f_{t}^{m} = \lim_{n\to m} f_{t}^{m,n}$$

- **Term or risk premium** of bond with maturity m at time t, where r_t is the instantaneous rate of return at time t:[59]

$$TP_{t}^{m} = \frac{-\log(Q_{t}^{m})}{m} - \frac{E_{t}\left[\int_{0}^{m} r_{t+s}ds\right]}{m}$$

[58]For convenience I refer to log return when I use the term return.

[59]Equivalent definitions are given in discrete time by Cochrane and Piazzesi (2009).

– If the term premium is zero for all m and t, this implies that the expected excess return from holding long-term bonds over any period is also 0. This can be seen from the following equivalent definition, where rx_t^m is the instantaneous excess return from holding a bond of maturity m:[60]

$$TP_t^m = \frac{E_t\Big[\int_0^m r_{t+s}^{m-s} - r_{t+s} d\tau\Big]}{m} \equiv \frac{E_t\Big[\int_0^m rx_{t+s}^{m-s} d\tau\Big]}{m}$$

– Here I have used the fact that:

$$\begin{aligned}
-\log(Q_t^m) =& \Big(-\log\left(Q_t^m\right) + \log\left(Q_{t+m/N}^{m-m/N}\right)\Big) + \Big(-\log\left(Q_{t+m/N}^{m-m/N}\right) + \log\left(Q_{t+2m/N}^{m-2m/N}\right)\Big) + \\
& \ldots + \Big(-\log\left(Q_{t+m-m/N}^{m/N}\right) + \underbrace{\log\left(Q_{t+m}^{0}\right)}_{=0}\Big) \\
=& \frac{m}{N}\left(r_{t,t+m/N}^m + r_{t+m/N,t+2m/N}^{m-m/N} + \ldots + r_{t+m-m/N,t+m}^{m/N}\right) \\
=& \int_0^m r_{t+s}^{m-s} ds
\end{aligned}$$

where N is some positive integer. The last line follows by N going to infinity, which means that the sum becomes an integral and the returns become instantaneous returns.

– Given that Q is the price of a bond that is perfectly liquid, the term premium does not include a liquidity premium.

- I also refer to the quantity used above:

$$\frac{E_t\Big[\int_0^m r_{t+s} ds\Big]}{m}$$

as **risk-neutral yield** of bond with maturity m.

- **Term or risk premium** of m-to-n year maturity forward at time t, where r_t is the instantaneous rate of return at time t:

$$TP_t^{m,n} = \frac{\log(Q_t^m) - \log(Q_t^n)}{n-m} - \frac{E_t\Big[\int_{t+m}^{t+n} r_\tau d\tau\Big]}{n-m}$$

- The second term on the right hand side of the equation above is the risk-neutral

[60] If the excess return were positive for any period, then the expected term premium for the remaining period would have to be negative. This violated the initial assumption.

m-to-n year forward.

- In the paper many of the variables introduced here depend on time only through the state variable. So they will be denoted instead as:

$$Q(x_t, m), y(x_t, m), R_s(x_t, m), r_s(x_t, m), r(x_t, m)$$

$$r(x_t), f(x_t, m, n), f(x_t, m), TP(x_t, m), TP(x_t, m, n)$$

- In the main paper I also refer to the **value of the risk-neutral bond**. This is the implied value attached to a bond by a risk-neutral investor and it can be defined based on the risk-neutral yield defined above:[61]

$$H(x_t, m) = e^{-E_t\left[\int_0^m r_{t+s} ds\right]}$$

- The **strong version of the Expectations Hypothesis** holds when:

$$TP_t^m = 0, \quad \text{for all } m$$

- The **weak version of the Expectations Hypothesis** holds when:

$$TP_t^m = g(m), \quad \text{for all } m$$

  where g is some function of maturity, independent of the state of the economy and independent of time.

- **Predictability** refers to the ability of predicting movements in excess returns. The prediction could be based on any information, but the literature has focused on using information in yields to predict subsequent yields in the future.

- **Excess volatility** of interest rates refers to long-term interest rate variations that are too large to be explained by the variation of the short rate alone, while keeping the discount rate constant.[62]

[61] In the main paper, I also present an equivalent definition in Section 4.7.3, which also shows the intuition regarding the calculation of the term premium in this paper.

[62] To be completely precise excess volatility needs to be defined in terms of some benchmark model. As, I do not investigate excess volatility directly, I do not provide such a definition.

B Explanatory power of the principal components of real interest rates

Apart from Figure 2, I also look at a series of regressions to demonstrate the strong dependence of nominal rates on real rates. In particular, I extract the first two principal components from a series of real yields with different maturities.[63] I only use two components because they explain more than 99.95% of the variance of real yields. Next, I regress nominal yields and nominal yield spreads on these two principal components.[64] Indeed, I find that the information contained within real rates explains most of the movements of nominal rates. The results are shown in Table 7. The coefficients are highly significant for both components, but more importantly the R-squared is high in these regressions. For the level regressions it ranges from 87% to 93%, while for the spread regressions it ranges from 69% to 79%. Thus, both the level and the spread of nominal rates is mostly explained by the information and hence the processes that generate the real term structure.

Table 7: Regressions of the level and the spread of nominal bonds on the principal components extracted from the real term structure

	5 yr	10 yr	5-10 yr spread	15 yr	5-15 yr spread	20 yr	5-20 yr spread
Intercept	2.94***	3.73***	0.79***	4.13***	1.19***	4.29***	1.35***
	(0.01)	(0.00)	(0.00)	(0.00)	(0.01)	(0.00)	(0.01)
comp1	0.28***	0.26***	-0.02***	0.25***	-0.04***	0.23***	-0.05***
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
comp2	0.43***	-0.24***	-0.66***	-0.55***	-0.97***	-0.67***	-1.09***
	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
R-squared	0.87	0.93	0.69	0.93	0.74	0.93	0.79
R-squared Adj.	0.87	0.93	0.69	0.93	0.74	0.93	0.79

[63]The principal components are extracted from yields of all yearly maturities from two to twenty years.

[64]A similar exercise in performed by Abrahams *et al.* (2016) and they also find similar results. In their case it is the real rates that are regressed on the principal components of the nominal rates. I do the inverse exercise because I ask how much nominal rates are explained by real rates.

C Explanation of the components of the pricing equation

This section provides an explanation for each part of the pricing equation (14) which I repeat here:

$$-Q_m - r(x_t)Q + \big(\log(\phi)x_t + A(x_t)\big)Q_x + \frac{\sigma_{xt}^2}{2}Q_{xx} = 0 \tag{24}$$

- In the simplest case $\phi = 1$, $A(x) = 0$ and $\sigma_x(x) = 0$ for all x. Then the equation is:

  $$Q_m = -r(x_t)Q = -x_tQ$$

  This corresponds to an economy with a constant state. Figure 8 shows that in this economy yields are always equal to the short rate, term premia are equal to 0 and given a state of the economy nothing will ever change.

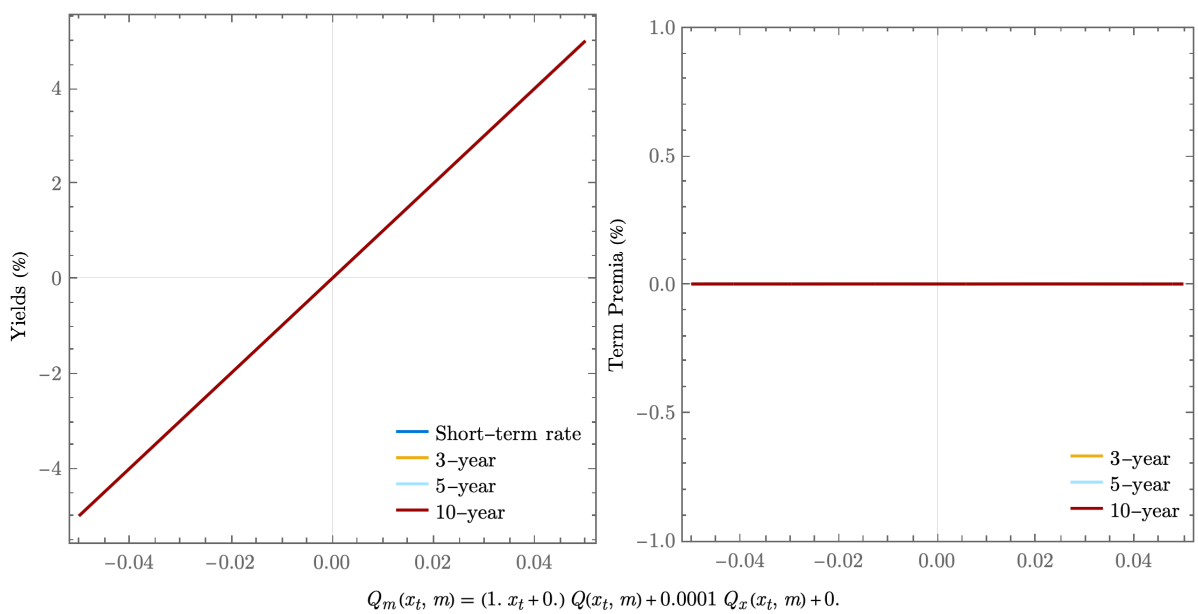


Figure 8: The left plot shows the short-term rate and yields of different maturities as a function of the state variable. The right plot shows the the term premia for different maturities as a function of the state variable.

- $\phi \neq 1$:

  $$Q_m = r(x_t)Q - \log(\phi)x_tQ_x = x_tQ - \log(0.9)x_tQ_x$$

  Here there is again no volatility of the state variable. Thus, this corresponds to a deterministic economy. However, the state is not constant, it drifts towards the state $x_t = 0$, which can be thought of as the steady state. This implies that long-term yields will lie between the contemporaneous short rate and the steady state short rate. As shown in Figure 9 this results in a characteristic picture, in which all yields intersect at the steady state. If the process moved towards the steady state faster (lower ϕ), then the yields would be more spread out. Given that there is no uncertainty, the term premia are again zero.

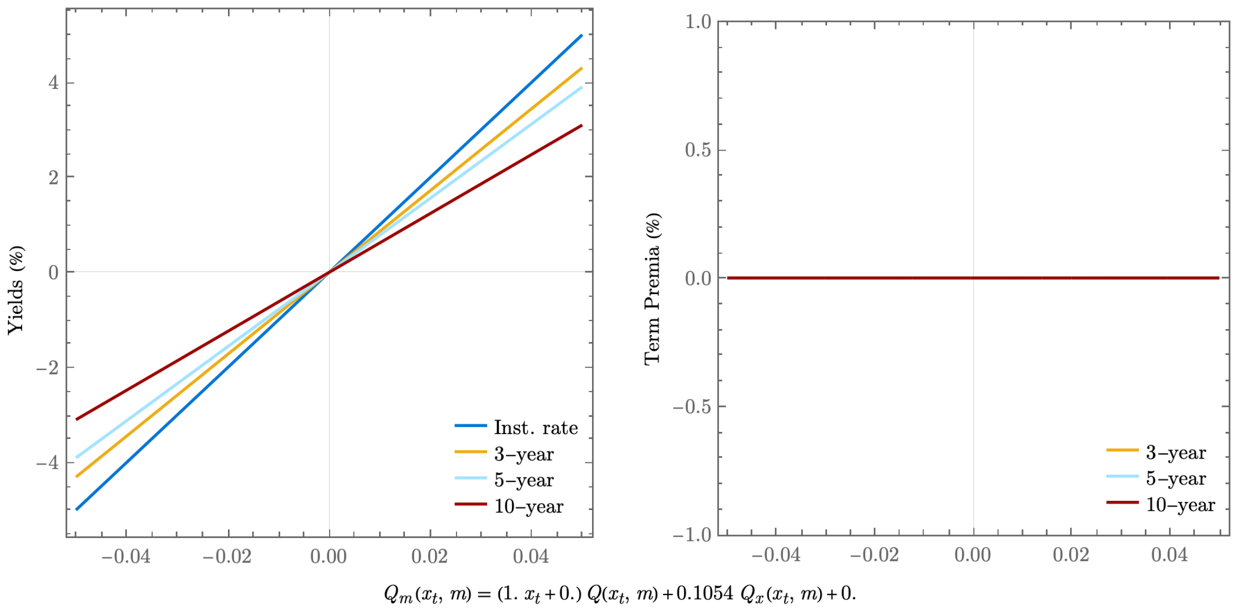


Figure 9: The left graph show the short-term rate, the five-year yield and the five-year risk-neutral yield as a function of CD. The right graphs shows the decomposition of the five-to-ten year forward into the term premium and the risk-neutral yield.

- $A(x_t) = c \neq 0$:

$$Q_m = r(x_t)Q - \big(\log(\phi)x_t + A(x_t)\big)Q_x = x_tQ - (\log(0.9)x_t + 0.01)Q_x$$

As stated in the main paper A generates term premia. This case does not directly correspond to some economic situation because, the state variable volatility is again 0, and in the actual economic models this also implies $A(x_t) = 0$. However, for the purposes of intuition I show the "yields" and "term premia" that arise. As Figure 10 shows, now the yields do not intersect at the steady state. Now the longer-term yields are higher at the steady state. This implies positive term premia and indeed as shown in the right panel, term premia are positive, proportional to the maturity of the bond and constant with respect to the state variable. The latter fact is due to $A(x_t)$ being constant for all x_t and the fact that yields are linear. Finally the term premia are positive, because A is positive and the short rate is increasing with respect to the state variable.

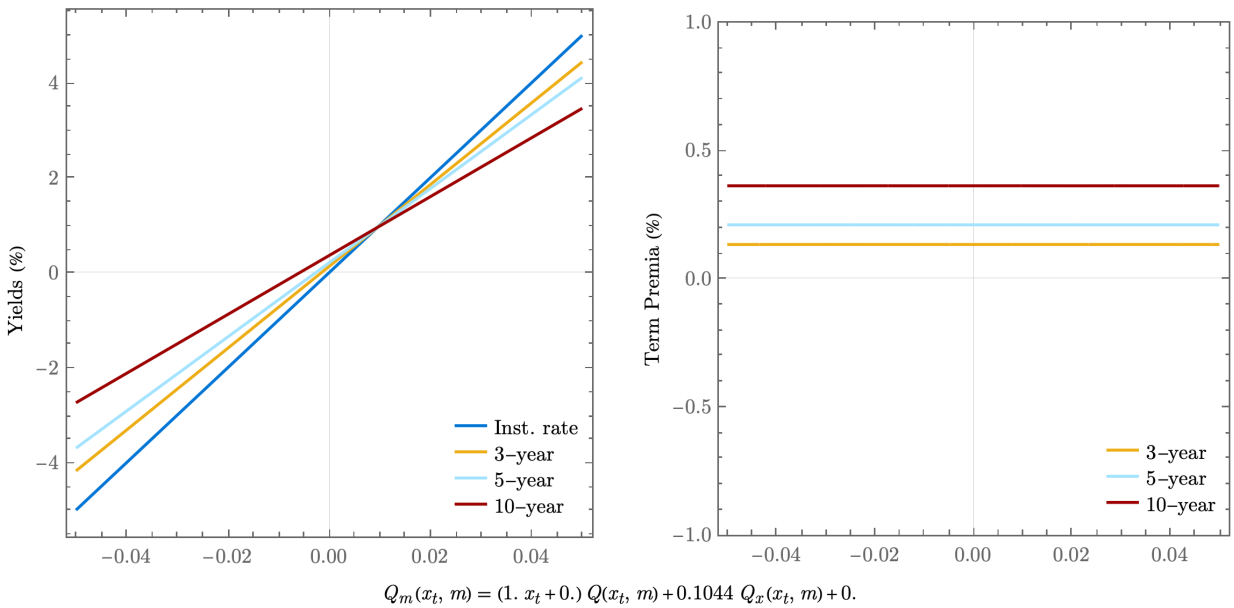


Figure 10: The left graph show the short-term rate, the five-year yield and the five-year risk-neutral yield as a function of CD. The right graphs shows the decomposition of the five-to-ten year forward into the term premium and the risk-neutral yield.

- $A(x_t) = 0.0005{+}0.02x_t$. This means that now A changes with the state variable. The result is shown in Figure 11. Term premia follow the behaviour of A. The correspondence would not be so close, if the short rate were a non-linear function of the state variable.

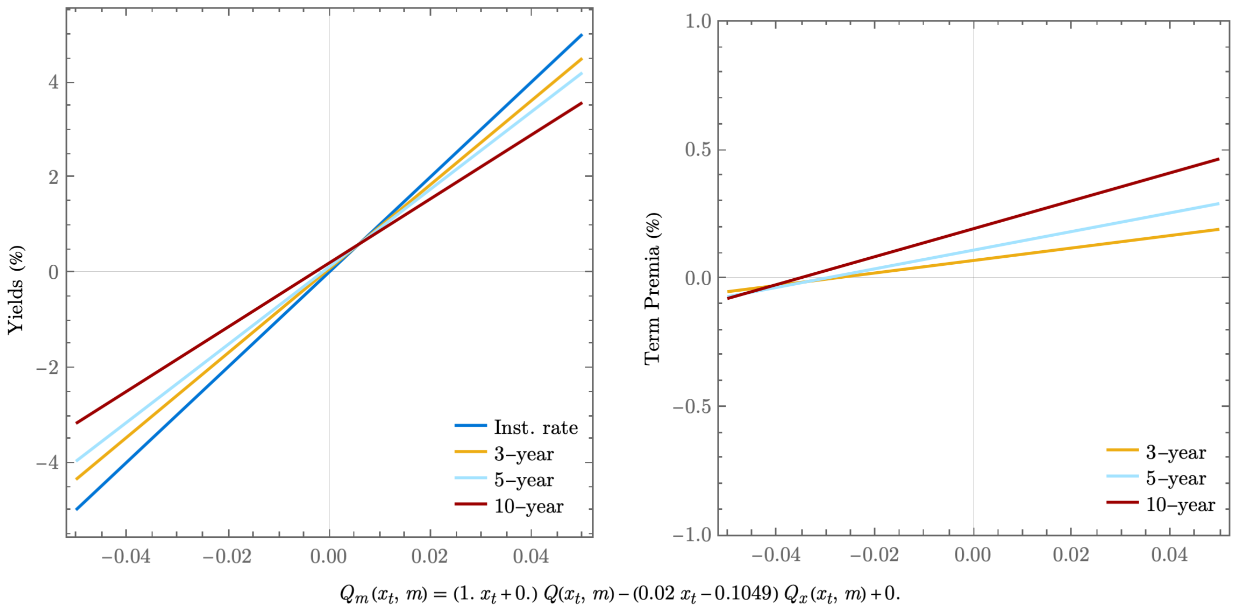


Figure 11: The left graph show the short-term rate, the five-year yield and the five-year risk-neutral yield as a function of CD. The right graphs shows the decomposition of the five-to-ten year forward into the term premium and the risk-neutral yield.

- $\sigma_{xt} \neq 0$:

$$Q_m = r(x_t)Q - \log(\phi)x_t Q_x + \frac{\sigma_{xt}^2}{2}Q_{xx} = x_t Q - \log(0.9)x_t Q_x - \frac{0.03^2}{2}Q_{xx}$$

Here $A(x_t) = 0$. Thus, the effect of volatility can be seen. This case corresponds to a case where there is volatility of the short rate, but there is again no priced risk. So there is no risk premium. This can be seen on the right panel of Figure 12.[65] Nevertheless, the yields are not the same as in the deterministic case with steady state reversion, as they do not intersect at the steady state. The long-term yields are pushed downwards, and, even though it might not be obvious, the effect of uncertainty increases more than linearly with maturity. This effect is due to so-called convexity that is common in finance. In particular, the price of the long-term bond is a convex decreasing function of the short-rate and this implies that lower interest rates have a higher effect on the price of the bond, especially for long maturities. Thus, given that there is variation and a chance for the short rate to reach lower levels, these will outweigh the high rates, and push long-term yields downward. Finally, this also means that a downward-sloping term structure does not necessarily imply negative term

[65] The term premia do not look completely flat because the Monte-Carlo calculation has some uncertainty in the calculation.

premia.

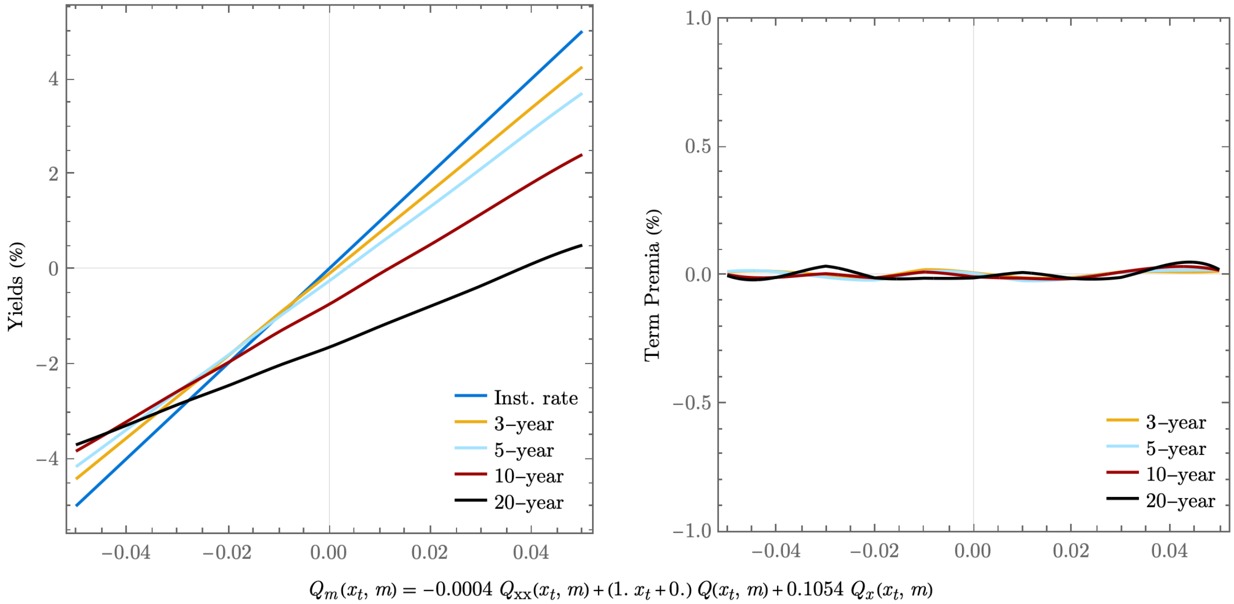


Figure 12: The left graph show the short-term rate, the five-year yield and the five-year risk-neutral yield as a function of considers. The right graphs shows the decomposition of the five-to-ten year forward into the term premium and the risk-neutral yield.

- full case:

$$Q_m = r(x_t)Q - \log(\phi)x_tQ_x + \frac{\sigma_{xt}^2}{2}Q_{xx} = x_tQ - \big(\log(0.9)x_t + 0.001\big)Q_x - \frac{0.005^2}{2}Q_{xx}$$

This case contains all the components. Unlike the previous case, as can be seen in Figure 13, the yields seem to intersect close to the steady state. Thus, the yield curve would often be flat in this economy. However, term premia are positive. The yields are close to flat at the steady state, because term premia and convexity largely cancel each other out.

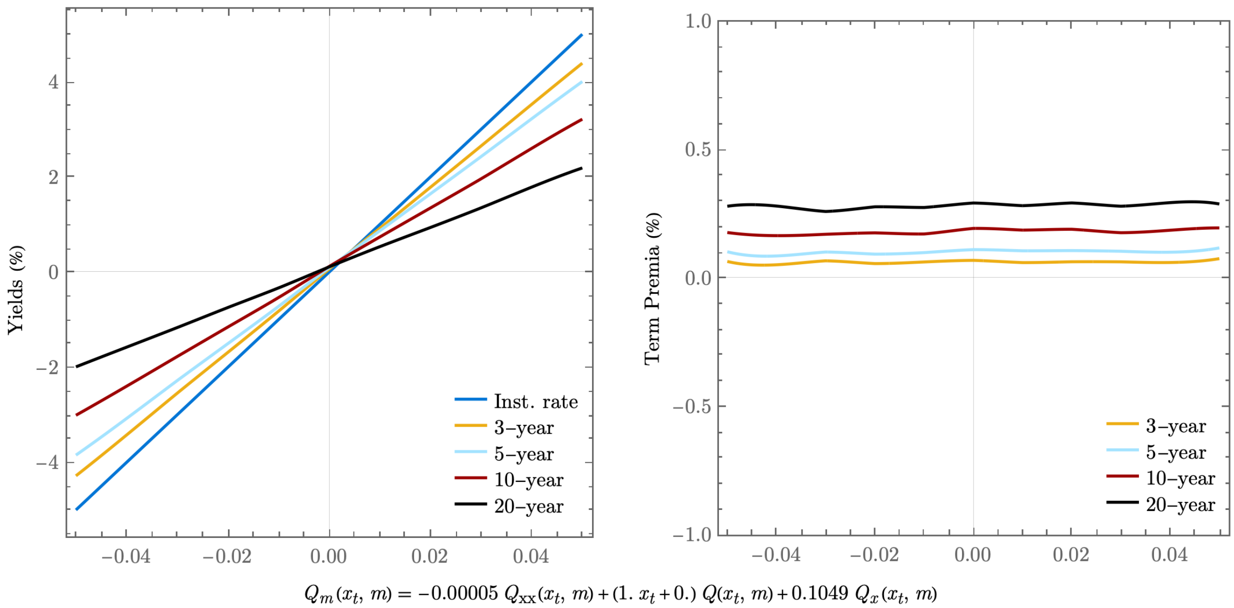


Figure 13: The left graph show the short-term rate, the five-year yield and the five-year risk-neutral yield as a function of CD. The right graphs shows the decomposition of the five-to-ten year forward into the term premium and the risk-neutral yield.

D Calibration of the state variable volatility

As mentioned in Subsection 5.1, the aim of the paper is to simultaneously match the variability of term premia and the variability of the short rate. I achieve this by calculating the range of the two-year TIPS security over the available sample in the Gürkaynak *et al.* (2010) dataset.[66] I find a range of 7.27%.[67] I then simulate time series with twelve year duration[68] for all the variations that I investigate. Based on these simulations I rank the range sizes and I aim for the tenth quantile to equal the range in the data. I do this for the models that are not able to produce highly variable term premia, in order to give these models the benefit of the doubt and the best chance to succeed. Namely, it is possible that the observed short rate volatility has been by chance relatively low and the underlying process is significantly more volatile. Thus, I want the model variations to be as volatile as possible in order to generate as large a time variability in term premia as possible. For the models that succeed in producing significantly time-varying term premia I again make sure that

[66] Two years is the shortest maturity in the data.

[67] This could be overestimating the plausible range as the maximum was achieved during the financial crisis, when the TIPS market was not behaving normally.

[68] This matches the length of the sample in Abrahams *et al.* (2016), but I should arguably change this to match the length of the sample in Gürkaynak *et al.* (2010). In any case the length of that sample is approximately 15 years.

the empirical volatility, as expressed by the observed range, falls within the model predictions, but I do not necessarily match the empirical with the tenth quantile. For each model variation, I show the value of the empirical range and the values of the model-implied tenth and ninetieth quantile ranges in the figures in Appendix F.

