Essays on Consumption-Based Asset Pricing

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ΚΑΙΑΛΛΟΥΧΑΡΙΝ

Aristotle, Nicomachean Ethics

Acknowledgments

Thank people who supported me.

Contents

List of Abbreviations

SDF	Stochastic Discount Factor
SDE	Stochastic Differential Equation

Essay

Include essay about thesis and cite papers like Gürkaynak, Sack and Swansonc (2005).

Bibliography

Gürkaynak, Refet S, Brian Sack, and Eric T Swansonc. 2005. "Do Actions Speak Louder Than Words? The Response of Asset Prices to Monetary Policy Actions and Statements." *International Journal of Central Banking.*

Chapter 1

Real Term Premia in Consumption-Based Models

Abstract

Can consumption-based mechanisms generate positive and time-varying real term premia as we see in the data? I show that only models with time-varying risk aversion or models with high consumption risk can independently produce these patterns. The latter explanation has not been analysed before with respect to real term premia, and it relies on a small group of investors exposed to high consumption risk. Additionally, it can give rise to a "consumption-based arbitrageur" story of term premia. In relation to preferences, I consider models with both time-separable and recursive utility functions. Specifically for recursive utility, I introduce a novel perturbation solution method in terms of the intertemporal elasticity of substitution. This approach has not been used before in such models, it is easy to implement, and it allows a wide range of values for the parameter of intertemporal elasticity of substitution.

1.1 Introduction

Risk-free bonds hold a central position in financial theory, and in practice government bonds hold a central position in financial markets. Yet we do not fully understand how risk premia are connected to the consumption of households. Understanding this connection would not only benefit consumption-based asset pricing but also other fields. For instance, it would facilitate households' investment decisions, and it would lead to a better understanding of the effects of monetary policy which are associated with changes in prices of real bonds.

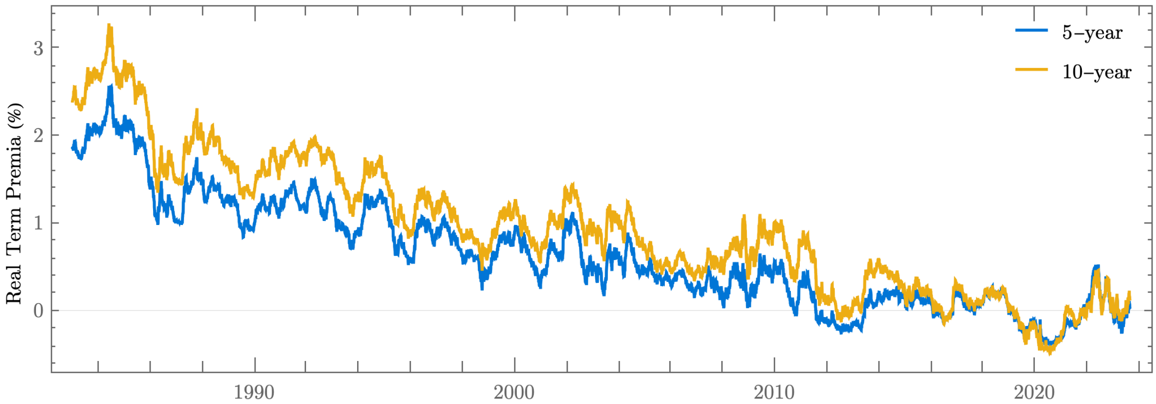


Figure 1.1: **Time series of real term premia for the US**
Data is taken from d 'Amico, Kim and Wei (2018), who decomposed nominal yields into risk-neutral real yields (expected short-term rates averaged over the corresponding period), real term premia, expected inflation, inflation premia and liquidity premia.
Data Source: `https://www.federalreserve.gov/econres/notes/feds-notes/tips-from-tips-update-and-discussions-20190521.html`

The focus of this paper is explaining real term premia, i.e., the risk premia of inflation-adjusted risk-free bonds over a specific holding period.[1] Thus, throughout this paper, terms like yields, returns, term premia etc. should be understood

[1] Term premia reflect the expected difference in log return from holding long-term bonds compared to short-term bonds over the same time period. On the contrary, risk premia usually refer to the same difference in expected returns taken over a single period (or instantaneously in continuous time). The exact definition of term premia along with the exact definition of all terms in this paper can be found in Appendix 1.A. Actual bonds may also have liquidity premia, which are deviations in the price of bonds due to their liquidity in the market. Liquidity premia are separate from term premia, or, in other words, my definition of term premia assumes that bonds are perfectly liquid.

as referring to their real counterparts, unless otherwise specified. Term premia in the data are mostly positive and significantly time-varying (Abrahams et al. 2016; d 'Amico, Kim and Wei 2018; Pflueger and Viceira 2016). Estimates from d 'Amico, Kim and Wei (2018) are shown in Figure 1.1. Overall, consumption-based models struggle to generate these main features. In the literature this is referred to as the bond premium puzzle (Backus, Gregory and Zin 1989).[2] The source of the puzzle is that consumption-based mechanisms typically generate small, negative, and often constant term premia, namely the exact opposite of what we see in the data. This is due to bond prices typically being counter-cyclical in these models, while consumption risk being relatively small and varying little with the business cycle. In addition, contrary to nominal term premia, it is not possible to explain real term premia by relying on an inflation premium, which arises due to risk associated with the inflation process.

In the literature it is known that models with time-varying risk aversion can generate positive and time-varying term premia. For instance, Wachter (2006) showed this in a model with an external habit following Campbell and Cochrane (1999). In this paper I perform a comprehensive investigation of term premia within separate consumption-based models, and my analysis shows that there is a second consumption-based model that can generate these features without employing a habit component or any time-varying risk aversion. To the best of my knowledge this result is new. The mechanism relies on: a) high time-varying consumption risk; and b) a negative correlation between consumption and the short-term interest rate. Firstly, the high and time-varying consumption risk generates term premia that are high in absolute value and time-varying. Given that we do not observe such high consumption risk in aggregate data, this is not the consumption process of the representative consumer. Instead, it is the consumption process of a small group of marginal investors.[3] Additionally, the mechanism is flexible and these investors can either be thought of as holding all bonds, or alternatively they can be thought of as holding any portion of total bonds. Secondly,

[2] The bond premium puzzle can also refer to nominal term premia. Nominal term premia have the same definition as real term premia with the underlying bonds not being inflation-adjusted.

[3] This means that the approach follows heterogeneous agent models, as a small group of investors have a different consumption process than the average in the economy. However, I do not examine a full heterogeneous agent model, as I restrict my analysis to marginal investors in the bond market. Furthermore, my approach does not follow the intermediary asset-pricing paradigm (He and Krishnamurthy 2013). In particular, despite the fact that a small group of investors is driving asset prices in the bond market, these investors do not act as intermediaries for the households, and the results do not stem from any intermediation constraints.

the negative correlation between the short-term rate and consumption implies a positive correlation between bond prices and consumption. Therefore, these marginal investors regard long-term bonds as risky, and they demand a positive term premium for them. In the main variation that I introduce, the intuition for the negative correlation between the short-term rate and consumption can be understood as follows: As the short-term rate falls (rises), marginal investors, who are also bond-holders, see an increase (decrease) in their wealth. In addition, they raise (lower) their net borrowing. Therefore, both consumption and consumption risk increase (decrease).[4] In the paper I do not argue that this mechanism is superior in explaining term premia compared to time-varying risk aversion. On the contrary, the focus of the paper is to introduce the new mechanism, and show that, within the constraints of my analysis, it is the only mechanism that generates the features of term premia without using time-varying risk aversion. My analysis employs standard consumption-based asset pricing, and I assume a single state variable following a stationary autoregressive process.[5]

The high consumption risk mechanism can also be adapted to tell a "consumption-based arbitrageur story" of the term structure of interest rates related to Vayanos and Vila (2021). They explained the term structure of interest rates by a preferred habitat model, in which so-called arbitrageurs integrate the yield curve by taking advantage of differences in expected return between different maturities of bonds. While Vayanos and Vila (2021) associate their arbitrageurs with banks and/or hedge funds, the authors do not take a position whether natural persons could correspond to arbitrageurs.[6] In this paper I show that arbitrageurs can be modelled as consumers, who drive positive and time-varying term premia, as long as consumption risk is high. In this adaptation the state variable corresponds to the magnitude of the risky investment opportunity, which could be due to long-term bonds having a higher expected return than short-term bonds. As the investment opportunity increases (decreases), arbitrageurs borrow more (less) and invest more (less), risk increases (decreases), expected returns increase (decrease),

[4] The intuition is similar to the mechanism in Schneider (2022).

[5] Thus, my analysis does not include models that are driven by higher order beliefs as in Angeletos, Collard and Dellas (2018). In addition, my analysis only includes steady-state-reverting autoregressive processes. However, there is literature suggesting that macro processes are more elaborate. For example, Bauer and Rudebusch (2020) decomposes the nominal yield curve by taking into account long-run macroeconomic trends, while there is a long literature investigating the time-series properties of interest rates (a survey is provided by Neely, Rapach et al. 2008). It would be interesting for further research to expand my analysis, in order to include more elaborate processes for the state variable.

[6] This could also be the case if arbitrageurs are investing on behalf of natural persons without significant intermediation distortions.

and thus consumption increases (decreases). The source of this investment opportunity is external to the arbitrageurs. For instance, it can be driven by demand pressure from preferred habitat investors as in Vayanos and Vila (2021) or by monetary policy. The arbitrageur approach shares the main characteristics of the baseline high consumption risk mechanism. Therefore, it also generates positive and substantially time-varying term premia.

Furthermore, my paper also contributes in the following ways. Firstly, I provide explicit values of term premia as a function of the state of the economy for a large range of model variations. This is useful, because consumption-based models in the literature often focus on nominal term premia, and even when they focus on real term premia, explicit state-dependent term premia are rarely displayed. In addition, I am providing the code to reproduce the calculations, which can easily be adapted for other specifications or calibrations. Secondly, apart from time-separable utility (TSU), my analysis also includes models with recursive utility (RU), and I contribute a novel perturbation method to easily and robustly solve such models. My perturbation method builds on the approach of Tsai and Wachter (2018). While they used an approximation to the value function that is constant in terms of the intertemporal elasticity of substitution (IES), and analytically correct only for IES equal to 1, I consider the full perturbation series in terms of the IES. This provides a global approximation in terms of the state variable of the economy that allows the easy solution of the model for most values of the IES that are economically interesting. It is also the first perturbation method in terms of the IES within recursive utility models.

The rest of the paper is organised as follows: In section 1.2, I provide more information regarding the literature on the bond premium puzzle. In section 1.3, I discuss interest rates in the data. In section 1.4, I present the setup that will allow me to price bonds in the context of TSU and RU. This includes the outline of the novel perturbation method. In section 1.5, I show and comment on the results for term premia. Finally, section 1.6 concludes.

1.2 Literature on the Bond Premium Puzzle

While I analyse real term premia, the bond premium puzzle originally referred to nominal term premia.[7] One of the first papers to address this was Backus, Gregory and Zin (1989). Utilising a consumption-based asset-pricing model of an

[7] Rudebusch and Swanson (2008) also offers a good summary of this extensive literature.

endowment economy, they discovered the model's inability to yield significant positive term premia. Subsequent studies by Donaldson, Johnsen and Mehra (1990) and Den Haan (1995) further indicated that standard real business cycle models also could not resolve the puzzle. Rudebusch and Swanson (2008) incorporated an external habit into DSGE models but found that the bond premium puzzle remains. Specifically, including a habit with non-flexible working hours can generate positive term premia, but at the cost of inducing volatile wages, prices and short-term interest rates. Duffee (2013) showed that basic properties of nominal yields cannot be explained macroeconomically, at least according to standard asset-pricing models. Also in a more generic contribution, Duffee (2002) shed light on the challenges of fitting both interest rate and term premium dynamics within affine models.

Next, a series of papers provided explanations that focused on nominal term premia, and not on real term premia. Notably, Piazzesi and Schneider (2006) showed that parameter uncertainty in a model where inflation brings bad news about future consumption growth can produce positive nominal term premia.[8] Gabaix (2012) and Tsai (2015), following Rietz (1988) and Barro (2006), showed that positive nominal term premia can be explained, if inflation is on average high during consumption disasters. Bansal and Shaliastovich (2013), following Bansal and Yaron (2004), demonstrated that the risk premium of a nominal bond can be positive in a model with long-run risk, as long as inflation is correlated with consumption trend. Rudebusch and Swanson (2012) used a similar model within a DSGE framework, which has real and nominal long-term risks, and they show that positive nominal term premia are generated; nevertheless real term premia are again negative in this model. Gomez-Cram and Yaron (2021) also used a model following Bansal and Yaron (2004), but they focused on explaining nominal term premia, using an inflation channel, while claiming that the apparent underperformance of their model with respect to real term premia should be expected due to liquidity premia in the TIPS market.

Alternatively, some articles also consider real term premia. For instance, Katagiri (2022) explored a model with monetary policy, in which consumption changes can be negatively correlated with consumption trend, and risk aversion is very high. As a result term premia can be positive, but the premia time variability is not examined. Ellison and Tischbirek (2021) went beyond standard rational

[8] Collin-Dufresne, Johannes and Lochstoer (2016) introduces a model with bayesian learning of parameters. However, this model does not emphasise bond term premia and it generates *negative* term premia.

expectations models by using a beauty contest mechanism as introduced by Angeletos, Collard and Dellas (2018), in which agents anticipate the expectations of other agents; their model generates positive term premia.

Using a similar approach to the current paper, some articles tackle the problem by deviating from the representative agent model. Vayanos and Vila (2021) suggested that term premia are generated by arbitrageurs interacting with so-called preferred habitat investors, namely investors that have a tendency to hold specific maturities of bonds. Kekre, Lenel and Mainardi (2022) built on Vayanos and Vila (2021), and showed that the characteristics of the arbitrageur portfolio can have important implications for the sign of term premia. Jappelli, Subrahmanyam and Pelizzon (2023) also built on Vayanos and Vila (2021) by integrating the repo market in their analysis. Schneider (2022) showed that positive term premia can arise in models with heterogeneous agents exhibiting different attitudes towards risk and different preferences to substituting consumption through time. Finally, returning to models with a representative agent, Wachter (2006) showed that term premia can be positive and time-varying, within a model with an external habit following Campbell and Cochrane (1999). Kliem and Meyer-Gohde (2022) used the same mechanism within a DSGE model, and they found positive term premia. Hsu, Li and Palomino (2021) also used this mechanism within a DSGE model, and they verified that a habit element is key in generating positive and time-varying term premia. Campbell, Pflueger and Viceira (2020) also used a habit model to explain the time-variability of term premia. More generally, a model with external habit can be classified as a model with time-varying effective risk aversion, and within this class of models, Lettau and Wachter (2011) showed that positive and time-varying term premia can be obtained, and Bekaert, Engstrom and Grenadier (2010) showed that time-varying term premia can be obtained. These papers all use time-varying risk aversion, which is to my knowledge the only mechanism in the literature that achieves positive and time-varying term premia within a rational representative agent model.[9]

[9] Yet, a utility with a time-varying degree of risk aversion may not be considered the most standard rational utility function.

1.3 Real Rates in the Data

1.3.1 TIPS as real rates

The first challenge regarding real rates is that they are not directly observable from standard bonds. The real interest rate is the yield of a nominal bond whose payoff is adjusted for inflation. So deducing real interest rates from nominal bonds requires at least the calculation of expected inflation, which is not trivial. The closest thing that we have in the data for real interest rates is inflation-adjusted government bonds. Such data are available for the UK and the US. In the UK, inflation-adjusted government bonds (inflation-adjusted GILTs) have been available since the 1980s. In the US, the corresponding securities are called TIPS (Treasury Inflation-Protected Securities) and corresponding price data are available for roughly twenty years (Gürkaynak, Sack and Wright 2010).[10] A severe limitation of TIPS is that they are not as liquid as normal US treasuries. For this reason, I focus on term premia measures produced by d 'Amico, Kim and Wei (2018) who computed risk-neutral yields and term premia, after taking account of the liquidity premia of TIPS over normal US treasuries.[11] As can be seen in Figure 1.2, in some periods liquidity premia of TIPS are considerable. Nevertheless, as has been discussed already and shown in Figure 1.1, term premia are still significantly time-varying.

1.3.2 Real rates as a component of nominal rates

Figure 1.2 shows real yields at the top and nominal yields at the bottom. The graph reveals several key conclusions. Firstly, both nominal and real interest rates are time-varying. In addition, different maturities have different yields and the term structure seems to be upward-sloping in both cases. In other words, longer maturities are associated with higher yields. The slope of the term structure is also not constant, as the spread between yields of different maturities varies. Secondly,

[10] Gürkaynak, Sack and Wright (2010) provides data starting from 1999. However, the full set of maturities is provided starting in 2002.

[11] Apart from liquidity issues related to TIPS, there is also a small concern (mostly with recently issued TIPS) that negative inflation is not correctly accounted for. This is because TIPS are guaranteed to pay investors at least the original principal value of the bond, even if the rate of inflation is negative. This makes inflation adjustment somewhat skewed. However, the effect will probably be small for securities that were issued several years prior, given that likely some inflation has already taken place and the probability that negative inflation will overcome it is small. Lastly, the accuracy of inflation adjustment can be debated, as the consumer price index might not capture the specific inflation concerns of investors.

it is clear from Figure 1.2 that nominal rates are highly correlated with real rates. Considering the Fisher equation:

$$y_t^{nom,m} \approx y_t^{real,m} + E[\pi_{t,t+m}] \tag{1.1}$$

where m denotes the maturity of the underlying bond. The nominal rate can be thought of as a composite rate that includes two separate components, the real rate and expected inflation.[12] Figure 1.2 also shows that real interest rates are a significant and non-trivial component of nominal interest rates. Namely, real rates are moving substantially and mostly in parallel to nominal rates. This means that in order to fully understand the movements in nominal rates, it is important to also consider the movements in real rates. In addition, models that seek to explain nominal rates solely or primarily based on processes related to inflation are not able to provide a comprehensive understanding of interest rates. This underscores the importance of finding models that can accurately describe real rates. In Appendix 1.B, I statistically verify that the information contained in the movements of in real rates explains a large proportion of the variation of nominal rates.

1.3.3 Empirical evidence regarding term premia

Empirical research has predominantly focused on nominal bonds in relation to term premia, *return predictability*, the *expectations hypothesis* (EH), and *excess volatility*. Specifically, predictability in nominal rates has been found by Fama and Bliss (1987) and Singleton (1980), who showed that yield spreads can partially predict excess returns of bonds over extended periods. This implies both the existence of term premia, and that they are time-varying. In addition, this is equivalent to a violation of the EH, which has been verified by Cochrane and Piazzesi (2005) among others. The existence of excess volatility (Shiller 1979) also indicates time-varying term premia, because the excess volatility is evidence that changing economic conditions affect the value of long-term bonds beyond what can be explained by movements in expected short rates. Even though the literature has focused less on real term premia, relatively recent studies have concluded that real term premia are also positive and time-varying, after accounting for liquidity premia. In particular, Abrahams et al. (2016) estimated the five-to-ten year real

[12] The Fisher equation can be made into an equality by adding an inflation risk premium.

forward term premium and find that it has ranged roughly from 0% to 4% between 2000 and 2014 (Figure 1.14 in Appendix 1.E).[13] d 'Amico, Kim and Wei (2018) find that the five-to-ten year forward term premium has ranged roughly from -0.5% to 4% between 1980 and 2022 (Figure 1.1).[14] Pflueger and Viceira (2016) have demonstrated the existence of predictability of real excess returns, also implying the existence of time-varying real term premia. The conclusion that real term premia are substantial and time-varying is significant, because it implies that the variability in nominal term premia is not exclusively (or even primarily) driven by inflation. Therefore, models cannot rely only on inflation to explain nominal term premia.

1.4 The Consumption-Based Framework

I adopt a consumption-based framework in continuous time, which can accommodate a range of model variations. The framework is built upon three main components from which everything else is derived: 1) an exogenous consumption process; 2) a utility specification; and 3) a process for the state variable. The state variable determines the state of the economy, and it is either connected with some component of the consumption process or with some components of the utility function. Specifically, in the variations in this paper, the state variable is either connected to consumption trend (otherwise referred to as consumption drift, CD), or connected to consumption volatility (CV), or connected to the external habit of the utility function. These three options in combination with different calibrations and utility specifications give rise to a long list of variations and interpretations. To keep things simple, I only use one state variable for each model variation. Utility will either be time-separable (TSU), or recursive (RU) following Duffie and Epstein (1992).

1.4.1 Naming the variations

As mentioned already, I analyse several model variations in the main text of this paper, and many more in Appendix 1.F. While I explain the models variations in detail in Sections 1.4 and 1.5, for convenience, I also provide abbreviations for the

[13] Shown in Figure 5 of Abrahams et al. (2016).

[14] For the period between 2000 and 2014, the results of d 'Amico, Kim and Wei (2018) imply a somewhat smaller variability of term premia compared to the results of Abrahams et al. (2016).

model variations in the following table. I use these abbreviations throughout the paper:

Model Variation Description	**Abbreviation**
Time-varying consumption drift with time-separable utility.	TSU-CD
Time-varying consumption volatility with time-separable utility.	TSU-CV
Time-varying habit with time-separable utility.	TSU-Habit
Time-varying consumption drift with recursive utility.	RU-CD
Time-varying consumption volatility with recursive utility.	RU-CV
Time-varying consumption drift and consumption volatility with recursive utility.	RU-Mixed
High time-varying consumption volatility with positive correlation $\rho_{cx} > 0$, and time-separable utility.	TSU-HCV
Arbitrageur case with short-term rate decreasing in the investment opportunity and positive correlation $\rho_{cx} > 0$, with time-separable utility.	Arb-DP
Arbitrageur case with short-term rate decreasing in the investment opportunity and negative correlation $\rho_{cx} > 0$, with time-separable utility.	Arb-IN

Table 1.1: Names of main model variations. The models are explained in Section 1.5.

1.4.2 Consumption process

Although consumption is often considered a fundamental choice variable for economic agents, it is assumed to be exogenous in this paper.[15] This approach is consistent with consumption having been decided at some earlier stage that is not explicitly modelled, and it significantly simplifies the analysis. In the most

[15] This is a standard choice in this literature. See for example Campbell and Cochrane (1999) and Bansal and Yaron (2004).

general form, consumption (C_t) follows the stochastic process expressed below:[16]

$$\mathrm{d}\log(C_t) = \mathrm{d}c_t = \mu_{ct}\mathrm{d}t + \sigma_{ct}\mathrm{d}W_{ct} \tag{1.2}$$

μ_{ct} denotes the CD at time t and σ_{ct} is the volatility coefficient of consumption growth at time t, which is multiplying the stochastic component $\mathrm{d}W_{ct}$.[17] In the remainder of the paper, CV refers to σ_{ct}.

1.4.3 Utility

Lifetime utility at time 0 takes the following form depending on the utility specification:

$$\underbrace{U_0 = E_0 \int_0^\infty e^{-\rho t} u(C_t, S_t)\mathrm{d}t,}_{\text{TSU}} \qquad \underbrace{V_0 = E_0 \int_0^\infty f(C_t, V_t)\mathrm{d}t}_{\text{RU}} \tag{1.3}$$

In both cases there is an infinite horizon, with ρ representing the time preference parameter. In the case of TSU, flow utility u depends on the consumption flow and potentially on the surplus consumption ratio S_t, which is connected to the external habit.[18] In the variations without habit, S_t is taken to be equal to 1. In the case of RU, the aggregator function f depends on the consumption flow and on the current lifetime utility V_t which in the context of RU is referred to as the value function. u and f take the following form:

$$\underbrace{u(C, S) = \frac{(CS)^{1-\gamma} - 1}{1-\gamma},}_{\text{TSU}} \qquad \underbrace{f(C, V) = \frac{\rho\,(1-\gamma)\,V}{1 - 1/\psi}\left(\left(\frac{C}{((1-\gamma)\,V)^{-1/(1-\gamma)}}\right)^{1-1/\psi} - 1\right)}_{\text{RU}} \tag{1.4}$$

γ is the risk aversion parameter, and in the standard TSU case it is equal to relative risk aversion, which also equals the inverse of IES. ψ is the IES parameter in the RU case.[19]

[16] It should be noted that I use the same parameter symbols for all model variations, and they should be distinguished by context. For example, in TSU-CD μ_{ct} is time-varying and a function of the state variable, while in TSU-CV μ_{ct} is a constant. The same applies to the symbols: σ_{ct} and σ_{xt}.

[17] W_{ct} is a standard Wiener Process associated with consumption such that $W_{ct} - W_{cs} \sim \text{Normal}(0, s-t)$.

[18] In the habit model of Campbell and Cochrane (1999), which is followed here, this variable is actually equal to $(C_t^a - X_t)/C_t^a$, where X_t is the level of habit and C^a is aggregate consumption.

[19] f has the form of a normalised aggregator as in Duffie and Epstein (1992).

1.4.4 State variable process

At time t, the state variable x_t follows the process:

$$\mathrm{d}x_t = -\log(\phi)(\mu_{x0} - x_t)\mathrm{d}t + \sigma_{xt}\mathrm{d}W_{xt} \tag{1.5}$$

This expression describes an autoregressive stochastic process that reverts to the steady state μ_{x0}.[20] The rate of reversion to the steady state is governed by ϕ, which is constrained to be between 0 and 1. Thus, $\log(\phi)$ is non-positive and it implies that when $x_t > \mu_x$ ($x_t < \mu_x$) the drift is downward-sloping (upward-sloping), always towards the steady state. $\mathrm{d}W_{xt}$ is also a standard Wiener process, and σ_{xt} is the volatility coefficient of the state variable and it is either a constant or it also depends on x_t. $\mathrm{d}W_{xt}$ can be correlated with $\mathrm{d}W_{ct}$, and the value of the correlation is captured by ρ_{cx}. In economic terms, the state variable plays a different role for each model variation. The full dependence of the model variations on the state variable is shown in Table 1.2:

Model variation			
TSU-CD:	$\mu_{ct} = \mu_{c0} + x_t$	$\sigma_{xt} = \sigma_{x0}$	$\mu_{x0} = 0$
TSU-CV:	$\sigma_{ct} = \sigma_{c1} x_t$	$\sigma_{xt} = \sigma_{x1}\sqrt{x_t}$	$\mu_{x0} = 1$
TSU-Habit:*	$S_t = S_0 e^{x_t}$	$\sigma_{xt} = \sigma_{ct}\lambda(x_t)$	$\mu_{x0} = 0$
RU-CD:	$\mu_{ct} = \mu_{c0} + x_t$	$\sigma_{xt} = \sigma_{x0}$	$\mu_{x0} = 0$
RU-CV:	$\sigma_{ct} = \sigma_{c1}\sqrt{x_t}$	$\sigma_{xt} = \sigma_{x1}\sqrt{x_t}$	$\mu_{x0} = 1$
TSU-HCV:	$\sigma_{ct} = \sigma_{c1} x_t$	$\sigma_{xt} = \sigma_{x1}\sqrt{x_t}$	$\mu_{x0} = 1$
Arb-IN:	$\mu_{ct} = \mu_{c1} x_t^{1/4}, \sigma_{ct} = \sigma_{c1}\sqrt{x_t}$	$\sigma_{xt} = \sigma_{x1}\sqrt{x_t}$	$\mu_{x0} = 1$
Arb-DP:	$\mu_{ct} = \mu_{c1} x_t^{3/2}, \sigma_{ct} = \sigma_{c1} x_t^{2/3}$	$\sigma_{xt} = \sigma_{x1}\sqrt{x_t}$	$\mu_{x0} = 1$

Table 1.2: Dependence on the state variable for each model variation

* Following Campbell and Cochrane (1999) and Wachter (2006), the exact form of $\lambda(\cdot)$ is:

$$\sigma_{xt} = \sigma_{ct}\lambda(x_t) = \begin{cases} \sigma_{ct}\left(\frac{\sqrt{1-2x_t}}{S_0} - 1\right) & \text{if } x_t < \frac{1-S_0^2}{2} \\ 0 & \text{if } x \geq \frac{1-S_0^2}{2} \end{cases}, \quad S_0 = \sqrt{\frac{\gamma}{-\log(\phi) - b/\gamma}} \tag{1.6}$$

In some variations the steady state is at $x_t = 0$, while in others it is at $x_t = 1$, and x_t is positive with probability 1. This specification is used for the variations in which CV σ_{ct} is proportional to the state variable, to ensure that σ_{ct} is positive.

[20] The steady state $x_t = \mu_{x0}$ does not necessarily coincide with the ergodic mean or median of the process when the diffusion of the process is not symmetric around the steady state value.

1.4.5 Stochastic discount factor

1.4.5.1 Time-separable utility Case

In the TSU case, the stochastic discount factor is the derivative of the flow utility with respect to consumption. In the general case the formula is the following:

$$\Lambda = e^{-\rho t}(C_t S_t)^{-\gamma} \tag{1.7}$$

where S_t is only relevant in the habit model. Using the above expression, along with the consumption process (Equation 1.2) and the state variable process (Equation 1.5), Ito's Lemma can be implemented to get the stochastic differential equation (SDE) of the SDF:

$$\begin{aligned}\frac{\mathrm{d}\Lambda}{\Lambda} =& \Big(-\rho - \gamma\mu_{ct} + \frac{\gamma^2}{2}\sigma_{ct}^2\Big)\mathrm{d}t - \gamma\sigma_{ct}\mathrm{d}W_{ct} \\ & \underbrace{+\Big(-\gamma\log(\phi)x_t + 2\rho_{cx}\sigma_{ct}\sigma_{xt} + \sigma_{xt}^2\Big)\mathrm{d}t - \gamma\sigma_{xt}\mathrm{d}W_{xt}}_{\text{habit model only}}\end{aligned} \tag{1.8}$$

For the details of the derivation, see Appendix 1.H.1.

1.4.5.2 Recursive utility case

In the case of RU, the stochastic process of the SDF is derived from the expressions for the value function and the aggregator function. The latter is given in Equation 1.4, and the value function turns out to have the following form:

$$V_t = \frac{C_t^{1-\gamma}e^{(1-\gamma)K(x_t)}}{1-\gamma} \tag{1.9}$$

V_t increases with K, which is a specific function of x_t that captures the full dependence of the value function on the state variable. At the end of this section, I show how the expression above is justified, and I compute a novel perturbation approximation that provides a formula for K. Given the expression for the value function, Ito's Lemma can be implemented to get to the SDE of the SDF. The calculation here follows Chen et al. (2009). In particular, the fundamental relationship is:

$$\frac{\mathrm{d}\Lambda}{\Lambda} = f_V(C,V)\mathrm{d}t + \frac{\mathrm{d}f_C(C,V)}{f_C(C,V)} \tag{1.10}$$

f_C and f_V denote partial derivatives of f with respect to consumption and the value function respectively. The first term on the right hand side is the derivative of the flow utility with respect to the value function. The second term can be computed by applying Ito's lemma on the derivative of flow utility with respect to consumption.[21] The result is the following:

$$\begin{aligned}\frac{\mathrm{d}\Lambda}{\Lambda} = & \Bigg(\frac{\rho\left(-(1-\gamma\psi)e^{\frac{(1-\psi 1)K[x_t]}{\psi}}-\gamma\psi+\psi\right)}{1-\psi} - \gamma\mu_{ct} + \frac{\gamma^2\sigma_{ct}^2}{2} + \frac{\gamma(\gamma\psi-1)\rho_{cx}\sigma_{xt}\sigma_{ct}K'(x_t)}{\psi} \\ & + \frac{(\gamma\psi-1)\left(2\psi(\mu_{x0}-x_t)\log(\phi)K'(x_t)+\sigma_{xt}^2\left((\gamma\psi-1)K'(x_t)^2-\psi K''(x_t)\right)\right)}{2\psi^2}\Bigg)\mathrm{d}t \\ & - \frac{(\gamma\psi-1)\sigma_{xt}K'(x_t)}{\psi}\mathrm{d}W_{xt} - \gamma\sigma_{ct}\mathrm{d}W_{ct}\end{aligned} \tag{1.11}$$

The details of the derivation can be found in Appendix 1.H.2. It is notable that the special case of $\gamma = 1/\psi$, is time-separable, and the equation above simplifies to the formula in Equation 1.8 from the standard TSU case. Also, the stochastic component relating to consumption ($-\gamma\sigma_{ct}\mathrm{d}W_{ct}$), is exactly the same as in TSU, and there is an extra component, namely $-\frac{(\gamma\psi-1)\sigma_{xt}K'(x_t)}{\psi}\mathrm{d}W_{xt}$, due to the direct dependence of the SDF on the state variable.

1.4.6 Instantaneous rate

From the SDF the short-term rate is derived as follows:

$$\text{TSU: } r(x_t)\mathrm{d}t = -\mathrm{E}_t\left[\frac{\mathrm{d}\Lambda}{\Lambda}\right] = \left(\rho+\gamma\mu_{ct}-\frac{\gamma^2}{2}\sigma_{ct}^2\right)\mathrm{d}t \underbrace{+\left(\gamma\log(\phi)x_t - 2\rho_{cx}\sigma_{ct}\sigma_{xt}-\sigma_{xt}^2\right)\mathrm{d}t}_{\text{habit model only}}$$

$$\begin{aligned}\text{RU: } r(x_t)\mathrm{d}t = -\mathrm{E}_t\left[\frac{\mathrm{d}\Lambda}{\Lambda}\right] = & \\ & \frac{\rho\left((1-\gamma\psi)e^{\frac{(1-\psi)K[x_t]}{\psi}}+\gamma\psi-\psi\right)}{1-\psi} + \gamma\mu_{ct} - \frac{\gamma^2\sigma_{ct}^2}{2} - \frac{\gamma(\gamma\psi-1)\rho_{cx}\sigma_{xt}\sigma_{ct}K'(x_t)}{\psi} \\ & - \frac{(\gamma\psi-1)\left(2\psi(\mu_{x0}-x_t)\log(\phi)K'(x_t)+\sigma_{xt}^2\left((\gamma\psi-1)K'(x_t)^2-\psi K''(x_t)\right)\right)}{2\psi^2}\end{aligned} \tag{1.12}$$

[21] This operation is performed by substituting the value function using Equation (1.20) and applying Ito's lemma based on consumption and the state variable as independent variables.

In the standard TSU case, the short rate depends on three components. The first is the time preference parameter ρ. The second is $\gamma\mu_{ct}$ and it relates to the consumption smoothing motive. As CD increases, agents try to borrow to increase current consumption, and in equilibrium the short rate increases. The third is $-\gamma^2\sigma_{ct}^2/2$ and it relates to the precautionary saving motive. As consumption becomes more risky, agents try to save, and in equilibrium the short rate decreases. In TSU-Habit, there are extra components that relate both to the consumption smoothing motive and the precautionary saving motive, and they are due to the state variable being part of the utility function. Thus, as the surplus consumption ratio falls, marginal consumption increases even more than in standard TSU. So, the agent has an even higher motive to smooth consumption. However, in the same state of the world, the surplus consumption ratio is also much more volatile and the agent also has a higher precautionary saving motive. In Campbell and Cochrane (1999) these two opposite effects on the short-term rate are regulated by a parameter denoted b. If $b = 0$, then the short rate becomes a constant. If $b > 0$ ($b < 0$), then the short rate is decreasing (increasing) in the surplus consumption ratio.

In the RU case, the short rate becomes more complicated. However, for the main calibrations the dominating additional effect comes from the fact that the marginal utility of consumption is expected to change as the state variable changes. The effect of this is that short rates are affected less by the consumption smoothing effect and the precautionary savings effect, and short rates under RU are less sensitive to the state variable than short rates under TSU.

1.4.7 Long-term bond

1.4.7.1 Bond pricing equation

Next, given the process of the SDF, the price of the long-term bond Q can be computed in the same way for both TSU and RU cases. The bond price is a function of the state variable x_t and its remaining maturity m. Thus, by using Ito's Lemma the stochastic process follows:[22]

$$dQ(x,m) = \Big(-\log(\phi)(\mu_{x0} - x_t)Q_x - Q_m + \frac{1}{2}\sigma_{xt}^2 Q_{xx}\Big)\mathrm{d}t + \sigma_{xt}Q_x\mathrm{d}W_{xt} \quad (1.13)$$

[22] Given the flow utility function, investors' decisions are not affected by the level of consumption. This implies that the long-term bond is not going to be a function of consumption itself (see for example Tsai and Wachter 2018).

In the equation above, subscripts $\cdot_x$ and $\cdot_m$, denote partial derivatives with respect to the corresponding variable. The next step is to derive the partial differential equation (PDE) that Q obeys in these models. Thus, I use the pricing equation following the approach in Cochrane (2009) and Chen, Cosimano and Himonas (2010):

$$\mathrm{E}\Big[d(\Lambda Q)\Big] = 0 \rightarrow \mathrm{E}\left[\frac{\mathrm{d}\Lambda}{\Lambda}Q + \mathrm{d}Q + \frac{\mathrm{d}\Lambda}{\Lambda}\mathrm{d}Q\right] = 0 \tag{1.14}$$

Substituting the expressions for Λ, $\mathrm{E}[\mathrm{d}\Lambda/\Lambda]$ and $\mathrm{d}Q$ from Equations (1.8), (1.12) and (1.13) respectively, gives rise to the PDE obeyed by Q:[23]

$$\begin{gathered} -Q_m - r(x_t)Q + \Big(-\log(\phi)(\mu_{x0} - x_t) + A(x_t)\Big)Q_x + \frac{\sigma_{xt}^2}{2}Q_{xx} = 0 \\ \text{where: } A(x_t)\mathrm{d}t = \frac{\mathrm{d}\Lambda}{\Lambda}\mathrm{d}Q \end{gathered} \tag{1.15}$$

The expression comprises five terms. The first is the derivative with respect to maturity Q_m. The second is the short rate term $r(x_t)Q$, which differs depending on the variation, as shown in Equation (1.12). The third is the expectation term $-\log(\phi)(\mu_{x0}-x_t)$, that captures the information that short rates may be expected to change in the future. The fourth is what I call the A term, and it is responsible for term premia, as it captures consumption-based risk. [24] The fifth is the diffusion term $\frac{\sigma_{xt}^2}{2}Q_{xx}$.[25] The solution of this equation is discussed next, while Appendix 1.C shows in more detail how these five terms affect the term structure of interest rates and its dynamics.

1.4.7.2 Solution of the pricing equation

Equation (1.15) is a PDE, and I solve it by making use of the Feynman-Kac formula, which re-expresses the solution of a PDE as an expectation of a stochastic process. In particular, the solution of Equation (1.15) is:

$$Q(m, x_t) = \mathrm{E}_t\left[\exp\left\{\int_m^0 r(\tilde{x}_{t+s})ds\right\}\right] = \mathrm{E}_t\left[\exp\left\{-\int_0^m r(\tilde{x}_{t+s})dt\right\}\right] \tag{1.16}$$

[23]This equation is similar to a Black-Scholes equation.

[24]Risk is understood in the context of consumption-based asset pricing. Therefore, if the the price of the bond does not co-vary with the SDF, then the A term is 0. The A term being 0 does not mean that the price of the bond is deterministic.

[25]This term is connected with the idea of *convexity* in finance.

where $\tilde{x}_0 = x_t$ and $\tilde{x}_t$ follows the modified stochastic process compared to the state variable:[26]

$$d\tilde{x}_t = \Big(-\log(\phi)(\mu_{x0} - \tilde{x}_t) + A(\tilde{x}_t) \Big) dt + \sigma_{xt}(\tilde{x}_t) dW_{xt} \tag{1.17}$$

The expectation is computed using Monte Carlo simulations.

1.4.7.3 Risk-neutral yield and term premium

Instead of using a modified process, the original process of the state variable can also be used in the Feynman-Kac formula:

$$H(m, x_t) = \mathrm{E}_t \left[\exp \left\{ \int_m^0 r(x_{t+s}) ds \right\} \right] = \mathrm{E}_t \left[\exp \left\{ - \int_0^m r(x_{t+s}) dt \right\} \right] \tag{1.18}$$

This is by definition the expected gross return from rolling over the short-term rate. Thus, $-\log(H(m, x_t))/m$ is by definition the risk-neutral yield, and the argument above shows that it corresponds to the solution of Equation (1.15), after setting $A(x_t) = 0$ for all x_t. In other words, the risk-neutral yield can be thought of as deriving from a bond priced by a risk-neutral investor. This also provides a natural way for computing term premia, which is:

$$TP(x_t, m) = \frac{-\log Q(x_t, m) - (-\log H(x_t, m))}{m} \tag{1.19}$$

Namely, the term premium is the difference between the yield of the bond and the risk-neutral yield. Unfortunately, there is no analytic expression for term premia, given that Q and H are computed numerically. However, there is an analytic expression for function A in Equation 1.15, and it can serve as a diagnostic of term premia, as it is the component that distinguishes Q from H. Especially when the short-term rate is linear in the state variable, the sign of A determines the sign of term premia,[27] the time variability of A determines the time variability of term premia, and the size of A determines the size of term premia.[28] In addition,

[26] Here I show the dependence of σ_{xt} on $\tilde{x}_t$, in order to clarify that it is the same function as before, but it takes the modified variable as the argument.

[27] In particular term premia have the sign of the product of A with the derivative of $-Q$ with respect to the state variable x_t, which usually has the same sign as the derivative of short-rete with respect ot the state variable x_t.

[28] To be precise, term premia depend on the entire pricing Equation (1.15). However, if the short-term rate is linear and the effect of the diffusion term is small, then the bulk of the time-varying behaviour of term premia is determined by A. In the explanation provided here, I assume that the diffusion term and non-linearities have a small effect on the yield spread. A detailed analysis is conducted in Appendix 1.C.

the size of A can easily be judged by comparison to the size of the expectation term $-\log(\phi)(\mu_{x0} - x_t)$, which also multiplies Q_x in the PDE. The expectation term and the A term are the two main drivers of the yield spread. Therefore, if the typical values of the expectation term are much larger than the typical values of the A term, this implies that the yield spread is due to expected changes of short-term rates in the future. On the other hand, if the values of the two terms are comparable in size, then the yield spread likely contains a component due to the term premium as large as a component due to the expectation term. This comparison is illustrated in practice in Section 1.5.

1.4.8 Perturbation approximation for K function in the recursive utility case

As mentioned in Subsection 1.4.5.2, given the process of the SDF it is possible to compute the price of bonds in the RU case in the same way as in the TSU case. This in turn requires an expression for the value function. This subsection is dedicated to explaining a novel perturbation method to approximate function K that was used in the RU value function.

As shown by Tsai and Wachter (2018), the value function follows:[29]

$$J(C, x) = V = \frac{C^{1-\gamma} e^{(1-\gamma)K(x)}}{1-\gamma} \tag{1.20}$$

where $K(x)$ solves the following ordinary differential equation (ODE):

$$-\frac{1}{2}\gamma\sigma_{ct}^2 + \frac{1}{2}\sigma_{xt}^2\left(K''(x) - (\gamma-1)K'(x)^2\right) - \log(\phi)(\mu_{x0} - x_t)K'(x) + \frac{\beta\left(e^{-\epsilon K(x)} - 1\right)}{\epsilon} + \mu_{ct} = 0 \tag{1.21}$$

where the substitution $\psi = \frac{1}{1-\epsilon}$ has been made. For $\psi = 1$ ($\epsilon = 0$) Equation (1.21) has an analytic solution. This can then be used to create a global perturbation solution in terms of the state variable, which I express in terms of ϵ.[30] In particular,

[29] A similar setup is used in Benzoni, Collin-Dufresne and Goldstein (2011) building upon earlier literature such as (Duffie and Epstein 1992). The setup here is simpler than Tsai and Wachter (2018), because there is only one state variable and there are no Poisson jumps in the consumption process and the state variable process.

[30] The approximation is global in terms of the state variable x, as the perturbation is done with respect to parameter ϵ. Nevertheless, it is not valid for all values of x. In particular, the approximation takes such a form, so that its validity depend on different regions of the state variable. In the region that it converges, the quality of the approximation is high for all values of x, but outside this region the series diverge.

Equation (1.21) can be expanded to:

$$-\frac{1}{2}\gamma\sigma_{ct}^2+\frac{1}{2}\sigma_{xt}^2(-(\gamma-1)\left(\epsilon^2K_2'(x)+\epsilon K_1'(x)+K_0'(x)\right)^2+\epsilon^2K_2''(x)+\epsilon K_1''(x)+K_0''(x))+\mu_{ct}$$
$$+\frac{\beta\left(e^{-\epsilon\left(\epsilon^2K_2(x)+\epsilon K_1(x)+K_0(x)\right)}-1\right)}{\epsilon}-\log(\phi)(\mu_{x0}-x_t)\left(\epsilon^2K_2'(x)+\epsilon K_1'(x)+K_0'(x)\right)\approx 0 \tag{1.22}$$

Here function K has been expanded up to second order, but it could also be expanded further. Given this expansion the equation admits a solution of the form:

$$\begin{aligned}K_0(x)&=a_{0,0}+a_{0,1}x\\ K_1(x)&=a_{1,0}+a_{1,1}x+a_{1,2}x^2\\ K_2(x)&=a_{2,0}+a_{2,1}x+a_{2,2}x^2+a_{2,3}x^3\\ &\dots\end{aligned} \tag{1.23}$$

This solution can by plugged into the ODE (1.22), and for each m, n, $a_{m,n}$ can be derived by setting each factor of $x^m\epsilon^n$ equal to zero. This leads to a linear equation for each coefficient.[31] Conveniently, these equations can be solved successively so that for each equation there is only one unknown. Unfortunately, as can be seen from equation (1.23), the full solution is a sum of polynomials in terms of x. For each successive order of ϵ, the order of the polynomial increases by one. While it is possible to compute many orders of approximation, eventually the computation becomes expensive, as each order of ϵ requires the solution of more linear equations, and each equation has an increasing complexity.

The solution in Tsai and Wachter (2018) only derived $K_0(\cdot)$ which is the first term in formula (1.23) and it is the "zeroth" order approximation in terms of ϵ or equivalently ψ. My approximation is useful, because it allows a much larger range of values for ψ, and it provides an analytic expression that is easy to include in the Monte Carlo simulations, that solve the pricing equation. My method is described in detail in Melissinos (2023).[32] On the contrary, given that the solution provided by Tsai and Wachter (2018) was analytically correct only for $\epsilon = 0$ ($\psi = 1$),

[31] Apart from coefficient $a_{0,1}$ which may require the solution of a second order equation.

[32] One limitation of the method, that my contribution does not overcome, is that the parameters of the processes should be at most linear functions of the state variable. In particular, σ_{xt}^2 and σ_{ct}^2 are linear in the state variable. Therefore, unlike in the TSU case where I set $\sigma_{ct}\propto x_t$, here I set $\sigma_{ct}\propto\sqrt{x_t}$. This is investigated in more detail in Melissinos (2023), but the main implication is that CV is relatively restricted in its variability.

implementing the method for other values of ψ is not easily justifiable, even if in practice it would generate qualitatively or even quantitatively similar results. It should also be noted that the full perturbation series, that I provide, is the exact solution to the ODE. Namely it is the unique perturbation series that represents the solution. Thus, it is highly likely that with some extra mathematical analysis, it can be re-expressed in terms of known special functions, and we can get an exact answer that is practically trivial to compute for arbitrary order. Nevertheless, the method in its current form allows the researcher to easily approximate the value function, while also practically checking its convergence. The value function can then be used in the pricing equation to directly get the price of assets, while being robust to a large range of parameters for the IES. Thus, this method can be implemented widely in RU models.

The details and the properties of the method are described further in Melissinos (2023), while the perturbation can be expanded in further research to more general setups including more than one state variables. Given that the full perturbation series represents the exact solution to the problem it can likely also give rise to an exact solution in terms of special mathematical functions. Such a solution would be significant, and it would likely help with the proof of existence and uniqueness of the solution, which has long been an open problem in the literature (Tsai and Wachter 2018).

1.4.9 Calibration

General paramaters	
Relative risk aversion γ	2.0
Rate of time preference ρ	0.012/yr
Steady state CD μ_{c0}	0.0252 /yr
Steady state CV σ_{c0}	0.02/yr
Steady state reversion $\log \phi$	log(0.92)/yr

Table 1.3: Calibration of common parameters

Given that the goal of the paper is to identify consumption-based mechanisms that are consistent with the patterns of term premia in the data, the emphasis is not on providing a perfect calibration. Instead, the focus is on finding the combination of the utility specification and the consumption process that generate the observed patterns in term premia. Thus, several parameter choices are explored, and the calibration of each model variation is reported in the corresponding figure showing the results. Nevertheless, there are shared parameter choices across model variations. These are reported in Table 1.3 and they follow Wachter (2013), while the risk aversion parameter γ is set equal to 2 following Wachter and Zhu (2019).

1.5 Results

1.5.1 How variations are evaluated

In evaluating the features of term premia, I also require that the model variations generate an empirically plausible short rate volatility.[33] Higher short rate volatility can give rise to higher term premia. Thus, I assume a relatively large short rate volatility in order to give these variations the best chance of success. Their performance is compared, by plotting model-implied term premia as a function of the state variable next to the time series of estimated term premia (these were already shown in Figure 1.1). In making the comparison, the focus is more on the variability than the level.[34] If the models generate roughly the same pattern of variability, then they are considered a success. In most cases it is obvious, when the models fail to generate the patterns of term premia in the data.

Table 1.4 shows information for function A for six separate variations with moderate CV, and Figures 1.3 and 1.5 show the corresponding five and ten-year term premia, which are discussed in Subsections 1.5.2.1 and 1.5.2.2 respectively. Further variations and calibrations are shown in Appendix 1.F. With these results, apart from illustrating the standard consumption-based mechanisms, I aim to provide a helpful reference, as in the literature state-dependent term premia are rarely provided. Furthermore, Table 1.5 and Figure 1.6 show variations with high CV, whose consequences have not been analysed before with respect to term premia. I am the first to show that these variations can generate the features of term premia in the data. These latter results are discussed in Subsection 1.5.3. While for my main results I use estimated premia from d 'Amico, Kim and Wei

[33] The calibration of the state variable is described in Appendix 1.D.

[34] For example, in the data, especially recently, term premia seem to also become negative. However, the successful models that I present here all have exclusively positive term premia. I do not consider this a large setback, as my focus is on the variability of term premia and the models investigated here only have one state variable. In a full explanation of term premia and interest rates more generally, at least two variables would be necessary, given that a principal component analysis of the yields and spreads requires at least two principal components to explain the bulk of the variation (this is shown in Appendix 1.B)

(2018), Abrahams et al. (2016) also estimate the term premium, and they provide a decomposition of the five-to-ten year forward. Estimations from both papers are shown in Appendix 1.E,[35] and the five-to-ten forward term premium generated by the variations analysed in the current paper are shown in Appendix 1.F.

1.5.2 Moderate consumption volatility

1.5.2.1 Time-separable utility

Model variation	$A(x_t)$	ρ_{cx}	Range of A term*	Range of expectation term*
TSU-CD	$-\gamma\rho_{cx}\sigma_c\sigma_x$	$+$	(-0.0002,-0.0002)	(0.0023, $-$0.0022)
TSU-CV	$-\gamma\rho_{cx}\sigma_{ct}\sigma_{xt}$	$-$	(0.0012, 0.0086)	(0.048, -0.052)
TSU-Habit	$-\gamma\rho_{cx}\sigma_c\sigma_{xt} - \gamma\sigma_{xt}^2$	$+$	(-0.083, -0.011)	(0.034, -0.030)
RU-CD	$-\gamma\rho_{cx}\sigma_c\sigma_x - \frac{(\gamma\psi-1)\sigma_x^2K'(x_t)}{\psi}$	$+$	(-0.001250, -0.001253)	(0.0023, -0.0022)
RU-CV	$-\gamma\rho_{cx}\sigma_{ct}\sigma_{xt} - \frac{(\gamma\psi-1)\sigma_{xt}^2K'(x_t)}{\psi}$	$-$	(0.0018, 0.0040)	(0.030, -0.034)
RU-Mixed	$-\gamma\rho_{cx}\sigma_{ct}\sigma_{xt} - \frac{(\gamma\psi-1)\sigma_{xt}^2K'(x_t)}{\psi}$	$+$	(-0.0045, -0.010)	(0.033, -0.031)

Table 1.4: Information on function A from Equation (1.15) in different model variations with moderate CV. The t-subscript has been dropped from the quantities that are not time-varying according to the variation.
* This range covers the typical values of the state variable. The values correspond to the dashed vertical lines in Figures 1.3 and 1.5.

As mentioned earlier, the three main mechanisms analysed are time-varying CD, time-varying CV, and time-varying surplus consumption ratio (in TSU-Habit).[36] The effect of these mechanisms on term premia can first be understood by looking

[35]The two measures are similar, with the term premium in Abrahams et al. (2016) reaching relatively higher values. In addition, the risk-neutral yield has much less variability in the Abrahams et al. (2016) estimation.

[36]The first two mechanisms can also be found in the long-run risk models introduced by Bansal and Yaron (2004), who used a recursive utility.

at function A for each of the variations. Table 1.4 shows the functional form of A, and which components are time-varying. It also shows the typical range for the size of the A term and the expectation term. As mentioned earlier, the sign of A, in conjunction with the slope of the short-term rate, determines the sign of term premia, while the size and variability of A also determine the size and variability of term premia. In TSU-CD the short rate is increasing with CD, due to the consumption smoothing motive.[37] As a result, in conjunction with $\rho_{cx} > 0$ the term premia are negative and constant in the state variable. The intuition for negative term premia is that the short-term rate goes up and bond prices go down when CD rises, which is also the time that consumption tends to increase (due to $\rho_{cx} > 0$). This means that long-term bonds act as a hedge, and they command a negative term premium. Apart from the negativity of term premia, A typically takes much smaller values in absolute value compared to the expectation term, implying that term premia should be very small. Thus, instead of positive, time-varying and sizeable, term premia are negative, constant and small. Alternatively, for TSU-CV, the short rate is decreasing in CV, due to the precautionary savings motive, and I assume $\rho_{cx} < 0$. Therefore, the A term is positive and time-varying in the state variable, as it includes CV σ_{ct} (in this specification σ_{xt} is also time-varying). As a result, the term premia are again negative (they have the same sign as the slope of the short-term rate), but in this case they are time-varying. However, the A term is much smaller in absolute value compared to the expectation term, so term premia apart from negative are again very small. Figure 1.3 shows the term premia for these two variations in comparison to the time series of term premia in the data.[38] It is evident from the figure that as the state of the economy changes, term premia would hardly move away from 0, and they would not be able to generate the variation estimated in the time-series. From the functional form of A it also follows that assuming a different sign for ρ_{cx}, would imply term premia of the opposite sign in both cases. However, for a representative consumption process it is reasonable that an increase of CD is associated with an increase in consumption itself, while an increase in CV is associated with a decrease in consumption.[39] In

[37]This means that the stochastic component of consumption is positively correlated with the stochastic component of the state variable, which is associated with CD. To avoid this long description, I will mostly use ρ_{cx}.

[38]In TSU-CV the short rate is also insensitive to CV.

[39]This is intuitive if the consumption process is thought of as a relatively independent consumption process that determines the short-term rate. However, if the short-term rate is the independent variable, and the consumption process is reacting, then it makes sense that as the short-term rate decreases, borrowing becomes cheaper and consumption temporarily increases. This can can either imply that CD decreases, as consumption comes back to its normal level,

Appendix 1.F the results above are verified for several different calibrations.

The mechanisms discussed above use the power utility setup. Here, I discuss the effect of including external habit in the utility function as in Campbell and Cochrane (1999). As shown by Wachter (2006), TSU-Habit can generate the basic patterns of term premia that we see in the data. As mentioned previously, models with time-varying risk aversion, like the habit model, belong to one of only two kinds of models that can explain the patterns of term premia in a consumption-based setup with a single stationary autoregressive process. Thus, I analyse this model within my setup, in order to comprehensively describe consumption-based explanations to real term premia, and delineate its main differences compared to the alternative explanation that I introduce in the next subsection. Table 1.4 shows that the habit model has an extra term in the functional form of A. It turns out that this second term is dominant because the state variable volatility is in most states much larger compared to CV ($\sigma_{xt} \gg \sigma_{ct}$).[40] As a result, the sign of A does not depend on ρ_{cx} (which in the canonical habit model is equal to 1 anyway, as consumption completely determines the habit variable.), and the sign of term premia is determined exclusively by the slope of the short-term rate as a function of the surplus consumption ratio. As discussed in Subsection 1.4.6, this relationship in TSU-Habit depends on parameter b, which is chosen positive so that the short-term rate is decreasing and the term premia are positive.[41] Furthermore, term premia are large, as the value of A is large compared to the expectation term. Lastly, term premia are time-varying, given that A includes σ_{xt}^2, which is time-varying. Namely, variability of term premia is due to the heteroskedasticity of the state variable, which is amplified because A includes the square of the volatility.[42] Therefore, term premia are positive, time-varying, and large. This is explicitly shown in Figure 1.3, and the typical amount of variability, captured between the dashed lines, matches closely the variability in the estimated term premia.

or that CV increases as the agent has less savings. In both cases, the sign of ρ_{cx} is the opposite compared to the first scenario. My conjecture is that this should not happen in a large economy with a short-term rate determined by the behaviour of a representative agent. However, it could also be argued that the short-term rate is the independent force in the economy, due to the actions of the monetary authority.

[40]The size of the two terms is shown in the right plot of Figure 1.4

[41]This was also the choice of Wachter (2006), while Campbell and Cochrane (1999) set $b = 0$ in the final version of their paper (in an earlier version they also investigated $b > 0$). In Appendix 1.F, I also derive the results of a variation in which $b < 0$. In this case the short rate is increasing in the surplus consumption ratio, and term premia are negative, time-varying and large in absolute value.

[42]In Appendix 1.F, I impose homoskedasticity, and this leads to constant term premia. Admittedly, this is contrary to the spirit of the habit model.

1.5.2.2 Recursive utility

In this subsection, the results are extended to RU. This case is arguably of higher interest, as it separates risk aversion and IES. Moreover, Bansal and Yaron (2004) were able to use this feature in conjunction with time-varying CD and CV in long-run risk models, to explain the equity premium puzzle. Indeed, similar to TSU-Habit, as is shown in Table 1.4, the RU variations have an extra term in function A. This term disappears for $\gamma = 1/\psi$ which coincides with the special case in which utility becomes time-separable. Similar to TSU-Habit, this term dominates the A function. Thus, the sign of term premia does not depend on the sign of ρ_{cx}, but on the slope of K, which turns out to match the slope of the short-term rate both in RU-CD and in RU-CV. This means that negative term premia are now a more robust prediction compared to TSU-CD and TSU-CV. However, in the case of RU-CD A is significantly larger compared to TSU-CD. Therefore, term premia are negative, constant, but can be somewhat sizeable in absolute value. In contrast, RU-CV shows the same patterns as TSU-CV. Given that the short-rate hardly exhibits variability in RU-CV, I also compute RU-Mixed which includes both time-varying CD and CV, governed by the same state variable. However, A in this variation is also quite small, and the term premia are small in absolute value. The term premia for RU-CD, RU-CV and RU-Mixed are shown in Figure 1.5, and it is clear that they cannot generate the variability in the estimated term premia. Appendix 1.F has further variations with different calibrations verifying these results.

Intuitively, RU models might be considered as good candidates for explaining term premia due to their flexibility in separating risk aversion and IES. However, term premia are constant in RU-CD and very small in RU-CV, while they are in both cases negative. This result is consistent with the literature. Specifically, Bansal and Shaliastovich (2013) study term premia in RU models, but they investigate the variability in *nominal* term premia and their mechanism involves inflation. Gomez-Cram and Yaron (2021) provide a similar explanation for nominal term premia using RU that also relies on inflation. Hence, the real term premia that they generate are not substantially time-varying. Van Binsbergen et al. (2012) also consider a RU setup with inflation, and they find that nominal term premia can be positive, for very high risk aversion values. However, they also find that real term premia are negative.

1.5.3 High consumption volatility

Model variation	$A(x_t)$	ρ_{cx}	Range of A term*	Range of steady state reversion term*
High CV	$-\gamma\rho_{cx}\sigma_{ct}\sigma_{xt}$	$+$	(-0.0079,-0.0057)	(0.0047, $-$0.0053)
Arb-IP	$-\gamma\rho_{cx}\sigma_{ct}\sigma_{xt}$	$-$	(-0.0069, -0.052)	(0.048, -0.052)
Arb-DN	$-\gamma\rho_{cx}\sigma_{c}\sigma_{xt}$	$+$	(0.0096, 0.045)	(0.047, -0.054)

Table 1.5: Information on function A from Equation (1.15) in different model variations with high CV.
* This range covers the typical values of the state variable. The values correspond to the dashed vertical lines in Figure 1.6.

In general, agents should be independently adjusting their investment and consumption. Thus, given the same asset-pricing processes, if optimising agents are heterogeneous in their utility function, they will have different consumption processes. Given a utility function the consistence of term premia with the consumption process can be checked independently for each consumer. Previously, I have shown that representative consumer explanations of term premia require time-varying risk aversion. This raises the question whether there is *any* consumer group whose consumption process is consistent with term premia, without assuming time-varying risk aversion. Given the negativity and the small size of term premia found in the previous subsection, it is reasonable to assume that the answer is again no. However, it turns out that there are other explanations that rely on the dynamics of the consumption process within TSU. Table 1.3 shows information on function A for these cases, while Figure 1.6 shows the corresponding state-dependent term premia. As has been shown previously, time-varying CV implies time-varying term premia. Thus, starting from time-varying CV, the way forward is in principle simple based on the expression for A. By changing the sign of ρ_{cx} and increasing the steady state level of σ_{ct}, term premia become positive and large. Indeed, this works in generating the amount of variability in the estimated term premia (Figure 1.6). This is noteworthy given the difficulty encountered previously in generating any amount of significant time-varying term premia. However, are these two necessary modifications economically sensible?

As shown in Figure 1.6 the typical variability of the state variable (area between dashed lines) ranges from 5% to more than 20% CV per year. Even accounting for potential mismeasurement of aggregate consumption, this range is excessively large. Therefore, this approach is not consistent with a representative consumer whose consumption coincides with aggregate consumption. However, this does not mean that the consumption process is too extreme for *any* consumer. Firstly, if financial markets are incomplete, and risk sharing is not possible, then idiosyncratic CV is relevant for asset prices (Constantinides and Duffie 1996). This means that aggregate CV could already be underestimating the CV that should be used in the models. Next, while 12% steady state CV is large compared to aggregate CV, it is not large compared to asset price volatility in financial markets. For people whose wealth lies in the financial sector, 11% wealth volatility is entirely plausible, and according to standard consumption-based portfolio theory, CV should follow wealth volatility. Lastly, there is also direct evidence that CV is much higher for some groups of consumers. While I do not take a position whether these investors are rich or poor, Ait-Sahalia, Parker and Yogo (2004) showed that the CV of rich individuals could be much higher compared to aggregate CV. In particular, while they reported that the annual standard deviation of non-durables and services was 2.3% according to the standard NIPA data, they measure an annual standard deviation of 19.6% for luxury retail sales and 20.4% for charitable contributions of wealthy individuals.[43] These values are both significantly larger than the steady state CV of the model variations in this subsection, which is equal to 12%.[44] Based on these results, Ait-Sahalia, Parker and Yogo (2004) also argued that the equity premium puzzle is less of a puzzle when considering the consumption process of rich consumers. In Appendix 1.G, I also show that TSU-HCV implies a sizeable equity premium. Lastly, Malloy, Moskowitz and Vissing-Jørgensen (2009) provided evidence that wealthy stockholders' consumption volatility is roughly three times higher compared to non-stockholders, while

[43]NIPA refers to the national income and product accounts produced by the Bureau of Economic Analysis of the US Department of Commerce. Ait-Sahalia, Parker and Yogo (2004) also include other measurements on the sales of luxury retail products.

[44]The standard deviation of consumption growth calculated from simulations also takes values similar to the CV of the model.

also showing that bond returns can be predicted by the covariance of wealthy stockholders' consumption growth with returns. This evidence is consistent with the idea that a small group of investors with high CV are driving term premia.

The second required assumption for the mechanism is that ρ_{cx} is positive. Previously, I have argued that this is not plausible for a representative consumer, because an increase in consumption risk should induce consumers to consume less and save more. However, the consumer-investors in TSU-HCV could be a small part of the overall population, and in this case $\rho_{cx} > 0$ can be justified. As CV increases, the short-term rate goes down, and this leads to an increase in bond prices. Thus, bondholders would then increase their consumption, given that their wealth also increases. An alternative intuition is that, as the short-term rate decreases, consumption increases due to borrowing, which in turn increases CV.

While I have shown the effect of high CV on term premia, I have only done so for $\gamma = 2$. Apart from further variations in Appendix 1.F, Figure 1.7 shows the different levels of term premia on the same scale for various values of γ and for various values of steady state CV. The results are interesting in several ways. Firstly, it stands out that different values of γ lead to huge changes in term premia, when consumption volatility is high. This means that term premia in TSU-HCV are highly sensitive to risk aversion levels. On the other hand, term premia are so small when CV is low, that moderate increases in risk aversion are not able to generate the required variability. Thus, even if $\gamma = 4$, CV needs to be able to reach at least 10%, so that time-variability in term premia is generated.

This subsection shows how some consumers could have consumption processes that are consistent with the main features of term premia. By restricting my attention to these investors, and not introducing a full heterogeneous agent model, I can examine many different variations. Nevertheless, it is important to also consider the potential behaviours of the remaining agents in the economy. For instance, they could be investing in the bond market, but their behaviour could be explained by more complicated or alternative models. It could also be the case that other investors in the bond market are entities, such as hedge funds and pension pension funds that are not appropriately modelled as consumers. The only requirement for the remaining investors is that they do not trade in such a way, that induces extensive risk sharing with high CV investors. If they did, then this would lead to a decrease in the CV of the high CV investors. Alternatively, many consumers may not be participating in the bond market at all.[45] In both cases the other agents can have moderate consumption processes, and be primarily responsible for aggregate consumption dynamics.[46]

TSU-HCV has been the simplest consumption-based variation that is able to generate large term premia. However, given a high CV, slightly more complicated variations can be examined, in which CD and CV are simultaneously changing. I refer to these as "arbitrageur" variations in relation to Vayanos and Vila (2021), who suggested that the term structure of interest rates is driven by "arbitrageurs", who take advantage of investment opportunities in the bond market. As these opportunities can be risky, arbitrageurs are not able to fully equate rates and eliminate the effect of the demand of idiosyncratic investors or "preferred habitat investors", as they are called in the article.[47] Here, I abstract from these latter investors and restrict my attention to arbitrageurs. They are marginal investors in the bond market. Their consumption process should be consistent with the observed term structure of interest rates, including term premia. I argue that the consumption process of the arbitrageurs has two main features. Firstly, their CV is high (similar to TSU-HCV). Secondly, as the investment opportunity increases, both CV and CD rise. This occurs because the higher investment opportunity offers higher expected returns, which implies a higher CD. At the same time, the higher investment opportunity brings more risk, and CV also rises. This setup can give rise to four separate variations depending on the behaviour of the short-term rate and the sign of ρ_{cx}. These are shown in Table 1.6. The movements in CD and CV have opposite effects on the short-term rate. Depending on the dominating component, the short-term rate can either be increasing or decreasing in the magnitude of the investment opportunity. In addition, the sign of function A is fully determined by ρ_{cx}, which in turn depends on the portfolio composition of arbitrageurs, and how its value fluctuates given the changing state of the economy.[48] These two binary choices give rise to the four possibilities shown in Table 1.6.

[45] Or they may not be marginal investors due to short-selling constraints. For instance, an investor who is constrained from shorting one end of the term structure could be holding some long-term bonds, but this does not make her a marginal investor of bonds in general.

[46] A fuller analysis would provide a full heterogeneous agent model explaining to what extent idiosyncratic consumption risk can be insured through financial markets.

[47] Given that there is risk, these investment opportunities fall under the category of "limited arbitrage".

[48] This is true to the extent that arbitrageurs do not have income external to their portfolio.

Short-term rate	Positive ρ_{cx}	Negative ρ_{cx}
Short-term rate Decreasing with CV (CV dominates)	Arb-DP, **positive** term premia	Arb-DN, **negative** term premia
Short-term rate Increasing with CV (CD dominates)	Arb-IP, **negative** term premia	Arb-IN, **positive** term premia

Table 1.6: **Term Premia Sign in Basic Arbitrageur Variations**

(back to text)

While each of these possibilities seems plausible, I focus on the two that generate positive term premia. In Arb-IN, term premia are positive and increasing with CV, as is the short-term rate. As shown in Figure 1.6, Arb-IN generates positive, time-varying and sizeable term premia. However, the size of the term premia is not as high as in TSU-Habit, TSU-HCV and Arb-DP. $\rho_{cx} < 0$ could be justified in Arb-IN, because an increase in the short-term rate could be inducing the arbitrageurs to invest more in the bond market and decrease their consumption. In addition, despite CD rising, an increase in the short-term rate can also imply a decrease in their wealth, if the arbitrageurs are bondholders.

