Essays on Consumption-Based Asset Pricing

Dissertation at the Graduate School of Economics, Finance, and Management in order to obtain the academic degree of Ph.D. from the Faculty of Economics and Business Administration at Johann Wolfgang Goethe-University Frankfurt am Main

Submitted by
Nikolaos-Errikos Melissinos
from Korinthos, Greece

Frankfurt am Main, 2024

First Evaluator: Prof. Volker Wieland

Second Evaluator:

Third Evaluator:

ΟΔΕΧΡΗΜΑΤΙΣΤΗ
ΣΒΙΑΙΟΣΤΙΣΕΣΤ
ΙΝΚΑΙΟΠΛΟΥΤΟΣ
ΔΗΛΟΝΟΤΙΟΥΤΟΖ
ΗΤΟΥΜΕΝΟΝΑΓΑΘ
ΟΝΧΡΗΣΙΜΟΝΓΑΡ
ΚΑΙΑΛΛΟΥΧΑΡΙΝ

Aristotle, Nicomachean Ethics

Acknowledgments

I am especially thankful to my supervisors, Prof. Volker Wieland and Prof. Christian Schlag. I am highly grateful for their support, as they have been instrumental during the time of the job market and in the preparation of my Ph.D. Additionally, I am thankful for the help provided to me by Prof. Alexander Hillert, Prof. Florian Heider, and Prof. Loriana Pelizzon. I am also thankful to Beate Rohé-Morgan for her assistance during the time of the job market.

For helpful comments, I thank Mariia Bondar, Julian Detemple, Fabio Girardi, Dennis Gram, Refet Gürkaynak, Michael Haliassos, Florian Heider, Alexander Hillert, Ruggero Jappeli, Martin Kliem, Holger Kraft, Nora Lamersdorf, Martin Lettau, Christoph Meinedring, Alexander Meyer-Gohde, Emanuel Mönch, Dmitry Mukhin, Christian Mücke, Loriana Pelizzon, Christian Schlag, Paul Schneider, Philip Schwedler, Nicolas Syrichas, Farah Tohme, Henning Weber, Volker Wieland, the participants of the SAFE Brown Bag Seminar and the participants of the first Bonn-Frankfurt-Mannheim Ph.D. Conference.

For providing a great working environment during my Ph.D., I am thankful to my colleagues, Dennis, Stephanie, Luciano, Philip, and all the people in SAFE, who make it a great place to work.

For their special academic support during my Ph.D., I am thankful to Mariia, Julian, Nicolas, Odysseas, Nora, Elias, and Dimitris.

For support beyond the confines of the academic campus, I am grateful for all the precious people in my life who have been next to me on this journey,
my parents,
my sister, Irini,
my brothers, Miltos and Takis,
my friends, Armin, Dimitris, Elias, Frieder, Georgy, Jelena, Julian, Kanellos, Mariia, Nicolas, Nikos, Nora, Odysseas, Spyros, Steffen,

and above all my fiancé, İdil.

Contents

List of Abbreviations

Arb	Arbitrageurs
CD	Consumption Drift
CRRA	Constant Relative Risk Aversion
CS	Cieslak and Schrimpf
CSV	Constant State Variable Volatility
CV	Consumption Volatility
DP	Decreasing Short Rate, Positive Correlation of State Variable and Consumption
DSGE	Dynamic Stochastic General Equilibrium
EH	Expectations Hypothesis
HCD	High Consumption Drift
HCor	High Correlation Between Consumption and State Variable
HCV	High Consumption Volatility
HImp	High Impatience in the Utility Function
HIES	High Intertemporal Elasticity of Substitution
IES	Intertemporal Elasticity of Substitution
JK	Jarocinski and Karadi
LIES	Low Intertemporal Elasticity of Substitution
LImp	Low Impatience in the Utility Function

HP	High State Variable Persistence
HRA	High Risk Aversion
LCor	Low Correlation Between Consumption and State Variable
LImp	Low Impatience in the Utility Function
LP	Low State Variable Persistence
NCor	No Correlation Between Consumption and State Variable
PDE	Partial Differential Equation
RU	Recursive Utility
SDF	Stochastic Discount Factor
SDE	Stochastic Differential Equation
TIPS	Treasury Inflation-Protected Securities
TSU	Time-Separable Utility
TV	Time-Varying

Essay

My contributions in this dissertation are connected to consumption-based asset pricing. The first chapter takes a deep dive into the predictions by standard consumption-based models for the term structure of interest rates. Such models have struggled to explain the main features of risk premia in real bonds (Backus, Gregory and Zin 1989).[1] While I verify the challenges that most models face, I also verify that models with time-varying effective risk aversion, like the model in Campbell and Cochrane (1999) and Wachter (2006) can explain these main features. In addition, I am the first to show that it is possible to explain these main features with a model that does not rely on time-varying risk aversion but only relies on a high degree of time-varying consumption volatility.

Yet, I do not claim that a model with high consumption volatility is necessarily better compared to a model with time-varying risk aversion. Instead, I show that these are the only two available alternatives, that explain the main features of real term premia within the consumption-based setup that I study. I claim this by studying a long list of model variations and exhausting the reasonable parameter space. Furthermore, I do not claim that consumption volatility is in general higher compared to what is usually assumed. Indeed, consumption volatility of most households is likely lower than what is required in the model to generate realistic levels of term premia. This suggests that interest rates may be primarily driven by specific kinds of investors whose consumption volatility, is considerably higher compared to the volatility of measured aggregate consumption. Alternatively, interest rates may be driven by other types of players who are not real consumers, such as banks or other financial institutions, that act as intermediaries (He and Krishnamurthy 2013). In this case, these agents may be acting in a way similar to a consumer with high time-varying consumption volatility.

[1] In particular, I study term premia which are closely connected to risk premia.

If, though, interest rates are driven by specific agents with high consumption volatility, this would provide a partial explanation. A full explanation would also answer why ordinary households do not participate in the bond market. This partial explanation is not surprising, given that ordinary households do have low participation in financial markets and many reasons have been suggested for this fact.[2] Still introducing a single model that can explain the behaviour of all households and how they interact would be valuable.

Directly related to the first chapter, the second chapter introduces a package within the Julia programming language that allows the computation of asset prices using a stochastic discount factor (SDF) as an input. Julia is a language that is fit for high-performance scientific programming and has a very advanced suite of tools to solve stochastic differential equations. The package leverages this functionality to compute prices of zero-coupon bonds and price-dividend ratios for dividend-paying securities. The package can handle general SDF processes, and it is ideal for (but not limited to) consumption-based SDFs with exogenous consumption as I employ in this dissertation.

In the third chapter, I provide a methodological contribution, by introducing a novel perturbation method to solve asset pricing models, in which utility is recursive (Duffie and Epstein 1992). While common perturbation methods in the broad economic literature perform the perturbation around the deterministic steady state, by perturbing the variance parameter around the value of zero, my method uses an alternative approach.[3] In particular, I perform the perturbation with respect to a parameter of the utility function, and this enables the easy and accurate solution of the problem, while also opening an avenue for further research by using a similar approach in other problems, where there is an analytic solution for a specific parameter value.

Leaving the detailed description of the chapters aside, in the remainder of this essay I explain my focus on consumption-based asset pricing. Especially, because standard consumption-based models give rise to a long list of puzzles,[4] and due to the difficulties in measuring consumption, one could argue that consumption-based models should be abandoned for the purpose of asset pricing altogether. This is the main concern that I address in what follows.

[2]See Van Rooij, Lusardi and Alessie (2011) and Lusardi and Mitchell (2023) for a review of the relevant literature.

[3]See Schmitt-Grohé and Uribe (2004) for a prominent example. Judd (1996) provides a general description of perturbation methods for economics and finance. Bender, Orszag and Orszag (1999) describes perturbation methods in applied mathematics more generally.

[4]In explaining such puzzles Gabaix (2012) provided a good review.

Interacting people compose all kinds of markets. This suggests that we should use models with the same types of agents. However, some human personalities may be more prevalent in financial markets compared to regular markets. In addition, markets are also enabled by institutions, and the design of institutions can incentivise specific types of actions. Moreover, markets are also affected by other factors, such as their size or the attributes of the traded products. These factors can also affect the types of agents that would offer effective descriptions of reality. Nevertheless, a model with the same types of agents that effectively describes a variety of markets would be a powerful tool. If such a model cannot be found, then we should use different types of agents in separate models. For example, investors and consumers may need to be modelled differently.

Science, though, is not a black-and-white process. Different problems may require different solutions. And some models may have a lot of descriptive power for some circumstances, but very little for others. Indeed some models may only apply to a very specific setting. For example, the best forecast of volatility in financial markets over the next year may be effectively generated by a statistical model that employs very little economic theory. However, the question of how volatility in financial markets will be affected by a large shift in fiscal policy, would most likely not be effectively answered by the same model. Instead, another model may be able to forecast volatility less effectively, but at the same time provide a connection to fiscal policy that the first model could not. Therefore, developing the second model would be a worthwhile endeavour.

A similar argument can be made for consumption-based asset pricing. For instance, an optimal explanation of risk premia in the bond market based on the current state of the literature would most likely not be provided by a consumption-based model.[5] However, what if we are interested in how risk premia in the bond market are affected by monetary policy, or what if we are interested in the welfare implications of different monetary policies that give rise to specific movements in risk premia? In such cases, consumption-based models naturally start gaining appeal.

Relying on such a model for these questions implies in some sense a sacrifice of explanatory power for the simple reason that the more powerful model does not provide any information about the questions at hand. Nevertheless, this

[5]For example, Abrahams et al. (2016) did not use a consumption-based model. Even, this model, though, is used for the specific use case of decomposing risk premia, inflation expectations, and expected short-term rates. For other questions about risk premia in the bond market, yet another model may be more appropriate.

sacrifice should still inspire some humility, stemming from the awareness that the model is not perfectly capturing the data. In the context of monetary policy, for example, consumption-based models are used, and the effect of monetary policy on consumption is mediated through interest rates.[6] However, at the same time we do not have a well-accepted asset pricing model establishing the connection between consumption and interest rates in general.[7]

Analogously, this implies that we should be humble about our ability to describe the effects of monetary policy on consumption and other similar variables. However, the questions are important and we should still try to answer them as best we can. Hence, it is valuable to develop an accurate consumption-based asset pricing model, to answer questions that involve both asset prices and consumption of different individuals, even if other more targeted models describe interest rates themselves better. While questions surrounding monetary policy are an important case (which also motivated me to study consumption-based asset pricing), many other questions involve both asset prices and consumption. And such questions can likely only be effectively tackled in models that honestly address both components.

Relevant literature on both asset pricing and macroeconomics highlights the need for this approach. Firstly, a large part of the appeal of heterogeneous agent New Keynesian models is that they can separately model different types of households, which are affected differently by monetary policy (Violante 2021). These models contain descriptions of both asset prices and the consumption of households. Next, the entire field of household finance relies on the ability to understand how consumption and investment are chosen given the available options. Even the literature on intermediary asset pricing, which is understood by some as a competitor to consumption-based asset pricing, attempts to address similar needs. In particular, it suggests a connection between the consumption of ordinary households to the decisions that move financial markets. This is provided by financial intermediaries, whose actions are to some extent misaligned with the interests of ordinary households. While intermediation frictions are the main focus of this literature, it nevertheless also offers a connection between consumption and asset prices, and it depends on the exact setup how much the consumption of different groups in the economy is affected by financial markets in these models.

Even though I have focused on consumption in this essay, a separate focus is

[6] This is the case in the standard new Keynesian model, see Woodford (2003) or Galí (2015) for a thorough treatment.

[7] A version of this concern can also be found in Canzoneri, Cumby and Diba (2007).

also possible, such as using wealth as the central variable. This approach may even be conceptually more appropriate for the modelling of the behaviour of financial institutions, such as banks or investment funds, given that these entities do not literally consume. However, these models are similar to consumption-based models and will likely lead to similar results. For instance, in Vayanos and Vila (2021) arbitrageurs try to optimise the mean and variance of their portfolio, but in my first chapter, I show that similar results can be obtained within a consumption-based framework.

Just like intermediary asset pricing is considered a competitor to consumption-based asset pricing, consumption-based asset pricing is also associated with a representative consumer or with the centrality of aggregate consumption. This should not be the case. Indeed consumption-based models could only describe a certain subset of investors. As known by equilibrium consumption-based asset pricing theory, there is a consumption-based stochastic discount factor for each optimising consumer participating in the financial market.[8] Given our difficulty in connecting asset prices to aggregate consumption, this poses the question of whether we can attempt to identify separate consumption processes, which individually give rise to consumption-based SDFs that are all consistent with asset prices. Then for those consumption processes that are not consistent with asset prices, we would have to explain why the corresponding consumers do not participate in the financial market, or why they do not invest optimally.

Economic and financial literature is arguably converging to such a view. Household finance has been studying the frictions and barriers that lead to limited participation in financial markets. Intermediary asset pricing complements this view by showing how financial institutions are moving markets, while also being connected to the consumption of ordinary households. Preferred habitat models argue that preferred habitat investors may have idiosyncratic preferences in holding some securities without strictly optimising, while "arbitrageurs" are optimising and reacting to the demand of the preferred habitat investors. Heterogeneous agent models also mirror these developments by describing the behaviours of different types of agents, who have different positions vis a vis financial markets.[9] Based on all these strands of literature, one can see a mosaic forming in which ideally all these components fit together, and allow an explanation of the behaviour of

[8]See for example Duffie (2010). This is the case even if the market is incomplete, as long as some technical requirements are fulfilled to ensure the existence of equilibrium.

[9]Heterogenous agent models themselves have been used in various fields of economics and finance, including monetary economics, as was already mentioned, or asset pricing for example in Gârleanu and Panageas (2023).

all basic types of economic agents. Such a model could then be used to answer broad questions, even if more specialised models are more powerful in answering specific questions.

When considering all these pieces that come together, to generate asset prices from different consumption processes and investment decisions, the reverse question naturally arises. Assuming a standard utility function, and given the observed processes followed by asset prices, what should the consumption process of an optimising agent look like? Even if aggregate consumption is difficult to connect to asst prices, just being rational and optimising should arguably imply a realistic consumption process, and identifying this process would be significant. Firstly, some subgroups in the economy could be following this process. Secondly, even if this consumption process is actually not followed by anyone, it would still act as a valuable benchmark with which all consumption processes could be compared. Indeed, the corresponding consumption-based SDF could be used as a benchmark even for financial institutions.

Endeavoring towards a better understanding, as outlined above, the three chapters of my dissertation contribute in a variety of ways. The first chapter provides a more theoretical contribution to consumption-based asset pricing, by showing that there is a consumption process that can explain the main features of observed real term premia. The second chapter provides a technical contribution, by providing software facilitating the solution of various kinds of asset pricing models, including consumption-based models. And the third chapter provides a methodological contribution, by introducing a novel solution method for consumption-based models with recursive utility, while the general approach can likely also be extended to other models.

Lastly, while the chapters in this dissertation are restricted compared to the broad vision outlined above, my hope is for them to become a building block toward the edifice of a fundamental and common understanding of people's and institutions' actions in economic markets.

Bibliography

Abrahams, Michael, Tobias Adrian, Richard K Crump, Emanuel Moench, and Rui Yu. 2016. "Decomposing Real and Nominal Yield Curves." *Journal of Monetary Economics*, 84: 182–200.

Backus, David K, Allan W Gregory, and Stanley E Zin. 1989. "Risk Premiums in the Term Structure: Evidence from Artificial Economies." *Journal of Monetary Economics*, 24(3): 371–399.

Bender, Carl M, Steven Orszag, and Steven A Orszag. 1999. *Advanced Mathematical Methods for Scientists and Engineers I: Asymptotic Methods and Perturbation Theory.* Vol. 1, Springer Science & Business Media.

Campbell, John Y, and John H Cochrane. 1999. "By Force of Habit: A Consumption-Based Explanation of Aggregate Stock Market Behavior." *Journal of Political Economy*, 107(2): 205–251.

Canzoneri, Matthew B, Robert E Cumby, and Behzad T Diba. 2007. "Euler Equations and Money Market Interest Rates: A Challenge for Monetary Policy Models." *Journal of Monetary Economics*, 54(7): 1863–1881.

Duffie, Darrell. 2010. *Dynamic Asset Pricing Theory.* Princeton University Press.

Duffie, Darrell, and Larry G Epstein. 1992. "Stochastic Differential Utility." *Econometrica: Journal of the Econometric Society*, 353–394.

Gabaix, Xavier. 2012. "Variable Rare Disasters: An Exactly Solved Framework for Ten Puzzles in Macro-Finance." *The Quarterly Journal of Economics*, 127(2): 645–700.

Galí, Jordi. 2015. *Monetary Policy, Inflation, and the Business Cycle: An Introduction to the New Keynesian Framework and its Applications.* Princeton University Press.

Gârleanu, Nicolae, and Stavros Panageas. 2023. "Heterogeneity and Asset Prices: An Intergenerational Approach." *Journal of Political Economy*, 131(4): 839–876.

He, Zhiguo, and Arvind Krishnamurthy. 2013. "Intermediary Asset Pricing." *American Economic Review*, 103(2): 732–770.

Judd, Kenneth L. 1996. "Approximation, Perturbation, and Projection Methods in Economic Analysis." *Handbook of Computational Economics*, 1: 509–585.

Lusardi, Annamaria, and Olivia S Mitchell. 2023. "The Importance of Financial Literacy: Opening a New Field." *Journal of Economic Perspectives*, 37(4): 137–154.

Schmitt-Grohé, Stephanie, and Martin Uribe. 2004. "Solving Dynamic General Equilibrium Models Using a Second-Order Approximation to the Policy Function." *Journal of Economic Dynamics and Control*, 28(4): 755–775.

Van Rooij, Maarten, Annamaria Lusardi, and Rob Alessie. 2011. "Financial Literacy and Stock Market Participation." *Journal of Financial Economics*, 101(2): 449–472.

Vayanos, Dimitri, and Jean-Luc Vila. 2021. "A Preferred-Habitat Model of the Term Structure of Interest Rates." *Econometrica*, 89(1): 77–112.

Violante, Giovanni L. 2021. "What have we learned from HANK models, thus far?" *Beyond the Pandemic: The Future of Monetary Policy*, 267.

Wachter, Jessica A. 2006. "A Consumption-Based Model of the Term Structure of Interest Rates." *Journal of Financial Economics*, 79(2): 365–399.

Woodford, Michael. 2003. "Interest and Prices: Foundations of a Theory of Monetary Policy."

Chapter 1

Real Term Premia in Consumption-Based Models

Abstract

Can consumption-based mechanisms generate positive and time-varying real term premia as we see in the data? I show that only models with time-varying risk aversion or models with high consumption risk can independently produce these patterns. The latter explanation has not been analysed before with respect to real term premia, and it relies on a small group of investors exposed to high consumption risk. Additionally, it can give rise to a "consumption-based arbitrageur" story of term premia. In relation to preferences, I consider models with both time-separable and recursive utility functions. Specifically for recursive utility, I introduce a novel perturbation solution method in terms of the intertemporal elasticity of substitution. This approach has not been used before in such models, it is easy to implement, and it allows a wide range of values for the parameter of intertemporal elasticity of substitution.

1.1 Introduction

Risk-free bonds hold a central position in financial theory, and in practice, government bonds hold a central position in financial markets. Yet we do not fully understand how risk premia are connected to the consumption of households. Understanding this connection would not only benefit consumption-based asset pricing but also other fields. For instance, it would facilitate households' investment decisions, and it would lead to a better understanding of the effects of monetary policy which are associated with changes in prices of real bonds. This paper

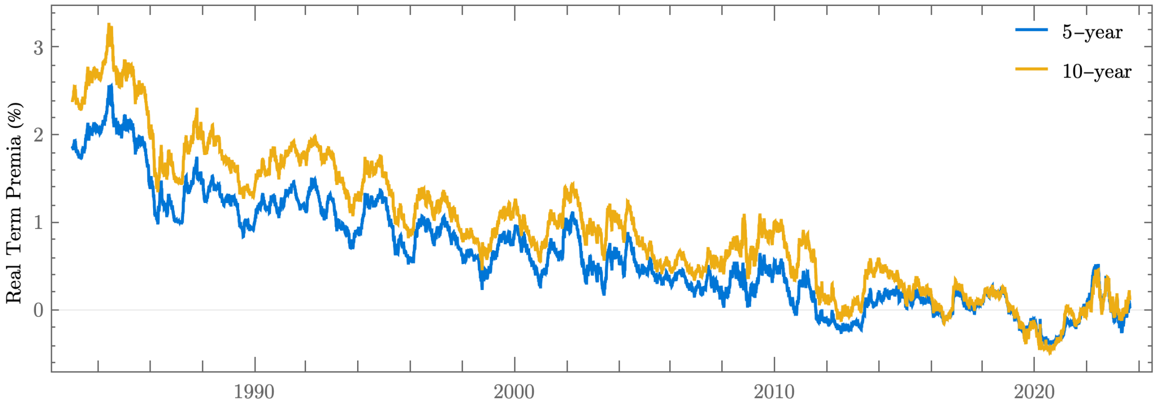


Figure 1.1: **Time series of real term premia for the US**
Data is taken from d 'Amico, Kim and Wei (2018), who decomposed nominal yields into risk-neutral real yields (expected short-term rates averaged over the corresponding period), real term premia, expected inflation, inflation premia and liquidity premia.
Data Source: https://www.federalreserve.gov/econres/notes/feds-notes/tips-from-tips-update-and-discussions-20190521.html

focuses on explaining real term premia, i.e. the risk premia of inflation-adjusted risk-free bonds over a specific holding period.[1] Thus, throughout this paper, terms like yields, returns, term premia etc. should be understood as referring to their real counterparts, unless otherwise specified. Term premia in the data are mostly positive and significantly time-varying (Abrahams et al. 2016; d 'Amico, Kim and

[1]Term premia reflect the expected difference in log return from holding long-term bonds compared to short-term bonds over the same period. On the contrary, risk premia usually refers to the same difference in expected returns taken over a single period (or instantaneously in continuous time). The exact definition of term premia along with the exact definition of all terms in this paper can be found in Appendix 1.A. Actual bonds may also have liquidity premia, which are deviations in the price of bonds due to their liquidity in the market. Liquidity premia are separate from term premia, or, in other words, my definition of term premia assumes that bonds are perfectly liquid.