E Term Premia Measures

E.1 Figure from Abrahams *et al.* (2016)

TIPS Term Premium

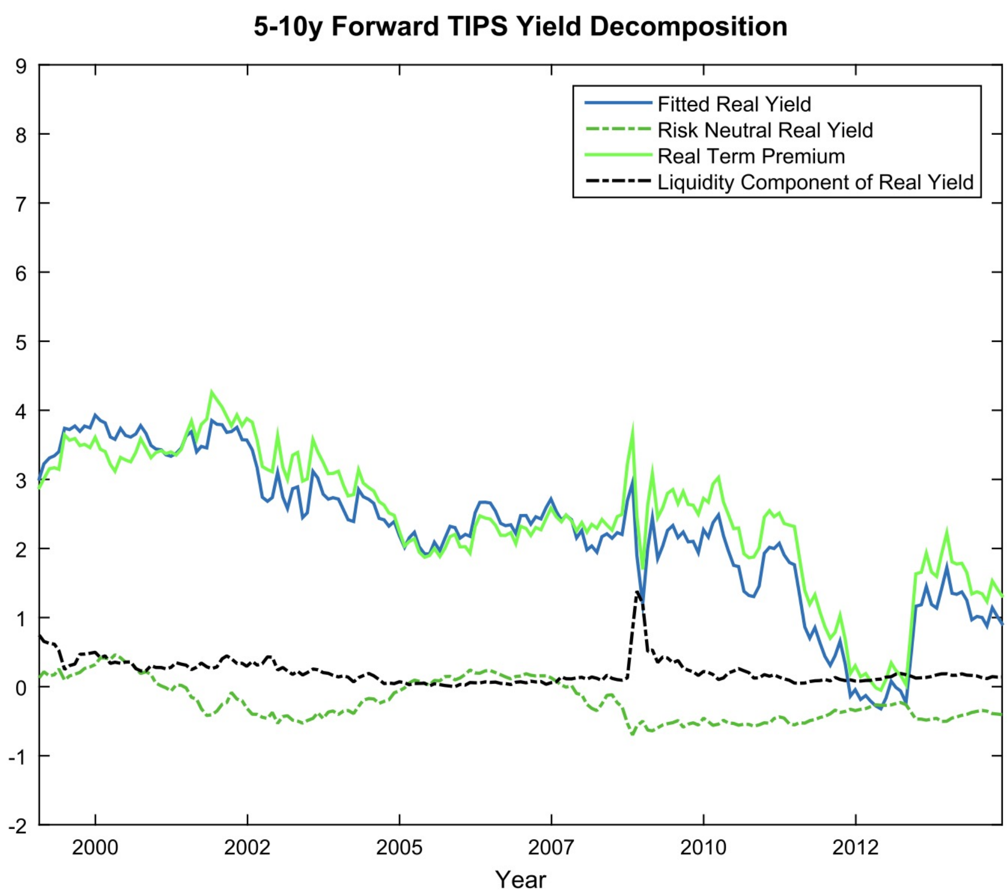


Figure 14: The figure shows the time series of the five-to-ten year forward term premium along with its decomposition to the risk-neutral yield, the term premium and the liquidity premium. I also show the same decomposition in the figures in this paper as a function of the state variable.

E.2 Term premium based on d 'Amico *et al.* (2018)

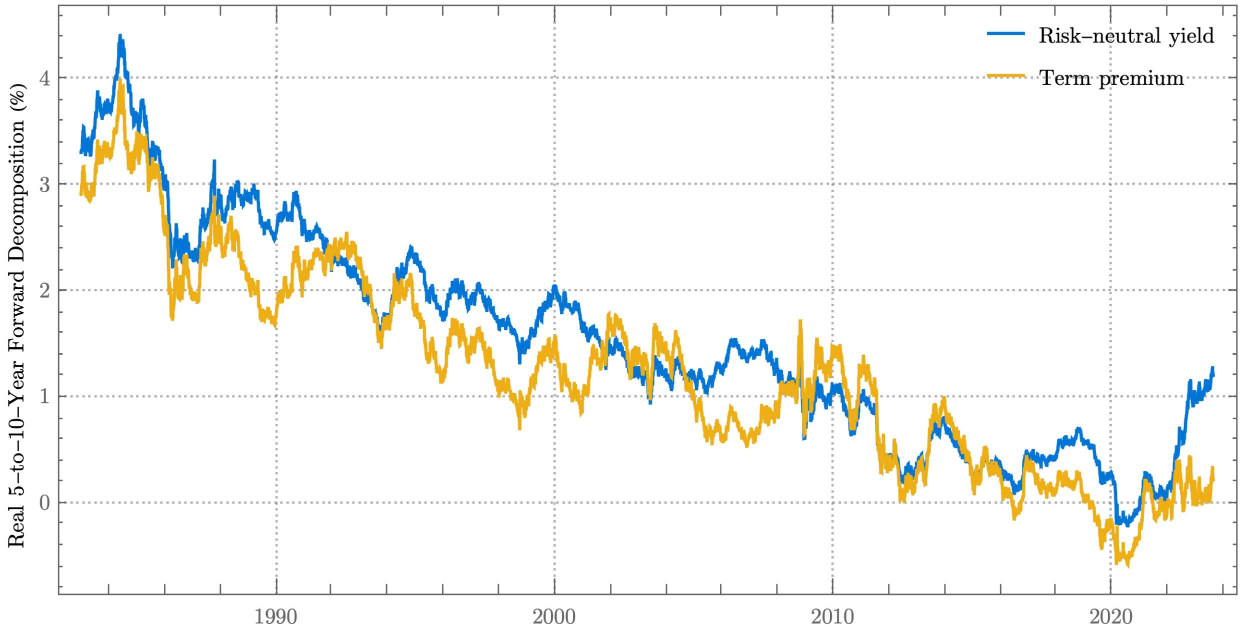


Figure 15: Time series of the forward 5-to-10 term premium for the US. (back to text)
This is the same quantity as the solid green line in Figure E.1 from Abrahams *et al.* (2016).
Data Source: `https://www.federalreserve.gov/econres/notes/feds-notes/tips-from-tips-update-and-discussions-20190521.html`

F Yields and Term Premia in Other Model Variations – Time-Separable Utility

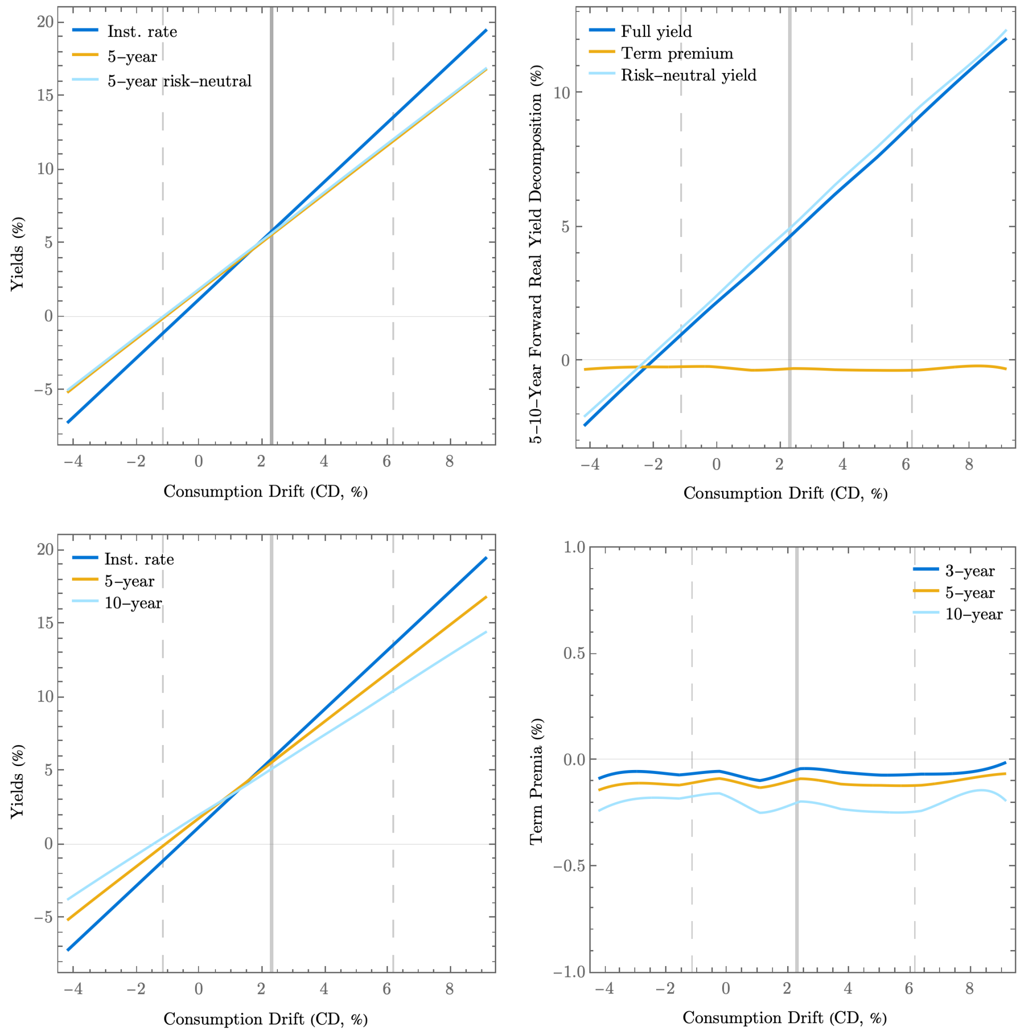
In this part I present more plots for the variations discussed in the main paper, and I also present results for other model variations. These are other variations should reinforce the conclusions in the main paper as a long series of calibrations is examined. The upper left and upper right plots are the same as in the main paper. The lower left plot shows the level of yields for different maturities as a function of the state variable. The lower right plot shows the level of the term premium for different maturities as a function of the state variable. Again each figure states the exact specification.

F.1 Contents

Names of all model variations shown in Appendix F. The abbreviations used here are: time-varying (tv), consumption drift (CD), consumption volatility (CV), time-separable utility (TSU), recursive utility (RU), intertemporal elasticity of substitution (IES).

Model Variation Description	**Abbreviation**	**References**
Tv CD with TSU.	TSU-CD	Figure 16
Tv CD with TSU and high risk aversion.	TSU-CD-HRA	Figure 17
Tv CD with TSU and low persistence.	TSU-CD-LP	Figure 18
Tv CD with TSU and high correlation ρ_{cx}.	TSU-CD-HCor	Figure 19
Tv CD with TSU and high impatience.	TSU-CD-HImp	Figure 20
Tv and high CD with TSU.	TSU-HCD	Figure 21
Tv CD with TSU and high CV.	TSU-CD-HCV	Figure 22
Tv CV with TSU.	TSU-CV	Figure 23
Tv CV with TSU and high risk aversion.	TSU-CV-HRA	Figure 24
Tv CV with TSU and high CD.	TSU-CV-HCD	Figure 25
Tv and high CV with TSU and positive correlation ρ_{cx}.	TSU-HCV	Figure 26
TV and high CV with TSU and negative correlation ρ_{cx}.	TSU-HCV-NCor	Figure 27
Both tv CD and CV, short-term rate decreasing in CV and ρ_{cx} positive.	TSU-Arb-DP	Figure 28
Both tv CD and CV, short-term rate increasing in CV and ρ_{cx} negative.	TSU-Arb-IN	Figure 29
Both tv CD and CV, short-term rate decreasing in CV and ρ_{cx} negative.	TSU-Arb-DN	Figure 30
Both tv CD and CV, short-term rate increasing in CV and ρ_{cx} positive.	TSU-Arb-IP	Figure 31
Tv external habit with TSU.	TSU-Habit	Figure 32
Tv external habit with TSU and low b.	TSU-Habit-L.b	Figure 33
Tv external habit with TSU and $b < 0$.	TSU-Habit-Neg.b	Figure 34
Tv external habit with TSU with constant state variable volatility.	TSU-Habit-ConstantSV	Figure 35
Tv CD with RU.	RU-CD	Figure 36
Tv CD with RU and high risk aversion.	RU-CD-HRA	Figure 37
Tv CD with RU with high IES.	RU-CD-HIES	Figure 38
Tv CD with RU with Low IES.	RU-CD-LIES	Figure 39
Tv CD with RU with high ρ_{cx}.	RU-CD-HCor	Figure 40
Tv CD with RU with ρ_{cx} negative.	RU-CD-NCor	Figure 41
Tv and high CD with RU.	RU-HCD	Figure 42
Tv CD with RU and high CV.	RU-CD-HCV	Figure 43
Tv and heteroskedastic CD with RU and ρ_{cx} positive.	RU-CD-Heterosk-PCor	Figure 44
Tv and heteroskedastic CD with RU and ρ_{cx} negative.	RU-CD-Heterosk-NCor	Figure 45
Tv CV with RU.	RU-CV	Figure 46
Tv CV with RU with high risk aversion.	RU-CV-HRA	Figure 47
Tv CV with RU and high persistence IES.	RU-CV-HP	Figure 48
Tv CV with RU and high IES.	RU-CV-HIES	Figure 49
Tv CV with RU and low IES.	RU-CV-LIES	Figure 50
Tv and high CV with RU and ρ_{cx} positive.	RU-HCV-PCor	Figure 51
Tv and high CV with RU and ρ_{cx} negative.	RU-HCV-NCor	Figure 52

F.2 TSU-CD, Calibration used in main paper, Figure 3



γ=2, ρ=0.012, $\mu_{\rm ct}=x_t+0.0252$, $\sigma_{\rm ct}$=0.02, ϕ=0.92, $\mu_{\rm xt}$=0, $\sigma_{\rm xt}$=0.0126445, $\rho_{\rm cx}$=0.5

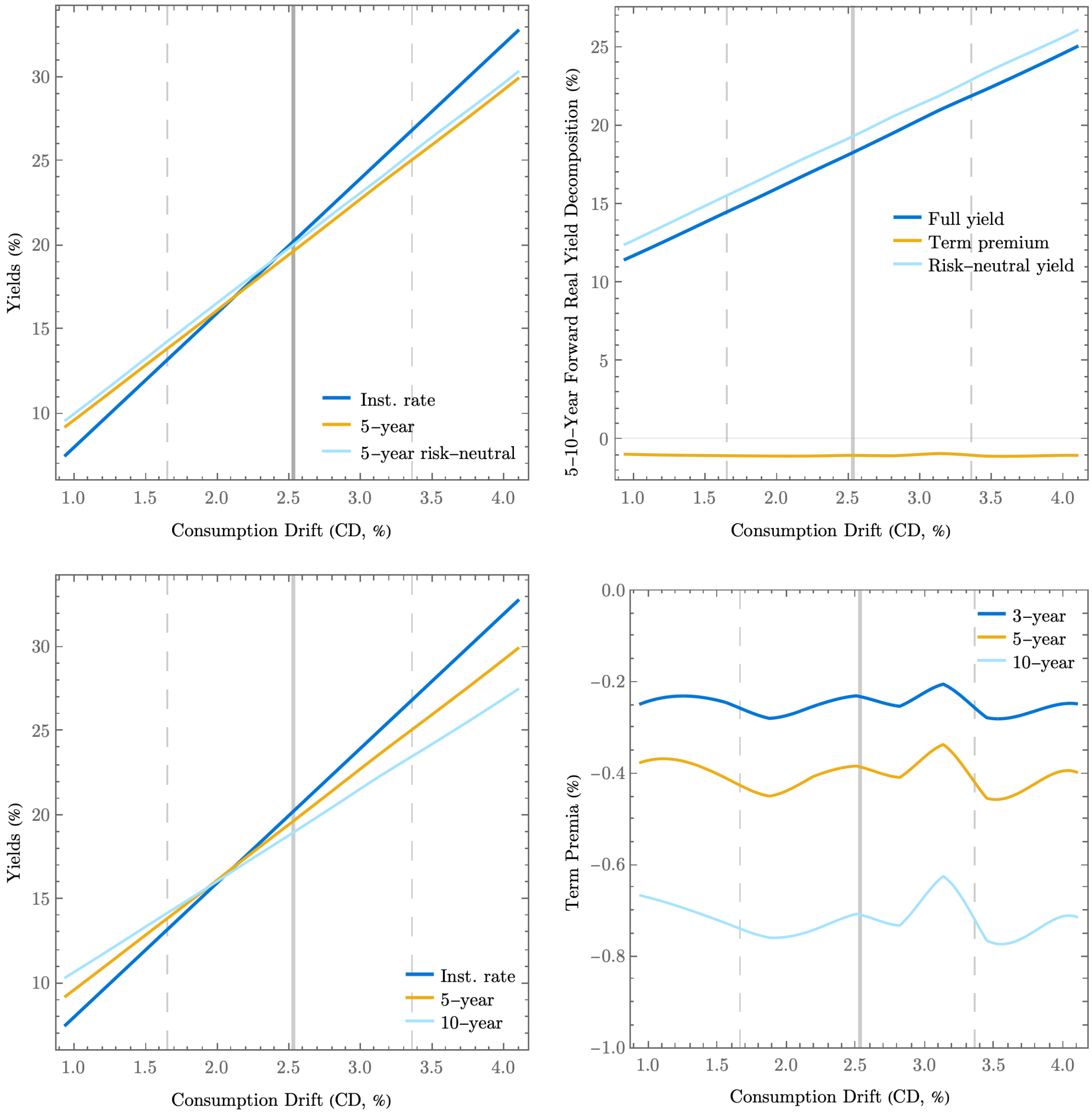
$\text{range}_{\text{low}}(\text{r}_{2,\text{model}})$=0.10958, $\text{range}_{\text{high}}(\text{r}_{2,\text{model}})$=0.202976, $\text{range}(\text{r}_{2,\text{data}})$=0.071748.

Figure 16: Time-varying consumption drift with time-separable utility.
The left plot shows the short-term rate, the five-year yield and the five-year risk-neutral yield as a function of consumption growth. The right plot shows the decomposition of the five-to-ten year forward into the term premium and the risk-neutral components. The solid vertical line shows the level of the ergodic median, the left and right dashed vertical lines show the median minimum and maximum value respectively over a series of simulations for 12 years. This means that half the simulated paths were below the right dashed line and half the simulated paths were above the left dashed line. The left and right boundaries are the 10th percentile of minimum values and the 90th percentile of maximum values from the same simulations. This means that 90% of simulated paths were above the left boundary and 90% of simulated paths were below the right boundary.

(variation overview)

F.3 TSU-CD-HRA, $\gamma = 8$

Term premia are a bit larger, but again negative and constant with respect to the state variable.



γ=8, ρ=0.012, μ_{ct}=x_t+0.0252, σ_{ct}=0.02, ϕ=0.92, μ_{xt}=0, σ_{xt}=0.00291989, ρ_{cx}=0.5

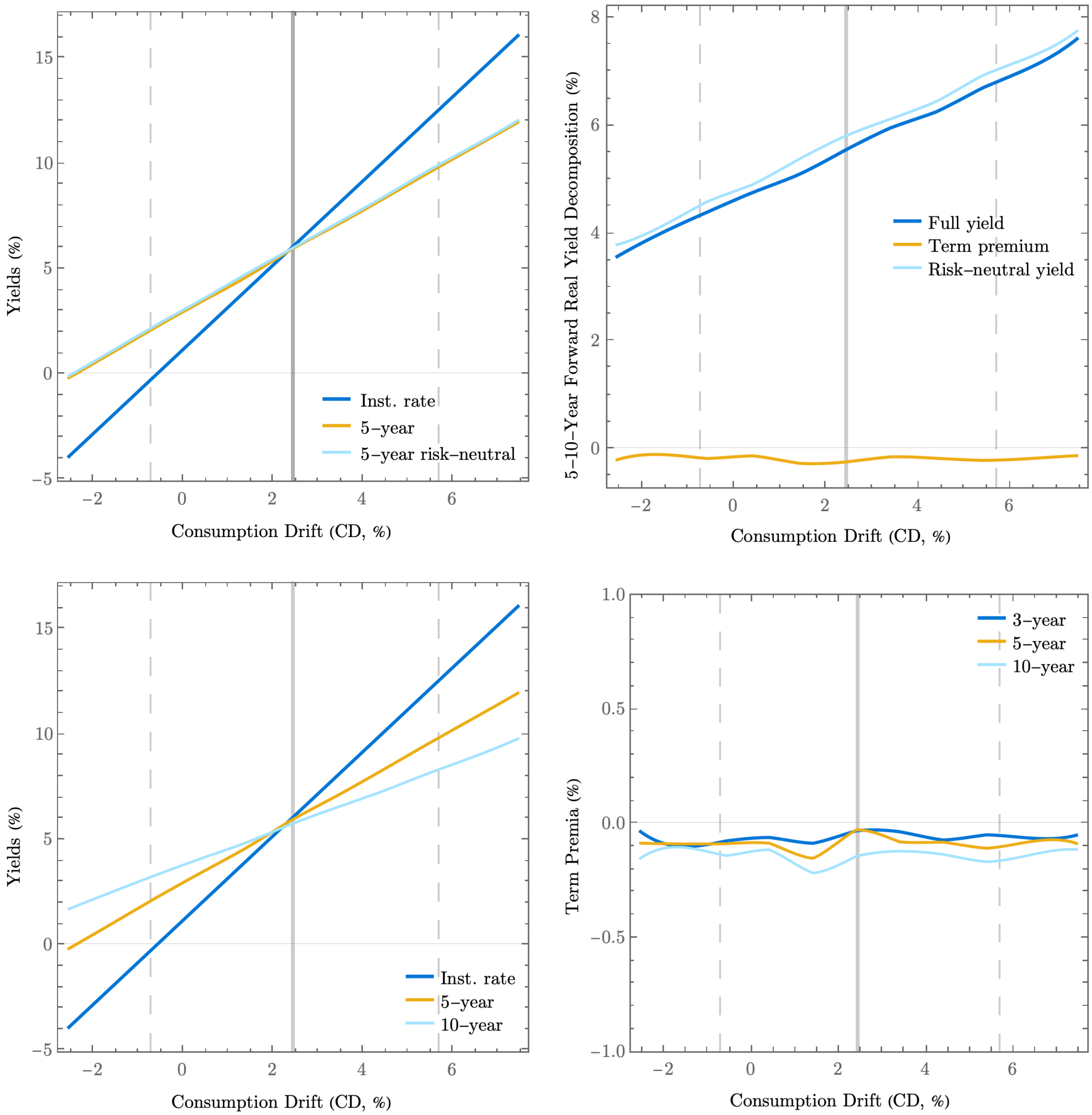
$\text{range}_{\text{low}}(\text{r}_{2,\text{model}})$=0.103026, $\text{range}_{\text{high}}(\text{r}_{2,\text{model}})$=0.18666, $\text{range}(\text{r}_{2,\text{data}})$=0.071748.

Figure 17: Time-varying consumption drift with time-separable utility and highr risk aversion.
See Figure 16 for more details about the plots.

(variation overview)

F.4 TSU-CD-LP, $\phi = 0.8$

Nothing changed in the term premia. There is larger separation between yields similar to the corresponding mechanism in Appendix C.



γ=2, ρ=0.012, μ_{ct}=$x_t + 0.0252$, σ_{ct}=0.02, ϕ=0.8, μ_{xt}=0, σ_{xt}=0.0126445, ρ_{cx}=0.5

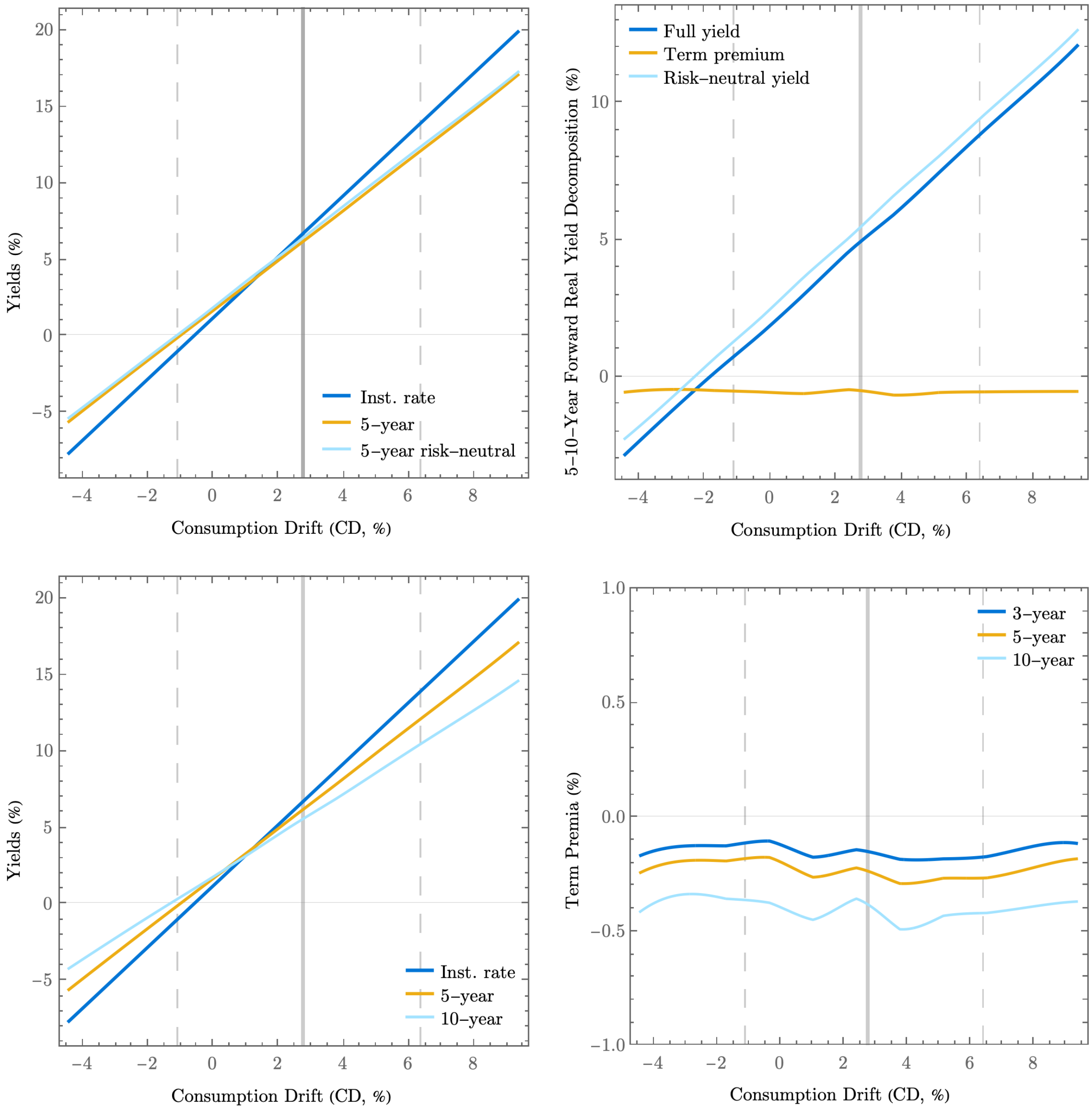
range$_{\mathrm{low}}$(r$_{2,\mathrm{model}}$)=0.0987483, range$_{\mathrm{high}}$(r$_{2,\mathrm{model}}$)=0.168236, range(r$_{2,\mathrm{data}}$)=0.071748.