In Arb-DP, $\rho_{cx} > 0$ can also be justified because it makes sense for consumption to increase when CD goes up. In addition, if the arbitrageurs are bond holders, then their wealth increases, as CV increases and the short rate goes down. This is also the variation that is most akin to the intuition provided in Vayanos and Vila (2021). As the short-term rate decreases, long-term bond yields underreact, and this leads to an increase in term premia. The arbitrageurs in Vayanos and Vila (2021) optimise between the mean and variance of their wealth, and consumption is not part of the analysis. To the best of my knowledge, I am the first to show that this behaviour can be rationalised within a consumption-based setup. Arb-DP also provides the characteristics of the consumption process that are consistent with term premia, and it shows that a low CV would not generate substantial variability in the arbitrageur setup. The mechanism driving term premia is basically the same as in TSU-HCV. Thus, explaining the main features of term premia requires high CV. Reaching a final conclusion whether actual bondholders' consumption process exhibits such volatility is not possible within this paper. However, my paper provides the theoretical prediction that can be evaluated and tested empirically. If such CV is judged to be too high, then arbitrageurs are likely not acting as consumers or on behalf of consumers. This would be evidence for the existence of frictions, such as the ones in the intermediary asset-pricing literature. Alternatively, if it is found that some bondholders have high CV as the model predicts, then it would be interesting to further research the reasons that distinguish these investors, and why they are not able to share their risk with the remaining population.

Apart from asset-pricing implications, the variations presented in this subsection are also significant for monetary policy, to the extent that monetary policy affects term premia (Beechey and Wright 2009). In particular, according to Arb-DP, central banks decreasing (increasing) interest rates is equivalent to increasing (decreasing) the CV of the marginal investors. An increasing CV implies higher term premia, and this mechanism hinges on stochastic consumption changes being positively correlated with CV. In addition, the effect on CV is very strong, as it can roughly range from 5% to 20%. On the contrary, the effect of monetary policy on the consumption process of non-investors might be muted, if they are indeed disconnected from the effects of bond markets. A full understanding of the effects of monetary policy on all agents in the economy would benefit from a full heterogeneous agent model that explains the investment behaviour of all households.[49]

Furthermore, the high CV and the arbitrageur variations have implications for household finance. In particular, the participants in these markets are assuming large consumption risks. Therefore, a usual household whose CV is low and whose utility function is similar to the utility function of the marginal investors, could benefit from investing in long-term bonds, when term premia are high. This is valid, as long as CV of the household does not become too volatile due to this investment. However, the benefit is conditional on the state of the economy, and it is not clear if the state of the economy is transparent to most households, as the current CV of marginal investors is not directly observable.[50] This advice would not be valid in the context of the habit model. In that case high term premia reflect states in which households have a high risk aversion, and investing in risky securities would not be appropriate.

[49] Schneider (2022) provides such a model, in which the state variable captures "aggregate conditions in the credit market". Similar models would be interesting, in which the state variable captures CD and CV.

[50] One could argue that the state of the economy is directly observable by the level of the short-term rate. However, here I have focused on explaining term premia, and I am using a single state variable. In a full explanation of the dynamics of interest rates, at least two state variables would be needed. Hence, the level of the short-term rate would most likely not directly imply the level of term premia.

1.6 Conclusion

In conclusion, consumption-based models encounter three key challenges in explaining the features of term premia. Firstly, they typically generate long-term bonds that provide a hedge against risk, which leads to negative instead of positive term premia. Specifically, for a representative consumer, it is reasonable that a rise in CD is associated with a stochastic consumption increase. Therefore, bond prices increase when CD decreases, and vice versa. Therefore, bonds are extra valuable, because they provide insurance against macroeconomic risk, and the associated term premia will be negative. Similarly, for an aggregate representative consumer it is reasonable that increased CV is associated with a stochastic consumption decrease. A similar argument implies that term premia are again negative. Secondly, time-varying CD generates constant instead of variable term premia. The paper shows that this turns out to be the case even in RU models. In contrast, time-varying CV always produces time-varying term premia, because by definition the state variable affects consumption uncertainty and, hence, risk. Thirdly, in calibrations according to an aggregate consumption process, term premia are typically very small in absolute value. The intuition for this is that consumption processes that are relatively stable give rise to term premia that are small. For term premia to be large it means that consumers are assuming large risks. Thus, given that aggregate consumption is relatively stable, the corresponding models imply low term premia. With the exception of the third shortcoming, these issues arise both in the TSU case and in the RU case.

However, I have identified model variations that do yield positive and significantly time-varying term premia. Firstly, a model with external habit, as in Campbell and Cochrane (1999) and Wachter (2006), produces better results. This occurs because a) the short rate is counter-cyclical, b) the state variable is relatively large and directly affects the utility function, and c) the state variable is heteroskedastic. These three factors respectively imply that the term premia are a) positive, b) large in absolute value and c) time-varying. In this variation the time-variability of term premia is directly related to the heteroskedasticity of the state variable, while a potential drawback is that effective risk aversion is highly volatile and takes extreme values.[51] Nevertheless, my analysis shows that the habit model (or models with time-varying risk aversion more generally) is the best model we have that is able to explain term premia while using a representative consumption process.

[51] This was also the main criticism of the habit model by Mehra et al. (2007).

I also demonstrate how large term premia can be explained by model variations that deviate from standard representative consumption processes. In particular, model variations for which a) CV is high, ranging for example from 5 to 20% per year, and for which b) stochastic consumption changes are positively correlated with CV, can generate positive and highly variable term premia. The first component contributes to term premia being high in absolute value, and the second component implies that term premia are positive. Apart from time-varying risk aversion this is the only available consumption-based mechanism to generate positive and substantially time-varying term premia. An important implication of this model is the high CV for many states of the economy. However, it is not ludicrously high. If the consumption-based setup were completely wrong and disconnected from the actual mechanisms generating term premia, then it could imply almost any value of CV for term premia to become highly variable. Moreover, the CV levels in these variations mirror return volatility levels in certain financial markets, and there is literature measuring a high CV in products consumed by rich households. An interesting empirical question would then be to ask, what the CV is for marginal optimising investors of the term structure of interest rates. Another important aspect is that a large part of the population does not actively participate in the bond market. Thus, maybe the consumption process of these households is not so relevant regarding the levels of term premia. In a separate variation, which also performs quite well, I combine high levels of CV with a time varying CD. In this variation there is a tradeoff between CV and CD, and I claim that this interpretation is similar to the arbitrageur story in Vayanos and Vila (2021), which to the best of my knowledge has not been implemented in a consumption-based framework. While I describe the consumption process of arbitrageurs, I do not take a stance whether its volatility is too high or not, and the final answer to this question probably requires further empirical research in the consumption process of direct and indirect bond holders. Whatever the answer is, further interesting questions emerge. If the CV implied by term premia is implausibly high, then arbitrageurs likely do not correspond to actual households, and they do not invest according to households' wishes, at least based on this high consumption risk explanation.[52] This could indicate the existence of intermediation constraints. Alternatively, if CV of actual bondholders is indeed high, then the question is why these bondholders do not engage in risk sharing with the rest

[52]The intuition for this statement comes from consumption-based portfolio selection theory, according to which the portfolio weights of risky assets should agree whether households are investing directly or through funds.

of the agents in the economy.

Finally, given that a couple of different mechanisms can generate the basic features of term premia, the question arises which one is the correct explanation. The answer requires further research. Nevertheless, one approach is to combine some of the explanations provided here within a full heterogeneous model that also accounts for households not participating in financial markets. Non-participation can be rationalised given the high volatility in financial markets and the existence of some friction. As a result, there would be reduced risk sharing, justifying CV being large. This setup is likely to jointly explain term premia, stock market non-participation, reduced risk sharing in the economy and the equity premium puzzle. Therefore, I consider it a promising direction for further research.

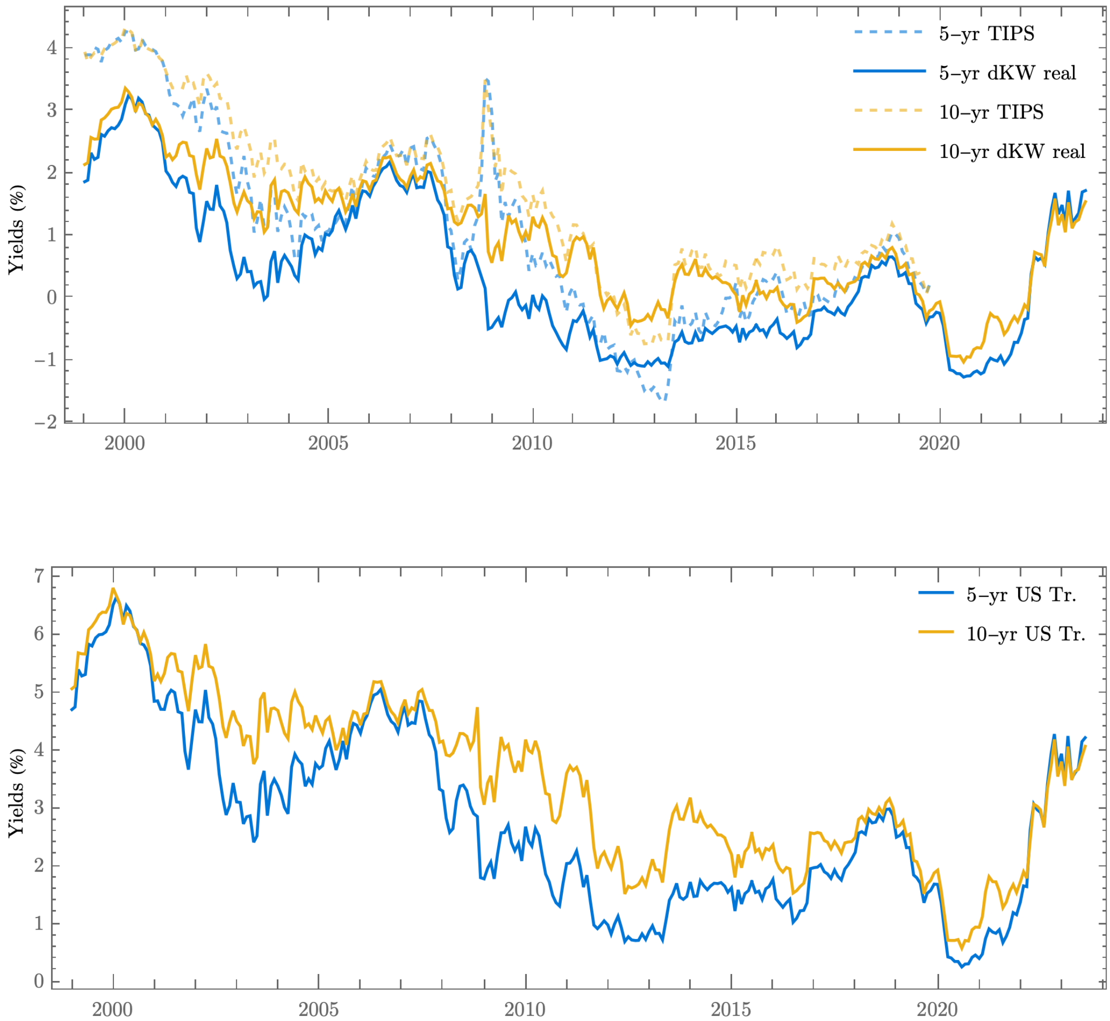


Figure 1.2: **Yields of US Treasuries**
TIPS data is taken from Gürkaynak, Sack and Wright (2010), normal US treasury yields data is taken from Gürkaynak, Sack and Wright (2007), and dKW real yields are taken from d 'Amico, Kim and Wei (2018). Real yields are the sum of risk-neutral yields and the real term premia. The difference between the dashed and solid lines are the liquidity premia of the TIPS over the normal treasuries. Thus, this assumes that normal treasuries are perfectly liquid.

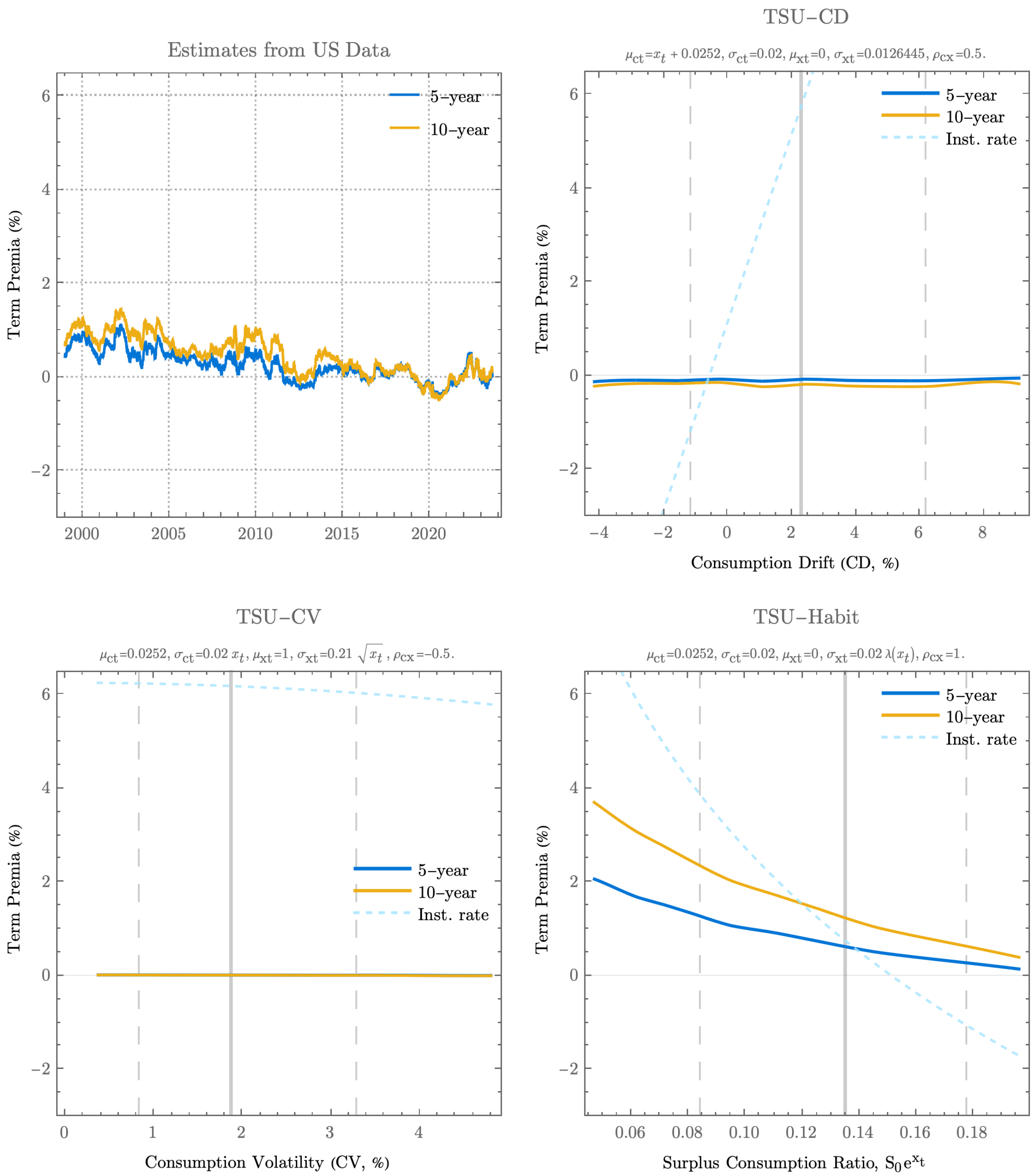


Figure 1.3: **Term Premia in standard models with Time-Separable Utility** The top left plot shows estimates of term premia according to d 'Amico, Kim and Wei (2018). The remaining plots show state-dependent term premia for three standard variations, namely variations with a) time-varying CD, b) time-varying CV, and c) an external habit in the utility function respectively. The dashed line shows the short-term rate.
The vertical dashed lines correspond to the typical values of the state variable based on simulations. The full range of the x-axis includes extreme values of the state variable, which are still possible (see Appendix 1.D or Figure 1.16 for the exact definition of the ranges).

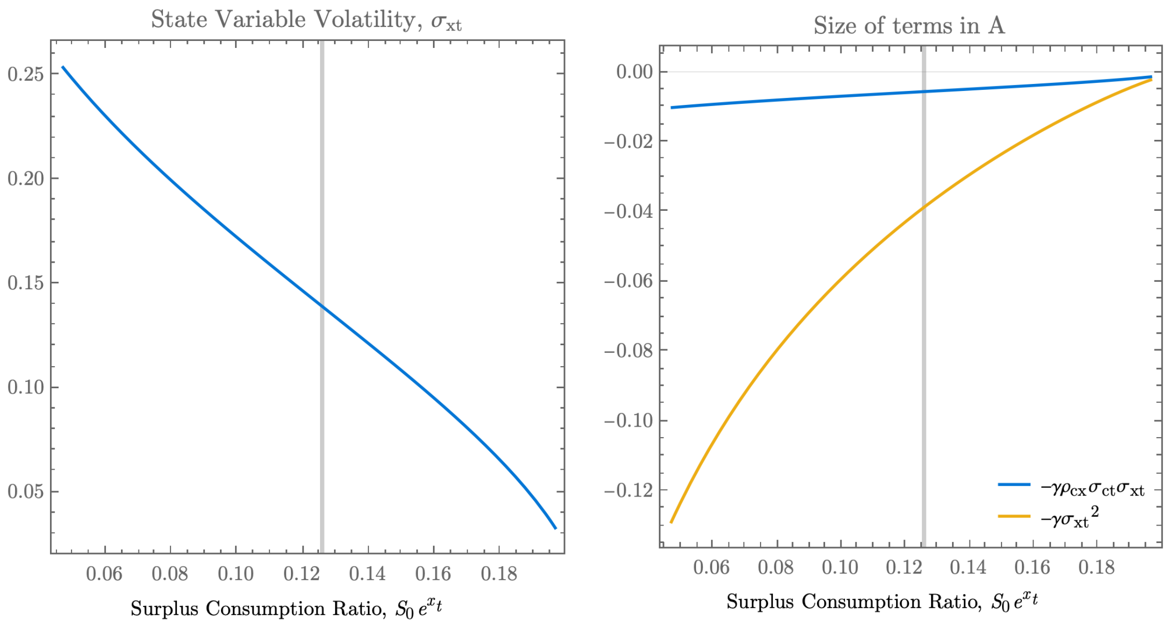


Figure 1.4: **Terms related to TSU-Habit**
The left plot shows the value of the volatility coefficient of the state variable in TSU-Habit. The right plot shows the magnitude of the two terms in the A function in TSU-Habit.

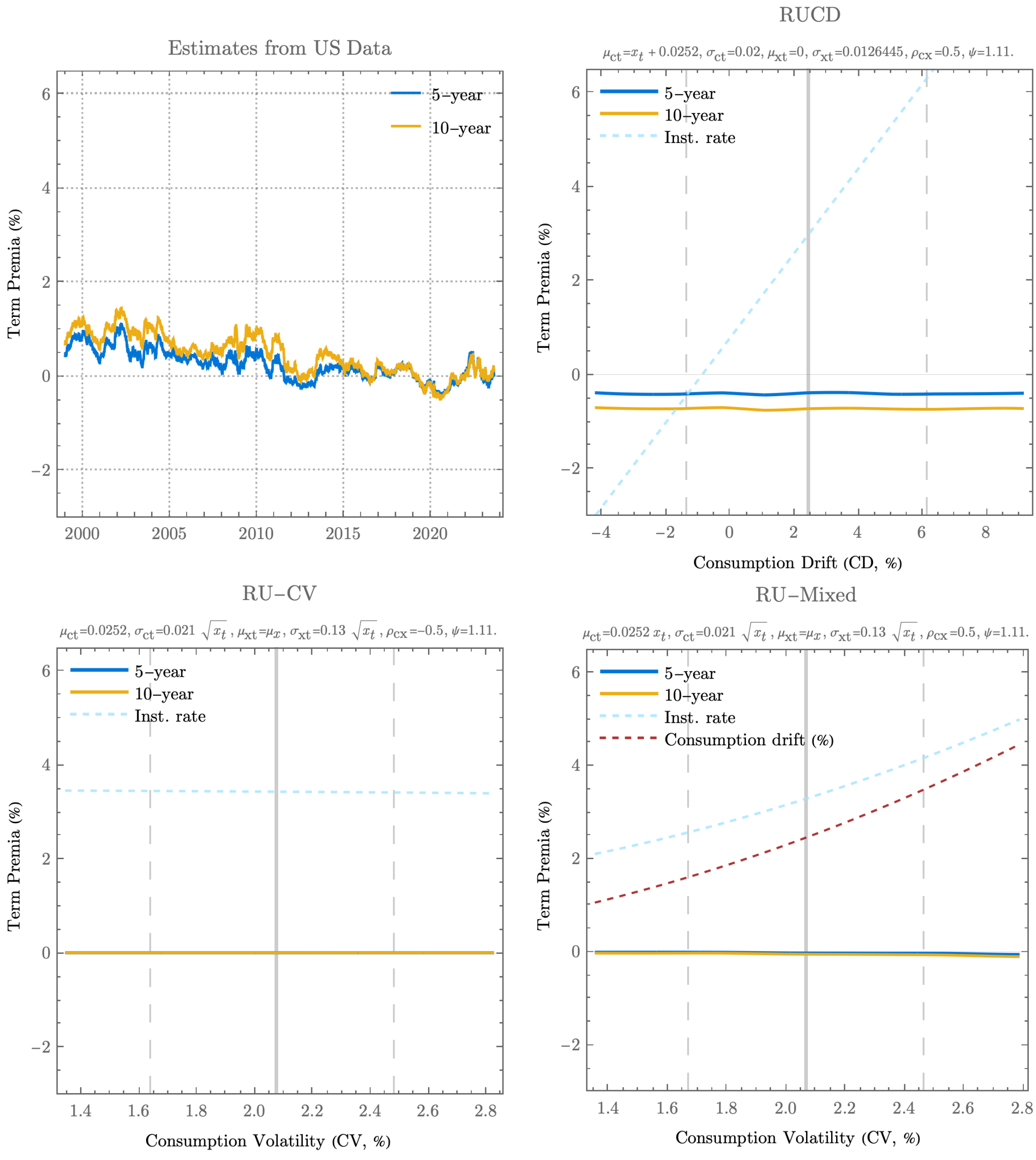


Figure 1.5: **Term Premia in standard models with Recursive Utility**
See Figure 1.3 for details.

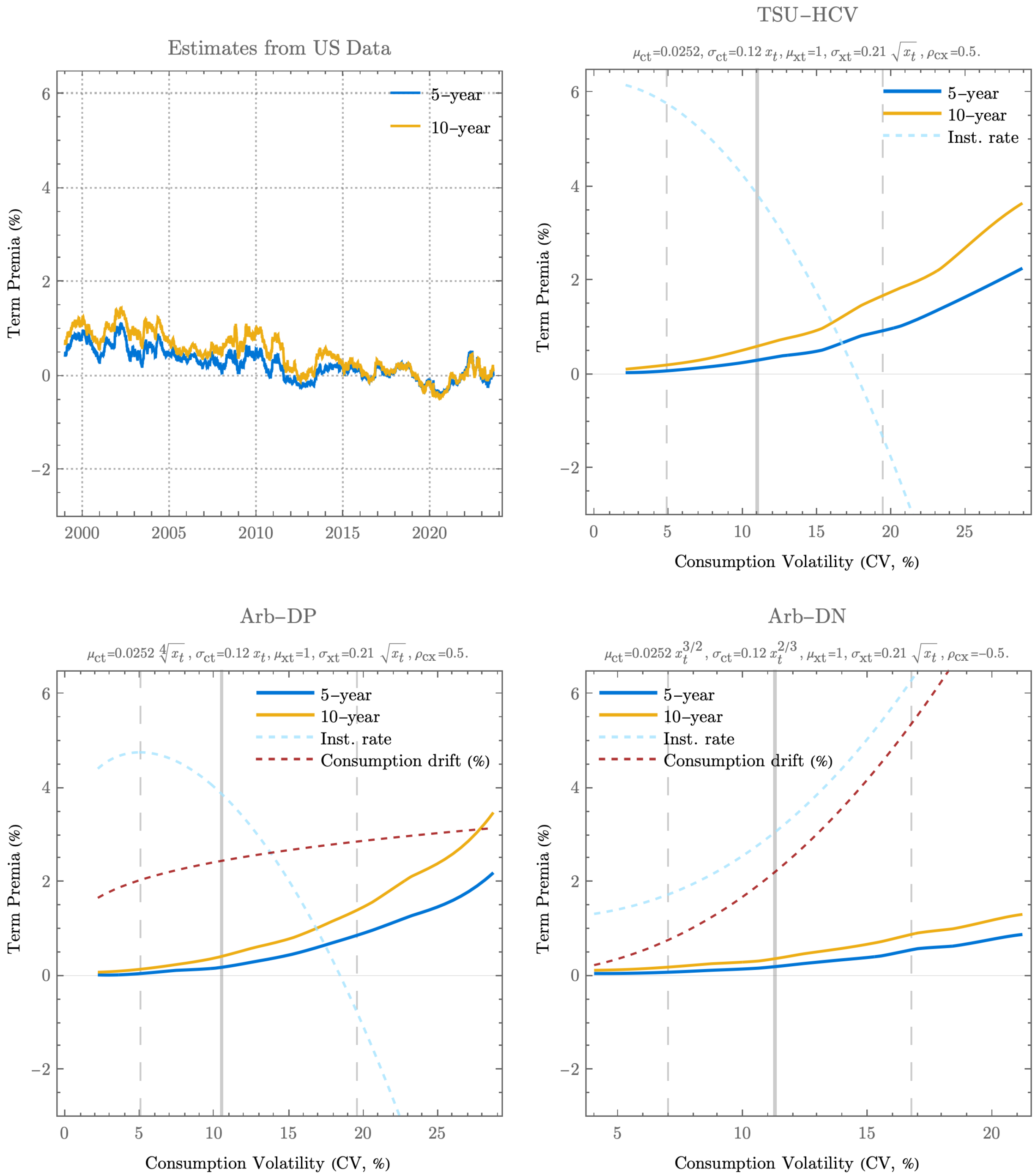


Figure 1.6: **Term Premia in models with Time-Separable Utility and High consumption volatility**
See Figure 1.3 for details.

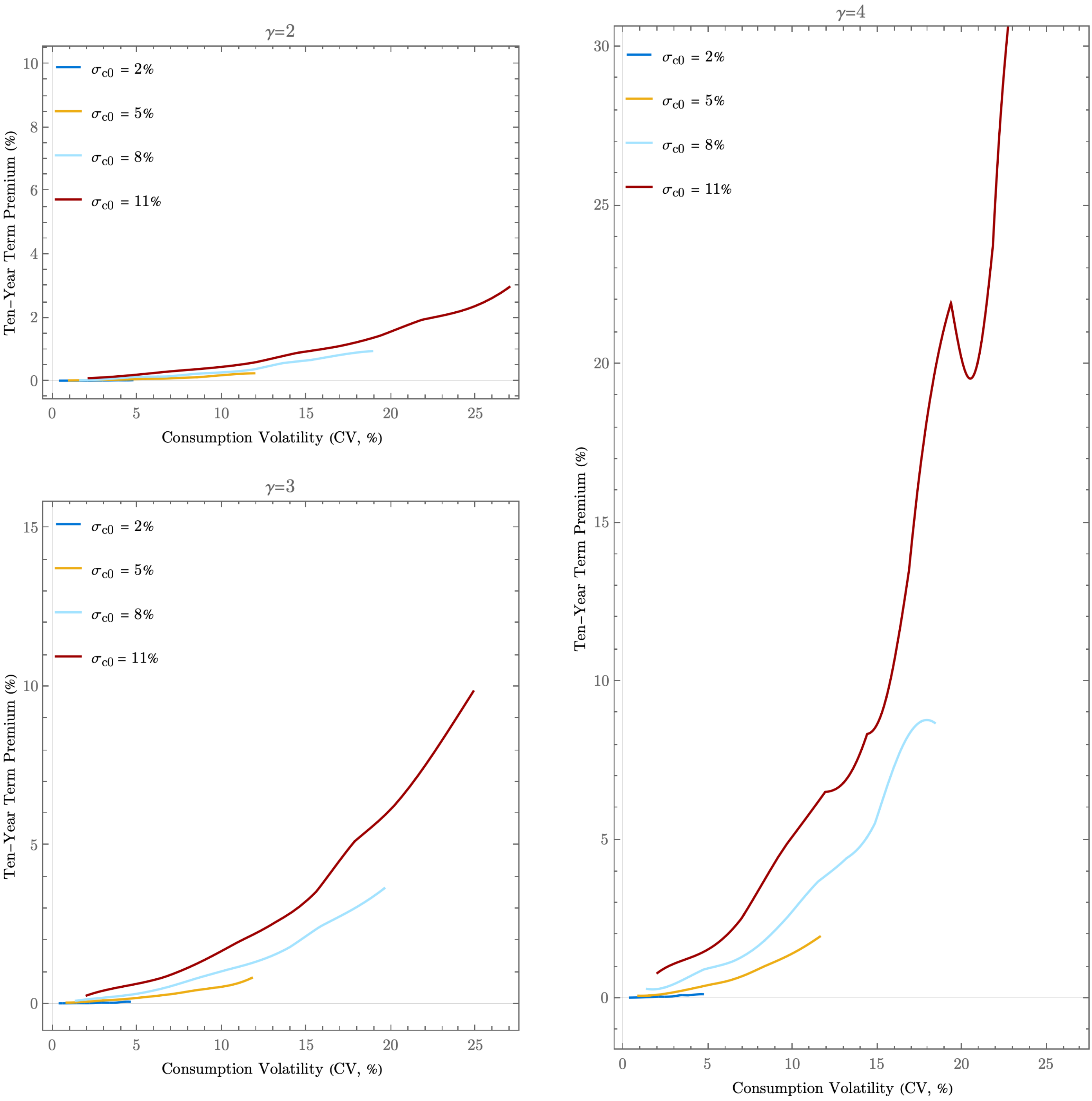


Figure 1.7: **Ten-Year Term Premium in the TSU-HCV Variation for Different Steady State CV σ_{c0}, Levels and Risk Aversion Levels**
The plots show the ten-year term premium for different variations, and they are drawn with the same scale. Each plot corresponds to a different value of the risk aversion paramter γ, and each line corresponds to a different value for the steady state value of CV. The range of CV over which the lines are drawn correspond to the values of CV that can reasonably be acquired (these are the same ranges as in the previous figures).

Appendix

1.A Definitions

In the following I provide a complete set of definitions.[53]

- Throughout the paper, terms like yields, returns, term premia etc. should be understood as referring to their real counterparts, unless otherwise specified. The distinction is still made explicit when necessary to avoid confusion.

- A **nominal zero-coupon bond** with maturity m is a security paying one unit of currency after m years.[54]

- A **real bond** with maturity m is a security paying one unit of currency times an adjustment, that corrects for the elapsed inflation from the time it was issued until its maturity. The payment occurs after m years. Equivalently, a real bond is a security that pays the value of some basket of goods[55] when it matures.[56]

- Q_t^m is the price of the bond with maturity m at time t.

- **Real (or nominal) yield** at time t of a real or (nominal) bond with maturity m years where Q_t^m is the price of the corresponding bond, which is perfectly liquid:[57]
$$y_t^m = \frac{-\log(Q_t^m)}{m}, \quad m > 0$$

- **Yield spread** at time t between maturity m and n, where typically $m > n$:
$$y_t^m - y_t^n$$

- **The yield curve or the term structure of interest rates** refers to yields as a function of maturity. The yield curve is sloping upward/downward (or the slope of the yield curve is positive/negative) when yields are an

[53] Including for some concepts to which I make reference in the main paper, without ever using in expressions.

[54] In the paper bonds always refer to zero-coupon bonds.

[55] Here there is an implicit assumption that individuals primarily care about this specific basket of goods. This basket of goods is also relevant for the calculation of inflation. Without this assumption the study of real interest rates would be significantly hindered.

[56] A real bond of maturity $m+1$ one year ago is also equivalent to a real bond with maturity m today up to a renormalisation so that the principals match.

[57] Actual bonds' prices may deviate from Q_t^m due to liquidity considerations.

increasing/decreasing function with respect to maturity. It is also possible that the slope is positive for some maturities and flat or negative for other maturities.

- **Annualised Gross Return** of a bond with maturity m from time t to $t+s$:

$$R^m_{t,t+s} = \sqrt[s]{Q^{m-s}_{t+s}/Q^m_t}$$

- **Log return or just return**[58] of a bond with maturity m from time t to $t+s$:

$$r^m_{t,t+s} = \log\left(R^m_{t,t+s}\right) = \frac{\log\left(Q^{m-s}_{t+s}\right) - \log\left(Q^m_t\right)}{s}$$

- **Instantaneous return** of a bond with maturity m at time t:

$$r^m_t = \lim_{s\to 0} r^m_{t,t+s}$$

- **Instantaneous short rate or just short rate** at time t:

$$r_t = \lim_{m\to 0} r^m_t = \lim_{m\to 0} y^m_t$$

- In the main paper yields are also referred to as *long-term interest rates*, whereas *interest rates* in general also include the short rate.

- m**-to-**n **year forward** at time t:

$$f^{m,n}_t = \frac{\log(Q^m_t) - \log(Q^n_t)}{n-m}$$

- **Instantaneous** m**-year forward** is:

$$f^m_t = \lim_{n\to m} f^{m,n}_t$$

- **Term or risk premium** of bond with maturity m at time t, where r_t is the instantaneous rate of return at time t:[59]

$$TP^m_t = \frac{-\log(Q^m_t)}{m} - \frac{E_t\left[\int_0^m r_{t+s}ds\right]}{m}$$

[58] For convenience I refer to log return when I use the term return.
[59] Equivalent definitions are given in discrete time by Cochrane and Piazzesi (2009).

- If the term premium is zero for all m and t, this implies that the expected excess return from holding long-term bonds over any period is also 0. This can be seen from the following equivalent definition, where rx_t^m is the instantaneous excess return from holding a bond of maturity m:[60]

$$TP_t^m = \frac{E_t\Big[\int_0^m r_{t+s}^{m-s} - r_{t+s} d\tau\Big]}{m} \equiv \frac{E_t\Big[\int_0^m rx_{t+s}^{m-s} d\tau\Big]}{m}$$

- Here I have used the fact that:

$$\begin{aligned}
-\log(Q_t^m) =& \Big(-\log\left(Q_t^m\right) + \log\left(Q_{t+m/N}^{m-m/N}\right)\Big) + \Big(-\log\left(Q_{t+m/N}^{m-m/N}\right) + \log\left(Q_{t+2m/N}^{m-2m/N}\right)\Big) + \\
&\ldots + \Big(-\log\left(Q_{t+m-m/N}^{m/N}\right) + \underbrace{\log\left(Q_{t+m}^{0}\right)}_{=0}\Big) \\
=& \frac{m}{N}\left(r_{t,t+m/N}^{m} + r_{t+m/N,t+2m/N}^{m-m/N} + \ldots + r_{t+m-m/N,t+m}^{m/N}\right) \\
=& \int_0^m r_{t+s}^{m-s} ds
\end{aligned}$$

where N is some positive integer. The last line follows by N going to infinity, which means that the sum becomes an integral and the returns become instantaneous returns.

- Given that Q is the price of a bond that is perfectly liquid, the term premium does not include a liquidity premium.

- I also refer to the quantity used above:

$$\frac{E_t\Big[\int_0^m r_{t+s} ds\Big]}{m}$$

as **risk-neutral yield** of bond with maturity m.

- **Term or risk premium** of m-to-n year maturity forward at time t, where r_t is the instantaneous rate of return at time t:

$$TP_t^{m,n} = \frac{\log(Q_t^m) - \log(Q_t^n)}{n-m} - \frac{E_t\Big[\int_{t+m}^{t+n} r_\tau d\tau\Big]}{n-m}$$

[60] If the excess return were positive for any period, then the expected term premium for the remaining period would have to be negative. This violated the initial assumption.

- The second term on the right hand side of the equation above is the risk-neutral m-to-n year forward.

- In the paper many of the variables introduced here depend on time only through the state variable. So they will be denoted instead as:

$$Q(x_t, m), y(x_t, m), R_s(x_t, m), r_s(x_t, m), r(x_t, m)$$

$$r(x_t), f(x_t, m, n), f(x_t, m), TP(x_t, m), TP(x_t, m, n)$$

- In the main paper I also refer to the **value of the risk-neutral bond**. This is the implied value attached to a bond by a risk-neutral investor and it can be defined based on the risk-neutral yield defined above:[61]

$$H(x_t, m) = e^{-E_t\left[\int_0^m r_{t+s}ds\right]}$$

- The **strong version of the Expectations Hypothesis** holds when:

$$TP_t^m = 0, \quad \text{for all } m$$

- The **weak version of the Expectations Hypothesis** holds when:

$$TP_t^m = g(m), \quad \text{for all } m$$

  where g is some function of maturity, independent of the state of the economy and independent of time.

- **Predictability** refers to the ability of predicting movements in excess returns. The prediction could be based on any information, but the literature has focused on using information in yields to predict subsequent yields in the future.

- **Excess volatility** of interest rates refers to long-term interest rate variations that are too large to be explained by the variation of the short rate alone, while keeping the discount rate constant.[62]

[61]In the main paper, I also present an equivalent definition in Section 1.4.7.3, which also shows the intuition regarding the calculation of the term premium in this paper.

[62]To be completely precise excess volatility needs to be defined in terms of some benchmark model. As, I do not investigate excess volatility directly, I do not provide such a definition.

1.B Explanatory power of the principal components of real interest rates

Apart from Figure 1.2, I also look at a series of regressions to demonstrate the strong dependence of nominal rates on real rates. In particular, I extract the first two principal components from a series of real yields with different maturities.[63] I only use two components because they explain more than 99.95% of the variance of real yields. Next, I regress nominal yields and nominal yield spreads on these two principal components.[64] Indeed, I find that the information contained within real rates explains most of the movements of nominal rates. The results are shown in Table 1.7. The coefficients are highly significant for both components, but more importantly the R-squared is high in these regressions. For the level regressions it ranges from 87% to 93%, while for the spread regressions it ranges from 69% to 79%. Thus, both the level and the spread of nominal rates is mostly explained by the information and hence the processes that generate the real term structure.

Table 1.7: Regressions of the level and the spread of nominal bonds on the principal components extracted from the real term structure

	5 yr	10 yr	5-10 yr spread	15 yr	5-15 yr spread	20 yr	5-20 yr spread
Intercept	2.94***	3.73***	0.79***	4.13***	1.19***	4.29***	1.35***
	(0.01)	(0.00)	(0.00)	(0.00)	(0.01)	(0.00)	(0.01)
comp1	0.28***	0.26***	-0.02***	0.25***	-0.04***	0.23***	-0.05***
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
comp2	0.43***	-0.24***	-0.66***	-0.55***	-0.97***	-0.67***	-1.09***
	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
R-squared	0.87	0.93	0.69	0.93	0.74	0.93	0.79
R-squared Adj.	0.87	0.93	0.69	0.93	0.74	0.93	0.79

[63] The principal components are extracted from yields of all yearly maturities from two to twenty years.

[64] A similar exercise in performed by Abrahams et al. (2016) and they also find similar results. In their case it is the real rates that are regressed on the principal components of the nominal rates. I do the inverse exercise because I ask how much nominal rates are explained by real rates.

1.C Explanation of the components of the pricing equation

This section provides an explanation for each part of the pricing equation (1.15) which I repeat here:

$$-Q_m - r(x_t)Q + \Big(\log(\phi)x_t + A(x_t)\Big)Q_x + \frac{\sigma_{xt}^2}{2}Q_{xx} = 0 \tag{1.24}$$

- In the simplest case $\phi = 1$, $A(x) = 0$ and $\sigma_x(x) = 0$ for all x. Then the equation is:

  $$Q_m = -r(x_t)Q = -x_tQ$$

  This corresponds to an economy with a constant state. Figure 1.8 shows that in this economy yields are always equal to the short rate, term premia are equal to 0 and given a state of the economy nothing will ever change.

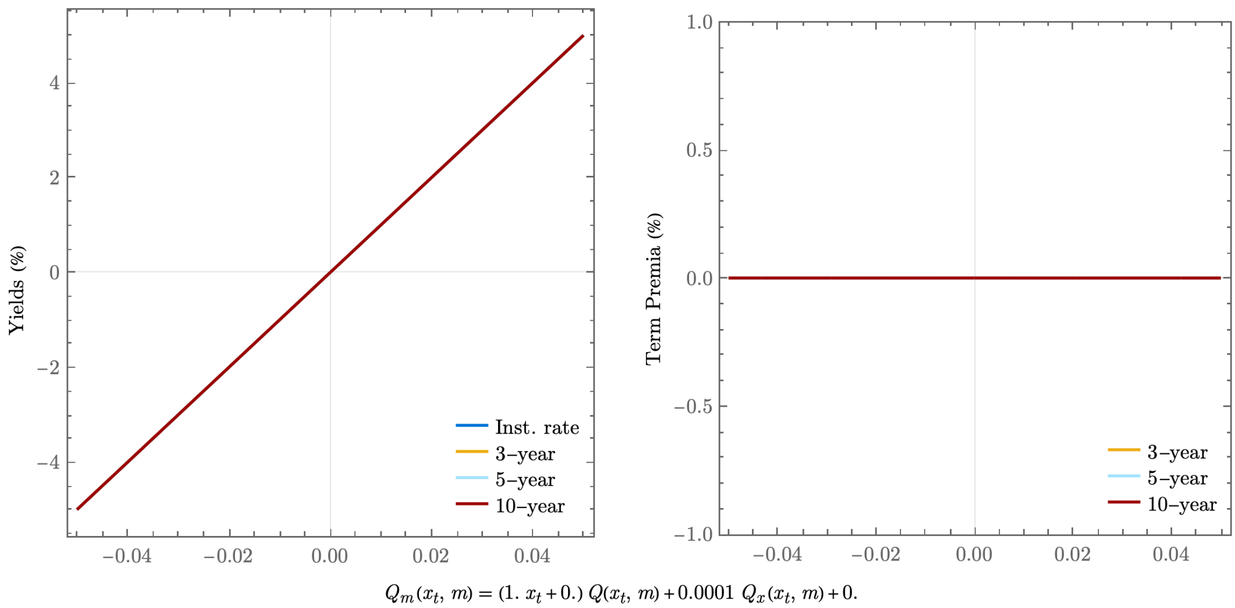


Figure 1.8: The left plot shows the short-term rate and yields of different maturities as a function of the state variable. The right plot shows the the term premia for different maturities as a function of the state variable.

- $\phi \neq 1$:

  $$Q_m = r(x_t)Q - \log(\phi)x_tQ_x = x_tQ - \log(0.9)x_tQ_x$$

  Here there is again no volatility of the state variable. Thus, this corresponds to a deterministic economy. However, the state is not constant, it drifts towards the state $x_t = 0$, which can be thought of as the steady state. This implies that long-term yields will lie between the contemporaneous short rate and the steady state short rate. As shown in Figure 1.9 this results in a characteristic picture, in which all yields intersect at the steady state. If the process moved towards the steady state faster (lower ϕ), then the yields would be more spread out. Given that there is no uncertainty, the term premia are again zero.

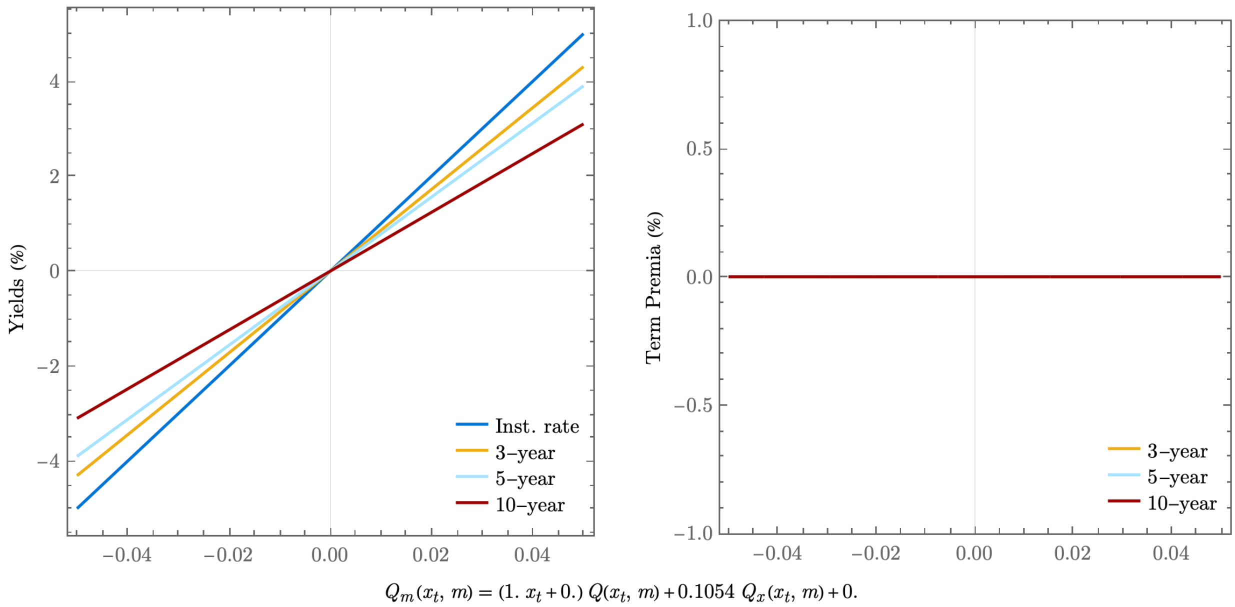


Figure 1.9: The left graph show the short-term rate, the five-year yield and the five-year risk-neutral yield as a function of CD. The right graphs shows the decomposition of the five-to-ten year forward into the term premium and the risk-neutral yield.

- $A(x_t) = c \neq 0$:

$$Q_m = r(x_t)Q - \Big(\log(\phi)x_t + A(x_t)\Big)Q_x = x_tQ - (\log(0.9)x_t + 0.01)Q_x$$

As stated in the main paper A generates term premia. This case does not directly correspond to some economic situation because, the state variable volatility is again 0, and in the actual economic models this also implies $A(x_t) = 0$. However, for the purposes of intuition I show the "yields" and "term premia" that arise. As Figure 1.10 shows, now the yields do not intersect at the steady state. Now the longer-term yields are higher at the steady state. This implies positive term premia and indeed as shown in the right panel, term premia are positive, proportional to the maturity of the bond and constant with respect to the state variable. The latter fact is due to $A(x_t)$ being constant for all x_t and the fact that yields are linear. Finally the term premia are positive, because A is positive and the short rate is increasing with respect to the state variable.

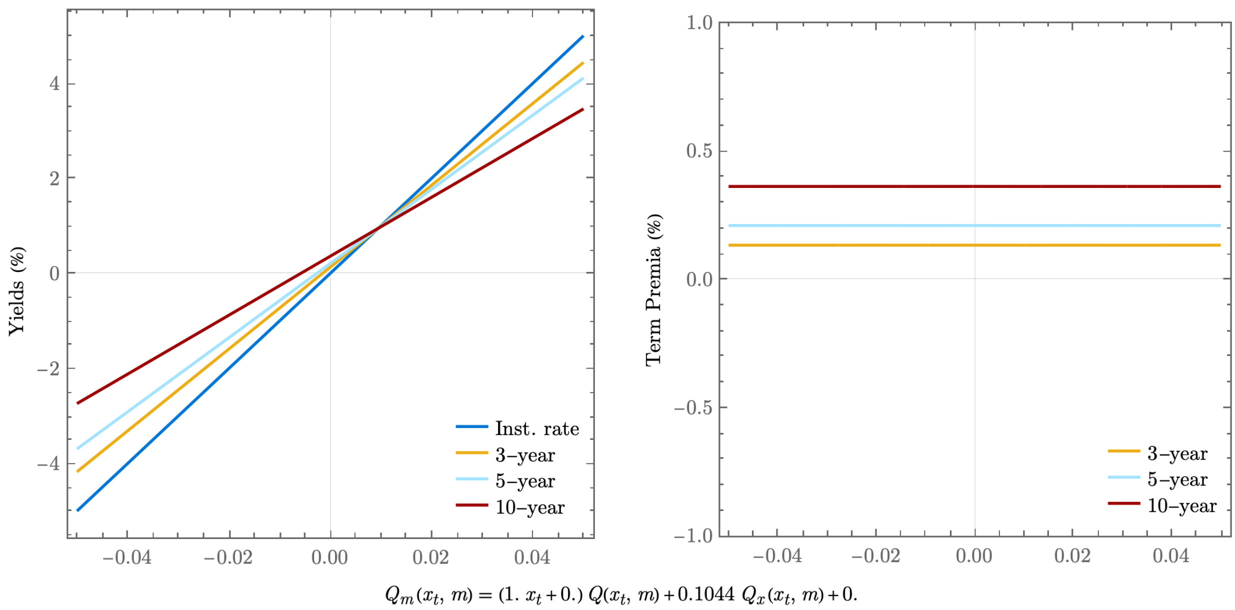


Figure 1.10: The left graph show the short-term rate, the five-year yield and the five-year risk-neutral yield as a function of CD. The right graphs shows the decomposition of the five-to-ten year forward into the term premium and the risk-neutral yield.

- $A(x_t) = 0.0005 + 0.02x_t$. This means that now A changes with the state variable. The result is shown in Figure 1.11. Term premia follow the behaviour of A. The correspondence would not be so close, if the short rate were a non-linear function of the state variable.

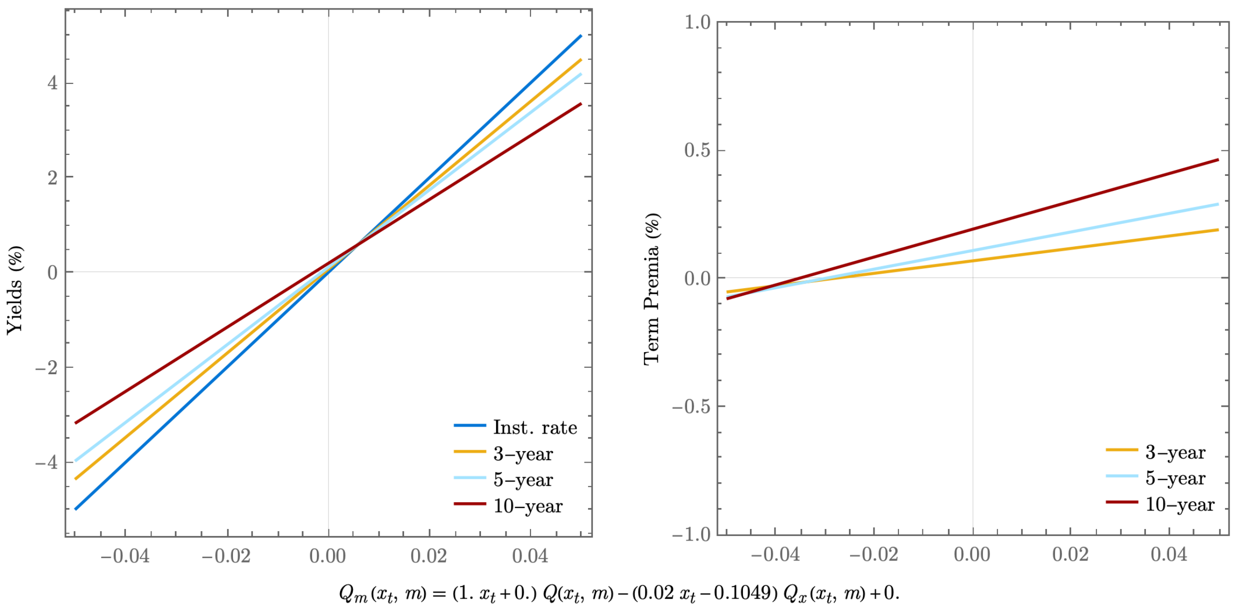


Figure 1.11: The left graph show the short-term rate, the five-year yield and the five-year risk-neutral yield as a function of CD. The right graphs shows the decomposition of the five-to-ten year forward into the term premium and the risk-neutral yield.

- $\sigma_{xt} \neq 0$:

$$Q_m = r(x_t)Q - \log(\phi)x_tQ_x + \frac{\sigma_{xt}^2}{2}Q_{xx} = x_tQ - \log(0.9)x_tQ_x - \frac{0.03^2}{2}Q_{xx}$$

Here $A(x_t) = 0$. Thus, the effect of volatility can be seen. This case corresponds to a case where there is volatility of the short rate, but there is again no priced risk. So there is no risk premium. This can be seen on the right panel of Figure 1.12.[65] Nevertheless, the yields are not the same as in the deterministic case with steady state reversion, as they do not intersect at the steady state. The long-term yields are pushed downwards, and, even though it might not be obvious, the effect of uncertainty increases more than linearly with maturity. This effect is due to so-called convexity that is common in finance. In particular, the price of the long-term bond is a convex decreasing function of the short-rate and this implies that lower interest rates have a higher effect on the price of the bond, especially for long maturities. Thus, given that there is variation and a chance for the short rate to

[65]The term premia do not look completely flat because the Monte-Carlo calculation has some uncertainty in the calculation.

reach lower levels, these will outweigh the high rates, and push long-term yields downward. Finally, this also means that a downward-sloping term structure does not necessarily imply negative term premia.

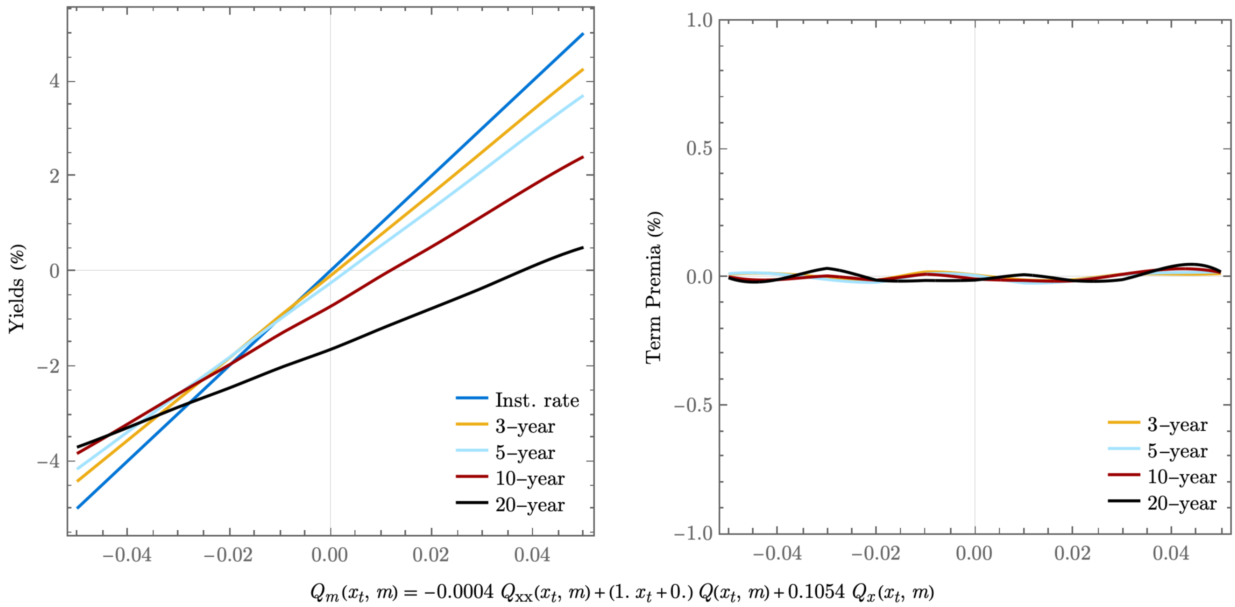


Figure 1.12: The left graph show the short-term rate, the five-year yield and the five-year risk-neutral yield as a function of considers. The right graphs shows the decomposition of the five-to-ten year forward into the term premium and the risk-neutral yield.

- full case:

$$Q_m = r(x_t)Q - \log(\phi)x_tQ_x + \frac{\sigma_{xt}^2}{2}Q_{xx} = x_tQ - \Big(\log(0.9)x_t + 0.001\Big)Q_x - \frac{0.005^2}{2}Q_{xx}$$

This case contains all the components. Unlike the previous case, as can be seen in Figure 1.13, the yields seem to intersect close to the steady state. Thus, the yield curve would often be flat in this economy. However, term premia are positive. The yields are close to flat at the steady state, because term premia and convexity largely cancel each other out.

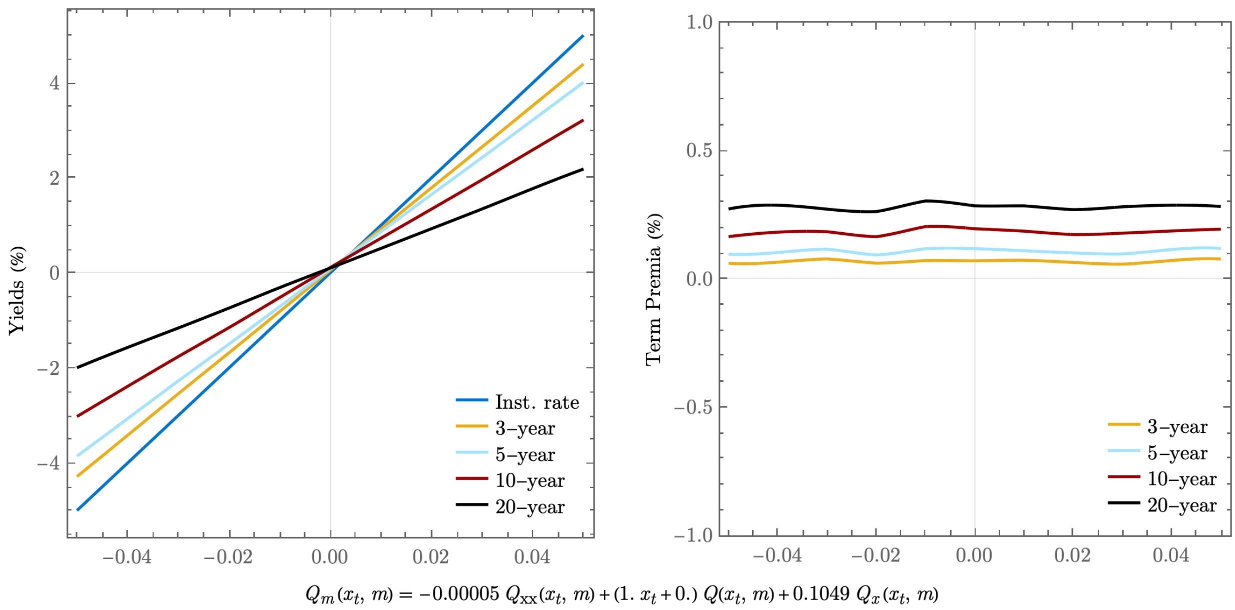


Figure 1.13: The left graph show the short-term rate, the five-year yield and the five-year risk-neutral yield as a function of CD. The right graphs shows the decomposition of the five-to-ten year forward into the term premium and the risk-neutral yield.

1.D Calibration of the state variable volatility

As mentioned in Subsection 1.5.1, the aim of the paper is to simultaneously match the variability of term premia and the variability of the short rate. I achieve this by calculating the range of the two-year TIPS security over the available sample in the Gürkaynak, Sack and Wright (2010) dataset.[66] I find a range of 7.27%.[67] I then simulate time series with twelve year duration[68] for all the variations that I investigate. Based on these simulations I rank the range sizes and I aim for the tenth quantile to equal the range in the data. I do this for the models that are not able to produce highly variable term premia, in order to give these models the benefit of the doubt and the best chance to succeed. Namely, it is possible that the observed short rate volatility has been by chance relatively low and the underlying process is significantly more volatile. Thus, I want the model variations to be as

[66] Two years is the shortest maturity in the data.

[67] This could be overestimating the plausible range as the maximum was achieved during the financial crisis, when the TIPS market was not behaving normally.

[68] This matches the length of the sample in Abrahams et al. (2016), but I should arguably change this to match the length of the sample in Gürkaynak, Sack and Wright (2010). In any case the length of that sample is approximately 15 years.

volatile as possible in order to generate as large a time variability in term premia as possible. For the models that succeed in producing significantly time-varying term premia I again make sure that the empirical volatility, as expressed by the observed range, falls within the model predictions, but I do not necessarily match the empirical with the tenth quantile. For each model variation, I show the value of the empirical range and the values of the model-implied tenth and ninetieth quantile ranges in the figures in Appendix 1.F.

1.E Term Premia Measures

1.E.1 Figure from Abrahams et al. (2016)

TIPS Term Premium

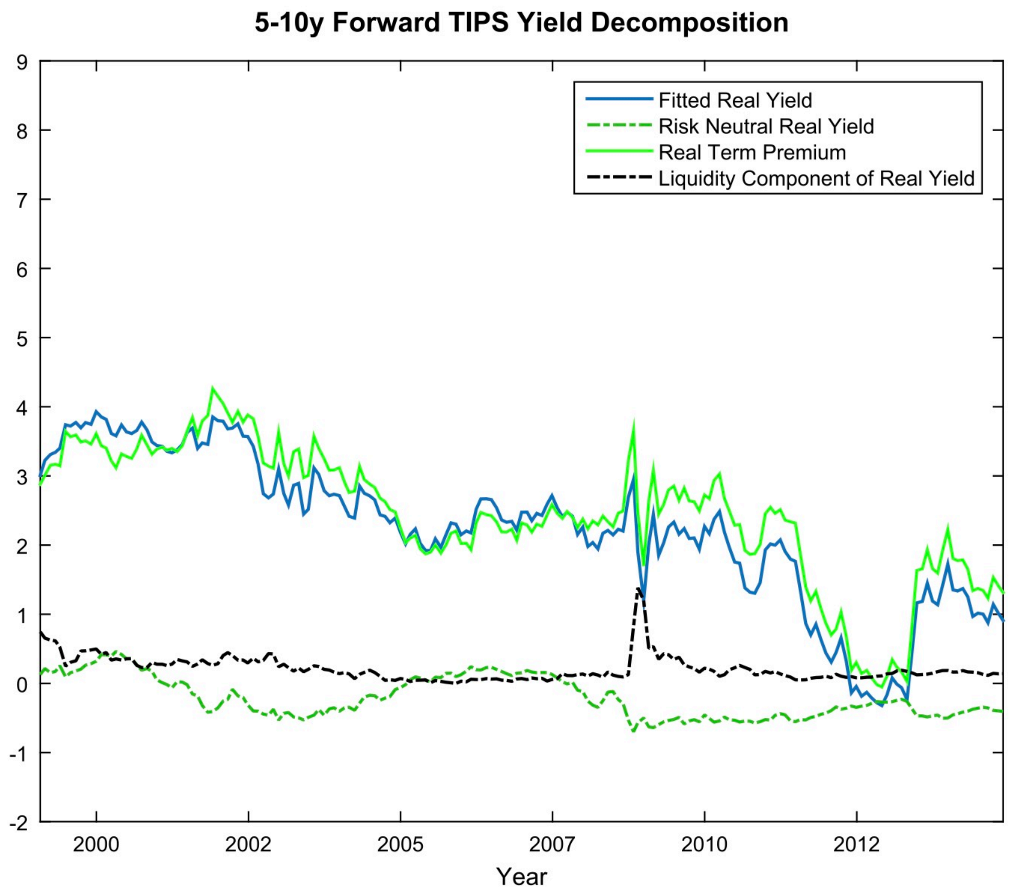


Figure 1.14: The figure shows the time series of the five-to-ten year forward term premium along with its decomposition to the risk-neutral yield, the term premium and the liquidity premium. I also show the same decomposition in the figures in this paper as a function of the state variable.

1.E.2 Term premium based on d' Amico et Al. (2018)

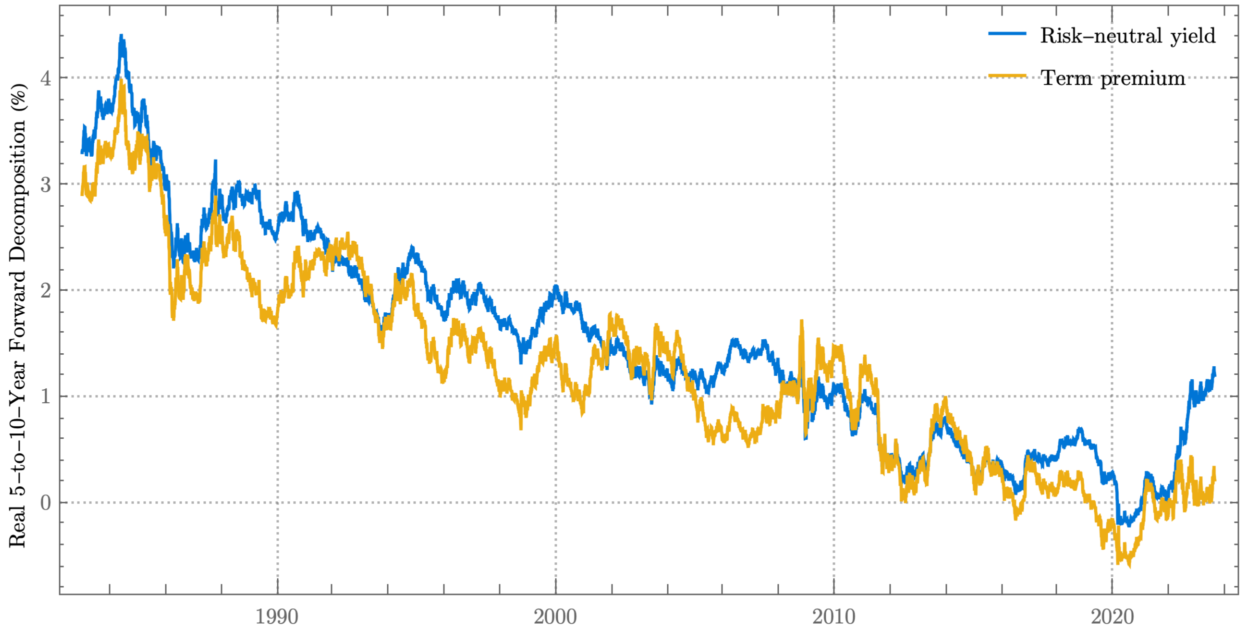


Figure 1.15: **Time series of the forward 5-to-10 term premium for the US. (back to text)**
This is the same quantity as the solid green line in Figure 1.E.1 from Abrahams et al. (2016).
Data Source: https://www.federalreserve.gov/econres/notes/feds-notes/tips-from-tips-update-and-discussions-20190521.html

1.F Yields and Term Premia in Other Model Variations – Time-Separable Utility

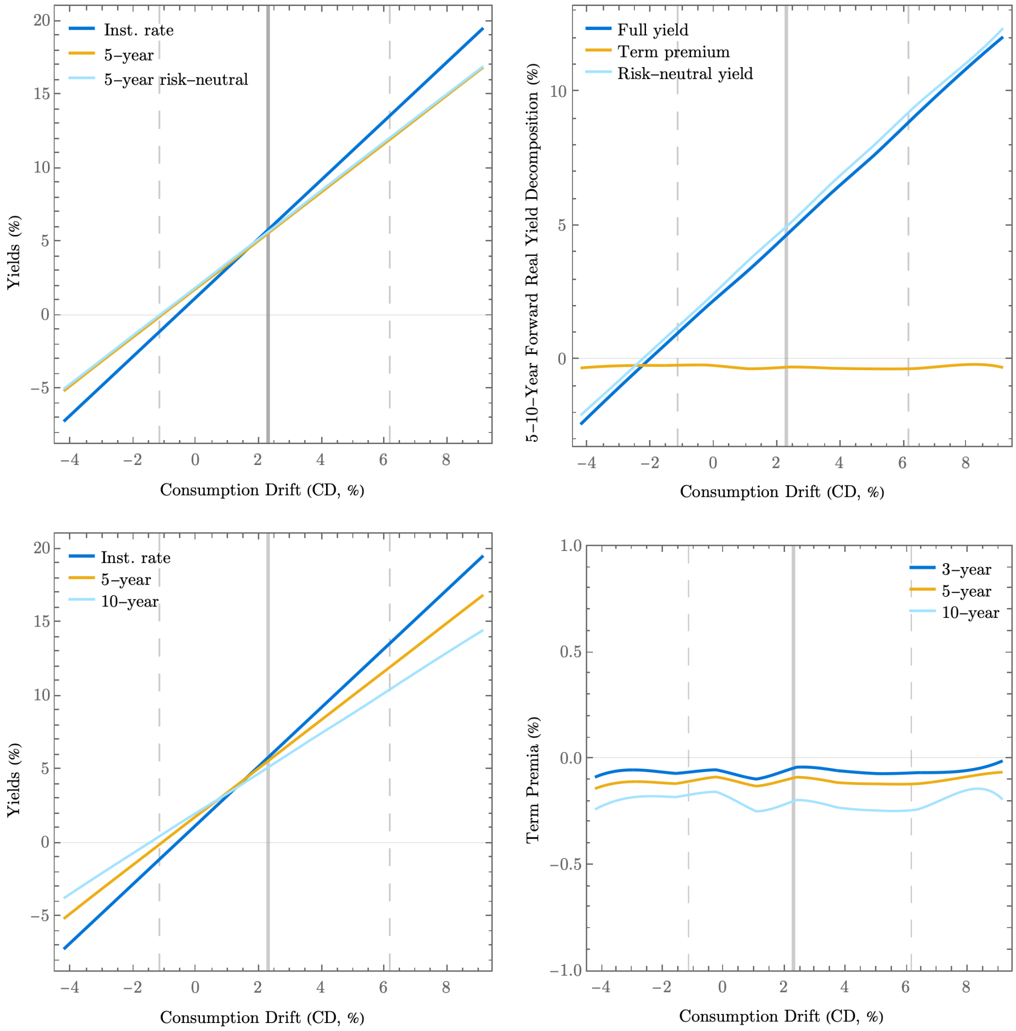
In this part I present more plots for the variations discussed in the main paper, and I also present results for other model variations. These are other variations should reinforce the conclusions in the main paper as a long series of calibrations is examined. The upper left and upper right plots are the same as in the main paper. The lower left plot shows the level of yields for different maturities as a function of the state variable. The lower right plot shows the level of the term premium for different maturities as a function of the state variable. Again each figure states the exact specification.