Wei 2018; Pflueger and Viceira 2016). Estimates from d 'Amico, Kim and Wei (2018) are shown in Figure 1.1.[2] Overall, consumption-based models struggle to generate these main features. In the literature, this is referred to as the bond premium puzzle (Backus, Gregory and Zin 1989).[3] The source of the puzzle is that consumption-based mechanisms typically generate small, negative, and often constant term premia, namely the exact opposite of what we see in the data. This is due to bond prices typically being counter-cyclical in these models, while consumption risk is relatively small and varies little with the business cycle. In addition, contrary to nominal term premia, it is not possible to explain real term premia by relying on an inflation premium, which arises due to the risk associated with the inflation process.

As is already known, models with time-varying risk aversion can generate positive and time-varying term premia. For instance, Wachter (2006) showed this in a model with an external habit following Campbell and Cochrane (1999). In this paper, I perform a comprehensive investigation of term premia within separate consumption-based models, and my analysis shows that there is a second consumption-based model that can generate these features without employing a habit component or any time-varying risk aversion. To the best of my knowledge, this result is new. The mechanism relies on a) high time-varying consumption risk, and b) negative correlation between consumption and the short-term interest rate.[4] Firstly, the high and time-varying consumption risk generates term premia that are high in absolute value and time-varying. Given that we do not observe such high consumption risk in aggregate data, this is not the consumption process of the representative consumer. Instead, it could be the consumption process of a small group of marginal investors.[5] Additionally, the mechanism is flexible and

[2]In these estimates a long-term downward trend stands out (this can be seen even clearer in the longer time-series in Figure 1.E.2). In this paper, I use a steady-state-reverting state variable. Therefore, I do not attempt to explain this component.

[3]The bond premium puzzle can also refer to nominal term premia. Nominal term premia have the same definition as real term premia with the underlying bonds not being inflation-adjusted.

[4]By correlation between these two variables, I mean the correlation between the high-frequency changes of the variables. The same will apply throughout the paper when I talk about the correlation of variables.

[5]This means that the approach follows heterogeneous agent models, as a small group of investors have a different consumption process than the average in the economy. However, I do not examine a full heterogeneous agent model, as I restrict my analysis to marginal investors in the bond market. Furthermore, my approach is similar to but does not strictly follow the intermediary asset-pricing paradigm (He and Krishnamurthy 2013). In particular, despite the fact that a small group of investors is driving asset prices in the bond market, these investors do not act as intermediaries for the households, and the results do not stem from any intermediation constraints.

these investors can either be thought of as holding all bonds, or alternatively they can be thought of as holding any portion of total bonds. Secondly, the negative correlation between the short-term rate and consumption implies a positive correlation between bond prices and consumption. Therefore, these marginal investors regard long-term bonds as risky, and they demand a positive term premium for them. In the main variation that I introduce, the intuition for the negative correlation between the short-term rate and consumption can be understood as follows: As the short-term rate falls (rises), marginal investors, who are also bondholders, see an increase (decrease) in their wealth. In addition, they raise (lower) their net borrowing. Therefore, both consumption and consumption risk increase (decrease).[6] In the paper I do not argue that this mechanism is superior in explaining term premia compared to time-varying risk aversion. On the contrary, the focus of the paper is to introduce the new mechanism, and show that, within the constraints of my analysis, it is the only mechanism that generates the features of term premia without using time-varying risk aversion. I can make this statement by analysing a long series of variations that exhaust the reasonable parameter space of the setup. My analysis employs standard consumption-based asset pricing, and I assume a single state variable following a stationary autoregressive process.[7]

This approach can also be adapted to fit a "consumption-based" arbitrageur story of the term structure of interest rates related to Vayanos and Vila (2021). In this paper, the term structure of interest rates is explained with a preferred habitat model, in which so-called arbitrageurs integrate the yield curve by taking advantage of differences in expected return between different maturities of bonds. While Vayanos and Vila (2021) associated their arbitrageurs with banks and/or hedge funds, the authors do not take a position on whether natural persons could correspond to arbitrageurs.[8] In this paper I show that arbitrageurs can be modelled as consumers, who drive positive and time-varying term premia, as long as consumption risk is high. In this adaptation, the state variable of the model corresponds to the magnitude of the risky investment opportunity, which could be

[6]The intuition is similar to the mechanism in Schneider (2022).

[7]Thus, my analysis does not include models that are driven by higher order beliefs as in Angeletos, Collard and Dellas (2018). In addition, my analysis only includes steady-state-reverting autoregressive processes. However, there is literature suggesting that macro processes are more elaborate. For example, Bauer and Rudebusch (2020) decomposed the nominal yield curve by taking into account long-run macroeconomic trends, while there is a long literature investigating the time-series properties of interest rates (a survey is provided by Neely, Rapach et al. 2008). It would be interesting for further research to expand my analysis, to include more elaborate processes for the state variable.

[8]This could also be the case if arbitrageurs are investing on behalf of natural persons without significant intermediation distortions.

generated by long-term bonds having a higher expected return than short-term bonds. As the investment opportunity increases (decreases), arbitrageurs borrow more (less) and invest more (less), risk increases (decreases), expected returns increase (decrease), and thus consumption increases (decreases). The source of this investment opportunity is external to the arbitrageurs. For instance, it can be driven by demand pressure from the central bank or other preferred habitat investors as in Vayanos and Vila (2021). The arbitrageur approach shares the main characteristics of the baseline high consumption risk mechanism. Therefore, it also generates positive and substantially time-varying term premia.

Furthermore, my paper also contributes in the following ways. Firstly, I provide explicit values of term premia as a function of the state of the economy for a large range of model variations. This is useful, because consumption-based models in the literature often focus on nominal term premia, and even when they focus on real term premia, explicit state-dependent term premia are rarely displayed. Secondly, apart from time-separable utility (TSU), my analysis also includes models with recursive utility (RU), and I contribute a novel perturbation method to easily and robustly solve such models. My perturbation method builds on the approach of Tsai and Wachter (2018). While they used an approximation to the value function that is constant in terms of the intertemporal elasticity of substitution (IES), and analytically correct only for IES equal to 1, I consider the full perturbation series in terms of the IES. This provides a global approximation in terms of the state variable of the economy that allows the easy solution of the model for most values of the IES that are economically interesting. It is also the first perturbation method in terms of the IES within RU models. This method is also explained in further detail in Melissinos (2023).

The rest of the paper is organised as follows: In section 1.2, I provide more information regarding the literature on the bond premium puzzle. In section 1.3, I discuss interest rates in the data. In section 1.4, I present the setup that will allow me to price bonds in the context of TSU and RU. This includes the outline of the novel perturbation method. In section 1.5, I show and comment on the results for term premia. Finally, section 1.6 concludes.

1.2 Literature on the Bond Premium Puzzle

While I analyse real term premia, the bond premium puzzle originally referred to nominal term premia.[9] One of the first papers to address this was Backus, Gregory and Zin (1989). Utilising a consumption-based asset-pricing model of an endowment economy, they discovered the model's inability to yield significant positive term premia. Subsequent studies by Donaldson, Johnsen and Mehra (1990) and Den Haan (1995) further indicated that standard real business cycle models also could not resolve the puzzle. Rudebusch and Swanson (2008) incorporated an external habit into DSGE models but found that the bond premium puzzle remains. Specifically, including a habit with non-flexible working hours can generate positive term premia, but at the cost of inducing volatile wages, prices and short-term interest rates. Duffee (2013) showed that basic properties of nominal yields cannot be explained macroeconomically, at least according to standard asset-pricing models. Also in a more generic contribution, Duffee (2002) shed light on the challenges of fitting both interest rate and term premium dynamics within affine models.

Next, a series of papers provided explanations that focused on nominal term premia, and not on real term premia. Notably, Piazzesi and Schneider (2006) showed that parameter uncertainty in a model where inflation brings bad news about future consumption growth can produce positive nominal term premia.[10] Gabaix (2012) and Tsai (2015), following Rietz (1988) and Barro (2006), showed that positive nominal term premia can be explained, if inflation is on average high during consumption disasters. Bansal and Shaliastovich (2013), following Bansal and Yaron (2004), demonstrated that the risk premium of a nominal bond can be positive in a model with long-run risk, as long as inflation is correlated with consumption trend. Rudebusch and Swanson (2012) used a similar model within a DSGE framework, which has real and nominal long-term risks, and they show that positive nominal term premia are generated; nevertheless real term premia are again negative in this model. Gomez-Cram and Yaron (2021) also used a model following Bansal and Yaron (2004), but they focused on explaining nominal term premia, using an inflation channel, while claiming that the apparent under-performance of their model concerning real term premia should be expected due to liquidity premia in the TIPS market.

[9] Rudebusch and Swanson (2008) also offered a good summary of this extensive literature.

[10] Collin-Dufresne, Johannes and Lochstoer (2016) introduced a model with Bayesian learning of parameters. However, this model does not emphasise bond term premia and it generates *negative* term premia.

Alternatively, some articles also consider real term premia. For instance, Katagiri (2022) explored a model with monetary policy, in which consumption changes can be negatively correlated with consumption trends, and risk aversion is very high. As a result, term premia can be positive, but the premia variability is not examined. Ellison and Tischbirek (2021) went beyond standard rational expectations models by using a beauty contest mechanism as introduced by Angeletos, Collard and Dellas (2018), in which agents anticipate the expectations of other agents; their model generates positive term premia.

Using a similar approach to the current paper, some articles tackle the problem by deviating from the representative agent model. Vayanos and Vila (2021) suggested that term premia are generated by arbitrageurs interacting with so-called preferred habitat investors, namely investors that tend to hold specific maturities of bonds. Kekre, Lenel and Mainardi (2022) built on Vayanos and Vila (2021), and showed that the characteristics of the arbitrageur portfolio can have important implications for the sign of term premia. Jappelli, Subrahmanyam and Pelizzon (2023) also built on Vayanos and Vila (2021) by integrating the repo market in their analysis. Schneider (2022) showed that positive term premia can arise in models with heterogeneous agents exhibiting different attitudes towards risk and different preferences to substituting consumption through time. Finally, returning to models with a representative agent, Wachter (2006) showed that term premia can be positive and time-varying, within a model with an external habit following Campbell and Cochrane (1999). Kliem and Meyer-Gohde (2022) used the same mechanism within a DSGE model, and they found positive term premia. Hsu, Li and Palomino (2021) also used this mechanism within a DSGE model, and they verified that a habit element is key in generating positive and time-varying term premia. Campbell, Pflueger and Viceira (2020) also used a habit model to explain the time-variability of term premia. More generally, a model with external habit can be classified as a model with time-varying effective risk aversion, and within this class of models, Lettau and Wachter (2011) showed that positive and time-varying term premia can be obtained, and Bekaert, Engstrom and Grenadier (2010) showed that time-varying term premia can be obtained. These papers all use time-varying risk aversion, which is to my knowledge the only mechanism in the literature that achieves positive and time-varying term premia within a rational representative agent model.[11]

[11] Yet, a utility with a time-varying degree of risk aversion may not be considered the most standard rational utility function.

1.3 Real Rates in the Data

1.3.1 TIPS as real rates

The first challenge regarding real rates is that they are not directly observable from standard bonds. The real interest rate is the yield of a nominal bond whose payoff is adjusted for inflation. So deducing real interest rates from nominal bonds requires at least the calculation of expected inflation, which is not trivial. The closest thing that we have in the data for real interest rates is inflation-adjusted government bonds. Such data are available for the UK and the US. In the UK, inflation-adjusted government bonds (inflation-adjusted GILTs) have been available since the 1980s. In the US, the corresponding securities are called TIPS (Treasury Inflation-Protected Securities) and corresponding price data are available for roughly twenty years (Gürkaynak, Sack and Wright 2010).[12] A severe limitation of TIPS is that they are not as liquid as normal US treasuries. For this reason, I focus on term premia measures produced by d 'Amico, Kim and Wei (2018) who computed risk-neutral yields and term premia, after taking account of the liquidity premia of TIPS over normal US treasuries.[13] As can be seen in Figure 1.2, in some periods liquidity premia of TIPS are considerable. Nevertheless, as shown in Figure 1.1, term premia are still significantly time-varying.

1.3.2 Real rates as a component of nominal rates

Figure 1.2 shows real yields at the top and nominal yields at the bottom. The plot reveals several key conclusions. Firstly, both nominal and real interest rates are time-varying. In addition, different maturities have different yields and the term structure seems to be upward-sloping in both cases. In other words, longer maturities are associated with higher yields. The slope of the term structure is also not constant, as the spread between yields of different maturities varies. Secondly, it is clear from Figure 1.2 that nominal rates are highly correlated with real rates.

[12] Gürkaynak, Sack and Wright (2010) has provided data starting from 1999. However, the full set of maturities is provided starting in 2002.

[13] Apart from liquidity issues related to TIPS, there is also a small concern (mostly with recently issued TIPS) that negative inflation is not correctly accounted for. This is because TIPS are guaranteed to pay investors at least the original principal value of the bond, even if the rate of inflation is negative. This makes inflation adjustment somewhat skewed. However, the effect will probably be small for securities that were issued several years prior, given that likely some inflation has already occurred and the probability that negative inflation will overcome it is small. Lastly, the accuracy of inflation adjustment can be debated, as the consumer price index might not capture the specific inflation concerns of investors.

Considering the Fisher equation:

$$y_t^{nom,m} \approx y_t^{real,m} + E[\pi_{t,t+m}] \tag{1.1}$$

where m denotes the maturity of the underlying bond. The nominal rate ($y_t^{nom,m}$) can be thought of as a composite rate that includes two separate components, the real rate $y_t^{real,m}$, and expected inflation $E[\pi_{t,t+m}]$.[14] Figure 1.2 also shows that real interest rates are a significant and non-trivial component of nominal interest rates. Namely, real rates are moving substantially and mostly in parallel to nominal rates. In Appendix 1.B, I statistically verify that the information contained in the movements of real rates explains a large proportion of the variation of nominal rates.

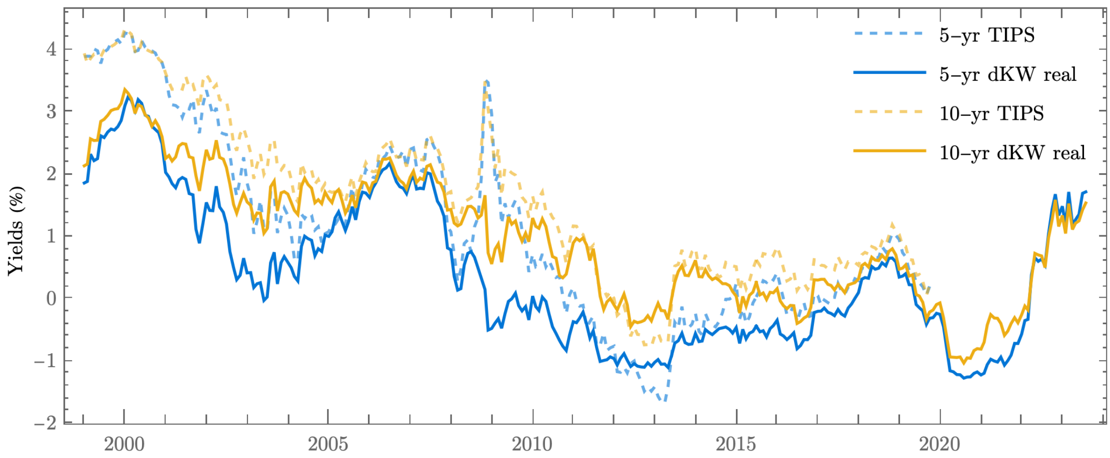
1.3.3 Empirical evidence regarding term premia

Empirical research has predominantly focused on nominal bonds in relation to concepts such as term premia, *return predictability*, the *expectations hypothesis* (EH), and *excess volatility*. Specifically, predictability in nominal rates has been found by Fama and Bliss (1987) and Singleton (1980), who showed that yield spreads can partially predict excess returns of bonds over extended periods. This implies both the existence of term premia and that they are time-varying. In addition, this is equivalent to a violation of the EH, which has been verified by Cochrane and Piazzesi (2005) among others. The existence of excess volatility (Shiller 1979) also indicated time-varying term premia, because the excess volatility is evidence that changing economic conditions affect the value of long-term bonds beyond what can be explained by movements in expected short rates. Even though the literature has focused less on real term premia, relatively recent studies have concluded that real term premia are also positive and time-varying, after accounting for liquidity premia. In particular, Abrahams et al. (2016) estimated the five-to-ten-year real forward term premium and found that it has ranged roughly from 0% to 4% between 2000 and 2014 (Figure 1.14 in Appendix 1.E).[15] d 'Amico, Kim and Wei (2018) found that the five-to-ten-year forward term premium has ranged roughly from -0.5% to 4% between 1980 and 2022 (Figure 1.1).[16] Pflueger and Viceira (2016) have demonstrated the existence of predictability of real excess

[14]The Fisher equation can be made into an equality by adding an inflation risk premium.

[15]Shown in Figure 5 of Abrahams et al. (2016).

[16]For the period between 2000 and 2014, the results of d 'Amico, Kim and Wei (2018) implied a somewhat smaller variability of term premia compared to the results of Abrahams et al. (2016).



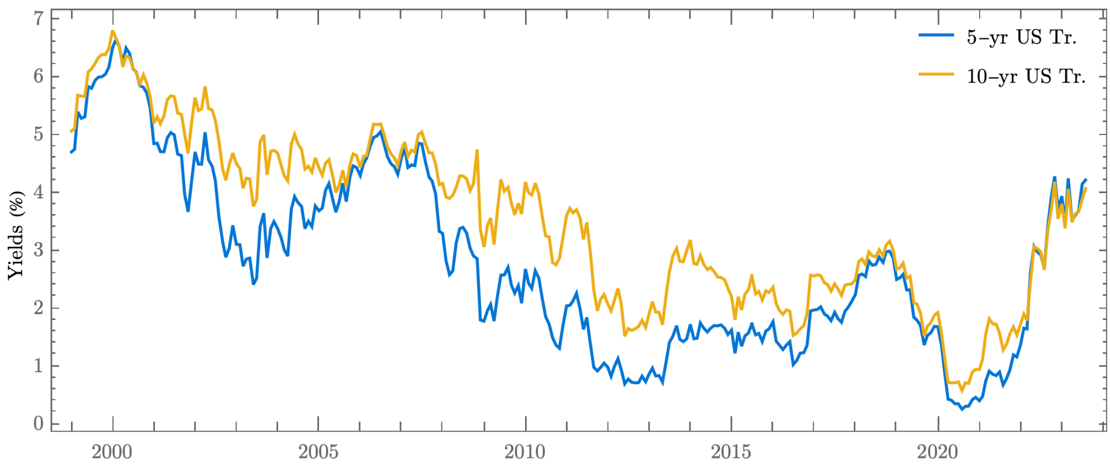


Figure 1.2: **Yields of US Treasuries**
TIPS data is taken from Gürkaynak, Sack and Wright (2010), normal US treasury yields data is taken from Gürkaynak, Sack and Wright (2007), and dKW real yields are taken from d 'Amico, Kim and Wei (2018). Real yields are the sum of risk-neutral yields and the real term premia. The difference between the dashed and solid lines corresponds to the liquidity premia of the TIPS over the normal treasuries. Thus, this assumes that normal treasuries are perfectly liquid.

returns, also implying the existence of time-varying real term premia. The conclusion that real term premia are substantial and time-varying is significant because it implies that models that exclusively use inflation processes are not sufficient to explain the dynamics of either nominal or real interest rates.

1.4 The Consumption-Based Framework

I adopt a consumption-based framework in continuous time, which can accommodate a range of model variations. The framework is built upon three main components from which everything else is derived: 1) an exogenous consumption process, 2) a utility specification, and 3) a process for the state variable. The state variable determines the state of the economy, and it is either connected with some component of the consumption process or with some components of the utility function. Specifically, in the variations in this paper, the state variable is either connected to consumption trend (otherwise referred to as CD) or connected to consumption volatility (CV), or connected to the external habit of the utility function. These three options in combination with different calibrations and utility specifications give rise to a long list of variations and interpretations. To keep things simple, I only use one state variable for each model variation. Utility is either time-separable (TSU) or recursive (RU), in the latter case following Duffie and Epstein (1992).

1.4.1 Naming the variations

As mentioned already, I analyse several model variations in the main text of this paper, and many more in Appendix 1.F. While I explain the models variations in detail in Sections 1.4 and 1.5, for convenience, I also provide abbreviations for the model variations in Table 1.1

1.4.2 Consumption process

Although consumption is often considered a fundamental choice variable for economic agents, it is assumed to be exogenous in this paper.[17] This approach is consistent with consumption having been decided at some earlier stage that is not explicitly modeled, and it significantly simplifies the analysis. In the most general form, consumption (C_t) follows the stochastic process expressed below:[18]

$$\mathrm{d}\log(C_t) = \mathrm{d}c_t = \mu_{ct}\mathrm{d}t + \sigma_{ct}\mathrm{d}W_{ct} \tag{1.2}$$

[17]This is a standard choice in this literature. See for example Campbell and Cochrane (1999) and Bansal and Yaron (2004).

[18]It should be noted that I use the same parameter symbols for all model variations, and they should be distinguished by context. For example, in TSU-CD μ_{ct} is time-varying and a function of the state variable, while in TSU-CV μ_{ct} is a constant. The same applies to the symbols: σ_{ct} and σ_{xt}.