Figure 18: Time-varying consumption drift with time-separable utility and low persistence
See Figure 16 for more details about the plots.

(variation overview)

F.5 TSU-CD-HCor, $\rho_{cx} = 1$

The term premia are larger in absolute value.



γ=2, ρ=0.012, μ_{ct}=x_t + 0.0252, σ_{ct}=0.02, ϕ=0.92, μ_{xt}=0, σ_{xt}=0.0126445, ρ_{cx}=1

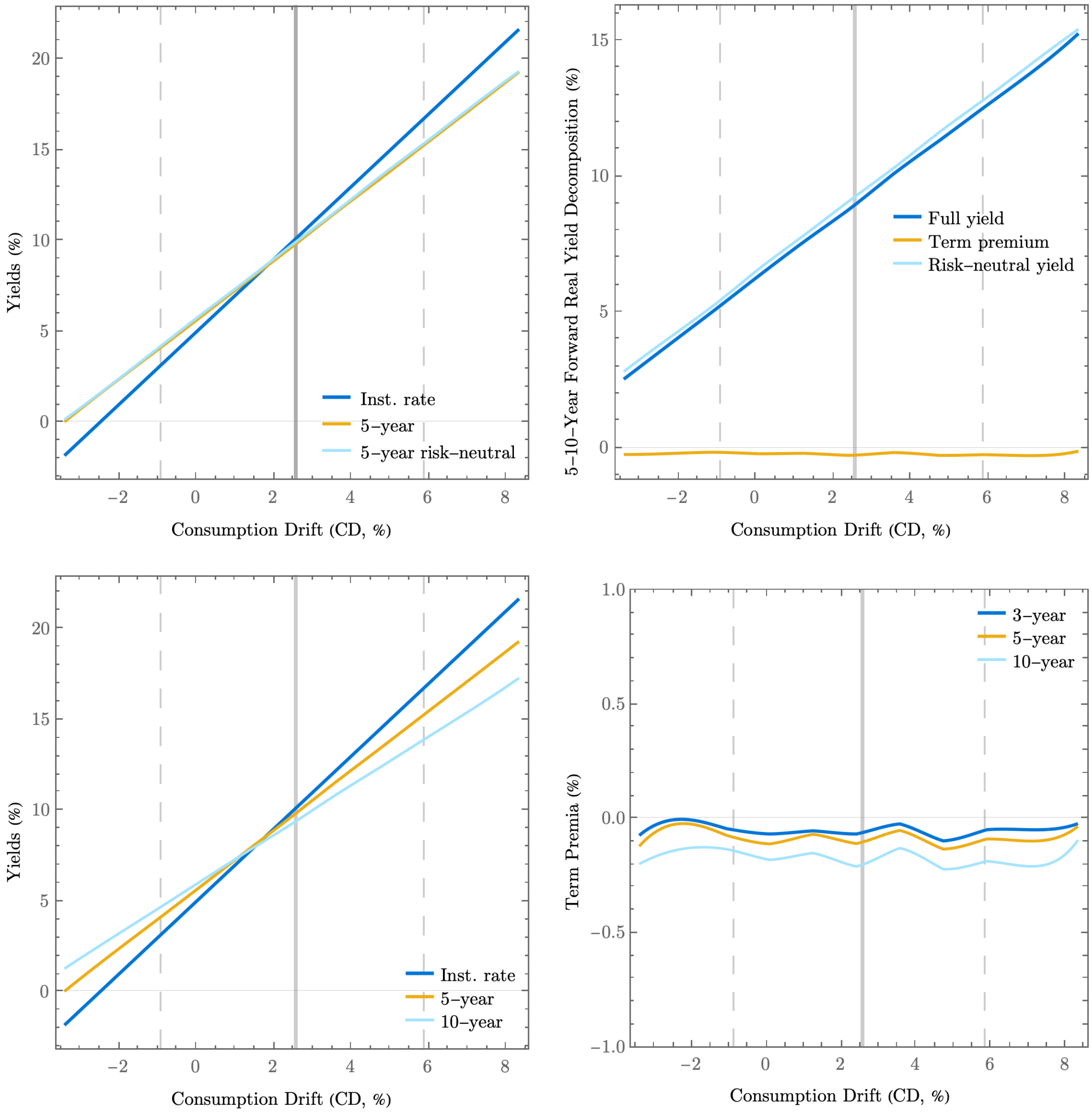
$\text{range}_{\text{low}}(\text{r}_{2,\text{model}})$=0.111377, $\text{range}_{\text{high}}(\text{r}_{2,\text{model}})$=0.203865, $\text{range}(\text{r}_{2,\text{data}})$=0.071748.

Figure 19: Time-varying consumption drift with time-separable utility and high correlation ρ_{cx}
See Figure 16 for more details about the plots.

(variation overview)

F.6 TSU-CD-HImp, $\rho = 0.05$

Yields move higher without any change in term premia.



γ=2, ρ=0.05, μ_{ct}=$x_t+0.0252$, σ_{ct}=0.02, ϕ=0.92, μ_{xt}=0, σ_{xt}=0.011495, ρ_{cx}=0.5

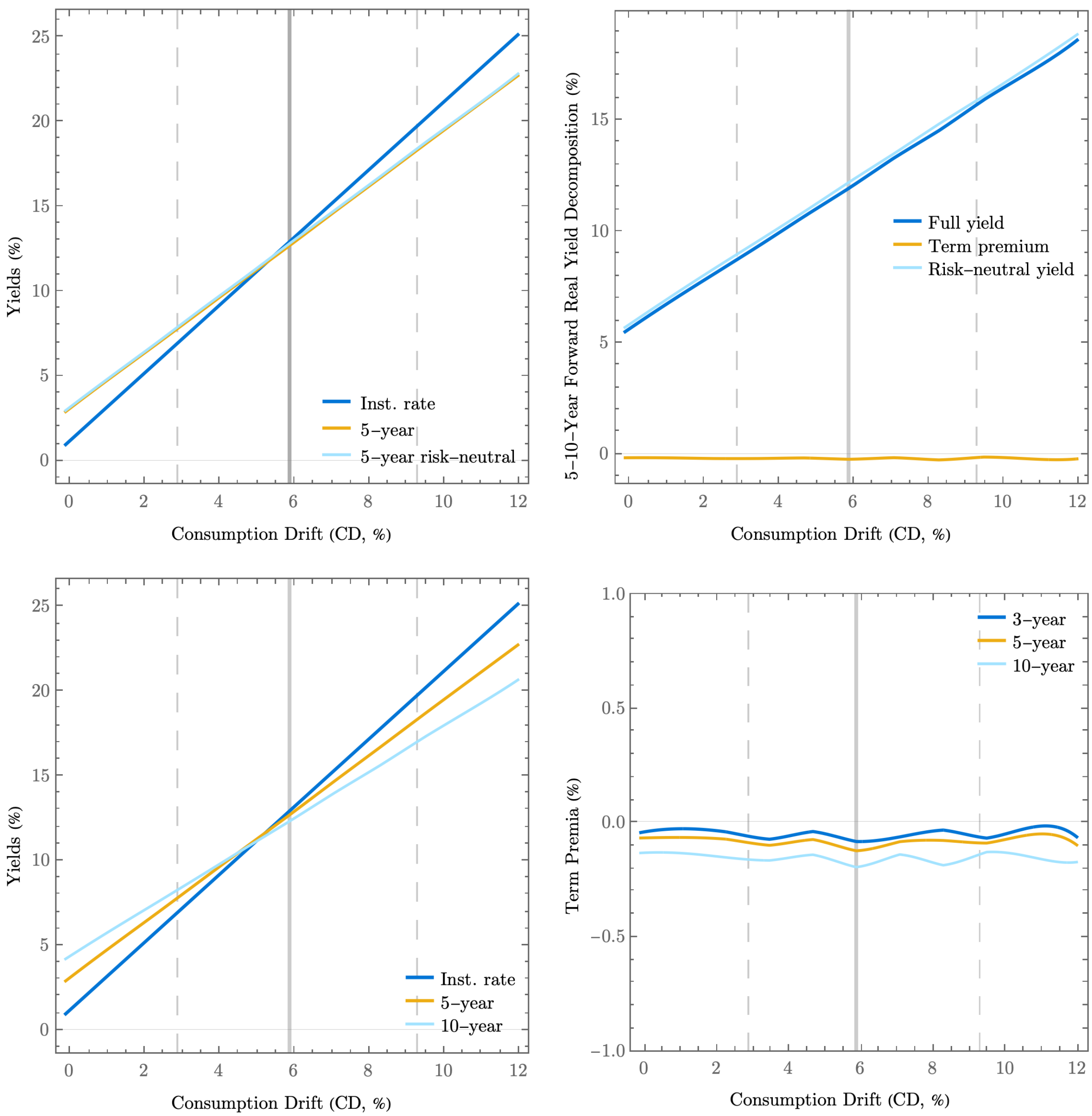
$\mathrm{range}_{\mathrm{low}}(\mathrm{r}_{2,\mathrm{model}})$=0.0973497, $\mathrm{range}_{\mathrm{high}}(\mathrm{r}_{2,\mathrm{model}})$=0.182416, $\mathrm{range}(\mathrm{r}_{2,\mathrm{data}})$=0.071748.

Figure 20: Time-varying consumption drift with time-separable utility and high impatience.
See Figure 16 for more details about the plots.

(variation overview)

F.7 TSU-HCD, $\mu_{c0} = 0.06$

Again, yields move higher without any change in term premia.



γ=2, ρ=0.012, $\mu_{\mathrm{ct}}=x_t+0.06$, σ_{ct}=0.02, ϕ=0.92, μ_{xt}=0, σ_{xt}=0.011495, ρ_{cx}=0.5

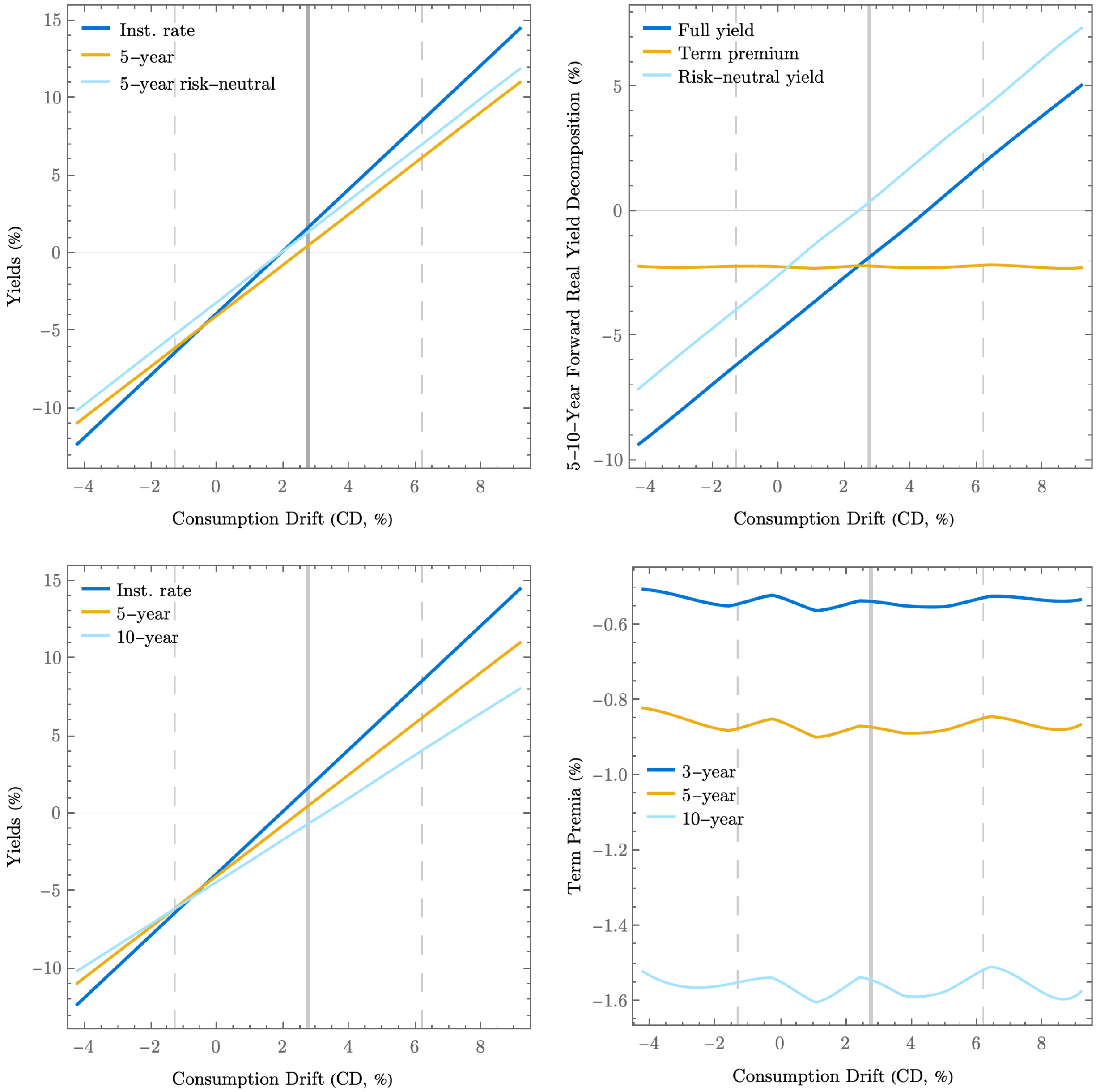
$\mathrm{range_{low}}(\mathrm{r_{2,model}})$=0.0977548, $\mathrm{range_{high}}(\mathrm{r_{2,model}})$=0.183359, $\mathrm{range}(\mathrm{r_{2,data}})$=0.071748.

Figure 21: Time-varying and high consumption drift with time-separable utility. See Figure 16 for more details about the plots.

(variation overview)

F.8 TSU-CD-HCV, $\sigma_{ct} = 0.16$

Yields move down and term premia increase in absolute value, but they are again constant.



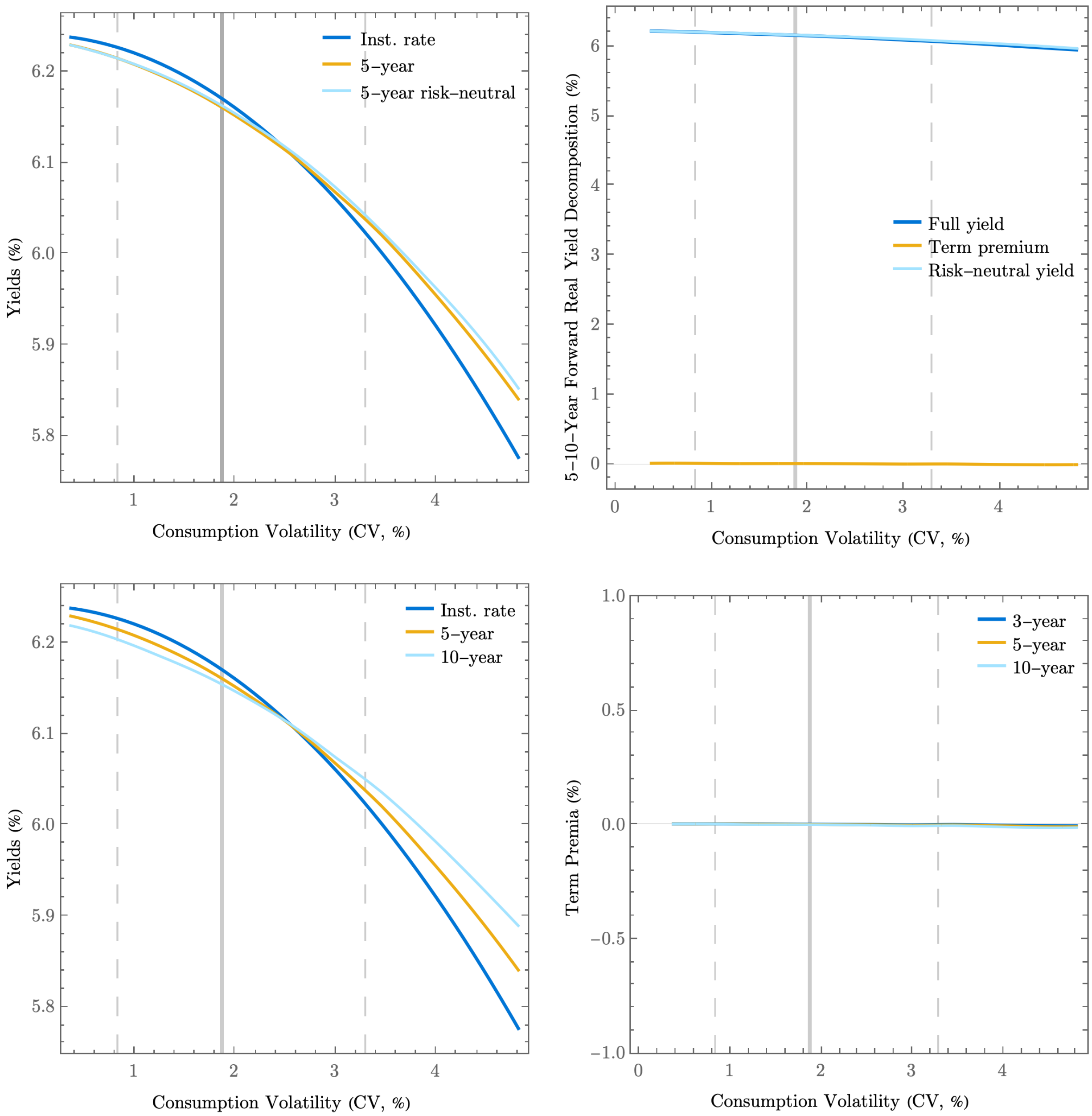
γ=2, ρ=0.012, μ_{ct}=x_t + 0.0252, σ_{ct}=0.16, ϕ=0.92, μ_{xt}=0, σ_{xt}=0.0126445, ρ_{cx}=0.5

$\text{range}_{\text{low}}(\text{r}_{2,\text{model}})$=0.107642, $\text{range}_{\text{high}}(\text{r}_{2,\text{model}})$=0.207586, $\text{range}(\text{r}_{2,\text{data}})$=0.071748.

Figure 22: Time-varying consumption drift with time-separable utility and high consumption volatility.
See Figure 16 for more details about the plots.

(variation overview)

F.9 TSU-CV, Calibration used in main paper, Figure 3



γ=2, ρ=0.012, μ_{ct}=0.0252, σ_{ct}=0.02 x_t, ϕ=0.92, μ_{xt}=1, σ_{xt}=0.21 $\sqrt{x_t}$, ρ_{cx}=−0.5

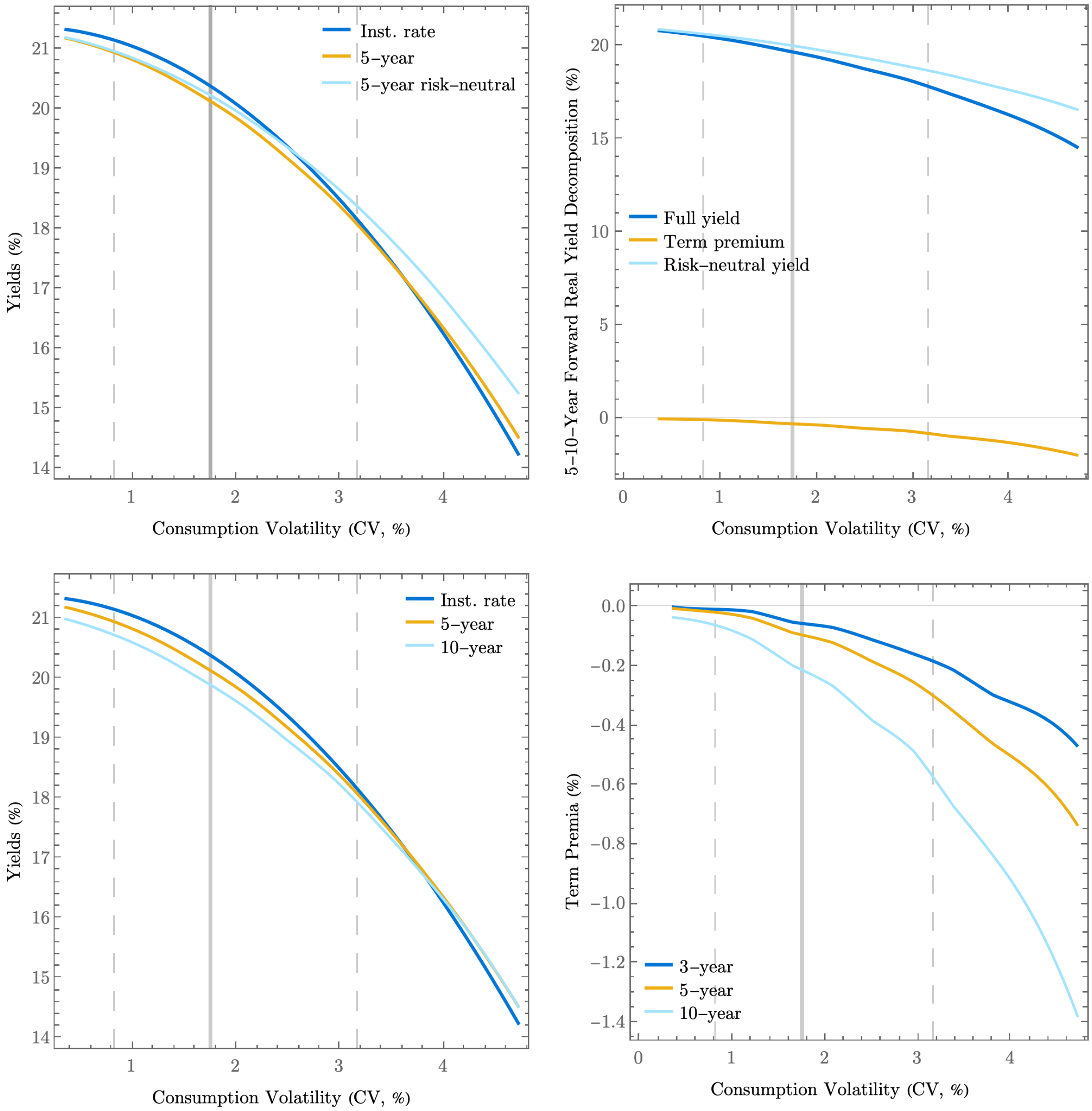
$\text{range}_{\text{low}}(\text{r}_{2,\text{model}})$=0.00103855, $\text{range}_{\text{high}}(\text{r}_{2,\text{model}})$=0.00408583, $\text{range}(\text{r}_{2,\text{data}})$=0.071748.

Figure 23: Time-varying consumption volatility with time-separable utility. See Figure 16 for more details about the plots.

(variation overview)

F.10 TSU-CV-HRA, $\gamma = 8$

Term premia increased in absolute value but not enough and yields moved very high.



γ=8, ρ=0.012, μ_{ct}=0.0252, σ_{ct}=0.02 x_t, ϕ=0.92, μ_{xt}=1, σ_{xt}=0.21 $\sqrt{x_t}$, ρ_{cx}=−0.5

range$_{low}$(r$_{2,model}$)=0.0157495, range$_{high}$(r$_{2,model}$)=0.0684834, range(r$_{2,data}$)=0.071748.

Figure 24: Time-varying consumption volatility with high risk aversion. See Figure 16 for more details about the plots.

(variation overview)

F.11 TSU-CV-HCD, $\mu_{c0} = 0.08$

Term premia did not change but yields moves implausibly high.