1.F.1 Contents

Names of all model variations shown in Appendix 1.F. The abbreviations used here are: time-varying (tv), consumption drift (CD), consumption volatility (CV), time-separable utility (TSU), recursive utility (RU), intertemporal elasticity of substitution (IES).

Model Variation Description	**Abbreviation**	**References**
Tv CD with TSU.	TSU-CD	Figure 1.16
Tv CD with TSU and high risk aversion.	TSU-CD-HRA	Figure 1.17
Tv CD with TSU and low persistence.	TSU-CD-LP	Figure 1.18
Tv CD with TSU and high correlation ρ_{cx}.	TSU-CD-HCor	Figure 1.19
Tv CD with TSU and high impatience.	TSU-CD-HImp	Figure 1.20
Tv and high CD with TSU.	TSU-HCD	Figure 1.21
Tv CD with TSU and high CV.	TSU-CD-HCV	Figure 1.22
Tv CV with TSU.	TSU-CV	Figure 1.23
Tv CV with TSU and high risk aversion.	TSU-CV-HRA	Figure 1.24
Tv CV with TSU and high CD.	TSU-CV-HCD	Figure 1.25
Tv and high CV with TSU and positive correlation ρ_{cx}.	TSU-HCV	Figure 1.26
TV and high CV with TSU and negative correlation ρ_{cx}.	TSU-HCV-NCor	Figure 1.27
Both tv CD and CV, short-term rate decreasing in CV and ρ_{cx} positive.	TSU-Arb-DP	Figure 1.28
Both tv CD and CV, short-term rate increasing in CV and ρ_{cx} negative.	TSU-Arb-IN	Figure 1.29
Both tv CD and CV, short-term rate decreasing in CV and ρ_{cx} negative.	TSU-Arb-DN	Figure 1.30
Both tv CD and CV, short-term rate increasing in CV and ρ_{cx} positive.	TSU-Arb-IP	Figure 1.31
Tv external habit with TSU.	TSU-Habit	Figure 1.32
Tv external habit with TSU and low b.	TSU-Habit-L.b	Figure 1.33
Tv external habit with TSU and $b < 0$.	TSU-Habit-Neg.b	Figure 1.34
Tv external habit with TSU with constant state variable volatility.	TSU-Habit-ConstantSV	Figure 1.35
Tv CD with RU.	RU-CD	Figure 1.36
Tv CD with RU and high risk aversion.	RU-CD-HRA	Figure 1.37
Tv CD with RU with high IES.	RU-CD-HIES	Figure 1.38
Tv CD with RU with Low IES.	RU-CD-LIES	Figure 1.39
Tv CD with RU with high ρ_{cx}.	RU-CD-HCor	Figure 1.40
Tv CD with RU with ρ_{cx} negative.	RU-CD-NCor	Figure 1.41
Tv and high CD with RU.	RU-HCD	Figure 1.42
Tv CD with RU and high CV.	RU-CD-HCV	Figure 1.43
Tv and heteroskedastic CD with RU and ρ_{cx} positive.	RU-CD-Heterosk-PCor	Figure 1.44
Tv and heteroskedastic CD with RU and ρ_{cx} negative.	RU-CD-Heterosk-NCor	Figure 1.45
Tv CV with RU.	RU-CV	Figure 1.46
Tv CV with RU with high risk aversion.	RU-CV-HRA	Figure 1.47
Tv CV with RU and high persistence IES.	RU-CV-HP	Figure 1.48
Tv CV with RU and high IES.	RU-CV-HIES	Figure 1.49
Tv CV with RU and low IES.	RU-CV-LIES	Figure 1.50
Tv and high CV with RU and ρ_{cx} positive.	RU-HCV-PCor	Figure 1.51
Tv and high CV with RU and ρ_{cx} negative.	RU-HCV-NCor	Figure 1.52

1.F.2 TSU-CD, Calibration used in main paper, Figure 1.3



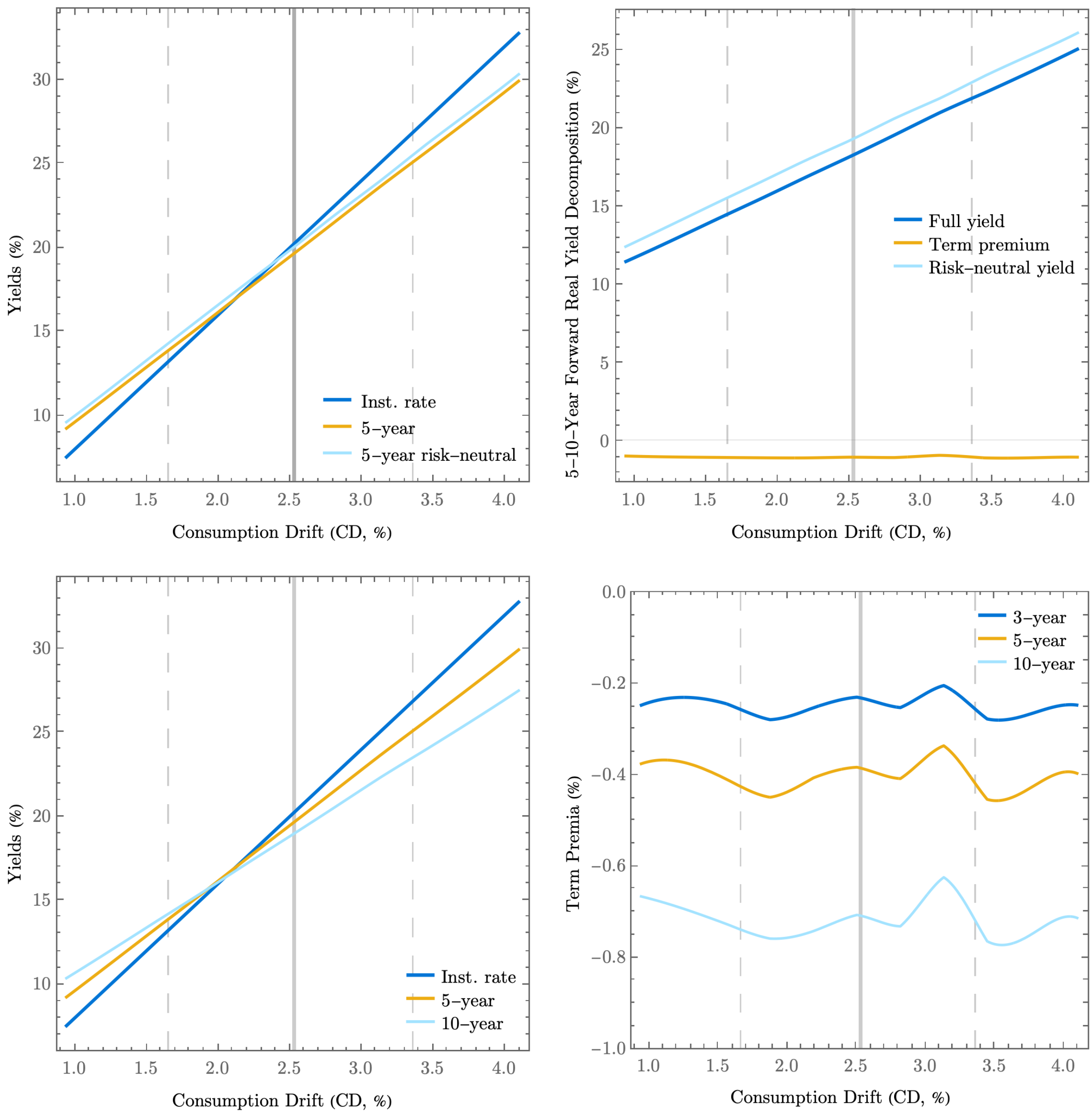
γ=2, ρ=0.012, $\mu_{\mathrm{ct}}=x_t+0.0252$, σ_{ct}=0.02, ϕ=0.92, μ_{xt}=0, σ_{xt}=0.0126445, ρ_{cx}=0.5

$\mathrm{range}_{\mathrm{low}}(\mathrm{r}_{2,\mathrm{model}})$=0.10958, $\mathrm{range}_{\mathrm{high}}(\mathrm{r}_{2,\mathrm{model}})$=0.202976, $\mathrm{range}(\mathrm{r}_{2,\mathrm{data}})$=0.071748.

Figure 1.16: Time-varying consumption drift with time-separable utility.
The left plot shows the short-term rate, the five-year yield and the five-year risk-neutral yield as a function of consumption growth. The right plot shows the decomposition of the five-to-ten year forward into the term premium and the risk-neutral components. The solid vertical line shows the level of the ergodic median, the left and right dashed vertical lines show the median minimum and maximum value respectively over a series of simulations for 12 years. This means that half the simulated paths were below the right dashed line and half the simulated paths were above the left dashed line. The left and right boundaries are the 10th percentile of minimum values and the 90th percentile of maximum values from the same simulations. This means that 90% of simulated paths were above the left boundary and 90% of simulated paths were below the right boundary.

1.F.3 TSU-CD-HRA, $\gamma = 8$

Term premia are a bit larger, but again negative and constant with respect to the state variable.



γ=8, ρ=0.012, $\mu_{\mathrm{ct}}=x_t+0.0252$, σ_{ct}=0.02, ϕ=0.92, μ_{xt}=0, σ_{xt}=0.00291989, ρ_{cx}=0.5

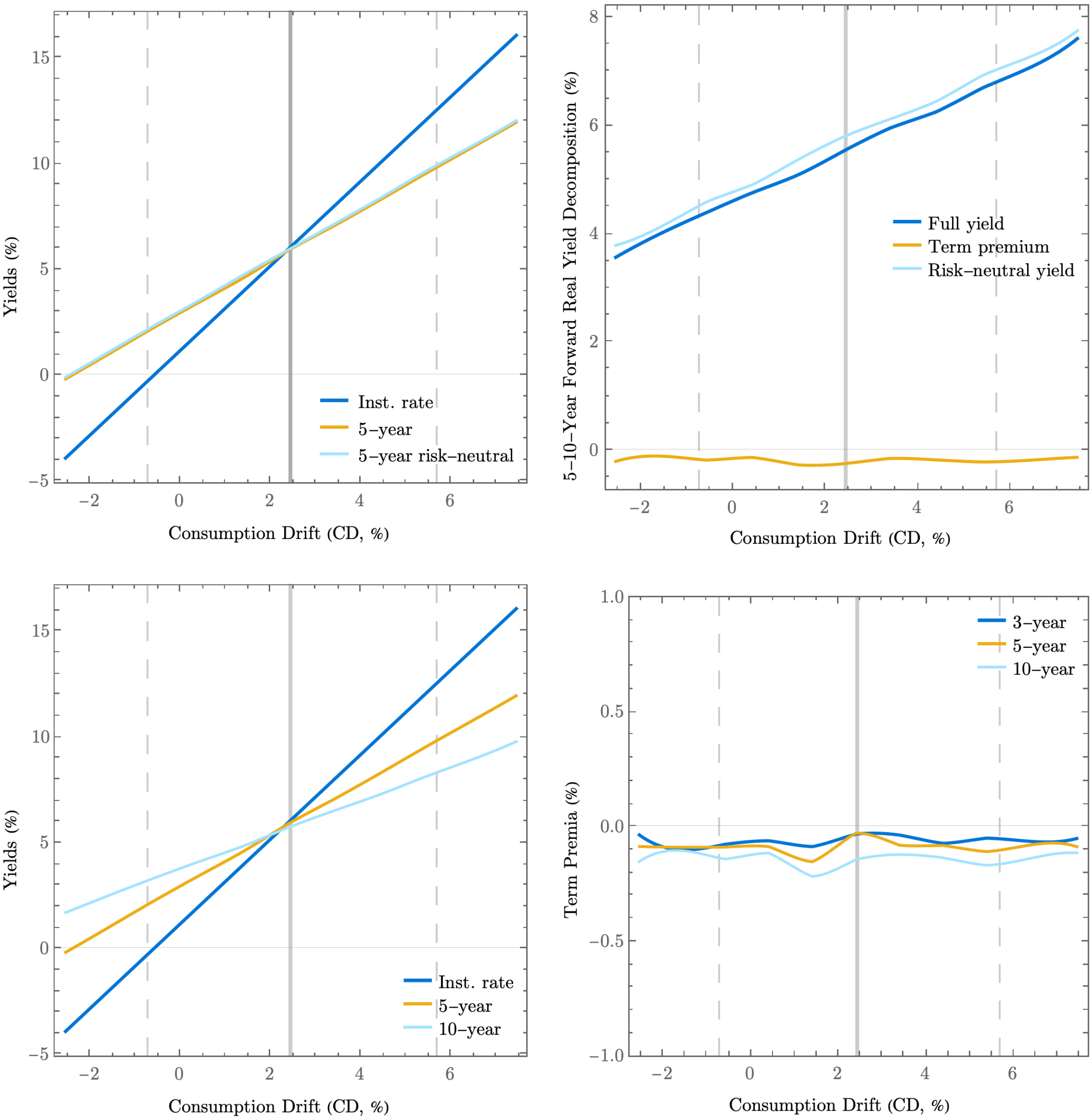
$\mathrm{range_{low}}(\mathrm{r_{2,model}})$=0.103026, $\mathrm{range_{high}}(\mathrm{r_{2,model}})$=0.18666, $\mathrm{range}(\mathrm{r_{2,data}})$=0.071748.

Figure 1.17: Time-varying consumption drift with time-separable utility and highr risk aversion.
See Figure 1.16 for more details about the plots.

(variation overview)

1.F.4 TSU-CD-LP, $\phi = 0.8$

Nothing changed in the term premia. There is larger separation between yields similar to the corresponding mechanism in Appendix 1.C.



$\gamma=2$, $\rho=0.012$, $\mu_{\text{ct}}=x_t+0.0252$, $\sigma_{\text{ct}}=0.02$, $\phi=0.8$, $\mu_{\text{xt}}=0$, $\sigma_{\text{xt}}=0.0126445$, $\rho_{\text{cx}}=0.5$

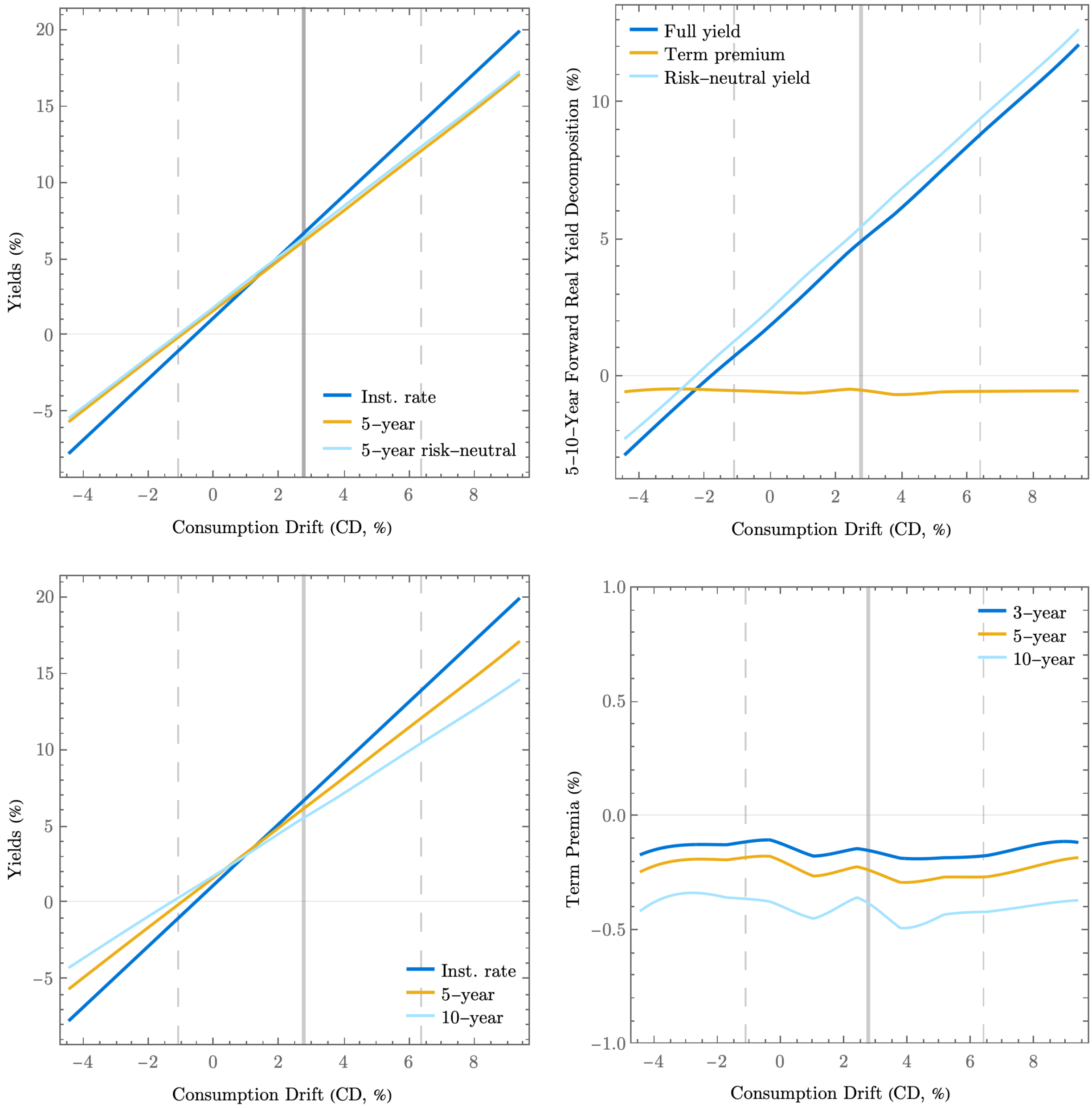
$\text{range}_{\text{low}}(\text{r}_{2,\text{model}})=0.0987483$, $\text{range}_{\text{high}}(\text{r}_{2,\text{model}})=0.168236$, $\text{range}(\text{r}_{2,\text{data}})=0.071748$.

Figure 1.18: Time-varying consumption drift with time-separable utility and low persistence
See Figure 1.16 for more details about the plots.

(variation overview)

1.F.5 TSU-CD-HCor, $\rho_{cx} = 1$

The term premia are larger in absolute value.



γ=2, ρ=0.012, μ_{ct}=x_t + 0.0252, σ_{ct}=0.02, ϕ=0.92, μ_{xt}=0, σ_{xt}=0.0126445, ρ_{cx}=1

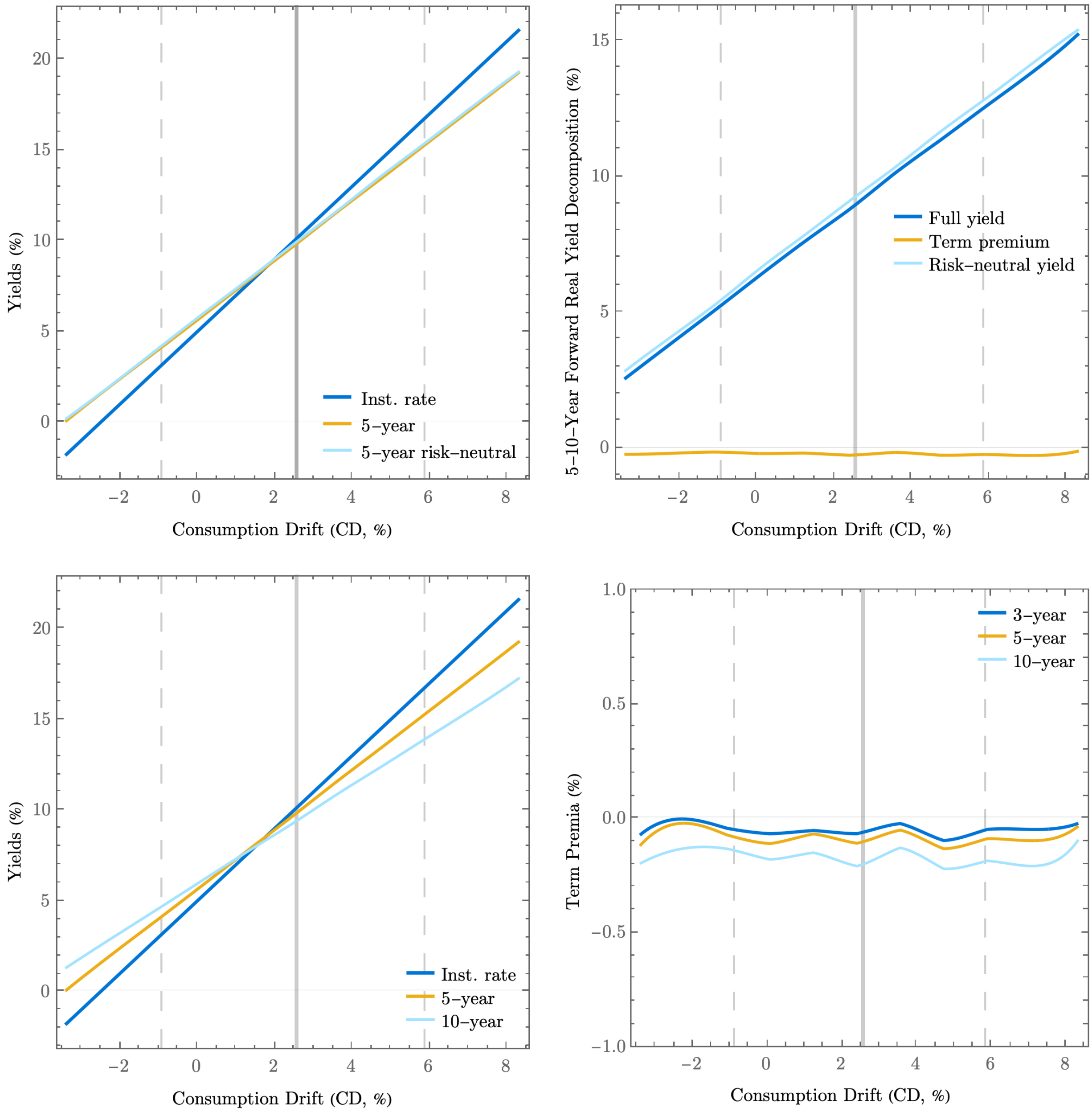
$\text{range}_{\text{low}}(\text{r}_{2,\text{model}})$=0.111377, $\text{range}_{\text{high}}(\text{r}_{2,\text{model}})$=0.203865, $\text{range}(\text{r}_{2,\text{data}})$=0.071748.

Figure 1.19: Time-varying consumption drift with time-separable utility and high correlation ρ_{cx}
See Figure 1.16 for more details about the plots.

(variation overview)

1.F.6 TSU-CD-HImp, $\rho = 0.05$

Yields move higher without any change in term premia.



γ=2, ρ=0.05, $\mu_{\mathrm{ct}}=x_t+0.0252$, σ_{ct}=0.02, ϕ=0.92, μ_{xt}=0, σ_{xt}=0.011495, ρ_{cx}=0.5

$\mathrm{range_{low}}(\mathrm{r_{2,model}})$=0.0973497, $\mathrm{range_{high}}(\mathrm{r_{2,model}})$=0.182416, $\mathrm{range}(\mathrm{r_{2,data}})$=0.071748.

Figure 1.20: Time-varying consumption drift with time-separable utility and high impatience.
See Figure 1.16 for more details about the plots.

(variation overview)

1.F.7 TSU-HCD, $\mu_{c0} = 0.06$

Again, yields move higher without any change in term premia.

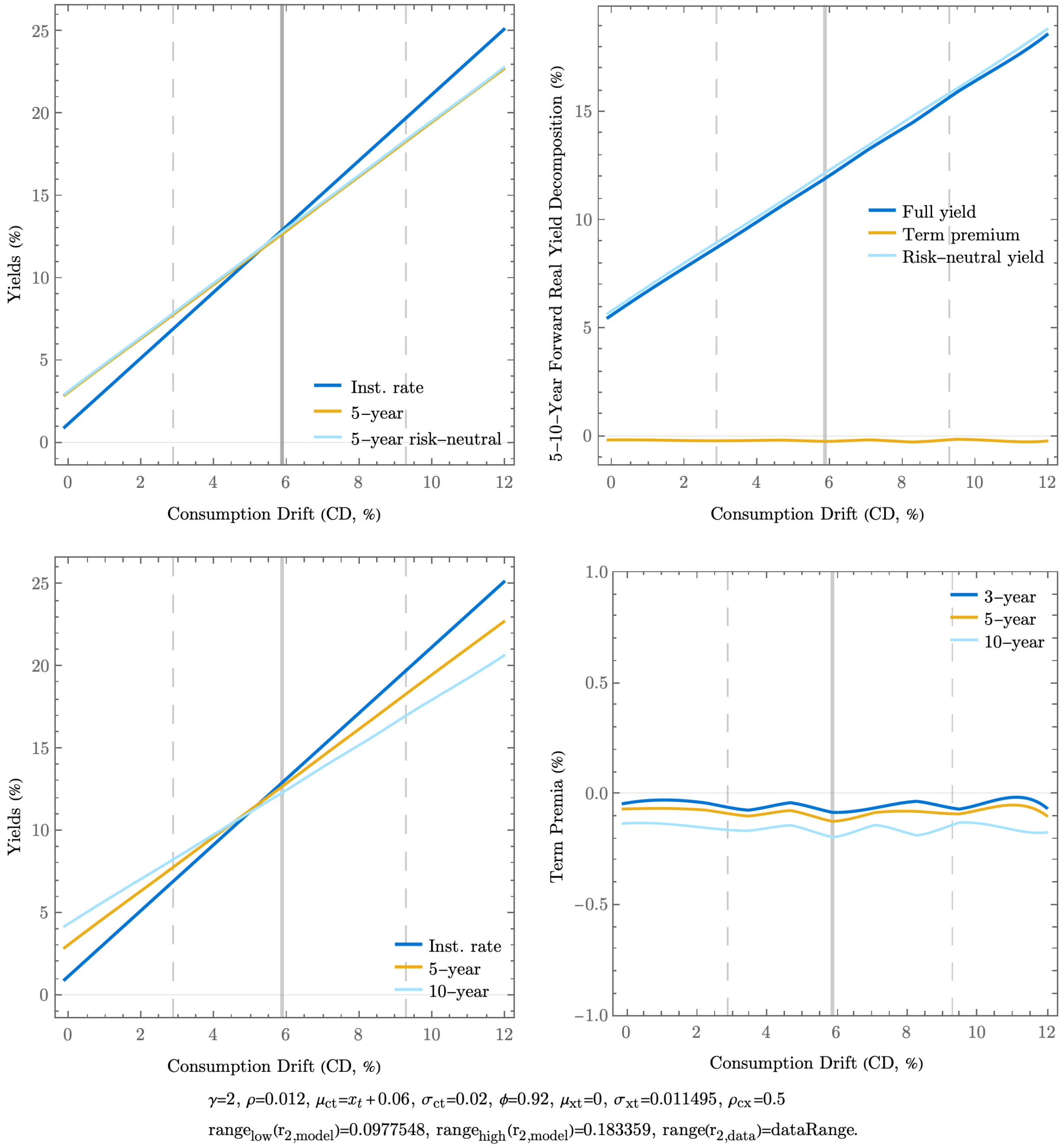


Figure 1.21: Time-varying and high consumption drift with time-separable utility.
See Figure 1.16 for more details about the plots.

(variation overview)

1.F.8 TSU-CD-HCV, $\sigma_{ct} = 0.16$

Yields move down and term premia increase in absolute value, but they are again constant.

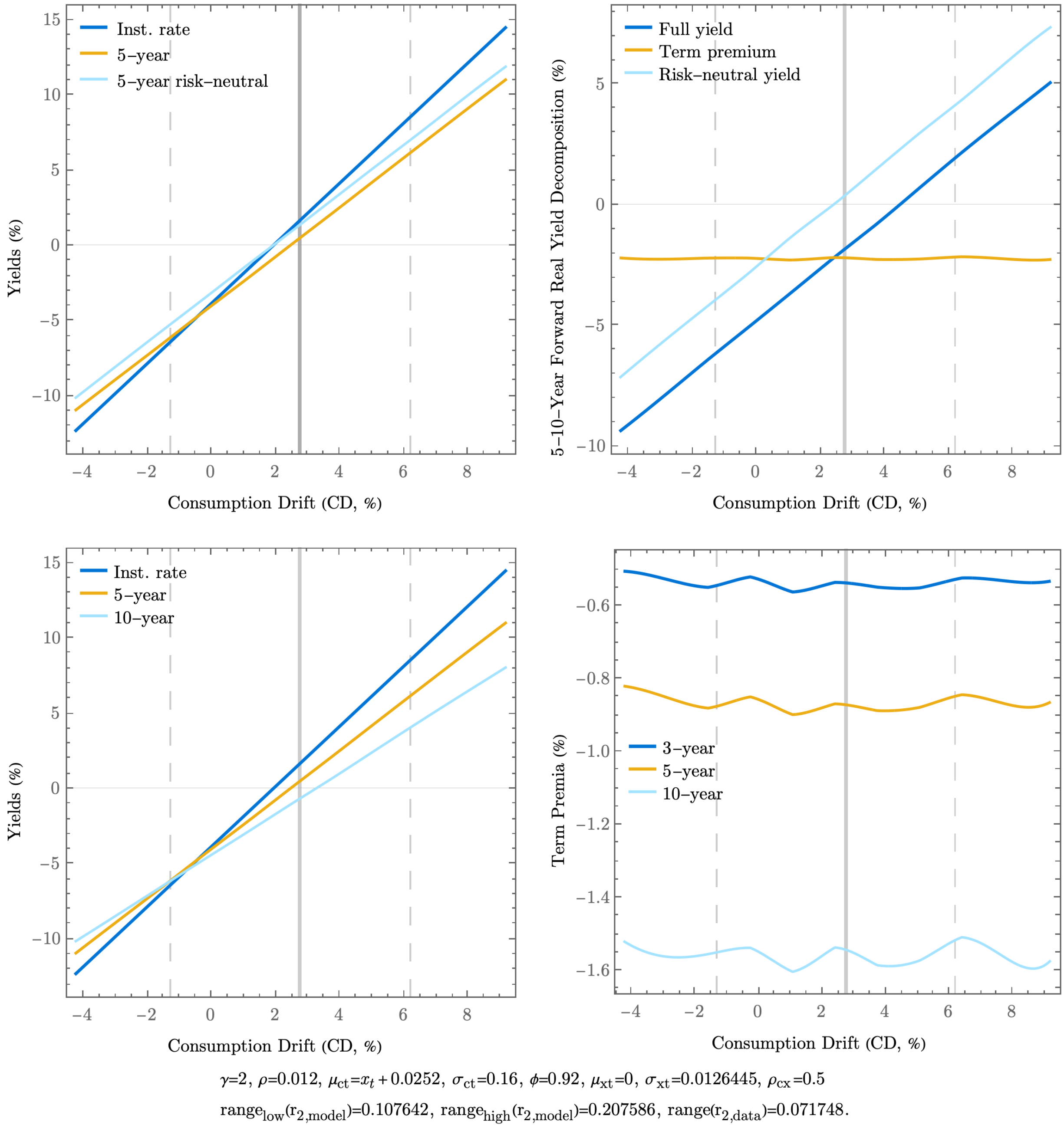
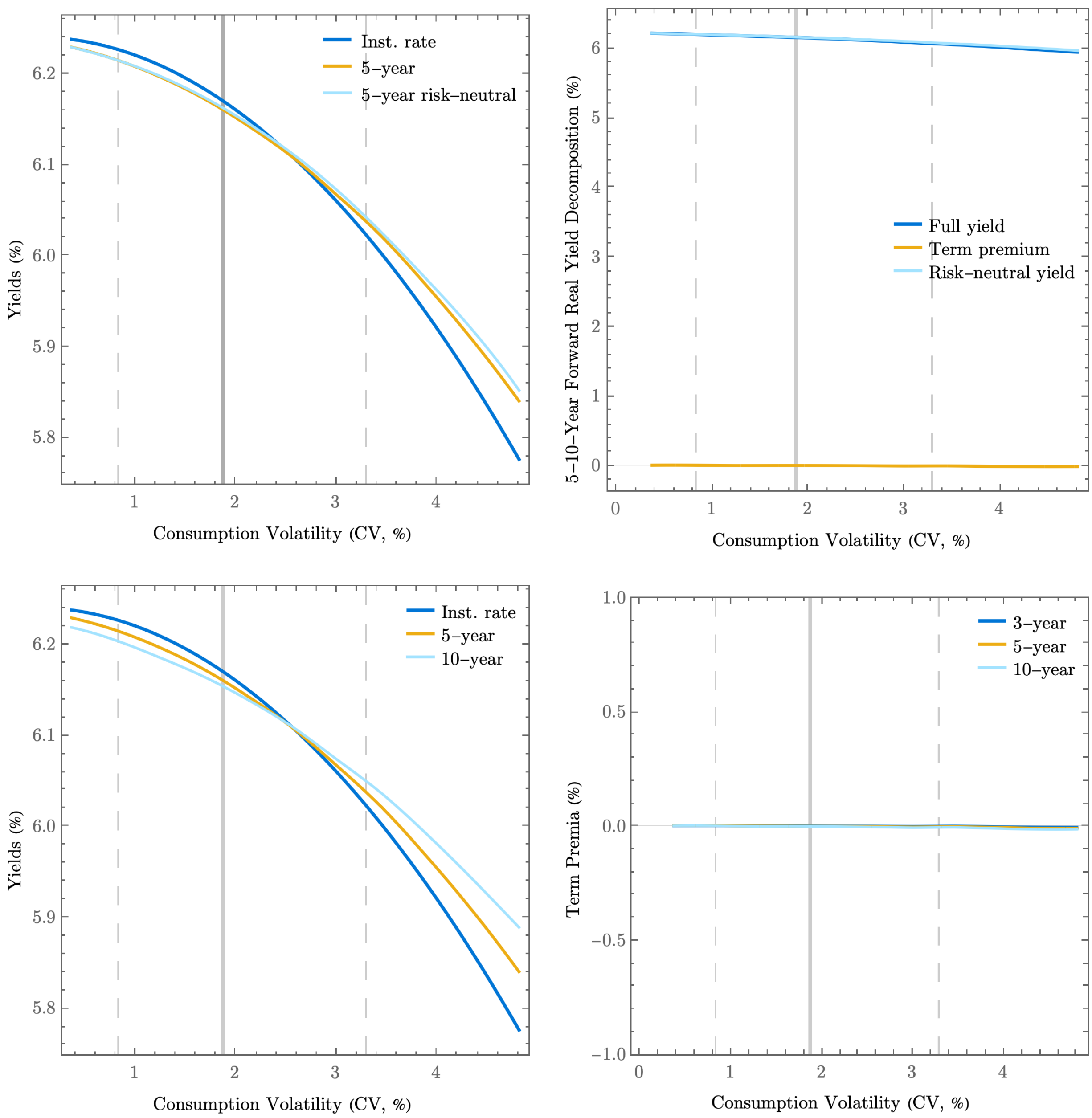


Figure 1.22: Time-varying consumption drift with time-separable utility and high consumption volatility.
See Figure 1.16 for more details about the plots.

(variation overview)

1.F.9 TSU-CV, Calibration used in main paper, Figure 1.3



γ=2, ρ=0.012, μ_{ct}=0.0252, σ_{ct}=0.02 x_t, ϕ=0.92, μ_{xt}=1, σ_{xt}=0.21 $\sqrt{x_t}$, ρ_{cx}=−0.5

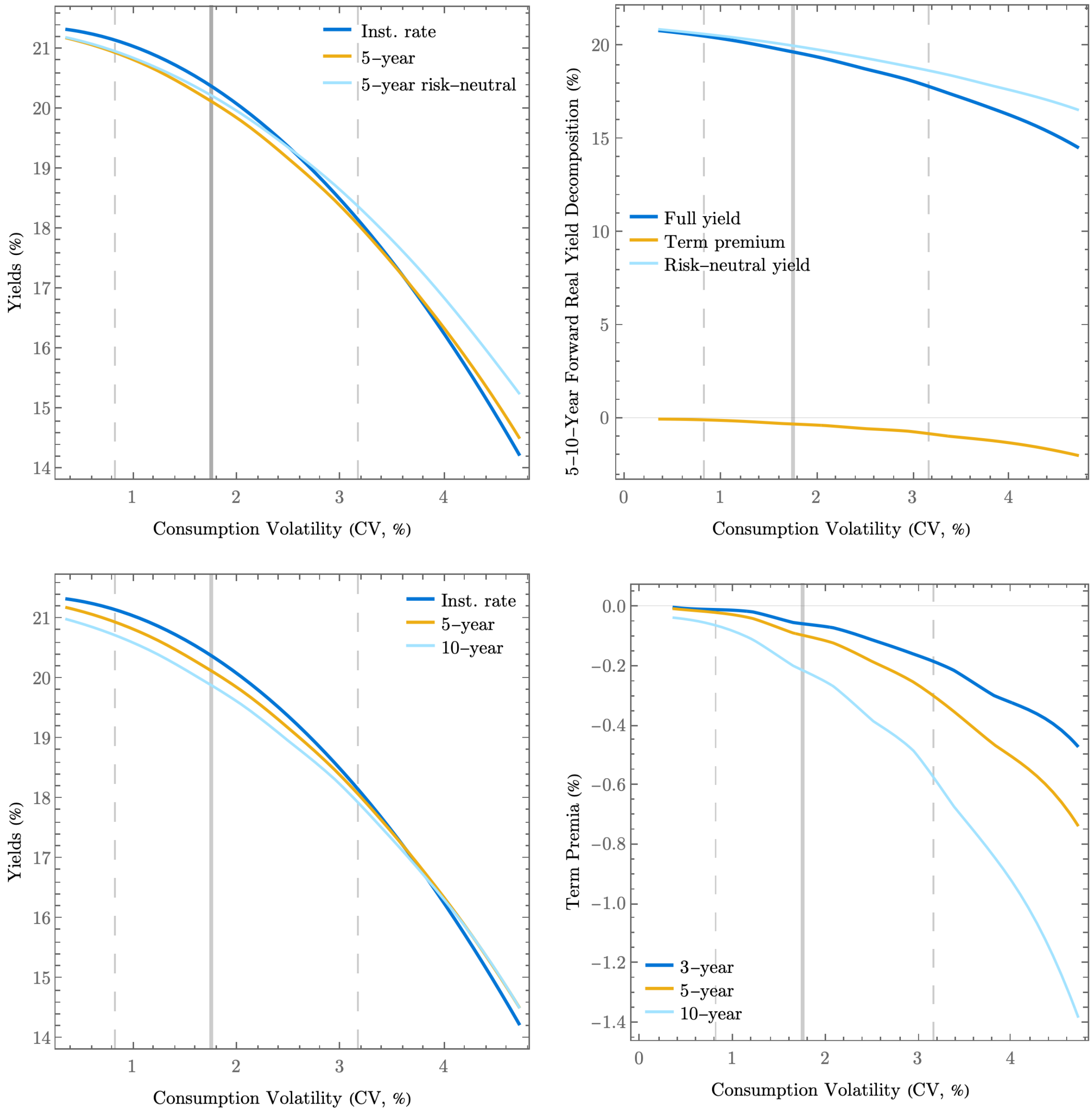
range$_{\mathrm{low}}$($\mathrm{r}_{2,\mathrm{model}}$)=0.00103855, range$_{\mathrm{high}}$($\mathrm{r}_{2,\mathrm{model}}$)=0.00408583, range($\mathrm{r}_{2,\mathrm{data}}$)=0.071748.

Figure 1.23: Time-varying consumption volatility with time-separable utility. See Figure 1.16 for more details about the plots.

(variation overview)

1.F.10 TSU-CV-HRA, $\gamma = 8$

Term premia increased in absolute value but not enough and yields moved very high.



γ=8, ρ=0.012, μ_{ct}=0.0252, σ_{ct}=0.02 x_t, ϕ=0.92, μ_{xt}=1, σ_{xt}=0.21 $\sqrt{x_t}$, ρ_{cx}=−0.5

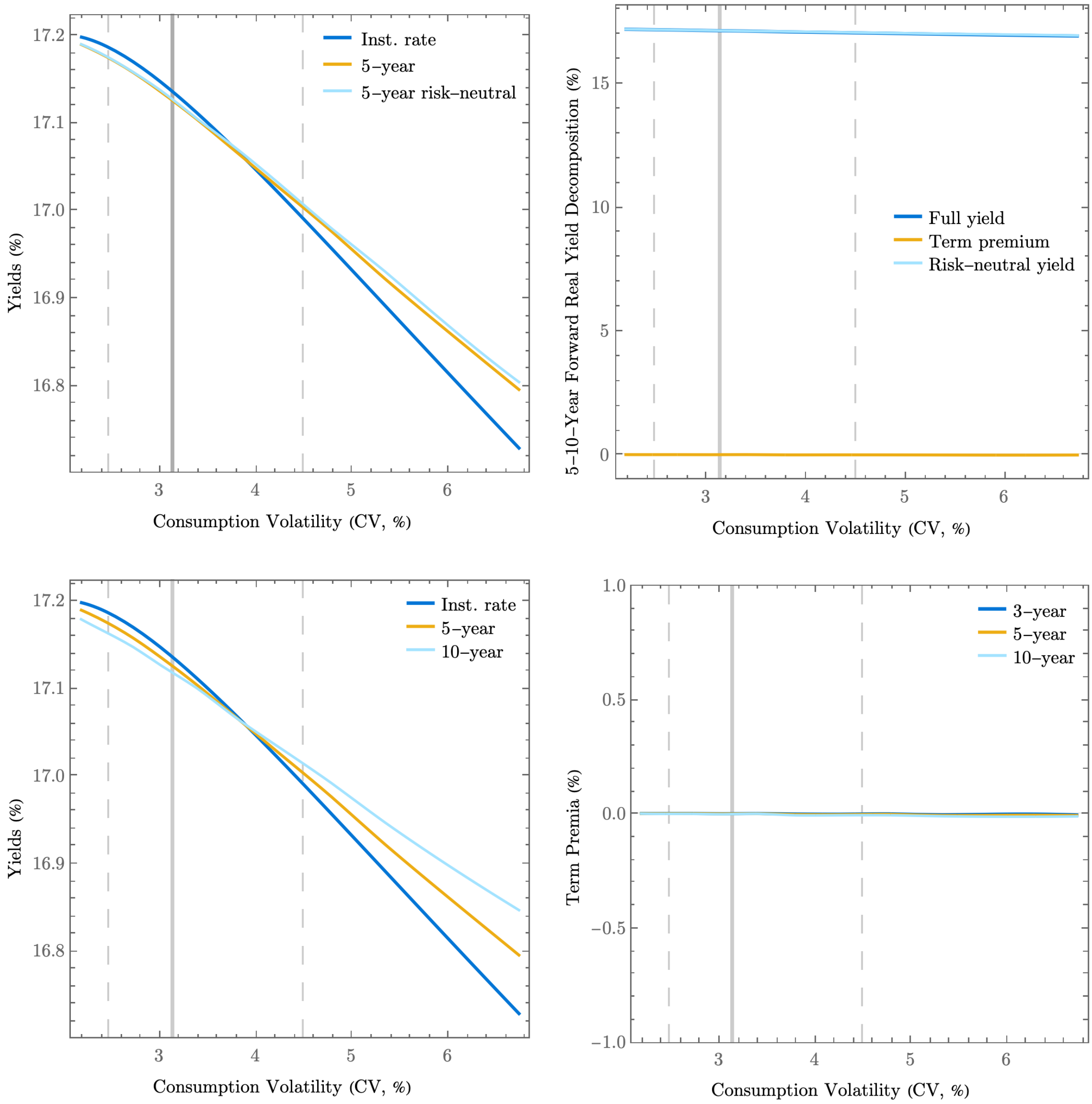
$\mathrm{range}_{\mathrm{low}}(\mathrm{r}_{2,\mathrm{model}})$=0.0157495, $\mathrm{range}_{\mathrm{high}}(\mathrm{r}_{2,\mathrm{model}})$=0.0684834, $\mathrm{range}(\mathrm{r}_{2,\mathrm{data}})$=0.071748.

Figure 1.24: Time-varying consumption volatility with high risk aversion. See Figure 1.16 for more details about the plots.

(variation overview)

1.F.11 TSU-CV-HCD, $\mu_{c0} = 0.08$

Term premia did not change but yields moves implausibly high.



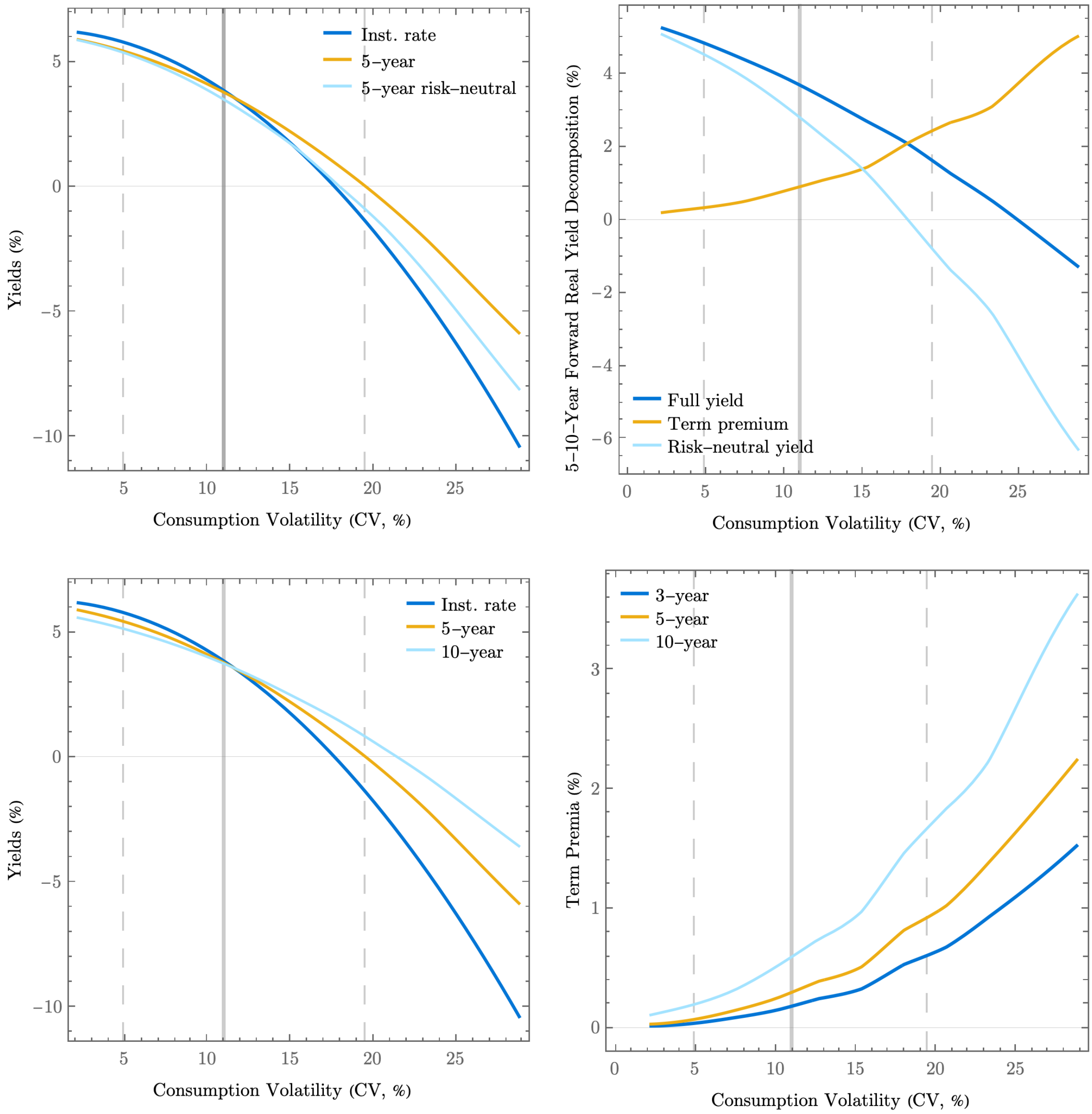
γ=2, ρ=0.012, μ_{ct}=0.08, σ_{ct}=0.02 x_t, ϕ=0.92, μ_{xt}=1, σ_{xt}=0.21 $\sqrt{x_t}$, ρ_{cx}=−0.5

$\text{range}_{low}(r_{2,model})$=0.000984241, $\text{range}_{high}(r_{2,model})$=0.00424656, $\text{range}(r_{2,data})$=0.071748.

Figure 1.25: Time-varying consumption volatility with time-separable utility and high consumption drift.
See Figure 1.16 for more details about the plots.

(variation overview)

1.F.12 TSU-HCV, Calibration used in main paper, Figure 1.6



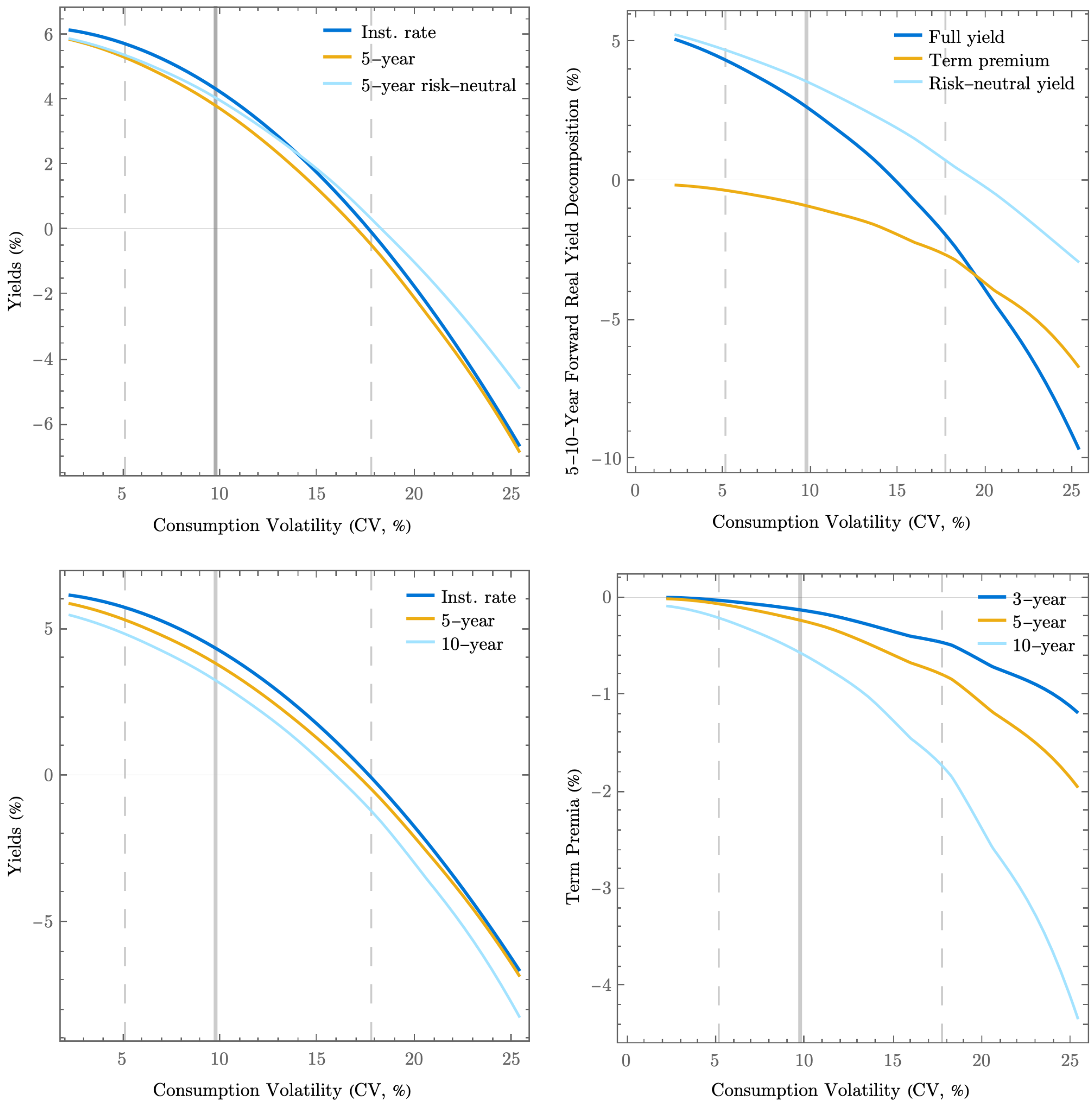
γ=2, ρ=0.012, μ_{ct}=0.0252, σ_{ct}=0.12 x_t, ϕ=0.92, μ_{xt}=1, σ_{xt}=0.21 $\sqrt{x_t}$, ρ_{cx}=0.5

$\text{range}_{\text{low}}(\text{r}_{2,\text{model}})$=0.0374496, $\text{range}_{\text{high}}(\text{r}_{2,\text{model}})$=0.149538, $\text{range}(\text{r}_{2,\text{data}})$=0.071748.

Figure 1.26: Time-varying and high consumption volatility with time-separable utility and positive ρ_{cx}.
See Figure 1.16 for more details about the plots.

(variation overview)

1.F.13 TSU-HCV-NCor, $\rho_{cx} < 0$



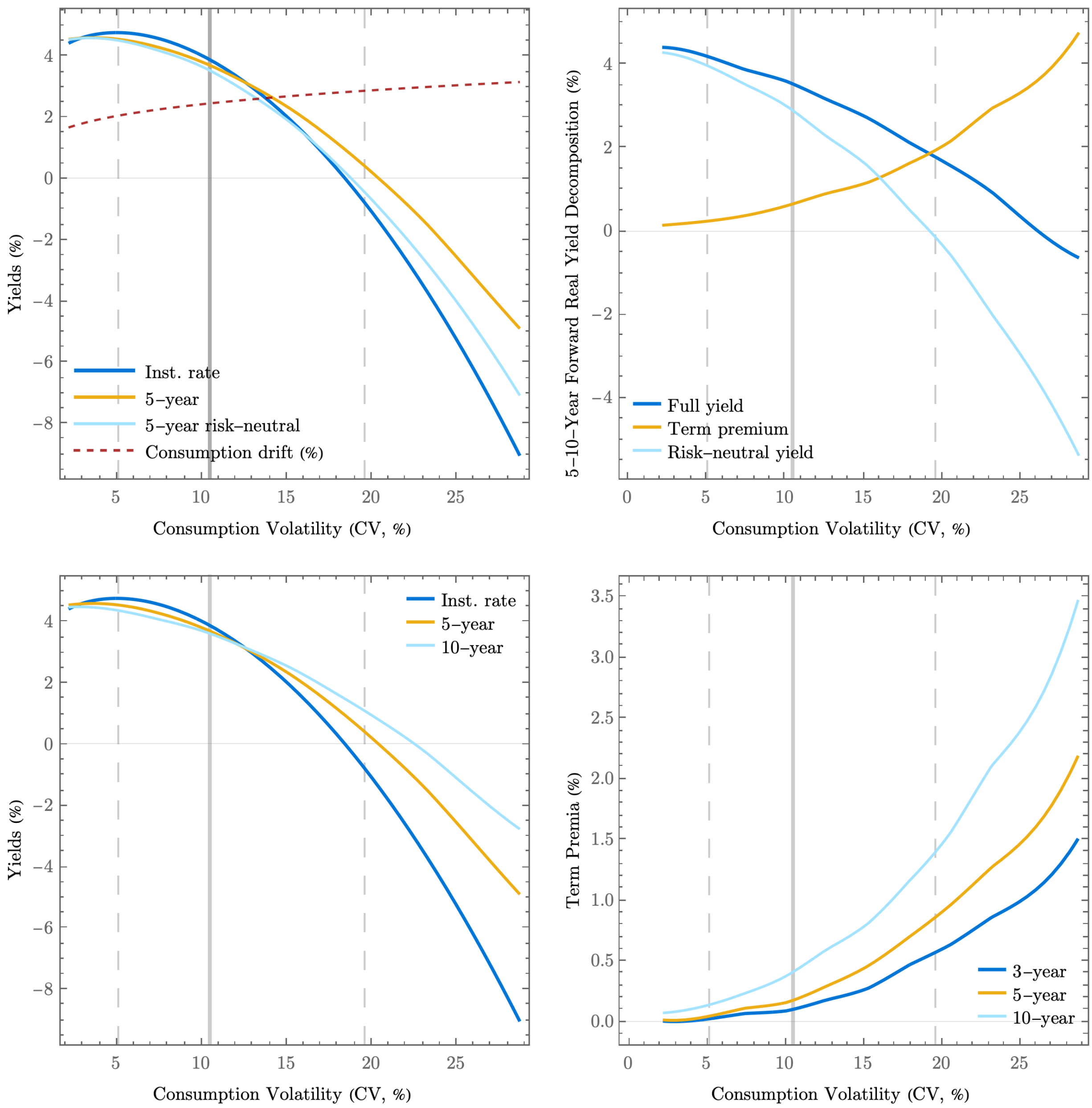
γ=2, ρ=0.012, μ_{ct}=0.0252, σ_{ct}=0.11 x_t, ϕ=0.92, μ_{xt}=1, σ_{xt}=0.2 $\sqrt{x_t}$, ρ_{cx}=−0.5

range$_{\text{low}}$($r_{2,\text{model}}$)=0.0306062, range$_{\text{high}}$($r_{2,\text{model}}$)=0.119326, range($r_{2,\text{data}}$)=0.071748.

Figure 1.27: Time-varying consumption volatility with time-separable utility and negative ρ_{cx}.
See Figure 1.16 for more details about the plots.

(variation overview)

1.F.14 Arb-DP, Calibration used in main paper, Figure 1.6



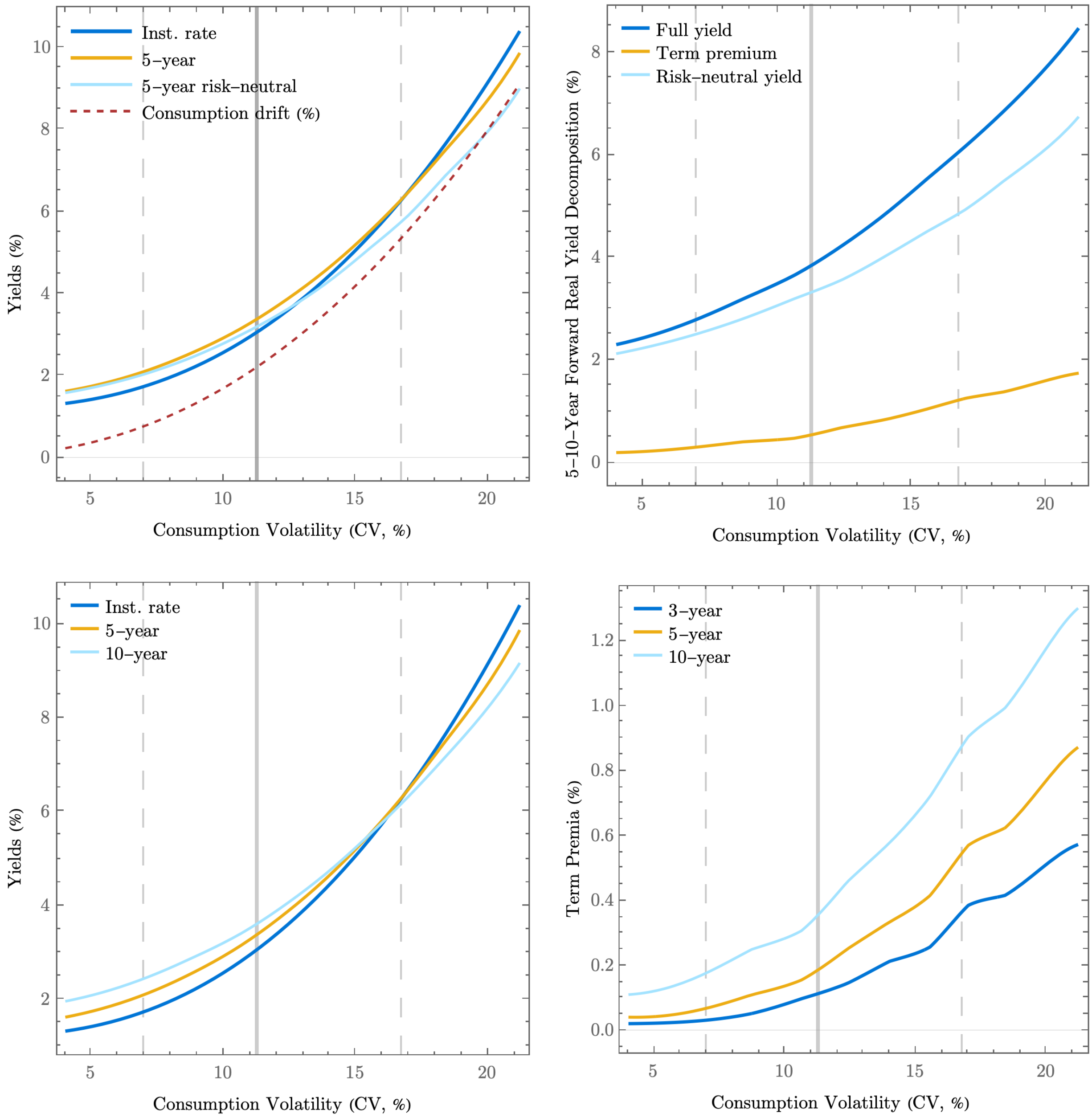
γ=2, ρ=0.012, μ_{ct}=0.0252 $\sqrt[4]{x_t}$, σ_{ct}=0.12 x_t, ϕ=0.92, μ_{xt}=1, σ_{xt}=0.21 $\sqrt{x_t}$, ρ_{cx}=0.5

$\mathrm{range_{low}}(\mathrm{r}_{2,\mathrm{model}})$=0.0222875, $\mathrm{range_{high}}(\mathrm{r}_{2,\mathrm{model}})$=0.132065, $\mathrm{range}(\mathrm{r}_{2,\mathrm{data}})$=0.071748.

Figure 1.28: Both time-varying consumption drift and consumption volatility with time-separable utility, short-term rate decreasing in consumption volatility and positive ρ_{xc}.
See Figure 1.6 for more details about the plots.

(variation overview)

1.F.15 Arb-IN, Calibration used in main paper, Figure 1.6



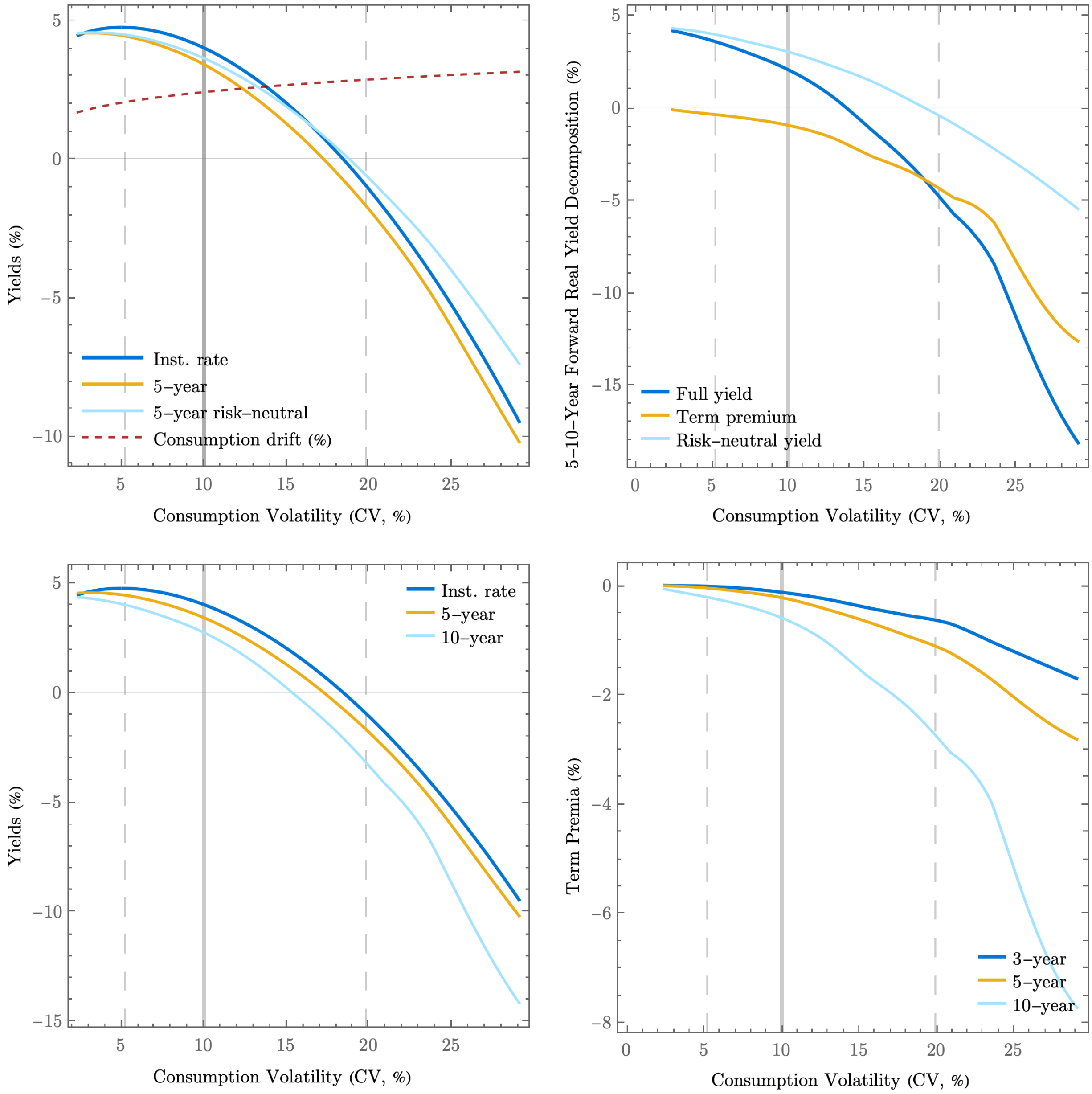
γ=2, ρ=0.012, μ_{ct}=0.0252 $x_t^{3/2}$, σ_{ct}=0.12 $x_t^{2/3}$, ϕ=0.92, μ_{xt}=1, σ_{xt}=0.21 $\sqrt{x_t}$, ρ_{cx}=−0.5

range$_{\text{low}}$(r$_{\text{2,model}}$)=0.0258998, range$_{\text{high}}$(r$_{\text{2,model}}$)=0.0818517, range(r$_{\text{2,data}}$)=dataRange.

Figure 1.29: Both time-varying consumption drift and consumption volatility with time-separable utility, short-term rate increasing in consumption volatility and negative ρ_{xc}.
See Figure 1.6 for more details about the plots.