Model Variation Description	Abbreviation
Time-varying CD with time-separable utility.	TSU-CD
Time-varying CV with time-separable utility.	TSU-CV
Time-varying habit with time-separable utility.	TSU-Habit
Time-varying CD with recursive utility.	RU-CD
Time-varying CV with recursive utility.	RU-CV
Time-varying CD and CV with recursive utility.	RU-Mixed
High time-varying CV with positive correlation $\rho_{cx} > 0$, and time-separable utility.	TSU-HCV
Arbitrageur case with short-term rate decreasing in the investment opportunity and positive correlation $\rho_{cx} > 0$, with time-separable utility.	Arb-DP
Arbitrageur case with short-term rate decreasing in the investment opportunity and negative correlation $\rho_{cx} > 0$, with time-separable utility.	Arb-IN

Table 1.1: **Names of main model variations**.
The models are explained in Section 1.5.

μ_{ct} denotes the CD at time t and σ_{ct} is the volatility coefficient of consumption growth at time t, which is multiplying the stochastic component $\mathrm{d}W_{ct}$.[19] In the remainder of the paper, CV refers to σ_{ct}.

1.4.3 Utility

Lifetime utility at time 0 takes the following form depending on the utility specification:

$$\underbrace{U_0 = \mathrm{E}_0 \int_0^\infty e^{-\rho t} u(C_t, S_t)\mathrm{d}t,}_{\text{TSU}} \qquad \underbrace{V_0 = \mathrm{E}_0 \int_0^\infty F(C_t, V_t)\mathrm{d}t}_{\text{RU}} \tag{1.3}$$

In both cases, there is an infinite horizon, with ρ representing the time preference parameter. In the case of TSU, flow utility u depends on the consumption flow (or just consumption for simplicity) and potentially on the surplus consumption ratio

[19] W_{ct} is a standard Wiener Process associated with consumption such that $W_{ct} - W_{cs} \sim \mathrm{Normal}(0, s-t)$.

S_t, which is connected to the external habit.[20] In the variations without habit, S_t is taken to be equal to 1. In the case of RU, the aggregator function FF depends on consumption and on the current lifetime utility V_t which in the context of RU is referred to as the value function. u and F take the following form:

$$\underbrace{u(C_t, S_t) = \frac{(C_t S_t)^{1-\gamma} - 1}{1-\gamma}}_{\text{TSU}}, \qquad \underbrace{F(C_t, V_t) = \frac{\rho\,(1-\gamma)\,V_t}{1-1/\psi}\left(\left(\frac{C_t}{((1-\gamma)\,V_t)^{-1/(1-\gamma)}}\right)^{1-1/\psi} - 1\right)}_{\text{RU}} \tag{1.4}$$

γ is the risk aversion parameter, and in the standard TSU case it is equal to relative risk aversion, which also equals the inverse of the IES. ψ is the IES parameter in the RU case.[21]

1.4.4 State variable process

At time t, the state variable x_t follows the process:

$$\mathrm{d}x_t = -\log(\phi)(\mu_{x0} - x_t)\mathrm{d}t + \sigma_{xt}\mathrm{d}W_{xt} \tag{1.5}$$

This expression describes an autoregressive stochastic process that reverts to the stochastic steady state (or steady state for simplicity), μ_{x0}.[22] The rate of reversion to the steady state is governed by ϕ, which is constrained to be between 0 and 1. Thus, $\log(\phi)$ is non-positive and it implies that when $x_t > \mu_{xt}$ ($x_t < \mu_{xt}$) the drift is downward-sloping (upward-sloping), always towards the steady state. $\mathrm{d}W_{xt}$ is also a standard Wiener process, and σ_{xt} is the volatility coefficient of the state variable and it is either a constant or it also depends on x_t. $\mathrm{d}W_{xt}$ can be correlated with $\mathrm{d}W_{ct}$, and the value of the correlation is captured by ρ_{cx}. In economic terms, the state variable plays a different role for each model variation. The full dependence of the model variations on the state variable is shown in Table 1.2: In some variations the steady state is at $x_t = 0$, while in others it is at $x_t = 1$, and x_t is positive with probability 1. This specification is used for the variations in which CV σ_{ct}, is proportional to the state variable, to ensure that σ_{ct} is positive.

[20] In the habit model of Campbell and Cochrane (1999), which is followed here, this variable is actually equal to $(C_t^a - X_t)/C_t^a$, where X_t is the level of habit and C^a is aggregate consumption.

[21] f has the form of a normalised aggregator as in Duffie and Epstein (1992).

[22] The steady state $x_t = \mu_{x0}$ does not necessarily coincide with the ergodic mean or median of the process when the diffusion of the process is not symmetric around the steady state value.

Model variation			
TSU-CD:	$\mu_{ct} = \mu_{c0} + x_t$	$\sigma_{xt} = \sigma_{x0}$	$\mu_{x0} = 0$
TSU-CV:	$\sigma_{ct} = \sigma_{c1} x_t$	$\sigma_{xt} = \sigma_{x1}\sqrt{x_t}$	$\mu_{x0} = 1$
TSU-Habit:*	$S_t = \bar{S}e^{x_t}$	$\sigma_{xt} = \sigma_{ct}\lambda(x_t)$	$\mu_{x0} = 0$
RU-CD:	$\mu_{ct} = \mu_{c0} + x_t$	$\sigma_{xt} = \sigma_{x0}$	$\mu_{x0} = 0$
RU-CV:	$\sigma_{ct} = \sigma_{c1}\sqrt{x_t}$	$\sigma_{xt} = \sigma_{x1}\sqrt{x_t}$	$\mu_{x0} = 1$
TSU-HCV:	$\sigma_{ct} = \sigma_{c1} x_t$	$\sigma_{xt} = \sigma_{x1}\sqrt{x_t}$	$\mu_{x0} = 1$
Arb-IN:	$\mu_{ct} = \mu_{c1} x_t^{1/4}, \sigma_{ct} = \sigma_{c1}\sqrt{x_t}$	$\sigma_{xt} = \sigma_{x1}\sqrt{x_t}$	$\mu_{x0} = 1$
Arb-DP:	$\mu_{ct} = \mu_{c1} x_t^{3/2}, \sigma_{ct} = \sigma_{c1} x_t^{2/3}$	$\sigma_{xt} = \sigma_{x1}\sqrt{x_t}$	$\mu_{x0} = 1$

Table 1.2: **Dependence on the state variable for each model variation**

* Following Campbell and Cochrane (1999) and Wachter (2006), the exact form of $\lambda(\cdot)$ is:

$$\sigma_{xt} = \sigma_{ct}\lambda(x_t) = \begin{cases} \sigma_{ct}\left(\frac{\sqrt{1-2x_t}}{\bar{S}} - 1\right) & \text{if } x_t < \frac{1-\bar{S}^2}{2} \\ 0 & \text{if } x \geq \frac{1-\bar{S}^2}{2} \end{cases}, \quad \bar{S} = \sqrt{\frac{\gamma}{-\log(\phi) - b/\gamma}} \tag{1.6}$$

1.4.5 Stochastic discount factor

1.4.5.1 Time-separable utility Case

In the TSU case, the stochastic discount factor (SDF) is the derivative of the flow utility with respect to consumption. In the general case the formula is the following:

$$\Lambda_t = e^{-\rho t}(C_t S_t)^{-\gamma} \tag{1.7}$$

where S_t is only relevant in the habit model. Using the above expression, along with the consumption process (Equation 1.2) and the state variable process (Equation 1.5), Ito's Lemma can be implemented to get the stochastic differential equation (SDE) of the SDF:

$$\begin{aligned} \frac{\mathrm{d}\Lambda_t}{\Lambda_t} =& \Big(-\rho - \gamma\mu_{ct} + \frac{\gamma^2}{2}\sigma_{ct}^2\Big)\mathrm{d}t - \gamma\sigma_{ct}\mathrm{d}W_{ct} \\ & \underbrace{+\Big(-\gamma\log(\phi)x_t + 2\rho_{cx}\sigma_{ct}\sigma_{xt} + \sigma_{xt}^2\Big)\mathrm{d}t - \gamma\sigma_{xt}\mathrm{d}W_{xt}}_{\text{habit model only}} \end{aligned} \tag{1.8}$$

For the details of the derivation, see Appendix 1.G.1.

1.4.5.2 Recursive utility case

In the case of RU, the stochastic process of the SDF is derived from the expressions for the value function and the aggregator function. The latter is given in Equation

1.4, and as shown by Tsai and Wachter (2018), the value function follows:[23]

$$V_t = \frac{C_t^{1-\gamma} e^{(1-\gamma)K(x_t)}}{1-\gamma} \tag{1.9}$$

V_t increases with K, which is a specific function of x_t that captures the full dependence of the value function on the state variable. At the end of this section, I show how the expression above is justified, and I compute a novel perturbation approximation that provides a formula for K. Given the expression for the value function, Ito's Lemma can be implemented to get to the SDE of the SDF. The calculation here follows Chen et al. (2009). In particular, the fundamental relationship is:

$$\frac{\mathrm{d}\Lambda_t}{\Lambda_t} = F_V(C_t, V_t)\mathrm{d}t + \frac{\mathrm{d}F_C(C_t, V_t)}{F_C(C_t, V_t)} \tag{1.10}$$

F_C and F_V denote partial derivatives of F with respect to consumption and the value function respectively. The first term on the right-hand side is the derivative of the flow utility with respect to the value function. The second term can be computed by applying Ito's lemma on the derivative of flow utility with respect to consumption.[24] The result is the following:

$$\begin{aligned}\frac{\mathrm{d}\Lambda_t}{\Lambda_t} =& \Bigg(\frac{\rho(-(1-\gamma\psi)e^{\frac{(1-\psi)K[x_t]}{\psi}} - \gamma\psi + \psi)}{1-\psi} - \gamma\mu_{ct} + \frac{\gamma^2\sigma_{ct}^2}{2} + \frac{\gamma(\gamma\psi-1)\rho_{cx}\sigma_{xt}\sigma_{ct}K'(x_t)}{\psi} \\ &+ \frac{(\gamma\psi-1)(2\psi(\mu_{x0}-x_t)\log(\phi)K'(x_t) + \sigma_{xt}^2((\gamma\psi-1)K'(x_t)^2 - \psi K''(x_t)))}{2\psi^2} \Bigg)\mathrm{d}t \\ &- \frac{(\gamma\psi-1)\sigma_{xt}K'(x_t)}{\psi}\mathrm{d}W_{xt} - \gamma\sigma_{ct}\mathrm{d}W_{ct}\end{aligned} \tag{1.11}$$

The details of the derivation can be found in Appendix 1.G.2. It is notable that in the special case of $\gamma = 1/\psi$, which corresponds to TSU, the equation above simplifies to the TSU formula in Equation 1.8. Also, the stochastic component relating to consumption ($-\gamma\sigma_{ct}\mathrm{d}W_{ct}$), is exactly the same as in TSU, and there is an extra component, namely $-\frac{(\gamma\psi-1)\sigma_{xt}K'(x_t)}{\psi}\mathrm{d}W_{xt}$, due to the direct dependence of the SDF on the state variable.

[23]Similar results are common in the literature, see for example Benzoni, Collin-Dufresne and Goldstein 2011; Kraft, Seiferling and Seifried 2017.

[24]This operation is performed by substituting the value function using Equation (1.9) and applying Ito's lemma based on consumption and the state variable as independent variables.

1.4.6 Instantaneous short-term rate

From the SDF the short-term rate is derived as follows:

$$\text{TSU: } r(x_t)\mathrm{d}t = -\mathrm{E}_t\left[\frac{\mathrm{d}\Lambda_t}{\Lambda_t}\right] = \left(\rho + \gamma\mu_{ct} - \frac{\gamma^2}{2}\sigma_{ct}^2\right)\mathrm{d}t\underbrace{+\left(\gamma\log(\phi)x_t - 2\rho_{cx}\sigma_{ct}\sigma_{xt} - \sigma_{xt}^2\right)\mathrm{d}t}_{\text{habit model only}}$$

$$\begin{aligned}\text{RU: } r(x_t)\mathrm{d}t = -\mathrm{E}_t\left[\frac{\mathrm{d}\Lambda_t}{\Lambda_t}\right] = \\ &\frac{\rho\left((1-\gamma\psi)e^{\frac{(1-\psi)K[x_t]}{\psi}} + \gamma\psi - \psi\right)}{1-\psi} + \gamma\mu_{ct} - \frac{\gamma^2\sigma_{ct}^2}{2} - \frac{\gamma(\gamma\psi-1)\rho_{cx}\sigma_{xt}\sigma_{ct}K'(x_t)}{\psi} \\ &- \frac{(\gamma\psi-1)\left(2\psi(\mu_{x0}-x_t)\log(\phi)K'(x_t) + \sigma_{xt}^2\left((\gamma\psi-1)K'(x_t)^2 - \psi K''(x_t)\right)\right)}{2\psi^2}\end{aligned} \tag{1.12}$$

In the standard TSU case, the short rate depends on three components. The first is the time preference parameter ρ. The second is $\gamma\mu_{ct}$, and it relates to the consumption smoothing motive. As CD increases, agents try to borrow to increase current consumption, and in equilibrium the short rate increases. The third is $-\gamma^2\sigma_{ct}^2/2$ and it relates to the precautionary savings motive. As consumption becomes more risky, agents try to save, and in equilibrium the short rate decreases. In TSU-Habit, there are extra components that relate both to the consumption smoothing motive and the precautionary savings motive, and they are due to the state variable being part of the utility function. Thus, as the surplus consumption ratio falls, marginal consumption increases even more than in standard TSU. So, the agent has an even higher motive to smooth consumption. However, in the same state of the world, the surplus consumption ratio is also much more volatile and the agent also has a higher precautionary savings motive. In Campbell and Cochrane (1999) these two opposite effects on the short-term rate are regulated by a parameter denoted b. If $b = 0$, then the short rate becomes a constant. If $b > 0$ ($b < 0$), then the short rate is decreasing (increasing) in the surplus consumption ratio.

In the RU case, the short rate becomes more complicated. However, for the main calibrations, the dominating additional effect comes from the fact that the marginal utility of consumption is expected to change as the state variable changes. As a consequence short rates are affected less by the consumption smoothing effect and the precautionary savings effect, and short rates under RU are less sensitive to the state variable than short rates under TSU.

1.4.7 Long-term bond

1.4.7.1 Bond pricing equation

Next, given the process of the SDF, the price of the long-term bond Q can be computed in the same way for both TSU and RU cases. The bond price is a function of the state variable x_t and its remaining maturity m. Thus, by using Ito's Lemma, the stochastic process follows:[25]

$$\mathrm{d}Q(x_t, m) = \Big(-\log(\phi)(\mu_{x0} - x_t)Q_x - Q_m + \frac{1}{2}\sigma_{xt}^2 Q_{xx}\Big)\mathrm{d}t + \sigma_{xt} Q_x \mathrm{d}W_{xt} \quad (1.13)$$

In the equation above, subscripts $\cdot_x$ and $\cdot_m$, denote partial derivatives with respect to the corresponding variable. The next step is to derive the partial differential equation (PDE) that Q obeys in these models. Thus, I use the pricing equation following the approach in Cochrane (2009) and Chen, Cosimano and Himonas (2010):

$$\mathrm{E}\Big[\mathrm{d}(\Lambda_t Q)\Big] = 0 \Rightarrow \mathrm{E}\left[\frac{\mathrm{d}\Lambda_t}{\Lambda_t}Q + \mathrm{d}Q + \frac{\mathrm{d}\Lambda_t}{\Lambda_t}\mathrm{d}Q\right] = 0 \quad (1.14)$$

Substituting the expressions for Λ_t, $\mathrm{E}[\mathrm{d}\Lambda_t/\Lambda_t]$ and $\mathrm{d}Q$ from Equations (1.8), (1.12) and (1.13) respectively, gives rise to the PDE obeyed by Q:[26]

$$-Q_m - r(x_t)Q + \Big(-\log(\phi)(\mu_{x0} - x_t) + A(x_t)\Big)Q_x + \frac{\sigma_{xt}^2}{2}Q_{xx} = 0$$
$$\text{where: } A(x_t)\mathrm{d}t = \frac{\mathrm{d}\Lambda_t}{\Lambda_t}\mathrm{d}Q \quad (1.15)$$

The expression comprises five terms. The first is the derivative with respect to maturity Q_m. The second is the short rate term $r(x_t)Q$, which differs depending on the variation, as shown in Equation (1.12). The third is the expectation term $-\log(\phi)(\mu_{x0} - x_t)$, which captures the information that short rates may be expected to change in the future. The fourth is what I call the A term, and it is responsible for term premia, as it captures consumption-based risk.[27] The fifth is the diffusion term $\frac{\sigma_{xt}^2}{2}Q_{xx}$.[28] The solution of this equation is discussed next, while Appendix 1.C shows in more detail how these five terms affect the term structure

[25]Given the flow utility function, investors' decisions are not affected by the level of consumption. This implies that the long-term bond is not going to be a function of consumption itself (see for example Tsai and Wachter 2018 who also commented on this).

[26]This equation is similar to a Black-Scholes equation.

[27]Risk is understood in the context of consumption-based asset pricing. Therefore, if the price of the bond does not co-vary with the SDF, then the A term is 0. The A term being 0 does not mean that the price of the bond is deterministic.

[28]This term is connected with the idea of *convexity* in finance.

of interest rates and their dynamics.

1.4.7.2 Solution of the pricing equation

Equation (1.15) is a PDE, and I solve it by making use of the Feynman-Kac formula, which re-expresses the solution of a PDE as an expectation of a stochastic process. In particular, the solution of Equation (1.15) is:

$$Q(m, x_t) = \mathrm{E}_t\left[\exp\left\{\int_m^0 r(\tilde{x}_{t+s})\mathrm{d}s\right\}\right] = \mathrm{E}_t\left[\exp\left\{-\int_0^m r(\tilde{x}_{t+s})\mathrm{d}t\right\}\right] \tag{1.16}$$

where $\tilde{x}_0 = x_t$ and $\tilde{x}_t$ follow the modified stochastic process compared to the state variable:[29]

$$\mathrm{d}\tilde{x}_t = \Big(-\log(\phi)(\mu_{x0} - \tilde{x}_t) + A(\tilde{x}_t)\Big)\mathrm{d}t + \sigma_{xt}(\tilde{x}_t)\mathrm{d}W_{xt} \tag{1.17}$$

The expectation is computed using Monte Carlo simulations.

1.4.7.3 Risk-neutral yield and term premium

Instead of using a modified process, the original process of the state variable can also be used in the Feynman-Kac formula:

$$H(m, x_t) = \mathrm{E}_t\left[\exp\left\{\int_m^0 r(x_{t+s})ds\right\}\right] = \mathrm{E}_t\left[\exp\left\{-\int_0^m r(x_{t+s})\mathrm{d}t\right\}\right] \tag{1.18}$$

This is by definition the expected gross return from rolling over the short-term rate. Thus, $-\log(H(m, x_t))/m$ is by definition the risk-neutral yield, and the argument above shows that it corresponds to the solution of Equation (1.15), after setting $A(x_t) = 0$ for all x_t. In other words, the risk-neutral yield can be thought of as deriving from a bond priced by a risk-neutral investor. This also provides a natural way for computing term premia, which is:

$$TP(x_t, m) = \frac{-\log Q(x_t, m) - \Big(-\log H(x_t, m)\Big)}{m} \tag{1.19}$$

Namely, the term premium is the difference between the regular yield and the risk-neutral yield. Unfortunately, there is no analytic expression for term premia, given that Q and H are computed numerically. However, there is an analytic expression for function A in Equation 1.15, and it can serve as a diagnostic of

[29] Here I show the dependence of σ_{xt} on $\tilde{x}_t$, in order to clarify that it is the same function as before, but it takes the modified variable as the argument.

term premia, as it is the component that distinguishes Q from H. Especially when the short-term rate is linear in the state variable, the sign of A determines the sign of term premia,[30] the time variability of A determines the time variability of term premia, and the size of A determines the size of term premia.[31] In addition, the size of A can easily be judged in comparison to the size of the expectation term $-\log(\phi)(\mu_{x0} - x_t)$, which also multiplies Q_x in the PDE. The expectation term and the A term are the two main drivers of the yield spread. Therefore, if the typical values of the expectation term are much larger than the typical values of the A term, this implies that the yield spread is due to expected changes in short-term rates in the future. On the other hand, if the values of the two terms are comparable in size, then the yield spread likely contains a component due to the term premium as large as a component due to the expectation term. This comparison is illustrated in practice in Section 1.5.

1.4.8 Perturbation approximation for K function in the recursive utility case

As mentioned in Subsection 1.4.5.2, given the process of the SDF it is possible to compute the price of bonds in the RU case in the same way as in the TSU case. This in turn requires an expression for the value function. This subsection is dedicated to explaining a novel perturbation method to approximate function K that was used in the RU value function. This novel method is explored in detail in Melissinos (2023).

Equation (1.7) shows how the value function can be written in a form that separates the dependence on consumption and the dependence on the state variable. Function K captures the dependence on the state variable, and it is the solution of the following ordinary differential equation (ODE):

$$-\frac{1}{2}\gamma\sigma_{ct}^2+\frac{1}{2}\sigma_{xt}^2\left(K''(x_t)-(\gamma-1)K'(x_t)^2\right)-\log(\phi)(\mu_{x0}-x_t)K'(x_t)+\frac{\beta\left(e^{-\epsilon K(x_t)}-1\right)}{\epsilon}+\mu_{ct}=0 \tag{1.20}$$

[30]In particular, term premia have the sign of the product of A with the derivative of Q with respect to the state variable x_t, which usually has the same sign as the derivative of the short rate with respect to the state variable x_t.