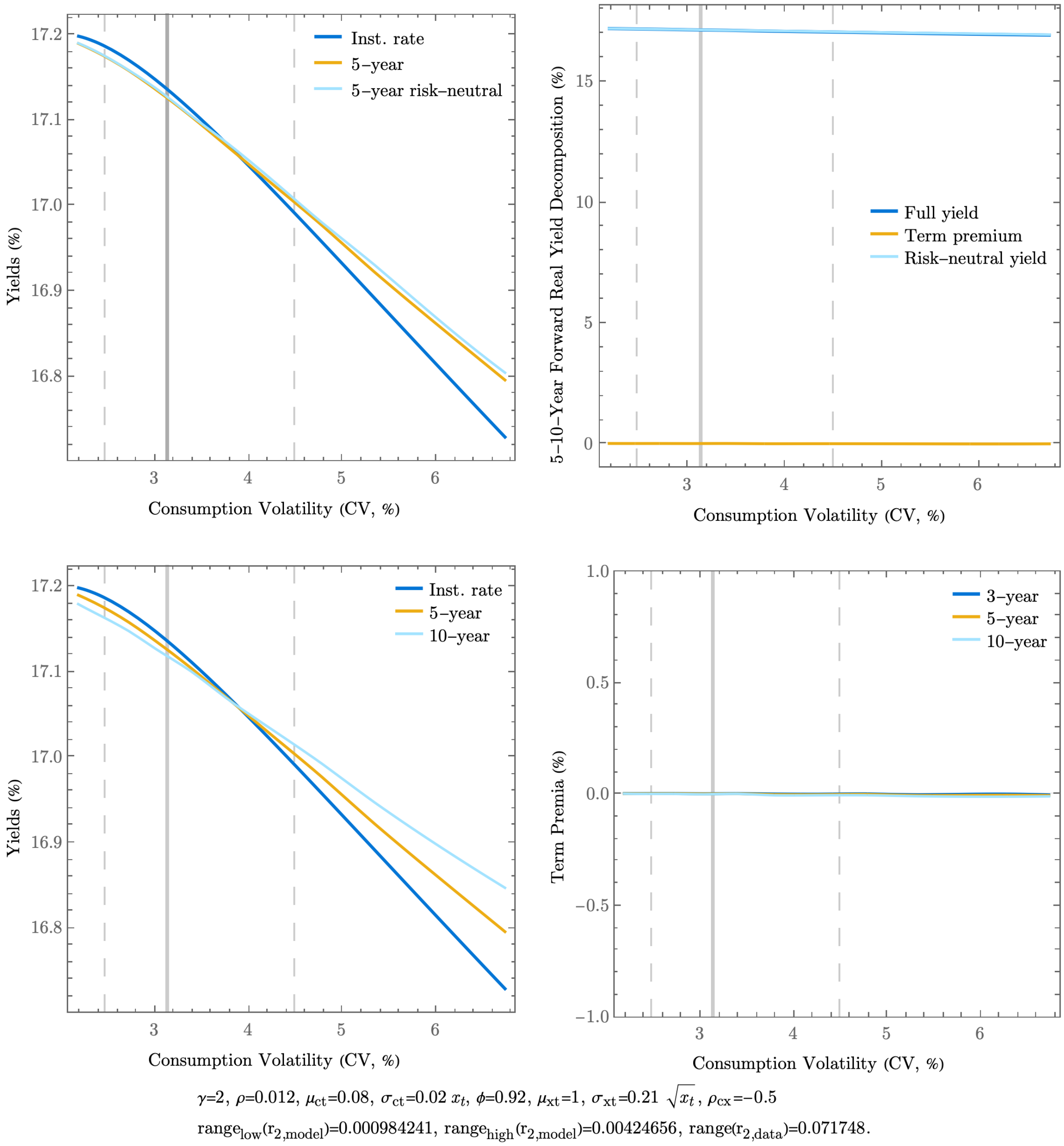
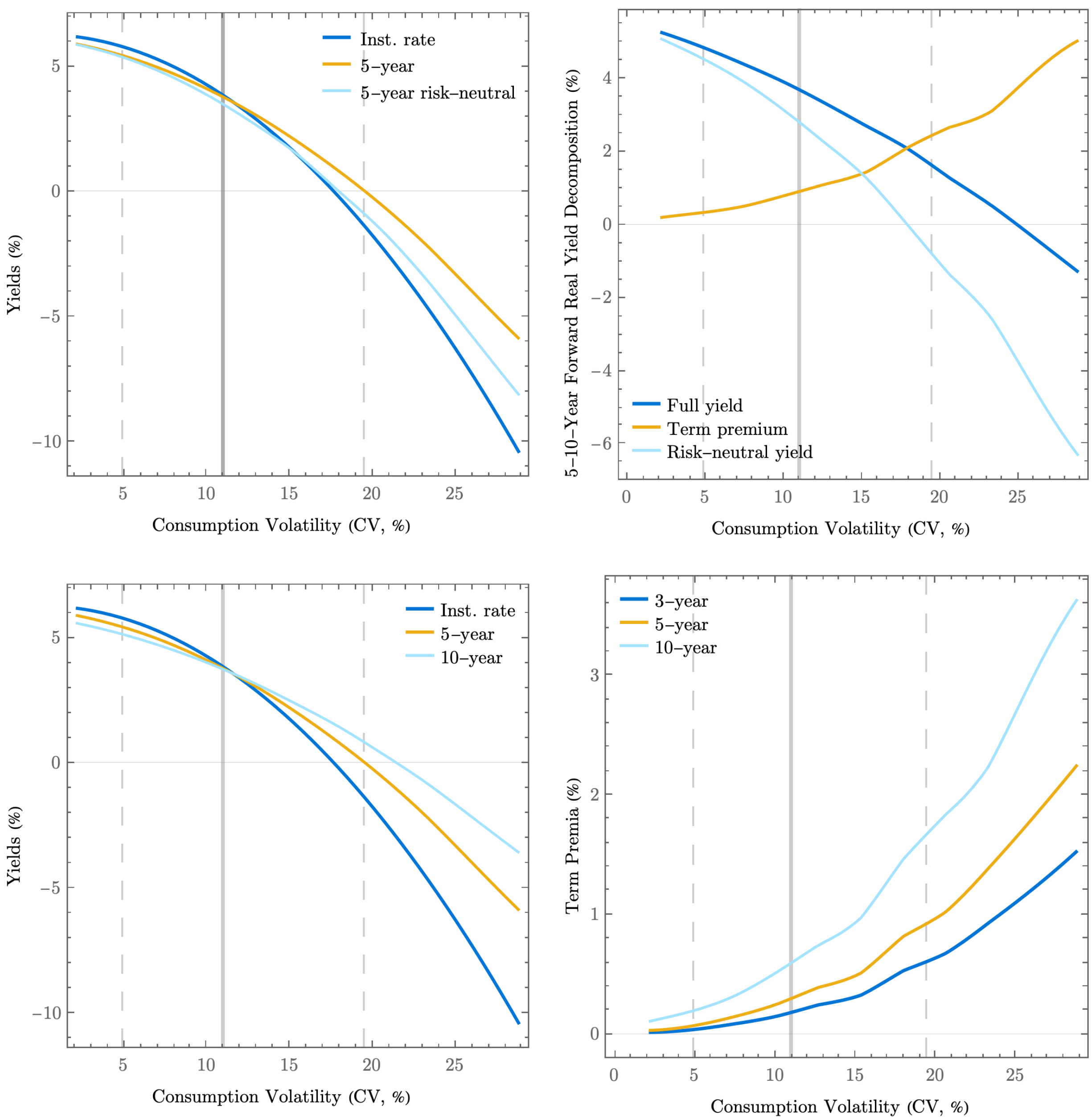


Figure 25: Time-varying consumption volatility with time-separable utility and high consumption drift.
See Figure 16 for more details about the plots.

(variation overview)

F.12 TSU-HCV, Calibration used in main paper, Figure 6



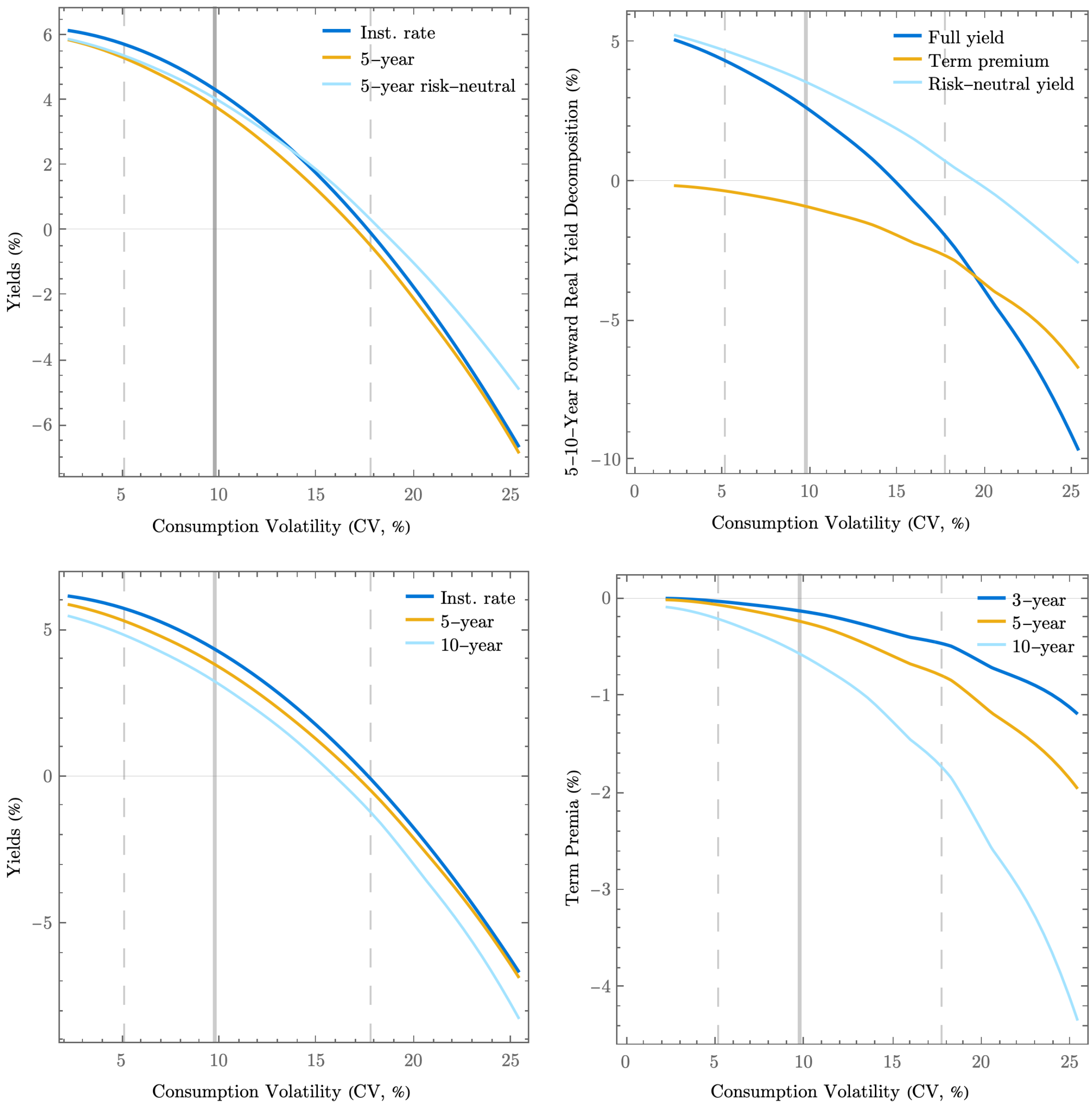
γ=2, ρ=0.012, μ_{ct}=0.0252, σ_{ct}=0.12 x_t, ϕ=0.92, μ_{xt}=1, σ_{xt}=0.21 $\sqrt{x_t}$, ρ_{cx}=0.5

$\text{range}_{\text{low}}(\text{r}_{2,\text{model}})$=0.0374496, $\text{range}_{\text{high}}(\text{r}_{2,\text{model}})$=0.149538, $\text{range}(\text{r}_{2,\text{data}})$=0.071748.

Figure 26: Time-varying and high consumption volatility with time-separable utility and positive ρ_{cx}.
See Figure 16 for more details about the plots.

(variation overview)

F.13 TSU-HCV-NCor, $\rho_{cx} < 0$



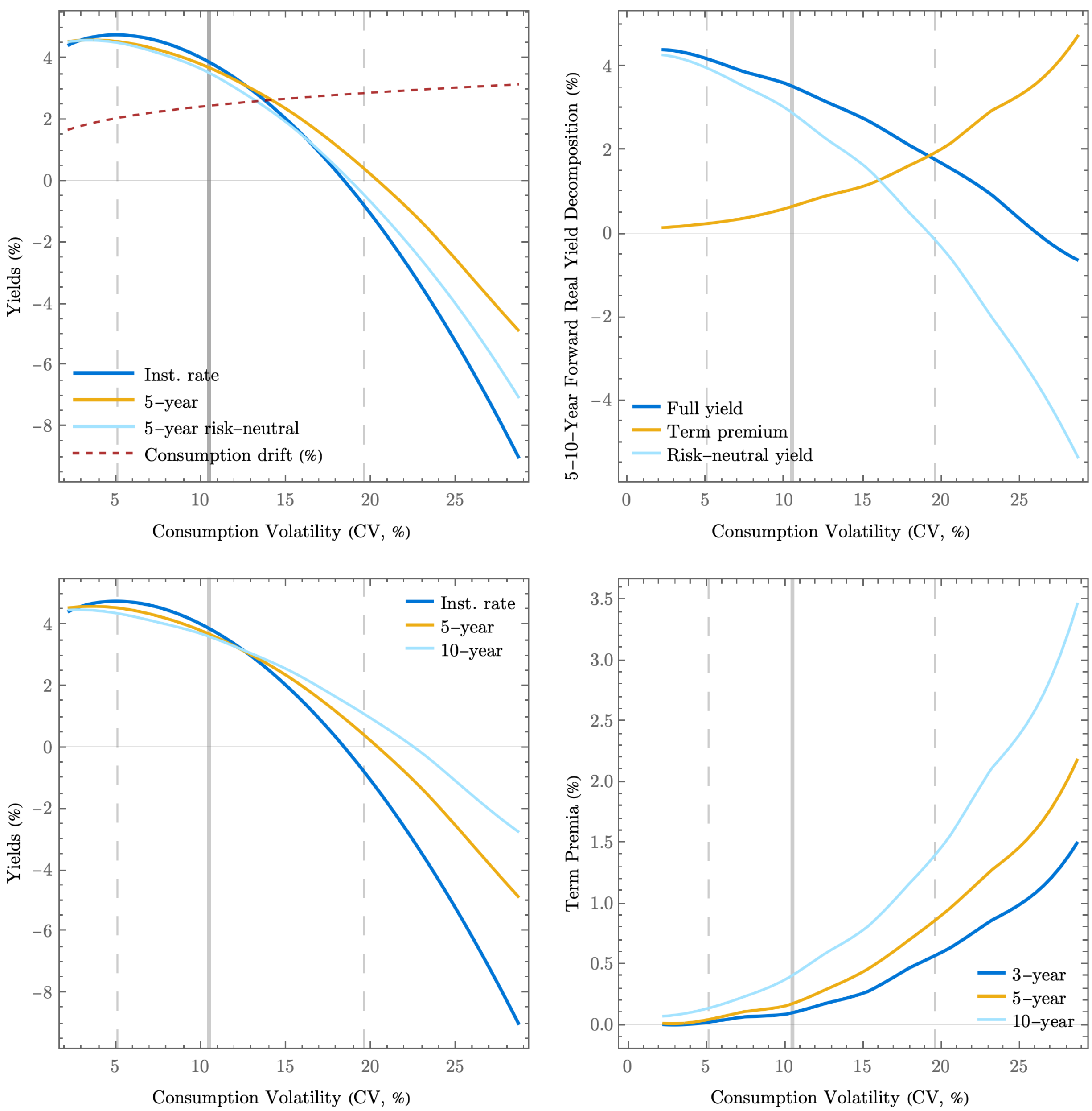
γ=2, ρ=0.012, μ_{ct}=0.0252, σ_{ct}=0.11 x_t, ϕ=0.92, μ_{xt}=1, σ_{xt}=0.2 $\sqrt{x_t}$, ρ_{cx}=−0.5

$\mathrm{range_{low}}(\mathrm{r_{2,model}})$=0.0306062, $\mathrm{range_{high}}(\mathrm{r_{2,model}})$=0.119326, $\mathrm{range}(\mathrm{r_{2,data}})$=0.071748.

Figure 27: Time-varying consumption volatility with time-separable utility and negative ρ_{cx}.
See Figure 16 for more details about the plots.

(variation overview)

F.14 Arb-DP, Calibration used in main paper, Figure 6



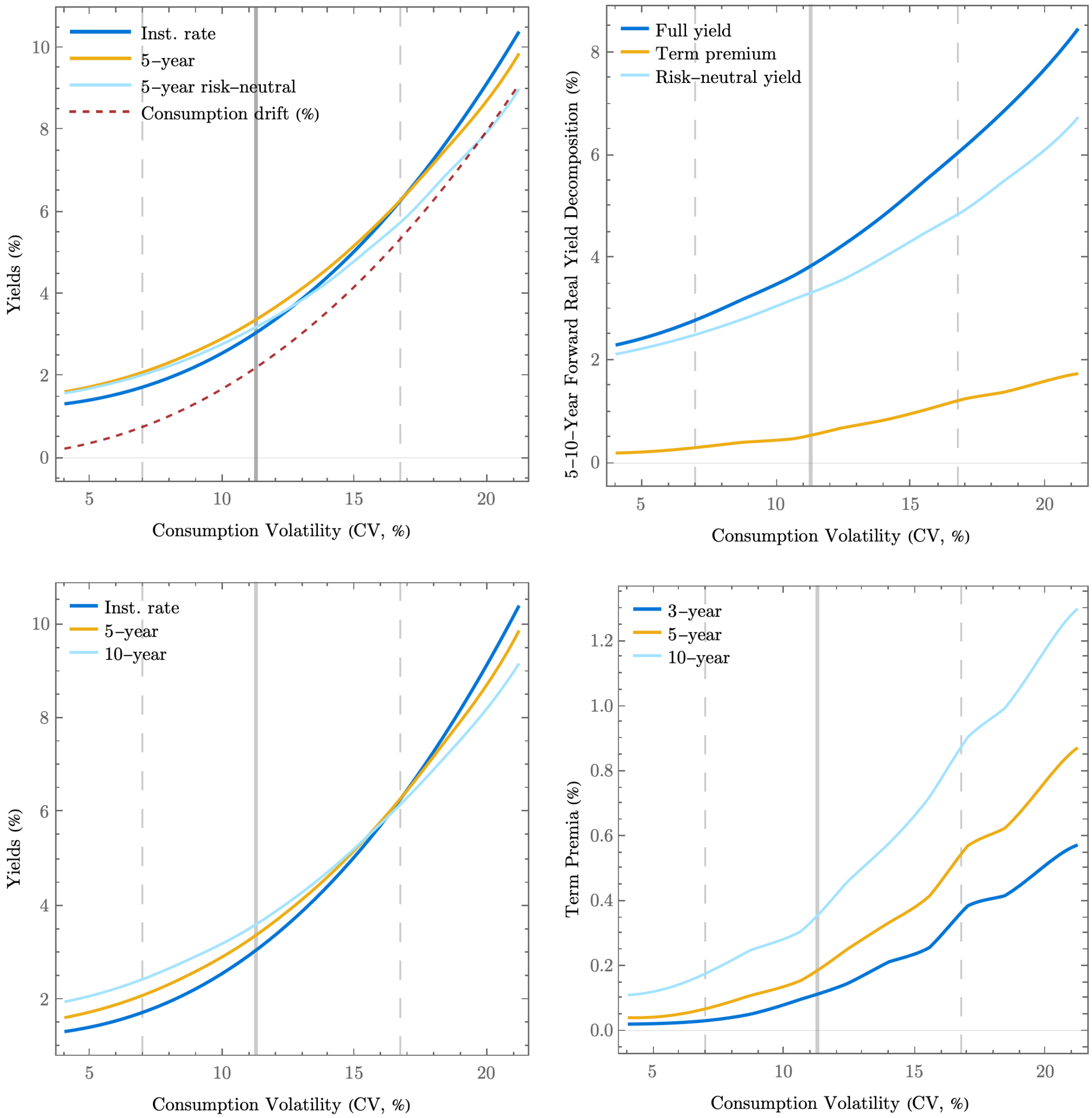
γ=2, ρ=0.012, μ_{ct}=0.0252 $\sqrt[4]{x_t}$, σ_{ct}=0.12 x_t, ϕ=0.92, μ_{xt}=1, σ_{xt}=0.21 $\sqrt{x_t}$, ρ_{cx}=0.5

$\mathrm{range_{low}}(\mathrm{r_{2,model}})$=0.0222875, $\mathrm{range_{high}}(\mathrm{r_{2,model}})$=0.132065, $\mathrm{range}(\mathrm{r_{2,data}})$=0.071748.

Figure 28: Both time-varying consumption drift and consumption volatility with time-separable utility, short-term rate decreasing in consumption volatility and positive ρ_{xc}. See Figure 6 for more details about the plots.

(variation overview)

F.15 Arb-IN, Calibration used in main paper, Figure 6



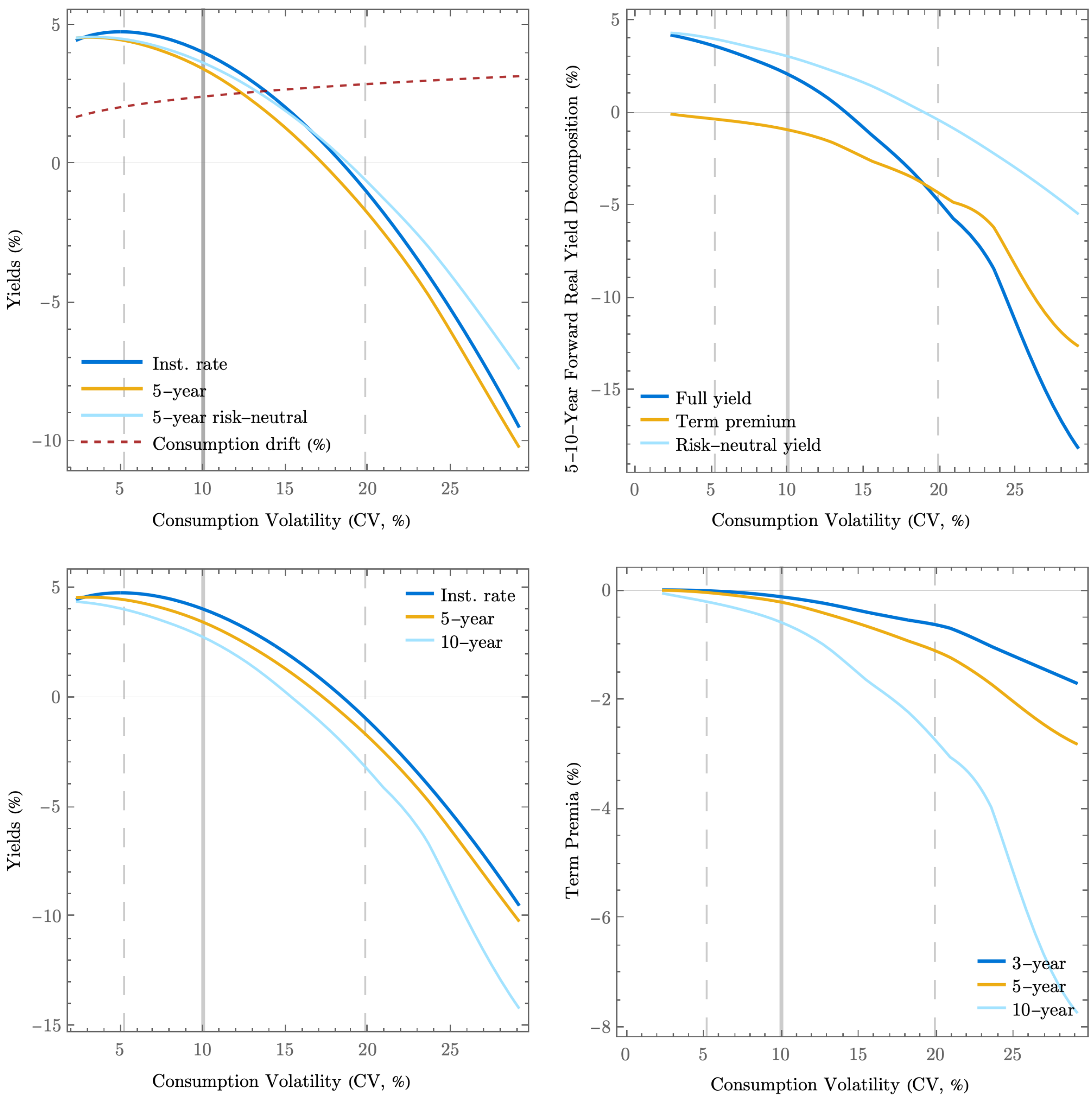
γ=2, ρ=0.012, μ_{ct}=0.0252 $x_t^{3/2}$, σ_{ct}=0.12 $x_t^{2/3}$, ϕ=0.92, μ_{xt}=1, σ_{xt}=0.21 $\sqrt{x_t}$, ρ_{cx}=−0.5

range$_{\mathrm{low}}$(r$_{2,\mathrm{model}}$)=0.0258998, range$_{\mathrm{high}}$(r$_{2,\mathrm{model}}$)=0.0818517, range(r$_{2,\mathrm{data}}$)=0.071748.

Figure 29: Both time-varying consumption drift and consumption volatility with time-separable utility, short-term rate increasing in consumption volatility and negative ρ_{xc}. See Figure 6 for more details about the plots.

(variation overview)

F.16 Arb-DN



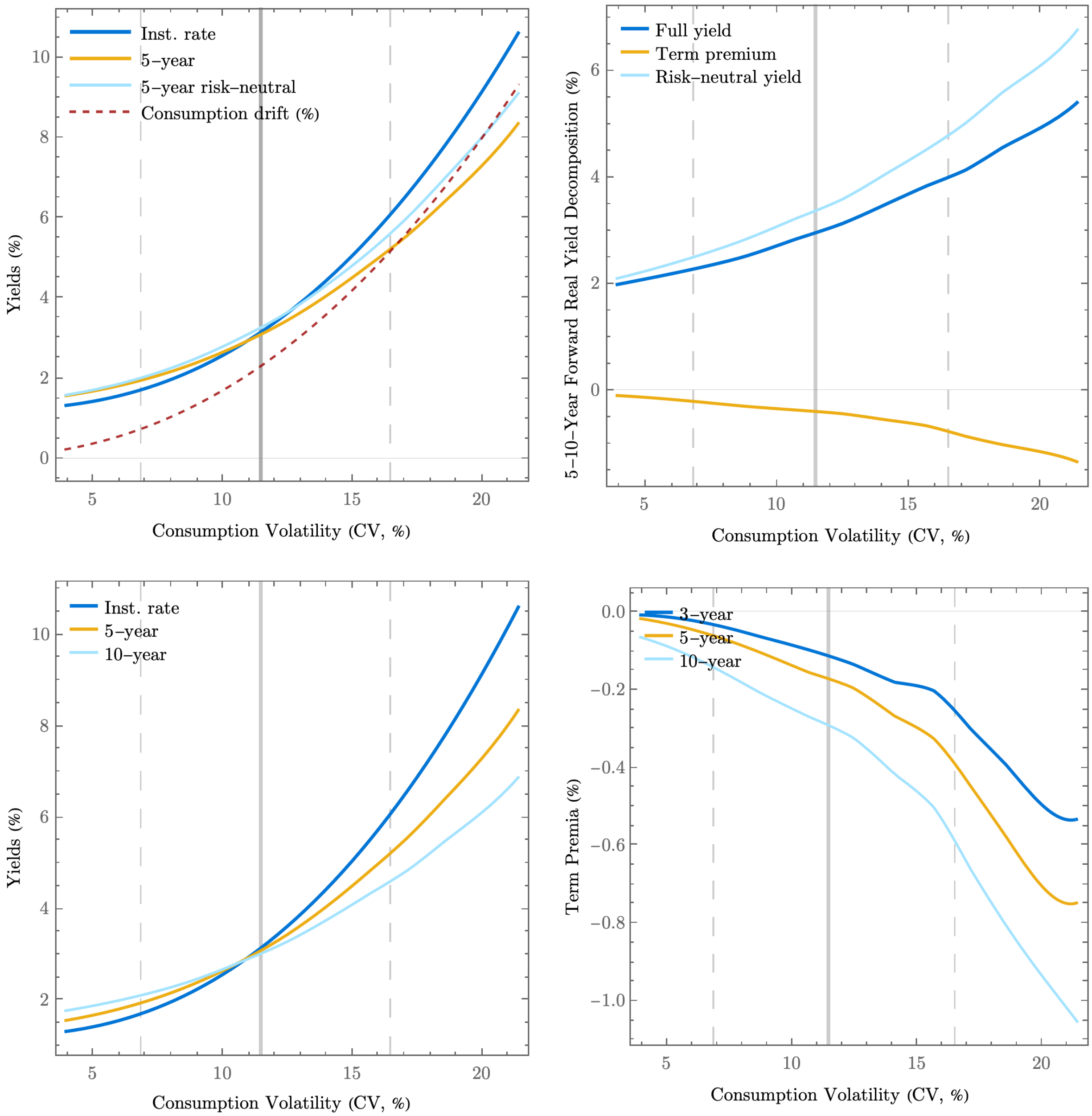
γ=2, ρ=0.012, μ_{ct}=0.0252 $\sqrt[4]{x_t}$, σ_{ct}=0.12 x_t, ϕ=0.92, μ_{xt}=1, σ_{xt}=0.21 $\sqrt{x_t}$, ρ_{cx}=−0.5

$\mathrm{range_{low}(r_{2,model})}$=0.0224998, $\mathrm{range_{high}(r_{2,model})}$=0.134553, $\mathrm{range(r_{2,data})}$=0.071748.

Figure 30: Both time-varying consumption drift and consumption volatility with time-separable utility, short-term rate decreasing in consumption volatility and negative ρ_{xc}.