(variation overview)

1.F.16 Arb-DN



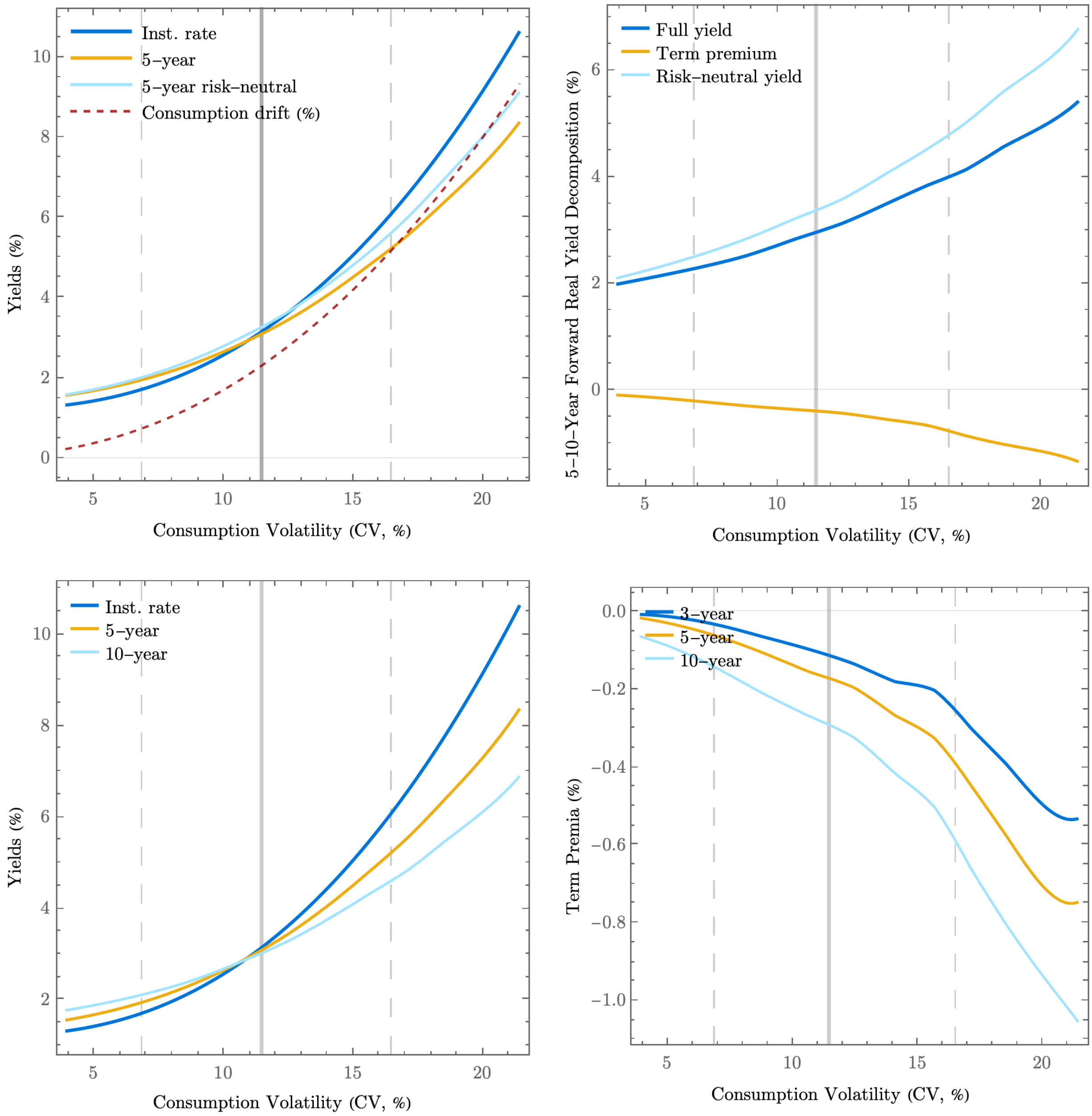
$\gamma=2$, $\rho=0.012$, $\mu_{\text{ct}}=0.0252\sqrt[4]{x_t}$, $\sigma_{\text{ct}}=0.12\,x_t$, $\phi=0.92$, $\mu_{\text{xt}}=1$, $\sigma_{\text{xt}}=0.21\sqrt{x_t}$, $\rho_{\text{cx}}=-0.5$

$\text{range}_{\text{low}}(\text{r}_{2,\text{model}})=0.0224998$, $\text{range}_{\text{high}}(\text{r}_{2,\text{model}})=0.134553$, $\text{range}(\text{r}_{2,\text{data}})=0.071748$.

Figure 1.30: Both time-varying consumption drift and consumption volatility with time-separable utility, short-term rate decreasing in consumption volatility and negative ρ_{xc}.

(variation overview)

1.F.17 Arb-IP



γ=2, ρ=0.012, $\mu_{\rm ct}$=0.0252 $x_t^{3/2}$, $\sigma_{\rm ct}$=0.12 $x_t^{2/3}$, ϕ=0.92, $\mu_{\rm xt}$=1, $\sigma_{\rm xt}$=0.21 $\sqrt{x_t}$, $\rho_{\rm cx}$=0.5

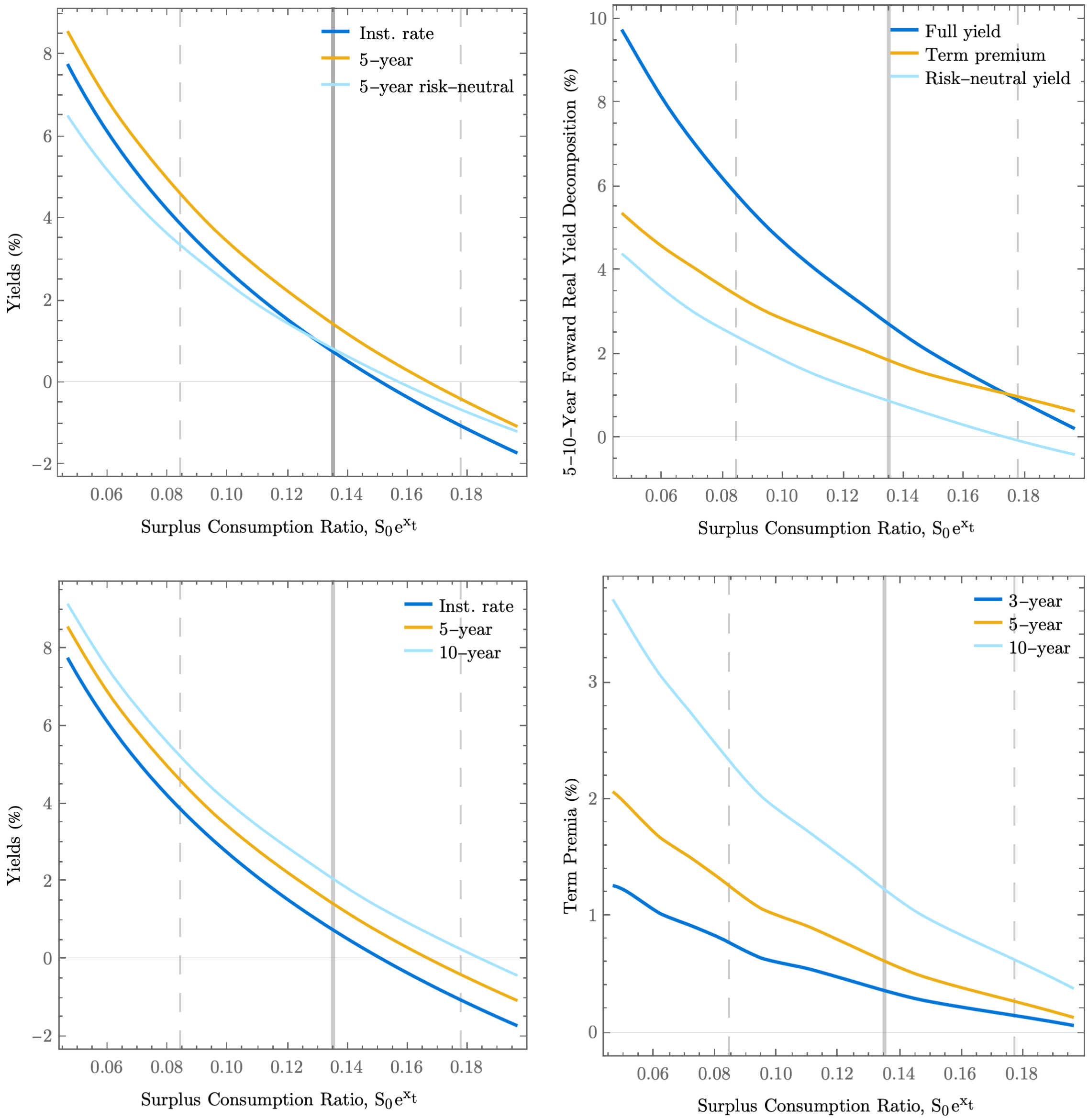
range$_{\rm low}$(r$_{\rm 2,model}$)=0.0263095, range$_{\rm high}$(r$_{\rm 2,model}$)=0.0794248, range(r$_{\rm 2,data}$)=0.071748.

Figure 1.31: Both time-varying consumption drift and consumption volatility with time-separable utility, short-term rate increasing in consumption volatility and positive ρ_{xc}.
See Figure 1.6 for more details about the plots.

(variation overview)

1.F.18 TSU-Habit

1.F.19 Calibration used in main paper, Figure 1.3



$$\gamma=2,\ \rho=0.012,\ \mu_{ct}=0.0252,\ \sigma_{ct}=0.02,\ \phi=0.92,\ \mu_{xt}=0,\ \sigma_{xt}=0.02\ \mathrm{If}\left[x_t > 0.492061,\ 0,\ \frac{\sqrt{1-2\,x_t}}{0.02\sqrt{\frac{2}{-\frac{0.066}{2}-\log(0.92)}}}-1\right],\ \rho_{cx}=1$$

$\mathrm{range}_{\mathrm{low}}(r_{2,\mathrm{model}})=0.0323706$, $\mathrm{range}_{\mathrm{high}}(r_{2,\mathrm{model}})=0.0831215$, $\mathrm{range}(r_{2,\mathrm{data}})=\mathrm{dataRange}$.

Figure 1.32: Time-varying extranl habit with time-separable utility. See Figure 1.16 for more details about the plots.

(variation overview)

1.F.20 TSU-Habit-L.b, b=0.033

Term premia did not change but yields became flatter. This is noteworthy because in Abrahams et al. (2016) the forward term premia are big while the forward risk-neutral yields are small in absolute value.

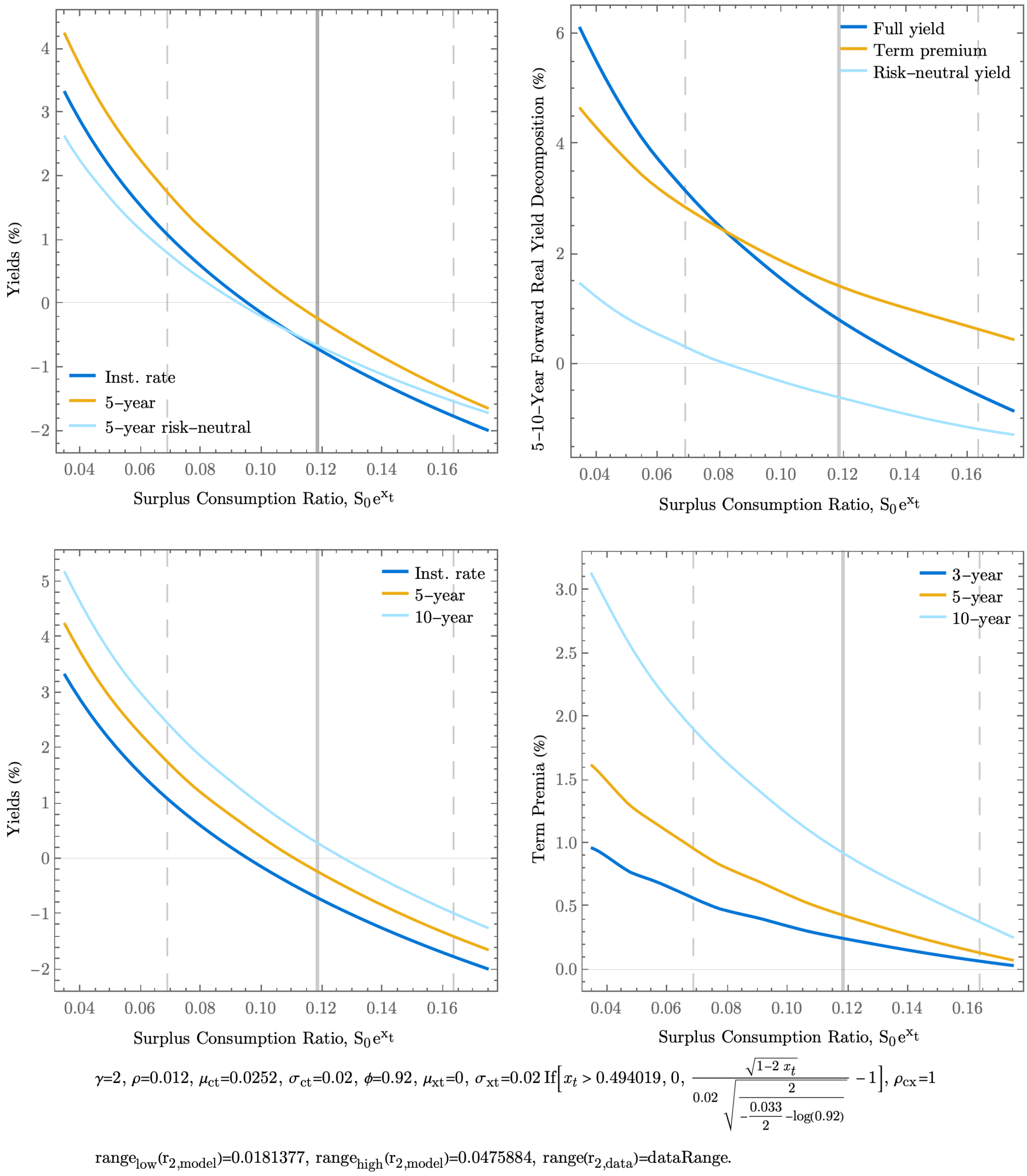
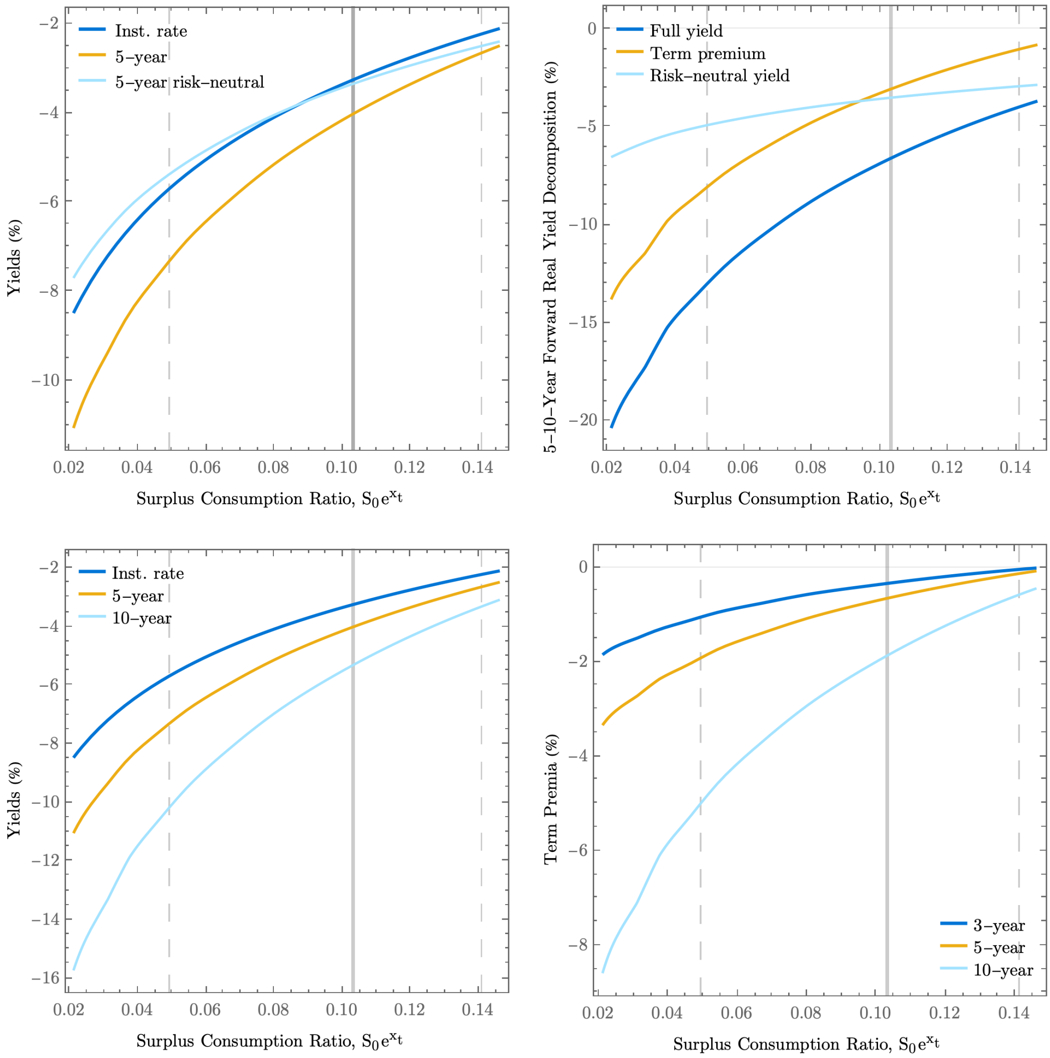


Figure 1.33: Time-varying extranl habit with time-separable utility. See Figure 1.16 for more details about the plots.

(variation overview)

1.F.21 TSU-Habit-Neg.b, b=-0.033

The short-term rate is now pro-cyclical and term premia are negative.



γ=2, ρ=0.012, μ_{ct}=0.0252, σ_{ct}=0.02, ϕ=0.92, μ_{xt}=0, σ_{xt}=0.02 If$\left[x_t > 0.495995,\ 0,\ \frac{\sqrt{1-2\,x_t}}{0.02\sqrt{\frac{2}{-\frac{0.033}{2}-\log(0.92)}}} - 1\right]$, ρ_{cx}=1

$\text{range}_{\text{low}}(r_{2,\text{model}})$=0.0191972, $\text{range}_{\text{high}}(r_{2,\text{model}})$=0.0644309, $\text{range}(r_{2,\text{data}})$=dataRange.

Figure 1.34: Time-varying extranl habit with time-separable utility and $b < 0$. See Figure 1.16 for more details about the plots.

(variation overview)

1.F.22 TSU-Habit-ConstantSV, $\sigma_{xt} = \lambda(0)\sigma_{c0}$

The term premia are now constant. This is partially contrary to the spirit of Campbell and Cochrane (1999), because the surplus consumption ratio does not get more volatile in bad states of the economy, but it illustrates how the heteroskedasticity is crucial for the generation of variable term premia.

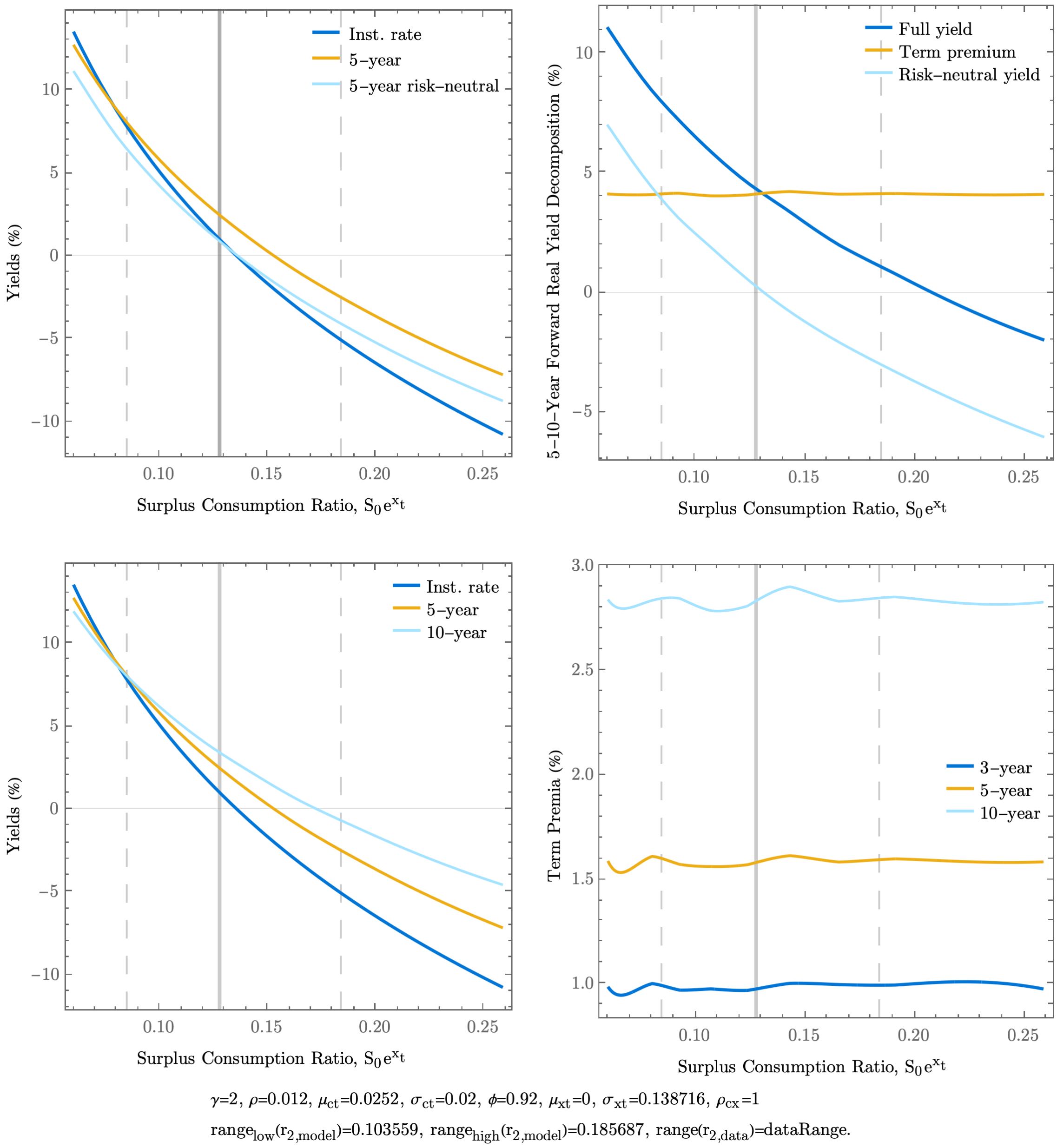
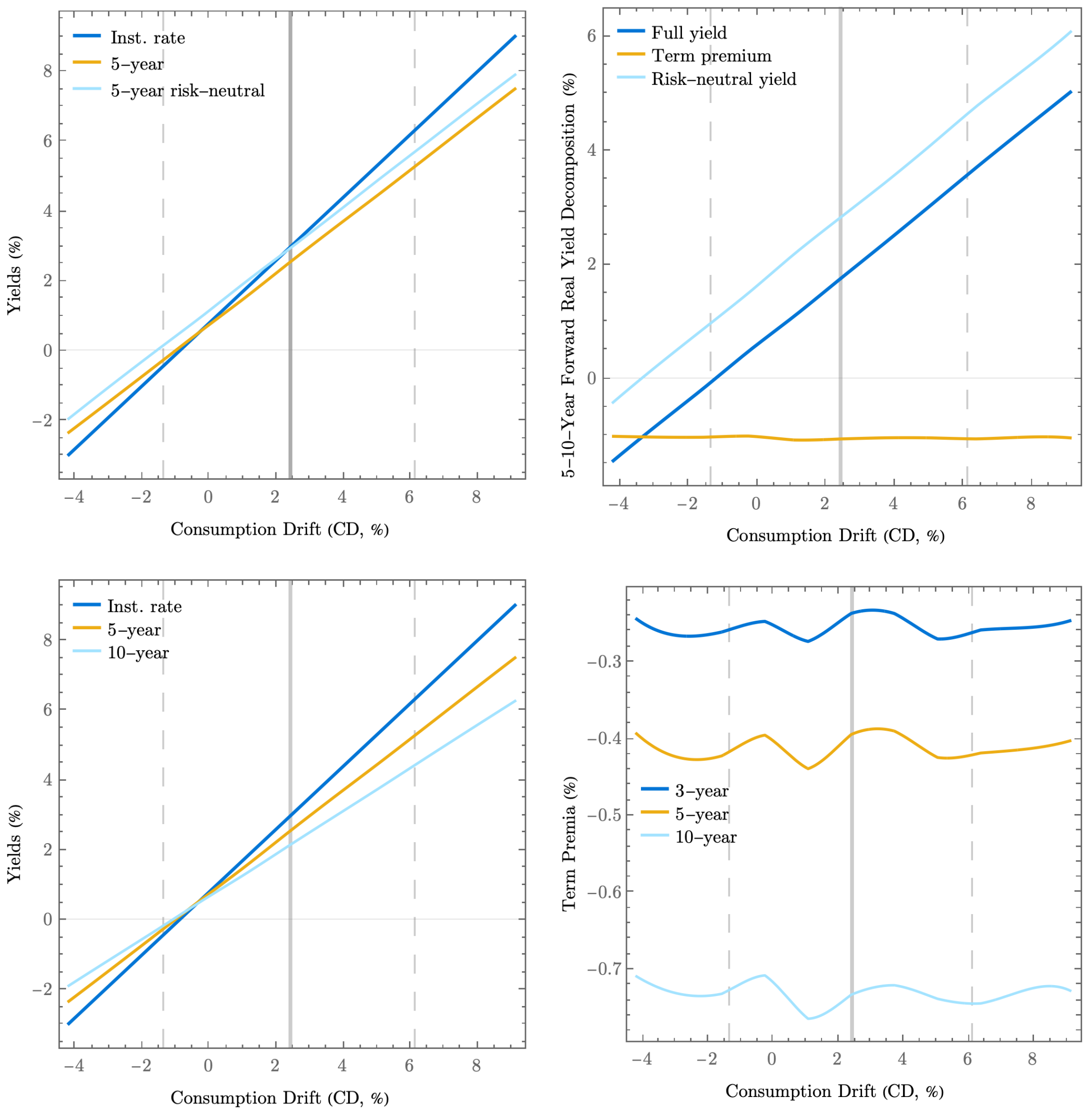


Figure 1.35: Time-varying extranl habit with time-separable utility and constant state variable volatility.
See Figure 1.16 for more details about the plots.

(variation overview)

1.F.23 RU-CD, Calibration used in main paper, Figure 1.5



$\gamma=2$, $\rho=0.012$, $\mu_{\text{ct}}=x_t+0.0252$, $\sigma_{\text{ct}}=0.02$, $\phi=0.92$, $\mu_{\text{xt}}=0$, $\sigma_{\text{xt}}=0.0126445$, $\rho_{\text{cx}}=0.5$, $\psi=1.11$, approx. order=8

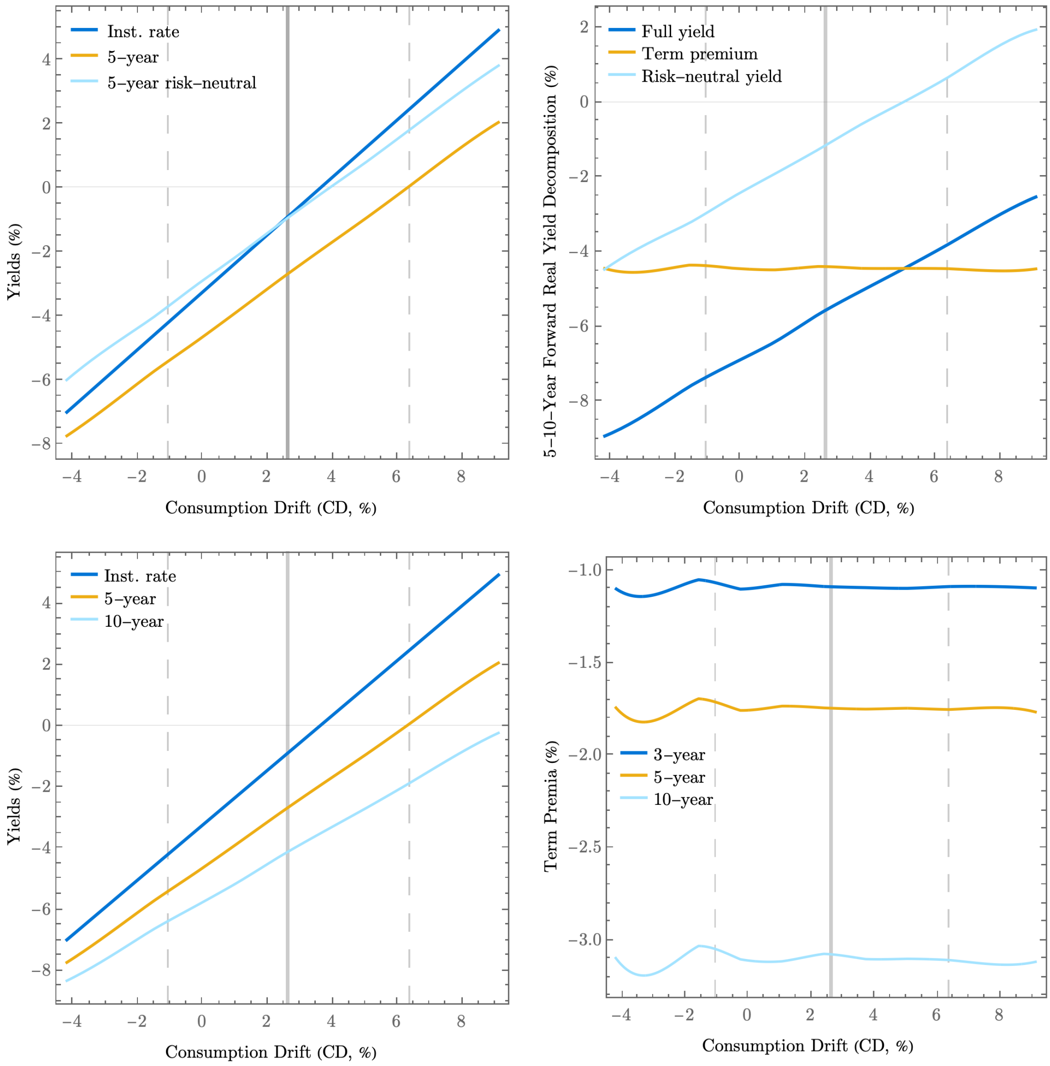
$\text{range}_{\text{low}}(\text{r}_{2,\text{model}})=0.050672$, $\text{range}_{\text{high}}(\text{r}_{2,\text{model}})=0.0927875$, $\text{range}(\text{r}_{2,\text{data}})=\text{dataRange}$.

Figure 1.36: Time-varying consumption drift with recursive utility. See Figure 1.16 for more details about the plots.

(variation overview)

1.F.24 RU-CD-HRA, $\gamma = 6$

Term premia stay constant and negative but it becomes significantly larger in absolute value. In this paper I do not consider a time-varying γ parameter, but this suggests that a time-varying risk aversion would be able to produce time-varying term premia. The habit model essentially provides a similar mechanism.



γ=6, ρ=0.012, μ_{ct}=x_t + 0.0252, σ_{ct}=0.02, ϕ=0.92, μ_{xt}=0, σ_{xt}=0.0126445, ρ_{cx}=0.5, ψ=1.11, approx. order=8

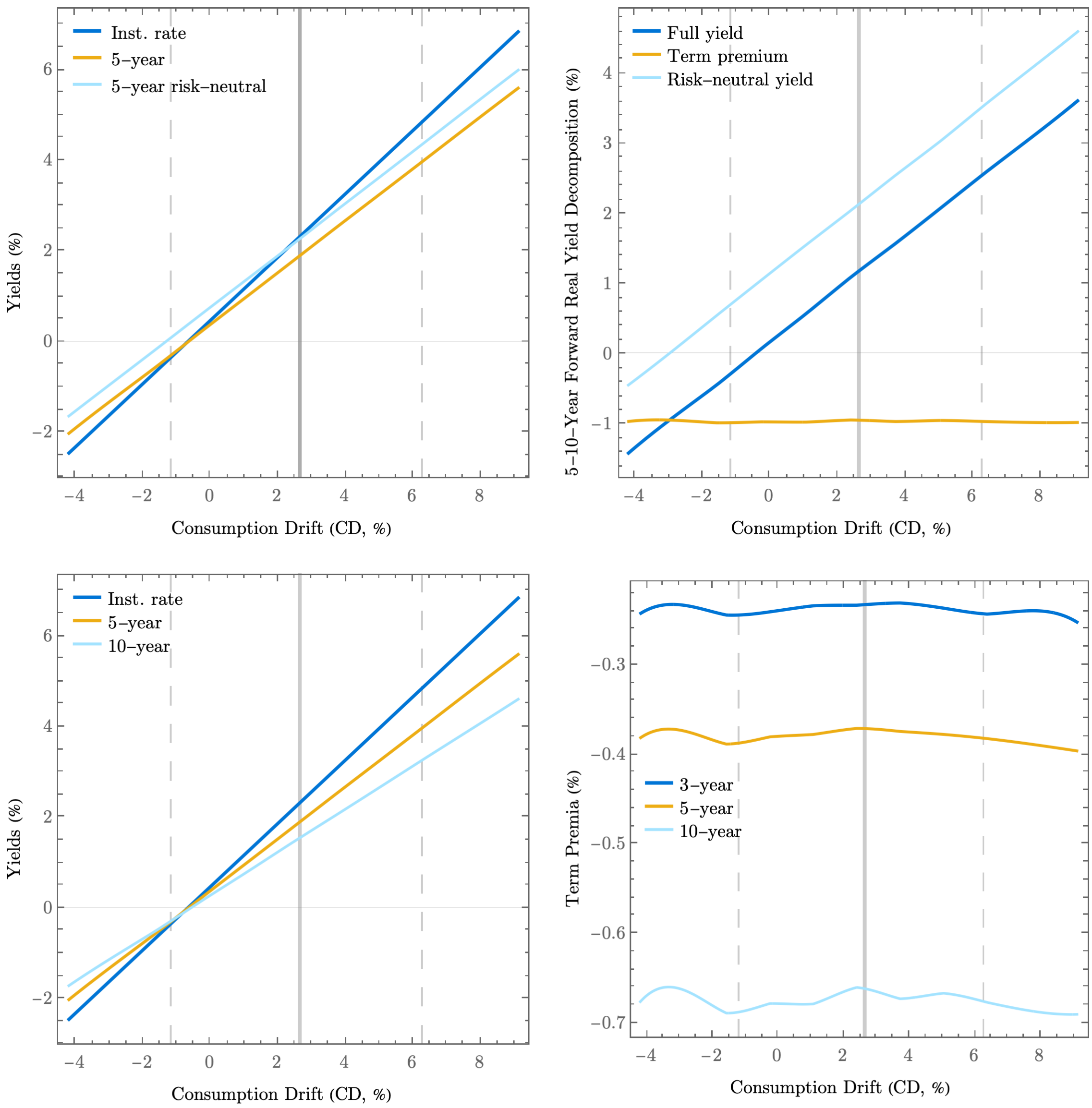
$\mathrm{range_{low}}(\mathrm{r_{2,model}})$=0.0491264, $\mathrm{range_{high}}(\mathrm{r_{2,model}})$=0.0893581, $\mathrm{range}(\mathrm{r_{2,data}})$=0.071748.

Figure 1.37: Time-varying consumption drift with recursive utility and high risk aversion.
See Figure 1.5 for more details about the plots.

(variation overview)

1.F.25 RU-CD-HIES, $\psi = 1.43$

Term premia do not seem to change significantly. The range of the short rate increases.



γ=2, ρ=0.012, $\mu_{\mathrm{ct}}=x_t+0.0252$, σ_{ct}=0.02, ϕ=0.92, μ_{xt}=0, σ_{xt}=0.0126445, ρ_{cx}=0.5, ψ=1.43, approx. order=8

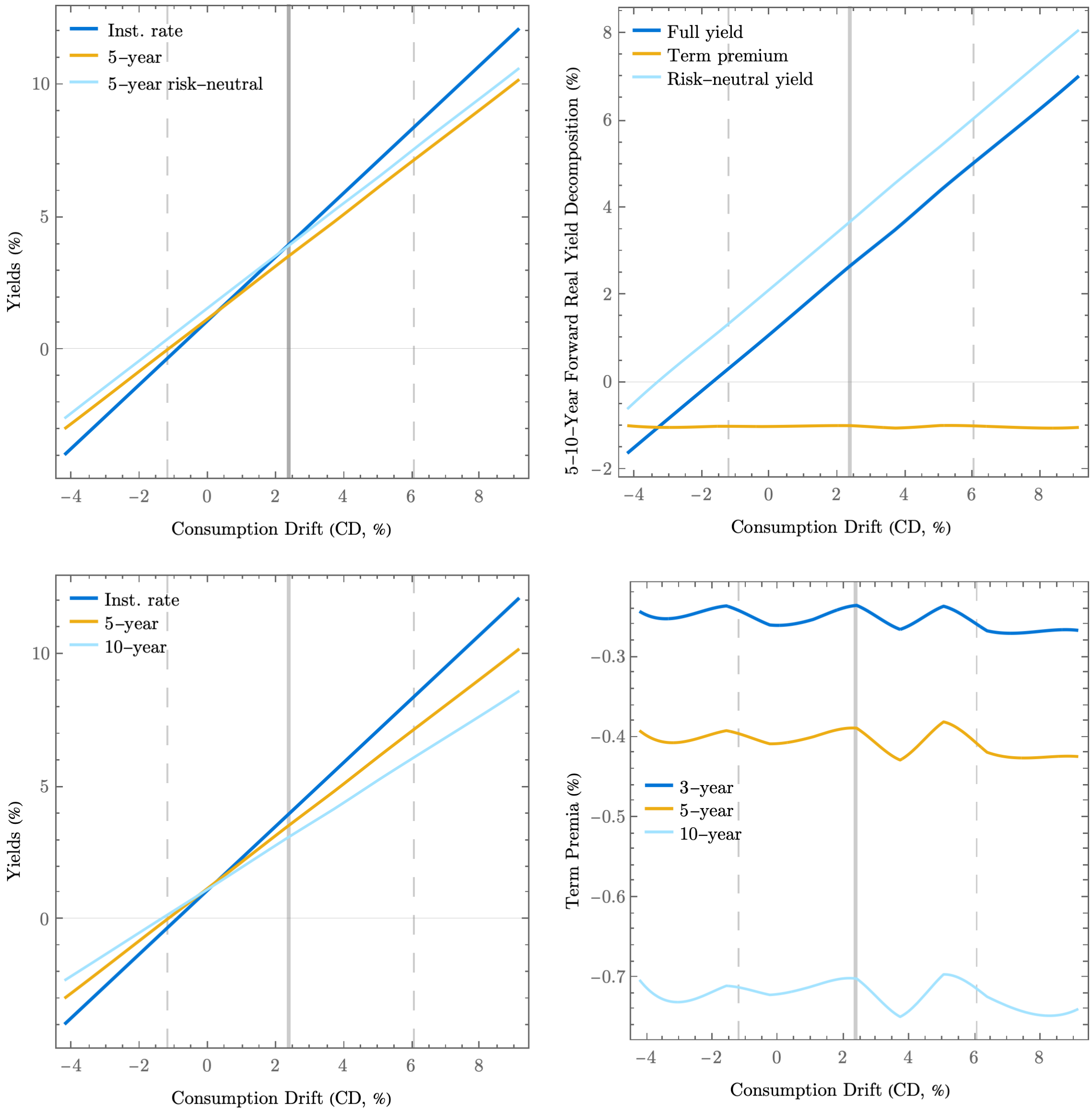
$\mathrm{range}_{\mathrm{low}}(\mathrm{r}_{2,\mathrm{model}})$=0.037872, $\mathrm{range}_{\mathrm{high}}(\mathrm{r}_{2,\mathrm{model}})$=0.071687, $\mathrm{range}(\mathrm{r}_{2,\mathrm{data}})$=0.071748.

Figure 1.38: Time-varying consumption drift with high intertemporal elasticity of substitution.
See Figure 1.16 for more details about the plots.

(variation overview)

1.F.26 RU-CD-LIES, $\psi = 0.83$

Term premia do not seem to change significantly. Curiously the range of the short rate increases again.



γ=2, ρ=0.012, μ_{ct}=x_t+0.0252, σ_{ct}=0.02, ϕ=0.92, μ_{xt}=0, σ_{xt}=0.0126445, ρ_{cx}=0.5, ψ=0.83, approx. order=11

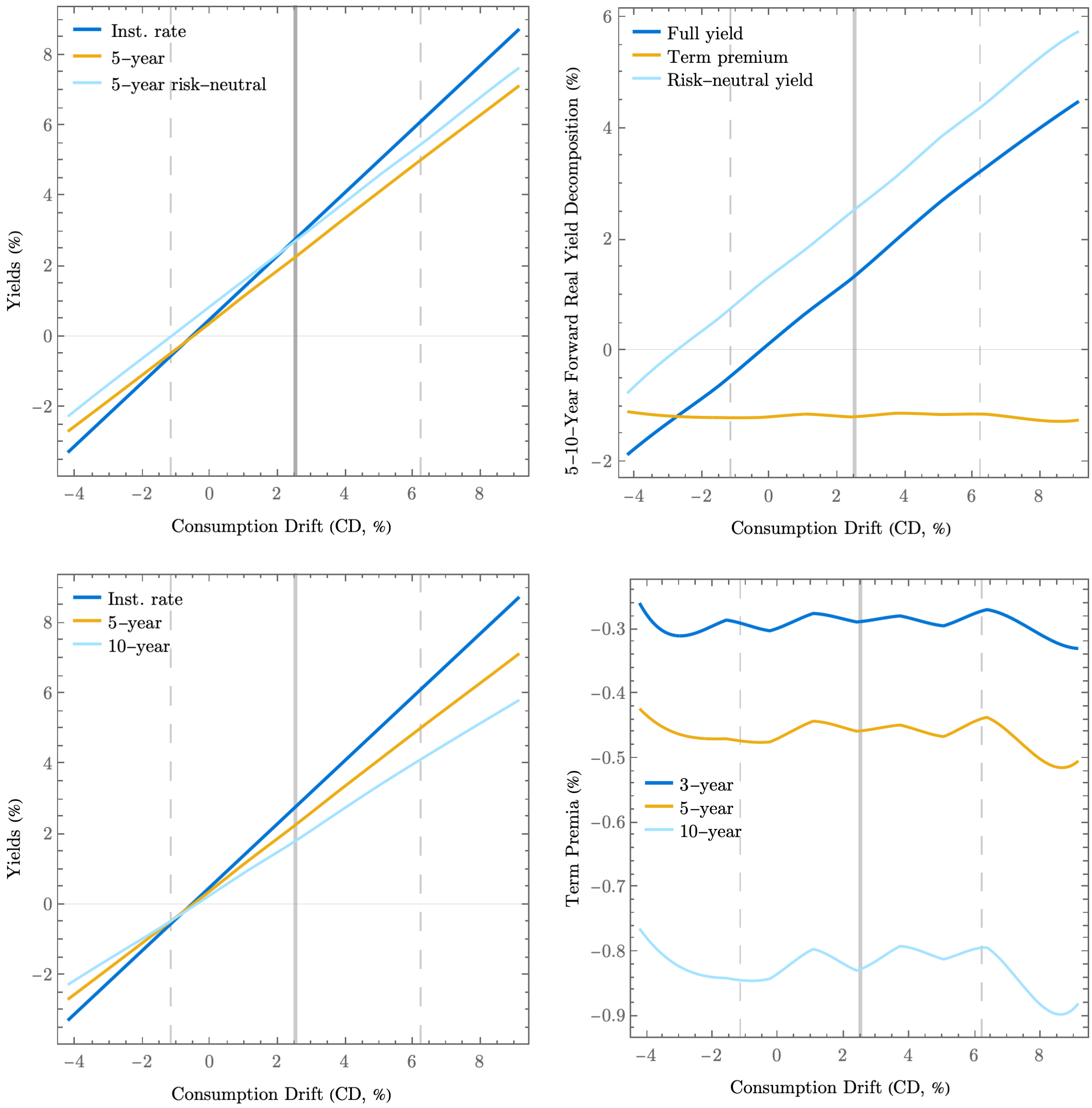
$\mathrm{range_{low}}(\mathrm{r_{2,model}})$=0.0651376, $\mathrm{range_{high}}(\mathrm{r_{2,model}})$=0.121441, $\mathrm{range}(\mathrm{r_{2,data}})$=0.071748.

Figure 1.39: Time-varying consumption drift with recursive utility with low intertemporal elasticity of substitution.
See Figure 1.16 for more details about the plots.

(variation overview)

1.F.27 RU-CD-HCor, $\rho_{cx} = 1$

Term premia increase in absolute value but do not double in size as did the correlation parameter.



γ=2, ρ=0.012, μ_{ct}=x_t + 0.0252, σ_{ct}=0.02, ϕ=0.92, μ_{xt}=0, σ_{xt}=0.0126445, ρ_{cx}=1, ψ=1.11, approx. order=8

$\mathrm{range}_{\mathrm{low}}(\mathrm{r}_{2,\mathrm{model}})$=0.0506132, $\mathrm{range}_{\mathrm{high}}(\mathrm{r}_{2,\mathrm{model}})$=0.0914971, $\mathrm{range}(\mathrm{r}_{2,\mathrm{data}})$=0.071748.

Figure 1.40: Time-varying consumption drift with recursive utility and high ρ_{cx}. See Figure 1.16 for more details about the plots.

(variation overview)

1.F.28 RU-CD-NCor, $\rho_{cx} < 1$

Term premia increase but they remain negative as in recursive utility term premia are dominated by the term not including ρ_{cx}.

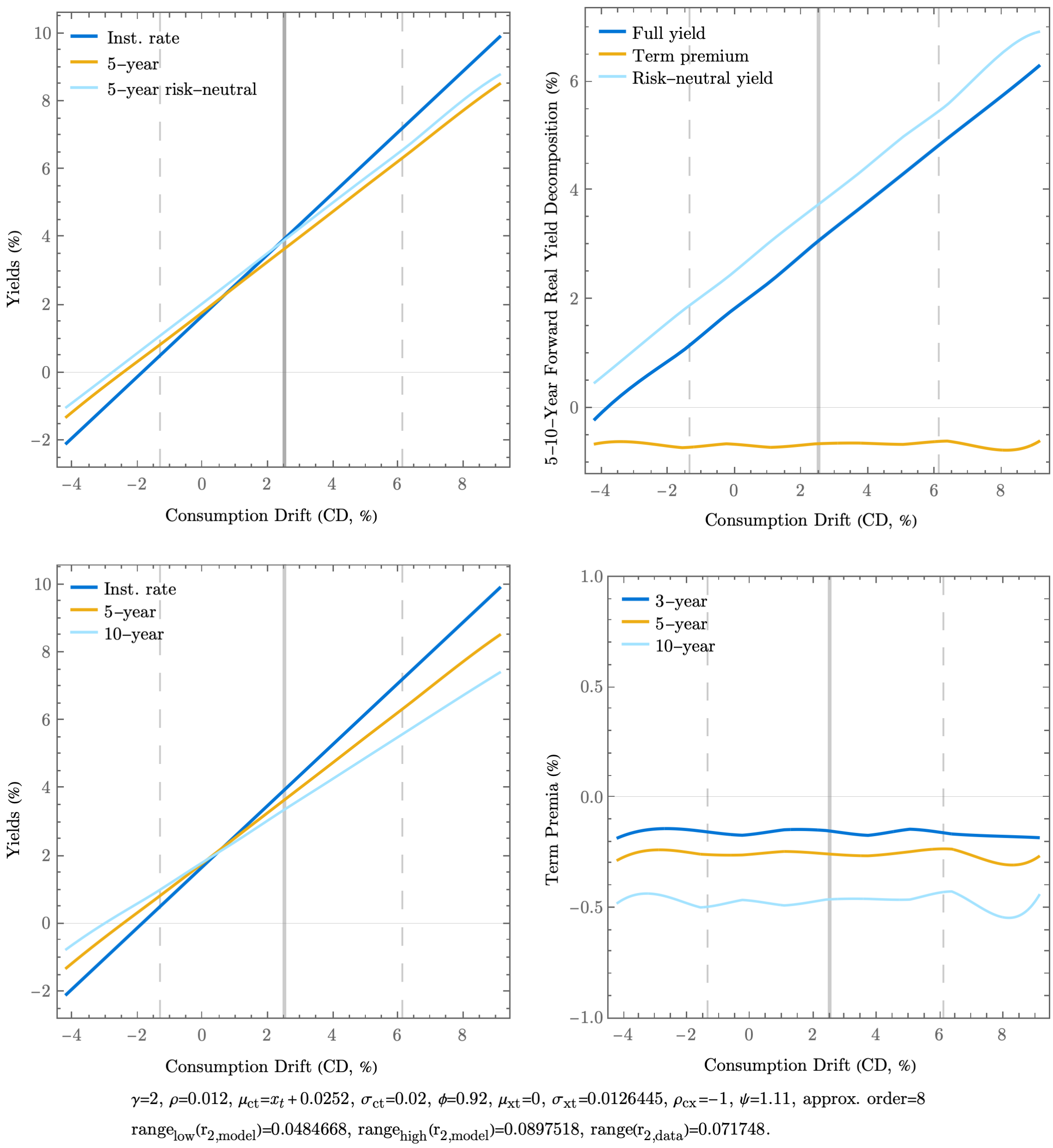


Figure 1.41: Time-varying consumption drift with recursive utility and negative ρ_{cx}.
See Figure 1.16 for more details about the plots.

(variation overview)

1.F.29 RU-HCD, $\mu_{x0} = 0.05$

Term premia do not change, but yields increase.

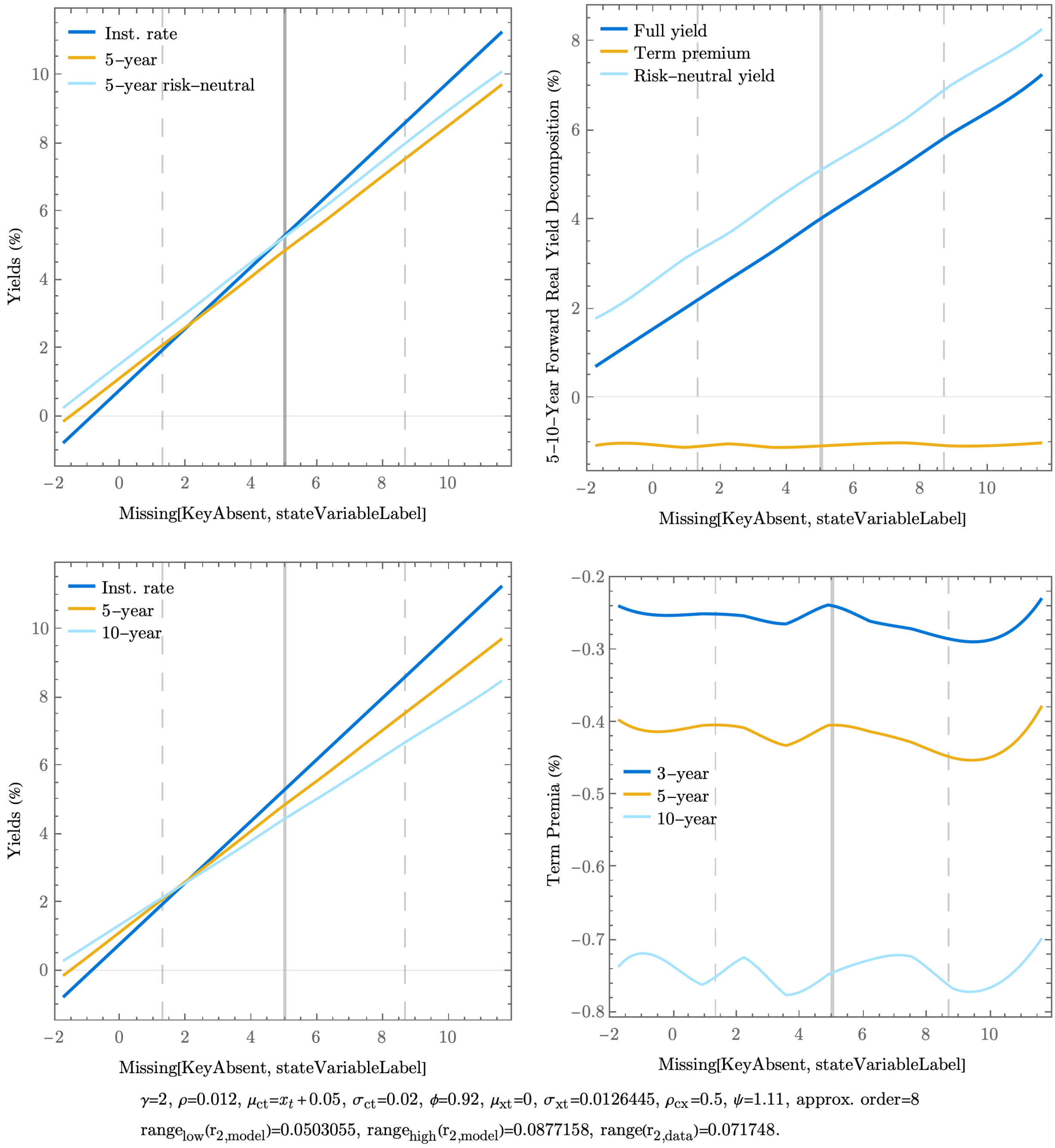
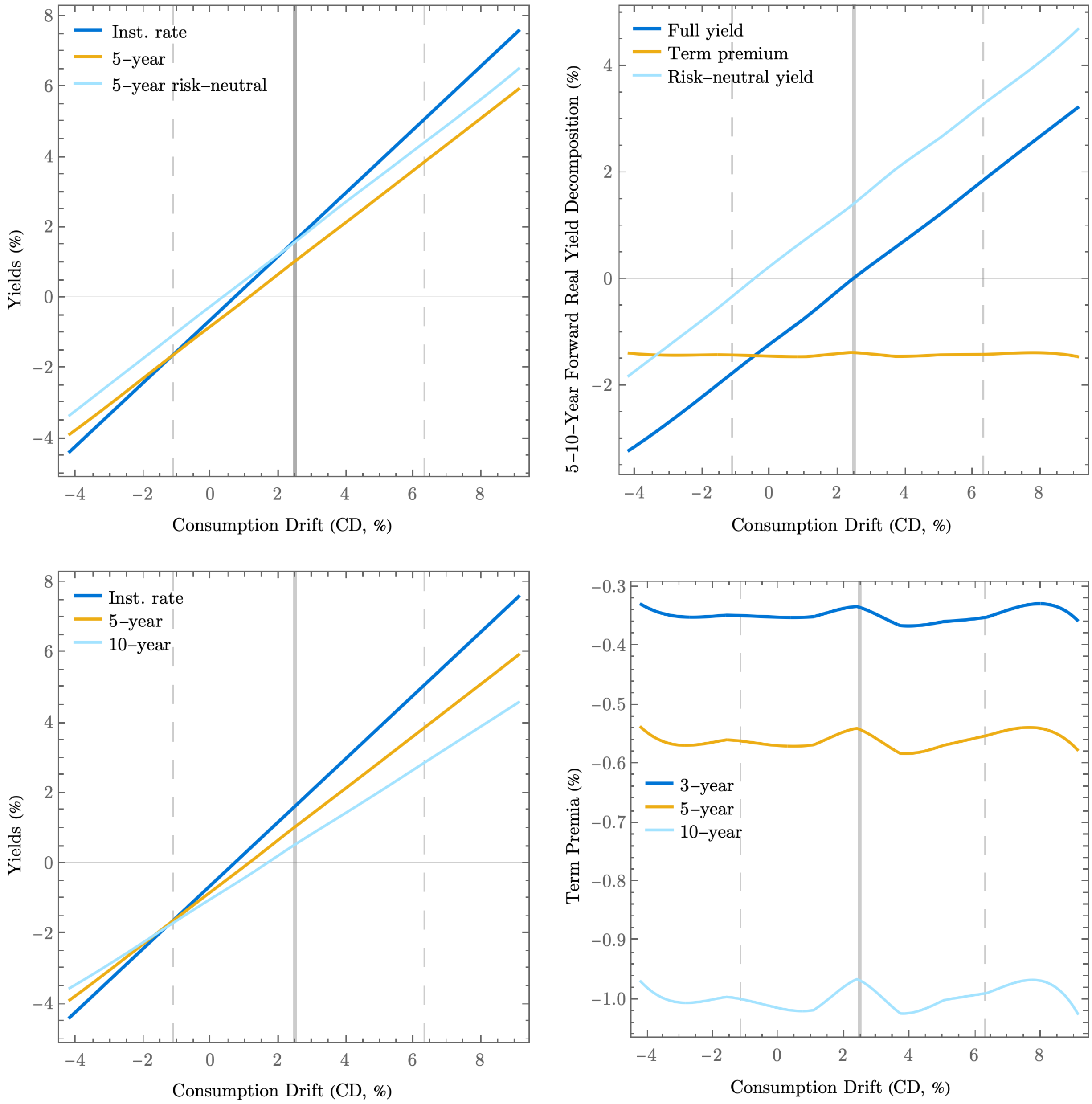


Figure 1.42: Time-varying and high consumption drift with recursive utility. See Figure 1.16 for more details about the plots.

(variation overview)

1.F.30 RU-CD-HCV, $\sigma_{ct} = 0.08$

Term premia do not change, but yields decrease.



γ=2, ρ=0.012, $\mu_{\mathrm{ct}}=x_t+0.0252$, σ_{ct}=0.08, ϕ=0.92, μ_{xt}=0, σ_{xt}=0.0126445, ρ_{cx}=0.5, ψ=1.11, approx. order=8

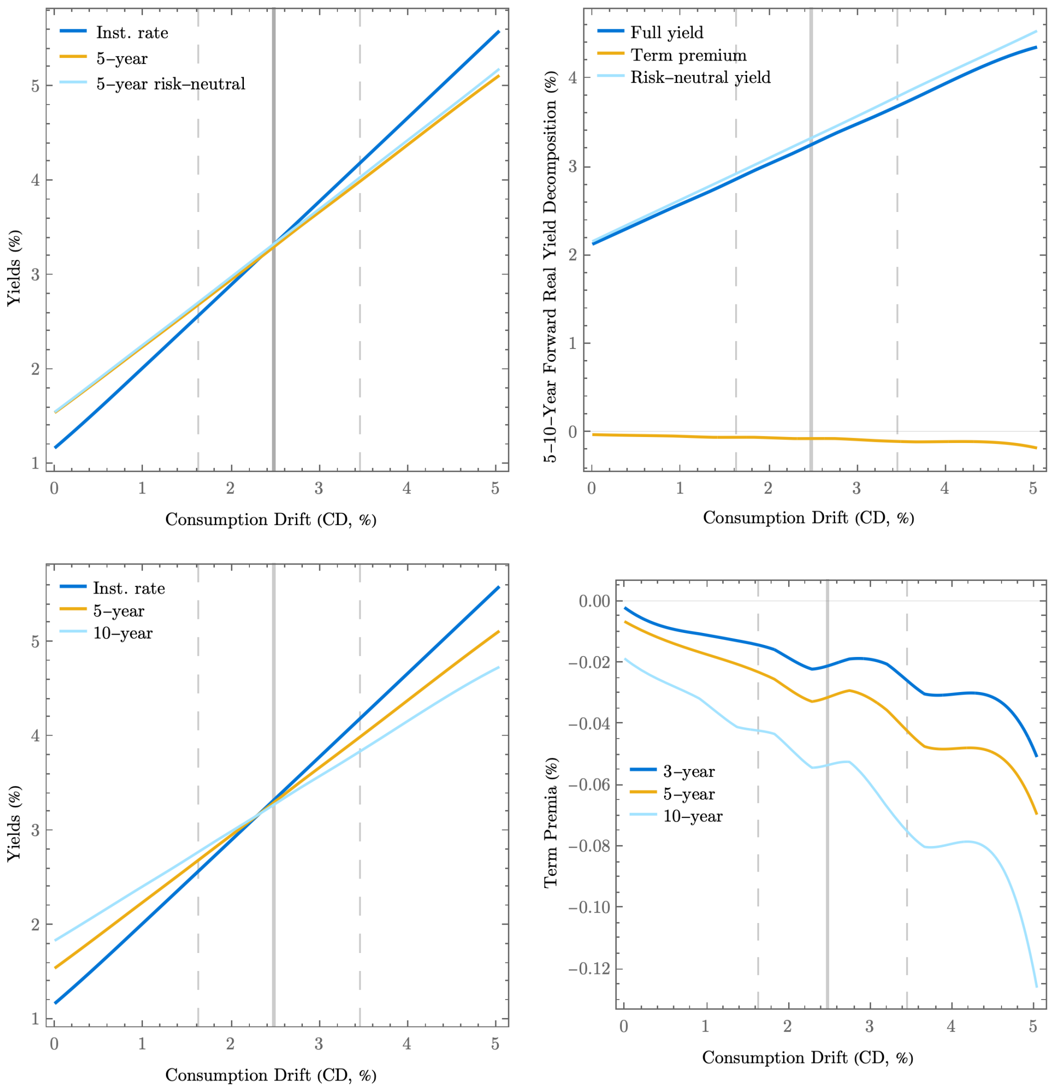
$\mathrm{range_{low}(r_{2,model})}$=0.0492211, $\mathrm{range_{high}(r_{2,model})}$=0.0952063, $\mathrm{range(r_{2,data})}$=0.071748.

Figure 1.43: Time-varying consumption drift with recursive utility and high consumption volatility.
See Figure 1.16 for more details about the plots.

(variation overview)

1.F.31 RU-CD-Heterk-PCor

When the state variable is heteroskedastic, term premia become time-varying. Here term premia are quite small, but this could change once a more volatile state variable is introduced. However, term premia are again negative.



γ=2, ρ=0.012, μ_{ct}=0.0252 x_t, σ_{ct}=0.021, ϕ=0.92, μ_{xt}=μ_x, σ_{xt}=0.12 $\sqrt{x_t}$, ρ_{cx}=0.5, ψ=1.11, approx. order=8

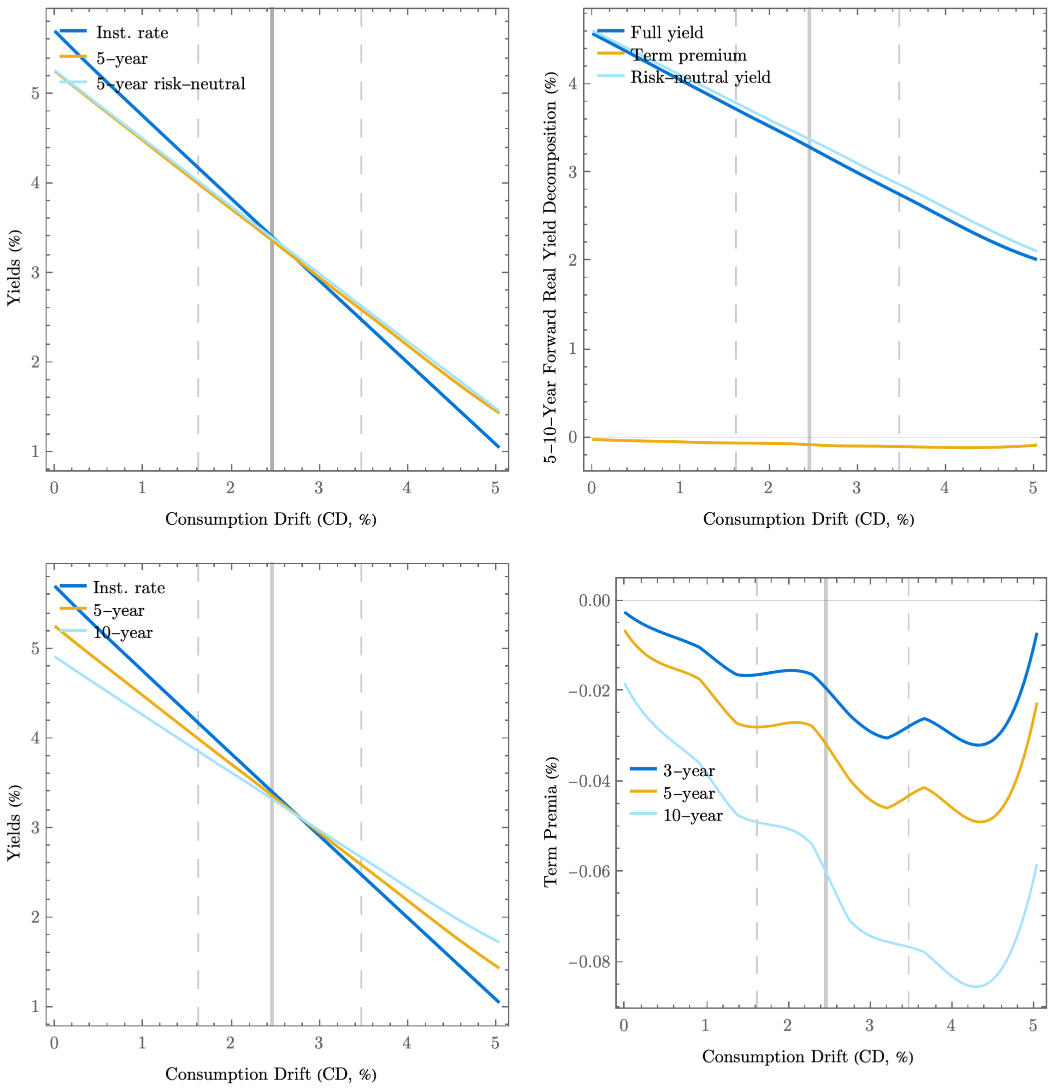
$\mathrm{range}_{\mathrm{low}}(\mathrm{r}_{2,\mathrm{model}})$=0.0115265, $\mathrm{range}_{\mathrm{high}}(\mathrm{r}_{2,\mathrm{model}})$=0.0221465, $\mathrm{range}(\mathrm{r}_{2,\mathrm{data}})$=0.071748.

Figure 1.44: Time-varying and heterskedastic consumption drift with recursive utility with positive ρ_{cx}.
See Figure 1.16 for more details about the plots.

(variation overview)

1.F.32 RU-CD-Heterk-PCor

Despite changing the correlation compared to the previous case term premia are still negative given that the dominant component in function A does not contain ρ_{cx}.



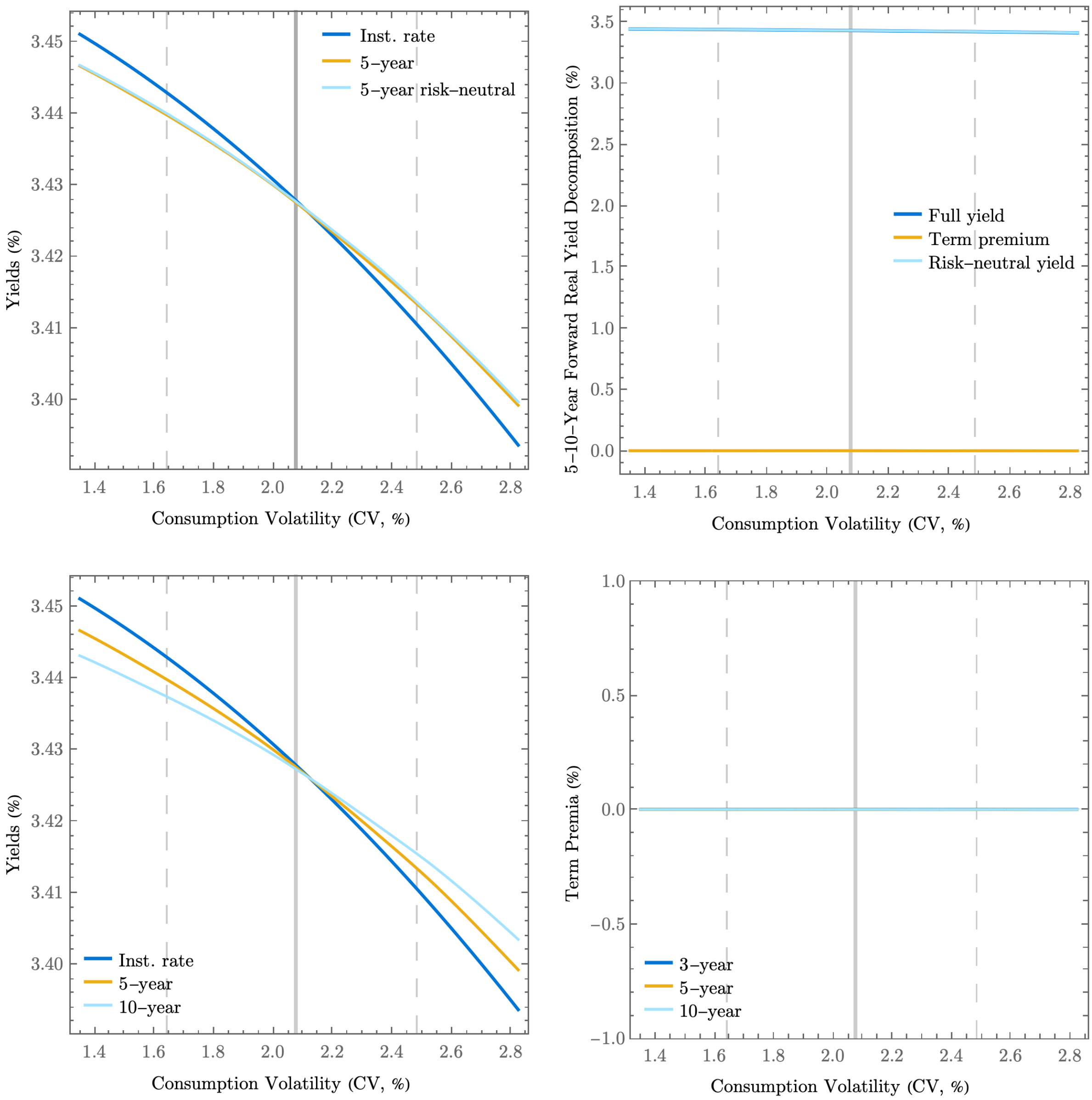
γ=2, ρ=0.012, μ_{ct}=0.0252 $(2 - x_t)$, σ_{ct}=0.021, ϕ=0.92, μ_{xt}=μ_x, σ_{xt}=0.12 $\sqrt{x_t}$, ρ_{cx}=−0.5, ψ=1.11, approx. order=8

$\mathrm{range_{low}}(\mathrm{r_{2,model}})$=0.0119906, $\mathrm{range_{high}}(\mathrm{r_{2,model}})$=0.0230955, $\mathrm{range}(\mathrm{r_{2,data}})$=0.071748.