[31]To be precise, term premia depend on the entire pricing Equation (1.15). However, if the short-term rate is linear and the effect of the diffusion term is small, then the bulk of the time-varying behaviour of term premia is determined by A. In the explanation provided here, I assume that the diffusion term and non-linearities have a small effect on the yield spread. A detailed analysis is conducted in Appendix 1.C.

where the substitution $\psi = \frac{1}{1-\epsilon}$ has been made. For $\psi = 1$ ($\epsilon = 0$) Equation (1.20) has an analytic solution. This can then be used to create a global perturbation solution in terms of the state variable, which I express in terms of ϵ.[32] In particular, Equation (1.20) can be expanded to:

$$-\frac{1}{2}\gamma\sigma_{ct}^2 + \frac{1}{2}\sigma_{xt}^2(-(\gamma-1)\left(\epsilon^2 K_2'(x_t) + \epsilon K_1'(x_t) + K_0'(x_t)\right)^2 + \epsilon^2 K_2''(x_t) + \epsilon K_1''(x_t) + K_0''(x_t)) + \mu_{ct}$$
$$+\frac{\beta\left(e^{-\epsilon\left(\epsilon^2 K_2(x_t)+\epsilon K_1(x_t)+K_0(x_t)\right)} - 1\right)}{\epsilon} - \log(\phi)(\mu_{x0} - x_t)\left(\epsilon^2 K_2'(x_t) + \epsilon K_1'(x_t) + K_0'(x_t)\right) \approx 0 \tag{1.21}$$

Here function K has been expanded up to second order, but it could also be expanded further. Given this expansion, the equation admits a solution of the form:

$$\begin{aligned} K_0(x_t) &= a_{0,0} + a_{0,1}x_t \\ K_1(x_t) &= a_{1,0} + a_{1,1}x_t + a_{1,2}x_t^2 \\ K_2(x_t) &= a_{2,0} + a_{2,1}x_t + a_{2,2}x_t^2 + a_{2,3}x_t^3 \\ &\cdots \end{aligned} \tag{1.22}$$

This solution can be plugged into the ODE (1.21), and for each m, n, $a_{m,n}$ can be derived by setting each factor of $x_t^m \epsilon^n$ equal to zero. This leads to a linear equation for each coefficient.[33] Conveniently, these equations can be solved successively so that for each equation there is only one unknown. As can be seen from equation (1.22), the full solution is a sum of polynomials in terms of x_t. For each successive order of ϵ, the order of the polynomial increases by one. While it is possible to compute many orders of approximation, eventually the computation becomes expensive, as each order of ϵ requires the solution of more linear equations, and each equation has an increasing complexity.

The solution in Tsai and Wachter (2018) only derived $K_0(\cdot)$ which is the first term in formula (1.22) and it is the “zeroth” order approximation in terms of ϵ or equivalently ψ. My approximation is useful because it allows a much larger range of values for ψ, and it provides an analytic expression that is easy to include

[32]The approximation is global in terms of the state variable x_t, as the perturbation is done with respect to parameter ϵ. Nevertheless, it is not valid for all values of x_t. In particular, the approximation takes such a form, so that its validity depends on different regions of the state variable. In the region where it converges, the quality of the approximation is high for all values of x_t, but outside this region, the series diverges.

[33]Apart from coefficient $a_{0,1}$ which may require the solution of a second order equation.

in the Monte Carlo simulations, that solve the pricing equation. Given that the solution provided by Tsai and Wachter (2018) is analytically correct only for $\epsilon = 0$ ($\psi = 1$), implementing the method for other values of ψ is not easily justifiable, even if in practice it would generate qualitatively or even quantitatively similar results. It should also be noted that the full perturbation series, that I provide, corresponds to the exact solution to the ODE, even if the series diverges. Namely, it is the unique perturbation series that represents the solution. Thus, it is highly likely that with some extra mathematical analysis, it can be re-expressed in terms of known special functions, and we can get an exact answer that is practically trivial to compute for arbitrary order. Nevertheless, the method in its current form allows the researcher to easily approximate the value function, while also practically checking its convergence. The value function can then be used in the pricing equation to directly get the price of assets while being robust to a large range of parameters for the IES. Thus, this method can be implemented widely in RU models. My method is described in detail in Melissinos (2023).[34]

1.4.9 Calibration

General parameters	
Relative risk aversion γ	2.0
Rate of time preference ρ	0.012/yr
Steady state CD μ_{c0}	0.0252 /yr
Steady state CV σ_{c0}	0.02/yr
Steady state reversion $\log\phi$	log(0.92)/yr

Table 1.3: **Calibration of common parameters**

Given that the goal of the paper is to identify consumption-based mechanisms that are consistent with the patterns of term premia in the data, the emphasis is not on providing a perfect calibration. Instead, the focus is on finding the combination of the utility specification and the consumption process that generates the observed patterns in term premia. Thus, several parameter choices are explored, and the calibration of each model variation is reported in the corresponding figure showing the results. Nevertheless, there are common parameter choices across

[34] One limitation of the method, that my contribution does not overcome, is that the parameters of the processes should be at most linear functions of the state variable. In particular, σ^2_{xt} and σ^2_{ct} are linear in the state variable. Therefore, unlike in the TSU case where I set $\sigma_{ct} \propto x_t$, here I set $\sigma_{ct} \propto \sqrt{x_t}$. This is investigated in more detail in Melissinos (2023), but the main implication is that CV is relatively restricted in its variability.

model variations. These are reported in Table 1.3 and they follow Wachter (2013), while the risk aversion parameter γ is set equal to 2 following Wachter and Zhu (2019).

1.5 Results

1.5.1 How variations are evaluated

In evaluating the features of term premia, I also require that the model variations generate empirically plausible short rate volatility.[35] Higher short rate volatility can give rise to higher-term premia. Thus, I assume a relatively large short rate volatility to give these variations the best chance of success. Their performance is compared, by plotting model-implied term premia as a function of the state variable next to the time series of estimated term premia (these were already shown in Figure 1.1). In making the comparison, the focus is more on the variability than the level.[36] If the models generate roughly the same pattern of variability, then they are considered a success. In most cases, it is obvious when the models fail to generate the patterns of term premia in the data.

Table 1.4 shows information for function A for six separate variations with moderate CV, and Figures 1.3 and 1.5 show the corresponding five and ten-year term premia, which are discussed in Subsections 1.5.2.1 and 1.5.2.2 respectively. Further variations and calibrations are shown in Appendix 1.F. With these results, apart from illustrating the standard consumption-based mechanisms, I aim to provide a helpful reference, as in the literature state-dependent term premia are rarely provided. Furthermore, Table 1.5 and Figure 1.6 show variations with high CV, whose consequences have not been analysed before with respect to term premia. I am the first to show that these variations can generate the features of term premia in the data. These latter results are discussed in Subsection 1.5.3. While for my main results I use estimated premia from d 'Amico, Kim and Wei (2018), Abrahams et al. (2016) also estimated the term premium, and they provide a decomposition of the five-to-ten-year forward. Estimations from both papers

[35]The calibration of the state variable is described in Appendix 1.D.

[36]For example, in the data, especially recently, term premia seem to also become negative. However, the successful models that I present here all have exclusively positive term premia. I do not consider this a large setback, as my focus is on the variability of term premia and the models investigated here only have one state variable. In a full explanation of term premia and interest rates more generally, at least two variables would be necessary, given that a principal component analysis of the yields and spreads requires at least two principal components to explain the bulk of the variation (this is shown in Appendix 1.B)

are shown in Appendix 1.E,[37] and the five-to-ten-year forward term premium generated by the variations analysed in the current paper is shown in Appendix 1.F.

1.5.2 Moderate consumption volatility

1.5.2.1 Time-separable utility

Model variation	$A(x_t)$	ρ_{cx}	Range of A term*	Range of expectation term*
TSU-CD	$-\gamma\rho_{cx}\sigma_c\sigma_x$	$+$	(-0.0002,-0.0002)	(0.0023, -0.0022)
TSU-CV	$-\gamma\rho_{cx}\sigma_{ct}\sigma_{xt}$	$-$	(0.0012, 0.0086)	(0.048, -0.052)
TSU-Habit	$-\gamma\rho_{cx}\sigma_c\sigma_{xt} - \gamma\sigma_{xt}^2$	$+$	(-0.083, -0.011)	(0.034, -0.030)
RU-CD	$-\gamma\rho_{cx}\sigma_c\sigma_x - \frac{(\gamma\psi-1)\sigma_x^2K'(x_t)}{\psi}$	$+$	(-0.001250, -0.001253)	(0.0023, -0.0022)
RU-CV	$-\gamma\rho_{cx}\sigma_{ct}\sigma_{xt} - \frac{(\gamma\psi-1)\sigma_{xt}^2K'(x_t)}{\psi}$	$-$	(0.0018, 0.0040)	(0.030, -0.034)
RU-Mixed	$-\gamma\rho_{cx}\sigma_{ct}\sigma_{xt} - \frac{(\gamma\psi-1)\sigma_{xt}^2K'(x_t)}{\psi}$	$+$	(-0.0045, -0.010)	(0.033, -0.031)

Table 1.4:
Information on function A from Equation (1.15) in different model variations with moderate CV. The t-subscript has been dropped from the quantities that are not time-varying according to the variation.
* This range covers the typical values of the state variable. The values correspond to the dashed vertical lines in Figures 1.3 and 1.5.

As mentioned earlier, the three main mechanisms analysed are time-varying CD, CV, and surplus consumption ratio (in TSU-Habit).[38] The effect of these mechanisms on term premia can first be understood by looking at function A for each of the variations. Table 1.4 shows the functional form of A, and which components are time-varying. It also shows the typical range for the size of the A term and the expectation term. As mentioned earlier, the sign of A, in conjunction

[37]The two measures are similar, with the term premium in Abrahams et al. (2016) reaching relatively higher values. In addition, the risk-neutral yield has much less variability in the Abrahams et al. (2016) estimation.

[38]The first two mechanisms can also be found in the long-run risk models introduced by Bansal and Yaron (2004), who used a RU.

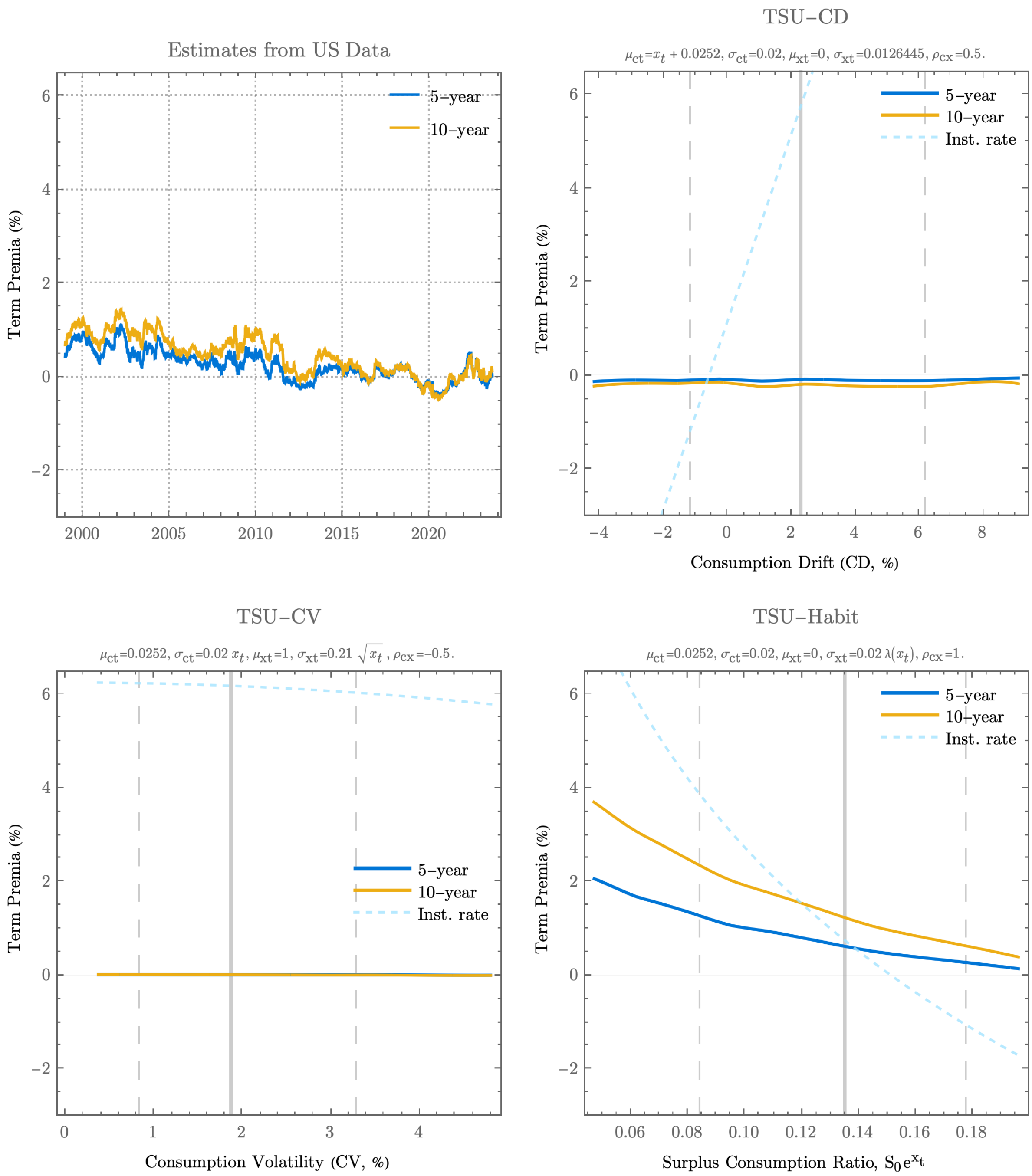


Figure 1.3: **Term premia in standard models with TSU**
The top left plot shows estimates of term premia according to d 'Amico, Kim and Wei (2018). The remaining plots show state-dependent term premia for three standard variations, namely variations with a) time-varying CD, b) time-varying CV, and c) an external habit in the utility function respectively. The dashed line shows the short-term rate.
The vertical dashed lines correspond to the typical values of the state variable based on simulations. The full range of the x_t-axis includes extreme values of the state variable, which are still possible (see Appendix 1.D or Figure 1.16 for the exact definition of the ranges).

with the slope of the short-term rate, determines the sign of term premia, while the size and variability of A also determine the size and variability of term premia. In TSU-CD the short rate is increasing with CD, due to the consumption smoothing motive.[39] As a result, in conjunction with $\rho_{cx} > 0$ the term premia are negative and constant in the state variable. The intuition for negative term premia is that the short-term rate goes up and bond prices go down when CD rises, which is also the time that consumption tends to increase (due to $\rho_{cx} > 0$). This means that long-term bonds act as a hedge, and they command a negative term premium. Apart from the negativity of term premia, A typically takes much smaller values in absolute value compared to the expectation term, implying that term premia should be very small. Thus, instead of positive, time-varying and sizeable, as in the data, term premia are negative, constant and small. Alternatively, for TSU-CV, the short rate is decreasing in CV, due to the precautionary savings motive, and I assume $\rho_{cx} < 0$. Therefore, the A term is positive and time-varying in the state variable, as it includes CV σ_{ct} (in this specification σ_{xt} is also time-varying). As a result, the term premia are again negative (they have the same sign as the slope of the short-term rate), but in this case, they are time-varying. However, the A term is much smaller in absolute value compared to the expectation term, so term premia apart from negative are again very small. Figure 1.3 shows the term premia for these two variations in comparison to the time series of term premia in the data.[40] It is evident from the figure that as the state of the economy changes, term premia would hardly move away from 0, and they would not be able to generate the variation estimated in the time-series. From the functional form of A it also follows that assuming a different sign for ρ_{cx}, would imply term premia of the opposite sign in both cases. However, for a representative consumption process it is reasonable that an increase of CD is associated with an increase in consumption itself, while an increase in CV is associated with a decrease in consumption.[41] In Appendix 1.F the results above are verified for several different

[39] This means that the stochastic component of consumption is positively correlated with the stochastic component of the state variable, which is associated with CD. To avoid this long description, I will mostly use ρ_{cx}.

[40] In TSU-CV the short rate is also insensitive to CV.

[41] This is intuitive if the consumption process is thought of as a relatively independent consumption process that determines the short-term rate. However, if the short-term rate is the independent variable, and the consumption process is reacting, then it makes sense that as the short-term rate decreases, borrowing becomes cheaper and consumption temporarily increases. This can can either imply that CD decreases, as consumption comes back to its normal level, or that CV increases as the agent has less savings. In both cases, the sign of ρ_{cx} is the opposite compared to the first scenario. I conjecture that this should not happen in a large economy with a short-term rate determined by the behaviour of a representative agent. However, it could also

calibrations.

The mechanisms discussed above use the power utility setup. Here, I discuss the effect of including external habit in the utility function as in Campbell and Cochrane (1999). As shown by Wachter (2006), TSU-Habit can generate the basic patterns of term premia that we see in the data. As mentioned previously, models with time-varying risk aversion, like the habit model, belong to one of only two kinds of models that can explain the patterns of term premia in a consumption-based setup with a single stationary autoregressive process. Thus, I analyse this model within my setup, in order to comprehensively describe consumption-based explanations to real term premia, and delineate its main differences compared to the alternative explanation that I introduce in the next subsection. Table 1.4 shows that the habit model has an extra term in the functional form of A. It turns out that this second term is dominant because the state variable volatility is in most states much larger compared to CV ($\sigma_{xt} \gg \sigma_{ct}$).[42] As a result, the sign of A does not depend on ρ_{cx} (which in the canonical habit model is equal to 1 anyway, as consumption completely determines the habit variable.), and the sign of term premia is determined exclusively by the slope of the short-term rate as a function of the surplus consumption ratio. As discussed in Subsection 1.4.6, this relationship in TSU-Habit depends on parameter b, which is chosen positive so that the short-term rate is decreasing and the term premia are positive.[43] Furthermore, term premia are large, as the value of A is large compared to the expectation term. Lastly, term premia are time-varying, given that A includes σ_{xt}^2, which is time-varying. Namely, the variability of term premia is due to the heteroskedasticity of the state variable, which is amplified because A includes the square of the volatility.[44] Therefore, term premia are positive, time-varying, and large. This is explicitly shown in Figure 1.3, and the typical amount of variability, captured between the dashed lines, matches closely the variability in the estimated term premia.

be argued that the short-term rate is the independent force in the economy, due to the actions of the monetary authority.

[42] The size of the two terms is shown in the right plot of Figure 1.4

[43] This was also the choice of Wachter (2006), while Campbell and Cochrane (1999) set $b = 0$ in the final version of their paper (in an earlier version they also investigated $b > 0$). In Appendix 1.F, I also derive the results of a variation in which $b < 0$. In this case, the short rate is increasing in the surplus consumption ratio, and term premia are negative, time-varying and large in absolute value.

[44] In Appendix 1.F, I impose homoskedasticity, and this leads to constant term premia. Admittedly, this is contrary to the spirit of the habit model.

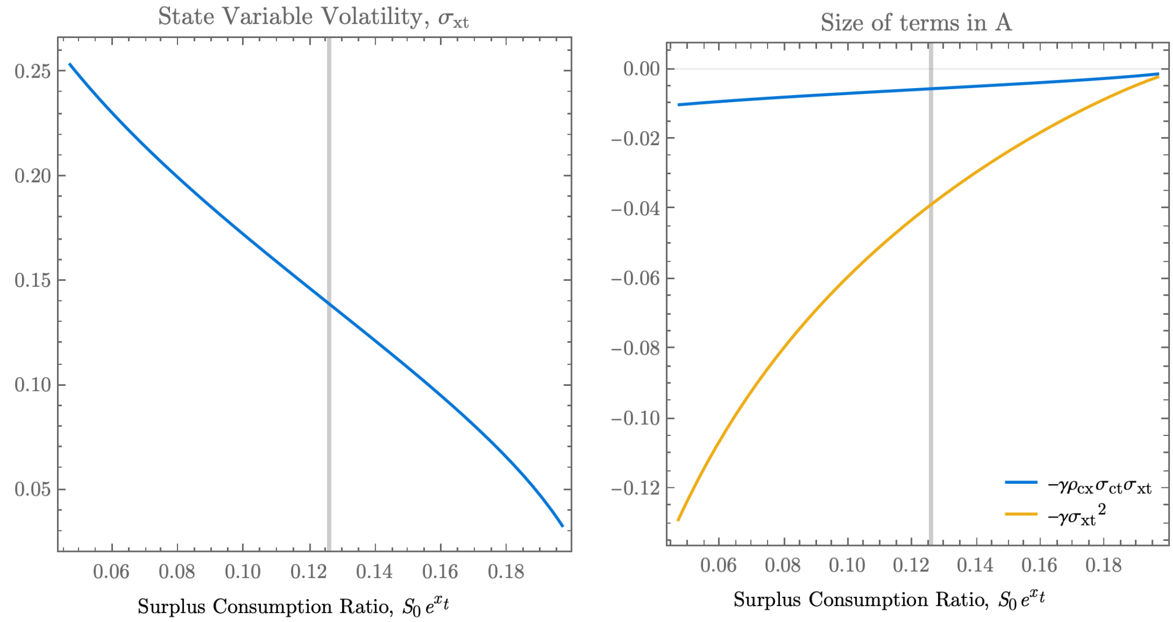


Figure 1.4: **Terms related to TSU-Habit**
The left plot shows the value of the volatility coefficient of the state variable in TSU-Habit. The right plot shows the magnitude of the two terms in the A function in TSU-Habit.

1.5.2.2 Recursive utility

In this subsection, the results are extended to RU. This case is arguably of higher interest, as it separates risk aversion and IES. Moreover, Bansal and Yaron (2004) were able to use this feature in conjunction with time-varying CD and CV in long-run risk models, to explain the equity premium puzzle. Indeed, similar to TSU-Habit, as is shown in Table 1.4, the RU variations have an extra term in function A. This term disappears for $\gamma = 1/\psi$ which coincides with the special case in which utility becomes time-separable. Similar to TSU-Habit, this term dominates the A function. Thus, the sign of term premia does not depend on the sign of ρ_{cx}, but on the slope of K, which turns out to match the slope of the short-term rate both in RU-CD and in RU-CV. This means that negative term premia are now a more robust prediction compared to TSU-CD and TSU-CV. However, in the case of RU-CD A is significantly larger compared to TSU-CD. Therefore, term premia are negative, and constant, but can be somewhat sizeable in absolute value. In contrast, RU-CV shows the same patterns as TSU-CV. Given that the short rate hardly exhibits variability in RU-CV, I also compute

RU-Mixed which includes both time-varying CD and CV, governed by the same state variable. However, A in this variation is also quite small, and the term premia are small in absolute value. The term premia for RU-CD, RU-CV and RU-Mixed are shown in Figure 1.5, and it is clear that they cannot generate the variability in the estimated term premia. Appendix 1.F has further variations with different calibrations verifying these results. Intuitively, RU models might be considered good candidates for explaining term premia due to their flexibility in separating risk aversion and IES. However, term premia are constant in RU-CD and very small in RU-CV, while they are in both cases negative. This result is consistent with the literature. Specifically, Bansal and Shaliastovich (2013) study term premia in RU models, but they investigate the variability in *nominal* term premia and their mechanism involves inflation. Gomez-Cram and Yaron (2021) provided a similar explanation for nominal term premia using RU that also relies on inflation. Hence, the real term premia that they generate are not substantially time-varying. Van Binsbergen et al. (2012) also considered a RU setup with inflation, and they find that nominal term premia can be positive, for very high risk aversion values. However, they also found that real term premia are negative.