(variation overview)

F.17 Arb-IP



γ=2, ρ=0.012, μ_{ct}=0.0252 $x_t^{3/2}$, σ_{ct}=0.12 $x_t^{2/3}$, ϕ=0.92, μ_{xt}=1, σ_{xt}=0.21 $\sqrt{x_t}$, ρ_{cx}=0.5

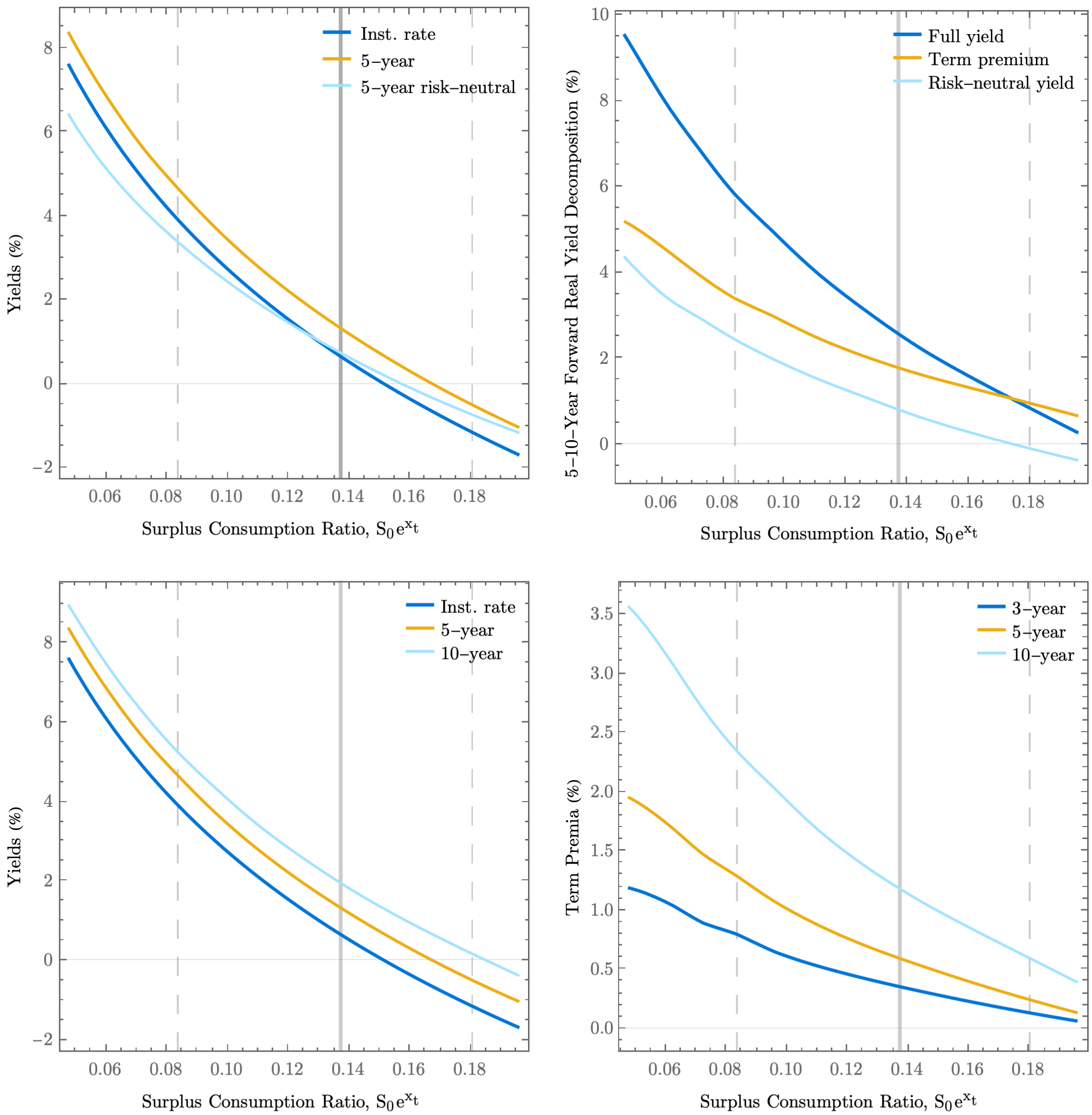
range$_{\mathrm{low}}$(r$_{2,\mathrm{model}}$)=0.0263095, range$_{\mathrm{high}}$(r$_{2,\mathrm{model}}$)=0.0794248, range(r$_{2,\mathrm{data}}$)=0.071748.

Figure 31: Both time-varying consumption drift and consumption volatility with time-separable utility, short-term rate increasing in consumption volatility and positive ρ_{xc}. See Figure 6 for more details about the plots.

(variation overview)

F.18 TSU-Habit

F.19 Calibration used in main paper, Figure 3



$$\gamma=2,\ \rho=0.012,\ \mu_{\mathrm{ct}}=0.0252,\ \sigma_{\mathrm{ct}}=0.02,\ \phi=0.92,\ \mu_{\mathrm{xt}}=0,\ \sigma_{\mathrm{xt}}=0.02\ \mathrm{If}\left[x_t > 0.492061,\ 0,\ \frac{\frac{\sqrt{1-2\,x_t}}{2}}{0.02\sqrt{-\frac{0.066}{2}-\log(0.92)}}-1\right],\ \rho_{\mathrm{cx}}=1$$

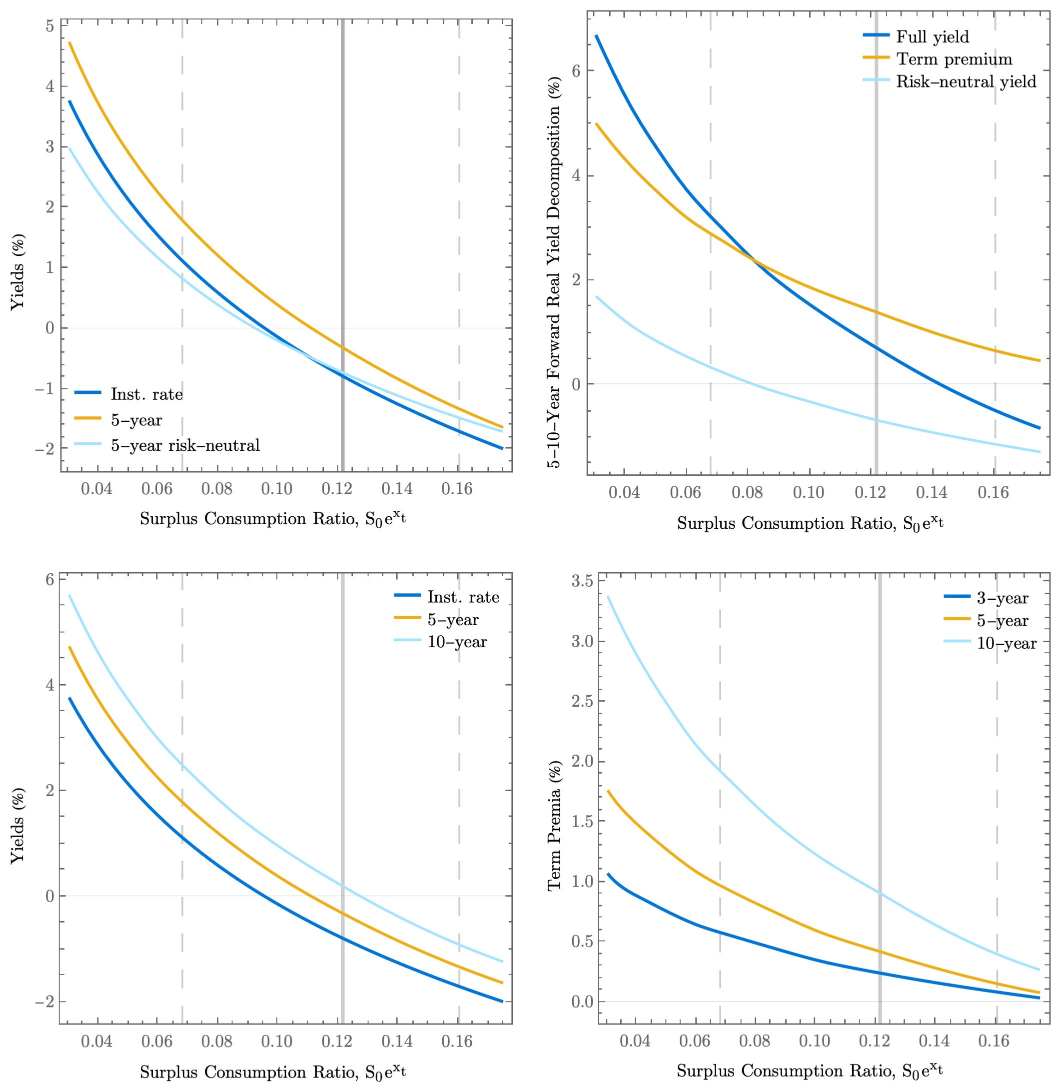
$\mathrm{range_{low}}(r_{2,\mathrm{model}})=0.0330628$, $\mathrm{range_{high}}(r_{2,\mathrm{model}})=0.0788834$, $\mathrm{range}(r_{2,\mathrm{data}})=0.071748$.

Figure 32: Time-varying extranl habit with time-separable utility. See Figure 16 for more details about the plots.

(variation overview)

F.20 TSU-Habit-L.b, b=0.033

Term premia did not change but yields became flatter. This is noteworthy because in Abrahams *et al.* (2016) the forward term premia are big while the forward risk-neutral yields are small in absolute value.



γ=2, ρ=0.012, μ_{ct}=0.0252, σ_{ct}=0.02, ϕ=0.92, μ_{xt}=0, σ_{xt}=0.02 $\mathrm{If}\left[x_t > 0.494019, 0, \frac{\sqrt{1-2\,x_t}}{0.02\sqrt{\frac{2}{-\frac{0.033}{2}-\log(0.92)}}} - 1\right]$, ρ_{cx}=1

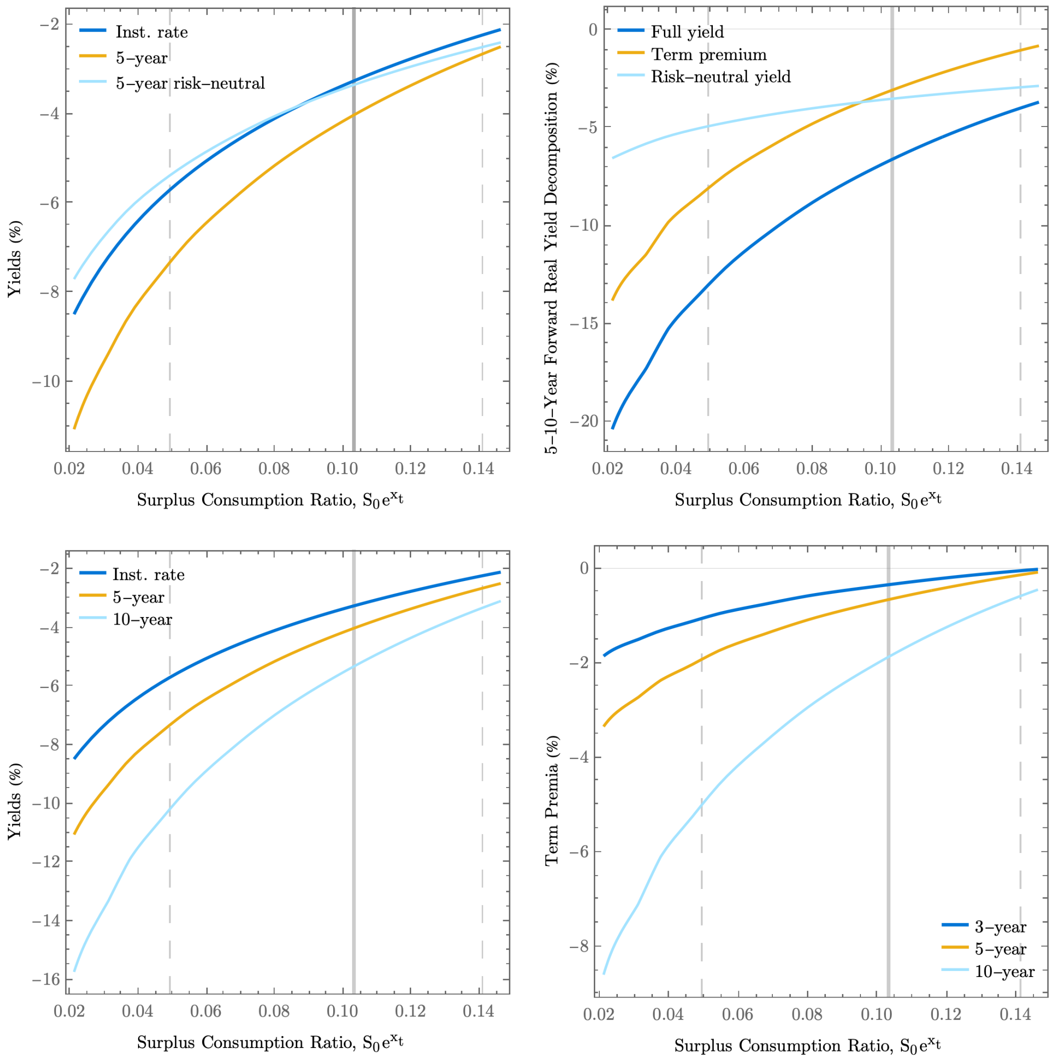
$\mathrm{range}_{\mathrm{low}}(r_{2,\mathrm{model}})$=0.0181462, $\mathrm{range}_{\mathrm{high}}(r_{2,\mathrm{model}})$=0.0476023, $\mathrm{range}(r_{2,\mathrm{data}})$=0.071748.

Figure 33: Time-varying extranl habit with time-separable utility. See Figure 16 for more details about the plots.

(variation overview)

F.21 TSU-Habit-Neg.b, b=-0.033

The short-term rate is now pro-cyclical and term premia are negative.



γ=2, ρ=0.012, μ_{ct}=0.0252, σ_{ct}=0.02, ϕ=0.92, μ_{xt}=0, σ_{xt}=0.02 $\text{If}\left[x_t > 0.495995,\ 0,\ \dfrac{\sqrt{\frac{1-2\,x_t}{2}}}{0.02\sqrt{-\frac{-0.033}{2}-\log(0.92)}} - 1\right]$, ρ_{cx}=1

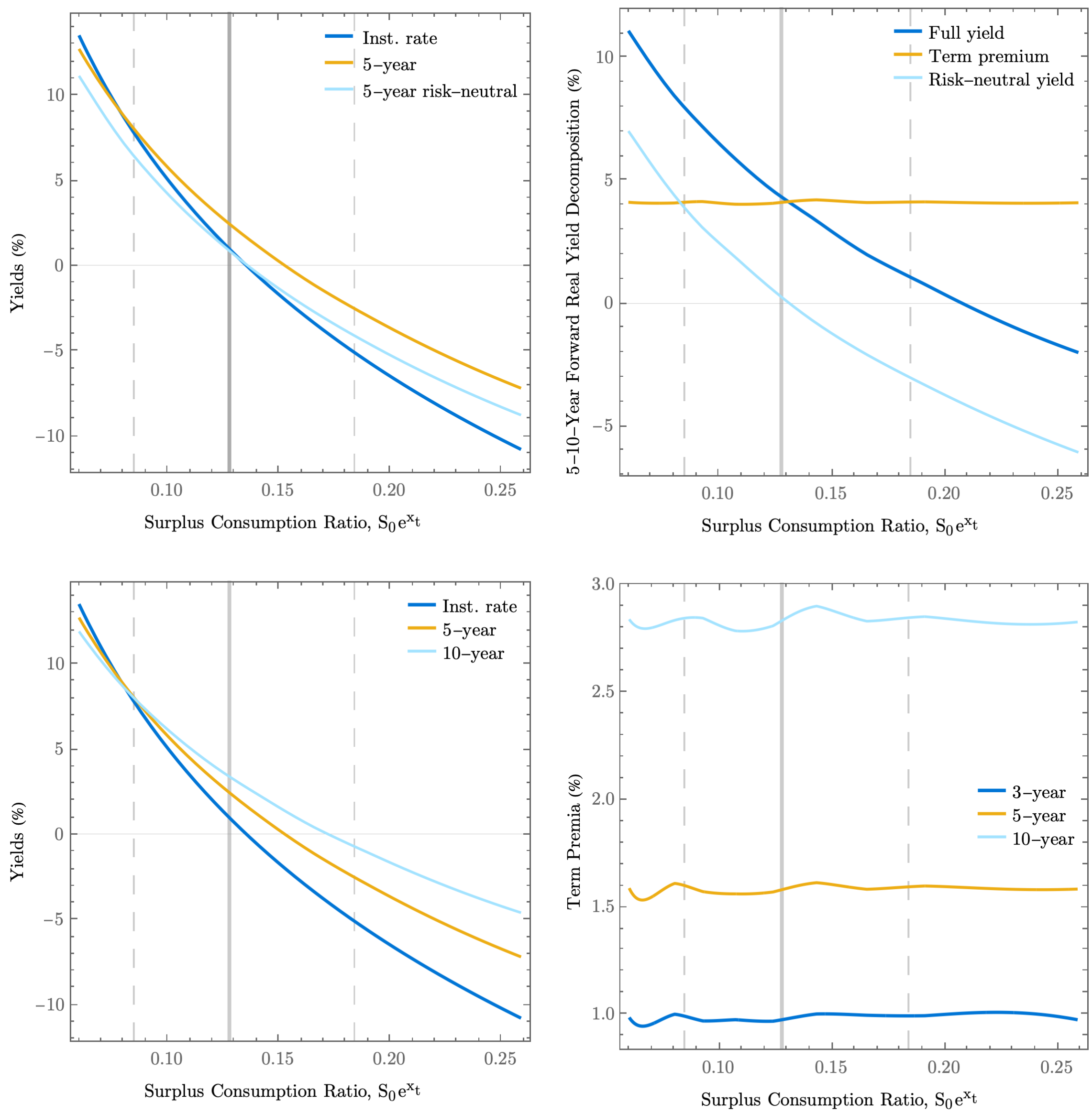
$\text{range}_{\text{low}}(r_{2,\text{model}})$=0.0191972, $\text{range}_{\text{high}}(r_{2,\text{model}})$=0.0644309, $\text{range}(r_{2,\text{data}})$=0.071748.

Figure 34: Time-varying extranl habit with time-separable utility and $b < 0$. See Figure 16 for more details about the plots.

(variation overview)

F.22 TSU-Habit-ConstantSV, $\sigma_{xt} = \lambda(0)\sigma_{c0}$

The term premia are now constant. This is partially contrary to the spirit of Campbell and Cochrane (1999), because the surplus consumption ratio does not get more volatile in bad states of the economy, but it illustrates how the heteroskedasticity is crucial for the generation of variable term premia.



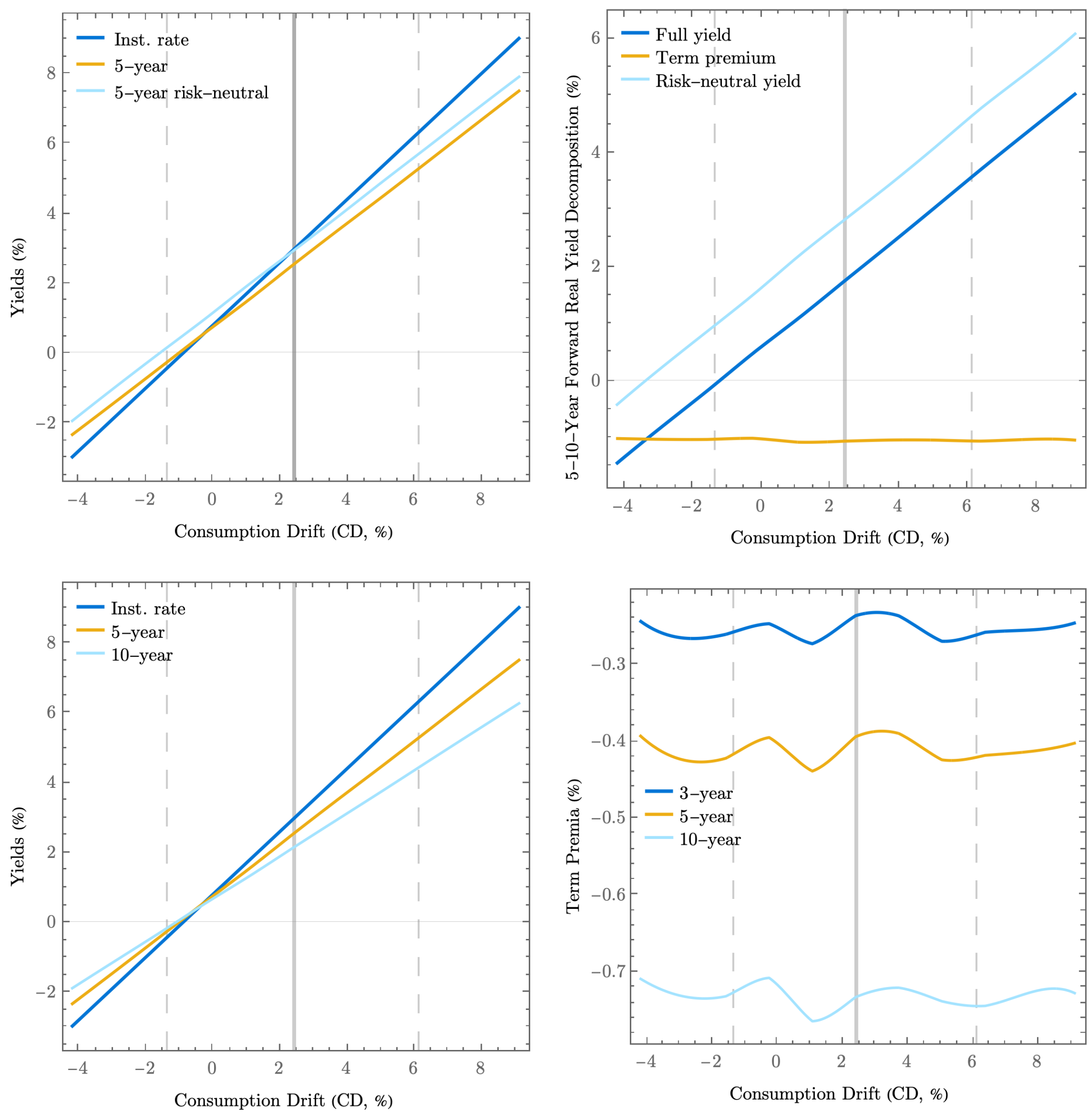
γ=2, ρ=0.012, μ_{ct}=0.0252, σ_{ct}=0.02, ϕ=0.92, μ_{xt}=0, σ_{xt}=0.138716, ρ_{cx}=1

$\text{range}_{\text{low}}(r_{2,\text{model}})$=0.103559, $\text{range}_{\text{high}}(r_{2,\text{model}})$=0.185687, $\text{range}(r_{2,\text{data}})$=0.071748.

Figure 35: Time-varying extranl habit with time-separable utility and constant state variable volatility.
See Figure 16 for more details about the plots.

(variation overview)

F.23 RU-CD, Calibration used in main paper, Figure 5



γ=2, ρ=0.012, μ_{ct}=x_t + 0.0252, σ_{ct}=0.02, ϕ=0.92, μ_{xt}=0, σ_{xt}=0.0126445, ρ_{cx}=0.5, ψ=1.11, approx. order=8

$\mathrm{range_{low}}(\mathrm{r}_{2,\mathrm{model}})$=0.050672, $\mathrm{range_{high}}(\mathrm{r}_{2,\mathrm{model}})$=0.0927875, $\mathrm{range}(\mathrm{r}_{2,\mathrm{data}})$=0.071748.

Figure 36: Time-varying consumption drift with recursive utility. See Figure 16 for more details about the plots.

(variation overview)

F.24 RU-CD-HRA, $\gamma = 6$

Term premia stay constant and negative but it becomes significantly larger in absolute value. In this paper I do not consider a time-varying γ parameter, but this suggests that a time-varying risk aversion would be able to produce time-varying term premia. The habit model essentially provides a similar mechanism.