Figure 1.45: Time-varying and heterskedastic consumption drift with recursive utility with negative ρ_{cx}.
See Figure 1.16 for more details about the plots.

(variation overview)

1.F.33 RU-CV, Calibration used in main paper, Figure 1.5



γ=2, ρ=0.012, μ_{ct}=0.0252, σ_{ct}=0.021 $\sqrt{x_t}$, ϕ=0.92, μ_{xt}=μ_x, σ_{xt}=0.13 $\sqrt{x_t}$, ρ_{cx}=−0.5, ψ=1.11, approx. order=8

$\text{range}_{\text{low}}(\text{r}_{2,\text{model}})$=0.000234865, $\text{range}_{\text{high}}(\text{r}_{2,\text{model}})$=0.000434075, $\text{range}(\text{r}_{2,\text{data}})$=dataRange.

Figure 1.46: Time-varying consumption volatility with recursive utility. See Figure 1.16 for more details about the plots.

(variation overview)

1.F.34 RU-CH-HRA, $\gamma = 6$

The term premia have hardly moved.

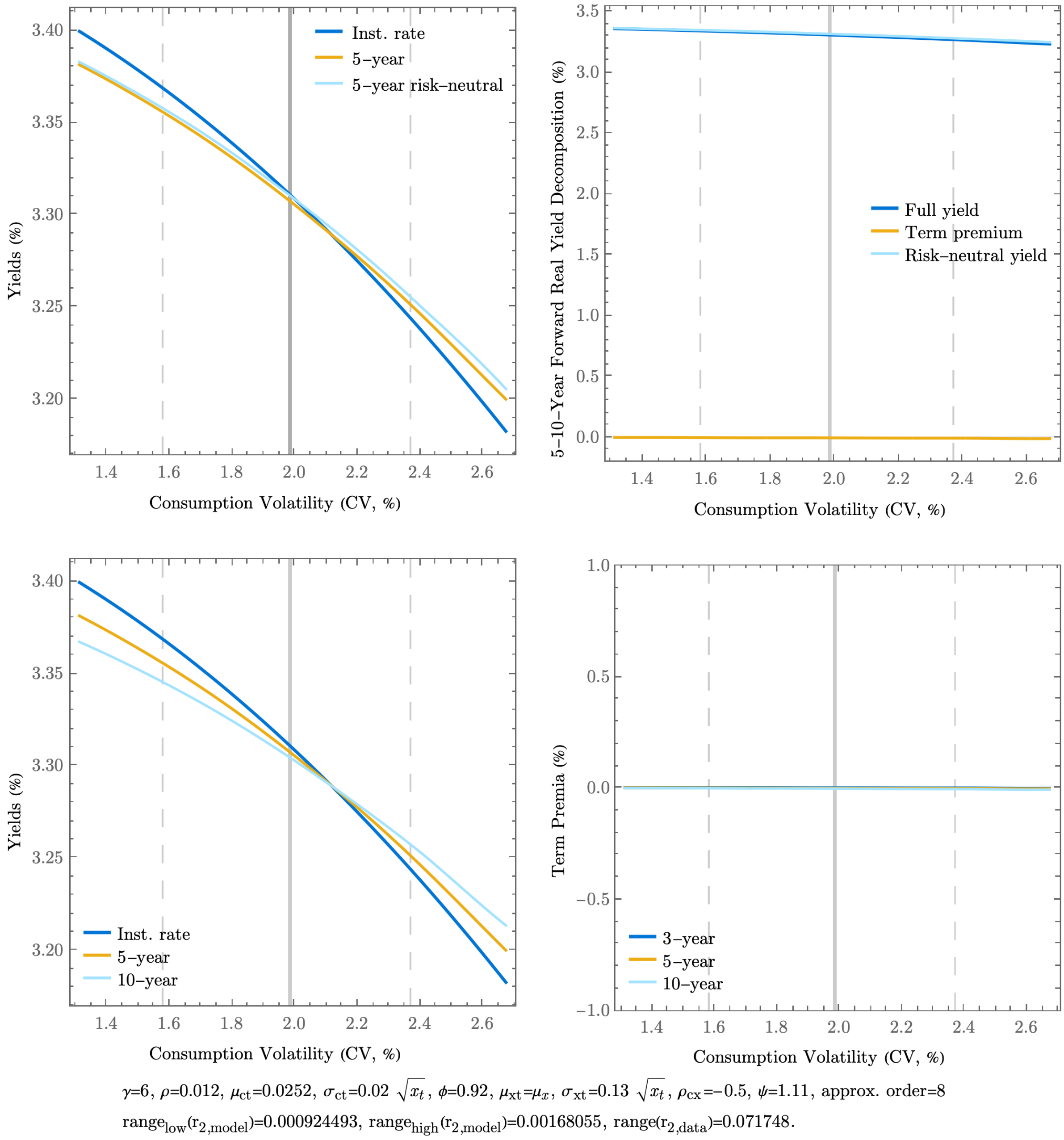


Figure 1.47: Time-varying consumption volatility with recursive utility and high risk aversion.
See Figure 1.16 for more details about the plots.

(variation overview)

1.F.35 RU-CV-HP, $\phi = 0.96$

The term premia have hardly moved and curiously the yields have become slightly more variable again.

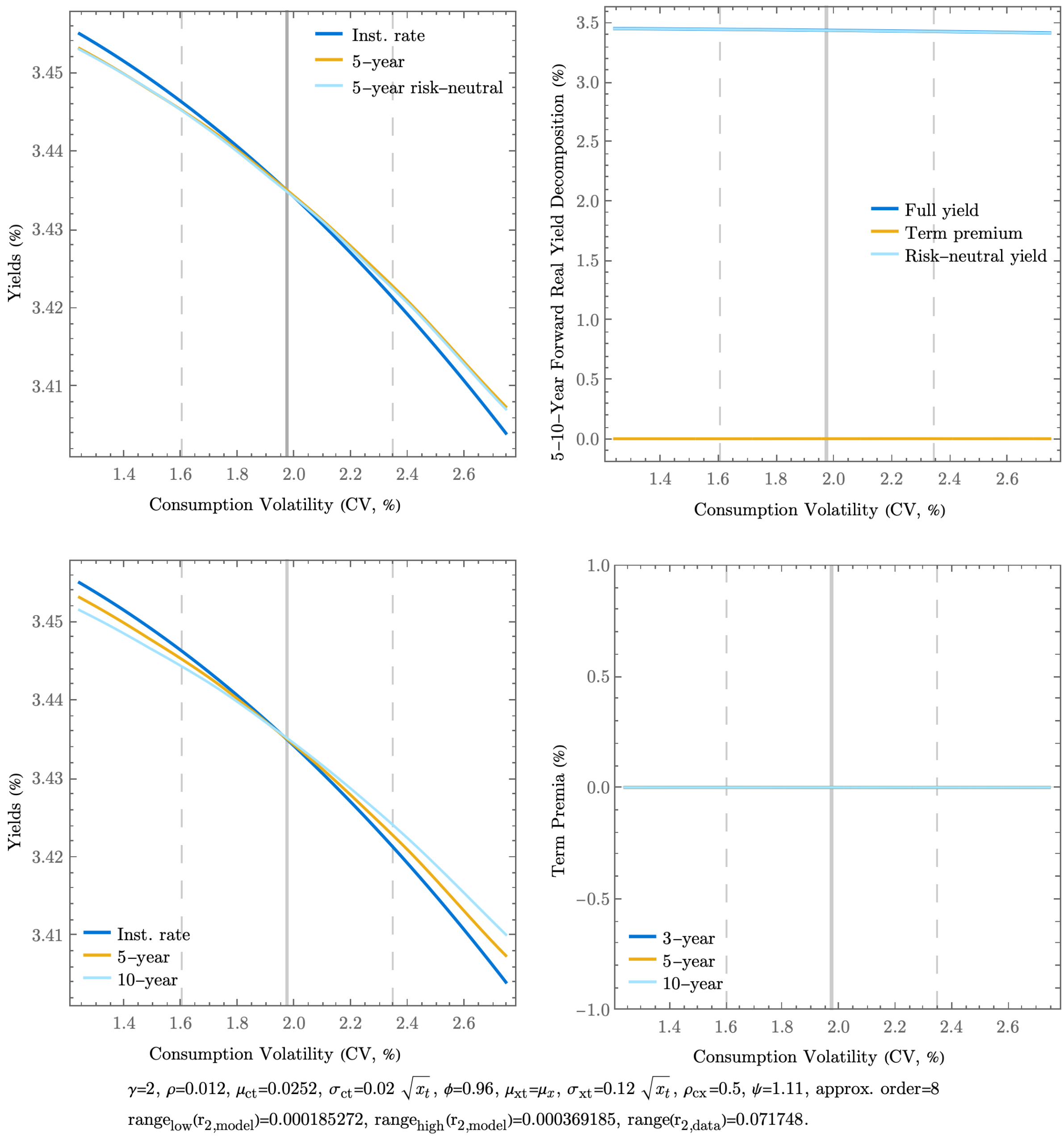
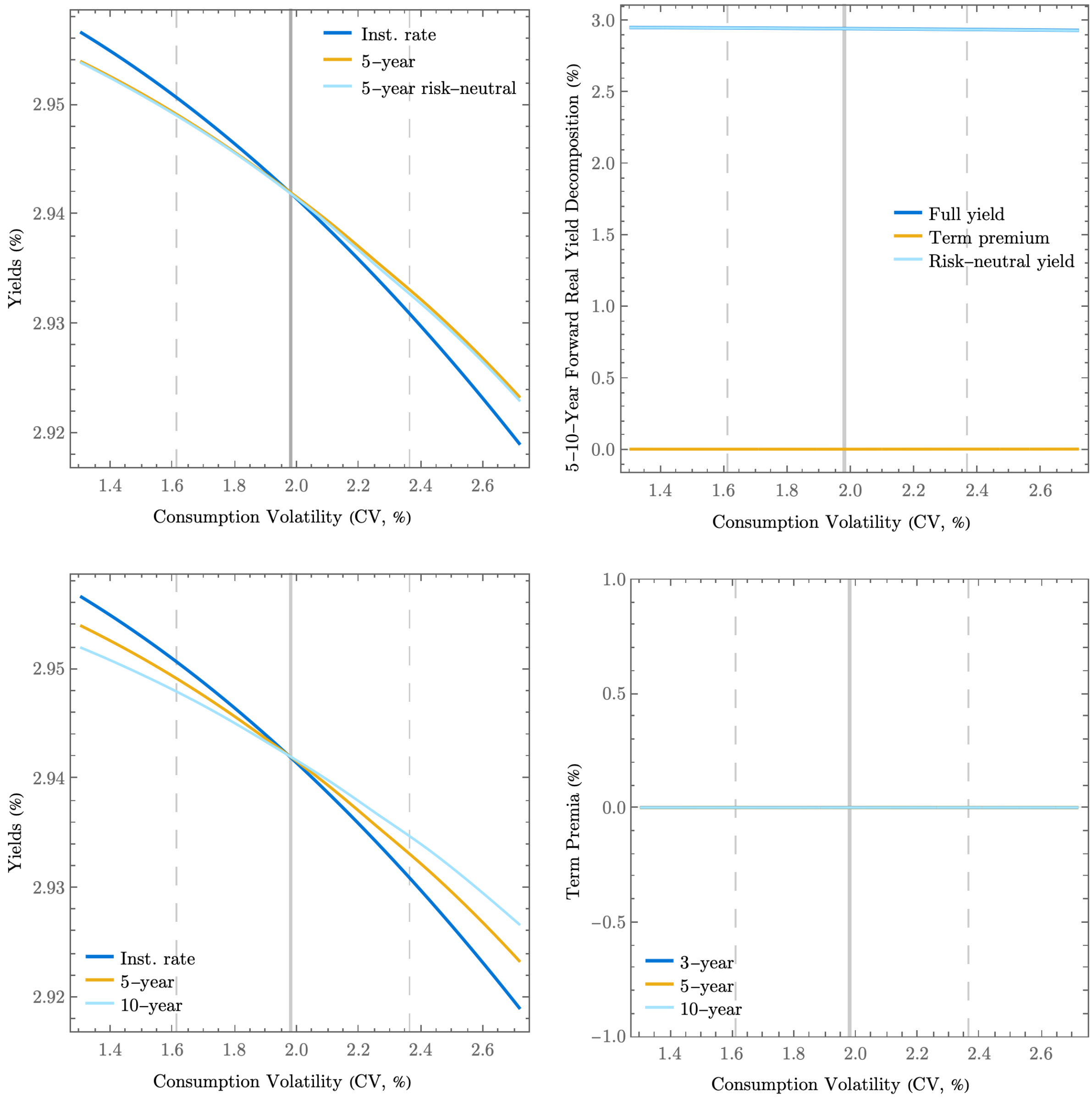


Figure 1.48: Time-varying consumption volatility with recursive utility and high persistence.
See Figure 1.16 for more details about the plots.

(variation overview)

1.F.36 RU-CV-HIES, $\psi = 1.43$

The term premia have hardly moved and the yields have become slightly more variable.



γ=2, ρ=0.012, μ_{ct}=0.0252, σ_{ct}=0.02 $\sqrt{x_t}$, ϕ=0.92, μ_{xt}=μ_x, σ_{xt}=0.13 $\sqrt{x_t}$, ρ_{cx}=0.5, ψ=1.43, approx. order=8

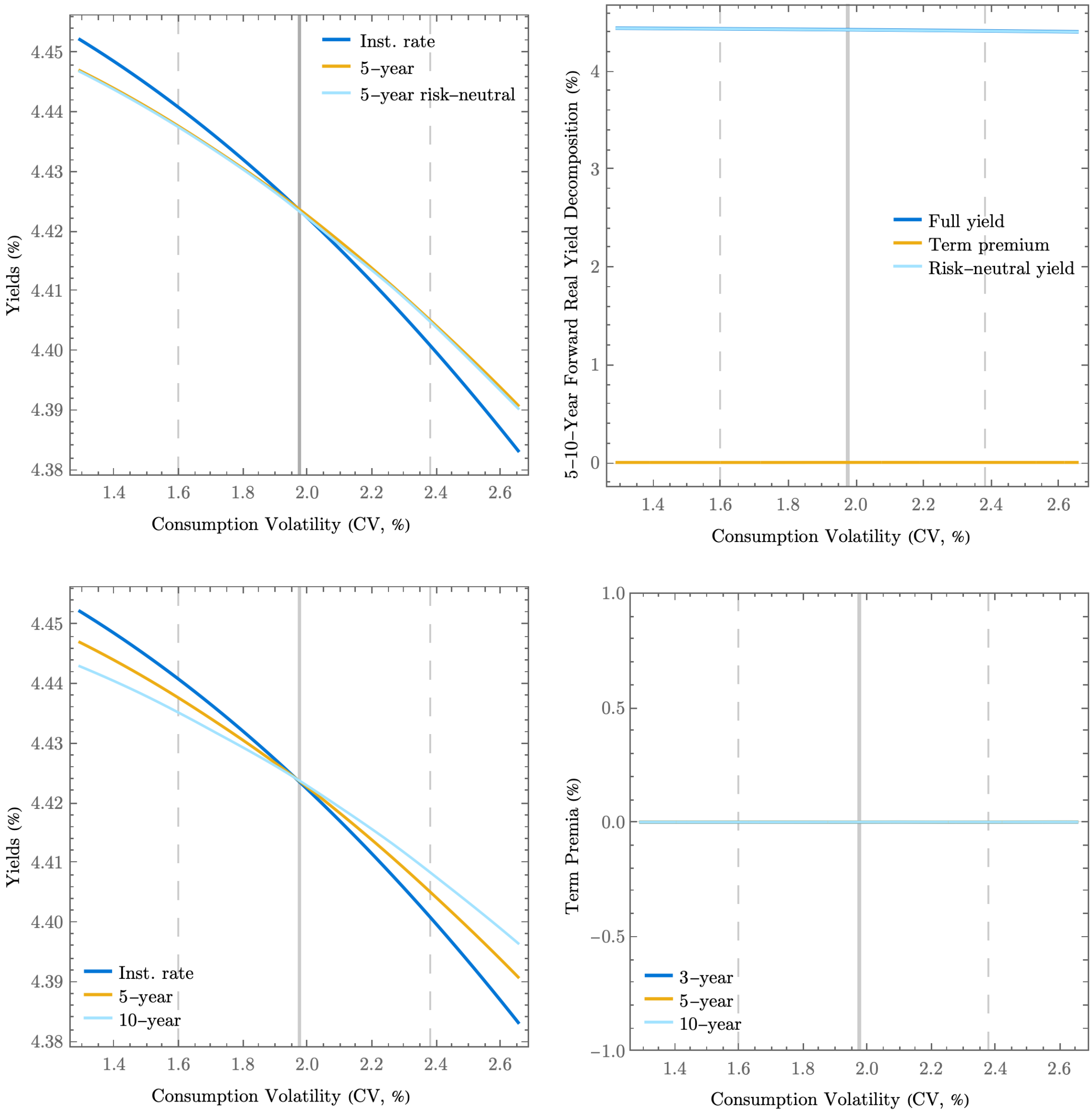
$\text{range}_{\text{low}}(\text{r}_{2,\text{model}})$=0.000147906, $\text{range}_{\text{high}}(\text{r}_{2,\text{model}})$=0.000289188, $\text{range}(\text{r}_{2,\text{data}})$=0.071748.

Figure 1.49: Time-varying consumption volatility with recursive utility and high intertemporal elasticity of substitution.
See Figure 1.16 for more details about the plots.

(variation overview)

1.F.37 RU-CV-LIES, $\psi = 0.77$

The term premia have hardly moved and curiously the yields have become slightly more variable again.



γ=2, ρ=0.012, μ_{ct}=0.0252, σ_{ct}=0.02 $\sqrt{x_t}$, ϕ=0.92, μ_{xt}=μ_x, σ_{xt}=0.13 $\sqrt{x_t}$, ρ_{cx}=0.5, ψ=0.77, approx. order=8

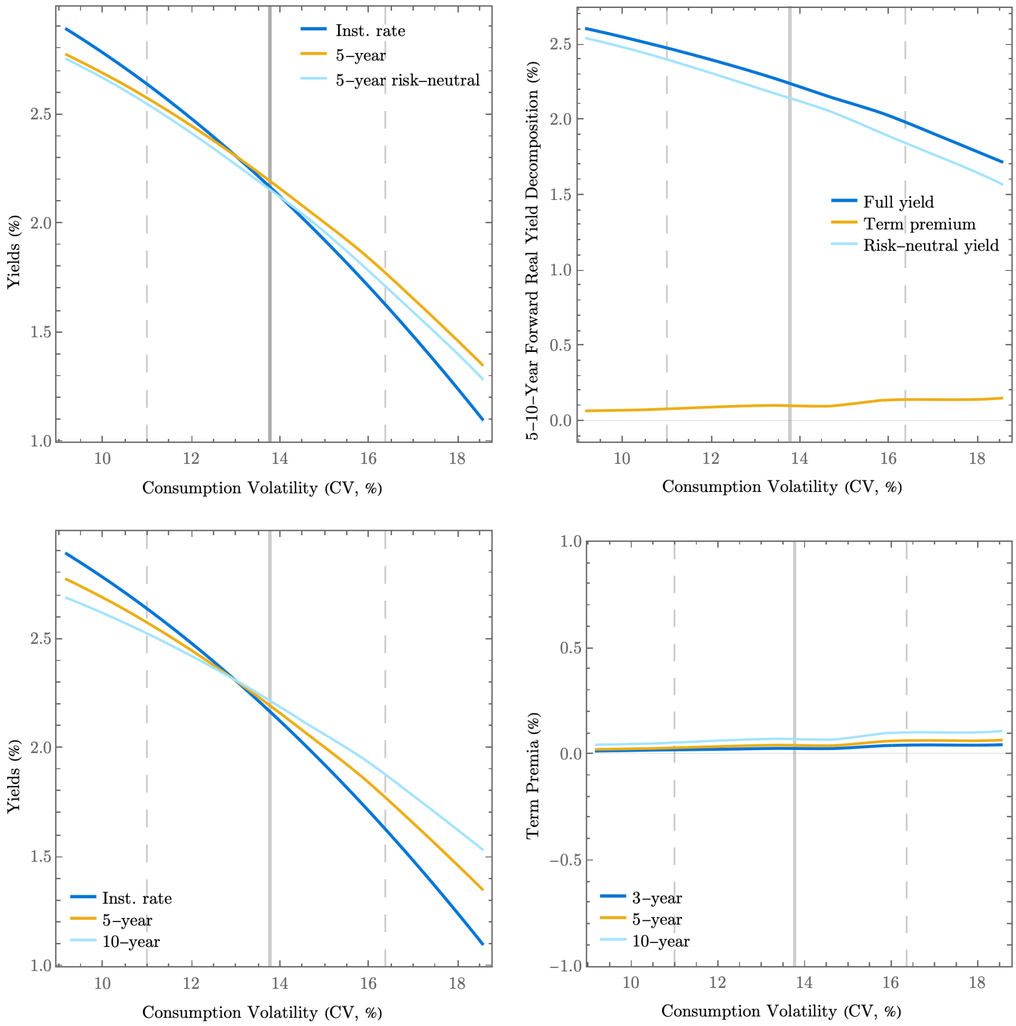
$\mathrm{range_{low}(r_{2,model})}$=0.000288099, $\mathrm{range_{high}(r_{2,model})}$=0.000549846, $\mathrm{range(r_{2,data})}$=0.071748.

Figure 1.50: Time-varying consumption volatility with recursive utility and low intertemporal elasticity of substitution.
See Figure 1.16 for more details about the plots.

(variation overview)

1.F.38 RU-HCV-PCor, $\sigma_{c0} = 0.14$, $\rho_{cx} = 0.5$

Here term premia are positive, which means that the first component of function A that contains ρ_{cx} has become dominant due to the increase in σ_{c0}. Nevertheless, term premia are still smaller than the corresponding term premia in the time-separable utility case, because the second term in function A is still negative.



γ=2, ρ=0.012, μ_{ct}=0.0252, σ_{ct}=0.14 $\sqrt{x_t}$, ϕ=0.92, μ_{xt}=μ_x, σ_{xt}=0.13 $\sqrt{x_t}$, ρ_{cx}=0.5, ψ=1.11, approx. order=8

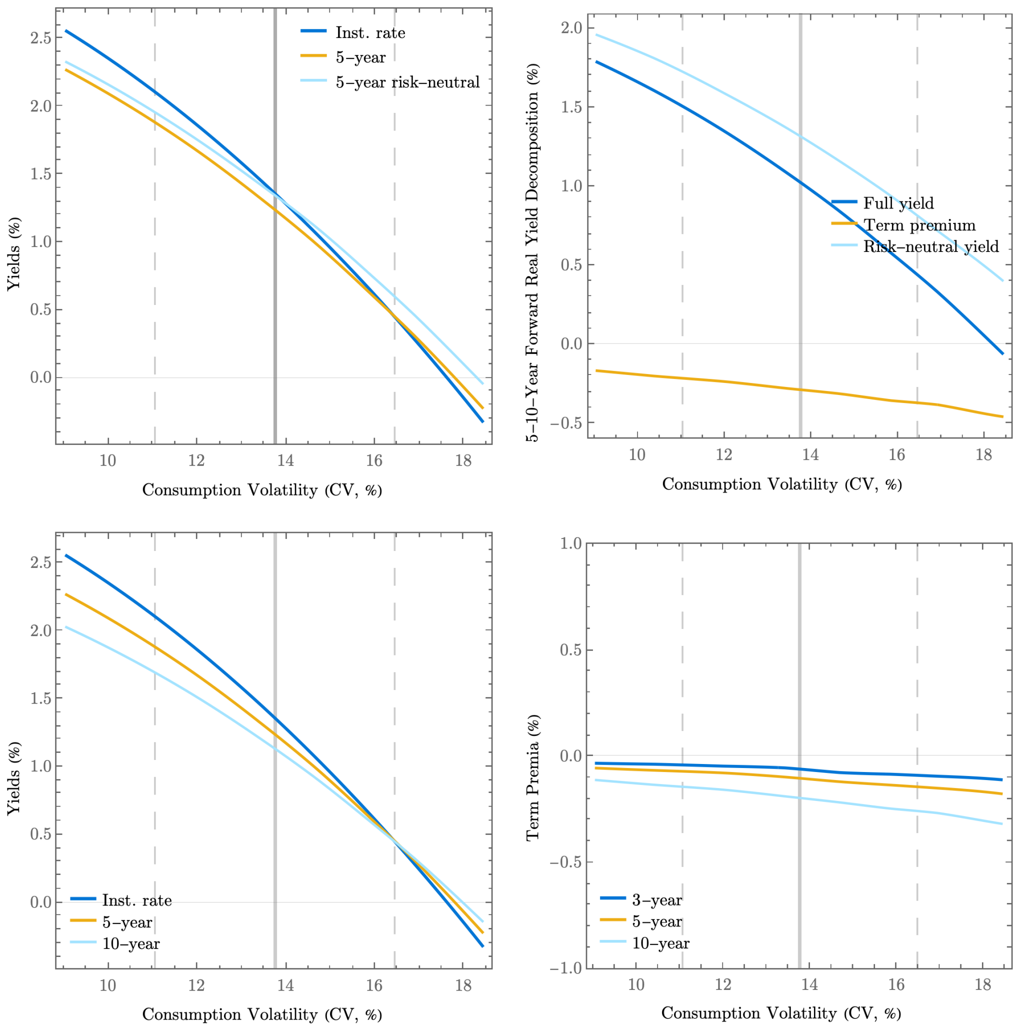
$\text{range}_{\text{low}}(\text{r}_{2,\text{model}})$=0.0074128, $\text{range}_{\text{high}}(\text{r}_{2,\text{model}})$=0.0146724, $\text{range}(\text{r}_{2,\text{data}})$=0.071748.

Figure 1.51: See Figure 1.16 for more details about the plots.

(variation overview)

1.F.39 RU-HCV-NCor, $\sigma_{c0} = 0.14, \rho_{cx} = 0.5$

Here both terms in function A are negative, so term premia are negative. They are also larger in absolute value than the corresponding term premia in RU-CV.



γ=2, ρ=0.012, μ_{ct}=0.0252, σ_{ct}=0.14 $\sqrt{x_t}$, ϕ=0.92, μ_{xt}=μ_x, σ_{xt}=0.13 $\sqrt{x_t}$, ρ_{cx}=−0.5, ψ=1.11, approx. order=8

$\text{range}_{\text{low}}(r_{2,\text{model}})$=0.0120834, $\text{range}_{\text{high}}(r_{2,\text{model}})$=0.0232408, $\text{range}(r_{2,\text{data}})$=0.071748.

Figure 1.52: See Figure 1.16 for more details about the plots.

(variation overview)

1.G Equity Premium

1.G.1 Derivation

Given that the focus of this paper is on interest rates, I do not present results on the equity premium. However, in this section I show that the variations with high CV, examined in the main paper also imply a sizeable equity premium.

Here, I outline how to derive the equity premium, based on the framework that I have introduced. Firstly, I show how to derive the price consumption ratio, which refers to the price of the consumption perpetuity over the consumption flow. By consumption perpetuity a mean security that pays dividend equal to the consumption flow forever. In order, to get the price consumption ratio, I first derive the zero coupon price consumption ratio. This is similar to the quantity described before, but instead of using the price of the consumption perpetuity in the numerator the price of a zero-coupon security that pays the consumption flow after a fixed duration is used. Then, by integrating all the zero-coupon consumption ratios, it is possible to get the original price consumption ratio of the consumption perpetuity.

$$p(x_t) = \int_0^\infty q(m, x_t)dm = \int_0^\infty \frac{P(m, x_t, C_t)}{C_t}dm \tag{1.25}$$

p, $P(\cdot,\cdot)$ q and $P(\cdot,\cdot,\cdot)$ are the price consumption ratio, the price of the consumption perpetuity, the zero-coupon price consumption ratio and the price of the zero-coupon security respectively. q can be derived by applying Ito's Lemma and combining the result with a transformation of the pricing equation:

$$\begin{aligned} E\Big[d(\Lambda P(m,x,C))\Big] = 0 \Rightarrow E\left[\frac{\mathrm{d}\Lambda}{\Lambda} + \frac{dP(m,x,C)}{P(m,x,C)} + \frac{\mathrm{d}\Lambda dP(m,x,C)}{\Lambda P(m,x,C)}\right] = 0 \\ \Rightarrow E\left[\frac{\mathrm{d}\Lambda}{\Lambda} + \frac{d\Big(q(m,x)C)\Big)}{q(m,x)C} + \frac{\mathrm{d}\Lambda d\Big(q(m,x)C)\Big)}{\Lambda q(m,x)C}\right] = 0 \\ \Rightarrow E\left[\frac{\mathrm{d}\Lambda}{\Lambda} + \frac{\mathrm{d}q}{q} + \frac{dC}{C} + \frac{\mathrm{d}\Lambda \mathrm{d}q}{\Lambda q} + \frac{\mathrm{d}\Lambda dC}{\Lambda C} + \frac{\mathrm{d}q dC}{qC}\right] = 0 \end{aligned} \tag{1.26}$$

Based on the values of q, p can be derived using Equation (1.25). In practise, the integral cannot be computed numerically up to infinity, so I implement a cutoff at two hundred years. By a further application of Ito's Lemma it is possible to derive the process for the return of the consumption perpetuity:

$$\left(\frac{dP(x,C)}{P(x,C)}+\frac{C}{P}\mathrm{d}t\right)/\mathrm{d}t \tag{1.27}$$

and the equity premium is:

$$\left(E\left[\frac{dP(x,C)}{P(x,C)}\right]+\frac{C}{P}\mathrm{d}t-r\mathrm{d}t\right)/\mathrm{d}t \tag{1.28}$$

The quantities above can be written in terms of p.

1.G.2 Results for high TSU-HCV cases

The graph shows the instantaneous annualised expected excess return of the consumption perpetuity for the models that correspond to Figures 1.26 and 1.28.

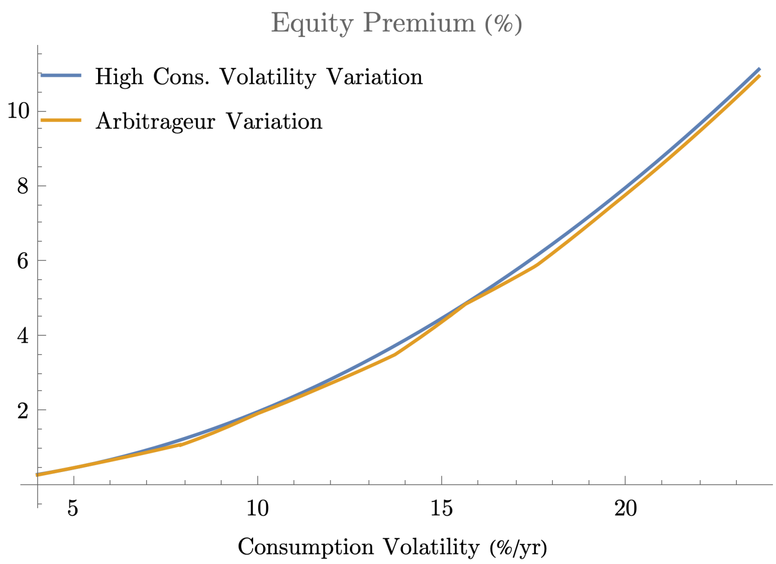


Figure 1.53: Equity premium for cases that exhibit high CV.

In both models the expected instantaneous return of the consumption perpetuity is close to be being constant based on the chosen calibration, and the variation of the equity premium is actually driven by the variation of the short-term rate. Nevertheless, the figure shows that the equity premium is considerable, ranging from below 2% to above 10%.

1.H Deriving the Stochastic Discount Factor

1.H.1 Derivation of the SDF with TSU

Here I derive the SDE of SDF, including the case of the habit model. I present the terms that only apply to the habit model in grey colour. The following is the regular form of the SDF, in which I have substituted the state variable and log consumption:

$$\Lambda = e^{-\rho t}(e^{c} S_0 e^{x})^{-\gamma} \tag{1.29}$$

Then, in order to get the SDE form, I apply Ito's Lemma:

$$\begin{aligned}
\mathrm{d}\Lambda &= \frac{\partial \Lambda}{\partial t}\mathrm{d}t + \frac{\partial \Lambda}{\partial c}dc + \underbrace{\frac{\partial \Lambda}{\partial x}dx}_{\text{habit model}} + \frac{1}{2}\left(\frac{\partial^2 \Lambda}{\partial c^2}(dc)^2 + \underbrace{\frac{\partial^2 \Lambda}{\partial x^2}(dx)^2 + \frac{\partial^2 \Lambda}{\partial x \partial c}dxdc}_{h.m.}\right) \\
&= -\rho\Lambda \mathrm{d}t - \gamma\Lambda dc - \underbrace{\gamma\Lambda dx}_{\text{h.m.}} + \frac{1}{2}\left(\gamma^2\Lambda(dc)^2 + \underbrace{\gamma^2\Lambda(dx)^2 + \gamma^2\Lambda dxdc}_{h.m.}\right)
\end{aligned}$$

$\Rightarrow$

$$\frac{\mathrm{d}\Lambda}{\Lambda} = \Bigg(-\rho - \gamma\mu_{ct} + \underbrace{\gamma\log(\phi)(\mu_{x0} - x_t)}_{h.m.} + \frac{\gamma^2\sigma_{ct}^2}{2} + \underbrace{\frac{\gamma^2\sigma_{xt}^2}{2} + \gamma^2\rho_{cx}\sigma_{xt}\sigma_{ct}}_{\text{h.m.}}\Bigg)\mathrm{d}t - \gamma\sigma_{ct}\mathrm{d}W_{ct} \underbrace{-\gamma\sigma_{xt}\mathrm{d}W_{xt}}_{\text{h.m.}} \tag{1.30}$$

1.H.2 Derivation of the SDF with RU

As mentioned in the main paper the SDE of the SDF can be derived based on the following expression:

$$\frac{\mathrm{d}\Lambda}{\Lambda} = f_V(C,V)\mathrm{d}t + \frac{df_C(C,V)}{f_C(C,V)} \tag{1.31}$$

thus, flow utility is a central component of the derivation:

$$f(C,V) = \frac{\beta}{1-1/\psi}\Big((1-\gamma)V\Big)\left(\Big(C((1-\gamma)V)^{-\frac{1}{1-\gamma}}\Big)^{1-1/\psi} - 1\right) \tag{1.32}$$

The partial derivative of f with respect to V is:

$$f_V(C,V) = \frac{\rho\left((\gamma-1)\psi + (1-\gamma\psi)\left(C(V-\gamma V)^{\frac{1}{\gamma-1}}\right)^{\frac{\psi-1}{\psi}}\right)}{\psi - 1} \tag{1.33}$$

The partial derivative of f with respect to C is:

$$f_C(C,V) = -\frac{(\gamma-1)\rho V \left(C(V-\gamma V)^{\frac{1}{\gamma-1}}\right)^{\frac{\psi-1}{\psi}}}{C} \tag{1.34}$$

As I implement Ito's Lemma directly using c_t and x_t as independent variables, I make the following replacements in the expressions above:

$$c_t = \log(C), \quad V = \frac{C^{1-\gamma}}{1-\gamma} e^{(1-\gamma)K(x_t)} \Rightarrow K(x_t) = \frac{\log\left(-\frac{C^{1-\gamma}}{(\gamma-1)V}\right)}{\gamma-1} \tag{1.35}$$

And after simplification they become:

$$\begin{aligned} f_V(C,V) \to g(c_t,x_t) =& \frac{\rho\left(-(1-\gamma\psi)e^{-\frac{(\psi-1)K[x_t]}{\psi}} - \gamma\psi + \psi\right)}{1-\psi} \\ f_C(C,V) \to h(c_t,x_t) =& \rho e^{\left(\frac{1}{\psi}-\gamma\right)K(x_t) - c_t\gamma} \end{aligned} \tag{1.36}$$

And I implement Ito's Lemma on g_2. The partial derivatives are:

$$\begin{aligned} \frac{\partial h(c_t,x_t)}{\partial c_t} =& \gamma\rho\left(-e^{\left(\frac{1}{\psi}-\gamma\right)K[x_t]-\gamma c_t}\right) = -\gamma h(c_t,x_t) \\ \frac{\partial h(c_t,x_t)}{\partial x_t} =& \rho\left(\frac{1}{\psi}-\gamma\right)K'(x_t)\, e^{\left(\frac{1}{\psi}-\gamma\right)K[x_t]-\gamma c_t} = \left(\frac{1}{\psi}-\gamma\right)K'(x_t)\, h(c_t,x_t) \\ \frac{\partial^2 h(c_t,x_t)}{\partial c_t^2} =& \gamma^2\rho e^{\left(\frac{1}{\psi}-\gamma\right)K[x_t]-\gamma c_t} = \gamma^2 h(c_t,x_t) \\ \frac{\partial^2 h(c_t,x_t)}{\partial x_t^2} =& \frac{\rho(\gamma\psi-1)\left((\gamma\psi-1)K'(x_t)^2 - \psi K''(x_t)\right) e^{\left(\frac{1}{\psi}-\gamma\right)K[x_t]-\gamma c_t}}{\psi^2} \\ =& \frac{(\gamma\psi-1)\left((\gamma\psi-1)K'(x_t)^2 - \psi K''(x_t)\right)}{\psi^2} h(c_t,x_t) \\ \frac{\partial h(c_t,x_t)}{\partial c_t \partial x_t} =& \frac{\gamma\rho(\gamma\psi-1)K'(x_t)\, e^{\left(\frac{1}{\psi}-\gamma\right)K[x_t]-\gamma c_t}}{\psi} = \frac{\gamma(\gamma\psi-1)K'(x_t)\, h(c_t,x_t)}{\psi} \end{aligned} \tag{1.37}$$

The expressions above should be plugged into the expression:

$$\begin{aligned} \frac{df_C}{f_C} =& \Bigg(\frac{\partial h(c_t,x_t)}{\partial c_t}\mu_{ct} + \frac{\partial h(c_t,x_t)}{\partial x_t}\Big(-\log(\phi)\Big)(\mu_{x0}-x_t) \\ &+ \frac{\sigma_{ct}^2}{2}\frac{\partial^2 h(c_t,x_t)}{\partial c_t^2} + \frac{\sigma_{xt}^2}{2}\frac{\partial^2 h(c_t,x_t)}{\partial x_t^2} + \frac{\rho_{cx}\sigma_{ct}\sigma_{xt}}{2}\frac{\partial^2 h(c_t,x_t)}{\partial c_t \partial x_t}\Bigg)\mathrm{d}t \\ &+ \frac{\partial h(c_t,x_t)}{\partial x_t}\sigma_{xt}\mathrm{d}W_{xt} + \frac{\partial h(c_t,x_t)}{\partial c_t}\sigma_{ct}\mathrm{d}W_{ct} \end{aligned} \tag{1.38}$$

Then everything is plugged into Equation (1.31) to give the final result:

$$\begin{aligned}\frac{\mathrm{d}\Lambda}{\Lambda} =& \Bigg(\frac{\gamma(\gamma\psi-1)\rho_{cx}\sigma_{xt}\sigma_{ct}K'\left(x_t\right)}{\psi} + \frac{\gamma^2\sigma_{ct}^2}{2} - \gamma\mu_{ct} \\ &+ \frac{(\gamma\psi-1)\left(2\psi(\mu_{x0}-x_t)\log(\phi)K'\left(x_t\right)+\sigma_{xt}^2\left((\gamma\psi-1)K'\left(x_t\right)^2-\psi K''\left(x_t\right)\right)\right)}{2\psi^2} \\ &\frac{\rho\left(-(1-\gamma\psi)e^{-\frac{(\psi-1)K[x_t]}{\psi}}-\gamma\psi+\psi\right)}{1-\psi}\Bigg)\mathrm{d}t \\ &- \frac{(\gamma\psi-1)\sigma_{xt}K'(x_t)}{\psi}\mathrm{d}W_{xt} - \gamma\sigma\mathrm{d}W_{ct}\end{aligned} \tag{1.39}$$

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Chapter 2

SDFPricing: A Julia Package for Asset Pricing Based on a Stochastic Discount Factor Process

Abstract

I introduce a package in the Julia programming language to perform asset pricing based on a stochastic discount factor in continuous time. Prices are computed through Monte Carlo simulations according to a pricing partial differential equation and the corresponding Feynman-Kac formula. At this stage it is possible to compute a) prices of zero-coupon fixed income securities and b) price-dividend ratios, which also allow the calculation of prices and returns of these securities. The package is focused on ease of use and is meant to be used in research and teaching. I illustrate the functionality of the package with examples and an application. In particular, I show how asset prices react after shifts in economic variables within a consumption-based model, and I discuss to what extent these shifts can be classified as monetary policy shocks or information shocks in connection to monetary policy announcements.

2.1 Introduction

The stochastic discount factor (SDF) is a fundamental concept in asset pricing. Given a joint process for the SDF and a payoff stream, the corresponding security can be priced. However, there is little software publicly available that is easy to use and allows the calculation of prices based on an SDF. This paper showcases a package in the Julia programming language that I developed to answer this need. The package is called *SDFPricing* and is available at Github In this first release the package has limited functionality compared to other projects that are being developed for more than a decade. Nevertheless, it allows the pricing of zero-coupon fixed income securities, which could then also be combined to price dividend-paying securities. In addition, it is possible to specify a process for a stochastic dividend stream and calculate the price-dividend ratio, which can then be used to calculate the return of the security and its price given the current level of the dividend.[1] While the pricing can be made arbitrarily accurate given the processes of the SDF and the dividend stream, calibration to market data is not directly possible. Therefore, for the time being this tool will not by itself be useful to price derivatives or any security in a way that is consistent with other market prices.

I have chosen to develop this package in Julia for two main reasons. Firstly, Julia being a high-level programming language is easy to use. In addition, its syntax is similar to Python and MATLAB, which are widely used languages, and it is well suited for scientific programming. Secondly, Julia includes the *DifferentialEquations* package, which is arguably the most comprehensive tool for solving differential equations, including stochastic differential equations which are at the core of the pricing problem. Furthermore, there is particular interest in Julia from the finance and economics communities. For example, the QuantEcon Organisation is actively engaged in the language.

The asset pricing package that I introduce is the first within the context of the Julia programming language. In other programming languages, the Computational Finance Suite of MATLAB contains resources that can be used for asset pricing, but unfortunately it requires a license.[2] Apart from MATLAB there is also Quantlib which is a free and open-source C++ library that can be used for asset pricing, while its functionality can also be leveraged by a dedicated Python

[1]For the time being the process of the dividend stream needs to obey a stochastic differential equation with no jumps.

[2]Information can be found on the official website `https://www.mathworks.com/solutions/computational-finance/computational-finance-suite.html`

library.[3] Both Quantlib and the resources available in Matlab are tools tailored to the industry and they they do not focus on the pricing of assets starting from a stochastic discount factor. The aim of my package is to be the go-to resource for those who want to price assets given an SDF as an "input".

In Section 2.2 I describe the package and its functionality. A central function of the package is *solve*, which takes a variable of type *Problem* and a variable of type *SolutionSettings* and returns a variable of type *Solution*, which contains the solution to the pricing problem described by the inputs. The solution is calculated using Monte Carlo simulations and the Feynman-Kac formula.

In Section 2.3 I provide concrete examples of how to use the package along with the results. I use both consumption-based and non-consumption-based SDFs. I use SDFs that depend both on one state variable and two state variables, and I compute, zero-coupon bond prices, price-consumption ratios, and returns of dividend-paying securities.

In Section 2.4 I perform an application using the functionality of the package. In particular, I analyse the effect of general monetary policy shocks on interest rates and asset prices in the context of a simple consumption-based model. This analysis is motivated by the literature on Delphic and Odyssean shocks (Campbell et al. 2012; Nakamura and Steinsson 2018; Jarociński and Karadi 2020; Andrade and Ferroni 2021; Altavilla et al. 2019), which are a classification of monetary policy surprises (Gürkaynak, Sack and Swansonc 2005; Kuttner 2001). Both kinds of shocks are caused by actions or announcements of central banks. However, Delphic shocks reveal information about the underlying state of the economy. In contrast, Odyssean shocks reveal information about the future conduct of monetary policy.[4] In the literature the type of shock is identified depending on how financial variables react after the monetary policy announcement. Using a simple consumption-based setup I analyse the possible reactions of interest rates and asset prices to monetary policy announcements. Importantly, monetary policy may not operate through one channel only. While a traditional monetary policy shock can be justified within a simple consumption-based model, my analysis shows

[3] Information can be found on the official website `https://www.quantlib.org`.

[4] The term Odyssean is due to the central bank committing to a future policy beforehand, as Odysseus did when he tied himself to the mast of his ship to avoid the Sirens. Originally, in Campbell et al. (2012) the emphasis was placed on whether the central bank is actually committed to the future policy or not. In subsequent literature the emphasis for the distinction between the two kinds of shocks was placed more on whether the shocks reveal important information about monetary policy actions as opposed to underlying macroeconomic conditions. In this papers Delphic shocks are identified with information shocks and Odyssean shocks are identified with monetary policy shocks.

that if monetary policy can also affect consumption volatility, then the response of financial variables can be much more unpredictable and it can give rise to movements in the short-term rate and in asset prices that are usually associated with an information shock. In addition, information shocks may also induce the effects that are associated with a standard monetary policy shock. In particular, this can happen, if the central bank *reveals* the kind of information about the consumption process that is usually assumed to *caused* by a standard monetary policy shock. These results suggest that accurate theoretical models may be required for the classification of monetary policy shocks.

Finally, Section 2.5 concludes.

2.2 Package Description

In the following description I assume a single state variable and a single Wiener process for simplicity of the formulas. However, it is also possible to have several state variables and several Wiener processes. The corresponding formulas are given in Appendix 2.A.

2.2.1 Theory

2.2.1.1 Zero Coupon Bond

Following Cochrane (2009) the pricing equation for an asset that pays no dividends in continuous time can be written as:

$$\mathrm{E}\Big[\mathrm{d}\Big(\Lambda Q(x_t,t,m)\Big)\Big] = 0, \quad Q(x_t,t,0) = g(t,x_t) \tag{2.1}$$

where m is the remaining maturity until the payoff is due, $g(\cdot)$ is a function for the terminal payoff for each time t, which can also depend on the state variable, but which typically is just equal to 1, Λ is the SDF and Q is the price function of the asset in terms of the state variables, current time and the remaining maturity of the asset. If the process for the SDF and the state variables is known, then this equation can be solved to derive the price function $Q(x_t,t,m)$ for any value of x, t, and m.[5] In particular, the state variables follow some process that can be

[5]Here, I show the dependence on time explicitly. Often it will be the case that there will be no dependence on time and the state variable and remaining maturity will be sufficient to determine the price of assets. The case without explicit time dependence is also easier to compute with the package.

written in stochastic differential equation (SDE) form as:

$$\mathrm{d}x = \mu(x_t, t)\mathrm{d}t + \sigma(x_t, t)\mathrm{d}W_t \tag{2.2}$$

where μ is the drift of the state variable, σ is the diffusion of the state variable, and W_t is the Wiener process. The process for the SDF can be written in SDE form as:

$$\frac{\mathrm{d}\Lambda}{\Lambda} = \mu_\Lambda(x_t, t)\mathrm{d}t + \sigma_\Lambda(x_t, t)\mathrm{d}W_t \tag{2.3}$$

Next, using Ito's Lemma, it is possible to express the SDE that the price function Q follows. This is given by:

$$\mathrm{d}Q = \left(\frac{\partial Q}{\partial t} - \frac{\partial Q}{\partial m} + \mu(x_t, t)\frac{\partial Q}{\partial x_t} + \frac{1}{2}\sigma(x_t, t)^2\frac{\partial^2 Q}{\partial x_t^2}\right)\mathrm{d}t + \sigma(x_t, t)\frac{\partial Q}{\partial x_t}\mathrm{d}W_t \tag{2.4}$$

where I have dropped the dependence of Q on the arguments for simplicity, and I have used that $\mathrm{d}m = -\mathrm{d}t$. This expression can now be inserted in Equation (2.1) to derive the pricing partial differential equation (SDE) that the price function Q follows:

$$\mathrm{E}\left[\frac{\mathrm{d}\Big(\Lambda Q\Big)}{\Lambda}\right] = 0 \Rightarrow \mathrm{E}\left[\frac{\mathrm{d}\Lambda}{\Lambda}Q + \mathrm{d}Q + \frac{\mathrm{d}\Lambda}{\Lambda}\mathrm{d}Q\right] = 0 \tag{2.5}$$

$$\Rightarrow - r(x_t, t)Q + \frac{\partial Q}{\partial t} - \frac{\partial Q}{\partial m} + \mu(x_t, t)\frac{\partial Q}{\partial x_t} + \frac{1}{2}\sigma(x_t, t)^2\frac{\partial^2 Q}{\partial x_t^2} + \sigma(x_t, t)\sigma_\Lambda\frac{\partial Q}{\partial x_t} = 0$$

where $r(x_t, t) = -\mathrm{E}[\mathrm{d}\Lambda/\Lambda]$ is the risk-free rate. This PDE can be solved using the Feynman-Kac formula, which states that the solution to the PDE is given by the expected value of the terminal payoff of the asset under the risk-neutral measure. This is given by:

$$Q(x_t, t, m) = \mathrm{E}\left[\exp\left(-\int_t^{t+m} r(\hat{x}_s, s)\mathrm{d}s\right)g(t+m, \hat{x}_{t+m})\middle|\hat{x}_t = x_t\right] \tag{2.6}$$

where $\hat{x}$ follows the modified process:

$$\mathrm{d}\hat{x}_t = \Big(\mu(\hat{x}_t, t) + \rho_{cx}\sigma(\hat{x}_t, t)\sigma_\Lambda\Big)\mathrm{d}t + \sigma(\hat{x}_t, t)\mathrm{d}W_t \tag{2.7}$$

This process has a modified process compared to the original process for the state variable. If the modification is equal to zero then the process for the state variable is the same as the original process, which implies that prices are set by risk-neutral investors (or equivalent to risk-neutral investors). If the modification is not equal to zero then there is a risk premium or a risk discount. By Monte Carlo simulations of the modified process in Equation (2.7), it is possible to compute the expectation in Equation (2.6), which gives the value of the zero-coupon bond. This allows to also derive the instantaneous return on bonds ($\mathrm{d}Q/Q$), which follows directly from Ito's Lemma as shown in Equation (2.4).

Finally, if the zero-coupon bond price is integrated up to infinity then this gives rise to the price of a perpetuity:

$$Z(x_t, t) \equiv \int_t^{\infty} Q(x_t, t, s)\mathrm{d}s \tag{2.8}$$

2.2.1.2 Price-Dividend Ratio

Based on the zero-coupon bonds it is also possible to derive the price-dividend ratio of a dividend-paying security. In particular, if at time t the price of the security is U_t and the dividend stream is D_t, then the price-dividend ratio is defined as:[6]

$$u_t(X_t) \equiv \frac{U_t}{D_t} \tag{2.9}$$

Where D_t follows the process:

$$\frac{\mathrm{d}D_t}{D_t} = \mu_D(x_t, t)\mathrm{d}t + \sigma_D(x_t, t)\mathrm{d}W_{Dt} \tag{2.10}$$

In order, to show the connection to zero-coupon bonds, I also define the dividend strip $Y_t(T)$, which pays an amount equal to the dividend of the security only at a specific time T ($Y_t(0) = D_t$). Then the strip price ratios are:

$$y_t(m) \equiv \frac{Y_t(m)}{D_t}, \quad y_t(0) \equiv 1 \tag{2.11}$$

By definition the price of the dividend paying security is the sum of the prices of the dividend strips:

$$U_t = \int_0^{\infty} Y_t(s)\mathrm{d}s \tag{2.12}$$

[6]Given the homotheticity of preferences, the price-dividend and price-consumption ratios are only functions of the state variable.

The price-dividend ratio of the dividend paying security is then:[7]

$$\frac{U_t}{D_t} \equiv u_t = \int_0^\infty \frac{Y_t(s)}{D_t} \mathrm{d}s = \int_0^\infty y_t(s) \mathrm{d}s \tag{2.13}$$

Given this setup we can again apply Equation (2.1) as before:[8]

$$\mathrm{E}\Big[\mathrm{d}\Big(\Lambda Y(x_t, t, m)\Big)\Big] = 0 \quad \forall t \in (t_0, T), \quad Y(x, t, 0) = g(t) \quad \forall x \tag{2.14}$$

This can then be expressed in terms of the ratio y_t as (where I stop showing the explicit dependence on the arguments for simplicity):[9]

$$\begin{aligned} &\mathrm{E}\Big[\mathrm{d}\Big(\Lambda y_t D_t\Big)\Big] = 0 \\ \Rightarrow &\mathrm{E}\Big[\frac{\mathrm{d}\Lambda}{\Lambda} y + \mathrm{d}y + \frac{\mathrm{d}D}{D_t} y + \frac{\mathrm{d}\Lambda}{\Lambda} \mathrm{d}y + \frac{\mathrm{d}\Lambda}{\Lambda}\frac{\mathrm{d}D}{D} y + \frac{\mathrm{d}D}{D} \mathrm{d}y\Big] = 0 \end{aligned} \tag{2.15}$$

In addition, Ito's lemma also applies to the price-dividend ratio, giving rise to an expression similar to Equation (2.4):

$$\mathrm{d}y_t = \left(\frac{\partial y_t}{\partial t} - \frac{\partial y_t}{\partial m} + \mu(x_t, t)\frac{\partial y_t}{\partial x_t} + \frac{1}{2}\sigma(x_t, t)^2 \frac{\partial^2 y_t}{\partial x_t^2}\right)\mathrm{d}t + \sigma(x_t, t)\frac{\partial y_t}{\partial x_t}\mathrm{d}W_{xt} \tag{2.16}$$

By inserting the expressions for y_t (Equation 2.16), D_t (Equation 2.10), and Λ_t (Equation 2.3) into Equation (2.15) we get:

$$\begin{aligned} &- r(x_t, t) y_t + \frac{\partial y_t}{\partial t} - \frac{\partial y_t}{\partial m} + \mu(x_t, t)\frac{\partial y_t}{\partial x_t} + \frac{1}{2}\sigma(x_t, t)^2 \frac{\partial^2 y_t}{\partial x_t^2} + \mu_D(x_t, t) y_t \\ &+ \rho_{cx}\sigma(x_t, t)\sigma_\Lambda \frac{\partial y_t}{\partial x_t} + \rho_{cD}\sigma_D(x_t, t)\sigma_\Lambda y_t + \rho_{xD}\sigma_D(x_t, t)\sigma(x_t, t)\frac{\partial y_t}{\partial x_t} = 0 \end{aligned} \tag{2.17}$$

which is similar to Equation (2.5) but with additional terms that depend on the dividend process. This PDE can also be solved using the Feynman-Kac formula:

$$y_t(x_t, t, m) = \mathrm{E}\left[\exp\left(-\int_t^{t+m} \tilde{r}(\tilde{x}_s, s)\mathrm{d}s\right) g(t+m) \middle| \tilde{x}_t = x_t\right] \tag{2.18}$$

where

$$\tilde{r}(x_t, t) = r(x_t, t) - \mu_D(x_t, t) - \rho_{cD}\sigma_D(x_t, t)\sigma_\Lambda \tag{2.19}$$

[7] A similar expression for discrete time is given in Wachter (2006).

[8] Now, I have adapted the notation as the price of the strip depends on the state variable and time explicitly.

[9] The derivation here is similar to Chen, Cosimano and Himonas (2010)

is adjusted due to the extra terms coming from the dividend stream. And the process for the modified state variable is given by:

$$\mathrm{d}\tilde{x}_{it} = \Big(\sigma(\tilde{x}_{it}, t)(\rho_{cx}\sigma_\Lambda + \rho_{xD}\sigma_D) + \mu(\tilde{x}_t, t)\Big)\mathrm{d}t + \sigma(\tilde{x}_t, t)\mathrm{d}W_{xt} \tag{2.20}$$

Using the ratios of the prices of the dividend strips over the current dividend for each maturity it is possible to integrate over all maturities to get the price-dividend ratio of the dividend-paying security as shown in Equation (2.13).

Finally, the return of the dividend-paying security can be computed:

$$\begin{aligned}
&\frac{\mathrm{d}U_t}{U_t} + \frac{D_t}{U_t}\mathrm{d}t = \frac{\mathrm{d}(u_t D_t)}{u_t D_t} + \frac{1}{u_t}\mathrm{d}t = \frac{\mathrm{d}u_t}{u_t} + \frac{\mathrm{d}D_t}{D_t} + \frac{\mathrm{d}u_t \mathrm{d}D_t}{u_t D_t} + \frac{1}{u_t}\mathrm{d}t \\
=&\frac{1}{u_t}\Bigg(\bigg(\frac{\partial u_t}{\partial t} + \mu(x_t, t)\frac{\partial u_t}{\mathrm{d}x_t} + \frac{1}{2}\sigma(x_t, t)^2\frac{\partial^2 u}{\partial x^2} + \mu_D(x_t, t)u_t + \frac{1}{2}\sigma_D(x_t, t)\sigma\frac{\partial u_t}{\partial x_t} + 1\bigg)\mathrm{d}t \\
&+ \bigg(\sigma(x_t, t)\frac{\partial u_t}{\partial x_t} + \sigma_D(x_t, t)\bigg)\mathrm{d}W_t\Bigg)
\end{aligned} \tag{2.21}$$

Where Ito's Lemma has been applied to u_t.

2.2.2 Implementation

The implementation of the package follows closely the logic of the Feynman-Kac formula. In order to get the price of a zero-coupon security, the modified state variable(s) needs to be simulated and the values of these simulations are used to simulate the stochastic integral of the corresponding short rate.[10] The user needs to specify formulas for the drift and diffusion of the state variables, while the for the stochastic the r function needs to be given as a drift and 0 should be given as the diffusion. Each simulation of the stochastic integral is expressed as:[11]

$$\mathcal{I}_i \approx -\int_t^{t+m} \underbrace{r(\hat{x}_s)}_{\text{drift for simulation}} \mathrm{d}s, \quad i = 1, 2, \ldots, N_s \tag{2.22}$$

[10]In the case of the continuous payoff security the modified short rate should be given by the user as defined in Equation 2.19.

[11]The simulation of the following integral is handled internally by the StochasticDiffEq.jl package, as is the simulation of the state variables.

where N_s is the number of simulations used. Given a large number of samples the price of the security is approximated by:

$$Q(x_t, t, m) = \mathrm{E}\left[\exp\left(-\int_t^{t+m} r(\hat{x}_s)\mathrm{d}s\right)\right] g(t+m, x_{t+m}) \approx \frac{1}{N_s}\sum_{i=1}^{N_s} \exp\left(\mathcal{I}_i\right) g(t+m, x_{t+m}) \tag{2.23}$$

This becomes a good approximation for a large enough number of simulations.

The main function of the package, which performs the computation above is *solve* which takes as input a variable of type *Problem* and a variable of type *SolutionSettings*. It then returns a variable of type *Solution*. The *Problem* type contains the information for the drift and the diffusion of the processes to be simulated, it also contains the terminal function for the payoff (typically just equal to one at maturity). The drift and diffusion functions for the processes that are meant to be simulated are given as they would be for the standard DifferentialEquations package in Julia.[12]. The *SolutionSettings* type contains information that is necessary for the solution of the problem, such as the grid for the state variable, the number of simulations to be performed, and the algorithm to be used for the simulations.[13] This type can also be given a specification that a continuous payoff variable is being simulated. In this case, prices of dividend strips are computed[14] and they are then also integrated to give the price-dividend ratio. Finally, the *Solution* type returns the result of the computation that can be called as a normal function. This means that it can be called with specific arguments for time and the state variable(s) to return the price of the zero-coupon security or the price-dividend ratio of the dividend-paying security, if a continuous payoff specification is given.[15]

Finally, a convenience function is also given to compute the derivatives of the price-dividend ratio with respect to the state variable. This can facilitate the computation of the return of the dividend-paying security as shown in Equation (2.21).[16]

A more detailed technical description of the package is provided in Appendix

[12]Or the more specialised package StochDiffEq.jl for stochastic differential equations.

[13]The algorithm is one of the algorithms offered in the standard DifferentialEquations.jl package. More information can be found in the documentation page.

[14]From the point of view of the code these are exactly the same as zero-coupon securities. The difference should be in the drift of the given process, that should be modified as specified in 2.13.

[15]The *Solution* type is further subdivided into *SinglePayoffSolution* and *ContinuousPayoff-Solution*.

[16]At the point of writing this function can only applies when the problem has one state variable.

2.B, while the following examples illustrate how the package can be used.

2.3 Examples

2.3.1 One State Variable

2.3.1.1 Time-Varying Consumption Drift – Zero-Coupon Security

While the package can be used with any process for theSDF, the examples in this paper are from the context of a consumption-based model, in which the investor has CRRA utility. The specific code and all the results for the examples are shown in Jupyter notebooks that are included in Appendix 2.C.[17] The consumption process is exogenous and given by:

$$\mathrm{d}\log C_t = \mathrm{d}c_t = \mu_c(x_t)\mathrm{d}t + \sigma_c \mathrm{d}W_t, \quad \mu_c(x_t) = \mu_{c0} + x_t \tag{2.24}$$

By Ito's Lemma and the fact that $\Lambda = e^{-\rho t}C^{-\gamma}$ the process for the SDF is given by:

$$\frac{\mathrm{d}\Lambda}{\Lambda} = \Big(-\rho - \gamma\mu_c(x_t) + \frac{1}{2}\gamma^2\sigma_c^2\Big)\mathrm{d}t - \gamma\sigma_c\mathrm{d}W_t \tag{2.25}$$

And this also provides the function for the short rate:

$$r(x_t) = -\mathrm{E}\left[\frac{\mathrm{d}\Lambda}{\Lambda}\right]\frac{1}{\mathrm{d}t} = \rho + \gamma\mu_c(x_t) - \frac{1}{2}\gamma^2\sigma_c^2 \tag{2.26}$$

The process for the state variable is given by:

$$\mathrm{d}x = -\log\phi(\bar{x} - x_t)\mathrm{d}t + \sigma_x\mathrm{d}W_t \tag{2.27}$$

where $\bar{x}$ is the point at which the process has a drift of zero, which I also call the *stochastic steady state*. So, based on Equation (2.7), the process for the modified state variable is:

$$\mathrm{d}\hat{x} = \underbrace{\Big(-\log\phi(\bar{x} - \hat{x}_t) - \gamma\rho_{cx}\sigma_x\sigma_c\Big)}_{\text{drift for simulation}}\mathrm{d}t + \underbrace{\sigma_x}_{\text{diffusion for simulation}}\mathrm{d}W_t \tag{2.28}$$

Apart from computing the regular price of the zero-coupon bond, I also compute the price of the risk-neutral zero-coupon bond. In this case I simulate the unmodified state variable of the problem. Then the term premium can be computed as

[17] These examples are also included as part of the code of the package to facilitate users.

the difference between the yield and the risk-neutral yield. As can be seen in the results, the term structure can be either upward or downward-sloping (positive or negative yield spread), depending on the value of the state variable. This is because the state variable is expected to revert to the stochastic steady state. So, long-term yields which are in some respect combinations of expected future short-rates are higher (lower) compared to short-term yields, when short rates are expected to increase (decrease). In addition, within this stylised model, even though consumption drift is significantly variable, the term premium is negative and tiny.

2.3.1.2 Time-Varying Consumption Diffusion - Zero-Coupon Security

In this example, instead of having a time-varying consumption drift as in the previous example, I have a time-varying consumption diffusion. The consumption process is given by:

$$\mathrm{d}\log C_t = \mathrm{d}c_t = \mu_c \mathrm{d}t + \sigma_c(x_t)\mathrm{d}W_t, \quad \sigma_c(x_t) = \begin{cases} \frac{2\sigma_{c0}}{1+\exp(-2x)} & \text{if } x < 0 \\ \frac{4\sigma_{c0}}{1+\exp(-x)} - 1 & \text{otherwise} \end{cases} \tag{2.29}$$

while the state variable follows the same process as in the previous example. The process could have also followed a CIR process and then be used as the consumption diffusion. This would also ensure that the consumption diffusion is positive. However, I use this relatively different process to show that the package can handle processes that are non-affine in terms of the state variable. In addition, I avoid using a simple exponential that would ensure the positivity of the consumption diffusion, because the exponential can increase too fast and make the some paths unstable. The functional form above ensures that the consumption diffusion is bounded between 0 and $3\sigma_{c0}$, where σ_{c0} is the value at the stochastic steady state. The modified state is then given by:

$$\mathrm{d}\hat{x} = \Big(- \log\phi(\bar{x} - \hat{x}_t) - \gamma\rho_{cx}\sigma_x\sigma_c(\hat{x}_t)\Big)\mathrm{d}t + \sigma_x \mathrm{d}W_{ct} \tag{2.30}$$

As can be seen in the results, the short term rate is slightly decreasing with consumption volatility due to the precautionary savings motive of agents and the term premium is negative and also very small.

2.3.1.3 Time-Varying Consumption Drift - Dividend-Paying Security

In this example, I show how the package can be used to compute the price-dividend ratio of a dividend-paying security for the same consumption process as in the first example. In general, it is possible for the dividend to follow any process. However, here I assume that the dividend process follows the same process as the consumption process. In this case the price-dividend ratio is called price-consumption ratio, and dividend strips are called consumption strips. Here, the modified process is not the same as in the case with a zero-coupon bond. Following Equation (2.20) the modified process is given by:

$$\mathrm{d}\tilde{x} = \underbrace{\Big(- \log\phi(\bar{x} - \tilde{x}_t) - (\gamma - 1)\rho_{cx}\sigma_x\sigma_c \Big)}_{\text{drift for simulation}} \mathrm{d}t + \underbrace{\sigma_x}_{\text{diffusion for simulation}} \mathrm{d}W_{ct} \tag{2.31}$$

Unlike the case of the zero-coupon bond for the consumption strip the short rate function that is used in the simulation is also modified. So, following Equation (2.19) the modified short rate is given by:

$$\tilde{r}(\tilde{x}_t, t) = \rho + \gamma\mu_c(\tilde{x}_t) - \frac{1}{2}\gamma^2\sigma_c^2 - \mu_c(\tilde{x}_t) + \gamma\sigma_c^2 \tag{2.32}$$

The results show that when consumption drift is significantly varying the price of the consumption perpetuity is significantly volatile. In addition, it is possible to use the resulting price-consumption ratio calculate its derivatives and then use these to get the return of the security as a function of the state variable.[18] The return of the security is very close to the short term rate verifying the idea behind the equity premium puzzle. In particular, this model would predict a very small equity premium, which is not consistent with the data.

2.3.1.4 Time-Varying Consumption Diffusion - Dividend-Paying Security

The dividend paying security can also be computed when consumption diffusion is time-varying. Again I assume that the dividend is following the same process as consumption. So, the modified process for the state variable is given by:

$$\tilde{r}(\tilde{x}_t, t) = \rho + \gamma\mu_c - \frac{1}{2}\gamma^2\sigma_c(x_t)^2 - \mu_c + \gamma\sigma_c(x_t)^2 \tag{2.33}$$

[18] Here, the derivatives work best when the solution is computed based on an interpolation function that is calculated based on the DataInterpolations package.

Interestingly for $\gamma = 2$ the function above becomes a constant which makes the price-consumption ratio also a constant. This implies that holding consumption constant and changing consumption volatility actually has no effect on prices and returns. However, the short rate is still a decreasing function of consumption volatility so the premium is increasing in the consumption diffusion.