1.5.3 High consumption volatility

Model variation	$A(x_t)$	ρ_{cx}	Range of A term*	Range of steady state reversion term*
High CV	$-\gamma\rho_{cx}\sigma_{ct}\sigma_{xt}$	$+$	(-0.0079,-0.0057)	$(0.0047, -0.0053)$
Arb-IP	$-\gamma\rho_{cx}\sigma_{ct}\sigma_{xt}$	$-$	(-0.0069, -0.052)	(0.048, -0.052)
Arb-DN	$-\gamma\rho_{cx}\sigma_{c}\sigma_{xt}$	$+$	(0.0096, 0.045)	(0.047, -0.054)

Table 1.5:
Information on function A from Equation (1.15) in different model variations with HCV.
* This range covers the typical values of the state variable. The values correspond to the dashed vertical lines in Figure 1.6.

In general, agents should be independently adjusting their investment and consumption. Thus, given the same asset-pricing processes, if optimising agents are heterogeneous in their utility functions, they will have different consumption processes. Given a utility function, the consistency of term premia with the con-

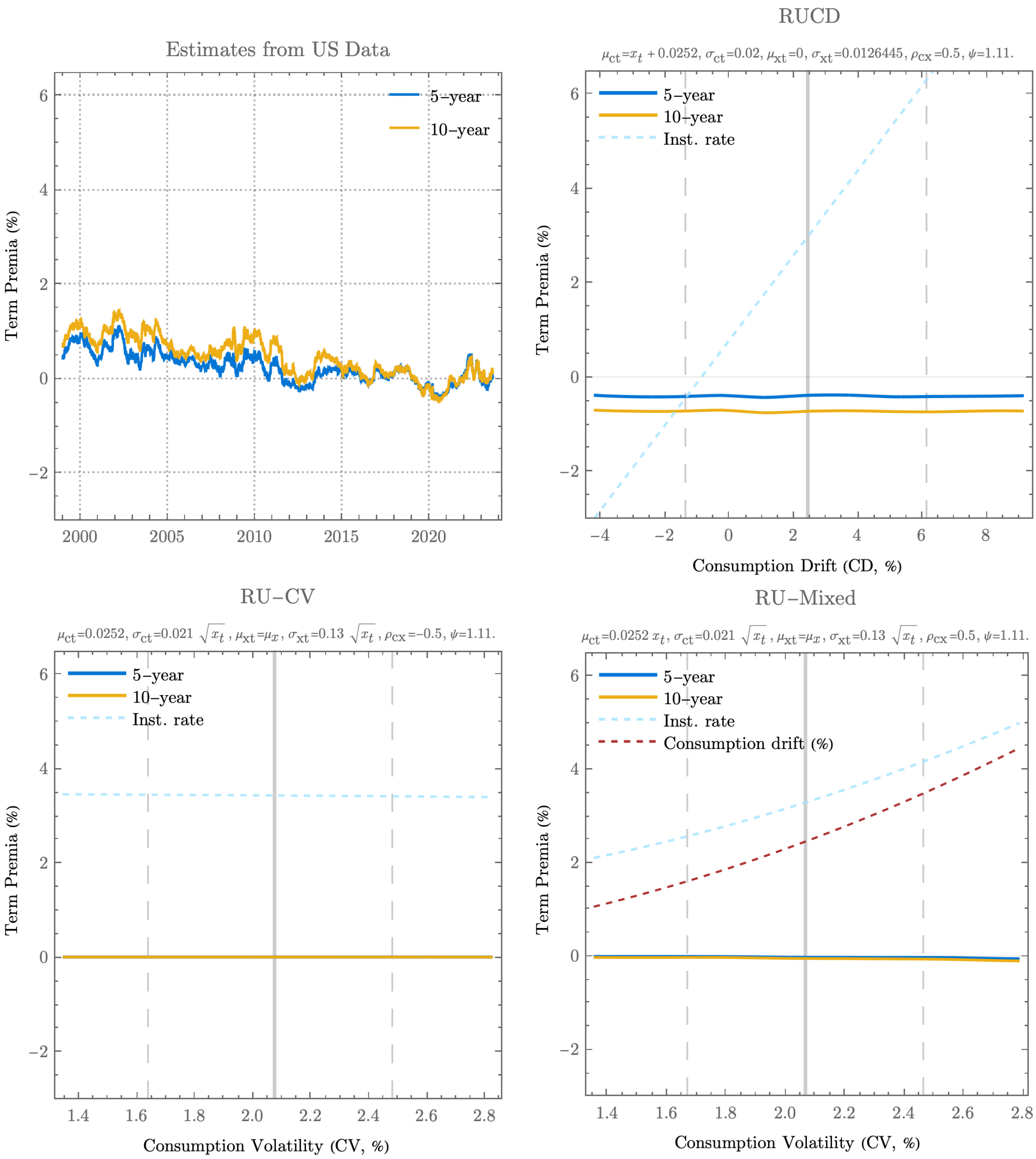


Figure 1.5: **Term premia in standard models with RU**
See Figure 1.3 for details. The variations are shown in the titles of the subplots.

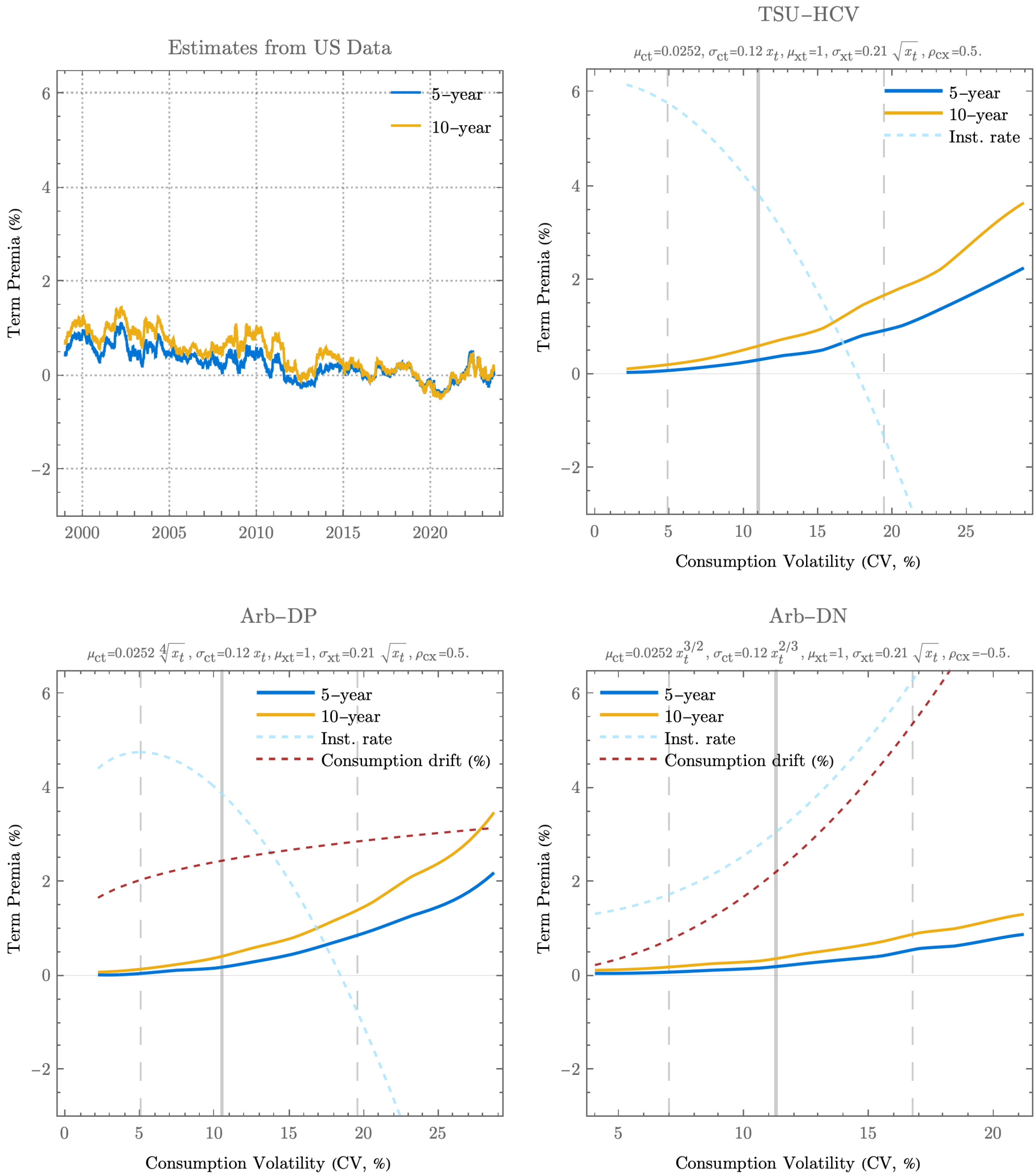


Figure 1.6: **Term premia in models with TSU and HCV**
See Figure 1.3 for details. The variations are shown in the titles of the subplots.

sumption process can be checked independently for each consumer. Previously, I have shown that representative consumer explanations of term premia require time-varying risk aversion. This raises the question of whether there is *any* consumer group whose consumption process is consistent with term premia, without assuming time-varying risk aversion. Given the negativity and the small size of term premia found in the previous subsection, it is reasonable to assume that the answer is again no. However, it turns out that other explanations rely on the dynamics of the consumption process within TSU. Table 1.3 shows information on function A for these cases, while Figure 1.6 shows the corresponding state-dependent term premia. As has been shown previously, time-varying CV implies time-varying term premia. Thus, starting from time-varying CV, the way forward is in principle simple based on the expression for A. By changing the sign of ρ_{cx} and increasing the steady state level of σ_{ct}, term premia become positive and large. Indeed, this works in generating the amount of variability in the estimated term premia (Figure 1.6). This is noteworthy given the difficulty encountered previously in generating any amount of significant time-varying term premia. However, are these two necessary modifications economically sensible?

As shown in Figure 1.6 the typical variability of the state variable (area between dashed lines) ranges from 5% to more than 20% CV per year. Even accounting for potential mismeasurement of aggregate consumption, this range is excessively large.[45] Therefore, this approach is not consistent with a representative consumer whose consumption coincides with aggregate consumption. However, this does not mean that the consumption process is too extreme for *any* consumer. Firstly, if financial markets are incomplete, and risk sharing is not possible, then idiosyncratic CV is relevant for asset prices (Constantinides and Duffie 1996). This means that aggregate CV could already be underestimating the CV that should be used in the models. Next, while 12% steady state CV is large compared to aggregate CV, it is not large compared to asset price volatility in financial markets. For people whose wealth lies in the financial sector, 11% wealth volatility is entirely plausible, and according to standard consumption-based portfolio theory, CV should follow wealth volatility. Lastly, there is also direct evidence that CV is much higher for some groups of consumers. While I do not take a position on whether these investors are rich or poor, Ait-Sahalia, Parker and Yogo (2004) showed that the CV of rich individuals could be much higher compared to aggregate CV. In particular, while they reported that the an-

[45] Savov (2011) suggested that due to mismeasurement, consumption volatility is underestimated.

nual standard deviation of non-durables and services was 2.3% according to the standard NIPA data, they measured an annual standard deviation of 19.6% for luxury retail sales and 20.4% for charitable contributions of wealthy individuals.[46] These values are both significantly larger than the steady state CV of the model variations in this subsection, which is equal to 12%.[47] Based on these results, Ait-Sahalia, Parker and Yogo (2004) also argued that the equity premium puzzle is less of a puzzle when considering the consumption process of rich consumers, as HCV also implies a sizeable equity premium.[48]. Similarly, Malloy, Moskowitz and Vissing-Jørgensen (2009) provided evidence that wealthy stockholders' CV is roughly three times higher compared to non-stockholders, while also showing that bond returns can be predicted by the covariance of wealthy stockholders' consumption growth with returns. Lastly, in a different strand of evidence, Carpenter et al. (2015) also showed that during the conduct of unconventional monetary policy, it was households that traded with the Fed, when it was trying to affect long-term yields.[49] All this evidence is consistent with the idea that a small group of investors with HCV are driving term premia.

The second required assumption for the mechanism is that ρ_{cx} is positive. Previously, I have argued that this is not plausible for a representative consumer, because an increase in consumption risk should induce consumers to consume less and save more. However, the consumer-investors in TSU-HCV could be a small part of the overall population, and in this case, $\rho_{cx} > 0$ can be justified. As CV increases, the short-term rate goes down, and this leads to an increase in bond prices. Thus, bondholders would then increase their consumption, given that their wealth also increases. An alternative intuition is that, as the short-term rate decreases, consumption increases due to borrowing, which in turn increases CV.

While I have shown the effect of HCV on term premia, I have only done so for $\gamma = 2$. Apart from further variations in Appendix 1.F, Figure 1.7 shows the different levels of term premia on the same scale for various values of γ and various values of steady-state CV. The results are interesting in several ways. Firstly, it stands out that different values of γ lead to huge changes in term premia, when CV is high. This means that term premia in TSU-HCV are highly sensitive to

[46]NIPA refers to the national income and product accounts produced by the Bureau of Economic Analysis of the US Department of Commerce. Ait-Sahalia, Parker and Yogo (2004) also included other measurements on the sales of luxury retail products.

[47]The standard deviation of consumption growth calculated from simulations also takes values similar to the CV of the model.

[48]This is also shown in an example in Melissinos (2024).

[49]This household classification is likely somewhat different compared to what the label implies, for instance, it usually includes hedge funds.

risk aversion levels. On the other hand, term premia are so small when CV is low, that moderate increases in risk aversion are not able to generate the required variability. Thus, even if $\gamma = 4$, CV needs to be able to reach at least 10%, so that time-variability in term premia is generated. This subsection shows how some consumers could have consumption processes that are consistent with the main features of term premia. By restricting my attention to these investors, and not introducing a full heterogeneous agent model, I can examine many different variations. Nevertheless, it is important to also consider the potential behaviours of the remaining agents in the economy. For instance, they could be investing in the bond market, but their behaviour could be explained by more complicated or alternative models. It could also be the case that other investors in the bond market are entities, such as hedge funds and pension funds that are not appropriately modelled as consumers. The only requirement for the remaining investors is that they do not trade in such a way, that induces extensive risk sharing with HCV investors. If they did, then this would lead to a decrease in the CV of the HCV investors. Alternatively, many consumers may not be participating in the bond market at all.[50] In both cases the other agents can have moderate consumption processes, and be primarily responsible for aggregate consumption dynamics.[51]

TSU-HCV has been the simplest consumption-based variation that can generate large term premia. However, given a high CV, slightly more complicated variations can be examined, in which CD and CV are simultaneously changing. I refer to these as "arbitrageur" variations as in Vayanos and Vila (2021), who suggested that the term structure of interest rates is driven by "arbitrageurs", who take advantage of investment opportunities in the bond market. As these opportunities can be risky, arbitrageurs are not able to fully equate rates and eliminate the effect of the demand of idiosyncratic investors or "preferred habitat investors", as they are called in the article.[52] Here, I abstract from these latter investors and restrict my attention to arbitrageurs. They are marginal investors in the bond market. Their consumption process should be consistent with the observed term structure of interest rates, including term premia. I argue that the consumption process of the arbitrageurs has two main features. Firstly, their CV

[50] Or they may not be marginal investors due to short-selling constraints. For instance, an investor who is constrained from shorting one end of the term structure could be holding some long-term bonds, but this does not make her a marginal investor of bonds in general.

[51] A fuller analysis would provide a full heterogeneous agent model explaining to what extent idiosyncratic consumption risk can be insured through financial markets.

[52] Given that there is risk, these investment opportunities fall under the category of "limited arbitrage".

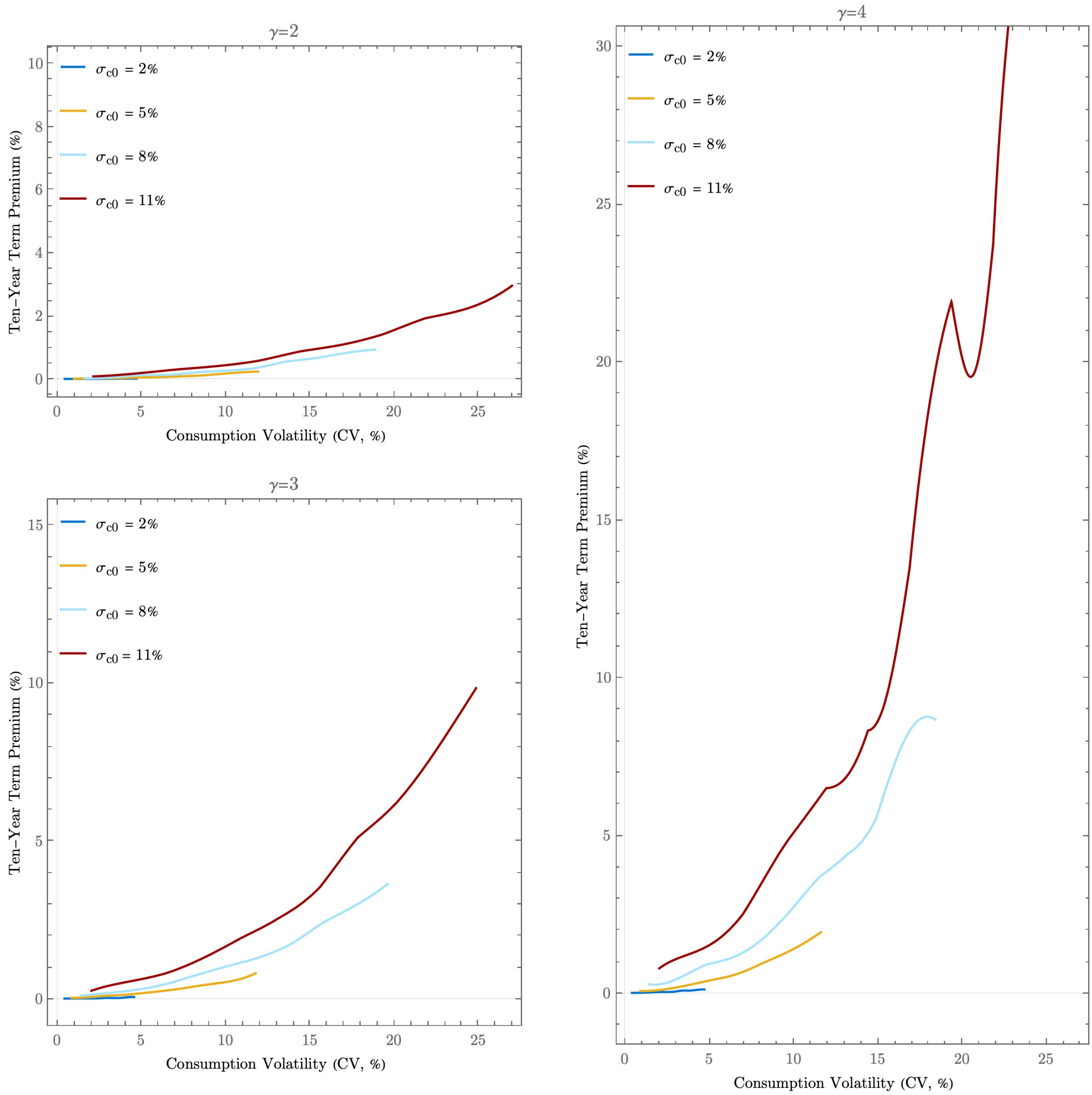


Figure 1.7: **Ten-year term premium in the TSU-HCV variation for different steady-state CV levels and risk aversion levels**
The plots show the ten-year term premium for different variations, and they are drawn with the same scale. Each plot corresponds to a different value of the risk aversion parameter γ, and each line corresponds to a different value for the steady state value of CV. The range of CV over which the lines are drawn, corresponds to the values of CV that can reasonably be acquired (these are the same ranges as in the previous figures).

is high (similar to TSU-HCV). Secondly, as the investment opportunity increases, both CV and CD rise. This occurs because the higher investment opportunity offers higher expected returns, which implies a higher CD. At the same time, the higher investment opportunity brings more risk, and CV also rises. This setup can give rise to four separate variations depending on the behaviour of the short-term rate and the sign of ρ_{cx}. These are shown in Table 1.6. The movements in CD and CV have opposite effects on the short-term rate. Depending on the dominating component, the short-term rate can either increase or decrease in the magnitude of the investment opportunity. In addition, the sign of function A is fully determined by ρ_{cx}, which in turn depends on the portfolio composition of arbitrageurs, and how its value fluctuates given the changing state of the economy.[53] These two binary choices give rise to the four possibilities shown in Table 1.6.