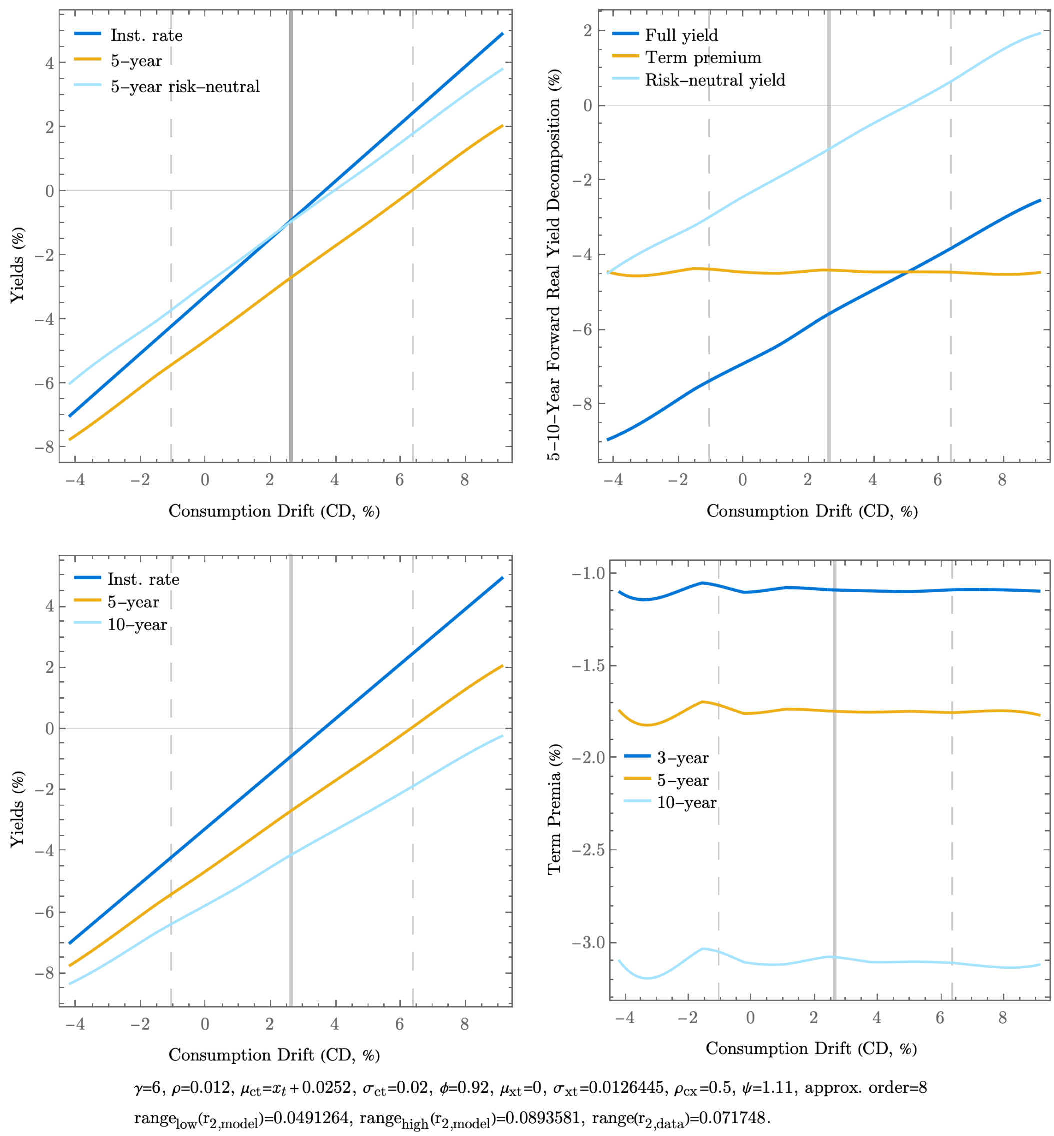
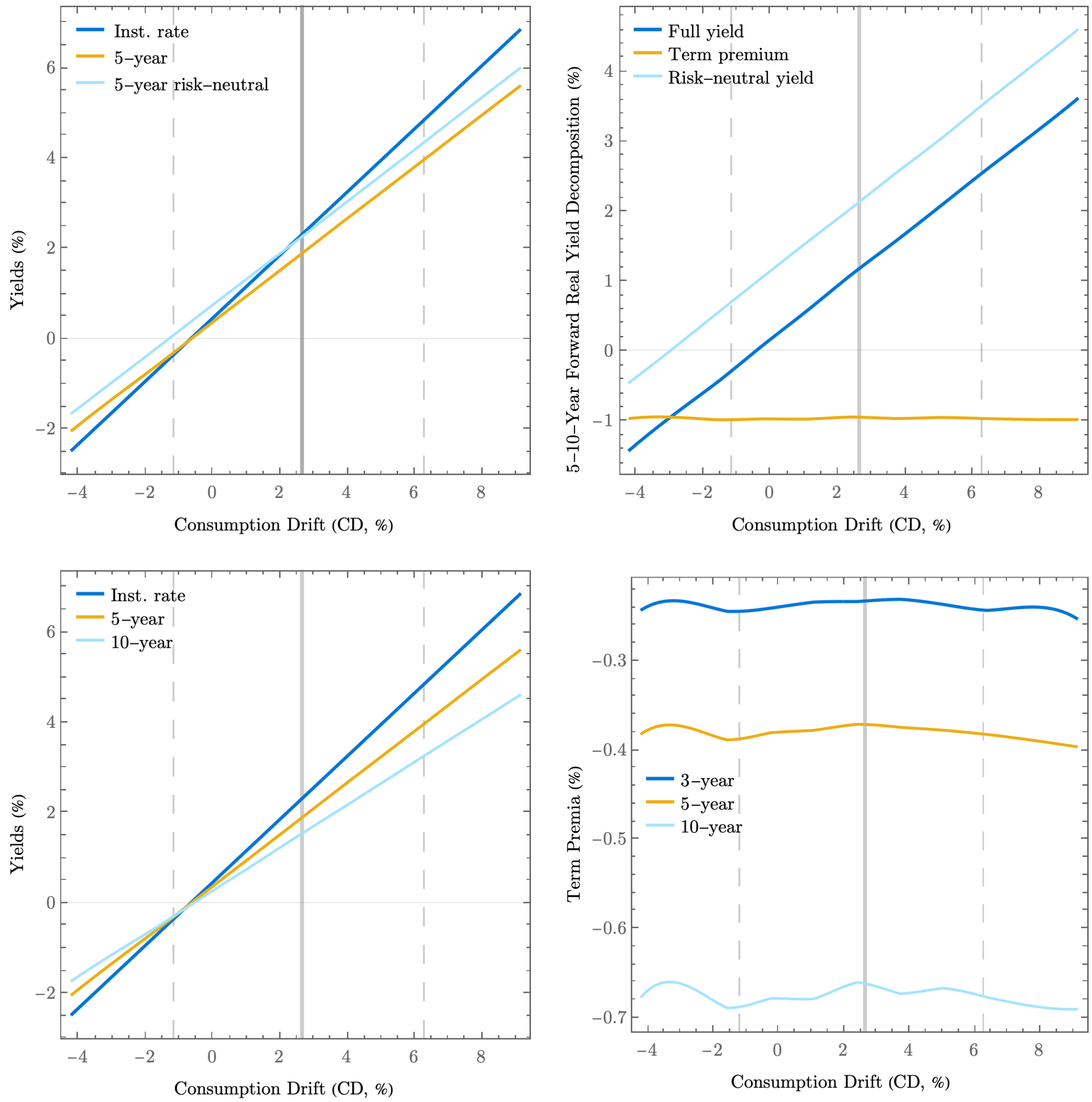


Figure 37: Time-varying consumption drift with recursive utility and high risk aversion.
See Figure 5 for more details about the plots.

(variation overview)

F.25 RU-CD-HIES, $\psi = 1.43$

Term premia do not seem to change significantly. The range of the short rate increases.



γ=2, ρ=0.012, μ_{ct}=$x_t + 0.0252$, σ_{ct}=0.02, ϕ=0.92, μ_{xt}=0, σ_{xt}=0.0126445, ρ_{cx}=0.5, ψ=1.43, approx. order=8

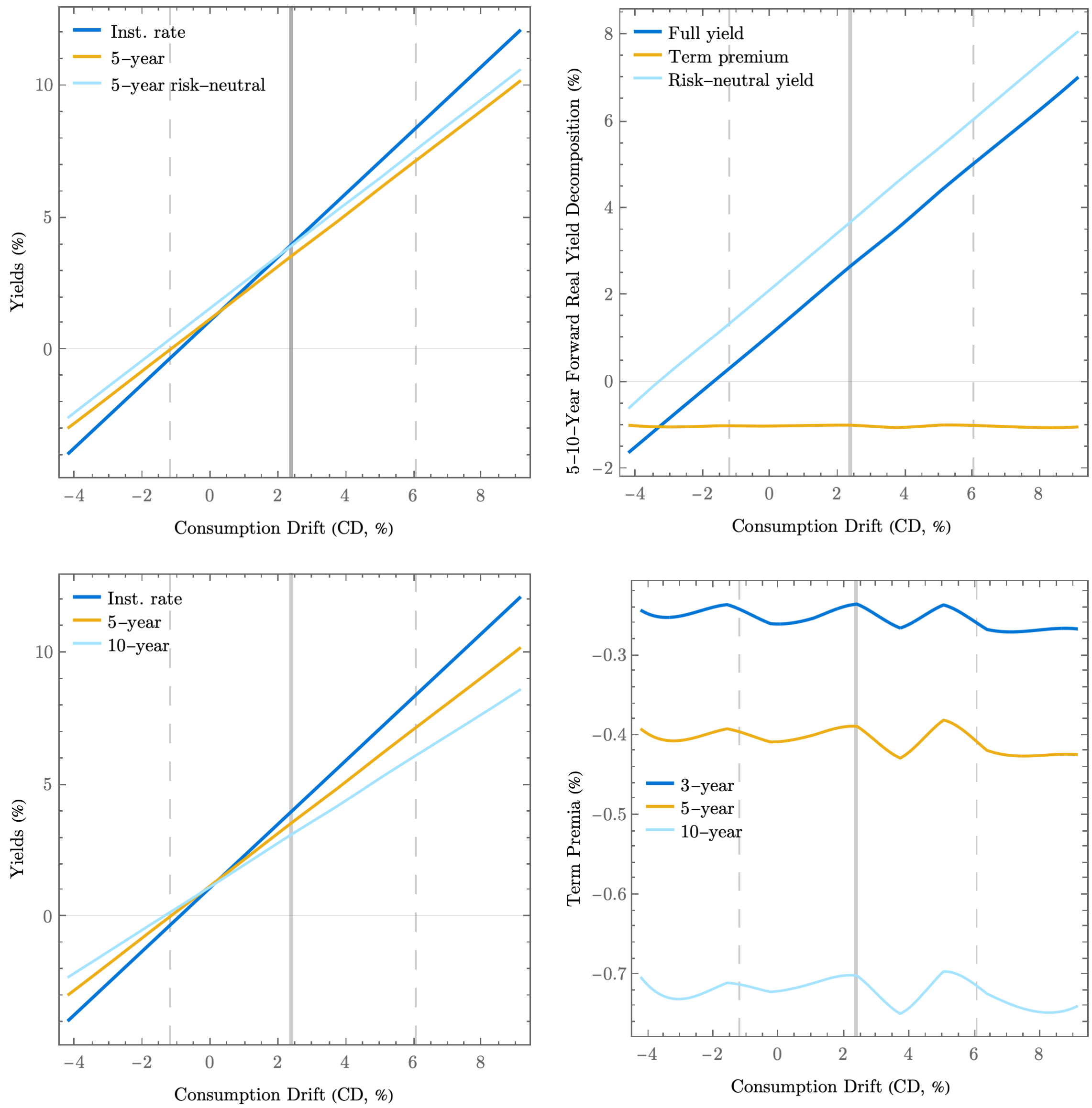
$\mathrm{range_{low}}(\mathrm{r_{2,model}})$=0.037872, $\mathrm{range_{high}}(\mathrm{r_{2,model}})$=0.071687, $\mathrm{range}(\mathrm{r_{2,data}})$=0.071748.

Figure 38: Time-varying consumption drift with high intertemporal elasticity of substitution.
See Figure 16 for more details about the plots.

(variation overview)

F.26 RU-CD-LIES, $\psi = 0.83$

Term premia do not seem to change significantly. Curiously the range of the short rate increases again.



γ=2, ρ=0.012, μ_{ct}=x_t+0.0252, σ_{ct}=0.02, ϕ=0.92, μ_{xt}=0, σ_{xt}=0.0126445, ρ_{cx}=0.5, ψ=0.83, approx. order=11

$\text{range}_{\text{low}}(r_{2,\text{model}})$=0.0651376, $\text{range}_{\text{high}}(r_{2,\text{model}})$=0.121441, $\text{range}(r_{2,\text{data}})$=0.071748.

Figure 39: Time-varying consumption drift with recursive utility with low intertemporal elasticity of substitution.
See Figure 16 for more details about the plots.

(variation overview)

F.27 RU-CD-HCor, $\rho_{cx} = 1$

Term premia increase in absolute value but do not double in size as did the correlation parameter.

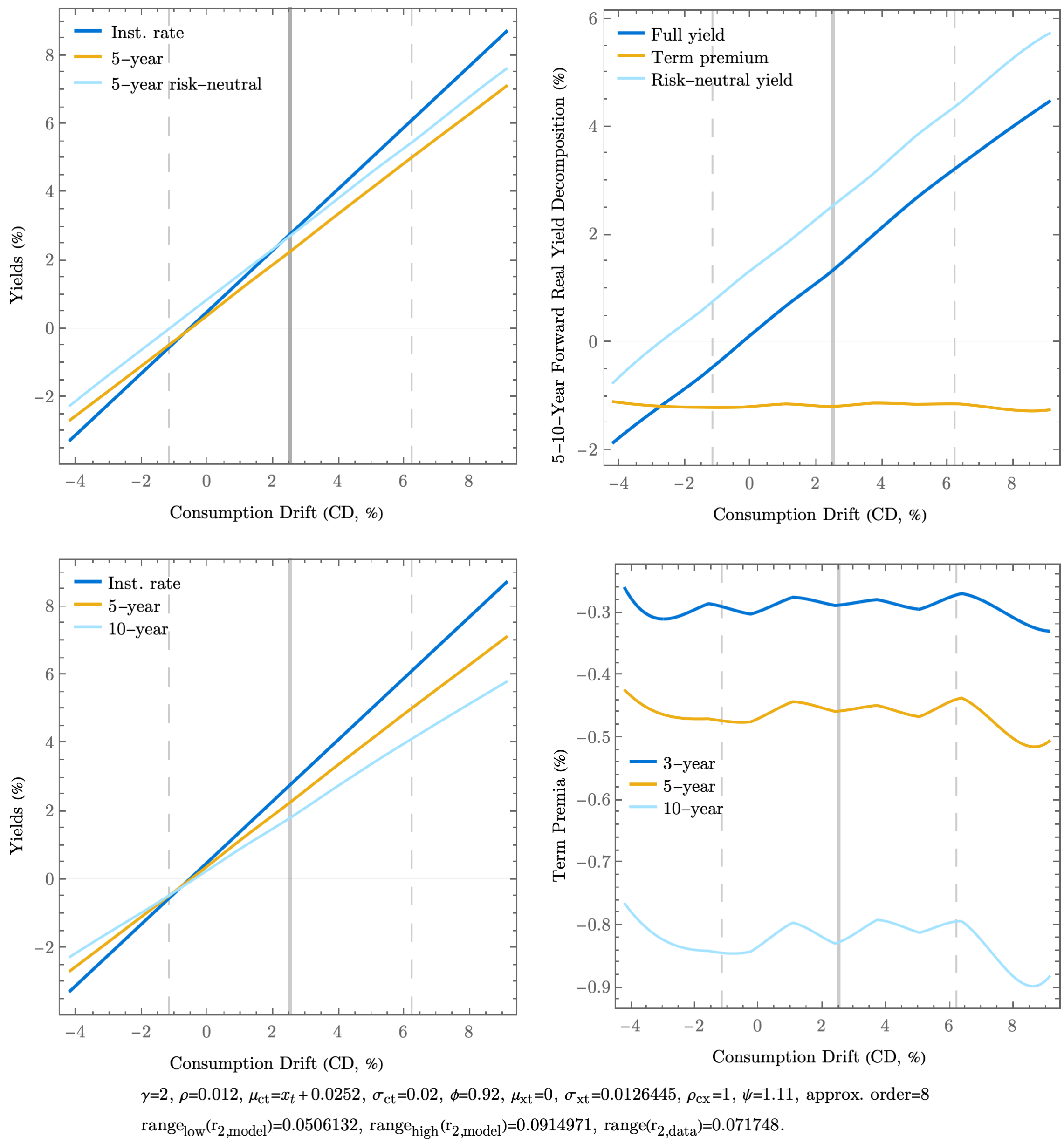
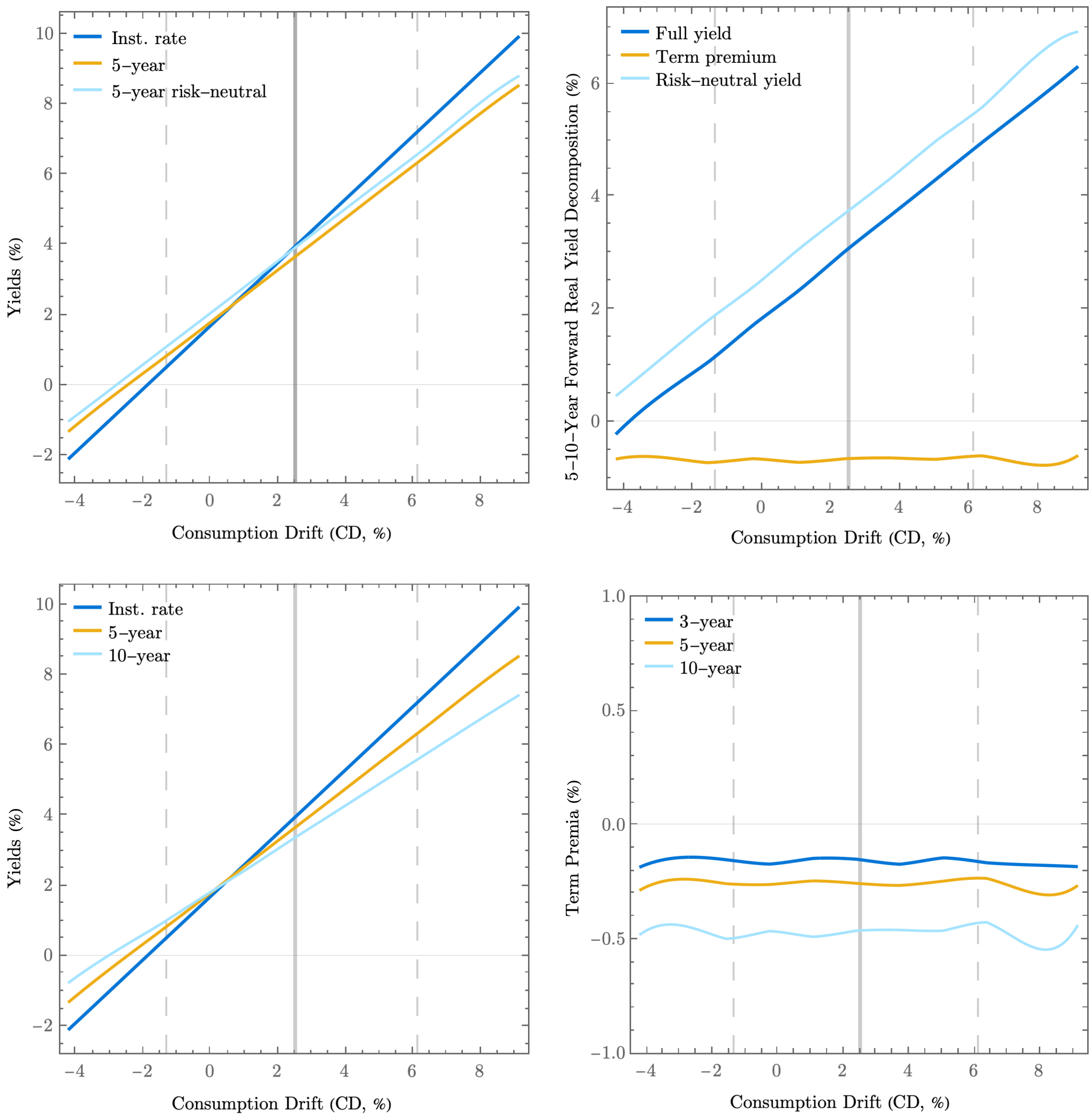


Figure 40: Time-varying consumption drift with recursive utility and high ρ_{cx}. See Figure 16 for more details about the plots.

(variation overview)

F.28 RU-CD-NCor, $\rho_{cx} < 1$

Term premia increase but they remain negative as in recursive utility term premia are dominated by the term not including ρ_{cx}.



γ=2, ρ=0.012, $\mu_{\rm ct}$=x_t+0.0252, $\sigma_{\rm ct}$=0.02, ϕ=0.92, $\mu_{\rm xt}$=0, $\sigma_{\rm xt}$=0.0126445, $\rho_{\rm cx}$=−1, ψ=1.11, approx. order=8

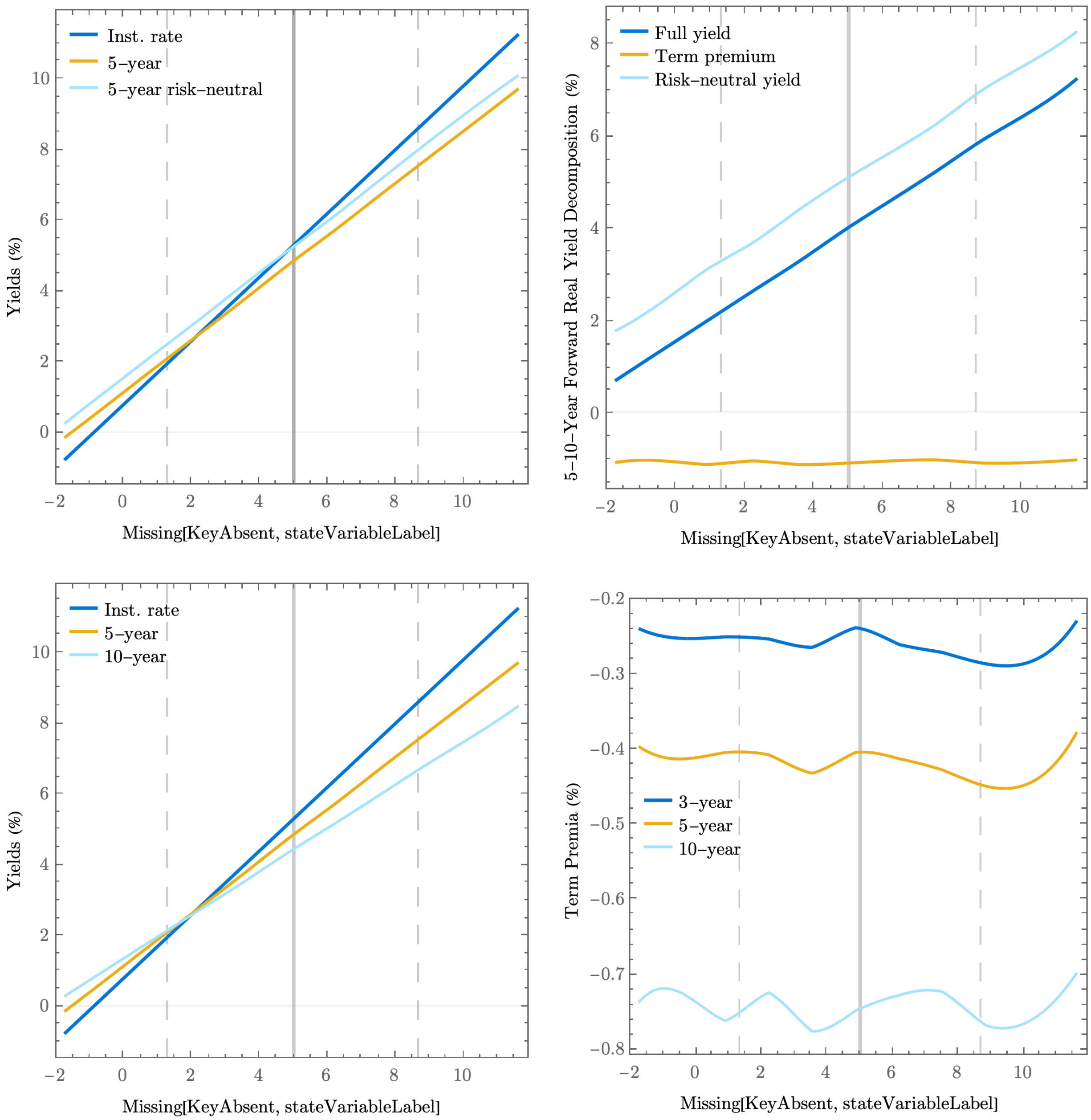
range$_{\rm low}$(r$_{\rm 2,model}$)=0.0484668, range$_{\rm high}$(r$_{\rm 2,model}$)=0.0897518, range(r$_{\rm 2,data}$)=0.071748.

Figure 41: Time-varying consumption drift with recursive utility and negative ρ_{cx}. See Figure 16 for more details about the plots.

(variation overview)

F.29 RU-HCD, $\mu_{x0} = 0.05$

Term premia do not change, but yields increase.



γ=2, ρ=0.012, μ_{ct}=x_t+0.05, σ_{ct}=0.02, ϕ=0.92, μ_{xt}=0, σ_{xt}=0.0126445, ρ_{cx}=0.5, ψ=1.11, approx. order=8

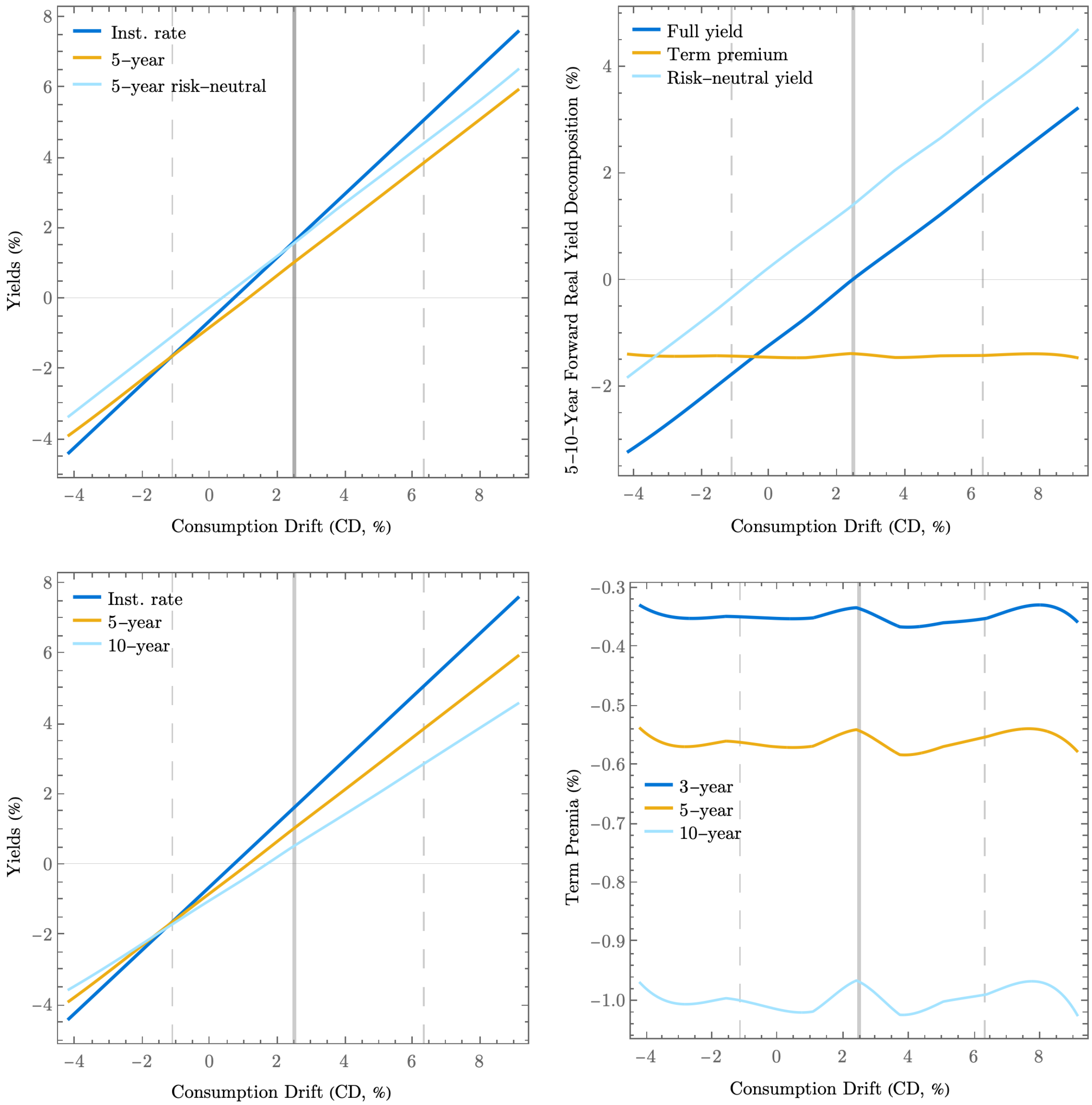
$\text{range}_{\text{low}}(\text{r}_{2,\text{model}})$=0.0503055, $\text{range}_{\text{high}}(\text{r}_{2,\text{model}})$=0.0877158, $\text{range}(\text{r}_{2,\text{data}})$=0.071748.

Figure 42: Time-varying and high consumption drift with recursive utility. See Figure 16 for more details about the plots.

(variation overview)

F.30 RU-CD-HCV, $\sigma_{ct} = 0.08$

Term premia do not change, but yields decrease.