2.3.2 Two State Variables

Finally, the package also works with more than one state variables. In this example, I allow consumption drift and consumption diffusion to vary independently. So, the consumption process is given by:

$$\mathrm{d}c_t = \mu_c(x_{1t})\mathrm{d}t + \sigma_c(x_{2t})\Big(1 - |\rho_{cx1}| - |\rho_{cx2}|\Big)\mathrm{d}W_{c1t} + \rho_{cx1}\sigma_c(x_{1t})\mathrm{d}W_{x1t} + \rho_{cx2}\sigma_c(x_{2t})\mathrm{d}W_{x2t} \tag{2.34}$$

Where $\mu_c(\cdot)$ and $\sigma_c(\cdot)$ are the same as in Subsections 2.3.1.1 and 2.3.1.2 respectively. And the processes of the state variables are given by:

$$\begin{aligned}
\mathrm{d}x_{1t} &= -\log\phi_1(\bar{x}_1 - x_{1t})\mathrm{d}t + \sigma_{x1}\frac{1}{1+\rho_{12}}\mathrm{d}W_{x1t} + \sigma_{x1}\frac{\rho_{12}}{1+\rho_{12}}\mathrm{d}W_{x2t} \\
\mathrm{d}x_{2t} &= -\log\phi_2(\bar{x}_2 - x_{2t})\mathrm{d}t + \sigma_{x2}\frac{\rho_{21}}{1+\rho_{21}}\mathrm{d}W_{x1t} + \sigma_{x1}\frac{\rho_{21}}{1+\rho_{21}}\mathrm{d}W_{x2t}
\end{aligned} \tag{2.35}$$

W_{ct}, W_{x1t}, and W_{x2t} are independent Wiener processes, but based on the structure above the correlations between the various components are:

$$\begin{aligned}
\mathrm{E}[\mathrm{d}c_t\mathrm{d}x_{1t}] &= \left(\rho_{cx1}\frac{1}{1+\rho_{12}} + \rho_{cx2}\frac{\rho_{12}}{1+\rho_{12}}\right)\sigma_c(x_{2t})\sigma_{x1}\mathrm{d}t \approxeq \rho_{cx1}\sigma_c\sigma_{x1}\mathrm{d}t \\
\mathrm{E}[\mathrm{d}c_t\mathrm{d}x_{2t}] &= \left(\rho_{cx2}\frac{1}{1+\rho_{21}} + \rho_{cx1}\frac{\rho_{21}}{1+\rho_{21}}\right)\sigma_c(x_{2t})\sigma_{x2}\mathrm{d}t \approxeq \rho_{cx2}\sigma_c(x_{2t})\sigma_{x_2}\mathrm{d}t \\
\mathrm{E}[\mathrm{d}x_{1t}\mathrm{d}x_{2t}] &= \sigma_{x1}\sigma_{x2}\frac{\rho_{21}+\rho_{12}}{1+\rho_{12}+\rho_{21}+\rho_{12}\rho_{21}}\mathrm{d}t \approxeq (\rho_{12}+\rho_{21})\sigma_{x1}\sigma_{x2}\mathrm{d}t
\end{aligned} \tag{2.36}$$

with the approximate equalities being valid when ρ_{12} and ρ_{21} are small. The benefit of this setup is that consumption diffusion is equal to $\sigma_c(\hat{x}_{2t})$, the diffusion of the state variables is close to σ_{x1} and σ_{x2}, when ρ_{12} and ρ_{21} are small, and a correlation structure can still be maintained between the consumption process and the state variables by an appropriate choice of parameters ρ_{cx1}, ρ_{cx2}, ρ_{12}, and ρ_{21}. For example, a negative correlation between consumption and consumption diffusion can be specified by letting ρ_{cx2} be negative, or a correlation between the two state variables can be specified without introducing correlation with consumption by letting ρ_{12} and ρ_{21} be different from zero. Similar to above the modified process is given by:

$$\mathrm{d}\hat{x}_{1t} = \Big(-\log\phi_1 \cdot (\bar{x}_1 - \hat{x}_{1t}) + \rho_{cx1}\sigma_c(x_t)\sigma_{x1}\Big)\mathrm{d}t + \sigma_{x1}\frac{1}{1+\rho_{12}}\mathrm{d}W_{x1t} + \sigma_{x1}\frac{\rho_{12}}{1+\rho_{12}}\mathrm{d}W_{x2t}$$
$$\mathrm{d}\hat{x}_{2t} = \Big(-\log\phi_2 \cdot (\bar{x}_2 - \hat{x}_{2t}) + \rho_{cx2}\sigma_c(x_t)\sigma_{x2}\Big)\mathrm{d}t + \sigma_{x2}\frac{\rho_{21}}{1+\rho_{21}}\mathrm{d}W_{x1t} + \sigma_{x2}\frac{1}{1+\rho_{21}}\mathrm{d}W_{x2t} \tag{2.37}$$

Finally, the short rate is unmodified and a function of two variables:

$$r(x_{1t}, x_{2t}) = -\mathrm{E}\left[\frac{\mathrm{d}\Lambda}{\Lambda}\right]\frac{1}{\mathrm{d}t} = \rho + \gamma\mu_c(x_{1t}) - \frac{1}{2}\gamma^2\sigma_c(x_{2t})^2 \tag{2.38}$$

Seen as a function of one state variable at a time the results are not significantly different compared to the examples before. However, the two-variable model can be used to show further moments including cross moments between different financial variables.

2.4 Application

The package allows the computation of asset prices in response to changes in the state variable and/or in consumption. As also mentioned in the introduction, the literature has focused on two kinds of shocks, Delphic and Odyssean. Delphic shocks reveal information about the underlying state of the economy, and Odyssean shocks introduce an unexpected monetary policy. Performing an analysis in the context of a consumption-based model highlights the fact that interest rates and asset prices ultimately only move when some component of the SDF changes (or is perceived to change). This holds regardless whether the central bank is revealing information or whether it is committing to a different monetary policy. In addition, the analysis highlights the importance of the channel through which monetary policy is conducted. For example, a standard monetary tightening is supposed to decrease output and this increases expected output growth as the economy is expected to revert to the steady state. This implies that the real short-term rate increases to counteract the increased consumption smoothing motive, and asset prices fall due to the higher discount rate. This literature makes the assumption that on the day of a monetary policy announcement the announcement itself is causing the changes in asset prices and not vice versa. And while this is reasonable and in most cases should be true, the observation of an increase in interest rates and a decrease in asset prices does not necessarily imply that the central bank has *caused* this effect by tightening monetary policy, if the central bank could just be revealing information. Indeed the central bank could be directly revealing that output growth has increased for other reasons (in the same way that it would have increased had the actual monetary policy changed), and this by definition should produce exactly the same response of interest rates and asset prices. So, even when we observe the "correct" pattern for monetary policy we cannot exclude the possibility of a Delphic/information shock.

Still one could claim that when the "wrong" pattern occurs, then it *is* due to a Delphic shock. However even in this case, monetary policy could be affecting the economy through different channels, even before the episodes of explicitly unconventional monetary policy. In this section I explore the behaviour of asset prices under when different components of the SDF change, and my results suggest that, whatever the pattern, it is not trivial to classify as Odyssean/monetary shocks and Delphic/information shocks. I analyse two main cases, in the first the state variable is consumption drift and in the second it is consumption diffusion. One could ask whether monetary policy can affect consumption diffusion. And while this channel is less standard, such a relationship can find support in at least two different strands of literature. Firstly, there is literature suggesting the importance of the "risk-taking channel" of monetary policy (Borio and Zhu 2012; Adrian and Shin 2010), and such a channel could be modelled as affecting consumption diffusion in the context of a consumption-based model.[19] Secondly, Vayanos and Vila (2021) has also suggested that the term structure of interest rates is driven by arbitrageurs taking on more or less risk. In their model, this would directly translate to more or less wealth volatility, which can be naturally modelled as consumption volatility within a model that has exogenous consumption.[20]

In the following cases that I examine, I will compare my results to the classi-

[19]A different approach within a consumption-based model would be to assume that the risk aversion parameter itself can stochastically change as in Lettau and Wachter (2011).

[20]In the original model there is no consumption and arbitrageurs are just optimising the mean and variance of their portfolio value. In particular, arbitrageurs are rather not associated with real consumers but with financial institutions. Here, I use a model with consumption without any explicit wealth, but this could be thought of as modelling either the behaviour of real consumers or the behaviour of financial institutions that use the a consumption-based SDF. While Vayanos and Vila (2021) is mostly associated with the conduct of unconventional monetary policy, it suggests a more general explanation of the term structure. So, assuming that monetary policy has played an important role in shaping the term structure of interest rates even before the introduction of unconventional monetary policies, it is arguable that arbitrageurs would be adjusting their levels of risk even before the introduction of unconventional monetary policies.

fication of monetary policy announcements in Jarociński and Karadi (2020) (JK) and Cieslak and Schrimpf (2019) (CS), which I summarize in Table 2.1. In JK the co-movement of stock prices with the short-term rate is exclusively considered as a criterion to classify shocks into pure monetary policy shocks and information shocks. In the case of CS the yields of long-term bonds are also considered, and essentially the difference is that information shocks are sub-categorised into "growth" and "risk premium" shocks, in the former (latter) short-term (long-term) yields react more aggressively than long-term (short-term) yields. The calibration for both cases is shown in Table 2.2.

	Shock	Yields short	Yields long	Stocks	Stock-Yield Co-movement
Jarociński and Karadi (2020)	Monetary policy:	$\uparrow$	-	$\downarrow$	-
	Information:	$\uparrow$	-	$\uparrow$	+
Cieslak and Schrimpf (2019)	Monetary policy:	$\uparrow$	$\scriptstyle\uparrow$	$\downarrow$	-
	Growth:	$\uparrow$	$\scriptstyle\uparrow$	$\uparrow$	+
	Risk premium:	$\scriptstyle\downarrow$	$\downarrow$	$\downarrow$	+

Table 2.1: This table is partly taken from a corresponding table in Cieslak and Schrimpf (2019), to which I have added the classification in Jarociński and Karadi (2020). In the latter paper the authors do not use long-term yields in the classification and they do not compare the size of the movements. The former paper uses both kinds of yields and compares the size of the movements. This is expressed through the size of the arrows in the table.

2.4.1 Case 1: Time-Varying Consumption Drift

In the first case that I analyse, the underlying model has a time-varying consumption drift. And the central bank is also able to "externally" affect the variables in the model. I assume that this takes place without adding an extra state variable.[21]

[21] Technically, the monetary policy variable should also have a distribution. However, for the stylized model I use here, I just assume that monetary policy can affect the economy in an unanticipated way. A different approach would be to introduce monetary policy as a separate state variable or assume that the state variable is already equivalent to monetary policy. The results would not be significantly different.

Parameter	Model	
	Time-varying consumption drift	Time-varying consumption diffusion
γ	1/2	1/2/2.5
ϕ	0.82	0.82
$\bar{x}$	0.0	0.0
ρ	0.02	0.02
μ_{c0}	0.01	-
μ_c	state variable	0.01
σ_{c0}	-	0.06
σ_c	0.01	state variable
σ_x	0.005	0.5
ρ_{cx}	0.3	0.3

Table 2.2: Calibration for the two cases

So, the processes evolve according to:

$$\begin{aligned} \mathrm{d}x_t &= -\log\phi(x_t - x_t)\mathrm{d}t + \sigma_x \mathrm{d}W_{xt} + \mathcal{M}_x \mathrm{d}q_t \\ \mathrm{d}c_t &= (\mu_{c0} + x_t)\mathrm{d}t + \sigma_c \mathrm{d}W_t + \mathcal{M}_c \mathrm{d}q_t \end{aligned} \tag{2.39}$$

Where $\mathrm{d}q_t$ is a Poisson shock assumed to be activated when the monetary policy announcement occurs. $\mathcal{M}_x$ and $\mathcal{M}_c$ express the size of the effect on the state variable and consumption respectively.[22] I focus on the effect of the effect of the shock on yields and on an asset that pays dividends equal to consumption. I call this asset *consumption perpetuity*.

If the announcement only affects the current level of consumption ($\mathcal{M}_x = 0, \mathcal{M}_c \neq 0$), then the short-term rate is unaffected, the price consumption ratio is unaffected, but the price of the consumption perpetuity undergoes a level shift that persists in time.[23] This is because there is only one state variable in the

[22] For mathematical consistency with the rest of the model, as also mentioned in the previous footnote, $\mathrm{d}q_t$ should have 0 intensity, which implies that the probability of such a change is equal to 0.

[23] When discussing how monetary policy announcements affect variables, this could be taking place either by monetary policy literally affecting the variable or by revealing information and

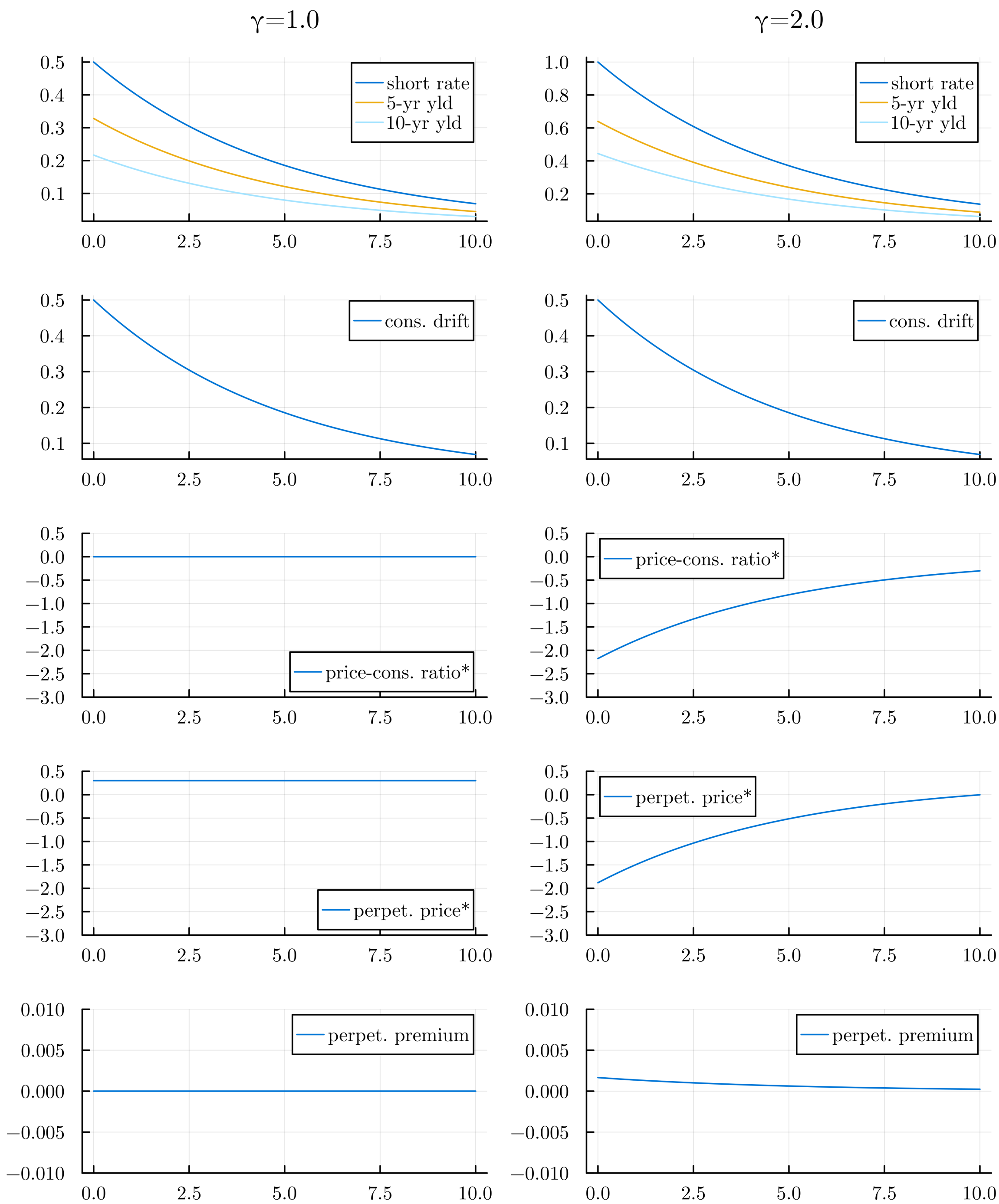


Figure 2.1: Impulse Responses to Consumption Drift Shock
These are responses after a consumption drift change. The period shown is ten years. The size of the shock corresponds to one standard deviation. I interpret these responses as taking place after a monetary policy announcement, and I assess to which type of shock they correspond. The short-term rate increases, but the response of the asset price is different depending on the risk aversion parameter. *Plots normally show percent deviation from the stochastic steady state. Plots with starred variables show percent relative deviations from the stochastic steady state.

model other than consumption, which by itself does not affect prices and the price-consumption ratios. In addition, the jump in consumption does not dissipate given the process chosen for consumption, which explains the persistence of the effect on the price of the consumption perpetuity. In a more realistic case in which the announcement affects both variables ($\mathcal{M}_x > 0, \mathcal{M}_c < 0$) the results are shown in Figure 2.1. For both risk aversion values the short-term rate increases, while longer-term yields increase also but less with maturity due to the mean reversion of the steady state. In addition, in both cases there is no significant change in the equity premium. While this change look more like a traditional monetary policy shock, it turns out that for the special case of $\gamma = 1$ the price-consumption ratio is constant as the higher discounting exactly offsets the higher expected dividends. The price of the asset still undergoes a level increase due to the increase in the dividends, which do not revert to the previous level. When $\gamma = 2.0$ the traditional pattern of monetary policy arises, in which the short rate goes up and the asset price goes down, with both effects subsequently dying out.[24] This is consistent with the monetary policy classification in both JK and CS. Nevertheless, these results highlight that ultimately the effects occur in connection to a shift in the consumption drift. So, if the central bank can affect the economy through an information channel, then the shift could be *caused* or alternatively it could be *revealed* by the central bank. Therefore, even in this conventional monetary policy case, a method is required to distinguish between the two alternatives.

2.4.2 Case 2: Time-Varying Consumption Diffusion

For the second case, I work on the model with time-varying consumption diffusion. Again I focus on yields and on the consumption perpetuity. Here the processes evolve according to:

$$\begin{aligned} \mathrm{d}x_t &= -\log\phi(x_t - x_t)\mathrm{d}t + \sigma_x \mathrm{d}W_{xt} + \mathcal{M}_x \mathrm{d}q_t \\ \mathrm{d}c_t &= \mu_c \mathrm{d}t + \sigma_c(x_t)\mathrm{d}W_t + \mathcal{M}_c \mathrm{d}q_t \end{aligned} \tag{2.40}$$

making investors aware that a variable has some value. For simplicity I do not explicitly and separately model perceived and real variables.

[24]Even in this case though there is a lasting residual effect on the price of the dividend-paying security that is due to the increase of the dividend which in this model does not revert to the previous level. There is also a tiny increase in the equity premium, significantly less than 0.01%, as the equity premium depends on the state variable, but only very slightly given that consumption volatility is constant.

with σ_c defined in Equation (2.29). The case where only consumption increases are the same as were discussed before. The effect of the monetary policy announcement affecting an increase in consumption volatility ($\mathcal{M}_x > 0, \mathcal{M}_c = 0$) is shown in Figure 2.2. Here, I use a higher average consumption volatility compared to the previous case, in order for consumption volatility changes to have a significant effect on the short rate and on returns. I also use a positive correlation between consumption volatility changes and consumption changes as was primarily done in Melissinos (2023).[25] For all three values of the risk aversion parameter ($\gamma = 1.0, 2.0.2.5$), the short-term rate decreases albeit with different intensity. This would be associated with an easing of monetary policy. In addition, despite consumption volatility and the equity premium rising considerably in all three cases, the shock is never classified as a risk premium shock of CS because long-term yields react less than short-term yields. This is due to the reversion of the state variable to the steady state, which implies that over the long run, the state variable should revert to its previous value. In addition, consistent with an information shock in JK and a growth shock in CS the price of the asset decreases when $\gamma = 1.0$ when the short-term rate also decreases.[26] Interestingly, this has nothing to do with a growth shock conceptually as by construction only consumption volatility is affected. On the contrary, the case of a growth shock (shift in consumption drift) was analysed in the previous subsection, and it had the effect of a monetary policy shock according to the classifications. Furthermore, there are two more cases for the risk aversion parameter that would imply a different classification. For $\gamma = 2.0$ the price of the asset remains constant, which is an alternative not considered by JK or CS. For $\gamma = 2.5$ the price of the asset increases, when the short-term rate decreases. So, this would actually be classified as a normal monetary policy shock both by JK and CS. The model variations analysed here are highly stylised, but the results are nevertheless suggestive. If monetary policy is inducing investors to take on more risk, by moving the interest rate, then the effect on asset prices may depend on the value of the risk aversion parameter of the investors who are responsible for pricing the assets. And there are some cases, in which, unlike what is assumed in most classifications of monetary policy shocks, the central bank *causes* the short-term rate and asset prices to move in the same direction.

[25]For this analysis using negative correlation would produce practically the same impulse response functions.

[26]This is also consistent with a corresponding result in Bansal and Yaron (2004).

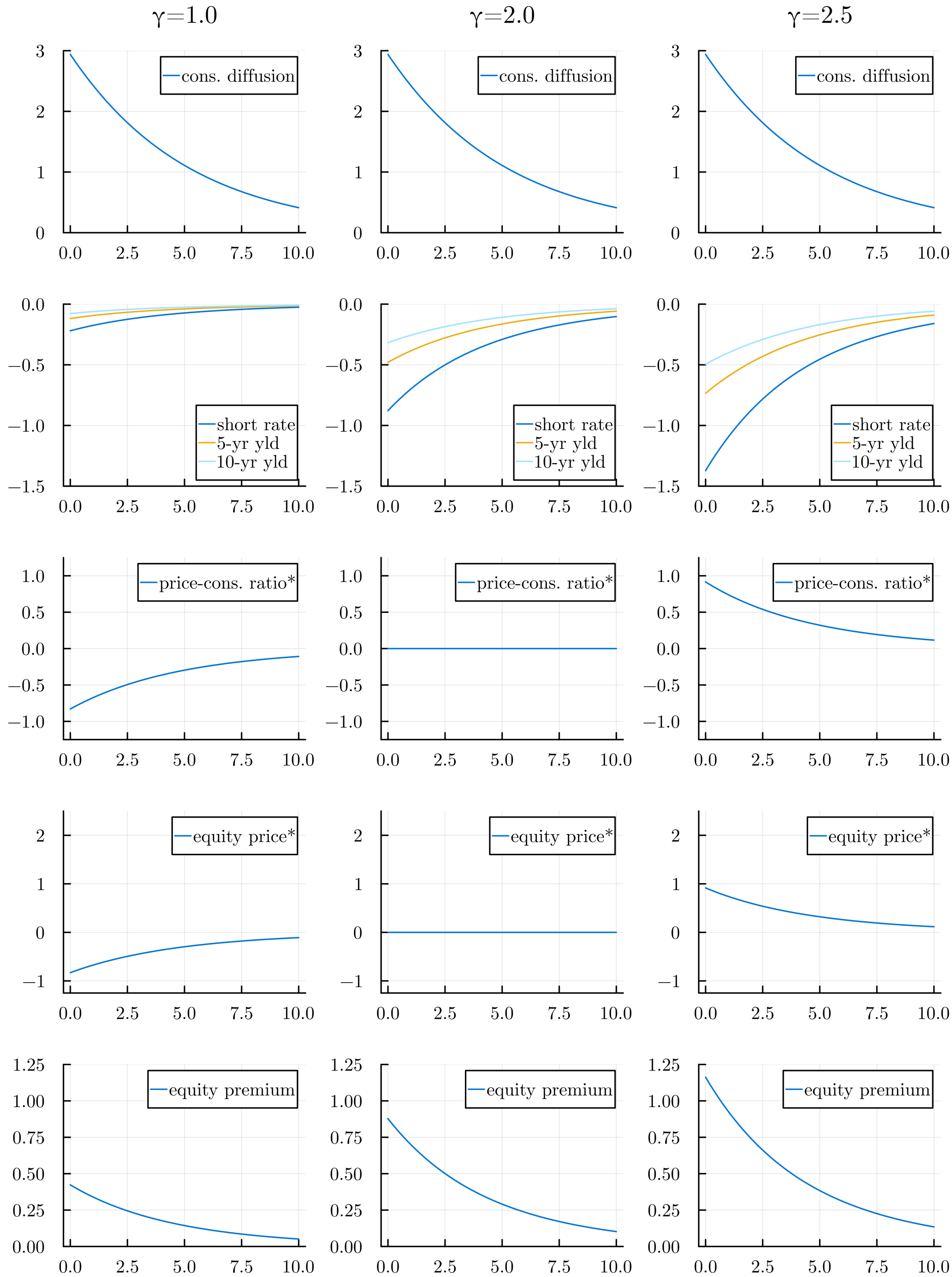


Figure 2.2: Impulse Responses to Consumption Diffusion Shock
These are responses after a consumption diffusion change. The period shown is ten years. The size of the shock corresponds to one standard deviation. I interpret these responses as taking place after a monetary policy announcement, and I assess to which type of shock they correspond. The short-term rate and the expected excess return (equity premium) always fall, but the response of the asset price is different depending on the risk aversion parameter.
*Plots normally show percent deviation from the stochastic steady state. Plots with starred variables show percent relative deviations from the stochastic steady state.

2.5 Conclusion

The primary goal of this paper is to introduce and explain a Julia package that is able to facilitate asset pricing in settings in which the process of the SDF is known. The package takes advantage of the already available DifferentialEquations.jl package, that is used to simulate SDEs. The package can then generate price functions for fixed income securities. In addition, the price-dividend ratio of dividend-paying securities can also be computed, as long as the joint processes of the SDF and the dividend are given. The package can handle single-variable and multi-variable problems for both fixed income and dividend-paying securities. In the case of single-variable problems the package also facilitates the calculation of the expected return. I illustrate the use of the package in fully worked out examples that have been explained in the paper and whose code is included in the Appendix.[27] The examples include consumption-based models in which the consumption process is exogenous and generates an SDF process. In the examples both one and two state variables are used.

The package is suited for academic research, and I illustrate this with an application, in which I study the possible effects of monetary policy announcements on interest rates and asset prices. While several papers have performed classification of monetary policy announcements, based on responses of financial markets, few provide an explicit theory that shows how these effects are channeled.[28] In general higher interest rates are associated with higher discounting and hence lower asset prices, but if monetary policy increases interest rates via a channel that also increases dividend flows, it becomes non-trivial whether asset prices should increase or decrease. In addition, if monetary policy runs through a risk-taking channel the effects on asset prices can again be ambiguous. My results highlight two things. Firstly, once it is accepted that the central bank can affect the economy by revealing information about the state of the economy, then it becomes difficult to identify a pure monetary policy shock even if financial variables follow the anticipated pattern. Secondly, given unconventional monetary policy or to the extent that monetary policy does not operate through the traditional channel, it is possible that "tightening" ("easing") does not necessarily lead to a decrease (increase) in asset prices. This suggests that the identification of information shocks should also be accompanied by specific theory that outlines the potential

[27]The examples are also included in the Github page for the package.

[28]An exception is Cieslak and Schrimpf (2019), who included a non-consumption-based macro-finance model.

channels of monetary policy.

In the future, I am planning to extend the functionality of the package. In particular, as first steps I am planning to a) add the possibility of discrete jumps of the relevant variables, b) include specialised functions to compute the prices of special payoff structures that correspond to different known financial securities, and c) facilitate the choice of underlying solution algorithms, in order to strike the desired balance between speed and accuracy for each kind of problem. Next, functions can be added for the calibration and/or estimation of models to macroeconomic and/or financial data.

Appendix

2.A Multivariable formulas

- Equation 2.2:

$$\mathrm{d}x_i = \mu_i(X_t, t)\mathrm{d}t + \sum_{j=1}^{M} \sigma_{i,j}(X_t, t)\mathrm{d}W_{jt}, \quad i = 1, 2, \ldots, N, \quad \mathrm{E}[\mathrm{d}W_{jt}\mathrm{d}W_k(t)] = \rho_{j,k}\mathrm{d}t$$

  where $X_t = (x_1, x_2, \ldots, x_N)$, μ_i is the drift of state variable i, W_{jt} is the j-th Wiener process, M is the number of Wiener processes, $\sigma_{i,j}$ is the diffusion of state variable i with respect to W_{jt}, and $\rho_{j,k}$ is the correlation between the jth and kth Wiener processes.

- Equation 2.3:

$$\frac{\mathrm{d}\Lambda}{\Lambda} = \mu_\Lambda(X_t, t)\mathrm{d}t + \sum_{j=1}^{M} \sigma_{\Lambda,j}(X_t, t)\mathrm{d}W_{jt}$$

- Equation 2.4:

$$\begin{aligned}
\mathrm{d}Q = &\left(\frac{\partial Q}{\partial t} - \frac{\partial Q}{\partial m} + \sum_{i=1}^{N} \mu_i(X_t, t)\frac{\partial Q}{\partial x_i} \right. \\
&\left. + \frac{1}{2}\sum_{a=1}^{N}\sum_{b=1}^{N}\sum_{c=1}^{M}\sum_{d=1}^{M} \rho_{c,d}\sigma_{a,c}(X_t, t)\sigma_{b,d}(X_t, t)\frac{\partial^2 Q}{\partial x_a \partial x_b} \right)\mathrm{d}t \\
&+ \sum_{i=1}^{N}\sum_{j=1}^{M} \sigma_{i,j}(X_t, t)\frac{\partial Q}{\partial x_i}\mathrm{d}W_{jt}
\end{aligned}$$

- Equation 2.5:

$$\begin{aligned}
&\mathrm{E}\left[\frac{\mathrm{d}\Big(\Lambda Q\Big)}{\Lambda}\right] = 0 \Rightarrow \mathrm{E}\left[\frac{\mathrm{d}\Lambda}{\Lambda}Q + \mathrm{d}Q + \frac{\mathrm{d}\Lambda}{\Lambda}\mathrm{d}Q\right] = 0 \\
&\Rightarrow -r(X, t)Q + \frac{\partial Q}{\partial t} - \frac{\partial Q}{\partial m} + \sum_{i=1}^{N} \mu_i(X_t, t)\frac{\partial Q}{\partial x_i} \\
&\quad + \frac{1}{2}\sum_{a=1}^{N}\sum_{b=1}^{N}\sum_{c=1}^{M}\sum_{d=1}^{M} \rho_{c,d}\sigma_{a,c}(X_t, t)\sigma_{b,d}(X_t, t)\frac{\partial^2 Q}{\partial x_a \partial x_b} \\
&\quad + \sum_{i=1}^{N}\sum_{j=1}^{M}\sum_{k=1}^{M} \rho_{j,k}\sigma_{i,j}(X_t, t)\sigma_{\Lambda,k}\frac{\partial Q}{\partial x_i} = 0
\end{aligned}$$

- Equation 2.6:

$$Q(X_t, t, m) = \mathrm{E}\left[\exp\left(-\int_t^{t+m} r(\hat{X}_s, s)\mathrm{d}s\right)g(t+m)\middle|\hat{X}_t = X_t\right]$$

where $\hat{X} = (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n)$.

- Equation 2.7:

$$\mathrm{d}\hat{x}_{it} = \left(\sum_{j=1}^{M}\sum_{k=1}^{M}\rho_{j,k}\sigma_{i,j}(\hat{X}_t, t)\sigma_{\Lambda,k} + \mu_i(\hat{X}_t, t)\right)\mathrm{d}t + \sum_{j=1}^{M}\sigma_{i,j}(\hat{X}_t, t)\mathrm{d}W_{jt}$$

- Equation 2.8:

$$Z(X_t, t) \equiv \int_t^{\infty} Q(X_t, t, s)\mathrm{d}s$$

- Equation 2.9:

$$u_t(X_t) \equiv \frac{U_t}{D_t(X_t)}$$

- Equation 2.10:

$$\frac{\mathrm{d}D_t}{D_t} = \mu_D(X_t, t)\mathrm{d}t + \sum_{j=1}^{M}\sigma_{D,j}(X_t, t)\mathrm{d}W_{jt}$$

- Equation 2.14:

$$\mathrm{E}\Big[\mathrm{d}\Big(\Lambda Y(X_t, t, m)\Big)\Big] = 0 \quad \forall t \in (t_0, T), \quad Y(X, t, 0) = g(t) \quad \forall X$$

- Equation 2.16:

$$\mathrm{d}y = \left(\frac{\partial y}{\partial t} - \frac{\partial y}{\partial m} + \sum_{i=1}^{N}\mu_i(X_t, t)\frac{\partial y}{\partial x_i} + \frac{1}{2}\sum_{a=1}^{N}\sum_{b=1}^{N}\sum_{c=1}^{M}\sum_{d=1}^{M}\rho_{c,d}\sigma_{a,c}(X_t, t)\sigma_{b,d}(X_t, t)\frac{\partial^2 y}{\partial x_a \partial x_b}\right)\mathrm{d}t$$
$$+ \sum_{i=1}^{N}\sum_{j=1}^{M}\sigma_{i,j}(X_t, t)\frac{\partial y}{\partial x_i}\mathrm{d}W_{jt}$$

- Equation 2.17:

$$
\begin{aligned}
&-r(X,t)y_t+\frac{\partial y_t}{\partial t}-\frac{\partial y_t}{\partial m}+\sum_{i=1}^{N}\mu_i(X_t,t)\frac{\partial y_t}{\partial x_i}+\frac{1}{2}\sum_{a=1}^{N}\sum_{b=1}^{N}\sum_{c=1}^{M}\sum_{d=1}^{M}\rho_{c,d}\sigma_{a,c}(X_t,t)\sigma_{b,d}(X_t,t)\frac{\partial^2 y_t}{\partial x_a\partial x_b}\\
&+\mu_D(X_t,t)y_t+\sum_{i=1}^{N}\sum_{j=1}^{M}\sum_{k=1}^{M}\rho_{j,k}\sigma_{i,j}(X_t,t)\sigma_{\Lambda,k}\frac{\partial y_t}{\partial x_i}+\sum_{j=1}^{M}\sum_{k=1}^{M}\rho_{j,k}\sigma_{D,j}(X_t,t)\sigma_{\Lambda,k}y_t\\
&+\sum_{i=1}^{N}\sum_{j=1}^{M}\sum_{k=1}^{M}\rho_{j,k}\sigma_{D,j}(X_t,t)\sigma_{i,j}(X_t,t)\frac{\partial y_t}{\partial x_i}=0
\end{aligned}
$$

- Equation 2.18:

$$
y_t(X_t,t,m)=\mathrm{E}\left[\exp\left(-\int_t^{t+m}\tilde{r}(\tilde{X}_s,s)\mathrm{d}s\right)g(t+m)\middle|\tilde{X}_t=X_t\right]
$$

- Equation 2.19:

$$
\tilde{r}(X_t,t)=r(X_t,t)+\mu_D(X_t,t)+\sum_{j=1}^{M}\sum_{k=1}^{M}\rho_{j,k}\sigma_{D,j}(X_t,t)\sigma_{\Lambda,k}
$$

- Equation 2.20:

$$
\mathrm{d}\tilde{x}=\left(\sum_{j=1}^{M}\sum_{k=1}^{M}\rho_{j,k}\sigma_{i,j}(X_t,t)(\sigma_{\Lambda,k}+\sigma_{D,k})+\mu_i(\tilde{X}_t,t)\right)\mathrm{d}t+\sum_{j=1}^{M}\sigma_{i,j}(\tilde{X}_t,t)\mathrm{d}W_{jt}
$$

- Equation 2.21:

$$
\begin{aligned}
\frac{\mathrm{d}U_t}{U_t}+\frac{D_t}{U_t}\mathrm{d}t=&\frac{\mathrm{d}(u_tD_t)}{u_tD_t}+\frac{1}{u_t}\mathrm{d}t=\frac{\mathrm{d}u_t}{u_t}+\frac{\mathrm{d}D_t}{D_t}+\frac{\mathrm{d}u_t\mathrm{d}D_t}{u_tD_t}+\frac{1}{u_t}\mathrm{d}t\\
=&\frac{1}{u_t}\Bigg(\Bigg(\frac{\partial u_t}{\partial t}+\sum_{i=1}^{N}\mu_i(X_t,t)\frac{\partial u_t}{\mathrm{d}x_i}+\frac{1}{2}\sum_{a=1}^{N}\sum_{b=1}^{N}\sum_{c=1}^{M}\sum_{d=1}^{M}\rho_{c,d}\sigma_{a,c}(X_t,t)\sigma_{b,d}(X_t,t)\frac{\partial^2 u}{\partial x_a\partial x_b}\\
&+\frac{\mu_D(X_t,t)}{D_t}u_t+\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{M}\sum_{k=1}^{M}\rho_{j,k}\sigma_{D,j}(X_t,t)\sigma_{i,k}\frac{\partial u_t}{\partial x_i}\Bigg)\mathrm{d}t\\
&+\left(\sum_{i=1}^{N}\sum_{j=1}^{M}\sigma_{i,j}(X_t,t)\frac{\partial u_t}{\partial x_i}+\sum_{j=1}^{M}\sigma_{D,j}(X_t,t)\right)\mathrm{d}W_{jt}\Bigg)
\end{aligned}
$$

2.B Documentation

2.C Examples

Example 1 – One state variable

Time-Varying Consumption Drift – Zero-Coupon Bond

The state variable is associated with the consumption drift. Given a CRRA utility function the SDF process can be computed, inserted in the pricing equation and then solved using a Feynman-Kac formula. The modified state variable follows the process:

$$\mathrm{d}\hat{x}_t = \big(-\log\phi(\bar{x}-\hat{x}_t) + \rho_{cx}\sigma_c\sigma_x\big)\mathrm{d}t + \sigma_x \mathrm{d}W_{xt}$$

While the state variable is not modified when there is no correlation between the process for consumption and the process for the state variable:

$$\mathrm{d}x_t = -\log\phi(\bar{x}-x_t)\mathrm{d}t + \sigma_x \mathrm{d}W_{xt}$$

In order to get the price of the zero-coupon security a process for the integral of the short-term rate will also be needed:

$$\mathrm{d}\mathcal{I} = r(\bar{x}_t)\mathrm{d}t$$

Import the packages

[1]:
```
import SDFPricing as sdf
import StochasticDiffEq as sde # this is needed in order to specify the algorithm
```

Define the parameters

[2]:
```
cs = (
    phi = 0.92, # mean reversion
    xbar = 0.0, # long-run mean
    rho = 0.01, # time preference parameter
    gamma = 2, # risk aversion
    muc0 = 0.005, # mean of consumption drift
    sigmac = 0.01, # consumption diffusion
    sigmax = 0.005, # state variable diffusion
    rhocx = 0.3 # correlation between consumption and state variable
);
```

Drift and Diffusion of the processes I also include the unmodified process which will correspond to "risk-neutral pricing". By comparing normal pricing with risk-neutral pricing it is possible to compute excess returns.

[3]:
```
mu0(x,c) = -log(c.phi)*(c.xbar-x) # drift of unmodified state
sigma(x,c) = c.sigmax; # diffusion of both modified and unmodified state
mu(x,c) = mu0(x,c)-c.rhocx*c.gamma*c.sigmac*sigma(x,c) # drift of modified state
```

[3]:
```
mu (generic function with 1 method)
```

Short-term rate function

[4]:
```
r(x,c) = c.rho+c.gamma*(c.muc0+x)-c.gamma^2*c.sigmac^2/2;
r(x) = r(x,cs); # define with one argument for convenience
```

Define setup consistent with SDE package in Julia

[5]:
```
function drift(du,u,p,t,c)
    du[1] = mu0(u[1],c)
    du[2] = mu(u[2],c)
    du[3] = r(u[1],c)
    du[4] = r(u[2],c)
end
drift(du,u,p,t) = drift(du,u,p,t,cs);
function diffusion(du,u,p,t,c)
    du[1] = sigma(u[1],c)
    du[2] = sigma(u[2],c)
    du[3] = 0.0
    du[4] = 0.0
end
diffusion(du,u,p,t) = diffusion(du,u,p,t,cs);
```

Define the Problem and SolutionSettings variables

[6]:
```
prob = sdf.
 ↪Problem(drift=drift,diffusion=diffusion,numNoiseVariables=1,outVariables=[3,4],
terminalFunction=(ik, x, y, z) -> exp(-x));
xRange = -0.05:0.01:0.05;
sett = sdf.SolutionSettings(xRanges=[xRange,], initialValues=[[x, x, 0.0, 0.0]␣
 ↪for x in xRange],
algorithm=sde.LambaEM(), pathsPerInitialValue=10000, tRange=0.0:1.0:10.0);
```

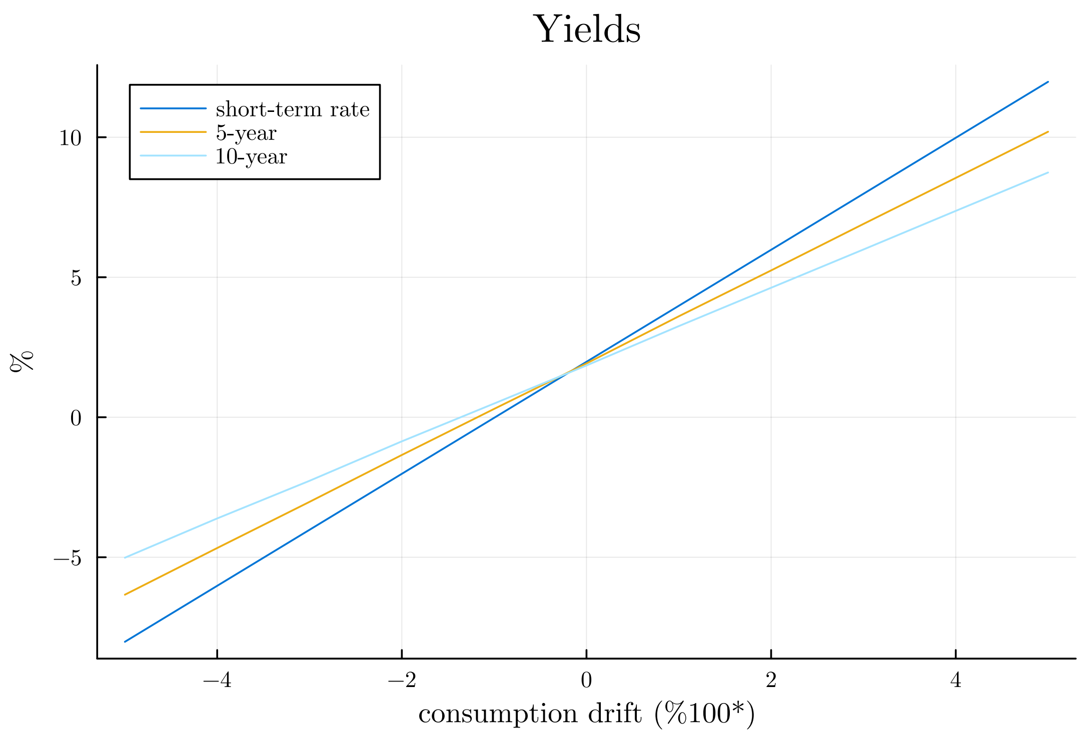
Solve Problem and Define Yield

[7]:
```
((bondPriceRiskNeutral,bondPrice),) = sdf.solve(prob, sett);
yld(t,x) = -log(bondPrice(t,x))/t;
yldRiskNeutral(t,x) = -log(bondPriceRiskNeutral(t,x))/t;
```

Plot the yield

[8]:
```
# colors: "#0075d6", "#edad14", "#a3e3ff", "#9c0000"
import Plots as plt
plt.default(titlefont= (14,"Computer Modern"),legendfont=(8,"Computer Modern"),
    tickfont=(8,"Computer Modern"),guidefont=(10,"Computer Modern"))
plt.plot(100*xRange, xRange .|> x->100*r(x), title="Yields",
    xlabel="consumption drift (%100*)",label="short-term␣
 ↪rate",color="#0075d6",ylabel="%")
plt.plot!(100*xRange, 100*yld.(5.0, xRange), label="5-year",color= "#edad14")
plt.plot!(100*xRange, 100*yld.(10.0, xRange), label="10-year",color="#a3e3ff")
```

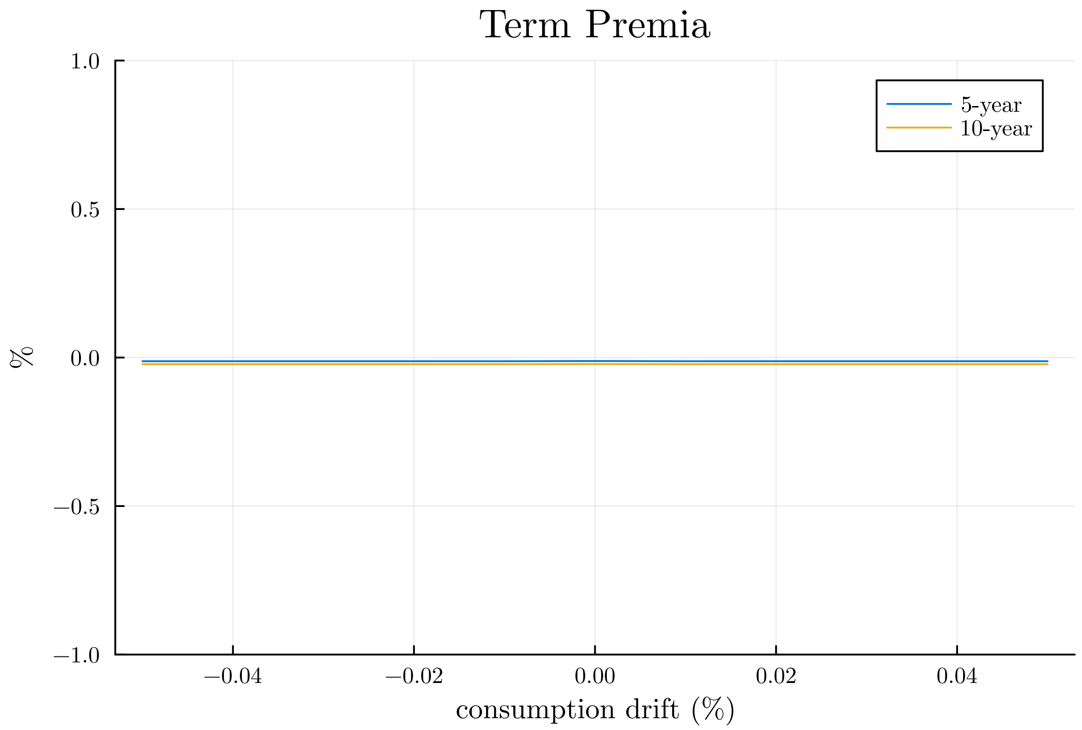
[8]:



Plot the term premium

[9]:
```
plt.plot(xRange, 100*(yld.(5.0, xRange) .- yldRiskNeutral.(5.0,␣
↪xRange)),title="Term Premia",
    xlabel="consumption drift (%)",label="5-year",ylims=(-0.005,0.
↪005),color="#0075d6",ylabel="%")
plt.plot!(xRange, 100*(yld.(10.0, xRange) .- yldRiskNeutral.(10.0, xRange)),
    label="10-year",ylims=(-1,1),color="#edad14")
```

[9]:



This shows that term premia in such a model with a time-varying consumption drift are negative, very small, and constant.

Example 2 – One state variable

Time-Varying Consumption Diffusion – Zero-Coupon Bond

The state variable is now associated with the consumption diffusion unlike example in which it was associated with consumption drift. Given a CRRA utility function the SDF process can be computed, inserted in the pricing equation and then solved using a Feynman-Kac formula. The modified state variable follows the process:

$$\mathrm{d}\hat{x}_t = \big(-\log\phi(\bar{x} - \hat{x}_t) + \rho_{cx}\sigma_{ct}\sigma_x \big)\mathrm{d}t + \sigma_x \mathrm{d}W_{xt}$$

While the state variable is not modified when there is no correlation between the process for consumption and the process for the state variable:

$$\mathrm{d}x_t = -\log\phi(\bar{x} - x_t)\mathrm{d}t + \sigma_x \mathrm{d}W_{xt}$$

In order to get the price of the zero-coupon security a process for the integral of the short-term rate will also be needed:

$$\mathrm{d}\mathcal{I} = r(\bar{x}_t)\mathrm{d}t$$

Import the packages

[1]:
```
import SDFPricing as sdf
import StochasticDiffEq as sde # this is needed in order to specify the algorithm
```

Define the parameters

[2]:
```
cs = (
    phi = 0.92, # mean reversion
    xbar = 0.0, # long-run mean
    rho = 0.01, # time preference parameter
    gamma = 2, # risk aversion
    muc0 = 0.005, # mean of consumption drift
    sigmac0 = 0.04, # consumption diffusion ###- higher compared to example 1
    sigmax = 0.5, # state variable diffusion ###- higher compared to example 1
    rhocx = -0.3 # correlation between consumption and state variable
);
```

Drift and Diffusion of the processes I also include the unmodified process which will correspond to "risk-neutral pricing". By comparing normal pricing with risk-neutral pricing it is possible to compute excess returns.

[3]:
```
###- now consumption diffusion is a non-linear function of the state,
###- given that it needs to be positive.
###- I use this function because the simple exponential can get too high for␣
↪some samples.
sigmac(x,c) = c.sigmac0*(x<0 ? 2/(1+exp(-2x)) : 4/(1+exp(-x))-1)
mu0(x,c) = -log(c.phi)*(c.xbar-x) # drift of unmodified state
```

```
sigma(x,c) = c.sigmax; # diffusion of modified and unmodified state
mu(x,c) = mu0(x,c)-c.rhocx*c.gamma*sigmac(x,c)*sigma(x,c) # drift of modified␣
↪state
```

[3]: mu (generic function with 1 method)

Short-term rate function

[4]:
```
r(x,c) = c.rho+c.gamma*c.muc0-c.gamma^2*sigmac(x,c)^2/2;
r(x) = r(x,cs);
```

Define setup consistent with SDE solution in Julia

[5]:
```
function drift(du,u,p,t,c)
    du[1] = mu0(u[1],c)
    du[2] = mu(u[2],c)
    du[3] = r(u[1],c)
    du[4] = r(u[2],c)
end
drift(du,u,p,t) = drift(du,u,p,t,cs);
function diffusion(du,u,p,t,c)
    du[1] = sigma(u[1],c)
    du[2] = sigma(u[2],c)
    du[3] = 0.0
    du[4] = 0.0
end
diffusion(du,u,p,t) = diffusion(du,u,p,t,cs);
```

Define the Problem and SolutionSettings variables

[6]:
```
prob = sdf.
 ↪Problem(drift=drift,diffusion=diffusion,numNoiseVariables=1,outVariables=[3,4],
terminalFunction=(ik, x, y, z) -> exp(-x));
xRange = -2.0:0.25:2.0;
sett = sdf.SolutionSettings(xRanges=[xRange,], initialValues=[[x, x, 0.0, 0.0]␣
 ↪for x in xRange],
algorithm=sde.LambaEM(), pathsPerInitialValue=20000, tRange=0.0:1.0:10.0);
```

Solve Problem and Define Yield

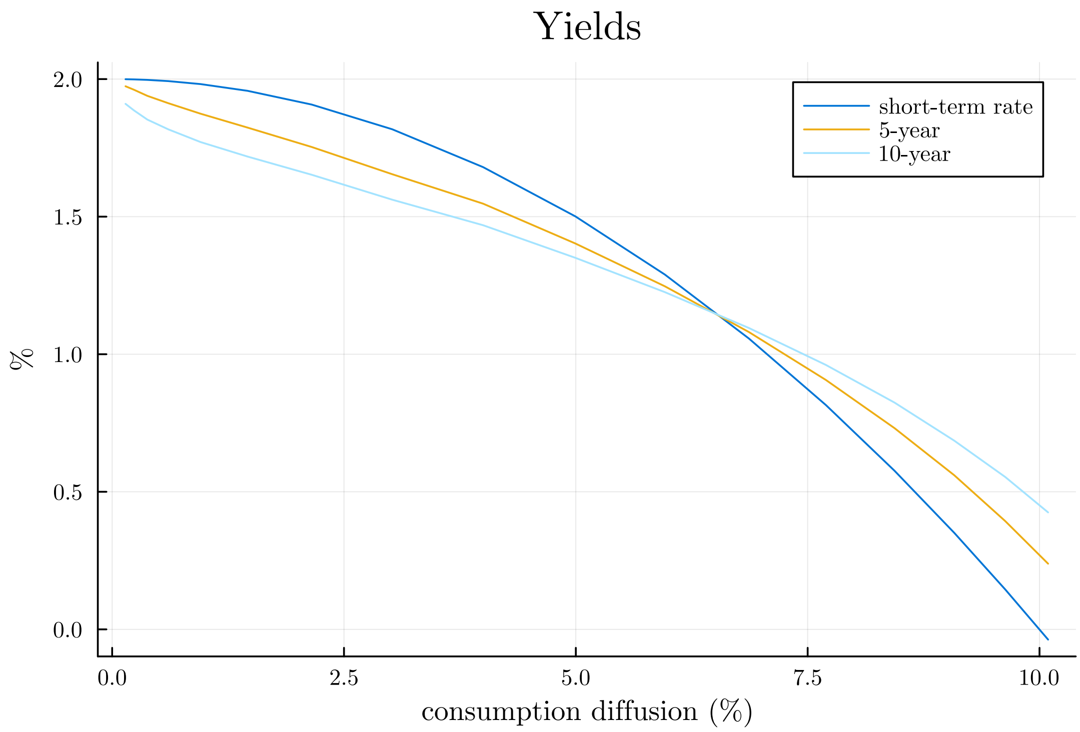
[7]:
```
((bondPriceRiskNeutral,bondPrice),) = sdf.solve(prob, sett);
yld(t,x) = -log(bondPrice(t,x))/t;
yldRiskNeutral(t,x) = -log(bondPriceRiskNeutral(t,x))/t;
```

Plot the yield

[8]:
```
# colors: "#0075d6", "#edad14", "#a3e3ff", "#9c0000"
import Plots as plt
plt.default(titlefont= (14,"Computer Modern"),legendfont=(8,"Computer Modern"),
    tickfont=(8,"Computer Modern"),guidefont=(10,"Computer Modern"))
plt.plot(xRange .|>x->100*sigmac(x,cs), xRange .|> x->100*r(x), title="Yields",
```

```
    xlabel="consumption diffusion (%)",label="short-term␣
↪rate",color="#0075d6",ylabel="%")
plt.plot!(xRange .|>x->100*sigmac(x,cs), 100*yld.(5.0, xRange),␣
↪label="5-year",color= "#edad14")
plt.plot!(xRange .|>x->100*sigmac(x,cs), 100*yld.(10.0, xRange),␣
↪label="10-year",color="#a3e3ff")
```

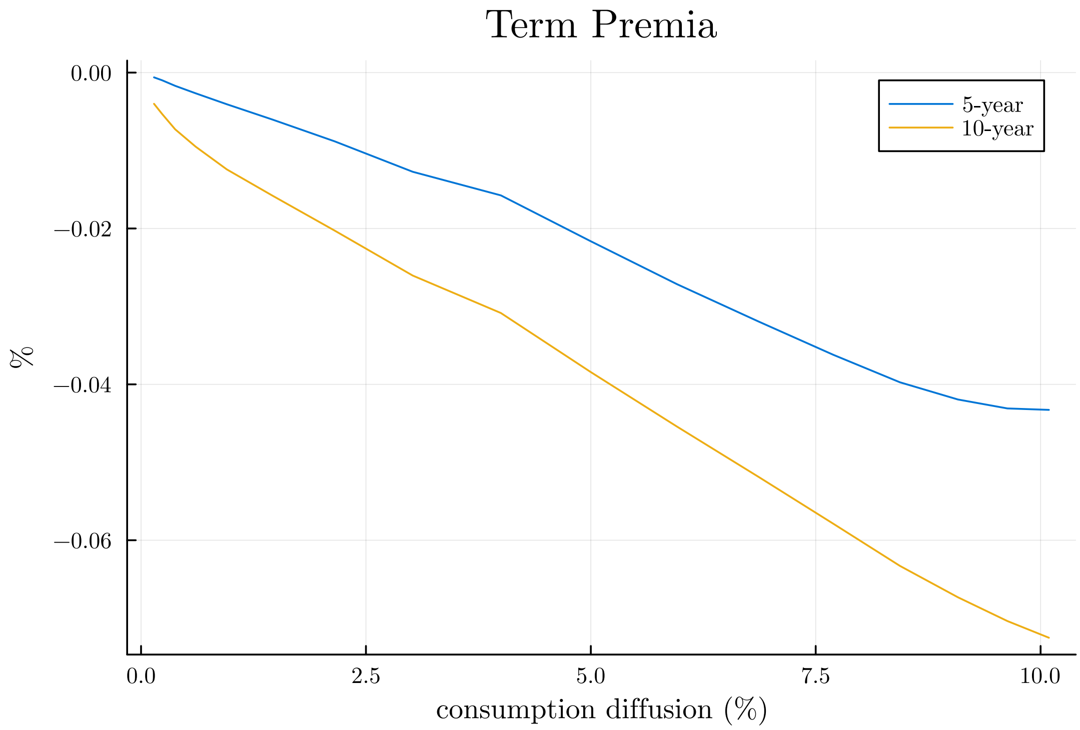
[8]:



Plot the term premium

[9]:
```
plt.plot(xRange .|>x->100*sigmac(x,cs), 100*(yld.(5.0, xRange) .- yldRiskNeutral.
↪(5.0, xRange)),title="Term Premia",
    xlabel="consumption diffusion (%)",label="5-year",color="#0075d6",ylabel="%")
plt.plot!(xRange .|>x->100*sigmac(x,cs), 100*(yld.(10.0, xRange) .-␣
↪yldRiskNeutral.(10.0, xRange)),
    label="10-year",color="#edad14")
```

[9]:



This shows that term premia are state-dependent when consumption diffusion is time-varying. They can also get larger in absolute value when consumption volatility is relatively high.

Example 3 – One state variable

Time-Varying Drift – Price Consumption Ratio

Here the setup is exactly the same as in example 1, but now I calculate the price consumption ratio instead of the price of zero coupon bond. By changing the values of the parameters it is also possible to compute a more general price-dividend ratio, for an asset that does not have the same dividend process as consumption. The modified process in this case is:

$$\mathrm{d}\tilde{x}_t = \big(-\log\phi(\bar{x}-\tilde{x}_t) + \rho_{cx}\sigma_c\sigma_x + \rho_{xD}\sigma_x\sigma_D\big)\mathrm{d}t + \sigma_x \mathrm{d}W_{xt}$$

In order to get the price of the zero-coupon security a process for the integral of the short-term rate will also be needed:

$$\mathrm{d}\mathcal{I} = r(\tilde{x}_t)\mathrm{d}t$$

Import the packages

[1]:
```
import SDFPricing as sdf
import StochasticDiffEq as sde # this is needed in order to specify the algorithm
```

Define the parameters

[2]:
```
cs = (
    phi = 0.92, # mean reversion
    xbar = 0.0, # long-run mean
    rho = 0.02, # time preference parameter
    gamma = 2, # risk aversion
    muc0 = 0.01, # mean of consumption drift
    sigmac = 0.01, # consumption diffusion
    sigmax = 0.005, # state variable diffusion
    rhocx = 0.3, # correlation between consumption and state variable
    # sigmaD = 0.02, # dividend diffusion ###- added compared to example 1
    # muD = 0.02, # dividend drift ###- added compared to example 1
    # rhoxD = 0.5, # correlation between dividends and state variable ###- added␣
↪compared to example 1
    # rhocD = 0.4 # correlation between dividends and consumption ###- added␣
↪compared to example 1
    sigmaD = 0.01, # dividend diffusion ###- case of consumption perpetuity
    muD = 0.01, # dividend drift ###- case of consumption perpetuity
    rhoxD = 0.3, # correlation between dividends and state variable ###- case of␣
↪consumption perpetuity
    rhocD = 1.0 # correlation between dividends and consumption ###- case of␣
↪consumption perpetuity
);
```

Drift and Diffusion of the processes I also include the unmodified process which will correspond to “risk-neutral pricing”. By comparing normal pricing with risk-neutral pricing it is possible to compute excess returns.

```
[3]: # diffusion of modified state
sigma(x,c) = c.sigmax;
# drift of modified state
mu(x,c) = -log(c.phi)*(c.xbar-x)-c.rhocx*c.gamma*c.sigmac*sigma(x,c)+c.
 ↪rhoxD*sigma(x,c)*c.sigmaD
```

```
[3]: mu (generic function with 1 method)
```

Short-term rate function

```
[4]: r(x,c) = c.rho+c.gamma*(c.muc0+x)-c.gamma^2*c.sigmac^2/2;
r(x) = r(x,cs);
muD(x) = cs.muD+x; #- case of consumption perpetuity
# muD(x) = cs.muD; #- perpetuity with constant dividend drift
rmod(x,c) = r(x,c)-(muD(x)-c.gamma*c.rhocD*c.sigmac*c.sigmaD);
rmod(x) = rmod(x,cs);
```

Define setup consistent with SDE solution in Julia

```
[5]: function drift(du,u,p,t,c)
    du[1] = mu(u[1],c)
    du[2] = rmod(u[1],c)
end
drift(du,u,p,t) = drift(du,u,p,t,cs);
function diffusion(du,u,p,t,c)
    du[1] = sigma(u[1],c)
    du[2] = 0.0
end
diffusion(du,u,p,t) = diffusion(du,u,p,t,cs);
```

Define the Problem and SolutionSettings variables In the theory I state that the price consumption ratio is computed from the integral over all consumption strip maturities. In practice it is not possible to integrate to infinity. So, I compute consumption strips up to a maturity of 300 years.

```
[6]: prob = sdf.
 ↪Problem(drift=drift,diffusion=diffusion,numNoiseVariables=1,outVariables=[2],
terminalFunction=(ik, x, y, z) -> exp(-x));
xRange = -0.05:0.006:0.05;
tRange = 0.0:5.0:300.0;
sett = sdf.SolutionSettings(xRanges=[xRange,], initialValues=[[x, 0.0] for x in␣
 ↪xRange],
algorithm=sde.LambaEM(), pathsPerInitialValue=5000, tRange=tRange);
# add the settings in order to compmute price-dividend ration of the continuous␣
 ↪payoff security
sett2 = sdf.SolutionSettings(sett; continuousPayoffVars=[2]);
```

Solve Problem, Get Yield and Price Consumption Ratio

[7]:
```
((consumptionStrip,),(priceConsumptionRatio,)) = sdf.solve(prob, sett2);
```

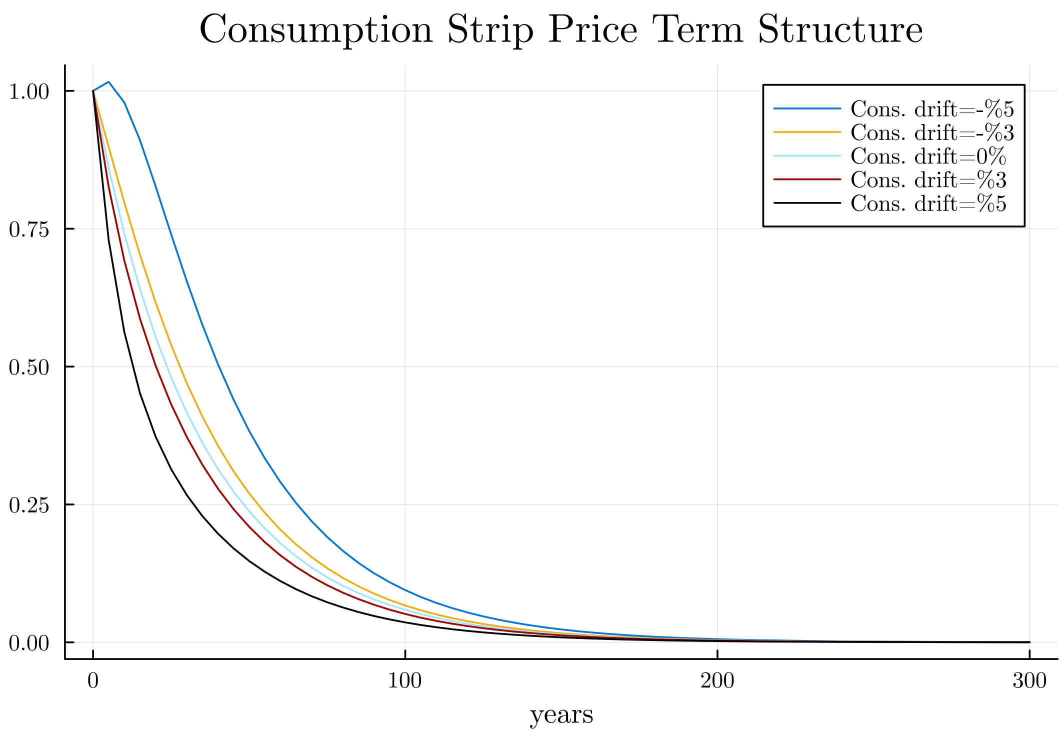
Get the Return of the Consumption Perpetuity The calculation requires the computation of the first and second derivatives of the price consumption ratio with respect to the state of the economy.

[8]:
```
(DPC,D2PC) = sdf.derivatives(priceConsumptionRatio);
ret(x) = (DPC(x)*(mu(x, cs) + sigma(x, cs) * cs.sigmac * cs.rhocx) +
    D2PC(x)* sigma(x, cs)^2/2.0 + 1.0)/priceConsumptionRatio(x) + muD(x);
```

Plot the Consumption Strip Term Structure

[9]:
```
import Plots as plt
# # colors: "#0075d6", "#edad14", "#a3e3ff", "#9c0000", "#000000"
plt.default(titlefont= (14,"Computer Modern"),legendfont=(8,"Computer Modern"),
    tickfont=(8,"Computer Modern"),guidefont=(10,"Computer Modern"))
plt.plot(tRange, consumptionStrip.(tRange, -0.04),color="#0075d6",
    title="Consumption Strip Price Term Structure", xlabel="years",label="Cons.␣
 ↪drift=-%5")
plt.plot!(tRange, consumptionStrip.(tRange, -0.01),label="Cons.␣
 ↪drift=-%3",color="#edad14")
plt.plot!(tRange, consumptionStrip.(tRange, 0.0),label="Cons.␣
 ↪drift=0%",color="#a3e3ff")
plt.plot!(tRange, consumptionStrip.(tRange, 0.01),label="Cons.␣
 ↪drift=%3",color="#9c0000")
plt.plot!(tRange, consumptionStrip.(tRange, 0.04),label="Cons.␣
 ↪drift=%5",color="#000000")
```

[9]:

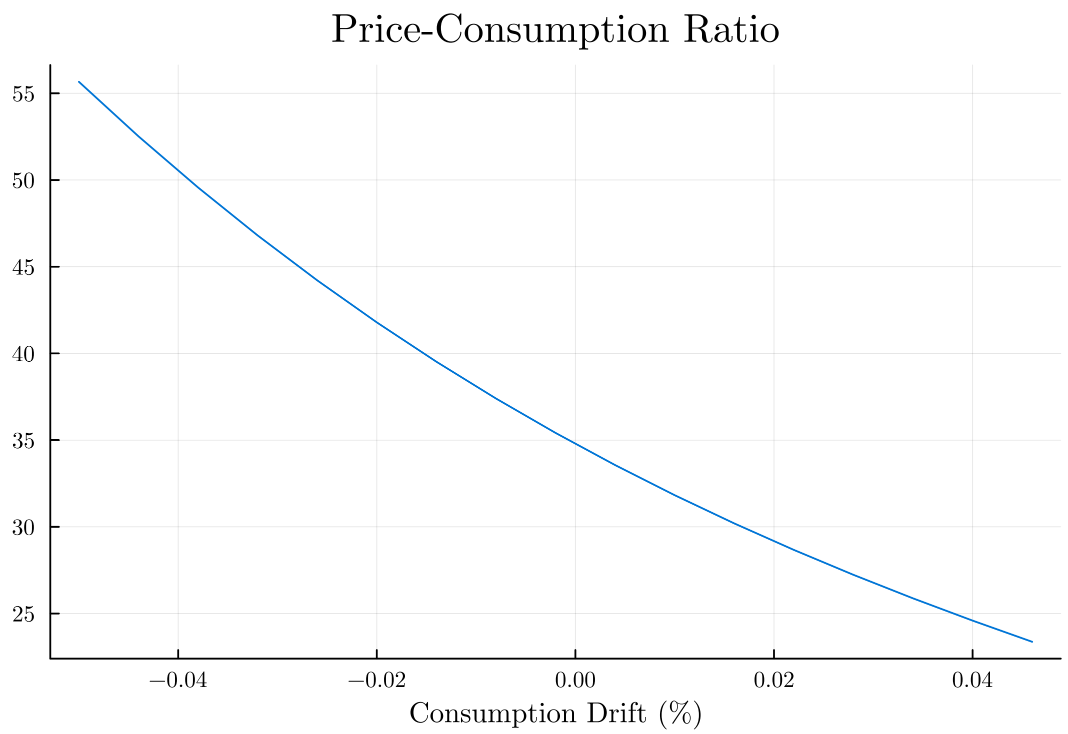


It can be seen that for all values of the state variable the price of the securities comes close to 0 for maturities as long as 300 years.