Short-term rate	Positive ρ_{cx}	Negative ρ_{cx}
Short-term rate Decreasing with CV (CV dominates)	Arb-DP, **positive** term premia	Arb-DN, **negative** term premia
Short-term rate Increasing with CV (CD dominates)	Arb-IP, **negative** term premia	Arb-IN, **positive** term premia

Table 1.6: **Term premia sign in basic arbitrageur variations**

While each of these possibilities seems plausible, I focus on the two that generate positive term premia. In Arb-IN, term premia are positive and increasing with CV, as is the short-term rate. As shown in Figure 1.6, Arb-IN generates positive, time-varying and sizeable term premia. However, the size of the term premia is not as high as in TSU-Habit, TSU-HCV and Arb-DP. $\rho_{cx} < 0$ could be justified in Arb-IN, because an increase in the short-term rate could be inducing the arbitrageurs to invest more in the bond market and decrease their consumption. In addition, despite CD rising, an increase in the short-term rate can also imply a decrease in their wealth, if the arbitrageurs are bondholders.

In Arb-DP, $\rho_{cx} > 0$ can also be justified because it makes sense for consumption to increase when CD goes up. In addition, if the arbitrageurs are bondholders, then their wealth increases, as CV increases and the short rate goes down. This is also the variation that is most akin to the intuition provided in Vayanos and Vila (2021). As the short-term rate decreases, long-term bond yields underreact, and

[53]This is true to the extent that arbitrageurs do not have income external to their portfolio.

this leads to an increase in term premia. The arbitrageurs in Vayanos and Vila (2021) optimise between the mean and variance of their wealth, and consumption is not part of the analysis. To the best of my knowledge, I am the first to show that this behaviour can be rationalised within a consumption-based setup. Arb-DP also provides the characteristics of the consumption process that are consistent with term premia, and it shows that a low CV would not generate substantial variability in the arbitrageur setup. The mechanism driving term premia is basically the same as in TSU-HCV. Thus, explaining the main features of term premia requires a high CV. Concluding on whether actual bondholders' consumption process exhibits such volatility is not possible within this paper. However, my paper provides a theoretical prediction that can be evaluated and tested empirically. If such CV is judged to be too high, then arbitrageurs are likely not acting as consumers or on behalf of consumers. This would be evidence for the existence of frictions, such as the ones in the intermediary asset-pricing literature. Alternatively, if it is found that some bondholders have HCV as the model predicts, then it would be interesting to further research the reasons that distinguish these investors, and why they are not able to share their risk with the remaining population.

Apart from asset-pricing implications, the variations presented in this subsection are also significant for monetary policy, to the extent that monetary policy affects term premia (Beechey and Wright 2009). In particular, according to Arb-DP, central banks decreasing (increasing) interest rates is equivalent to increasing (decreasing) the CV of the marginal investors. An increasing CV implies higher term premia, and this mechanism hinges on stochastic consumption changes being positively correlated with CV. In addition, the effect on CV is very strong, as it can roughly range from 5% to 20%. On the contrary, the effect of monetary policy on the consumption process of non-investors might be muted, if they are indeed disconnected from the effects of bond markets. A full understanding of the effects of monetary policy on all agents in the economy would benefit from a full heterogeneous agent model that explains the investment behaviour of all households.[54]

Furthermore, the HCV and the arbitrageur variations have implications for household finance. In particular, the participants in these markets are assuming large consumption risks. Therefore, a usual household whose CV is low and whose utility function is similar to the utility function of the marginal investors could

[54]Schneider (2022) provided such a model, in which the state variable captures "aggregate conditions in the credit market". Similar models would be interesting, in which the state variable captures CD and CV.

benefit from investing in long-term bonds, when term premia are high. This is valid, as long as CV of the household does not become too volatile due to this investment. However, the benefit is conditional on the state of the economy, and it is not clear if the state of the economy is transparent to most households, as the current CV of marginal investors is not directly observable.[55] This advice would not be valid in the context of the habit model. In that case high term premia reflect states in which households have a high risk aversion, and investing in risky securities would not be appropriate.

1.6 Conclusion

In conclusion, consumption-based models encounter three key challenges in explaining the features of term premia. Firstly, they typically generate long-term bonds that provide a hedge against risk, which leads to negative instead of positive term premia. Specifically, for a representative consumer, it is reasonable that a rise in CD is associated with a stochastic consumption increase. Therefore, bond prices increase when CD decreases, and vice versa. Therefore, bonds are extra valuable, because they provide insurance against macroeconomic risk, and the associated term premia are negative. Similarly, for an aggregate representative consumer, it is reasonable that increased CV is associated with a stochastic consumption decrease. A similar argument implies that term premia are again negative. Secondly, time-varying CD generates constant instead of variable term premia. The paper shows that this turns out to be the case even in RU models. In contrast, time-varying CV always produces time-varying term premia, because by definition the state variable affects consumption uncertainty and, hence, risk. Thirdly, in calibrations according to an aggregate consumption process, term premia are typically very small in absolute value. The intuition for this is that consumption processes that are relatively stable give rise to term premia that are small. For term premia to be large it means that consumers are assuming large risks. Thus, given that aggregate consumption is relatively stable, the corresponding models imply low term premia. With the exception of the third shortcoming, these issues arise both in the TSU case and in the RU case.

[55] One could argue that the state of the economy is directly observable by the level of the short-term rate. However, here I have focused on explaining term premia, and I am using a single state variable. In a full explanation of the dynamics of interest rates, at least two state variables would be needed. Hence, the level of the short-term rate would most likely not directly imply the level of term premia.

However, I have identified model variations that do yield positive and significantly time-varying term premia. Firstly, a model with an external habit, as in Campbell and Cochrane (1999) and Wachter (2006), produces better results. This occurs because a) the short rate is counter-cyclical, b) the state variable is relatively large and directly affects the utility function, and c) the state variable is heteroskedastic. These three factors respectively imply that the term premia are a) positive, b) large in absolute value and c) time-varying. In this variation the time-variability of term premia is directly related to the heteroskedasticity of the state variable, while a potential drawback is that effective risk aversion is highly volatile and takes extreme values.[56] Nevertheless, my analysis shows that the habit model (or models with time-varying risk aversion more generally) is the best model we have that can explain term premia while using a representative consumption process.

I also demonstrate how large term premia can be explained by model variations that deviate from standard representative consumption processes. In particular, model variations for which a) CV is high, ranging for example from 5 to 20% per year, and for which b) stochastic consumption changes are positively correlated with CV, can generate positive and highly variable term premia. The first component contributes to term premia being high in absolute value, and the second component implies that term premia are positive. Apart from time-varying risk aversion, this is the only available consumption-based mechanism to generate positive and substantially time-varying term premia. An important implication of this model is the HCV for many states of the economy. However, it is not ludicrously high. If the consumption-based setup were completely wrong and disconnected from the actual mechanisms generating term premia, then it could imply almost any value of CV for term premia to become highly variable. Moreover, the CV levels in these variations mirror return volatility levels in certain financial markets, and there is literature measuring a HCV in products consumed by rich households. An interesting empirical question would then be to ask, what the CV is for marginal optimising investors of the term structure of interest rates. Another important aspect is that a large part of the population does not actively participate in the bond market. Thus, maybe the consumption process of these households is not so relevant regarding the levels of term premia. In a separate variation, which also performs quite well, I combine high levels of CV with a time-varying CD. In this variation, there is a tradeoff between CV and CD, and I claim that

[56] This was also the main criticism of the habit model by Mehra et al. (2007).

this interpretation is similar to the arbitrageur story in Vayanos and Vila (2021), which to the best of my knowledge has not been implemented in a consumption-based framework. While I describe the consumption process of arbitrageurs, I do not take a stance on whether its volatility is too high or not, and the final answer to this question probably requires further empirical research in the consumption process of direct and indirect bondholders. Whatever the answer is, further interesting questions emerge. If the CV implied by term premia is implausibly high, then arbitrageurs likely do not correspond to actual households, and they do not invest according to households' wishes, at least based on this high consumption risk explanation.[57] This could indicate the existence of intermediation constraints. Alternatively, if CV of actual bondholders is indeed high, then the question is why these bondholders do not engage in risk sharing with the rest of the agents in the economy.

Finally, given that a couple of different mechanisms can generate the basic features of term premia, the question arises which one is the correct explanation. The answer requires further research. Nevertheless, one approach is to combine some of the explanations provided here within a full heterogeneous model that also accounts for households not participating in financial markets. Non-participation can be rationalised given the high volatility in financial markets and the existence of some friction. As a result, there would be reduced risk sharing, justifying CV being large. This setup is likely to jointly explain term premia, stock market non-participation, reduced risk sharing in the economy and the equity premium puzzle. Therefore, I consider it a promising direction for further research.

[57] The intuition for this statement comes from consumption-based portfolio selection theory, according to which the portfolio weights of risky assets should agree whether households are investing directly or through funds.

Appendix

1.A Definitions

In the following, I provide a set of definitions for the concepts used in the paper.[58]

- Throughout the paper, terms like yields, returns, term premia etc. should be understood as referring to their real counterparts, unless otherwise specified. The distinction is made explicit when necessary to avoid confusion.

- A **nominal zero-coupon bond** with maturity m is a security paying one unit of currency after m years.[59]

- A **real bond** with maturity m is a security paying one unit of currency times an adjustment, that corrects for the elapsed inflation from the time it was issued until its maturity. The payment occurs after m years. Equivalently, a real bond is a security that pays the value of some basket of goods[60] when it matures.[61]

- Q_t^m is the price of the bond with maturity m at time t.

- **Real (or nominal) yield** at time t of a real or (nominal) bond with maturity m years where Q_t^m is the price of the corresponding bond, which is perfectly liquid:[62]
$$y_t^m = \frac{-\log(Q_t^m)}{m}, \quad m > 0$$

- **Yield spread** at time t between maturity m and n, where typically $m > n$:
$$y_t^m - y_t^n$$

- **The yield curve or the term structure of interest rates** refers to yields as a function of maturity. The yield curve is sloping upward/downward (or the slope of the yield curve is positive/negative) when yields are an

[58] Including for some concepts to which I refer in the main paper, without ever using in expressions.

[59] In the paper bonds always refer to zero-coupon bonds.

[60] Here there is an implicit assumption that individuals primarily care about this specific basket of goods. This basket of goods is also relevant for the calculation of inflation. Without this assumption, the study of real interest rates would be significantly hindered.

[61] A real bond of maturity $m+1$ one year ago is also equivalent to a real bond with maturity m today up to a renormalisation so that the principals match.

[62] Actual bonds' prices may deviate from Q_t^m due to liquidity considerations.

increasing/decreasing function of maturity. It is also possible that the slope is positive for some maturities and flat or negative for other maturities.

- **Annualised Gross Return** of a bond with maturity m from time t to $t+s$:

$$R^m_{t,t+s} = \sqrt[s]{Q^{m-s}_{t+s}/Q^m_t}$$

- **Log return or just return**[63] of a bond with maturity m from time t to $t+s$:

$$r^m_{t,t+s} = \log\left(R^m_{t,t+s}\right) = \frac{\log\left(Q^{m-s}_{t+s}\right) - \log\left(Q^m_t\right)}{s}$$

- **Instantaneous return** of a bond with maturity m at time t:

$$r^m_t = \lim_{s\to 0} r^m_{t,t+s}$$

- **Instantaneous short rate or just short rate** at time t:

$$r_t = \lim_{m\to 0} r^m_t = \lim_{m\to 0} y^m_t$$

- In the main paper yields are also referred to as *long-term interest rates*, whereas *interest rates* in general also include the short rate.

- **m-to-n year forward** at time t:

$$f^{m,n}_t = \frac{\log(Q^m_t) - \log(Q^n_t)}{n-m}$$

- **Instantaneous m-year forward** is:

$$f^m_t = \lim_{n\to m} f^{m,n}_t$$

- **Term or risk premium** of bond with maturity m at time t, where r_t is the instantaneous short-term rate of return at time t:[64]

$$TP^m_t = \frac{-\log(Q^m_t)}{m} - \frac{E_t\left[\int_0^m r_{t+s}ds\right]}{m}$$

[63]For convenience I refer to log return when I use the term return.
[64]Equivalent definitions are given in discrete time by Cochrane and Piazzesi (2009).

– If the term premium is zero for all m and t, this implies that the expected excess return from holding long-term bonds over any period is also 0. This can be seen from the following equivalent definition, where rx_t^m is the instantaneous excess return from holding a bond of maturity m:[65]

$$TP_t^m = \frac{E_t\Big[\int_0^m r_{t+s}^{m-s} - r_{t+s} d\tau\Big]}{m} \equiv \frac{E_t\Big[\int_0^m rx_{t+s}^{m-s} d\tau\Big]}{m}$$

– Here I have used the fact that:

$$\begin{aligned} -\log(Q_t^m) =& \Big(-\log\left(Q_t^m\right) + \log\left(Q_{t+m/N}^{m-m/N}\right)\Big) + \Big(-\log\left(Q_{t+m/N}^{m-m/N}\right) + \log\left(Q_{t+2m/N}^{m-2m/N}\right)\Big) + \\ & \ldots + \Big(-\log\left(Q_{t+m-m/N}^{m/N}\right) + \underbrace{\log\left(Q_{t+m}^{0}\right)}_{=0}\Big) \\ =& \frac{m}{N}\left(r_{t,t+m/N}^{m} + r_{t+m/N,t+2m/N}^{m-m/N} + \ldots + r_{t+m-m/N,t+m}^{m/N}\right) \\ \approx& \int_0^m r_{t+s}^{m-s} ds \end{aligned} \tag{1.23}$$

where N is some positive integer. The last line follows by N going to infinity, which means that the sum becomes an integral and the returns become instantaneous returns.

– Since Q represents the price of a bond that is perfectly liquid, the term premium does not include a liquidity premium.

- I also refer to the quantity used above:

$$\frac{E_t\Big[\int_0^m r_{t+s} ds\Big]}{m}$$

as **risk-neutral yield** of bond with maturity m.

- **Term or risk premium** of m-to-n year maturity forward at time t, where r_t is the instantaneous short-term rate of return at time t:

$$TP_t^{m,n} = \frac{\log(Q_t^m) - \log(Q_t^n)}{n-m} - \frac{E_t\Big[\int_{t+m}^{t+n} r_\tau d\tau\Big]}{n-m}$$

[65] If the excess return were positive for any period, then the expected term premium for the remaining period would be negative. This violated the initial assumption.

- The second term on the right-hand side of the equation above is the risk-neutral m-to-n year forward.

- In the paper many of the variables introduced here depend on time only through the state variable. So they will be denoted instead as:

$$Q(x_t, m), r(x_t), TP(x_t, m)$$

- In the main paper I also refer to the **value of the risk-neutral bond**. This is the implied value attached to a bond by a risk-neutral investor and it can be defined based on the risk-neutral yield defined above:[66]

$$H(x_t, m) = e^{-E_t\left[\int_0^m r_{t+s}ds\right]}$$

- The **strong version of the Expectations Hypothesis** holds when:

$$TP_t^m = 0, \quad \text{for all } m$$

- The **weak version of the Expectations Hypothesis** holds when:

$$TP_t^m = g(m), \quad \text{for all } m$$

  where g is some function of maturity, independent of the state of the economy and independent of time.

- **Predictability** refers to the ability to predict movements in excess returns. The prediction could be based on any information, but the literature has focused on using information in yields to predict subsequent yields in the future.

- **Excess volatility** of interest rates refers to long-term interest rate variations that are too large to be explained by the variation of the short rate alone while keeping the discount rate constant.[67]

[66]In the main paper, I also present an equivalent definition in Section 1.4.7.3, which also shows the intuition regarding the calculation of the term premium in this paper.

[67]To be completely precise excess volatility needs to be defined in terms of some benchmark model. As I do not investigate excess volatility directly, I do not provide such a definition.

1.B Explanatory power of the principal components of real interest rates

Apart from Figure 1.2, I also look at a series of regressions to demonstrate the strong dependence of nominal rates on real rates. In particular, I extract the first two principal components from a series of real yields with different maturities.[68] I only use two components because they explain more than 99.95% of the variance of real yields. Next, I regress nominal yields and nominal yield spreads on these two principal components.[69] Indeed, I find that the information contained within real rates explains most of the movements of nominal rates. The results are shown in Table 1.7. The coefficients are highly significant for both components, but more importantly, the R-squared is high in these regressions. For the level regressions, it ranges from 87% to 93%, while for the spread regressions, it ranges from 69% to 79%. Thus, both the level and the spread of nominal rates are mostly explained by the information and hence the processes that generate the real term structure.

Table 1.7: Regressions of the level and the spread of nominal bonds on the principal components extracted from the real term structure

	5 yr	10 yr	5-10 yr spread	15 yr	5-15 yr spread	20 yr	5-20 yr spread
Intercept	2.94***	3.73***	0.79***	4.13***	1.19***	4.29***	1.35***
	(0.01)	(0.00)	(0.00)	(0.00)	(0.01)	(0.00)	(0.01)
comp1	0.28***	0.26***	-0.02***	0.25***	-0.04***	0.23***	-0.05***
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
comp2	0.43***	-0.24***	-0.66***	-0.55***	-0.97***	-0.67***	-1.09***
	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
R-squared	0.87	0.93	0.69	0.93	0.74	0.93	0.79
R-squared Adj.	0.87	0.93	0.69	0.93	0.74	0.93	0.79

[68] The principal components are extracted from yields of all yearly maturities from two to twenty years.

[69] A similar exercise in performed by Abrahams et al. (2016) and they also find similar results. In their case, it is the real rates that are regressed on the principal components of the nominal rates. I do the inverse exercise because I ask how much nominal rates are explained by real rates.

1.C Components of the pricing equation

This section provides an explanation for each part of the pricing equation (1.15) which I repeat here:

$$-Q_m - r(x_t)Q + \Big(\log(\phi)x_t + A(x_t)\Big)Q_x + \frac{\sigma_{xt}^2}{2}Q_{xx} = 0 \tag{1.24}$$

- In the simplest case $\phi = 1$, $A(x_t) = 0$ and $\sigma_x(x_t) = 0$ for all x_t. Then the equation is:

  $$Q_m = -r(x_t)Q = -x_t Q$$

  This corresponds to an economy with a constant state. Figure 1.8 shows that in this economy yields are always equal to the short rate, term premia are equal to 0 and given the state of the economy nothing will ever change.

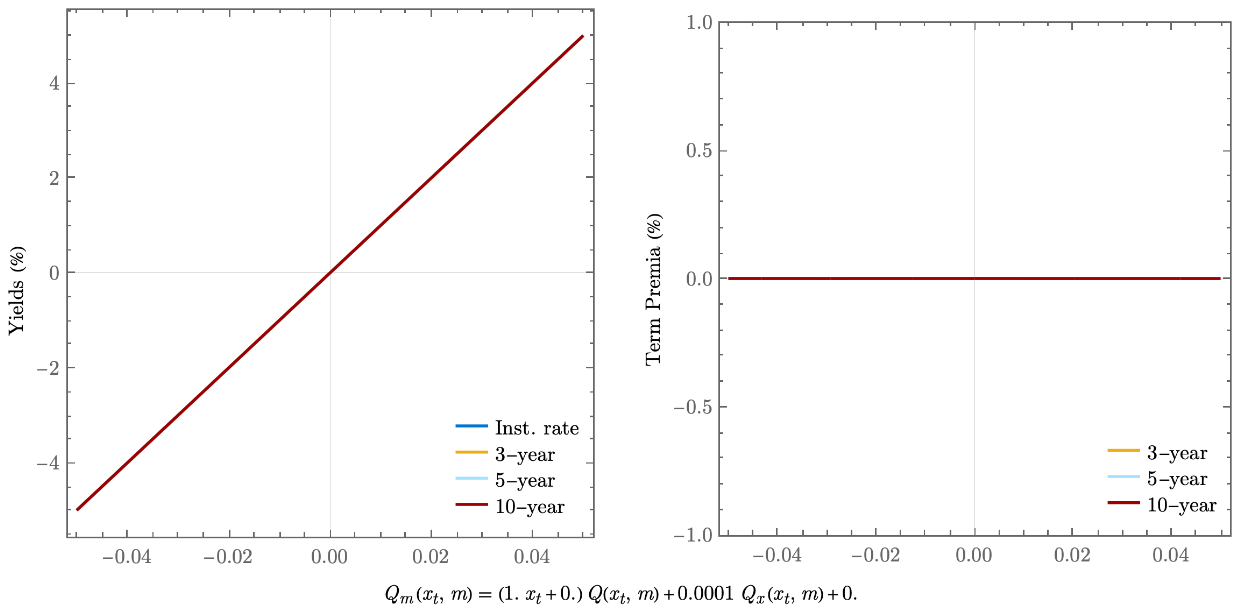


Figure 1.8: The left plot shows the short-term rate and yields of different maturities as a function of the state variable. The right plot shows the term premia for different maturities as a function of the state variable.

- $\phi \neq 1$:

  $$Q_m = r(x_t)Q - \log(\phi)x_t Q_x = x_t Q - \log(0.9)x_t Q_x$$

  Here there is again no volatility of the state variable. Thus, this corresponds to a deterministic economy. However, the state is not constant, it drifts towards the state $x_t = 0$, which can be thought of as the steady state. This implies that long-term yields will lie between the contemporaneous short rate and the steady-state short rate. As shown in Figure 1.9 this results in a characteristic picture, in which all yields intersect at the steady state. If the process moved towards the steady state faster (lower ϕ), then the yields would be more spread out. Given that there is no uncertainty, the term premia are again zero.