γ=2, ρ=0.012, μ_{ct}=x_t + 0.0252, σ_{ct}=0.08, ϕ=0.92, μ_{xt}=0, σ_{xt}=0.0126445, ρ_{cx}=0.5, ψ=1.11, approx. order=8

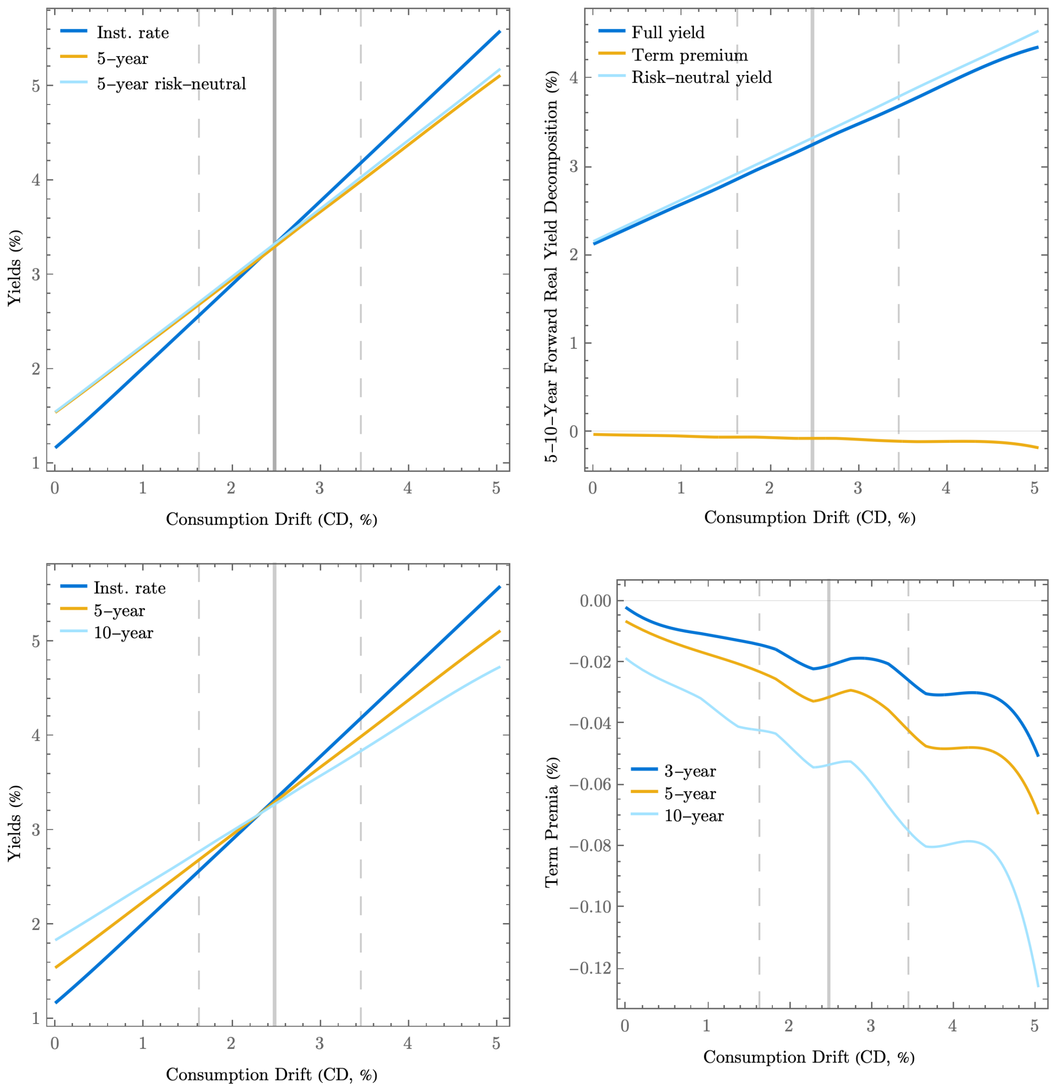
range$_{low}$(r$_{2,model}$)=0.0492211, range$_{high}$(r$_{2,model}$)=0.0952063, range(r$_{2,data}$)=0.071748.

Figure 43: Time-varying consumption drift with recursive utility and high consumption volatility.
See Figure 16 for more details about the plots.

(variation overview)

F.31 RU-CD-Heterk-PCor

When the state variable is heteroskedastic, term premia become time-varying. Here term premia are quite small, but this could change once a more volatile state variable is introduced. However, term premia are again negative.



γ=2, ρ=0.012, μ_{ct}=0.0252 x_t, σ_{ct}=0.021, ϕ=0.92, μ_{xt}=μ_x, σ_{xt}=0.12 $\sqrt{x_t}$, ρ_{cx}=0.5, ψ=1.11, approx. order=8

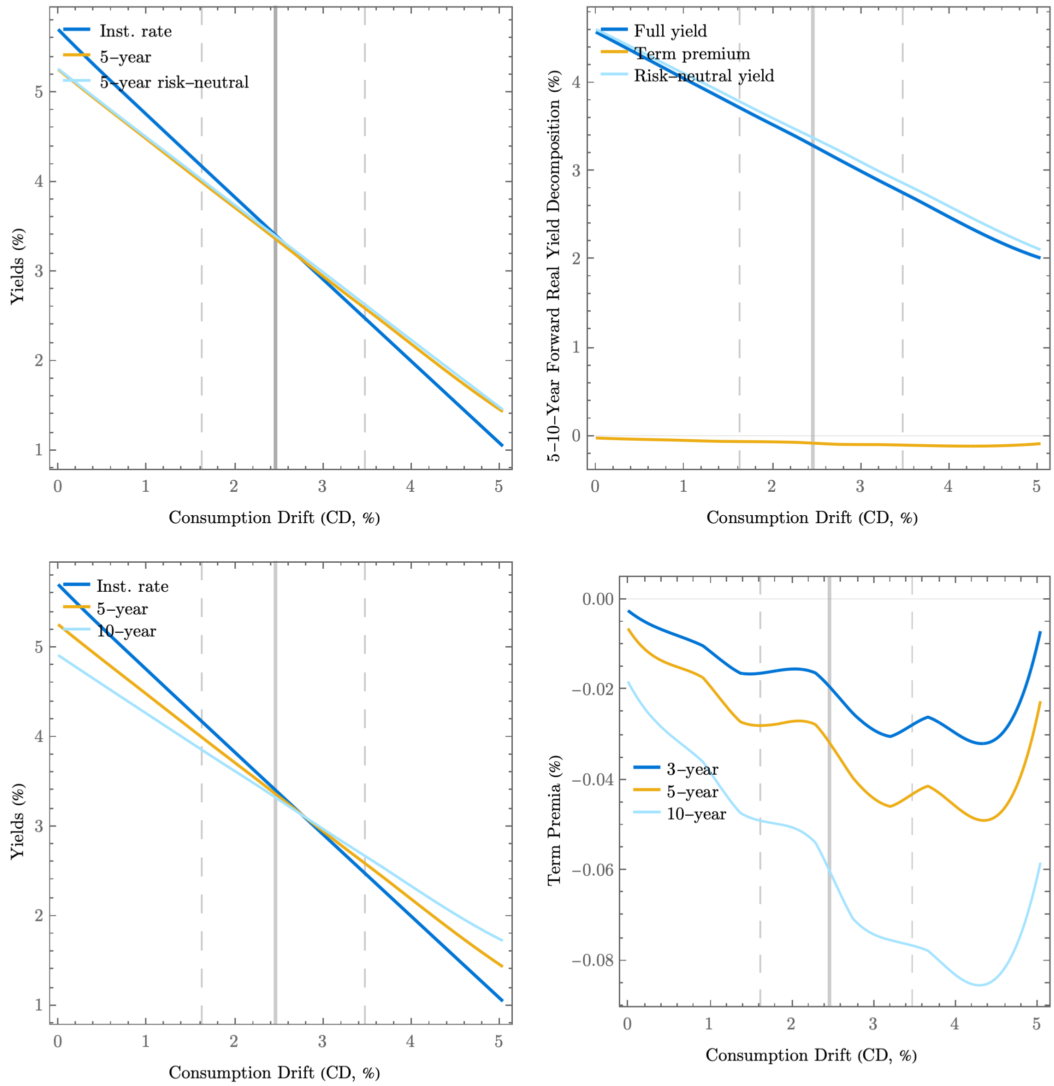
range$_{\mathrm{low}}$($\mathrm{r}_{2,\mathrm{model}}$)=0.0115265, range$_{\mathrm{high}}$($\mathrm{r}_{2,\mathrm{model}}$)=0.0221465, range($\mathrm{r}_{2,\mathrm{data}}$)=0.071748.

Figure 44: Time-varying and heterskedastic consumption drift with recursive utility with positive ρ_{cx}.
See Figure 16 for more details about the plots.

(variation overview)

F.32 RU-CD-Heterk-PCor

Despite changing the correlation compared to the previous case term premia are still negative given that the dominant component in function A does not contain ρ_{cx}.



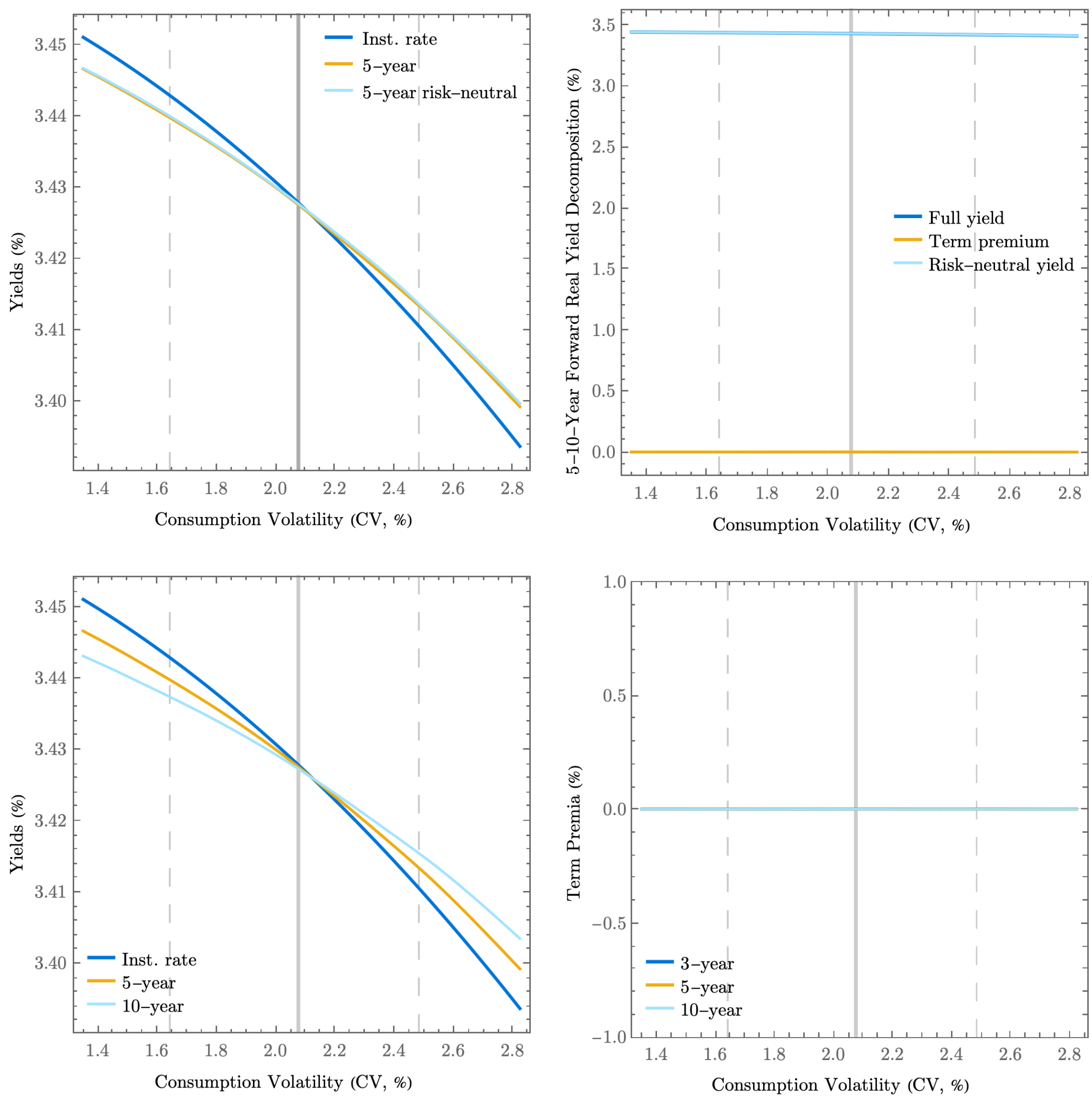
γ=2, ρ=0.012, μ_{ct}=0.0252 $(2-x_t)$, σ_{ct}=0.021, ϕ=0.92, $\mu_{\mathrm{xt}}=\mu_x$, σ_{xt}=0.12 $\sqrt{x_t}$, ρ_{cx}=−0.5, ψ=1.11, approx. order=8

$\mathrm{range}_{\mathrm{low}}(\mathrm{r}_{2,\mathrm{model}})$=0.0119906, $\mathrm{range}_{\mathrm{high}}(\mathrm{r}_{2,\mathrm{model}})$=0.0230955, $\mathrm{range}(\mathrm{r}_{2,\mathrm{data}})$=0.071748.

Figure 45: Time-varying and heterskedastic consumption drift with recursive utility with negative ρ_{cx}.
See Figure 16 for more details about the plots.

(variation overview)

F.33 RU-CV, Calibration used in main paper, Figure 5



γ=2, ρ=0.012, μ_{ct}=0.0252, σ_{ct}=0.021 $\sqrt{x_t}$, ϕ=0.92, μ_{xt}=μ_x, σ_{xt}=0.13 $\sqrt{x_t}$, ρ_{cx}=−0.5, ψ=1.11, approx. order=8

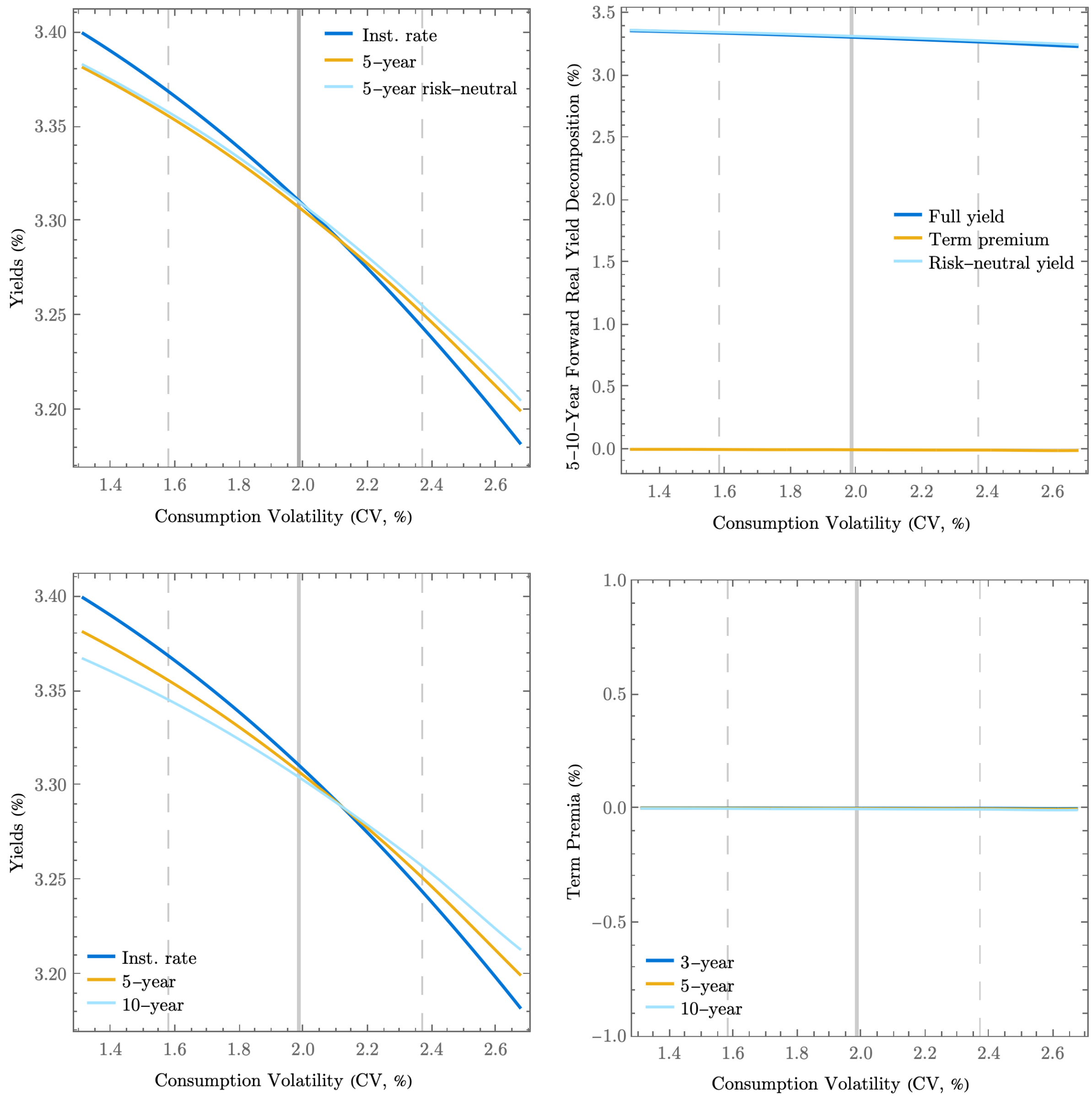
$\mathrm{range_{low}}(\mathrm{r_{2,model}})$=0.000234865, $\mathrm{range_{high}}(\mathrm{r_{2,model}})$=0.000434075, $\mathrm{range}(\mathrm{r_{2,data}})$=0.071748.

Figure 46: Time-varying consumption volatility with recursive utility.
See Figure 16 for more details about the plots.

(variation overview)

F.34 RU-CH-HRA, $\gamma = 6$

The term premia have hardly moved.



γ=6, ρ=0.012, μ_{ct}=0.0252, σ_{ct}=0.02 $\sqrt{x_t}$, ϕ=0.92, μ_{xt}=μ_x, σ_{xt}=0.13 $\sqrt{x_t}$, ρ_{cx}=−0.5, ψ=1.11, approx. order=8

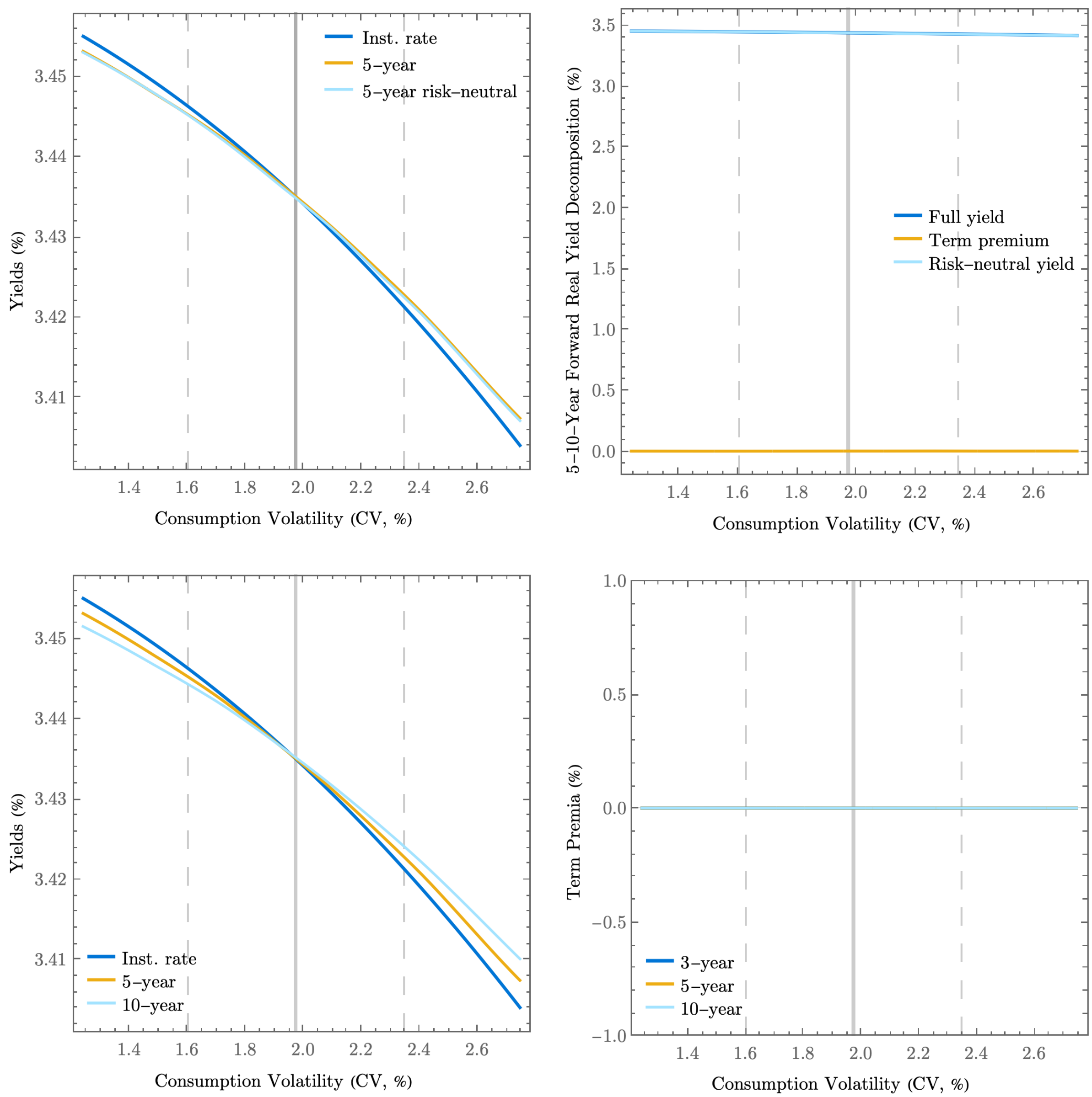
$\text{range}_{\text{low}}(\text{r}_{2,\text{model}})$=0.000924493, $\text{range}_{\text{high}}(\text{r}_{2,\text{model}})$=0.00168055, $\text{range}(\text{r}_{2,\text{data}})$=0.071748.

Figure 47: Time-varying consumption volatility with recursive utility and high risk aversion.
See Figure 16 for more details about the plots.

(variation overview)

F.35 RU-CV-HP, $\phi = 0.96$

The term premia have hardly moved and curiously the yields have become slightly more variable again.



γ=2, ρ=0.012, μ_{ct}=0.0252, σ_{ct}=0.02 $\sqrt{x_t}$, ϕ=0.96, μ_{xt}=μ_x, σ_{xt}=0.12 $\sqrt{x_t}$, ρ_{cx}=0.5, ψ=1.11, approx. order=8

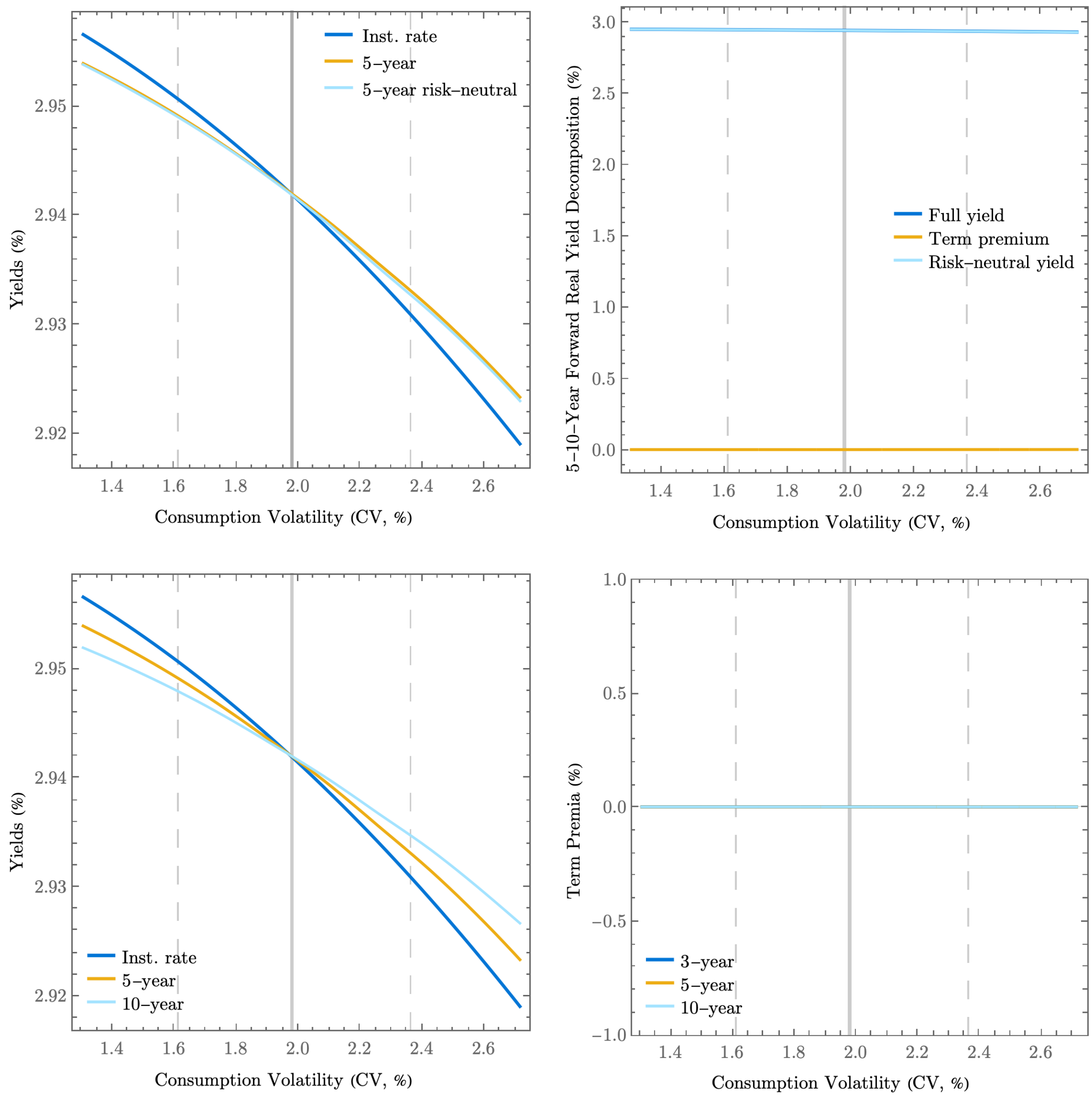
range$_{\mathrm{low}}$($r_{2,\mathrm{model}}$)=0.000185272, range$_{\mathrm{high}}$($r_{2,\mathrm{model}}$)=0.000369185, range($r_{2,\mathrm{data}}$)=0.071748.

Figure 48: Time-varying consumption volatility with recursive utility and high persistence.
See Figure 16 for more details about the plots.

(variation overview)

F.36 RU-CV-HIES, $\psi = 1.43$

The term premia have hardly moved and the yields have become slightly more variable.