Plot the Price Consumption Ratio

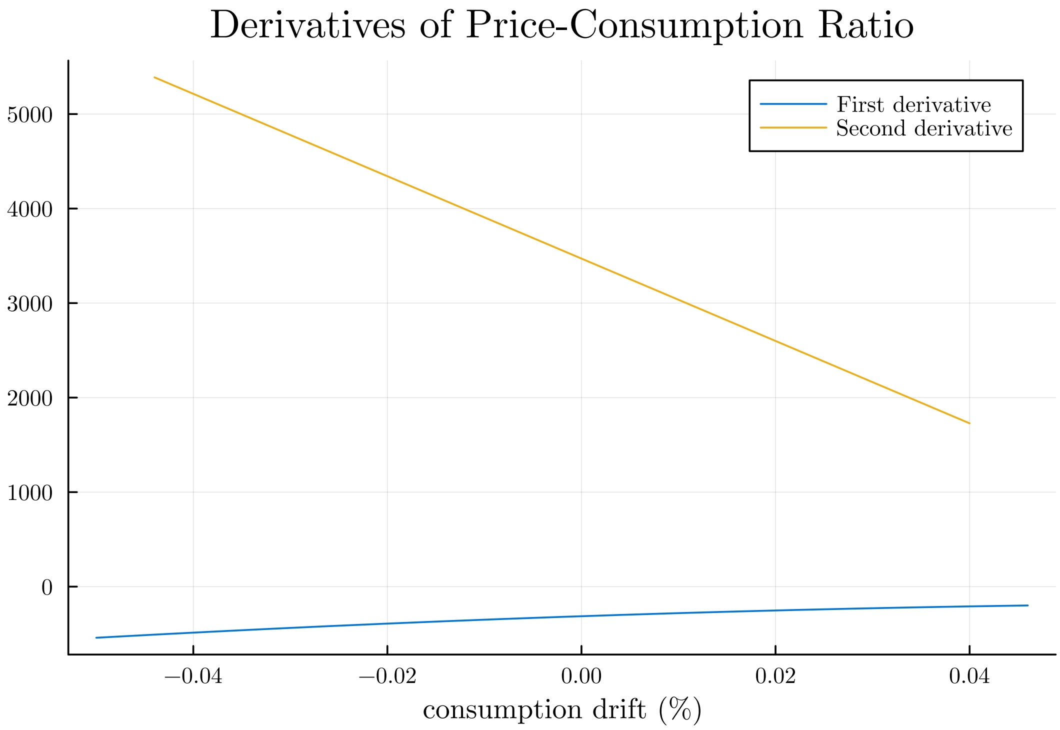
[10]:
```
plt.plot(xRange, priceConsumptionRatio(xRange), legend=false,
    title="Price-Consumption Ratio",color="#0075d6", xlabel="Consumption Drift␣
↪(%)")
```

[10]:



[11]:
```
plt.plot(xRange, DPC.(xRange),
    title="Derivatives of Price-Consumption Ratio",label="First␣
↪derivative",color="#0075d6")
plt.plot!(xRange[2:end-1], D2PC.(xRange[2:end-1]),label="Second␣
↪derivative",color="#edad14",xlabel="consumption drift (%)")
```

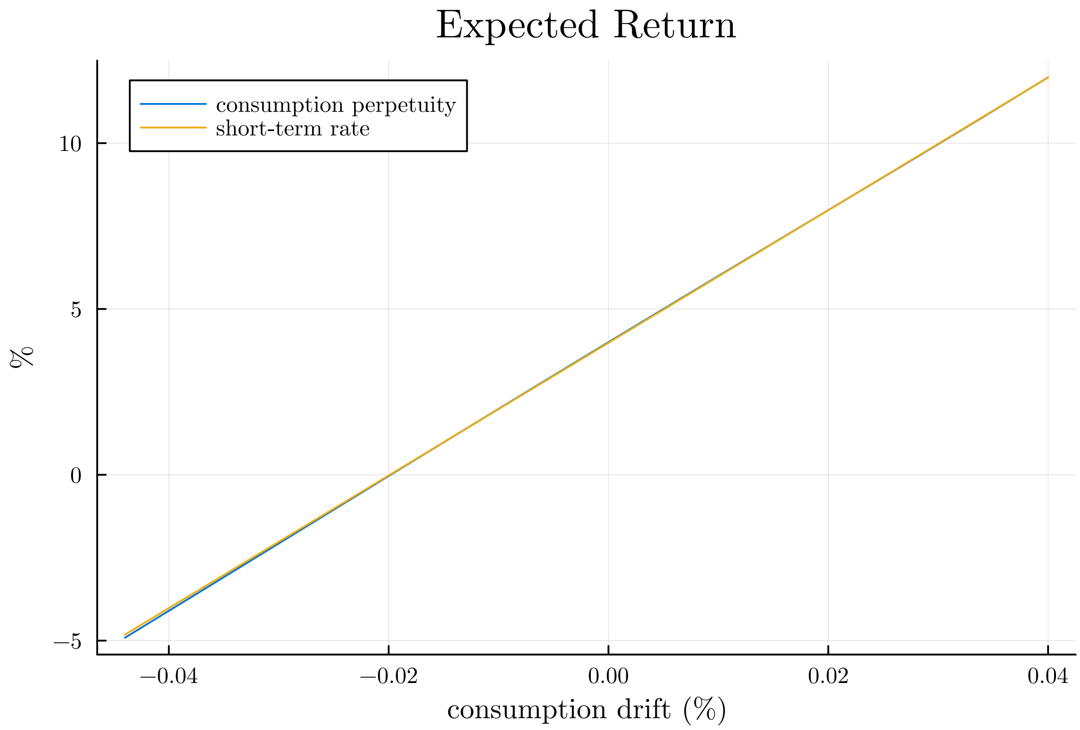
[11]:



Plot the Return

[12]:
```
plt.plot(xRange[2:end-1], 100*ret.(xRange[2:end-1]), label="consumption␣
↪perpetuity",
    title="Expected Return", color="#0075d6",xlabel="consumption drift␣
↪(%)",ylabel="%")
plt.plot!(xRange[2:end-1], 100*r.(xRange[2:end-1]),
    label="short-term rate",color="#edad14")
```

[12]:



As expected in the standard model with time-varying consumption drift the premium compared to the short-term rate is almost zero.

Example 4 – One state variable

Time-Varying Consumption Drift – Price Consumption Ratio

Here the setup is exactly the same as in example 2, but now I calculate the price consumption ratio instead of the price of zero coupon bond. The modified process in this case is:

$$\mathrm{d}\tilde{x}_t = \big(- \log \phi(\bar{x}_0 - \tilde{x}_t) + \rho_{cx}\sigma_{ct}\sigma_x + \rho_{xD}\sigma_x\sigma_D \big)\mathrm{d}t + \sigma_x \mathrm{d}W_{xt}$$

In order to get the price of the zero-coupon security a process for the integral of the short-term rate will also be needed:

$$\mathrm{d}\mathcal{I} = r(\tilde{x}_t)\mathrm{d}t$$

Import the packages

[1]:
```
import SDFPricing as sdf
import StochasticDiffEq as sde # this is needed in order to specify the algorithm
```

Define the parameters

[2]:
```
cs = (
    phi = 0.92, # mean reversion
    xbar = 0.0, # long-run mean
    rho = 0.02, # time preference parameter
    gamma = 2.0, # risk aversion
    muc0 = 0.005, # mean of consumption diffusion
    sigmac = 0.08, # consumption diffusion
    sigmax = 0.5, # state variable diffusion
    rhocx = -0.3, # correlation between consumption and state variable
    # sigmaD = 0.10, # dividend diffusion ###- added compared to example 1
    # muD = 0.01, # dividend drift ###- added compared to example 1
    # rhoxD = -0.5, # correlation between dividends and state variable ###-␣
↪added compared to example 1
    # rhocD = 0.4 # correlation between dividends and consumption ###- added␣
↪compared to example 1
    sigmaD = 0.08, # dividend diffusion ###- case of consumption perpetuity
    muD = 0.005, # dividend drift ###- case of consumption perpetuity
    rhoxD = -0.3, # correlation between dividends and state variable ###- case␣
↪of consumption perpetuity
    rhocD = 1.0 # correlation between dividends and consumption ###- case of␣
↪consumption perpetuity
);
```

Drift and Diffusion of the processes I also include the unmodified process which will correspond to "risk-neutral pricing". By comparing normal pricing with risk-neutral pricing it is possible to compute excess returns.

[3]:
```
###- now consumption diffusion is a non-linear function of the state,
###- given that it needs to be positive.
###- I use this function because the simple exponential can get too high for␣
↪some samples.
sigmac(x,c) = c.sigmac*(x<0 ? 2/(1+exp(-2x)) : 4/(1+exp(-x))-1);
sigmac(x) = sigmac(x,cs);
# sigmaD(x,c) = defineSomeFunctionOf(x,c); #- general dividend diffusion
sigmaD(x,c) = sigmac(x,c); #- case of consumption perpetuity
sigma(x,c) = c.sigmax; # diffusion of modified state
mu(x,c) = -log(c.phi)*(c.xbar-x)-c.rhocx*c.gamma*sigmac(x,c)*sigma(x,c)+c.
↪rhoxD*sigma(x,c)*sigmaD(x,c) ; # drift of unmodified state
```

Short-term rate function

[4]:
```
r(x,c) = c.rho+c.gamma*c.muc0-c.gamma^2*sigmac(x,c)^2/2;
r(x) = r(x,cs);
muD(x) = cs.muD; # perpetuity with constant dividend drift
rmod(x,c) = r(x,c)-(muD(x)-c.gamma*c.rhocD*sigmac(x,c)*sigmaD(x,c));
rmod(x) = rmod(x,cs);
```

Define setup consistent with SDE solution in Julia

[5]:
```
function drift(du,u,p,t,c)
    du[1] = mu(u[1],c)
    du[2] = rmod(u[1],c)
end
drift(du,u,p,t) = drift(du,u,p,t,cs);
function diffusion(du,u,p,t,c)
    du[1] = sigma(u[1],c)
    du[2] = 0.0
end
diffusion(du,u,p,t) = diffusion(du,u,p,t,cs);
```

Define the Problem and SolutionSettings variables

[6]:
```
prob = sdf.
↪Problem(drift=drift,diffusion=diffusion,numNoiseVariables=1,outVariables=[2],
terminalFunction=(ik, x, y, z) -> exp(-x));
xRange = -2.0:0.4:2.0;
tRange = 0.0:5.0:300.0;
sett = sdf.SolutionSettings(xRanges=[xRange,], initialValues=[[x, 0.0] for x in␣
↪xRange],
algorithm=sde.LambaEM(), pathsPerInitialValue=5000, tRange=tRange);
# add the settings in order to compmute price-dividend ration of the continuous␣
↪payoff security
sett2 = sdf.SolutionSettings(sett; continuousPayoffVars=[2]);
```

Solve Problem and Define Yield

[7]:
```
((consumptionStrip,),(priceConsumptionRatio,)) = sdf.solve(prob, sett2);
```

Get the Return of the Consumption Perpetuity The calculation requires the computation of the first and second derivatives of the price consumption ratio with respect to the state of the economy.

[8]:
```
(DPC,D2PC) = sdf.derivatives(priceConsumptionRatio);
ret(x) = (DPC(x)*(mu(x, cs) + sigma(x, cs) * sigmac(x,cs) * cs.rhocx) +
    D2PC(x)* sigma(x, cs)^2/2.0 + 1.0)/priceConsumptionRatio(x) + muD(x);
```

In the theory I state that the price consumption ratio is computed from the integral over all consumption strip maturities. In practice it is not possible to integrate to infinity. So, I compute consumption strips up to a maturity of 300 years.

It can be seen that for all values of the state variable the price of the securities comes close to 0 for maturities as long as 300 years.

Plot the Price Consumption Ratio

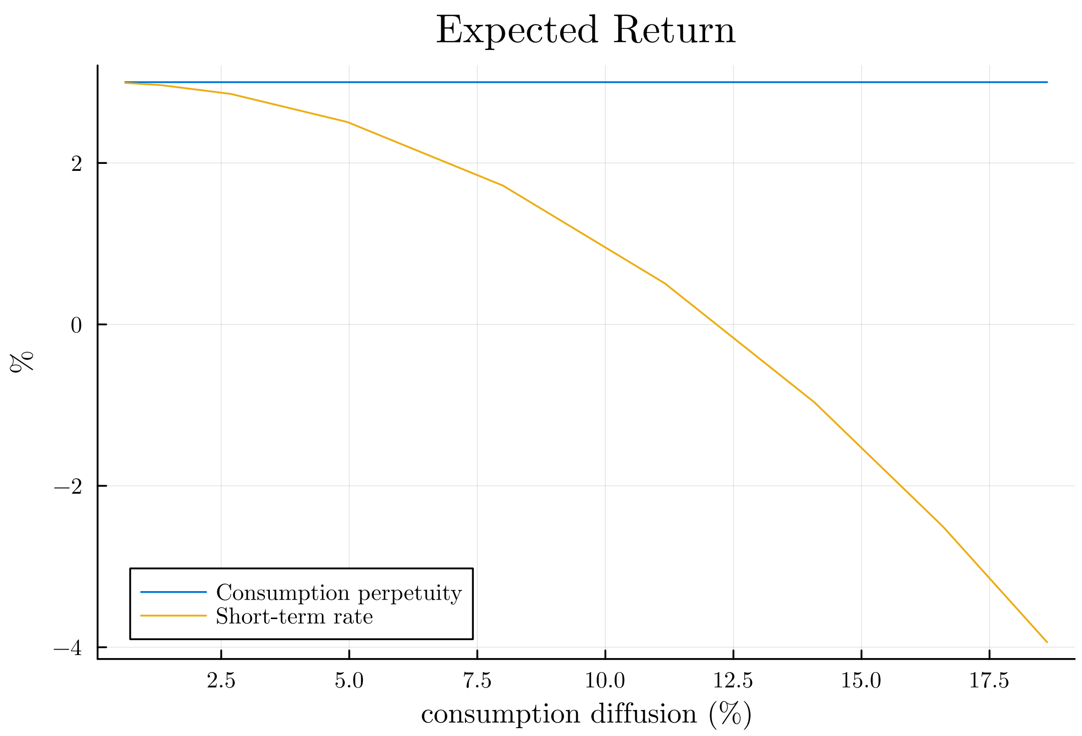
[9]:
```
import Plots as plt
plt.default(titlefont= (14,"Computer Modern"),legendfont=(8,"Computer Modern"),
    tickfont=(8,"Computer Modern"),guidefont=(10,"Computer Modern"))
plt.plot(100*sigmac.(xRange), priceConsumptionRatio(xRange), legend=false,
    title="Price-Consumption Ratio",color="#0075d6",ylims = (0.0, 50.0),␣
 ↪xlabel="consumption diffusion (%)")
```

[9]:

Plot the Return

[10]:
```
plt.plot(100*sigmac.(xRange[2:end-1]), 100*ret.(xRange[2:end-1]),␣
↪label="Consumption perpetuity",
    title="Expected Return", color="#0075d6", xlabel="consumption diffusion␣
↪(%)",ylabel="%")
plt.plot!(100*sigmac.(xRange[2:end-1]), 100*r.(xRange[2:end-1]),
    label="Short-term rate",color="#edad14")
```

[10]:



It turns out that for a value of gamma=2 the effects cancel out and the price consumption ratio constant. However, there is still an expected return due to the dividend and the equity premium is large and increasing as volatility increases, due to the falling risk-free rate.

Example 5 – Two state variables

Time-Varying Drift and Diffusion – Zero-Coupon Bond

Here the setup is more complicated, having two state variables. Both consumption drift and diffusion are simultaneously time-varying. The modified processes that will give the price of the zero coupon security are the following (where $\hat{x}_1$ and $\hat{x}_2$ revert to $\bar{x}_1$ and $\bar{x}_2$ respectively):

$$\mathrm{d}c_t = \mu_{ct}\mathrm{d}t + \sigma_{ct}(1 - |\rho_{c1}| - |\rho_{c2}|)\mathrm{d}W_{ct} + \sigma_{ct}\rho_{cx1}\mathrm{d}W_{x1t} + \sigma_{ct}\rho_{cx1}\mathrm{d}W_{x2t}\mathrm{d}\hat{x}_{1t} = \left(-\log\phi\cdot(\bar{x}_1 - \hat{x}_{1t}) + \rho_{cx1}\sigma_{ct}\sigma_{1x}\right)\mathrm{d}t + \sigma_{x1}\frac{1}{1+}$$

where W_{c1}, W_{x1} and W_{x2} are independent and:

$$\mathrm{E}[\mathrm{d}c_t\mathrm{d}\bar{x}_{1t}] = \left(\rho_{cx1}\frac{1}{1+\rho_{12}} + \rho_{cx2}\frac{\rho_{12}}{1+\rho_{12}}\right)\sigma_{ct}\sigma_{x1}\mathrm{d}t \approxeq \rho_{cx1}\sigma_{ct}\sigma_{x1}\mathrm{d}t$$

$$\mathrm{E}[\mathrm{d}c_t\mathrm{d}\bar{x}_{2t}] = \left(\rho_{cx2}\frac{1}{1+\rho_{21}} + \rho_{cx1}\frac{\rho_{21}}{1+\rho_{21}}\right)\sigma_{ct}\sigma_{x2}\mathrm{d}t \approxeq \rho_{cx2}\sigma_{ct}\sigma_{x2}\mathrm{d}t$$

$$\mathrm{E}[\mathrm{d}\bar{x}_{1t}\mathrm{d}\bar{x}_{2t}] = \sigma_{x1}\sigma_{x2}\frac{\rho_{21}+\rho_{12}}{1+\rho_{12}+\rho_{21}+\rho_{12}\rho_{21}}\mathrm{d}t \approxeq (\rho_{12}+\rho_{21})\sigma_{x1}\sigma_{x2}\mathrm{d}t$$

and the approximate equations are valid if ρ_{12} and ρ_{21} are small.

In order to get the price of the zero-coupon security a process for the integral of the short-term rate will also be needed:

$$\mathrm{d}\mathcal{I} = r(\hat{x}_{1t}, \hat{x}_{2t})\mathrm{d}t$$

Import the packages

[1]:
```
import SDFPricing as sdf
import StochasticDiffEq as sde # this is needed in order to specify the algorithm
```

```
[ Info: Precompiling SDFPricing
[8f91c045-db67-4ada-b18f-1d80840c3158]
```

Define the parameters

[2]:
```
cs = (
    phi1 = 0.91, # mean reversion
    phi2 = 0.96, # mean reversion
    xbar1 = 0.0, # long-run mean
    xbar2 = 0.0, # long-run mean
    rho = 0.02, # time preference parameter
    gamma = 2.0, # risk aversion
    muc = 0.005, # mean of consumption drift
    sigmac = 0.08, # mean of consumption diffusion
    sigmax1 = 0.005, # volatility
    sigmax2 = 0.2, # volatility
    rhocx1 = 0.0, # correlation parameter
```

```
    rhocx2 = -0.6, # correlation parameter
    rho12 = 0.1, # correlation parameter
    rho21 = 0.1 # correlation parameter
    );
```

Drift and Diffusion of the processes I also include the unmodified process which will correspond to "risk-neutral pricing". By comparing normal pricing with risk-neutral pricing it is possible to compute excess returns.

[3]:
```
###- now consumption diffusion is a non-linear function of the state,
###- given that it needs to be positive.
###- I use this function because the simple exponential can get too high for␣
 ↪some samples.
sigmac(x,c) = c.sigmac*(x<0 ? 2/(1+exp(-2x)) : 4/(1+exp(-x))-1);
muc(x,c) = c.muc + x;
sigmac(x) = sigmac(x,cs);
sigma1_1(x,c) = c.sigmax1/(1+c.rho12);
sigma1_2(x,c) = c.sigmax1*c.rho12/(1+c.rho12);
sigma1(x,c) = sigma1_1(x,c)+sigma1_2(x,c);
sigma2_1(x,c) = c.sigmax2*c.rho21/(1+c.rho21);
sigma2_2(x,c) = c.sigmax2/(1+c.rho21);
sigma2(x,c) = sigma2_1(x,c)+sigma2_2(x,c);
mu1(x,c) = -log(c.phi1)*(c.xbar1-x)-c.gamma*sigmac(x,c)*(sigma1_1(x,c)*c.
 ↪rhocx1+sigma1_2(x,c)*c.rhocx2); # drift of modified state
mu2(x,c) = -log(c.phi2)*(c.xbar2-x)-c.gamma*sigmac(x,c)*(sigma2_1(x,c)*c.
 ↪rhocx1+sigma2_2(x,c)*c.rhocx2); # drift of modified state
mu10(x,c) = -log(c.phi1)*(c.xbar1-x); # drift of unmodified state
mu20(x,c) = -log(c.phi2)*(c.xbar2-x); # drift of unmodified state
```

Short-term rate function

[4]:
```
r(x1,x2,c) = c.rho+c.gamma*muc(x1,c)-c.gamma^2*sigmac(x2,c)^2/2;
r(x1,x2) = r(x1,x2,cs);
```

Define setup consistent with SDE solution in Julia

[5]:
```
function drift(du,u,p,t,c)
    du[1] = mu1(u[1],c)
    du[2] = mu2(u[2],c)
    du[3] = mu10(u[3],c)
    du[4] = mu20(u[4],c)
    du[5] = r(u[1],u[2],c)
    du[6] = r(u[3],u[4],c)
end
drift(du,u,p,t) = drift(du,u,p,t,cs);
function diffusion(du,u,p,t,c)
    du[1, 1] = sigma1_1(u[1], c)
    du[1, 2] = sigma1_2(u[1], c)
```

```
    du[2, 1] = sigma2_1(u[2], c)
    du[2, 2] = sigma2_2(u[2], c)
    du[3, 1] = sigma1_1(u[3], c)
    du[3, 2] = sigma1_2(u[3], c)
    du[4, 1] = sigma2_1(u[4], c)
    du[4, 2] = sigma2_2(u[4], c)
    du[5, 1] = 0.0
    du[5, 2] = 0.0
    du[6, 1] = 0.0
    du[6, 2] = 0.0
end
diffusion(du,u,p,t) = diffusion(du,u,p,t,cs);
```

Define the Problem and SolutionSettings variables

[6]:
```
prob = sdf.
 ↪Problem(drift=drift,diffusion=diffusion,numNoiseVariables=2,outVariables=[5,6],
terminalFunction=(ik, x, y, z) -> exp(-x),diagonalNoise=false);
xRanges = [-0.03:0.005:0.03,-2.0:0.4:2.0];
tRange = 0.0:1.0:20.0;
sett = sdf.SolutionSettings(xRanges=xRanges, initialValues=vcat([[x, y,x,y, 0.
 ↪0,0.0] for y in xRanges[2] for x in xRanges[1]]),
algorithm=sde.LambaEM(), pathsPerInitialValue=5000, tRange=tRange);
```

Solve Problem and Define Yield

[7]:
```
((bondPrice,riskNeutralPrice),) = sdf.solve(prob, sett);
yld(t,x1,x2) = -log(bondPrice(t,x1,x2))/t;
yldRiskNeutral(t,x1,x2) = -log(riskNeutralPrice(t,x1,x2))/t;
```

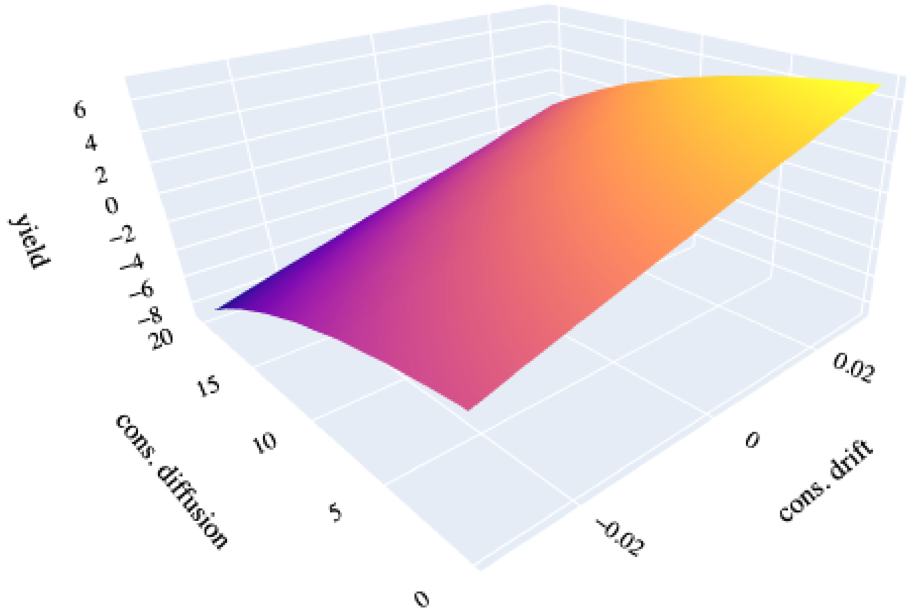
Plot the Yield in 3D

[10]:
```
import Plots as plt
import PlotlyJS as pltjs
coordinates = pltjs.surface(
    z=[100*yld(10.0, x1, x2) for x1 in xRanges[1], x2 in xRanges[2]],␣
 ↪x=xRanges[1],
    y=100*sigmac.(xRanges[2]),
    showscale=false)

layout = pltjs.Layout(
    width=800, height=350,
    title_x=0.5,
    titlefont_size="16",
    scene_aspectratio=pltjs.attr(x=1, y=1, z=0.5),
    scene=pltjs.attr(
        xaxis=pltjs.attr(title="cons. drift"),
        yaxis=pltjs.attr(title="cons. diffusion"),
        zaxis=pltjs.attr(title="yield"),
```

```
        camera=pltjs.attr(
            center=pltjs.attr(x=0.3, y=0, z=-0.40),
            eye=pltjs.attr(x=-.95, y=-1.25, z=0.65)
        )
    ),
    font=pltjs.attr(family="Computer Modern", size=12, color="black"),
    margin=pltjs.attr(l=0, r=0, b=0, t=0, pad=0))
pltjs.plot([coordinates], layout)
```

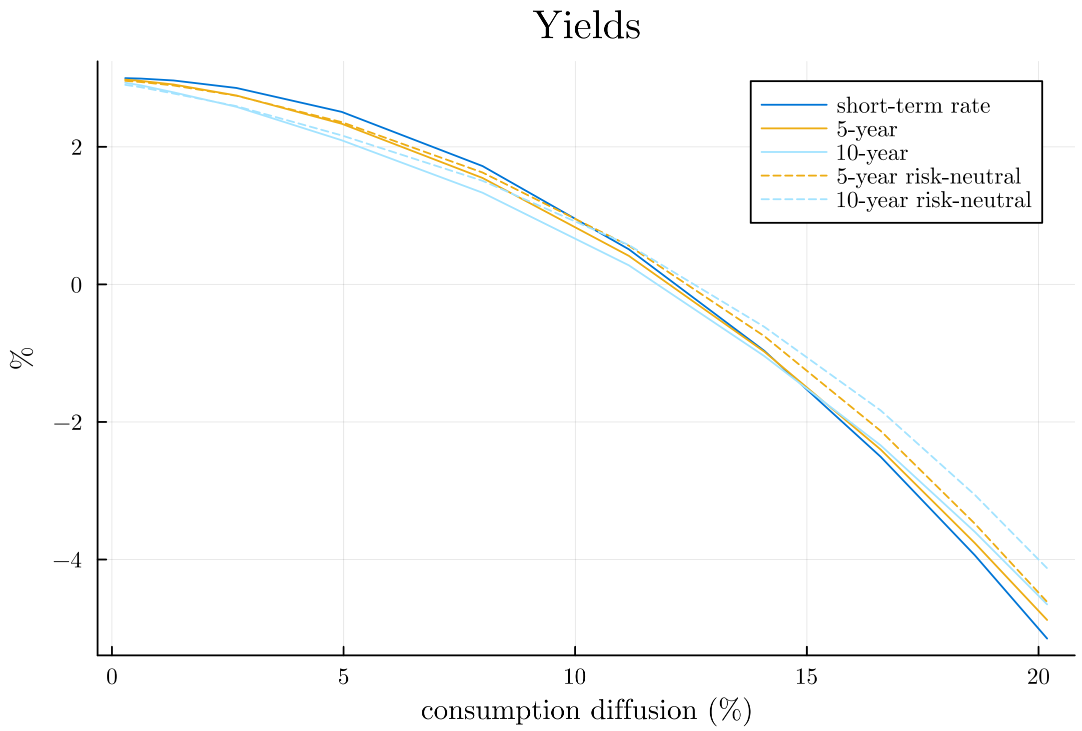
[10]:



[9]:

```
# colors: "#0075d6", "#edad14", "#a3e3ff", "#9c0000"
import Plots as plt
x1 = 0.0
plt.default(titlefont= (14,"Computer Modern"),legendfont=(8,"Computer Modern"),
    tickfont=(8,"Computer Modern"),guidefont=(10,"Computer Modern"))
plt.plot(100*sigmac.(xRanges[2]), xRanges[2] .|> x2->100*r(x1,x2),␣
 ↪title="Yields",
    xlabel="consumption diffusion (%)",ylabel="%",label="short-term␣
 ↪rate",color="#0075d6")
plt.plot!(100*sigmac.(xRanges[2]), 100*yld.(5.0, x1,xRanges[2]),␣
 ↪label="5-year",color= "#edad14")
plt.plot!(100*sigmac.(xRanges[2]), 100*yld.(10.0, x1,xRanges[2]),␣
 ↪label="10-year",color="#a3e3ff")
plt.plot!(100*sigmac.(xRanges[2]), 100*yldRiskNeutral.(5.0, x1,xRanges[2]),␣
 ↪label="5-year risk-neutral",color= "#edad14",style=:dash)
plt.plot!(100*sigmac.(xRanges[2]), 100*yldRiskNeutral.(10.0, x1,xRanges[2]),␣
 ↪label="10-year risk-neutral",color="#a3e3ff",style=:dash)
```

[9]:



Focusing on one state variable shows that the results are similar to the single variable case.

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Chapter 3

A Perturbation Solution Method for Models with Recursive Utility

Abstract

I illustrate a novel method for pricing assets within recursive utility models in continuous time, that has first been used in Melissinos (2023). My method builds on the analytic solution of Tsai and Wachter (2018). While their solution is valid for a value of the intertemporal elasticity of substitution, ψ, equal to 1, I provide the full perturbation series in terms of ψ, which gives rise to a global perturbation approximation. This allows the pricing of assets for a much larger range of values for ψ, which are economically meaningful. I comment on the convergence properties of the perturbation series, and I show that the method provides a straightforward and reliable approach to asset pricing. I employ my method to derive prices of long-term bonds, the price consumption ratio and the instantaneous return of the consumption perpetuity.

3.1 Introduction

Even though utility functions with time-separable utility are most often used in the literature, recursive utility models are also popular, as they can describe more general preferences and behaviours. Indeed some models seem to require the use of recursive utility, for example Bansal and Yaron (2004) and Wachter (2013), in order to produce important features of the data (yyy). Recursive utility models in continuous time were introduced by Duffie and Epstein (1992*b*) and an important literature describing their properties has since followed, for example Duffie and Epstein 1992*a*; Duffie, Schroder and Skiadas 1996; Schroder and Skiadas 1999. However, solving these models in continuous time remains challenging. Recently, Tsai and Wachter (2018) suggested a method for pricing long-term assets using recursive utility in continuous time. In this paper, I introduce a perturbation method that is based on the analytical solution that Tsai and Wachter (2018) found.

The main underlying assumption of the method by Tsai and Wachter (2018) is that the intertemporal elasticity of substitution (IES) is equal to 1. While the authors claim that their solution can also be used for other IES values, it is not a priori obvious under which conditions this is true. In this paper I use their analytical solution as a base case and then perform a perturbation expansion around the IES value of 1. This allows me to get a solution that is valid for a large range of IES values and evaluate the accuracy of the approximation offered by Tsai and Wachter (2018). My approach is based on the perturbation theory described in Bender, Orszag and Orszag (1999), which has also been advanced in economics and finance by Judd (1996). Related literature includes Caldara et al. (2012), who solve DSGE models with recursive utility in discrete time. To the best of my knowledge, apart from Melissinos (2023) which uses the method described in this paper to solve long-run risk models and rare disaster models, Leisen (2016) is only one other paper that uses perturbation theory to solve recursive utility models in continuous time in a similar setup to what I examine here. However, unlike the current paper, Leisen (2016) looks at a model that also includes portfolio selection and the IES parameter is not the basis of the perturbation.

Once the value function is derived based on the perturbation approximation, I can derive the stochastic differential equation form of the stochastic discount factor (SDF), following the derivation of Chen et al. (2009). Then I can proceed to solve the partial differential equation that is associated with the pricing of the long-term zero-coupon bond. In addition, quantities like the price-dividend ratio,

the wealth-consumption ratio and the price of dividend-bearing securities can also be computed. I solve the pricing equation for the long-term bond by using the Feynman-Kac formula, which I implement through Monte Carlo simulations.
The rest of the paper is organised as follows. Section 2 introduces the general framework including the recursive utility component, Section 3 introduces the perturbation expansion, Section 4 performs the pricing of the long-term bond based on the previous results, and Section 5 concludes.

3.2 Recursive Utility Framework

This section closely follows the framework introduced by Tsai and Wachter (2018). For simplicity, I introduce my method using only one state variable. Introducing multiple state variables is straightforward based on the single-variable case.

3.2.1 Consumption process

The consumption process has two components: a deterministic trend and a Brownian motion component:[1]

$$\frac{dC_t}{C_t} = d\log(C_t) = dc_t = \mu_{ct}dt + \sigma_{ct}dZ_{ct} \tag{3.1}$$

where the t subscript denotes variables at time t,[2] C_t is consumption flow, c_t is log consumption flow, x_t is the state variable characterising the economy, μ_{ct} is the deterministic consumption trend, σ_{ct} determines consumption volatility and Z_{ct} is the Brownian motion component. μ_{ct} and $\sigma_c t$ are either parameters or they depend on the state variable.

3.2.2 State variable

Similar to consumption, the state variable also has two components:

$$dx_t = \mu_{xt}dt + \sigma_{xt}dZ_{xt} \tag{3.2}$$

[1]Here, for simplicity, I assume that consumption does not undergo discontinuous jumps (with probability 1), but my solution method can be extended to the case, in which the consumption process includes Poisson jumps.

[2]For generality I use a subscript t for all symbols that can correspond to variables. However, in some variations these symbols may also correspond to parameters.

Here, x_t denotes the state variable, and the functions and parameters are analogous to the case of consumption.

3.2.3 Utility

Lifetime utility at time t_0 is:[3]

$$V_{t_0} = E_t\left[\int_{t_0}^{\infty} f(C_t, V_t)dt\right] \tag{3.3}$$

This equation highlights why utility is referred to as ”recursive”, as the integrand depends on the current value of the agent at each point in time. The combination of the consumption flow with the concurrent lifetime utility takes place via the so-called aggregator function:[4]

$$f(C,V) = \frac{(1-\gamma)\rho V\left(\left(C((1-\gamma)V)^{-\frac{1}{1-\gamma}}\right)^{1-\frac{1}{\psi}} - 1\right)}{1-\frac{1}{\psi}} \tag{3.4}$$

This function represents a flow which depends both on current consumption flow, C_t, and on the current level of the value, V_t. ρ denotes a time preference parameter, γ denotes the risk aversion parameter and ψ denotes the intertemporal elasticity of substitution (IES). Recursive utility is a useful modelling tool because it allows the separation of the risk aversion parameter from the IES parameter. This utility specification reduces to the more familiar time-additive case when $\gamma = 1/\psi$. In this case the agent is indifferent about when uncertainty is resolved. For more general parameter specifications the agent may exhibit a preference for late or early resolution of uncertainty. In particular, for $\gamma > 1/\psi$ ($\gamma < 1/\psi$) there is a preference for early (late) resolution of uncertainty. The intuition for this mechanism can also be explained differently. In particular, $\gamma > 1/\psi$ implies that:

$$\frac{\partial^2 f(C,V)}{\partial C \partial V} < 0 \text{ and } \frac{\partial^3 f(C,V)}{\partial C^2 \partial V} > 0 \tag{3.5}$$

$$\Rightarrow E_t\left[\frac{\partial f(C,V_t)}{\partial C}\right] < E_t\left[\frac{\partial f(C,E_{t+1}[V_t])}{\partial C}\right] \tag{3.6}$$

[3]Following Tsai and Wachter (2018) I do not prove existence and uniqueness of my solution. Hence, I use the infinite horizon case for simplicity. When considering a proof of existence and/or uniqueness, a finite horizon may be easier to deal with.

[4]This is the normalised form of the aggregator in Duffie and Epstein (1992*b*)

On the right-hand side, the notation means that the agent has been given early information about the state of the world in $t+1$, while on the left-hand side this is not the case. It follows that the ex-ante expectation of these two situations leads to a preference for early resolution of uncertainty, as the utility flow is expected to be higher, in a state where the agent has early knowledge. The opposite is true in the case of a preference for late resolution of uncertainty. The mathematics of the situation is similar to the case, in which a risk-averse agent prefers a safe sum of money to a risky lottery with the same expected value. So, it is crucial for the preference of early resolution of uncertainty that consumption becomes less enjoyable as the value of the agent increases. Notice that this is not the familiar diminishing marginal utility of consumption. Instead, this implies that the same quantity of consumption is less enjoyable when the agent becomes happier for reasons that are not related to current consumption, for example, she may have learned that expected consumption in the future has increased and this makes her current consumption less enjoyable.

3.2.4 Decomposition of the value function

In the recursive utility framework, there exists a scale invariance property Duffie and Epstein (1992*b*), which allows us to express the value of the agent in a way that separates the dependence on consumption from the dependence on the state variable. As shown in Benzoni, Collin-Dufresne and Goldstein (2011) and Tsai and Wachter (2018), the value function can be written as:

$$V = \frac{C^{1-\gamma} e^{K(x)(1-\gamma)}}{1-\gamma} \tag{3.7}$$

Where $K(x)$ satisfies the following differential equation:[5]

$$\rho \frac{\psi}{1-\psi}\left(1-e^{(1/\psi-1)K(x)}\right)-\frac{\gamma\sigma_{ct}^2}{2}+\mu_{ct}+\frac{\sigma_{xt}^2}{2}K''(x)+\mu_{xt}K'(x)+\frac{(1-\gamma)\sigma_{xt}^2}{2}K'(x)^2=0 \tag{3.8}$$

A proof of this result is included in Appendix 3.A.[6]

[5]This equation is also valid for $\psi = 1$, in which case the expressions are replaced by their limits.

[6]A very similar result is also proven by Tsai and Wachter (2018).

3.2.5 Functional forms for consumption and state variable processes

The asset pricing problem based on the above setup is generally not easy to solve. However, Tsai and Wachter (2018) showed that significant progress can be made under the following specification for the consumption process and the process of the state variable:

$$
\begin{aligned}
\mu_{ct} &= \mu_{c0} + \mu_{c1} x_t \\
\sigma_{ct} &= \sqrt{\sigma_{c0} + \sigma_{c1} x_t} \\
\mu_{xt} &= -\log(\phi)(\mu_{x0} - x_t) \\
\sigma_{xt} &= \sqrt{\sigma_{x0} + \sigma_{x1} x_t}
\end{aligned}
\tag{3.9}
$$

These parameters on the right hand side can be chosen. Notably, this specification is particularly useful because plugging these expressions into Equation (3.8) produces linear terms in x.

3.3 Method Description

3.3.1 Exact Solution for ψ=1

As Tsai and Wachter (2018) Equation 3.8 has an exact solution for $\psi = 1$. In particular, after the parametrisation from Expressions (3.9) is used, the differential equation becomes:

$$
\begin{aligned}
-\rho K(x) - \frac{1}{2}\gamma\left(\sigma_{\mathrm{c0}} + x\sigma_{\mathrm{c1}}\right) + \mu_{\mathrm{c0}} + x\mu_{\mathrm{c1}} + \frac{1}{2}K''(x)\left(x\sigma_{\mathrm{x1}} + \sigma_{\mathrm{x0}}\right) - \log(\phi)K'(x)\left(\mu_{\mathrm{x0}} - x\right) \\
-\frac{1}{2}\gamma K'(x)^2\left(x\sigma_{\mathrm{x1}} + \sigma_{\mathrm{x0}}\right) + \frac{1}{2}K'(x)^2\left(x\sigma_{\mathrm{x1}} + \sigma_{\mathrm{x0}}\right) = 0
\end{aligned}
\tag{3.10}
$$

and the solution takes the form, $K(x) = a_{0,0} + a_{0,1}x$. The coefficients, $a_{0,0}$ and $a_{0,1}$ can be solved by sequentially solving the following equations:

$$
\begin{aligned}
0 &= -\rho a_{0,0} - \frac{1}{2}\gamma a_{0,1}^2\sigma_{\mathrm{x0}} - a_{0,1}\mu_{\mathrm{x0}}\log(\phi) + \frac{1}{2}a_{0,1}^2\sigma_{\mathrm{x0}} - \frac{\gamma\sigma_{\mathrm{c0}}}{2} + \mu_{\mathrm{c0}} \\
0 &= -\rho a_{0,1} - \frac{1}{2}\gamma a_{0,1}^2\sigma_{\mathrm{x1}} + \frac{1}{2}a_{0,1}^2\sigma_{\mathrm{x1}} + a_{0,1}\log(\phi) - \frac{\gamma\sigma_{\mathrm{c1}}}{2} + \mu_{\mathrm{c1}}
\end{aligned}
\tag{3.11}
$$

The second equation is solved first as it only includes parameter $a_{0,1}$. Then using this solution the first equation can also be solved.[7]

$$
\begin{aligned}
a_{0,1} &= -\frac{\rho - \log(\phi) \pm \sqrt{2(\gamma-1)\mu_{\mathrm{c1}}\sigma_{\mathrm{x1}} - (\gamma-1)\gamma\sigma_{\mathrm{c1}}\sigma_{\mathrm{x1}} + (\rho - \log(\phi))^2}}{(\gamma-1)\sigma_{\mathrm{x1}}} \\
&\quad \text{or if } \sigma_{x1} = 0 \\
a_{0,1} &= \frac{2\mu_{\mathrm{c1}} - \gamma\sigma_{\mathrm{c1}}}{2\rho - 2\log(\phi)} \\
a_{0,0} &= -\frac{\gamma a_{0,1}^2\sigma_{\mathrm{x0}}}{2\rho} - \frac{a_{0,1}\mu_{\mathrm{x0}}\log(\phi)}{\rho} + \frac{a_{0,1}^2\sigma_{\mathrm{x0}}}{2\rho} - \frac{\gamma\sigma_{\mathrm{c0}}}{2\rho} + \frac{\mu_{\mathrm{c0}}}{\rho}
\end{aligned}
\tag{3.12}
$$

Tsai and Wachter (2018) use this solution to derive analytical expressions for the pricing of long-term assets, when $\psi = 1$. They also use these results to derive approximate expressions for the case when $\psi \neq 1$.

3.3.2 Extension of the method to the case $\psi \neq 1$

3.3.2.1 General description

In this paper, I extend the above solution method, in order to allow the parameter for IES to take a large range of values. As is common with perturbation solutions, instead of solving the problem for a specific value of ψ for which there is no analytic solution, the problem is redefined and solved for arbitrary ψ. This may seem as more difficult, but since the solution for $\psi = 1$ is already known, the perturbation solution provides a way to start from the solution that is known, and then gradually move towards a solution that is valid for any ψ. I achieve this by re-expressing ψ in terms of ϵ and expanding the problem in a series of ϵ. In particular, ψ is replaced by $\frac{1}{1-\epsilon}$ and the expansion of ψ in terms of ϵ is:

$$
\psi = \frac{1}{1-\epsilon} = 1 + \epsilon + \epsilon^2 + \epsilon^3 + \dots \tag{3.13}
$$

The redefinition in terms of ϵ is convenient because the analytic solution occurs for $\epsilon = 0$, and this makes the power series of $K(\cdot)$ considerably simpler. As above,

[7] When $\sigma_{x1} \neq 0$, then $a_{0,1}$ has a double solution. However, only one of the two solutions is economically meaningful. This duplicity is explained by the existence of the square root and by the fact that the state variable can be defined to be an increasing or decreasing function of the state variable.

I proceed by guessing the series solution of the differential equation (3.10):

$$\begin{aligned}
K(x,\epsilon) =& \sum_{n=0}^{\infty} \epsilon^n \left(\sum_{m=0}^{n+1} a_{n,m} x^m \right) \\
=& \left(a_{0,0} + a_{0,1} x\right) \\
& + \epsilon \left(a_{1,0} + a_{1,1} x + a_{1,2} x^2\right) \\
& + \epsilon^2 \left(a_{2,0} + a_{2,1} x + a_{2,2} x^2 + a_{2,3} x^3\right) \\
& + \epsilon^3 \left(a_{3,0} + a_{3,1} x + a_{3,2} x^2 + a_{3,3} x^3 + a_{3,4} x^4\right) \\
& \quad \ldots \\
=& K_0(x) + K_1(x)\epsilon + K_2(x)\epsilon^2 + \ldots
\end{aligned} \tag{3.14}$$

In the remaining of the paper I refer to the approximations according to the highest power of ϵ. For example the approximation that only maintains the first line of Equation (3.14) is the "zeroth" order approximation. The approximation that maintains the first two lines is the "first" order approximation and so on. The structure of the solution is a polynomial both in terms of x and in terms of ϵ. In particular, for each successive order of ϵ the order of the polynomial in terms of x that is multiplying it increases by one. As it turns out, the solution of Tsai and Wachter (2018) corresponds to the zeroth order of the series. This makes sense given that $\epsilon = 0$ simplifies to the previous case, namely $\psi = 1$. The other higher polynomials in x are new and they will show what the effect is from moving away from IES equal to 1. Thus, by replacing ψ with $\frac{1}{1-\epsilon}$, then plugging in the guess and expanding in terms of ϵ, Equation (3.10) becomes:

$$\begin{aligned}
0 =& -\frac{1}{2}\gamma\left(\sigma_{\mathrm{c0}} + x\sigma_{\mathrm{c1}}\right) + \mu_{\mathrm{c0}} + x\mu_{\mathrm{c1}} - \rho K_0(x) - \log(\phi) K_0'(x)\left(\mu_{\mathrm{x0}} - x\right) - \frac{1}{2}\gamma K_0'(x)^2\left(x\sigma_{\mathrm{x1}} + \sigma_{\mathrm{x0}}\right) \\
& \qquad + \frac{1}{2}K_0'(x)^2\left(x\sigma_{\mathrm{x1}} + \sigma_{\mathrm{x0}}\right) + \frac{1}{2}K_0''(x)\left(x\sigma_{\mathrm{x1}} + \sigma_{\mathrm{x0}}\right)\Big) \\
& \epsilon\Big(-\rho K_1(x) - \log(\phi) K_1'(x)\left(\mu_{\mathrm{x0}} - x\right) - \gamma K_0'(x) K_1'(x)\left(x\sigma_{\mathrm{x1}} + \sigma_{\mathrm{x0}}\right) + K_0'(x) K_1'(x)\left(x\sigma_{\mathrm{x1}} + \sigma_{\mathrm{x0}}\right) \\
& \qquad + \frac{1}{2}K_1''(x)\left(x\sigma_{\mathrm{x1}} + \sigma_{\mathrm{x0}}\right)\Big) \\
& \epsilon^2\Big(-\rho K_2(x) - \log(\phi) K_2'(x)\left(\mu_{\mathrm{x0}} - x\right) - \frac{1}{2}\gamma K_1'(x)^2\left(x\sigma_{\mathrm{x1}} + \sigma_{\mathrm{x0}}\right) \\
& \quad - \gamma K_0'(x) K_2'(x)\left(x\sigma_{\mathrm{x1}} + \sigma_{\mathrm{x0}}\right) + \frac{1}{2}K_1'(x)^2\left(x\sigma_{\mathrm{x1}} + \sigma_{\mathrm{x0}}\right) + K_0'(x) K_2'(x)\left(x\sigma_{\mathrm{x1}} + \sigma_{\mathrm{x0}}\right) \\
& \qquad + \frac{1}{2}K_2''(x)\left(x\sigma_{\mathrm{x1}} + \sigma_{\mathrm{x0}}\right)\Big)
\end{aligned} \tag{3.15}$$

In the expression above I have still not inserted the $a_{\cdot,\cdot}$ parameters in detail, in order to not clutter the overall expression too much. Nevertheless, it can be seen that for this equation to hold for all values of ϵ, we need the coefficient for each power of ϵ to be equal to 0. Subsequently, each of these coefficients includes the K_n's and their derivatives, which contain polynomials of x. Following the same strategy, for these polynomials to be equal to 0 for all values of x, the corresponding coefficients also need to equal 0. Combining the two stages, this implies that for each pair of powers for ϵ and x, that show up in Equation (3.14), there is a corresponding equation that allows us to compute the respective coefficient.[8] In addition, each of these equations turns out to be linear and sequentially solvable given the solutions of the previous equations.[9] The order for solving the equation increases with the powers of ϵ and decreases with the powers of x. Namely, the parameters are found in the following order:

$$a_{0,1}, a_{0,0}, a_{1,2}, a_{1,1}, a_{1,0}, a_{2,3}, a_{2,2}, a_{2,1}, a_{2,0}, \ldots \tag{3.16}$$

Given that each parameter requires the solution of a linear equation, it is easy to solve the model for high orders of approximation. However, for each order of ϵ the number of parameters increases by one and the equation become increasingly complicated. As a result, roughly fifteen orders of approximation in terms of ϵ can be found relatively quickly (this corresponds to 135 distinct parameter values).

3.3.2.2 First order approximation

Finding the first order approximation requires the solution of the following equations:

$$\begin{aligned}
0 &= -\rho a_{1,2} - 2\gamma a_{0,1} a_{1,2} \sigma_{\mathrm{x}1} + 2 a_{0,1} a_{1,2} \sigma_{\mathrm{x}1} + 2 a_{1,2} \log(\phi) \\
0 &= -\rho a_{1,1} - 2\gamma a_{0,1} a_{1,2} \sigma_{\mathrm{x}0} - 2 a_{1,2} \mu_{\mathrm{x}0} \log(\phi) + 2 a_{0,1} a_{1,2} \sigma_{\mathrm{x}0} - \gamma a_{0,1} a_{1,1} \sigma_{\mathrm{x}1} \\
&\qquad + a_{0,1} a_{1,1} \sigma_{\mathrm{x}1} + a_{1,2} \sigma_{\mathrm{x}1} + a_{1,1} \log(\phi) \\
0 &= -\rho a_{1,0} - \gamma a_{0,1} a_{1,1} \sigma_{\mathrm{x}0} - a_{1,1} \mu_{\mathrm{x}0} \log(\phi) + a_{0,1} a_{1,1} \sigma_{\mathrm{x}0} + a_{1,2} \sigma_{\mathrm{x}0}
\end{aligned} \tag{3.17}$$

[8] Another way to think of this is the following: For each n power of ϵ, there is a linear second order differential equation for $K_n(\cdot)$ which can be solved sequentially using the solutions of the previous differential equations.

[9] The only exception is parameter $a_{0,1}$ which was already mentioned above and might require the solution of a second order equation.

whose solutions are:

$$
\begin{aligned}
a_{1,2} =& \frac{\rho a_{0,1}^2}{2\left(2\gamma a_{0,1}\sigma_{\mathrm{x}1} - 2a_{0,1}\sigma_{\mathrm{x}1} + \rho - 2\log(\phi)\right)} \\
a_{1,1} =& \frac{\rho a_{0,0}a_{0,1} - 2\gamma a_{1,2}a_{0,1}\sigma_{\mathrm{x}0} - 2a_{1,2}\mu_{\mathrm{x}0}\log(\phi) + 2a_{1,2}a_{0,1}\sigma_{\mathrm{x}0} + a_{1,2}\sigma_{\mathrm{x}1}}{\gamma a_{0,1}\sigma_{\mathrm{x}1} - a_{0,1}\sigma_{\mathrm{x}1} + \rho - \log(\phi)} \\
a_{1,0} =& \frac{\rho a_{0,0}^2 - 2\gamma a_{0,1}a_{1,1}\sigma_{\mathrm{x}0} - 2a_{1,1}\mu_{\mathrm{x}0}\log(\phi) + 2a_{0,1}a_{1,1}\sigma_{\mathrm{x}0} + 2a_{1,2}\sigma_{\mathrm{x}0}}{2\rho}
\end{aligned} \tag{3.18}
$$

As can be seen above, the values of all parameters can be found by plugging in the previous solutions.

3.3.2.3 Second order approximation

The solution of the second order proceeds similarly. The equations to be solved are the following:

$$
\begin{aligned}
0 =& \frac{1}{6}\gamma\rho a_{0,0}^3 - \frac{1}{6}\rho a_{0,0}^3 - \gamma\rho a_{1,0}a_{0,0} + \rho a_{1,0}a_{0,0} + \frac{1}{2}\gamma^2\sigma_{\mathrm{x}0}a_{1,1}^2 - \gamma\sigma_{\mathrm{x}0}a_{1,1}^2 + \frac{1}{2}\sigma_{\mathrm{x}0}a_{1,1}^2 + \gamma\rho a_{2,0} \\
&- \rho a_{2,0} + \gamma\log(\phi)\mu_{\mathrm{x}0}a_{2,1} - \log(\phi)\mu_{\mathrm{x}0}a_{2,1} + \gamma^2\sigma_{\mathrm{x}0}a_{0,1}a_{2,1} - 2\gamma\sigma_{\mathrm{x}0}a_{0,1}a_{2,1} + \sigma_{\mathrm{x}0}a_{0,1}a_{2,1} \\
&- \gamma\sigma_{\mathrm{x}0}a_{2,2} + \sigma_{\mathrm{x}0}a_{2,2} \\
0 =& \frac{1}{2}\sigma_{\mathrm{x}1}a_{1,1}^2\gamma^2 + 2\sigma_{\mathrm{x}0}a_{1,1}a_{1,2}\gamma^2 + \sigma_{\mathrm{x}1}a_{0,1}a_{2,1}\gamma^2 + 2\sigma_{\mathrm{x}0}a_{0,1}a_{2,2}\gamma^2 - \sigma_{\mathrm{x}1}a_{1,1}^2\gamma + \frac{1}{2}\rho a_{0,0}^2a_{0,1}\gamma \\
&- \rho a_{0,1}a_{1,0}\gamma - \rho a_{0,0}a_{1,1}\gamma - 4\sigma_{\mathrm{x}0}a_{1,1}a_{1,2}\gamma + \rho a_{2,1}\gamma - \log(\phi)a_{2,1}\gamma - 2\sigma_{\mathrm{x}1}a_{0,1}a_{2,1}\gamma \\
&+ 2\log(\phi)\mu_{\mathrm{x}0}a_{2,2}\gamma - \sigma_{\mathrm{x}1}a_{2,2}\gamma - 4\sigma_{\mathrm{x}0}a_{0,1}a_{2,2}\gamma - 3\sigma_{\mathrm{x}0}a_{2,3}\gamma + \frac{1}{2}\sigma_{\mathrm{x}1}a_{1,1}^2 - \frac{1}{2}\rho a_{0,0}^2a_{0,1} \\
&+ \rho a_{0,1}a_{1,0} + \rho a_{0,0}a_{1,1} + 2\sigma_{\mathrm{x}0}a_{1,1}a_{1,2} - \rho a_{2,1} + \log(\phi)a_{2,1} + \sigma_{\mathrm{x}1}a_{0,1}a_{2,1} - 2\log(\phi)\mu_{\mathrm{x}0}a_{2,2} \\
&+ \sigma_{\mathrm{x}1}a_{2,2} + 2\sigma_{\mathrm{x}0}a_{0,1}a_{2,2} + 3\sigma_{\mathrm{x}0}a_{2,3} \\
0 =& 2\sigma_{\mathrm{x}0}a_{1,2}^2\gamma^2 + 2\sigma_{\mathrm{x}1}a_{1,1}a_{1,2}\gamma^2 + 2\sigma_{\mathrm{x}1}a_{0,1}a_{2,2}\gamma^2 + 3\sigma_{\mathrm{x}0}a_{0,1}a_{2,3}\gamma^2 + \frac{1}{2}\rho a_{0,0}a_{0,1}^2\gamma \\
&- 4\sigma_{\mathrm{x}0}a_{1,2}^2\gamma - \rho a_{0,1}a_{1,1}\gamma - \rho a_{0,0}a_{1,2}\gamma - 4\sigma_{\mathrm{x}1}a_{1,1}a_{1,2}\gamma + \rho a_{2,2}\gamma - 2\log(\phi)a_{2,2}\gamma \\
&- 4\sigma_{\mathrm{x}1}a_{0,1}a_{2,2}\gamma + 3\log(\phi)\mu_{\mathrm{x}0}a_{2,3}\gamma - 3\sigma_{\mathrm{x}1}a_{2,3}\gamma - 6\sigma_{\mathrm{x}0}a_{0,1}a_{2,3}\gamma - \frac{1}{2}\rho a_{0,0}a_{0,1}^2 \\
&+ 2\sigma_{\mathrm{x}0}a_{1,2}^2 + \rho a_{0,1}a_{1,1} + \rho a_{0,0}a_{1,2} + 2\sigma_{\mathrm{x}1}a_{1,1}a_{1,2} - \rho a_{2,2} + 2\log(\phi)a_{2,2} + 2\sigma_{\mathrm{x}1}a_{0,1}a_{2,2} \\
&- 3\log(\phi)\mu_{\mathrm{x}0}a_{2,3} + 3\sigma_{\mathrm{x}1}a_{2,3} + 3\sigma_{\mathrm{x}0}a_{0,1}a_{2,3} \\
0 =& \frac{1}{6}\gamma\rho a_{0,1}^3 - \frac{1}{6}\rho a_{0,1}^3 - \gamma\rho a_{1,2}a_{0,1} + \rho a_{1,2}a_{0,1} + 3\gamma^2\sigma_{\mathrm{x}1}a_{2,3}a_{0,1} - 6\gamma\sigma_{\mathrm{x}1}a_{2,3}a_{0,1} + 3\sigma_{\mathrm{x}1}a_{2,3}a_{0,1} \\
&+ 2\gamma^2\sigma_{\mathrm{x}1}a_{1,2}^2 - 4\gamma\sigma_{\mathrm{x}1}a_{1,2}^2 + 2\sigma_{\mathrm{x}1}a_{1,2}^2 + \gamma\rho a_{2,3} - \rho a_{2,3} - 3\gamma\log(\phi)a_{2,3} + 3\log(\phi)a_{2,3}
\end{aligned} \tag{3.19}
$$

And the solutions are:

$$
\begin{aligned}
a_{2,3} =& -\frac{\rho a_{0,1}^3}{6\left(3\gamma a_{0,1}\sigma_{\mathrm{x}1} - 3a_{0,1}\sigma_{\mathrm{x}1} + \rho - 3\log(\phi)\right)} + \frac{\rho a_{1,2}a_{0,1}}{3\gamma a_{0,1}\sigma_{\mathrm{x}1} - 3a_{0,1}\sigma_{\mathrm{x}1} + \rho - 3\log(\phi)} \\
& - \frac{2\gamma a_{1,2}^2\sigma_{\mathrm{x}1}}{3\gamma a_{0,1}\sigma_{\mathrm{x}1} - 3a_{0,1}\sigma_{\mathrm{x}1} + \rho - 3\log(\phi)} + \frac{2a_{1,2}^2\sigma_{\mathrm{x}1}}{3\gamma a_{0,1}\sigma_{\mathrm{x}1} - 3a_{0,1}\sigma_{\mathrm{x}1} + \rho - 3\log(\phi)} \\
a_{2,2} =& -\frac{3a_{2,3}\mu_{\mathrm{x}0}\log(\phi)}{2\gamma a_{0,1}\sigma_{\mathrm{x}1} - 2a_{0,1}\sigma_{\mathrm{x}1} + \rho - 2\log(\phi)} + \frac{3(1-\gamma)a_{2,3}a_{0,1}\sigma_{\mathrm{x}0}}{2\gamma a_{0,1}\sigma_{\mathrm{x}1} - 2a_{0,1}\sigma_{\mathrm{x}1} + \rho - 2\log(\phi)} \\
& + \frac{2(1-\gamma)a_{1,2}^2\sigma_{\mathrm{x}0}}{2\gamma a_{0,1}\sigma_{\mathrm{x}1} - 2a_{0,1}\sigma_{\mathrm{x}1} + \rho - 2\log(\phi)} - \frac{\rho a_{0,0}a_{0,1}^2}{2\left(2\gamma a_{0,1}\sigma_{\mathrm{x}1} - 2a_{0,1}\sigma_{\mathrm{x}1} + \rho - 2\log(\phi)\right)} \\
& + \frac{\rho a_{1,1}a_{0,1}}{2\gamma a_{0,1}\sigma_{\mathrm{x}1} - 2a_{0,1}\sigma_{\mathrm{x}1} + \rho - 2\log(\phi)} + \frac{2(1-\gamma)a_{1,1}a_{1,2}\sigma_{\mathrm{x}1}}{2\gamma a_{0,1}\sigma_{\mathrm{x}1} - 2a_{0,1}\sigma_{\mathrm{x}1} + \rho - 2\log(\phi)} \\
& + \frac{\rho a_{0,0}a_{1,2}}{2\gamma a_{0,1}\sigma_{\mathrm{x}1} - 2a_{0,1}\sigma_{\mathrm{x}1} + \rho - 2\log(\phi)} + \frac{3a_{2,3}\sigma_{\mathrm{x}1}}{2\gamma a_{0,1}\sigma_{\mathrm{x}1} - 2a_{0,1}\sigma_{\mathrm{x}1} + \rho - 2\log(\phi)} \\
a_{2,1} =& -\frac{2a_{2,2}\mu_{\mathrm{x}0}\log(\phi)}{\gamma a_{0,1}\sigma_{\mathrm{x}1} - a_{0,1}\sigma_{\mathrm{x}1} + \rho - \log(\phi)} + \frac{a_{2,2}\sigma_{\mathrm{x}1}}{\gamma a_{0,1}\sigma_{\mathrm{x}1} - a_{0,1}\sigma_{\mathrm{x}1} + \rho - \log(\phi)} \\
& + \frac{2(1-\gamma)a_{1,1}a_{1,2}\sigma_{\mathrm{x}0}}{\gamma a_{0,1}\sigma_{\mathrm{x}1} - a_{0,1}\sigma_{\mathrm{x}1} + \rho - \log(\phi)} + \frac{2(1-\gamma)a_{0,1}a_{2,2}\sigma_{\mathrm{x}0}}{\gamma a_{0,1}\sigma_{\mathrm{x}1} - a_{0,1}\sigma_{\mathrm{x}1} + \rho - \log(\phi)} \\
& + \frac{3a_{2,3}\sigma_{\mathrm{x}0}}{\gamma a_{0,1}\sigma_{\mathrm{x}1} - a_{0,1}\sigma_{\mathrm{x}1} + \rho - \log(\phi)} + \frac{(1-\gamma)a_{1,1}^2\sigma_{\mathrm{x}1}}{2\left(\gamma a_{0,1}\sigma_{\mathrm{x}1} - a_{0,1}\sigma_{\mathrm{x}1} + \rho - \log(\phi)\right)} \\
& - \frac{\rho a_{0,1}a_{0,0}^2}{2\left(\gamma a_{0,1}\sigma_{\mathrm{x}1} - a_{0,1}\sigma_{\mathrm{x}1} + \rho - \log(\phi)\right)} + \frac{\rho a_{1,1}a_{0,0}}{\gamma a_{0,1}\sigma_{\mathrm{x}1} - a_{0,1}\sigma_{\mathrm{x}1} + \rho - \log(\phi)} \\
& + \frac{\rho a_{0,1}a_{1,0}}{\gamma a_{0,1}\sigma_{\mathrm{x}1} - a_{0,1}\sigma_{\mathrm{x}1} + \rho - \log(\phi)} \\
a_{2,0} =& -\frac{a_{2,1}\mu_{\mathrm{x}0}\log(\phi)}{\rho} + \frac{(1-\gamma)a_{1,1}^2\sigma_{\mathrm{x}0}}{2\rho} + \frac{(1-\gamma)a_{0,1}a_{2,1}\sigma_{\mathrm{x}0}}{\rho} + \frac{a_{2,2}\sigma_{\mathrm{x}0}}{\rho} - \frac{1}{6}a_{0,0}^3 + a_{1,0}a_{0,0}
\end{aligned}
\tag{3.20}
$$

As can be seen above, the expressions become complicated fast. However, it is straightforward to use a computer to derive higher orders of approximation, at least up to the point that it is also too much for the computer.

3.3.3 Convergence

Regarding convergence the problem constitutes a regular perturbation problem (Bender, Orszag and Orszag 1999). So, the series has a non-vanishing radius of converges around $\epsilon = 0$ for each x. This is known rigorously but it is not clear exactly what the radius of convergence is. Based on the definition of K in Equation (3.14) some conclusions can be drawn. Firstly, a finite order of approximation will never work for all x. As x goes to $\pm\infty$, the highest power of x will necessarily blow up, in a way that does not correspond to the approximated function, as the highest power of x changes for each order of approximation. Furthermore, something can also be said regarding the convergence of the series. On the one hand, for small x, only parameters of the form $a_{n,0}$, $n = 0, 1, 2, 3, \ldots$ matter for the approximation. So, if their growth rate is slower than the decay rate of ϵ^n, then the approximation will converge. On the other hand, if x is not very small, then the parameters of the form $a_{n,n+1}$, $n = 0, 1, 2, 3, \ldots$ will matter for convergence, and the approximation will converge, if the growth rate of these parameters is slower than the decay rate of $(\epsilon \times x)^n$. So, then the question is whether we can deduce the growth rate of these parameters.

Indeed the parameters follow some patterns, that can already be seen in the expression that I provided in the previous subsection. Firstly, the parameters of the form $a_{n,n+1}$, with $n = 0, 1, 2, \ldots$ are determined recursively based on other parameters of the same form, $a_{n',n'+1}$, with $n = 0, 1, 2, \ldots$ and $n' < n$. Thus, these "diagonal" parameters can be computed independently. In addition, the pattern of products implies that any parameter of the form $a_{n,n+1}$ for each n includes terms containing $a_{0,1}$ raised at most to the power $n+1$. In fact, something similar holds for all parameters of the nth order approximation. In particular, when a parameter $a_{n,m}$ is written in terms of $a_{0,0}$ and $a_{0,1}$, each term in the corresponding expression contains combinations of powers of these two initial parameters, and the sum of the powers is always less or equal to $n + 1$. This can be seen, for instance, in Figure 3.1.[10]

This indicates that as long as the following conditions jointly hold the series will converge:

- ϵ^n decays faster than the growth of $a_{0,1}^{(n+1)}$
- ϵ^n decays faster than the growth of $a_{0,0}^n$
- $(\epsilon x)^n$ decays faster than the growth of $a_{0,1}^{n+1}$

Even though these conditions are mostly reliable, they are not exact, as the series can converge when the conditions are violated, and it can also diverge when the conditions hold. In order to prove the exact conditions for convergence, a more extensive analysis needs to be made of the combinatorial structure of the problem,

[10] Here I set $\gamma = 1$. This facilitates some simplifications. In the general case and as long as the exact growth properties are considered an extra factor would arise, as in each step going from $m + 1$ to 0 there are terms that contain an extra power of $a_{0,1}$ both in the numerator and in the denominator. This would affect the third case that I show below.