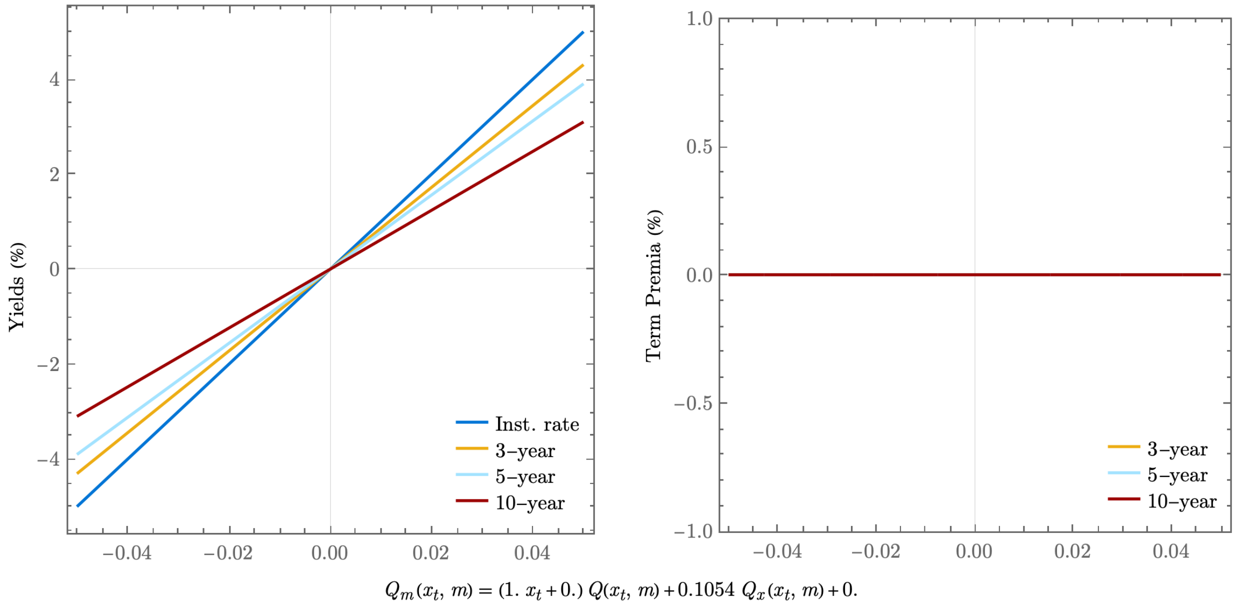


Figure 1.9: The left plots show the short-term rate, the five-year yield and the five-year risk-neutral yield as a function of CD. The right plot shows the term premia for different maturities as a function of the state variable.

- $A(x_t) = c \neq 0$:

$$Q_m = r(x_t)Q - \Big(\log(\phi)x_t + A(x_t)\Big)Q_x = x_tQ - (\log(0.9)x_t + 0.01)Q_x$$

As stated in the main paper A generates term premia. This case does not directly correspond to some economic situation because, the state variable volatility is again 0, and in the actual economic models this also implies $A(x_t) = 0$. However, for intuition, I show the “yields” and “term premia” that arise. As Figure 1.10 shows, now the yields do not intersect at the steady state. Now the longer-term yields are higher at the steady state. This implies positive term premia and indeed as shown in the right panel, term premia are positive, proportional to the maturity of the bond and constant as a function of the state variable. The latter fact is due to $A(x_t)$

being constant for all x_t and the fact that yields are linear. Finally, the term premia are positive, because A is positive and the short rate is increasing with respect to the state variable.

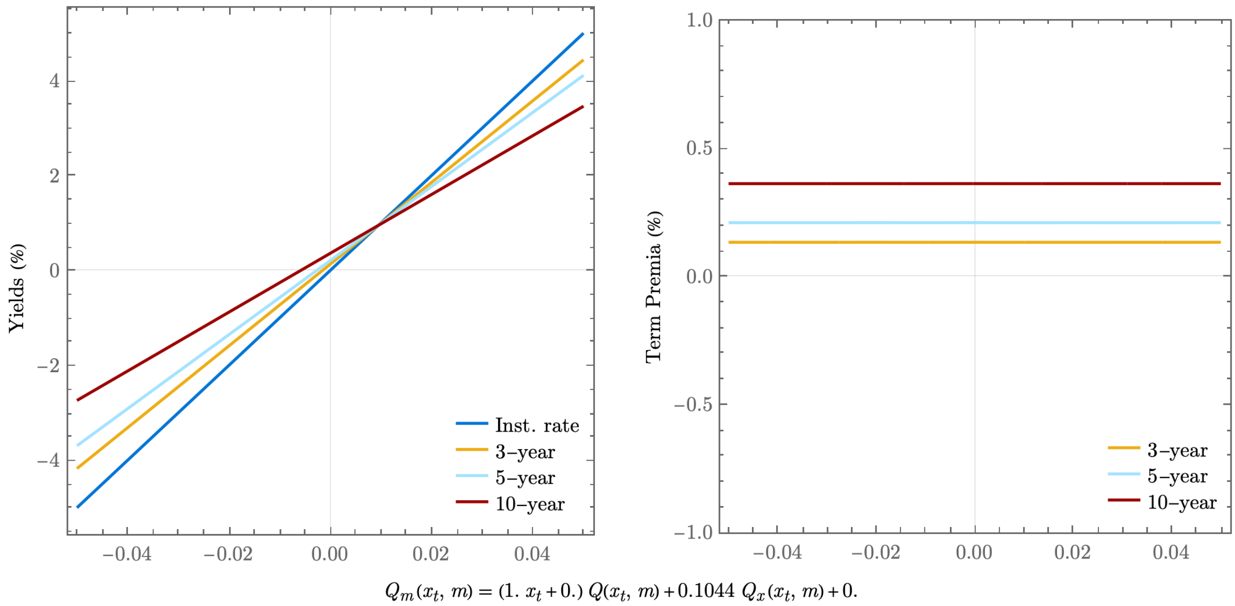


Figure 1.10: The left plots show the short-term rate, the five-year yield and the five-year risk-neutral yield as a function of CD. The right plot shows the term premia for different maturities as a function of the state variable.

- $A(x_t) = 0.0005 + 0.02x_t$. This means that now A changes with the state variable. The result is shown in Figure 1.11. Term premia follow the behaviour of A. The correspondence would not be so close if the short rate were a non-linear function of the state variable.

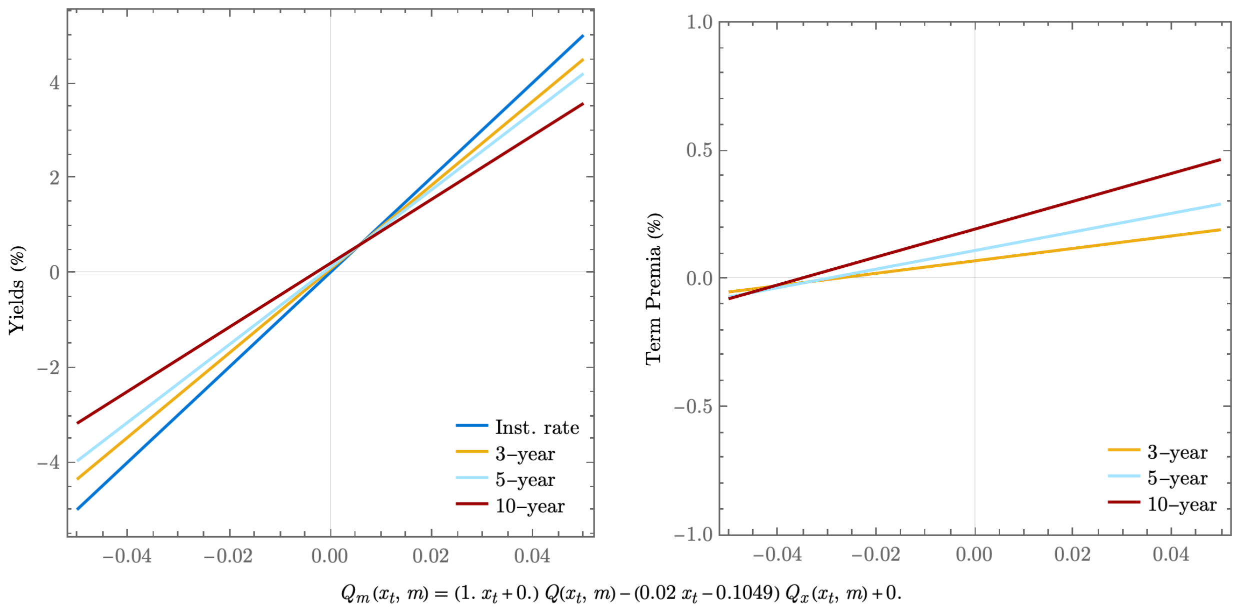


Figure 1.11: The left plots show the short-term rate, the five-year yield and the five-year risk-neutral yield as a function of CD. The right plot shows the term premia for different maturities as a function of the state variable.

- $\sigma_{xt} \neq 0$:

$$Q_m = r(x_t)Q - \log(\phi)x_tQ_x + \frac{\sigma_{xt}^2}{2}Q_{xx} = x_tQ - \log(0.9)x_tQ_x - \frac{0.03^2}{2}Q_{xx}$$

Here $A(x_t) = 0$. Thus, the effect of volatility can be seen. This case corresponds to a case where there is volatility of the short rate, but there is again no priced risk. So there is no risk premium. This can be seen on the right panel of Figure 1.12.[70] Nevertheless, the yields are not the same as in the deterministic case with steady-state reversion, as they do not intersect at the steady state. The long-term yields are pushed downwards, and, even though it might not be obvious, the effect of uncertainty increases more than linearly with maturity. This effect is due to the so-called convexity that is common in finance. In particular, the price of the long-term bond is a decreasing convex function of the short rate and this implies that lower interest rates have a higher effect on the price of the bond, especially for long maturities. Thus, given that there is variation and a chance for the short rate to reach lower levels, these will outweigh the high rates, and push long-

[70] The term premia do not look completely flat because the Monte-Carlo calculation has some uncertainty in the calculation.

term yields downward. Finally, this also means that a downward-sloping term structure does not necessarily imply negative term premia.

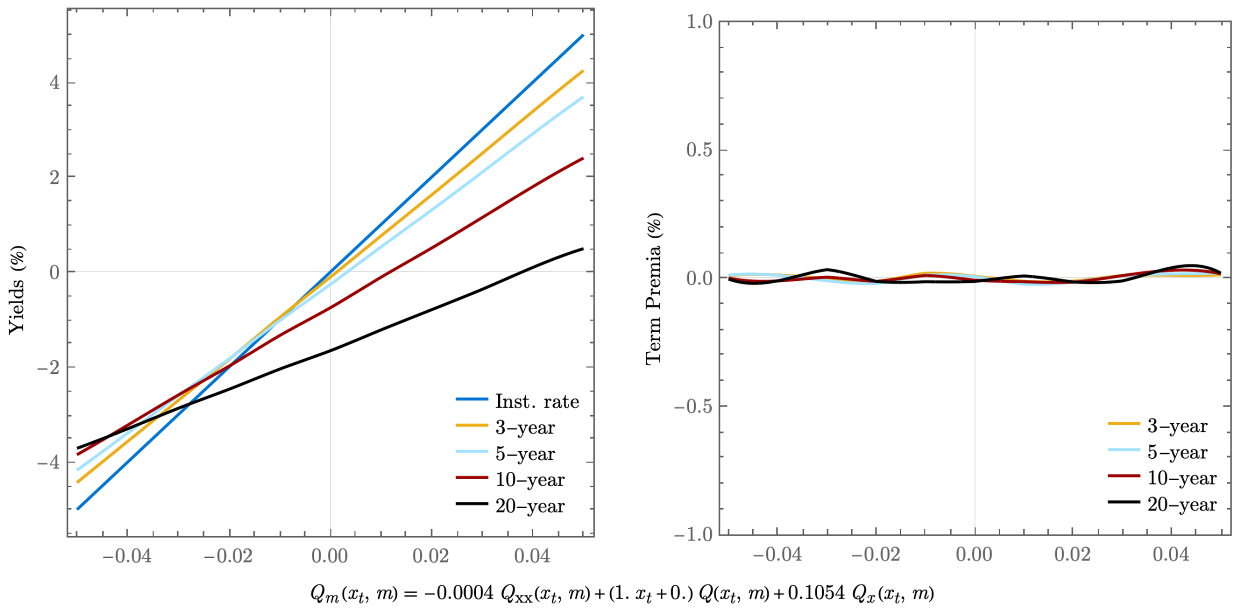


Figure 1.12: The left plots show the short-term rate, the five-year yield and the five-year risk-neutral yield as a function of considers. The right plot shows the term premia for different maturities as a function of the state variable.

- full case:

$$Q_m = r(x_t)Q - \log(\phi)x_tQ_x + \frac{\sigma_{xt}^2}{2}Q_{xx} = x_tQ - \Big(\log(0.9)x_t + 0.001\Big)Q_x - \frac{0.005^2}{2}Q_{xx}$$

This case contains all the components. Unlike the previous case, as can be seen in Figure 1.13, the yields seem to intersect close to the steady state. Thus, the yield curve would often be flat in this economy. However, term premia are positive. The yields are close to flat at the steady state because term premia and convexity largely cancel each other out.

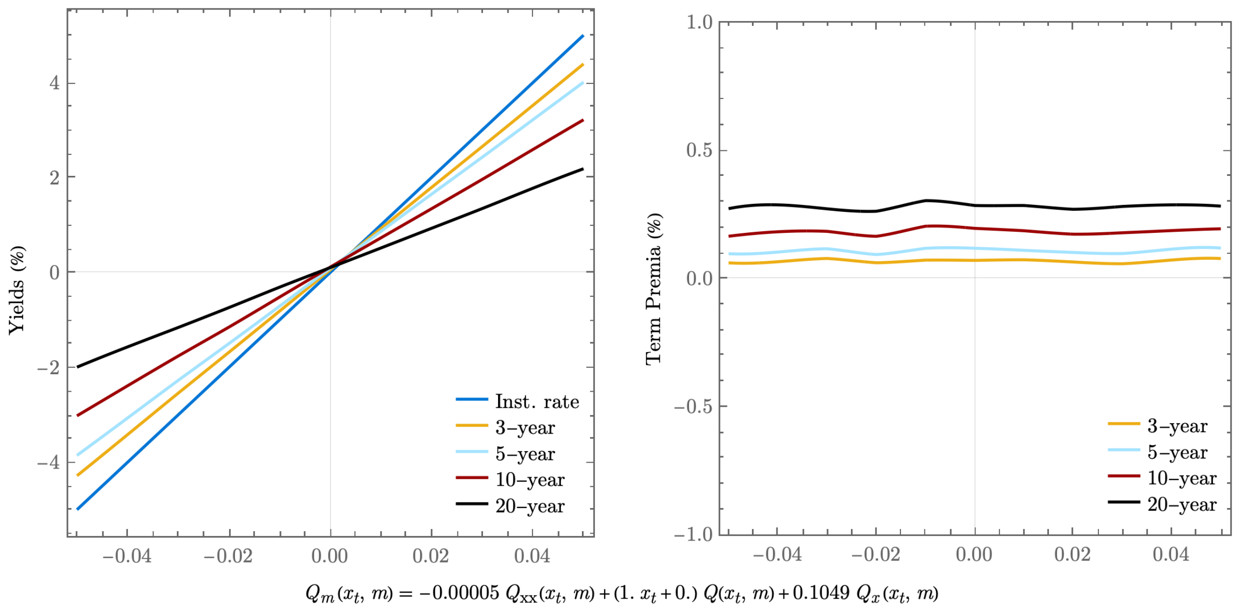


Figure 1.13: The left plots show the short-term rate, the five-year yield and the five-year risk-neutral yield as a function of CD. The right plot shows the term premia for different maturities as a function of the state variable.

1.D Calibration of the state variable volatility

As mentioned in Subsection 1.5.1, the paper aims to simultaneously match the variability of term premia and the variability of the short rate. I achieve this by calculating the range of the two-year TIPS security over the available sample in the Gürkaynak, Sack and Wright (2010) dataset.[71] I find a range of 7.27%.[72] I then simulate time series with twelve-year duration[73] for all the variations that I investigate. Based on these simulations I rank the range sizes and I aim for the tenth quantile to equal the range in the data. I do this for the models that are not able to produce highly variable term premia, to give these models the benefit of the doubt and the best chance to succeed. Namely, it is possible that the observed short rate volatility has been by chance relatively low and the underlying process is significantly more volatile. Thus, I want the model variations to be as volatile as possible to generate as large a time variability in term premia as possible. For

[71] Two years is the shortest maturity in the data.

[72] This could be overestimating the plausible range as the maximum was achieved during the financial crisis when the TIPS market was not behaving normally.

[73] This matches the length of the sample in Abrahams et al. (2016), but I should arguably change this to match the length of the sample in Gürkaynak, Sack and Wright (2010). In any case, the length of that sample is approximately 15 years.

the models that succeed in producing significantly time-varying term premia, I again make sure that the empirical volatility, as expressed by the observed range, falls within the model predictions. For each model variation, I show the value of the empirical range and the values of the model-implied tenth and ninetieth quantile ranges in the figures in Appendix 1.F.

1.E Term premia measures

1.E.1 Figure from Abrahams et al. (2016)

TIPS Term Premium

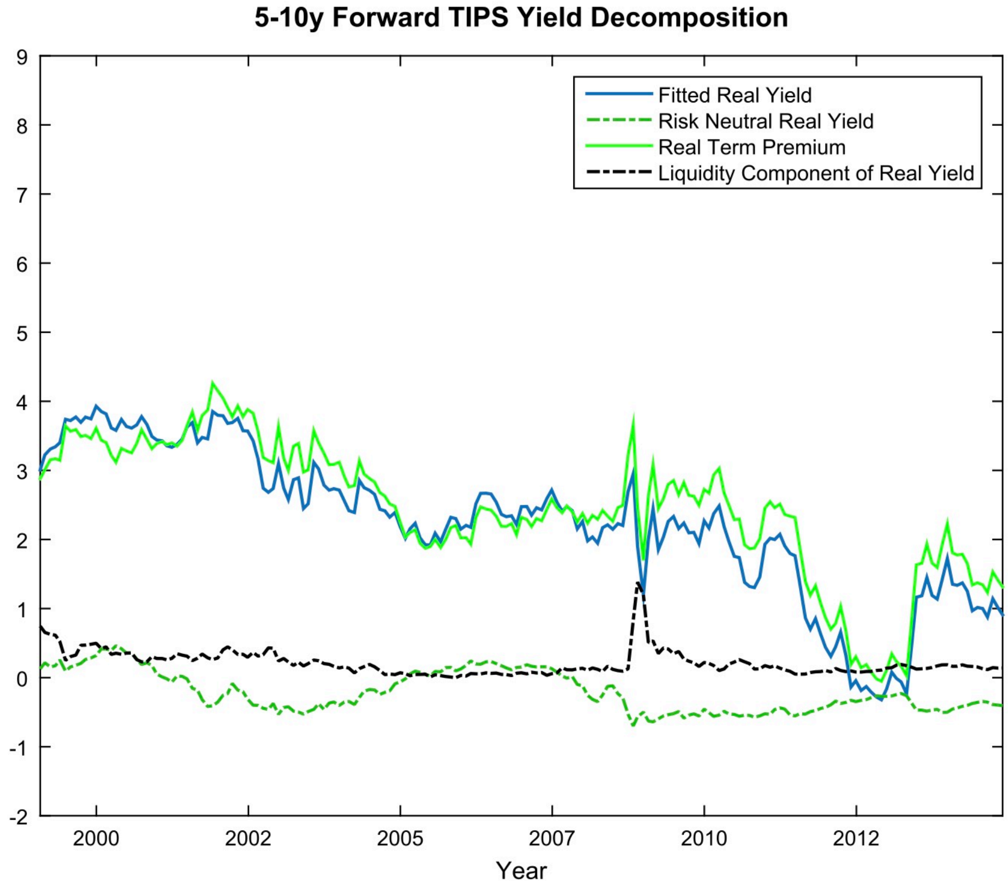


Figure 1.14: The figure shows the time series of the five-to-ten-year forward term premium along with its decomposition to the risk-neutral yield, the term premium and the liquidity premium. I also show the same decomposition in the figures in this paper as a function of the state variable.