γ=2, ρ=0.012, μ_{ct}=0.0252, σ_{ct}=0.02 $\sqrt{x_t}$, ϕ=0.92, μ_{xt}=μ_x, σ_{xt}=0.13 $\sqrt{x_t}$, ρ_{cx}=0.5, ψ=1.43, approx. order=8

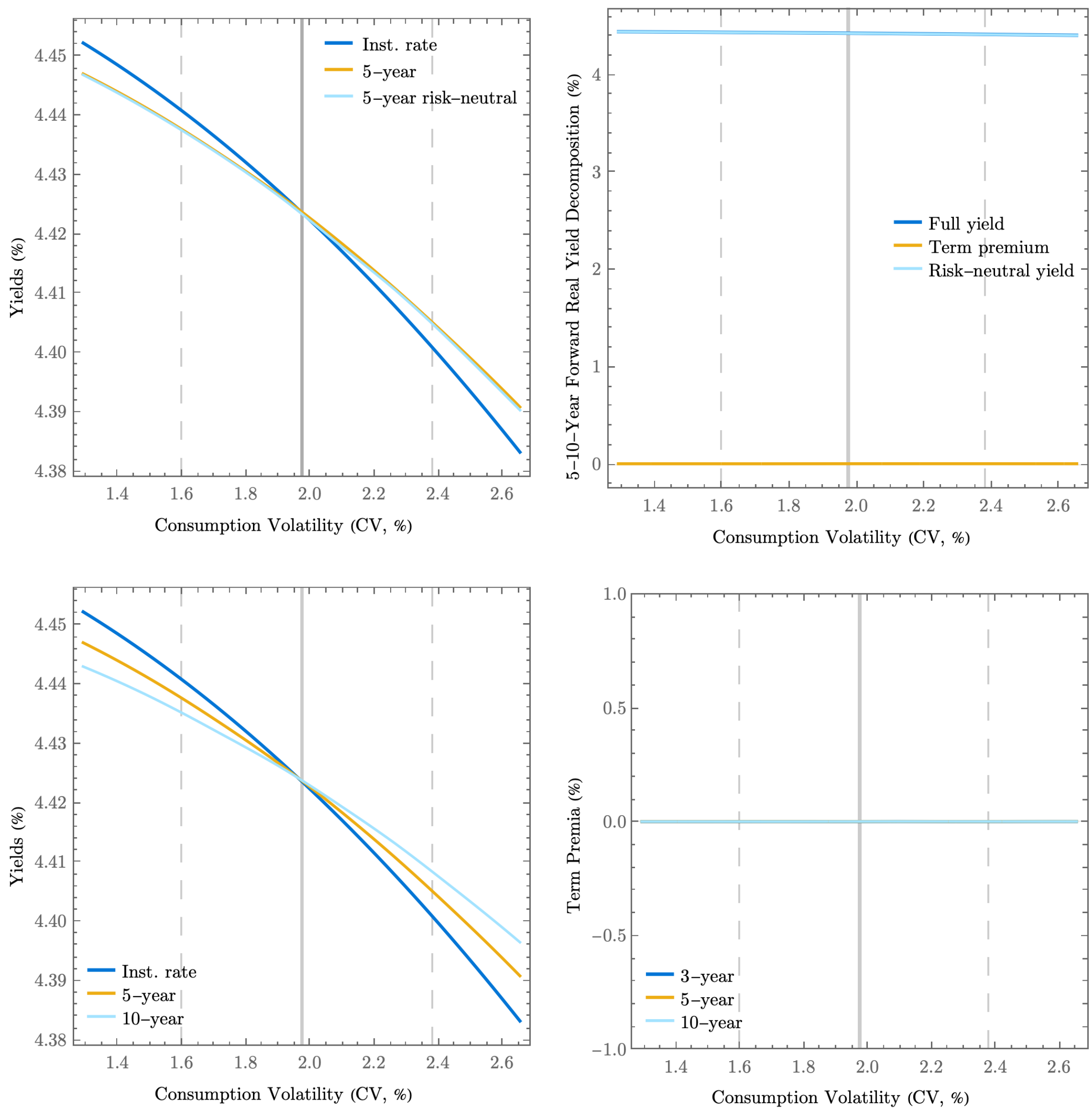
range$_{\text{low}}$(r$_{2,\text{model}}$)=0.000147906, range$_{\text{high}}$(r$_{2,\text{model}}$)=0.000289188, range(r$_{2,\text{data}}$)=0.071748.

Figure 49: Time-varying consumption volatility with recursive utility and high intertemporal elasticity of substitution.
See Figure 16 for more details about the plots.

(variation overview)

F.37 RU-CV-LIES, $\psi = 0.77$

The term premia have hardly moved and curiously the yields have become slightly more variable again.



γ=2, ρ=0.012, μ_{ct}=0.0252, σ_{ct}=0.02 $\sqrt{x_t}$, ϕ=0.92, μ_{xt}=μ_x, σ_{xt}=0.13 $\sqrt{x_t}$, ρ_{cx}=0.5, ψ=0.77, approx. order=8

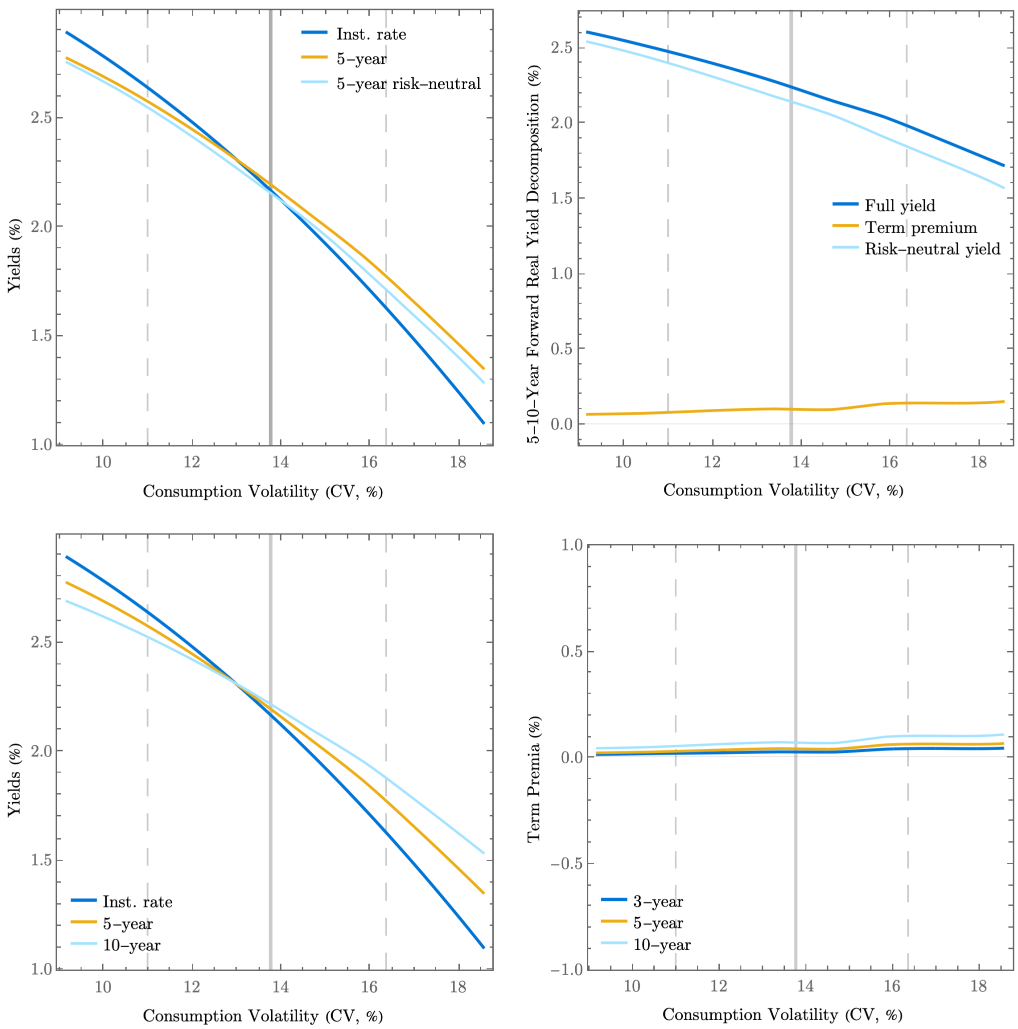
$\mathrm{range}_{\mathrm{low}}(\mathrm{r}_{2,\mathrm{model}})$=0.000288099, $\mathrm{range}_{\mathrm{high}}(\mathrm{r}_{2,\mathrm{model}})$=0.000549846, $\mathrm{range}(\mathrm{r}_{2,\mathrm{data}})$=0.071748.

Figure 50: Time-varying consumption volatility with recursive utility and low intertemporal elasticity of substitution.
See Figure 16 for more details about the plots.

(variation overview)

F.38 RU-HCV-PCor, $\sigma_{c0} = 0.14, \rho_{cx} = 0.5$

Here term premia are positive, which means that the first component of function A that contains ρ_{cx} has become dominant due to the increase in σ_{c0}. Nevertheless, term premia are still smaller than the corresponding term premia in the time-separable utility case, because the second term in function A is still negative.



γ=2, ρ=0.012, μ_{ct}=0.0252, σ_{ct}=0.14 $\sqrt{x_t}$, ϕ=0.92, μ_{xt}=μ_x, σ_{xt}=0.13 $\sqrt{x_t}$, ρ_{cx}=0.5, ψ=1.11, approx. order=8

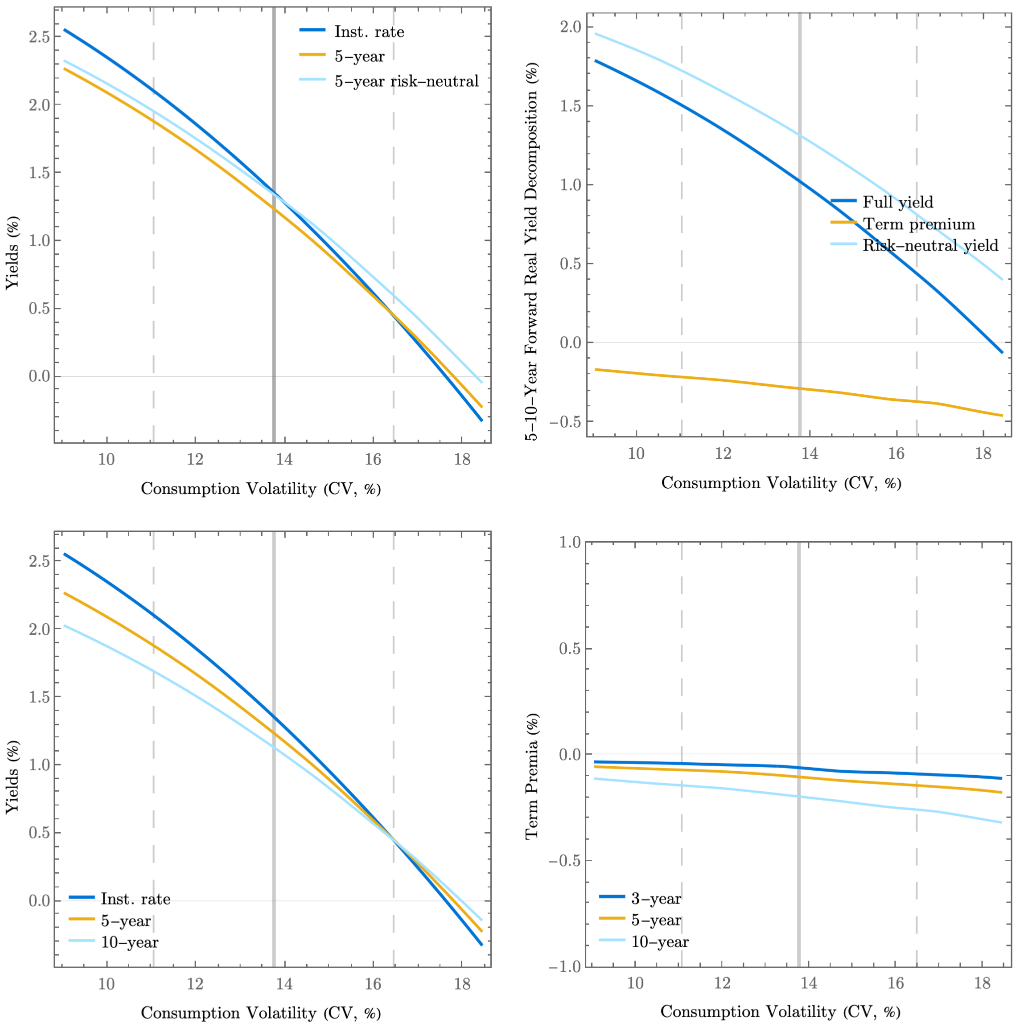
range$_{low}$(r$_{2,model}$)=0.0074128, range$_{high}$(r$_{2,model}$)=0.0146724, range(r$_{2,data}$)=0.071748.

Figure 51: See Figure 16 for more details about the plots.

(variation overview)

F.39 RU-HCV-NCor, $\sigma_{c0} = 0.14, \rho_{cx} = 0.5$

Here both terms in function A are negative, so term premia are negative. They are also larger in absolute value than the corresponding term premia in RU-CV.



γ=2, ρ=0.012, μ_{ct}=0.0252, σ_{ct}=0.14 $\sqrt{x_t}$, ϕ=0.92, μ_{xt}=μ_x, σ_{xt}=0.13 $\sqrt{x_t}$, ρ_{cx}=−0.5, ψ=1.11, approx. order=8

$\text{range}_{\text{low}}(\text{r}_{2,\text{model}})$=0.0120834, $\text{range}_{\text{high}}(\text{r}_{2,\text{model}})$=0.0232408, $\text{range}(\text{r}_{2,\text{data}})$=0.071748.

Figure 52: See Figure 16 for more details about the plots.

(variation overview)

G Equity Premium

G.1 Derivation

Given that the focus of this paper is on interest rates, I do not present results on the equity premium. However, in this section I show that the variations with high CV, examined in the main paper also imply a sizeable equity premium.

Here, I outline how to derive the equity premium, based on the framework that I have introduced. Firstly, I show how to derive the price consumption ratio, which refers to the price of the consumption perpetuity over the consumption flow. By consumption perpetuity a mean security that pays dividend equal to the consumption flow forever. In order, to get the price consumption ratio, I first derive the zero coupon price consumption ratio. This is similar to the quantity described before, but instead of using the price of the consumption perpetuity in the numerator the price of a zero-coupon security that pays the consumption flow after a fixed duration is used. Then, by integrating all the zero-coupon consumption ratios, it is possible to get the original price consumption ratio of the consumption perpetuity.

$$p(x_t) = \int_0^\infty q(m, x_t)dm = \int_0^\infty \frac{P(m, x_t, C_t)}{C_t} dm \tag{25}$$

p, $P(\cdot,\cdot)$ q and $P(\cdot,\cdot,\cdot)$ are the price consumption ratio, the price of the consumption perpetuity, the zero-coupon price consumption ratio and the price of the zero-coupon security respectively. q can be derived by applying Ito's Lemma and combining the result with a transformation of the pricing equation:

$$\begin{aligned} E\big[d(\Lambda P(m,x,C))\big] = 0 &\Rightarrow E\left[\frac{\mathrm{d}\Lambda}{\Lambda} + \frac{dP(m,x,C)}{P(m,x,C)} + \frac{\mathrm{d}\Lambda dP(m,x,C)}{\Lambda P(m,x,C)}\right] = 0 \\ &\Rightarrow E\left[\frac{\mathrm{d}\Lambda}{\Lambda} + \frac{d\Big(q(m,x)C\Big)}{q(m,x)C} + \frac{\mathrm{d}\Lambda d\Big(q(m,x)C\Big)}{\Lambda q(m,x)C}\right] = 0 \\ &\Rightarrow E\left[\frac{\mathrm{d}\Lambda}{\Lambda} + \frac{\mathrm{d}q}{q} + \frac{dC}{C} + \frac{\mathrm{d}\Lambda \mathrm{d}q}{\Lambda q} + \frac{\mathrm{d}\Lambda dC}{\Lambda C} + \frac{\mathrm{d}q dC}{qC}\right] = 0 \end{aligned} \tag{26}$$

Based on the values of q, p can be derived using Equation (25). In practise, the integral cannot be computed numerically up to infinity, so I implement a cutoff at two hundred years. By a further application of Ito's Lemma it is possible to derive the process for the return of the consumption perpetuity:

$$\left(\frac{dP(x,C)}{P(x,C)} + \frac{C}{P}\mathrm{d}t\right)/\mathrm{d}t \tag{27}$$

and the equity premium is:

$$\left(E\left[\frac{dP(x,C)}{P(x,C)}\right] + \frac{C}{P}\mathrm{d}t - r\mathrm{d}t\right)/\mathrm{d}t \tag{28}$$

The quantities above can be written in terms of p.

G.2 Results for high TSU-HCV cases

The graph shows the instantaneous annualised expected excess return of the consumption perpetuity for the models that correspond to Figures 26 and 28.

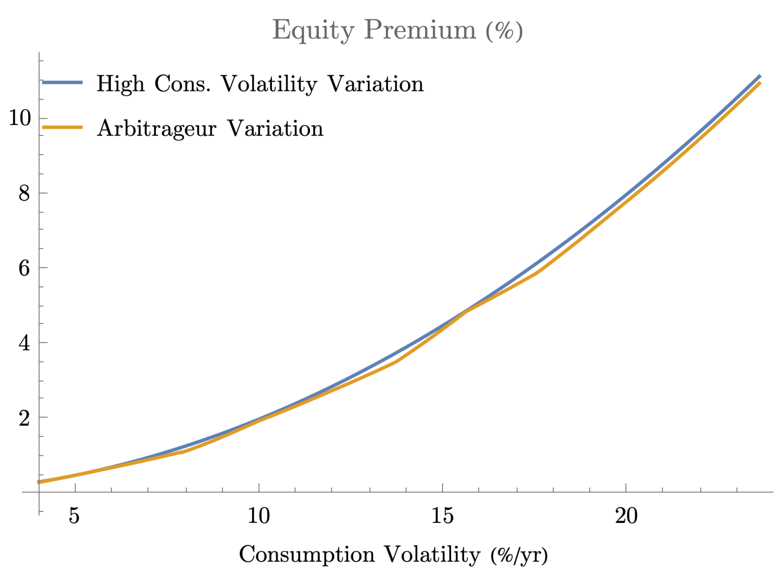


Figure 53: Equity premium for cases that exhibit high CV.

In both models the expected instantaneous return of the consumption perpetuity is close to be being constant based on the chosen calibration, and the variation of the equity premium is actually driven by the variation of the short-term rate. Nevertheless, the figure shows that the equity premium is considerable, ranging from below 2% to above 10%.

H Deriving the Stochastic Discount Factor

H.1 Derivation of the SDF with TSU

Here I derive the SDE of SDF, including the case of the habit model. I present the terms that only apply to the habit model in grey colour. The following is the regular form of the SDF, in which I have substituted the state variable and log consumption:

$$\Lambda = e^{-\rho t}(e^c S_0 e^x)^{-\gamma} \tag{29}$$

Then, in order to get the SDE form, I apply Ito's Lemma:

$$\begin{aligned}
\mathrm{d}\Lambda &= \frac{\partial \Lambda}{\partial t}\mathrm{d}t + \frac{\partial \Lambda}{\partial c}dc + \underbrace{\frac{\partial \Lambda}{\partial x}dx}_{\text{habit model}} + \frac{1}{2}\left(\frac{\partial^2 \Lambda}{\partial c^2}(dc)^2 + \underbrace{\frac{\partial^2 \Lambda}{\partial x^2}(dx)^2 + \frac{\partial^2 \Lambda}{\partial x \partial c}dxdc}_{h.m.}\right) \\
&= -\rho\Lambda \mathrm{d}t - \gamma\Lambda dc - \underbrace{\gamma\Lambda dx}_{\text{h.m.}} + \frac{1}{2}\left(\gamma^2\Lambda(dc)^2 + \underbrace{\gamma^2\Lambda(dx)^2 + \gamma^2\Lambda dxdc}_{h.m.}\right)
\end{aligned}$$

$$\Rightarrow$$

$$\frac{\mathrm{d}\Lambda}{\Lambda} = \Big(-\rho - \gamma\mu_{ct} + \underbrace{\gamma\log(\phi)(\mu_{x0} - x_t)}_{h.m.} + \frac{\gamma^2\sigma_{ct}^2}{2} + \underbrace{\frac{\gamma^2\sigma_{xt}^2}{2} + \gamma^2\rho_{cx}\sigma_{xt}\sigma_{ct}}_{\text{h.m.}}\Big)\mathrm{d}t - \gamma\sigma_{ct}\mathrm{d}W_{ct}\,\underbrace{-\gamma\sigma_{xt}\mathrm{d}W_{xt}}_{\text{h.m.}} \tag{30}$$

H.2 Derivation of the SDF with RU

As mentioned in the main paper the SDE of the SDF can be derived based on the following expression:

$$\frac{\mathrm{d}\Lambda}{\Lambda} = f_V(C,V)\mathrm{d}t + \frac{df_C(C,V)}{f_C(C,V)} \tag{31}$$

thus, flow utility is a central component of the derivation:

$$f(C,V) = \frac{\beta}{1-1/\psi}\big((1-\gamma)V\big)\left(\left(C((1-\gamma)V)^{-\frac{1}{1-\gamma}}\right)^{1-1/\psi} - 1\right) \tag{32}$$

The partial derivative of f with respect to V is:

$$f_V(C,V) = \frac{\rho\left((\gamma-1)\psi + (1-\gamma\psi)\left(C(V-\gamma V)^{\frac{1}{\gamma-1}}\right)^{\frac{\psi-1}{\psi}}\right)}{\psi - 1} \tag{33}$$

The partial derivative of f with respect to C is:

$$f_C(C,V) = -\frac{(\gamma-1)\rho V\left(C(V-\gamma V)^{\frac{1}{\gamma-1}}\right)^{\frac{\psi-1}{\psi}}}{C} \tag{34}$$

As I implement Ito's Lemma directly using c_t and x_t as independent variables, I make the following replacements in the expressions above:

$$c_t = \log(C), \quad V = \frac{C^{1-\gamma}}{1-\gamma}e^{(1-\gamma)K(x_t)} \Rightarrow K(x_t) = \frac{\log\left(-\frac{C^{1-\gamma}}{(\gamma-1)V}\right)}{\gamma-1} \tag{35}$$

And after simplification they become:

$$\begin{aligned} f_V(C,V) \to g(c_t,x_t) =& \frac{\rho\left(-(1-\gamma\psi)e^{-\frac{(\psi-1)K[x_t]}{\psi}} - \gamma\psi+\psi\right)}{1-\psi} \\ f_C(C,V) \to h(c_t,x_t) =& \rho e^{\left(\frac{1}{\psi}-\gamma\right)K(x_t)-c_t\gamma} \end{aligned} \tag{36}$$

And I implement Ito's Lemma on g_2. The partial derivatives are:

$$\begin{aligned} \frac{\partial h(c_t,x_t)}{\partial c_t} =& \gamma\rho\left(-e^{\left(\frac{1}{\psi}-\gamma\right)K[x_t]-\gamma c_t}\right) = -\gamma h(c_t,x_t) \\ \frac{\partial h(c_t,x_t)}{\partial x_t} =& \rho\left(\frac{1}{\psi}-\gamma\right)K'(x_t)\,e^{\left(\frac{1}{\psi}-\gamma\right)K[x_t]-\gamma c_t} = \left(\frac{1}{\psi}-\gamma\right)K'(x_t)\,h(c_t,x_t) \\ \frac{\partial^2 h(c_t,x_t)}{\partial c_t^2} =& \gamma^2\rho e^{\left(\frac{1}{\psi}-\gamma\right)K[x_t]-\gamma c_t} = \gamma^2 h(c_t,x_t) \\ \frac{\partial^2 h(c_t,x_t)}{\partial x_t^2} =& \frac{\rho(\gamma\psi-1)\left((\gamma\psi-1)K'(x_t)^2-\psi K''(x_t)\right)e^{\left(\frac{1}{\psi}-\gamma\right)K[x_t]-\gamma c_t}}{\psi^2} \\ =& \frac{(\gamma\psi-1)\left((\gamma\psi-1)K'(x_t)^2-\psi K''(x_t)\right)}{\psi^2}h(c_t,x_t) \\ \frac{\partial h(c_t,x_t)}{\partial c_t\partial x_t} =& \frac{\gamma\rho(\gamma\psi-1)K'(x_t)\,e^{\left(\frac{1}{\psi}-\gamma\right)K[x_t]-\gamma c_t}}{\psi} = \frac{\gamma(\gamma\psi-1)K'(x_t)\,h(c_t,x_t)}{\psi} \end{aligned} \tag{37}$$

The expressions above should be plugged into the expression:

$$\begin{aligned} \frac{df_C}{f_C} =& \Bigg(\frac{\partial h(c_t,x_t)}{\partial c_t}\mu_{ct} + \frac{\partial h(c_t,x_t)}{\partial x_t}\big(-\log(\phi)\big)(\mu_{x0}-x_t) \\ &+\frac{\sigma_{ct}^2}{2}\frac{\partial^2 h(c_t,x_t)}{\partial c_t^2} + \frac{\sigma_{xt}^2}{2}\frac{\partial^2 h(c_t,x_t)}{\partial x_t^2} + \frac{\rho_{cx}\sigma_{ct}\sigma_{xt}}{2}\frac{\partial^2 h(c_t,x_t)}{\partial c_t\partial x_t}\Bigg)\mathrm{d}t \\ &+\frac{\partial h(c_t,x_t)}{\partial x_t}\sigma_{xt}\mathrm{d}W_{xt} + \frac{\partial h(c_t,x_t)}{\partial c_t}\sigma_{ct}\mathrm{d}W_{ct} \end{aligned} \tag{38}$$

Then everything is plugged into Equation (31) to give the final result:

$$
\begin{aligned}
\frac{\mathrm{d}\Lambda}{\Lambda} =& \Bigg(\frac{\gamma(\gamma\psi-1)\rho_{cx}\sigma_{xt}\sigma_{ct}K'\left(x_t\right)}{\psi} + \frac{\gamma^2\sigma_{ct}^2}{2} - \gamma\mu_{ct} \\
& + \frac{(\gamma\psi-1)\left(2\psi(\mu_{x0}-x_t)\log(\phi)K'\left(x_t\right) + \sigma_{xt}^2\left((\gamma\psi-1)K'\left(x_t\right)^2 - \psi K''\left(x_t\right)\right)\right)}{2\psi^2} \\
& \frac{\rho\left(-(1-\gamma\psi)e^{-\frac{(\psi-1)K[x_t]}{\psi}} - \gamma\psi + \psi\right)}{1-\psi} \Bigg)\mathrm{d}t \\
& - \frac{(\gamma\psi-1)\sigma_{xt}K'(x_t)}{\psi}\mathrm{d}W_{xt} - \gamma\sigma\mathrm{d}W_{ct}
\end{aligned}
\tag{39}
$$