	1	x	x^2	x^3	x^4	x^5
1	$a_{0,0}$	$a_{0,1}$	0	0	0	0
ϵ	$0. + 0.5\, a_{0,0}^2 + 0.806542\, a_{0,0}\, a_{0,1} + 0.360086\, a_{0,1}^2$	$0. + 0.193458\, a_{0,0}\, a_{0,1} + 0.0863705\, a_{0,1}^2$	$0. + 0.0535437\, a_{0,1}^2$	0	0	0
ϵ^2	$0. + 0.333333\, a_{0,0}^3 + 0.962574\, a_{0,0}^2\, a_{0,1} + 0.898055\, a_{0,0}\, a_{0,1}^2 + 0.271482\, a_{0,1}^3$	$0. + 0.037426\, a_{0,0}^2\, a_{0,1} + 0.129038\, a_{0,0}\, a_{0,1}^2 + 0.0651179\, a_{0,1}^3$	$0. - 0.0270929\, a_{0,0}\, a_{0,1}^2 - 0.00196799\, a_{0,1}^3$	$0. - 0.00837498\, a_{0,1}^3$	0	0
ϵ^3	$0. + 0.25\, a_{0,0}^4 + 0.99276\, a_{0,0}^3\, a_{0,1} + 1.46067\, a_{0,0}^2\, a_{0,1}^2 + 0.936459\, a_{0,0}\, a_{0,1}^3 + 0.219814\, a_{0,1}^4$	$0. + 0.00724036\, a_{0,0}^3\, a_{0,1} + 0.056934\, a_{0,0}^2\, a_{0,1}^2 + 0.0898408\, a_{0,0}\, a_{0,1}^3 + 0.0371744\, a_{0,1}^4$	$0. - 0.0176065\, a_{0,0}^2\, a_{0,1}^2 - 0.0332772\, a_{0,0}\, a_{0,1}^3 - 0.0129955\, a_{0,1}^4$	$0. - 0.000645249\, a_{0,0}\, a_{0,1}^3 - 0.00244501\, a_{0,1}^4$	$0. + 0.00044994\, a_{0,1}^4$	0
ϵ^4	$0. + 0.2\, a_{0,0}^5 + 0.998599\, a_{0,0}^4\, a_{0,1} + 1.98805\, a_{0,0}^3\, a_{0,1}^2 + 1.96372\, a_{0,0}^2\, a_{0,1}^3 + 0.958123\, a_{0,0}\, a_{0,1}^4 + 0.184172\, a_{0,1}^5$	$0. + 0.00140071\, a_{0,0}^4\, a_{0,1} + 0.0182958\, a_{0,0}^3\, a_{0,1}^2 + 0.0591872\, a_{0,0}^2\, a_{0,1}^3 + 0.0625563\, a_{0,0}\, a_{0,1}^4 + 0.0207276\, a_{0,1}^5$	$0. - 0.00634651\, a_{0,0}^3\, a_{0,1}^2 - 0.0311549\, a_{0,0}^2\, a_{0,1}^3 - 0.0346041\, a_{0,0}\, a_{0,1}^4 - 0.0113813\, a_{0,1}^5$	$0. + 0.00403845\, a_{0,0}^2\, a_{0,1}^3 + 0.00259804\, a_{0,0}\, a_{0,1}^4 + 0.000192267\, a_{0,1}^5$	$0. + 0.00136972\, a_{0,0}\, a_{0,1}^4 + 0.000603844\, a_{0,1}^5$	$0. + 0.000153175\, a_{0,1}^5$

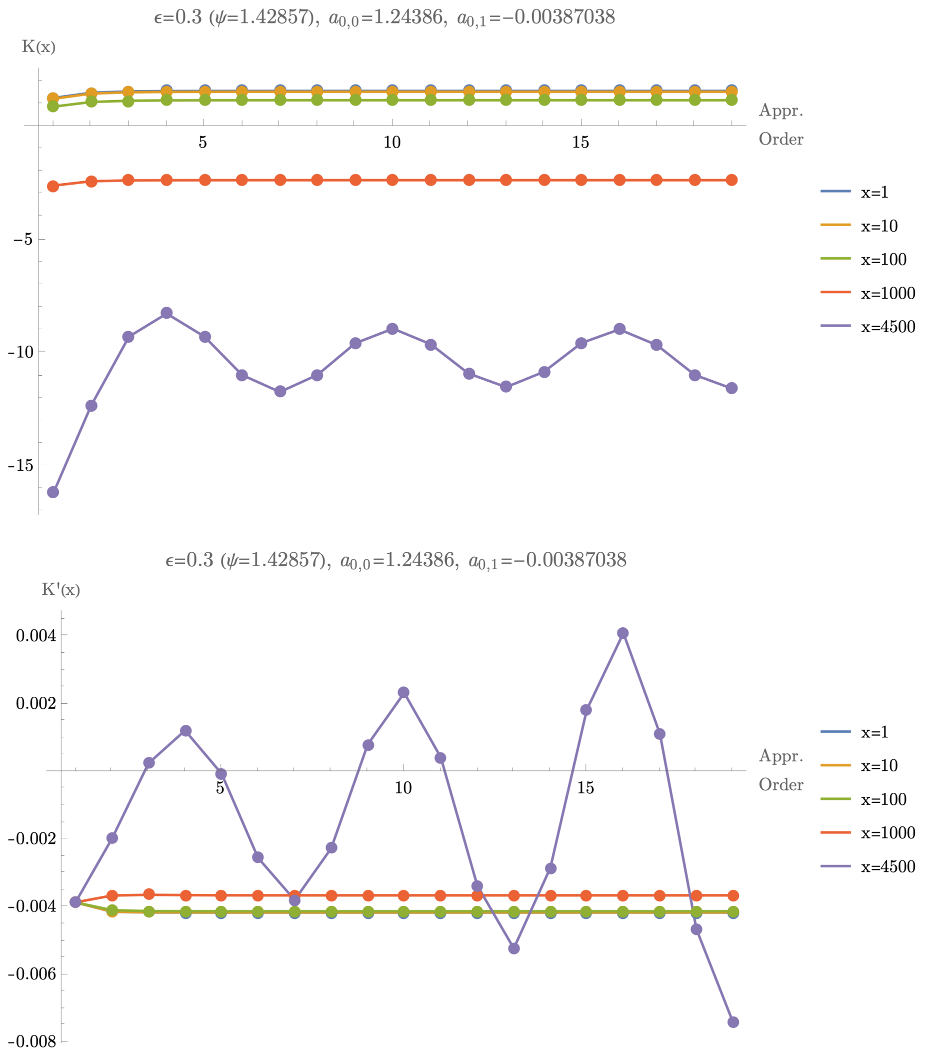
$\{\gamma=1,\ \rho=0.02,\ \mu_{c0}=0.0252,\ \sigma_{c0}=0,\ \sigma_{c1}=0.0004,\ \phi=0.92,\ \sigma_{ct}=\sqrt{\sigma_{c0}+\sigma_{c1}\, x_t},\ \sigma_{x1}=0.0169,\ \mu_{x0}=1,\ \rho cx=\rho\rho_{cx}\ \&,\ \rho_{cx}=-0.5\}$

Figure 3.1: **Series Coefficients – Variation: Time-varying consumption volatility.**
This shows the value of the parameters in terms of $a_{0,0}$ and $a_{0,1}$. The first row and first column show the corresponding power of x and ϵ respectively. The nth power of ϵ and mth power of x correspond to $a_{n,m}$. It can be seen that the highest power of $a_{0,0}$, $a_{0,1}$ or the higher sum of their powers for the parameters in the nth order approximation is $n+1$. The calibration used is also labeled.

so that the growth rate of the coefficients can be exactly determined. However, in practise these conditions are good indications regarding convergence, which can practically be checked by looking at the first partial sums of the series. Figure 3.2 shows these partial sums for the same calibration as in Figure 3.1. The top plot shows convergence of the series for $K(x)$ for different values of x according to the approximate conditions expressed previously, the series should converge as long as $|x \times \epsilon| < 1/a_{0,1}$, that is less than 238 approximately. In fact, the series seems to converge for much larger values also, and it starts diverging when $|x \times \epsilon|$ is about 1200. These numbers are huge, as in this calibration the state variable would practically never take values larger than 10, which would mean that consumption volatility is ten times larger compared to the steady state. The bottom plot shows the convergence of the series for the derivative of $K(x)$ for different values of x, and it is clear that the convergence of the derivative follows the same pattern. This is reasonable given that in this case convergence is regulated by the terms that have a high power in terms of x, and these terms appear in both K and its derivative.

Figure 3.3 shows convergence for different values of ϵ using the same calibration.[11] According to the approximate conditions, the series should converge, if the absolute value of ϵ is less than 0.8. Indeed, as can be seen in the figure the series begins to diverge both for $\epsilon = 0.8$ and for $\epsilon = -0.8$. The figure also shows the corresponding values of ψ for each value of ϵ, and in this example the figurer indicates that the zeroth order approximation used by Tsai and Wachter (2018) can indeed be used for a significant range of ψ values. However, using higher orders of approximation leads to a much larger range of ψ values becoming usable. The second and third plots of Figure 3.3 show respectively that the first and second derivative of K converge for all the values of ϵ, that I have chosen, including the values of ϵ for which K itself diverged. This can be explained, because the convergence of the series depends on the terms containing the lowest powers of x, as terms with higher powers of x will almost certainly be smaller. Hence, the derivatives are much more likely to converge compared to the original function. This means that for some quantities that only rely on the derivatives, the approximation may be used even when K itself diverges.

[11] Apart from γ which is now equal to 2.



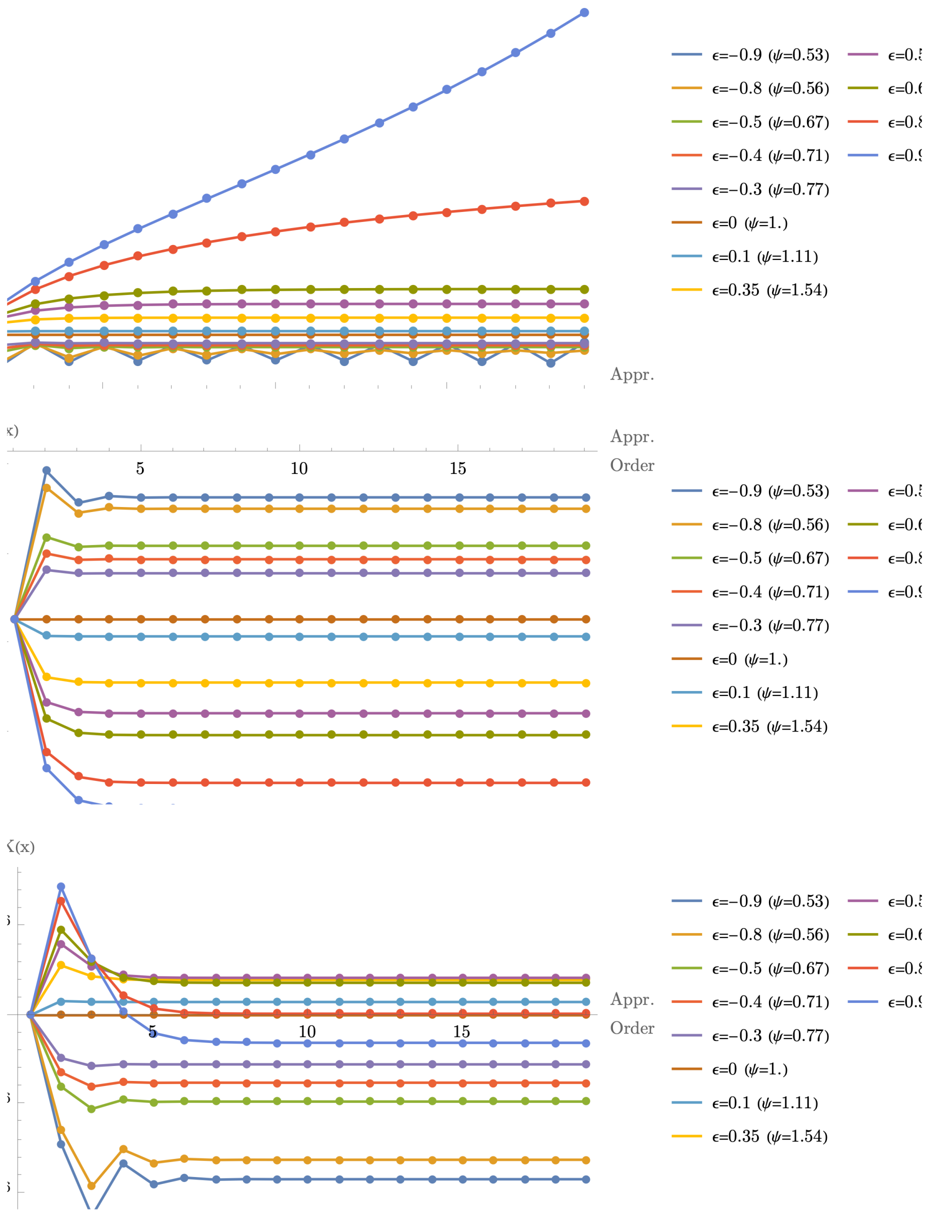
$\{\gamma=2,\ \rho=0.02,\ \mu_{c0}=0.0252,\ \sigma_{c0}=0,\ \sigma_{c1}=0.0004,\ \phi=0.92,\ \sigma_{ct}=\sqrt{\sigma_{c0}+\sigma_{c1}\,x_t}\ ,\ \sigma_{x1}=0.0169,\ \mu_{x0}=1,\ \rho cx=\rho\rho_{cx}\ \&,\ \rho_{cx}=-0.5\}$

Figure 3.2: **Variation: Time-varying consumption volatility.** This shows the convergence of the problem for different values of x.

3.4 Pricing

3.4.1 Process of the stochastic discount factor

As has been shown already the method can provide reliable approximations for a large range of ψ values. The next step is to use the approximation to perform the pricing of securities. This requires the derivation of the process of the SDF. In particular, given the expression for the value function, Ito's Lemma can be implemented to get to the stochastic differential equation that governs the SDF.



$\{\gamma=2,\ \rho=0.02,\ \mu_{c0}=0.0252,\ \sigma_{c0}=0,\ \sigma_{c1}=0.0004,\ \phi=0.92,\ \sigma_{ct}=\sqrt{\sigma_{c0}+\sigma_{c1}\,x_t}\,,\ \sigma_{x1}=0.0169,\ \mu_{x0}=1,\ \rho cx=\rho\rho_{cx}\ \&,\ \rho_{cx}=-0.5\}$

Figure 3.3: **Variation: Time-varying consumption volatility.**
This shows the convergence of the problem for different values of ϵ. The plots correspond to the series of K and its two derivatives.

The calculation here follows Chen et al. (2009). In particular, the fundamental relationship is:

$$\frac{d\Lambda}{\Lambda} = f_V(C,V)dt + \frac{df_C(C,V)}{f_C(C,V)} \tag{3.21}$$

This can be computed relatively easily. The first term is just the derivative of the flow utility with respect to the value function. The second term can be computed with an application of Ito's lemma on the derivative of flow utility with respect to consumption.[12] The result is the following:

$$\begin{aligned}\frac{d\Lambda}{\Lambda} =& \Bigg(\frac{\rho\left(-(1-\gamma\psi)e^{-\frac{(\psi-1)K[x_t]}{\psi}} - \gamma\psi + \psi\right)}{1-\psi} - \gamma\mu_{ct} + \frac{\gamma^2\sigma_{ct}^2}{2} + \frac{\gamma(\gamma\psi-1)\rho_{cx}\sigma_{xt}\sigma_{ct}K'(x_t)}{\psi} \\ &+ \frac{(\gamma\psi-1)\Big(2\psi(\mu_{x0}-x_t)\log(\phi)K'(x_t) + \sigma_{xt}^2\left((\gamma\psi-1)K'(x_t)^2 - \psi K''(x_t)\right)\Big)}{2\psi^2}\Bigg)dt \\ &+ \frac{(1-\gamma\psi)\sigma_{xt}K'(x_t)}{\psi}dZ_{xt} - \gamma\sigma_{ct}dZ_{ct}\end{aligned} \tag{3.22}$$

where ρ_{cx} is the correlation between consumption and the state variable. The time-separable case arises when $\gamma = 1/\psi$. As can be seen by the expression above the dependence of the SDF on $K(x)$ and on x disappears in this case. The stochastic component that is related to consumption $(-\gamma\sigma_{ct}dZ_{ct})$ is the same as in the time-separable case. On the contrary, the stochastic component that is related to the state variable $((1-\gamma\psi)\sigma_{xt}K'(x_t)/\psi dZ_{xt})$ does not appear in the time-separable case. So, given that these stochastic components are ultimately responsible for the generation of risk premia, recursive utility introduces an extra mechanism by which premia can be generated. The sign of this mechanism depends on the sign of $(1-\gamma\psi)$, which corresponds to a preference for late or early resolution of uncertainty.

Furthermore, based on the SDF expression the risk-free rate can also be de-

[12] It is possible to do this operation after substituting the value function using Equation (3.7) and applying Ito's lemma based on consumption and the state variable as independent variables.

duced:

$$
\begin{aligned}
r(x) =& - E\frac{d\Lambda}{\Lambda}\frac{1}{dt} = \\
& \gamma\mu_{ct} - \frac{\gamma^2\sigma_{ct}^2}{2} + \frac{\rho\left((1-\gamma\psi)e^{-\frac{(\psi-1)K[x_t]}{\psi}} + \gamma\psi - \psi\right)}{1-\psi} + \frac{\gamma(1-\gamma\psi)\rho_{cx}\sigma_{xt}\sigma_{ct}K'(x_t)}{\psi} \\
& + \frac{(1-\gamma\psi)\Big(2\psi(\mu_{x0}-x_t)\log(\phi)K'(x_t) + \sigma_{xt}^2\left((1-\gamma\psi)K'(x_t)^2 - \psi K''(x_t)\right)\Big)}{2\psi^2}
\end{aligned}
\tag{3.23}
$$

The short rate is also affected by recursive utility. While the consumption smoothing motive ($\gamma\mu_{ct}$) and the precautionary savings motive ($-\frac{\gamma^2\sigma_{ct}^2}{2}$) is the same as in the time-separable case, the time preference parameter is multiplied by a new factor, and the remaining terms are all new.

3.4.2 Long-term bonds

The process for the SDF can be inserted in the pricing differential equation as in Cochrane (2009) and Chen et al. (2009):

$$
E\Big[d(\Lambda Q)\Big] = 0 \Rightarrow E\left[\frac{d\Lambda}{\Lambda} + \frac{dQ}{Q} + \frac{d\Lambda dQ}{\Lambda Q} = 0\right] = 0 \tag{3.24}
$$

Here $Q(m, x)$ is the price of the zero-coupon bond with maturity m when the state of the economy is x.[13] By Ito's Lemma:

$$
dQ(x, m) = \Big(-\log(\phi)(\mu_{x0} - x)Q_x - Q_m + \frac{1}{2}\sigma_x^2 Q_{xx}\Big)dt + \sigma_{xt}Q_x dZ_{xt} \tag{3.25}
$$

This can be directly plugged in Equation 3.24 and the result is:

$$
\begin{aligned}
0 =& - Q_m + r(x)Q + (-\log(\phi)(\mu_{x0} - x) + A(x))\,Q_x + \frac{1}{2}Q_{xx}\sigma_{xt}^2 \\
A(x) =& \,(\gamma + \epsilon - 1)\sigma_{xt}^2 K^{(1,0)}(x, \epsilon) + \gamma\rho_{cx}\sigma_{ct}\sigma_{xt}
\end{aligned}
\tag{3.26}
$$

The subscripts $\cdot_m$ and $\cdot_x$ denote partial derivatives with respect to maturity, m, and with respect to the state variable, x, respectively. In the above equation $K(x)$ appears in $r(x)$, but the coefficients of Q_x and Q_{xx} only contain $K^{(1,0)}(x)$. This is noteworthy because Equations (3.25) and (3.23)imply that the expected

[13] In the formulas I use Q instead of $Q(m, x)$ to avoid cluttering.

instantaneous excess return obeys the following relationship:

$$E\left[\frac{dQ}{Q}\right]-r(x_t)dt=-E\left[\frac{d\Lambda dQ}{\Lambda Q}\right]=A(x)dt=\Big((\gamma+\epsilon-1)\sigma_{xt}^2K^{(1,0)}(x,\epsilon)+\gamma\rho_{cx}\sigma_{ct}\sigma_{xt}\Big)dt \tag{3.27}$$

So, the term premium also primarily depends on $K'(x)$ and not $K(x)$ itself. This implies that my approximation may be able to provide useful information about term premia even when it diverges, given the result in Section 3.3.3 that the derivative of K can converge even when K diverges.[14]

Continuing with the pricing of the long-term bond, according to the Feynman-Kac method Equation (3.26) can be solved by Monte Carlo simulations. In particular:

$$Q(m,x_t)=E\left[\exp\left\{\int_m^0 r(\tilde{x}_{t+s})ds\right\}\right]=E\left[\exp\left\{-\int_0^m r(\tilde{x}_{t+s})dt\right\}\right] \tag{3.28}$$

where $\tilde{x}_0=x$ and $\tilde{x}_t$ follows the modified process:

$$d\tilde{x}=\Big(-\log(\phi)(\mu_{x0}-\tilde{x})+(\gamma+\epsilon-1)\sigma_x(\tilde{x})^2K'(\tilde{x})+\gamma\rho_{cx}\sigma_c(\tilde{x})\sigma_x(\tilde{x})\Big)\,dt+\sigma_x(\tilde{x}_t)dZ_{xt} \tag{3.29}$$

This is a modified process because, while it is similar to the regular state variable of the model, the trend component of the modified process has extra terms coming from the interaction of the SDF with the stochastic components of the state variable process.[15] Based on function Q and Equation (3.25), it is also easy to derive the instantaneous expected return of long-term bonds.

If, instead of using the modified process, the original state variable is used:

$$H(m,x_t)=E\left[\exp\left\{\int_m^0 r(x_{t+s})ds\right\}\right]=E\left[\exp\left\{-\int_0^m r(x_{t+s})dt\right\}\right] \tag{3.30}$$

The result is the price of the *risk-neutral bond*, namely a bond priced by a risk-neutral investor with the same consumption process and utility function as in the original model. The difference between the yields of Q and H can be defined as the term premium for the corresponding maturity:

$$TP(m,x_t)=-\frac{\log\big(Q(m,x_t)\big)}{m}-\left(-\frac{\log\big(H(m,x_t)\big)}{m}\right) \tag{3.31}$$

[14] For brave researchers this could suggest the use of this approximation even when the original series diverges, when the item of interest is the risk premium, which is determined by the derivative of $K(x)$.

[15] In this expression, instead of σ_{ct} and σ_{xt}, I am using $\sigma_c(\tilde{x})$ and $\sigma_x(\tilde{x})$ in order to make explicit that these quantities can now be functions of the modified process $\tilde{x}$.

3.4.3 Price-consumption ratio

The price consumption-ratio is a concept similar to the price-dividend ratio. It is a ratio, whose numerator is the price of the consumption perpetuity, a security that continuously pays the consumption flow for an infinite horizon, and its denominator is the concurrent consumption flow. Wachter (2006) derived the price consumption ratio in discrete time for the habit model of Campbell and Cochrane (1999). Here I use the same approach adapted for continuous time. So, I build up the price-consumption ratio from zero-coupon securities that pay the value of consumption after m periods. These securities have a price $P(m, x_t, C_t)$ at time t. The value of these securities depends on the current value of the sate variable and the current value of consumption. In order to avoid the dependence on consumption, I divide these securities by current consumption. This leads to a *zero-coupon m-year price consumption ratio*, $q(m, x) = P(m, x, C_t)/C_t$. Then the combination of these zero-coupon securities leads to the full price-consumption ratio:[16]

$$p(x_t) = \int_0^\infty q(m, x_t)dm = \int_0^\infty \frac{P(m, x_t, C_t)}{C_t}dm \tag{3.32}$$

Furthermore, one can also define price-consumption *annuity* ratio. The consumption annuity is similar to the consumption perpetuity, but it only pays coupons for a finite period of time M. For example:

$$p_M(x_t) = \int_0^M q(m, x_t)dm = \int_0^M \frac{P(m, x_t, C_t)}{C_t}dm \tag{3.33}$$

If M is large, this quantity likely behaves similar to the price-consumption ratio, but in practice this may be easier to compute as it does not require the calculation of the integral for an infinite horizon.

Moving on, for simplicity I drop the time subscript in the following expressions. In order to derive $q(m, x)$, I follow an approach similar to Chen, Cosimano and Himonas (2010), who use the pricing equation to calculate the price-consumption ratio directly. Unlike them I first calculate the q's and I then build up the price-consumption ratio. This is arguably more complicated as it involves the solution of a partial differential equation and the computation of an integral, instead of the solution of an ordinary differential equation only. However, my approach does not require the specification of initial conditions and it determines the price-

[16] This assumes that the integral is finite.

consumption ratio uniquely. Thus, the pricing equation can be re-written:

$$
\begin{aligned}
E\Big[d(\Lambda P(m,x,C))\Big] = 0 \Rightarrow & E\Bigg[\frac{d\Lambda}{\Lambda} + \frac{dP(m,x,C)}{P(m,x,C)} + \frac{d\Lambda dP(m,x,C)}{\Lambda P(m,x,C)}\Bigg] = 0 \\
\Rightarrow & E\Bigg[\frac{d\Lambda}{\Lambda} + \frac{d\Big(q(m,x)C\Big)}{q(m,x)C} + \frac{d\Lambda d\Big(q(m,x)C\Big)}{\Lambda q(m,x)C}\Bigg] = 0 \\
\Rightarrow & E\Bigg[\frac{d\Lambda}{\Lambda} + \frac{dq}{q} + \frac{dC}{C} + \frac{d\Lambda dq}{\Lambda q} + \frac{d\Lambda dC}{\Lambda C} + \frac{dqdC}{qC}\Bigg] = 0
\end{aligned}
\tag{3.34}
$$

In the final line I do not show the dependence of p for simplicity. Similar to above, by Ito's Lemma:

$$
dq = \Big(- \log(\phi)(\mu_{x0} - x)q_x - q_m + \frac{1}{2}\sigma_x^2 q_{xx}\Big)dt + \sigma_{xt} q_x dZ_{xt} \tag{3.35}
$$

So the processes for the SDF, for the zero-coupon consumption security and for consumption can all be substituted in the equation above and this will again generate a partial differential equation that can be solved, by computing the Feynman-Kac formula through Monte Carlo simulations. The pricing equation is:

$$
\begin{aligned}
0 = & \underbrace{-r(x)}_{d\Lambda/\Lambda} + \underbrace{\Big(- \log(\phi)(\mu_{x0} - x)\frac{q_x}{q} - \frac{q_m}{q} + \frac{1}{2}\frac{q_{xx}}{q}\sigma_{xt}^2\Big)}_{dq/q} + \underbrace{\mu_{ct}}_{dC/C} \\
& + \underbrace{\frac{(1-\gamma\psi)\rho_{cx}\sigma_{xt}\sigma_{ct}K'(x_t)}{\psi} - \gamma\sigma_{ct}^2}_{d\Lambda dC_t/(\Lambda C_t)} + B(x)\frac{q_x}{q} \\
B(x) = & \underbrace{\frac{(1-\gamma\psi)\sigma_{xt}^2 K'(x_t)}{\psi} - \gamma\rho_{cx}\sigma_{xt}\sigma_{ct}}_{d\Lambda dq/(\Lambda q)} + \underbrace{\rho_{cx}\sigma_{xt}\sigma_{ct}}_{dqdC_t/(qC_t)}
\end{aligned}
\tag{3.36}
$$

The brackets show where the expressions in the equations come from. The equation can be rewritten as:

$$
\begin{aligned}
0 = & -\left(r(x) - \mu_{ct} - \frac{(1-\gamma\psi)\rho_{cx}\sigma_{xt}\sigma_{ct}K'(x_t)}{\psi} + \gamma\sigma_{ct}^2\right)q - q_m + \frac{\sigma_{xt}^2}{2}q_{xx} \\
& + \Big(- \log(\phi)(\mu_{x0} - x) + \frac{(1-\gamma\psi)\sigma_{xt}^2 K'(x_t)}{\psi} - \gamma\rho_{cx}\sigma_{xt}\sigma_{ct} + \rho_{cx}\sigma_{xt}\sigma_{ct}\Big)q_x
\end{aligned}
\tag{3.37}
$$

The corresponding Feynman-Kac formula is:

$$q(m, x_t) = E\left[\exp\left\{\int_m^0 \bar{r}(\bar{x}_{t+s})ds\right\}\right] = E\left[\exp\left\{-\int_0^m \bar{r}(\bar{x}_{t+s})dt\right\}\right] \tag{3.38}$$

where

$$\bar{r}(\bar{x}) = r(\bar{x}) - \mu_c(\bar{x}) - \frac{(1-\gamma\psi)\rho_{cx}\sigma_x(\bar{x})\sigma_c(\bar{x})K'(\bar{x})}{\psi} + \gamma\sigma_c(\bar{x})^2 \tag{3.39}$$

$\bar{x}_0 = x_0$ and $\bar{x}$ follows another modified process:[17]

$$d\bar{x} = \left(-\log(\phi)(\mu_{x0} - \bar{x}) + \frac{(1-\gamma\psi)\sigma_x(\bar{x})^2 K'(x_t)}{\psi} + (1-\gamma)\rho_{cx}\sigma_x(\bar{x})\sigma_c(\bar{x})\right)dt + \sigma_x(\bar{x}_t)dZ_{xt} \tag{3.40}$$

Given the price-consumption ratio as a function of the state variable and the given stochastic process of the price-consumption ratio that can be written as follows:

$$dp = \Big(-\log(\phi)(\mu_{x0} - x)p_x + \frac{1}{2}\sigma_x^2 p_{xx}\Big)dt + \sigma_{xt}p_x dZ_{xt} \tag{3.41}$$

The return of the consumption perpetuity can be derived:[18]

$$\begin{aligned} \frac{dP}{P} + \frac{C}{P}dt =& \frac{d(Cp)}{Cp} + \frac{1}{p} = \frac{dC}{C} + \frac{dp}{p} + \frac{dCdp}{Cp} + \frac{1}{p}dt \\ =& \mu_{ct}dt + \sigma_{ct}dZ_{ct} - \log(\phi)(\mu_{x0} - x)\frac{p_x}{p}dt + \frac{\sigma_{xt}^2}{2}\frac{p_{xx}}{p}dt \end{aligned} \tag{3.42}$$

$$+ \sigma_{xt}\frac{p_x}{p}dZ_{xt} + \rho_{cx}\sigma_{ct}\sigma_{xt}\frac{p_x}{p}dt + \frac{1}{p}dt$$

$$= \left(\mu_{ct} - \log(\phi)(\mu_{x0} - x)\frac{p_x}{p} + \frac{\sigma_{xt}^2}{2}\frac{p_{xx}}{p} + \rho_{cx}\sigma_{ct}\sigma_{xt}\frac{p_x}{p} + \frac{1}{p}\right)dt + \sigma_{ct}dZ_{ct} + \sigma_{xt}\frac{p_x}{p}dZ_{xt}$$

Finally the expected return is:

$$E\left[\frac{dP}{P}\right] + \frac{C}{P}dt = \left(\mu_{ct} - \log(\phi)(\mu_{x0} - x)\frac{p_x}{p} + \frac{\sigma_{xt}^2}{2}\frac{p_{xx}}{p} + \rho_{cx}\sigma_{ct}\sigma_{xt}\frac{p_x}{p} + \frac{1}{p}\right)dt \tag{3.43}$$

[17]Similar to above, in this expression, instead of σ_{ct} and σ_{xt}, I am using $\sigma_c(\bar{x})$ and $\sigma_x(\bar{x})$ in order to make explicit that these quantities can now be functions of the modified process $\bar{x}$.

[18]This calculation only applies for the consumption perpetuity. For the case of the consumption annuity the calculation would require an extra component that accounts for the fact that the annuity at time t has infinitesimally lower duration compared to the annuity at time $t + dt$. Numerically, I only calculate annuities. So, I only apply this formula for long-lived annuities, for which the security's price should not change significantly with duration.

3.5 Applications

3.5.1 Time-varying consumption drift

Given the setup introduced in the previous section, real interest rates and the price consumption ratio can be determined. Here, I show the results for the case when consumption drift is time-varying. This is the result of setting $\mu_{c1} = 1$, which means that the consumption drift is proportional to the state variable. In this variation consumption volatility is constant, because $\sigma_{c1} = 0$, and the process is homoskedastic, because $\sigma_{x1} = 0$. Figure 3.4 shows the results, while comparing the eighth order approximation, using the method introduced in this paper, to the zeroth order approximation, that is equivalent to the method of Tsai and Wachter (2018). In addition, here $\epsilon = 0.1$ ($\psi = 1.11$) which is close to the analytic solution for $\epsilon = 0$ ($\psi = 1$). The results verify that the basic approximation can be accurate for $\psi \neq 1$. The fist row shows the instantaneous rate and the ten-year yield as a function of the state of the economy, which is reflected by consumption drift. The results for the two approximations are very similar. The second row shows the inverse price consumption ratio as a function of the consumption drift. The inverse price consumption ratio is equivalent to the dividend yield of the security. As I have already mentioned, I calculate the price consumption ratio numerically by integrating the zero coupon price consumption ratios. However, I cannot numerically integrate to infinity. So, I put the cutoff at 200 years. This means that technically I am calculating price consumption ratio for the 200 year consumption annuity. The second plot of the second row shows the value of the inverse price consumption ratio for different cutoff points and it can be seen that at 200 hundred years it is relatively close to being converged. Similar to above, the price consumption ratio for the two approximations is quite close, even though in this case the price consumption ratio is not very sensitive to the value of consumption drift. So, the difference appears larger in the figure. Finally in the third row, I also show the instantaneous expected return of the consumption perpetuity, which is very similar to the instantaneous short-term rate in this variation.

In Figure 3.5, I show the results for $\epsilon = 0.7$, which is not so close to the analytic solution of $\epsilon = 0$, and as can be seen in Figure 3.9, the value function varies significantly for different orders of approximation. This example illustrates the value of my method, as it shows that for some interesting values for the intertemporal elasticity of substitution ($\psi = 3.3$ in this case, but in other examples it can also be lower), the value function deviates significantly between the different orders of approximation. And this has consequences for the implied value of the short-term rate, the yield and the price consumption ratio. The short-term rate and the ten-year yield appear linear as functions of the consumption drift for both approximations, but the slope is different and at the steady state there is a difference of roughly 1% for the short-term rate and a bit lower for the ten-year yield. The difference between the two approximations is also higher for the price consumption ratio, which is now also more sensitive to the consumption drift.

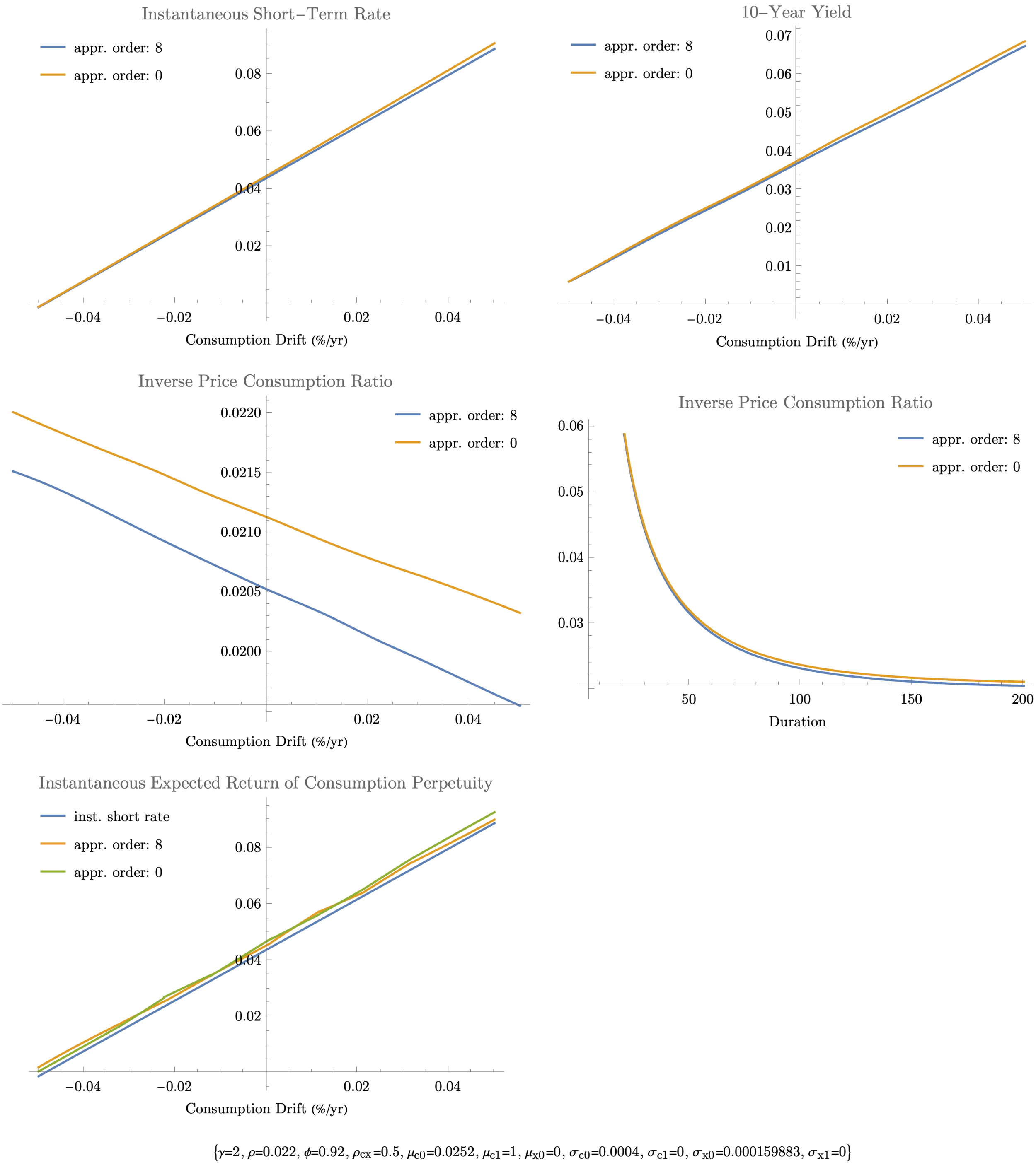


Figure 3.4: **Variation: Time-varying consumption drift** $-\ \epsilon = 0.1$
The first row shows the .

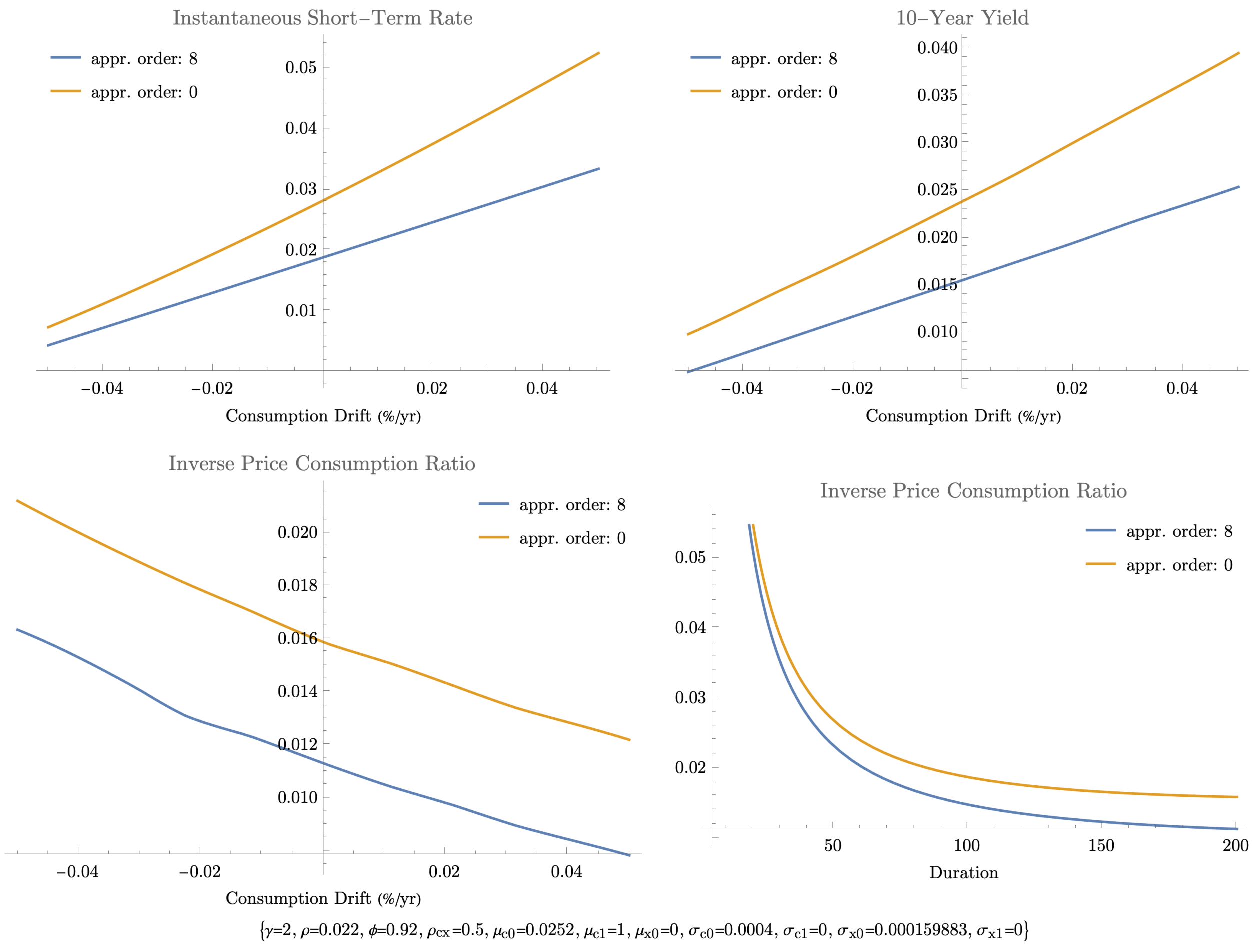


Figure 3.5: **Variation: Time-varying consumption drift** $- \epsilon = 0.1$
The first row shows the .

3.5.2 Time-varying consumption volatility

As a second application, I introduce the case when consumption volatility is time-varying. This is the result of setting $\sqrt{\sigma_{c1}} \neq= 0$, while consumption drift is constant because $\mu_{x1} = 0$. The state variable is also heteroskedastic, and it is guaranteed to be positive, because $\sigma_{x0} = 0$ and $\sqrt{\sigma_{x1}} \neq 0$. Figure 3.6 shows the case where $\epsilon = 0.1$, and all rates are not very sensitive to consumption volatility. Nevertheless, the figure demonstrates that the zeroth order approximation is roughly within 15 basis points compared to the higher approximation. Depending on the application, this difference could be considered negligible. In addition, while there is a difference in the level of the rates, the difference in the slope of the rates with respect to consumption volatility is not noticeable.

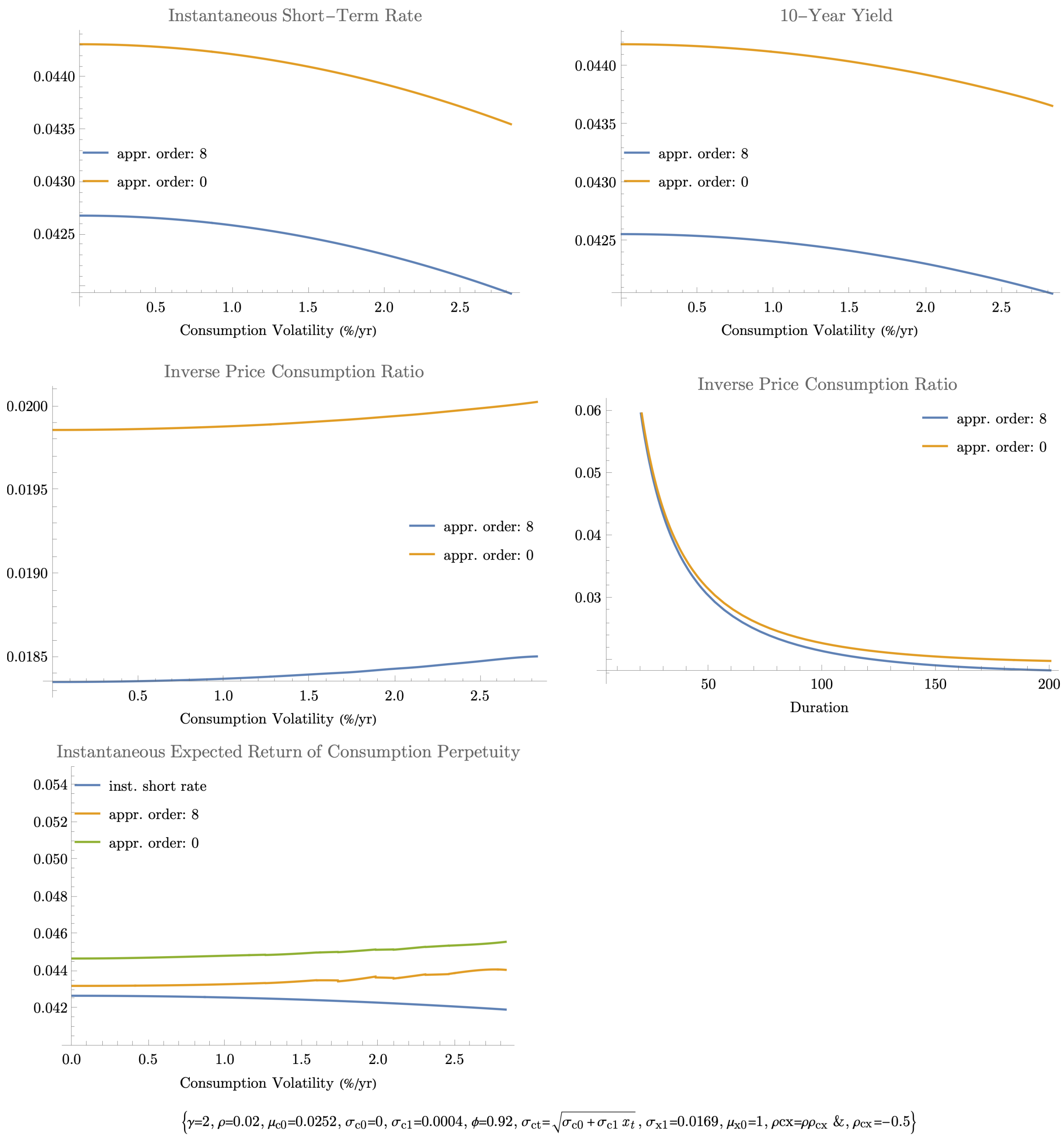
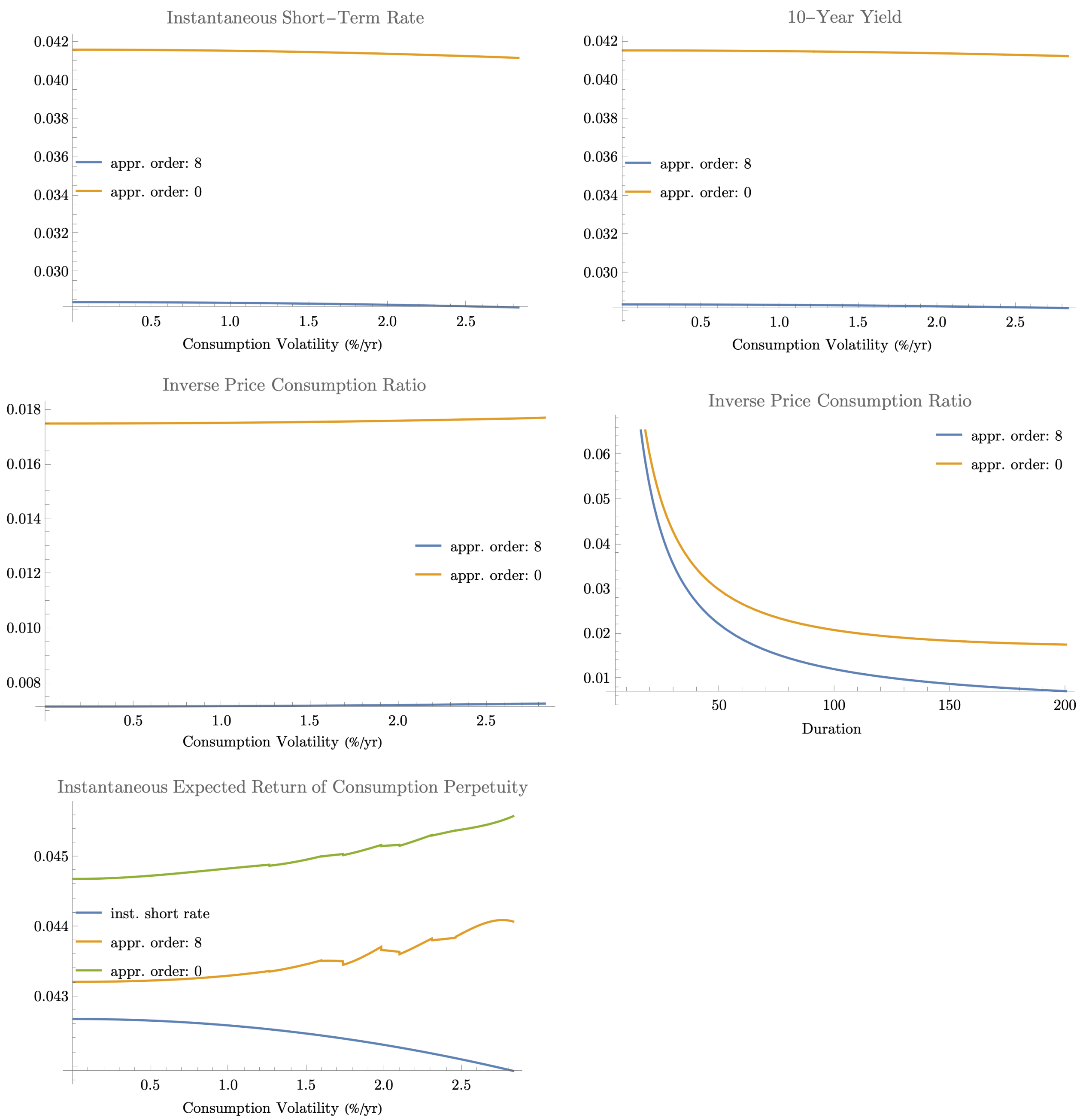


Figure 3.6: **Variation: Time-varying consumption drift** $- \epsilon = 0.1$
The first row shows the .

Figure 3.7 shows the case where $\epsilon = 0.7$. Now, the difference in the rates is certainly not negligible, as it ranges around 100 basis points. Nevertheless the slopes still appear the same. This is the result of the derivatives of K being very well approximated by the zeroth order approximation (Figure 3.9).



$\left\{\gamma=2, \rho=0.02, \mu_{c0}=0.0252, \sigma_{c0}=0, \sigma_{c1}=0.0004, \phi=0.92, \sigma_{ct}=\sqrt{\sigma_{c0}+\sigma_{c1}\, x_t}, \sigma_{x1}=0.0169, \mu_{x0}=1, \rho\text{cx}=\rho\rho_{cx}\ \&, \rho_{cx}=-0.5\right\}$

Figure 3.7: **Variation: Time-varying consumption drift** $-\ \epsilon = 0.7$
The first row shows the.

3.6 Conclusion

In conclusion, I have introduced a new method, based on perturbation theory, to express the value function when the agent exhibits recursive utility. The value function is expressed as a series in terms of ϵ and it constitutes a global perturbation solution. The value of ϵ is determined by the value of ψ, which represents the IES in the problem. The first term in the series (which multiplies ϵ^0) gives the solution for $\psi = 1$. Each further order of approximation only requires the solution of linear equations. Computing the first fifteen orders of approximation is relatively easy, but higher orders are typically computationally demanding as the number of coefficients increases by one for each order of approximation and the equations become increasingly complicated.

The method is useful for a wide range of calibrations. Tsai and Wachter (2018) only use the zeroth order approximation. I have shown that this can produce accurate results for a relatively low absolute value of ϵ, but the approximation can deteriorate as the absolute value of ϵ increases. Higher order approximations using my method can solve this issue, and this applies both for models with time-varying consumption drift and time-varying consumption volatility. The paper can also be extended to include multiple state variables and Poisson jump components in the consumption process. I have used the perturbation series to derive both the price of long-term bonds and the price consumption ratio of zero coupon consumption securities, consumption annuities and the consumption perpetuity. Furthermore, I have also derived the expected instantaneous return of the consumption perpetuity. Apart from being easy to implement, my method allows to easily check whether the results are accurate and how many orders of approximation are necessary for an acceptable solution. Despite not having derived exact convergence conditions, I have sketched the behaviour of the series for different orders of approximation, and I hope to derive more exact results in future versions of this paper.

For future research, it would be important to derive results that would guarantee the existence of a solution to this problem, as also discussed in Tsai and Wachter (2018), but such a task may not be easy. In addition, concentrating on my method, the perturbation series uniquely determines the value function, even if it is a divergent asymptotic series for some combinations of parameters and values of x. This means that further work following this approach, using more sophisticated mathematical analysis, could provide an expression of the solution that is easily computable and uniformly convergent, possibly in terms of special mathematic functions.[19]

[19] Applying a Padé approximation to the problem did not yield converging results in the regions that were diverging under the regular approximation.

Appendix

3.A Proof of result in Equation (3.8)

The expression is derived from the Hamilton-Jacobi-Bellman equation, $\mathcal{D}V + f(C,V) = 0$, after the relevant quantities have been substituted. By applying Ito's Lemma to V, which is a function of C and x, the result is:

$$\frac{\mathcal{D}V}{V} = -\frac{1}{2}(\gamma-1)\left(-\gamma\sigma_{\mathrm{ct}}^2 + 2\mu_{\mathrm{ct}} + \sigma_{\mathrm{xt}}^2 K''(x) - \gamma\sigma_{\mathrm{xt}}^2 K'(x)^2 + 2\mu_{\mathrm{xt}}K'(x) + \sigma_{\mathrm{xt}}^2 K'(x)^2\right) \tag{3.44}$$

Here, I can substitute the guessed expression for the value function, $V = \frac{C^{1-\gamma}e^{(1-\gamma)K(x)}}{1-\gamma}$, which I will verify later, in the previous expression and in the expression for flow utility:

$$\frac{\mathcal{D}V}{V} = (1-\gamma)\left(\mu_{\mathrm{ct}} + \mu_{\mathrm{xt}}K'(x) - \frac{\gamma\sigma_{\mathrm{ct}}^2}{2} + \frac{(1-\gamma)\sigma_{\mathrm{xt}}^2}{2}K'(x)^2 + \frac{\sigma_{\mathrm{xt}}^2}{2}K''(x)\right)$$

$$f(C,V) = \frac{(1-\gamma)\rho V\left(\left(C((1-\gamma)V)^{-\frac{1}{1-\gamma}}\right)^{1-\frac{1}{\psi}} - 1\right)}{1-\frac{1}{\psi}} = (1-\gamma)\rho\frac{\psi\left(1-e^{\left(\frac{1}{\psi}-1\right)K[x]}\right)}{1-\psi} \tag{3.45}$$

After plugging these two expressions in the JHB equation, the result is:

$$\rho\frac{\psi\left(1-e^{\left(\frac{1}{\psi}-1\right)K[x]}\right)}{1-\psi} + \mu_{\mathrm{ct}} + \mu_{\mathrm{xt}}K'(x) - \frac{\gamma\sigma_{\mathrm{ct}}^2}{2} + \frac{(1-\gamma)\sigma_{\mathrm{xt}}^2}{2}K'(x)^2 + \frac{\sigma_{\mathrm{xt}}^2}{2}K''(x) = 0 \tag{3.46}$$

This is Equation (3.8) in the main text. By the fact that this is the result of the HJB equation, assuming that the solution exists, the guess is verified.

3.B Derivation of the SDF with time-recursive utility

As mentioned in the paper the stochastic differential equation of the SDF can be derived based on the following expression:

$$\frac{d\Lambda}{\Lambda} = f_V(C,V)dt + \frac{df_C(C,V)}{f_C(C,V)} \tag{3.47}$$

So, flow utility is a central component of the derivation:

$$f(C,V) = \frac{\beta}{1-1/\psi}\Big((1-\gamma)V\Big)\left(\left(C((1-\gamma)V)^{-\frac{1}{1-\gamma}}\right)^{1-1/\psi} - 1\right) \tag{3.48}$$

The partial derivative of f with respect to V is:

$$f_V(C,V) = \frac{\rho\left((\gamma-1)\psi + (1-\gamma\psi)\left(C(V-\gamma V)^{\frac{1}{\gamma-1}}\right)^{\frac{\psi-1}{\psi}}\right)}{\psi-1} \tag{3.49}$$

The partial derivative of f with respect to C is:

$$f_C(C,V) = -\frac{(\gamma-1)\rho V\left(C(V-\gamma V)^{\frac{1}{\gamma-1}}\right)^{\frac{\psi-1}{\psi}}}{C} \tag{3.50}$$

As I implement Ito's Lemma directly using c_t and x_t as independent variables, I make the following replacements in the expressions above:

$$c_t = \log(C), \quad V = \frac{C^{1-\gamma}}{1-\gamma}e^{(1-\gamma)K(x_t)} \Rightarrow K(x_t) = \frac{\log\left(-\frac{C^{1-\gamma}}{(\gamma-1)V}\right)}{\gamma-1} \tag{3.51}$$

So, they become after simplification:

$$\begin{aligned} f_V(C,V) \to g(c_t,x_t) =& \frac{\rho\left(-(1-\gamma\psi)e^{-\frac{(\psi-1)K[x_t]}{\psi}} - \gamma\psi + \psi\right)}{1-\psi} \\ f_C(C,V) \to h(c_t,x_t) =& \rho e^{\left(\frac{1}{\psi}-\gamma\right)K(x_t) - c_t\gamma} \end{aligned} \tag{3.52}$$

And I implement Ito's Lemma on g_2. The partial derivatives are:

$$
\begin{aligned}
\frac{\partial h(c_t,x_t)}{\partial c_t} =&\gamma\rho\left(-e^{\left(\frac{1}{\psi}-\gamma\right)K[x_t]-\gamma c_t}\right) = -\gamma h(c_t,x_t)\\
\frac{\partial h(c_t,x_t)}{\partial x_t} =&\rho\left(\frac{1}{\psi}-\gamma\right)K'\left(x_t\right)e^{\left(\frac{1}{\psi}-\gamma\right)K[x_t]-\gamma c_t} = \left(\frac{1}{\psi}-\gamma\right)K'\left(x_t\right)h\left(c_t,x_t\right)\\
\frac{\partial^2 h(c_t,x_t)}{\partial c_t^2} =&\gamma^2\rho e^{\left(\frac{1}{\psi}-\gamma\right)K[x_t]-\gamma c_t} = \gamma^2 h\left(c_t,x_t\right)\\
\frac{\partial h(c_t,x_t)}{\partial x_t^2} =&\frac{\rho(\gamma\psi-1)\left((\gamma\psi-1)K'\left(x_t\right)^2-\psi K''\left(x_t\right)\right)e^{\left(\frac{1}{\psi}-\gamma\right)K[x_t]-\gamma c_t}}{\psi^2}\\
=&\frac{(\gamma\psi-1)\left((\gamma\psi-1)K'\left(x_t\right)^2-\psi K''\left(x_t\right)\right)}{\psi^2}h\left(c_t,x_t\right)\\
\frac{\partial h(c_t,x_t)}{\partial c_t\partial x_t} =&\frac{\gamma\rho(\gamma\psi-1)K'\left(x_t\right)e^{\left(\frac{1}{\psi}-\gamma\right)K[x_t]-\gamma c_t}}{\psi} = \frac{\gamma(\gamma\psi-1)K'\left(x_t\right)h\left(c_t,x_t\right)}{\psi}
\end{aligned}
\tag{3.53}
$$

The expressions above should be plugged into the expression:

$$
\begin{aligned}
\frac{df_C}{f_C} =&\Bigg(\frac{\partial h(c_t,x_t)}{\partial c_t}\mu_{ct} + \frac{\partial h(c_t,x_t)}{\partial x_t}\Big(-\log(\phi)\Big)(\mu_{x0}-x_t)\\
&+\frac{\sigma_{ct}^2}{2}\frac{\partial^2 h(c_t,x_t)}{\partial c_t^2} + \frac{\sigma_{xt}^2}{2}\frac{\partial^2 h(c_t,x_t)}{\partial x_t^2} + \frac{\rho_{cx}\sigma_{ct}\sigma_{xt}}{2}\frac{\partial^2 h(c_t,x_t)}{\partial c_t\partial x_t}\Bigg)dt\\
&+\frac{\partial h(c_t,x_t)}{\partial x_t}\sigma_{xt}dZ_{xt} + \frac{\partial h(c_t,x_t)}{\partial c_t}\sigma_{ct}dZ_{ct}
\end{aligned}
\tag{3.54}
$$

Then everything is plugged into Equation (3.47) to give the final result:

$$
\begin{aligned}
\frac{d\Lambda}{\Lambda} =&\Bigg(\frac{\gamma(\gamma\psi-1)\rho_{cx}\sigma_{xt}\sigma_{ct}K'\left(x_t\right)}{\psi} + \frac{\gamma^2\sigma_{ct}^2}{2} - \gamma\mu_{ct}\\
&+\frac{(\gamma\psi-1)\Big(2\psi(\mu_{x0}-x_t)\log(\phi)K'\left(x_t\right)+\sigma_{xt}^2\left((\gamma\psi-1)K'\left(x_t\right)^2-\psi K''\left(x_t\right)\right)\Big)}{2\psi^2}\\
&+\frac{\rho\left(-(1-\gamma\psi)e^{-\frac{(\psi-1)K[x_t]}{\psi}}-\gamma\psi+\psi\right)}{1-\psi}\Bigg)dt\\
&-\frac{(\gamma\psi-1)\sigma_{xt}K'(x_t)}{\psi}dZ_{xt} - \gamma\sigma_{ct}dZ_{ct}
\end{aligned}
\tag{3.55}
$$

3.C Convergence – Time-varying consumption drift

In the main paper I show convergence for the case, in which consumption volatility is time-varying. Here, I show the case when the consumption drift is time-varying. The convergence properties are similar in this case.

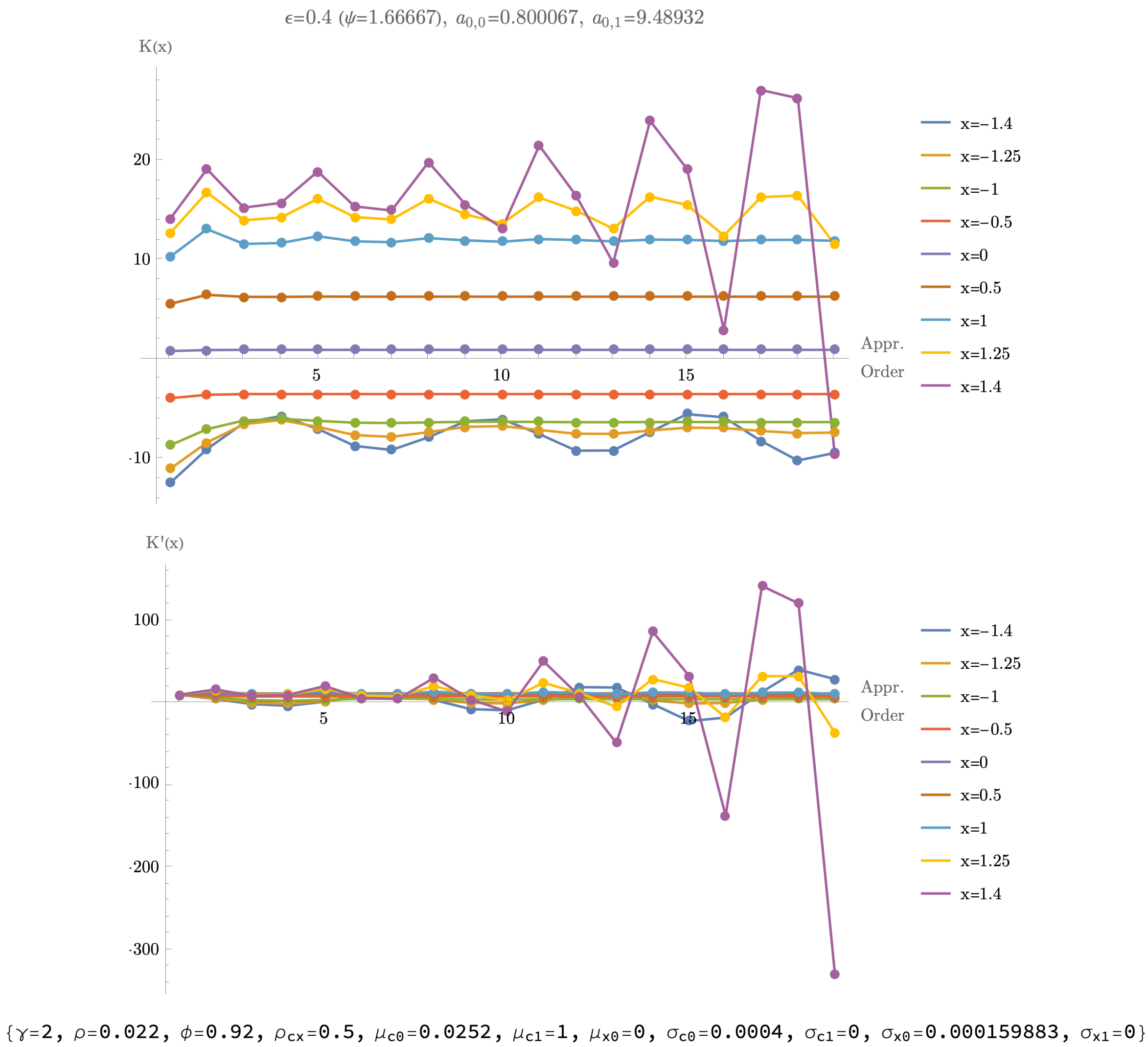
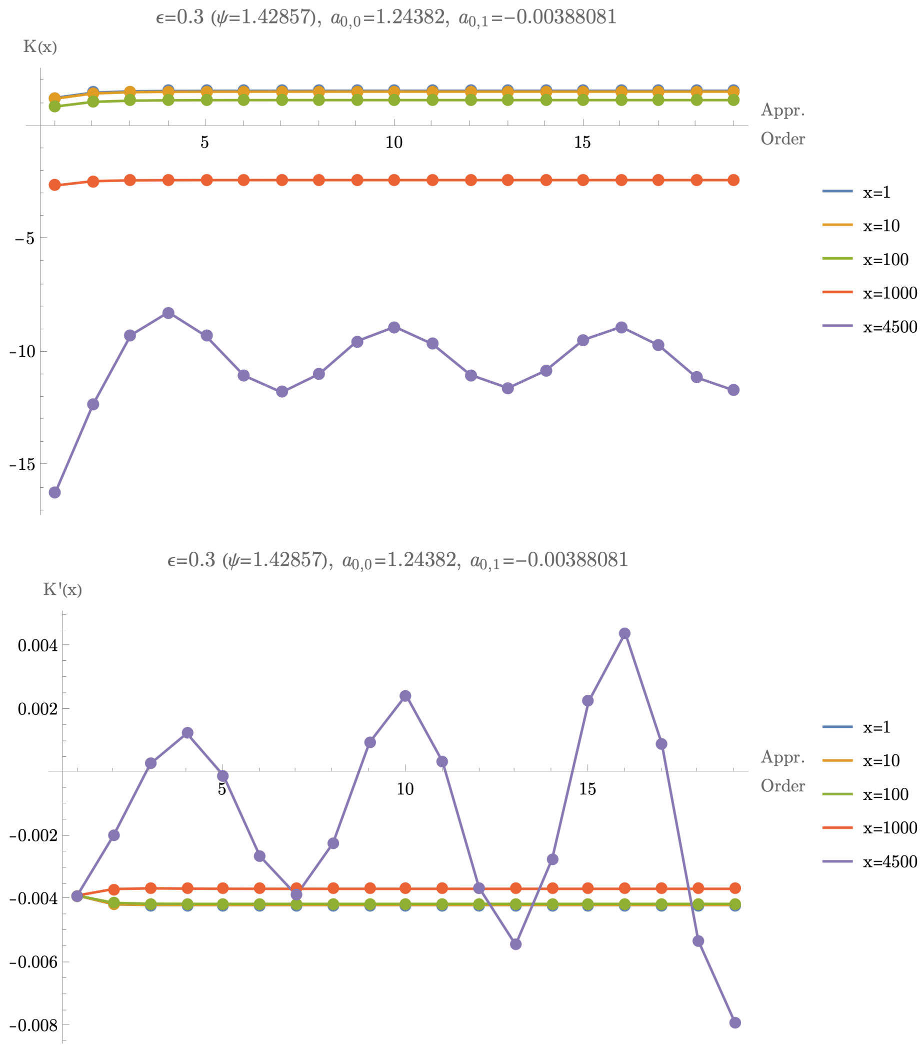


Figure 3.8: **Variation: Time-varying consumption drift.**
This shows the convergence of the problem for different values of x.

In this case, the series converges for all values of ϵ between 0 and 1. this means that the approximation works quite effectively for all values of $\psi > 1$.



$\{\gamma=2,\ \rho=0.02,\ \mu_{c0}=0.0252,\ \sigma_{c0}=0,\ \sigma_{c1}=0.0004,\ \phi=0.92,\ \sigma_{ct}=\sqrt{\sigma_{c0}+\sigma_{c1}\,x_t}\ ,\ \sigma_{x1}=0.16,\ \mu_{x0}=1,\ \rho cx=\rho\rho_{cx}\ \&,\ \rho_{cx}=-0.5\}$

Figure 3.9: **Variation: Time-varying consumption drift.**
This shows the convergence of the problem for different values of ϵ. The plots correspond to the series of K and its two derivatives.

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Education

2016 – 2024 (expected)	PhD in Economics, Goethe University Frankfurt
2013 - 2014	Master of Economics and Finance, Barcelona School of Economics
2007 - 2012	Bachelor of Laws, University of Athens

Research Interests

I am interested in macroeconomics and finance, with a special focus on monetary economics, asset pricing and solution methods.

Research Projects

Working Papers

Real Term Premia in Consumption-Based Models (job market paper)

A Perturbation Solution Method for Models with Recursive Utility

Measuring On-the-job Learning Rates in Multidimensional Skills, *with Mariia Bondar*

Work in Progress

Jointly Explaining Stock Market Non-Participation and the Equity Premium

Monetary Policy Evaluation in Real Models

Phillips Curve Relationship at the Regional Level during the War in Ukraine and the Covid-19 pandemic, *with Henning Weber*

Research Experience

2018 - now	Research Assistant, Leibniz Institute - SAFE, Data Center Among other tasks, I contributed to the management of financial databases, I organized seminars taught by data providers such as Bloomberg and Refinitiv, I provided relevant support to researchers, and I assisted with the grading of a Textual Analysis course.
2020 – 2022	Research Assistant, Deutsche Bundesbank, Research Center Among other tasks I assisted in the paper: **The case for a positive Euro Area inflation target: Evidence from France, Germany and Italy** by *K. Adam, E. Gautier, S. Santoro and H. Weber*, Journal of Monetary Economics
2022 - 2024	Visitor Researcher, Deutsche Bundesbank, Research Center Working with the official German micro-price data.

Teaching

05/2022	Webinar: Pandas in Python for Economics and Finance, SAFE
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10/2020	Webinar: Web-Scraping, SAFE
11/2019	Seminar: Python APIs for WRDS, Bloomberg and Eikon (Refinitiv), SAFE
09/2019	PhD Pre-Semester Course: Real Analysis, GSEFM

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2023	Bonn-Frankfurt-Mannheim PhD Conference
2022	15th RGS Doctoral Conference in Economics (presented by co-author)
2021	16th BiGSEM Doctoral Workshop on Economics and Management (presented by co-author)
2021	Frankfurt-Mannheim Macro Workshop

Other Experience

09/2014 – 08/2016	Lawyer, Independent Practice
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