1.E.2 Term premium based on d‘ Amico et Al. (2018)

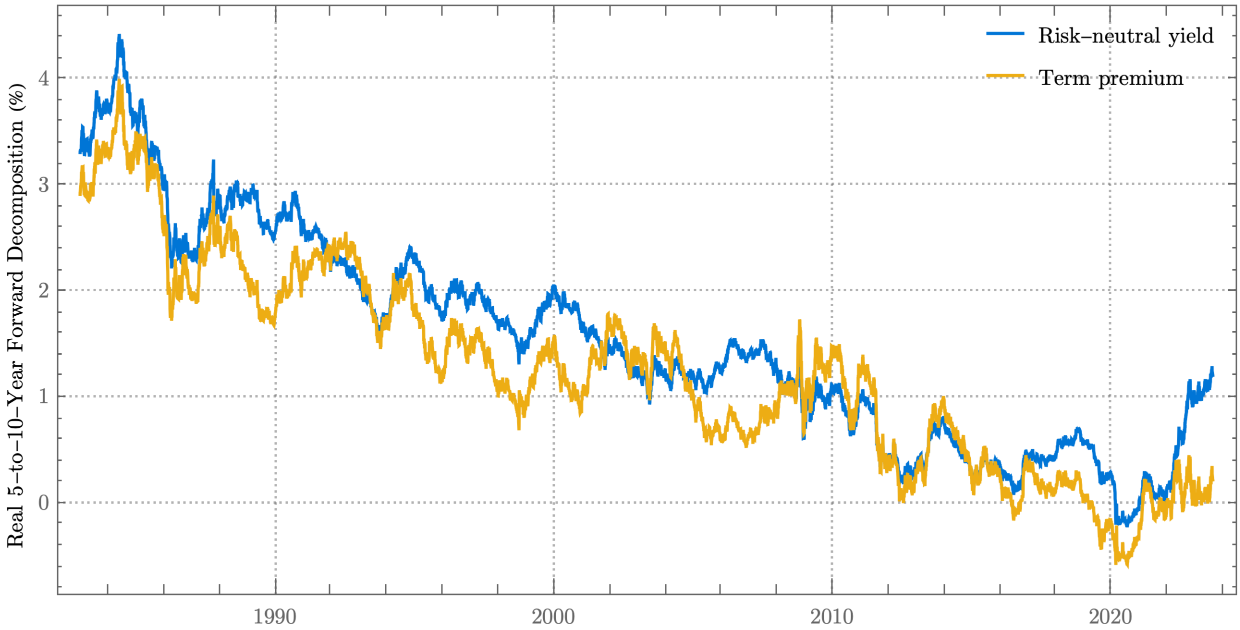


Figure 1.15: Time series of the forward 5-to-10-year term premium for the US. This is the same quantity as the solid green line in Figure 1.E.1 from Abrahams et al. (2016).
Data Source:
https://www.federalreserve.gov/econres/notes/feds-notes/tips-from-tips-update-and-discussions-20190521.html

1.F Yields and term premia in other model variations – Time-separable utility

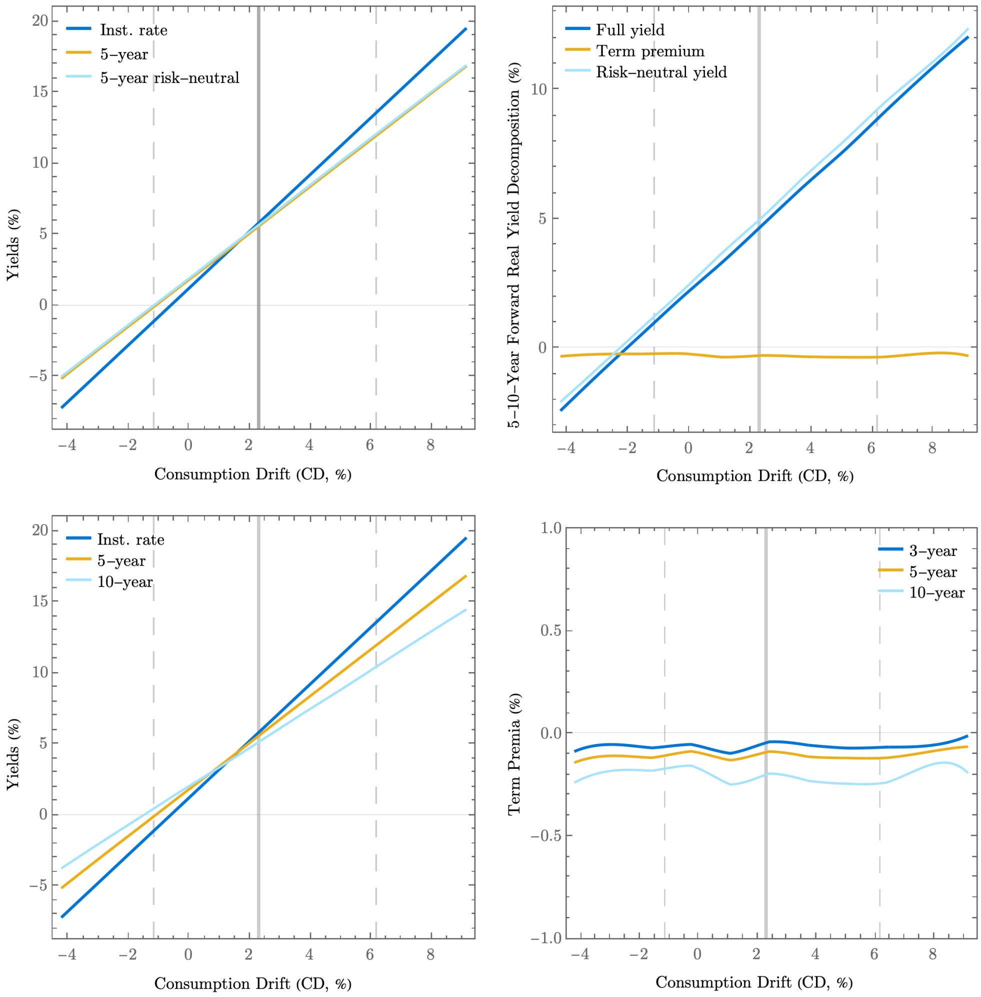
In this part, I present more plots for the variations discussed in the main paper, and I also present results for other model variations. These other variations should reinforce the conclusions in the main paper as a long series of calibrations is examined. The upper left and upper right plots are the same as in the main paper. The lower left plot shows the level of yields for different maturities as a function of the state variable. The lower right plot shows the level of the term premium for different maturities as a function of the state variable. Again each figure states the exact specification.

1.F.1 Model index

Names of all model variations are shown in Appendix 1.F. The abbreviations used here are time-varying (TV), consumption drift (CD), consumption volatility (CV), time-separable utility (TSU), recursive utility (RU), intertemporal elasticity of substitution (IES).

Model Variation Description	**Abbreviation**	**References**
TV CD with TSU.	TSU-CD	Figure 1.16
TV CD with TSU and high risk aversion.	TSU-CD-HRA	Figure 1.17
TV CD with TSU and low persistence.	TSU-CD-LP	Figure 1.18
TV CD with TSU and high correlation ρ_{cx}.	TSU-CD-HCor	Figure 1.19
TV CD with TSU and high impatience.	TSU-CD-HImp	Figure 1.20
TV and high CD with TSU.	TSU-HCD	Figure 1.21
TV CD with TSU and high CV.	TSU-CD-HCV	Figure 1.22
TV CV with TSU.	TSU-CV	Figure 1.23
TV CV with TSUand high risk aversion.	TSU-CV-HRA	Figure 1.24
TV CV with TSU and high CD.	TSU-CV-HCD	Figure 1.25
TV and HCV with TSU and positive correlation ρ_{cx}.	TSU-HCV	Figure 1.26
TV and HCV with TSU and negative correlation ρ_{cx}.	TSU-HCV-NCor	Figure 1.27
Both TV CD and CV, short-term rate decreasing in CV and ρ_{cx} positive.	TSU-Arb-DP	Figure 1.28
Both TV CD and CV, short-term rate increasing in CV and ρ_{cx} negative.	TSU-Arb-IN	Figure 1.29
Both TV CD and CV, short-term rate decreasing in CV and ρ_{cx} negative.	TSU-Arb-DN	Figure 1.30
Both TV CD and CV, short-term rate increasing in CV and ρ_{cx} positive.	TSU-Arb-IP	Figure 1.31
TV external habit with TSU.	TSU-Habit	Figure 1.32
TV external habit with TSU and low b.	TSU-Habit-Low.b	Figure 1.33
TV external habit with TSU and $b < 0$.	TSU-Habit-Neg.b	Figure 1.34
TV external habit with TSU with constant state variable volatility.	TSU-Habit-CSV	Figure 1.35
TV CD with RU.	RU-CD	Figure 1.36
TV CD with RU and high risk aversion.	RU-CD-HRA	Figure 1.37
TV CD with RU with high IES.	RU-CD-HIES	Figure 1.38
TV CD with RU with Low IES.	RU-CD-LIES	Figure 1.39
TV CD with RU with high ρ_{cx}.	RU-CD-HCor	Figure 1.40
TV CD with RU with ρ_{cx} negative.	RU-CD-NCor	Figure 1.41
TV and high CD with RU.	RU-HCD	Figure 1.42
TV CD with RU and high CV.	RU-CD-HCV	Figure 1.43
TV and heteroskedastic CD with RU and ρ_{cx} positive.	RU-CD-Heterosk-PCor	Figure 1.44
TV and heteroskedastic CD with RU and ρ_{cx} negative.	RU-CD-Heterosk-NCor	Figure 1.45
TV CV with RU.	RU-CV	Figure 1.46
TV CV with RU with high risk aversion.	RU-CV-HRA	Figure 1.47
TV CV with RU and high persistence IES.	RU-CV-HP	Figure 1.48
TV CV with RU and high IES.	RU-CV-HIES	Figure 1.49
TV CV with RU and low IES.	RU-CV-LIES	Figure 1.50
TV and HCV with RU and ρ_{cx} positive.	RU-HCV-PCor	Figure 1.51
TV and HCV with RU and ρ_{cx} negative.	RU-HCV-NCor	Figure 1.52

1.F.2 TSU-CD, calibration used in the main paper, Figure 1.3



γ=2, ρ=0.012, $\mu_{\mathrm{ct}}=x_t+0.0252$, σ_{ct}=0.02, ϕ=0.92, μ_{xt}=0, σ_{xt}=0.0126445, ρ_{cx}=0.5

$\mathrm{range}_{\mathrm{low}}(\mathrm{r}_{2,\mathrm{model}})$=0.10958, $\mathrm{range}_{\mathrm{high}}(\mathrm{r}_{2,\mathrm{model}})$=0.202976, $\mathrm{range}(\mathrm{r}_{2,\mathrm{data}})$=0.071748.

Figure 1.16: Time-varying CD with TSU.
The left plot shows the short-term rate, the five-year yield and the five-year risk-neutral yield as a function of consumption growth. The right plot shows the decomposition of the five-to-ten-year forward into the term premium and the risk-neutral components. The solid vertical line shows the level of the ergodic median, and the left and right dashed vertical lines show the median minimum and maximum value respectively over a series of simulations for 12 years. This means that half the simulated paths were below the right dashed line and half the simulated paths were above the left dashed line. The left and right boundaries are the 10th percentile of minimum values and the 90th percentile of maximum values from the same simulations. This means that 90% of simulated paths were above the left boundary and 90% of simulated paths were below the right boundary.

(variation overview)

1.F.3 TSU-CD-HRA, $\gamma = 8$

Term premia are a bit larger, but again negative and constant with respect to the state variable.

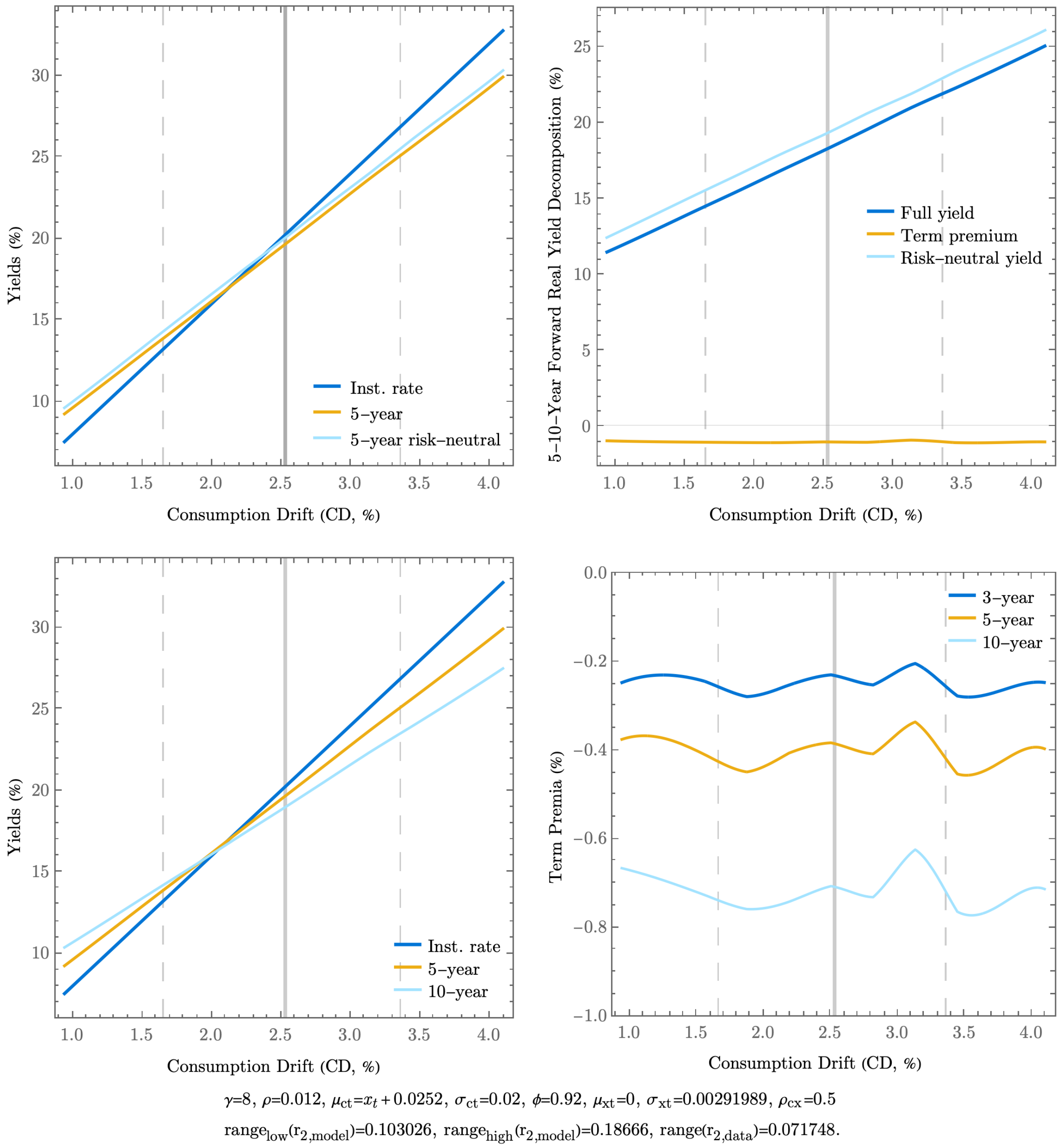
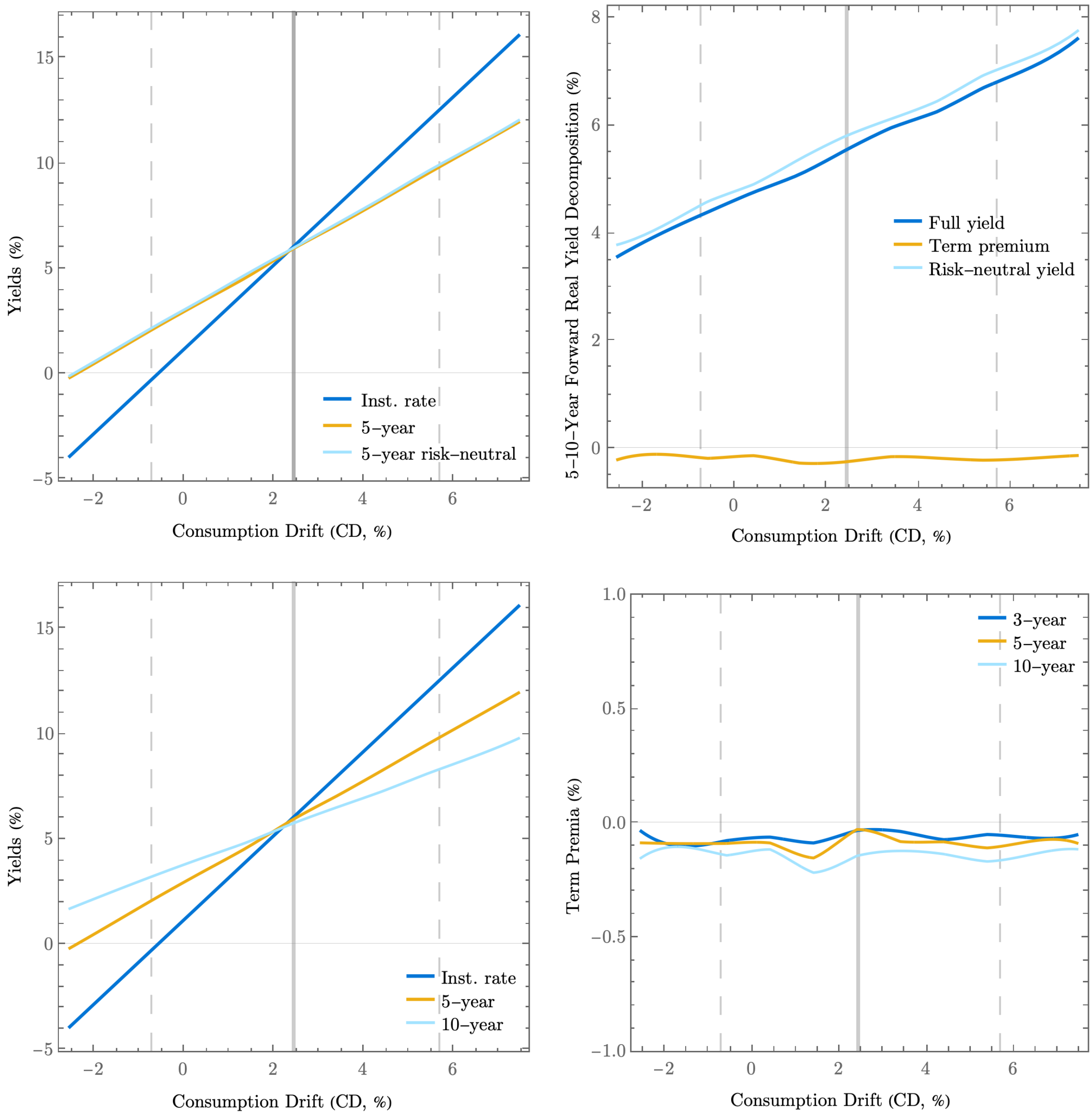


Figure 1.17: Time-varying CD with TSU and higher risk aversion.
See Figure 1.16 for more details about the plots.

(variation overview)

1.F.4 TSU-CD-LP, $\phi = 0.8$

Nothing changed in the term premia. There is a larger separation between yields similar to the corresponding mechanism in Appendix 1.C.



γ=2, ρ=0.012, $\mu_{\mathrm{ct}}=x_t+0.0252$, σ_{ct}=0.02, ϕ=0.8, μ_{xt}=0, σ_{xt}=0.0126445, ρ_{cx}=0.5

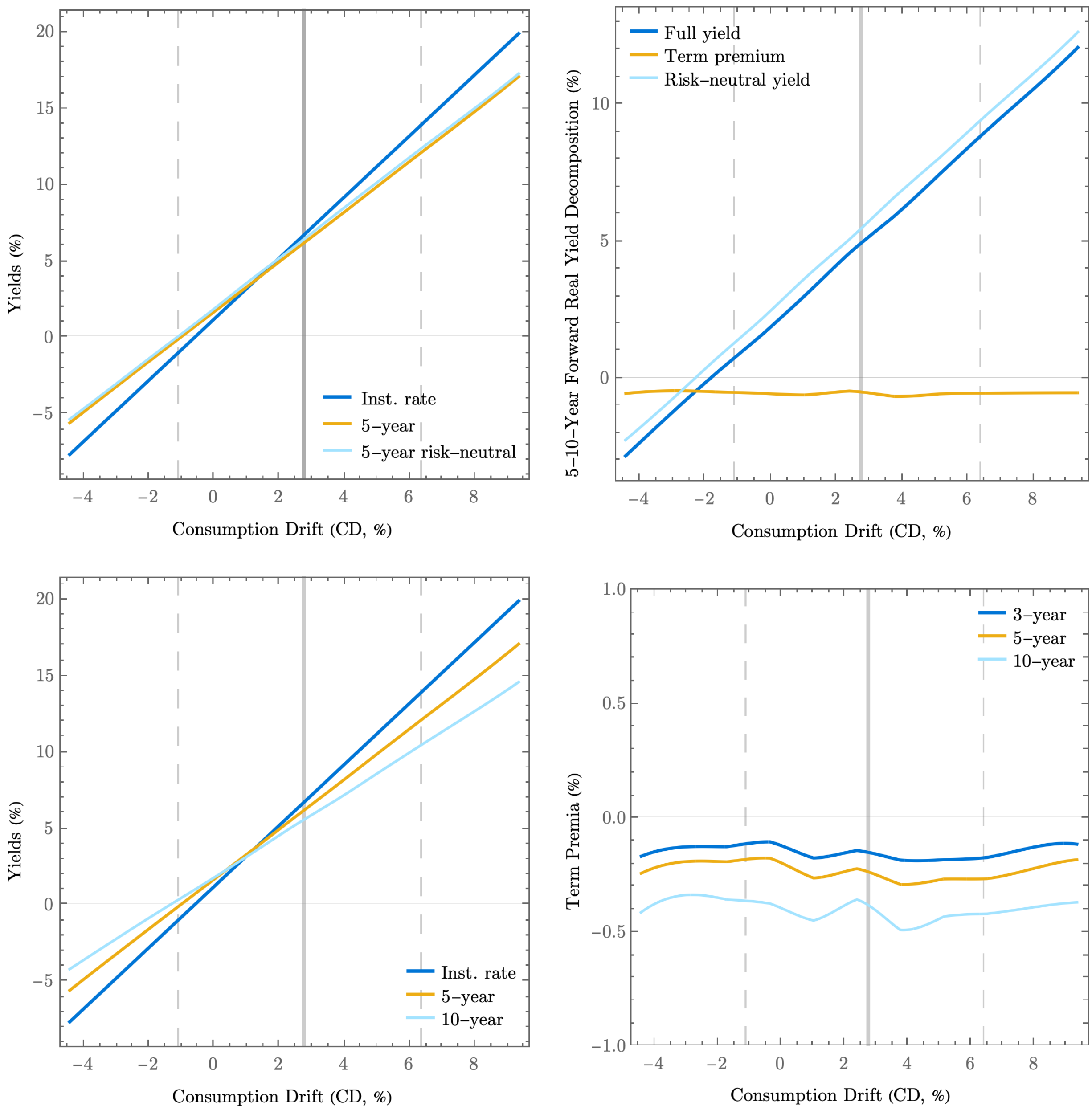
$\mathrm{range}_{\mathrm{low}}(\mathrm{r}_{2,\mathrm{model}})$=0.0987483, $\mathrm{range}_{\mathrm{high}}(\mathrm{r}_{2,\mathrm{model}})$=0.168236, $\mathrm{range}(\mathrm{r}_{2,\mathrm{data}})$=0.071748.

Figure 1.18: Time-varying CD with TSU and low persistence.
See Figure 1.16 for more details about the plots.

(variation overview)

1.F.5 TSU-CD-HCor, $\rho_{cx} = 1$

The term premia are larger in absolute value.



γ=2, ρ=0.012, $\mu_{\mathrm{ct}}=x_t+0.0252$, σ_{ct}=0.02, ϕ=0.92, μ_{xt}=0, σ_{xt}=0.0126445, ρ_{cx}=1

$\mathrm{range_{low}}(\mathrm{r_{2,model}})$=0.111377, $\mathrm{range_{high}}(\mathrm{r_{2,model}})$=0.203865, $\mathrm{range}(\mathrm{r_{2,data}})$=0.071748.

Figure 1.19: Time-varying CD with TSU and high correlation ρ_{cx}
See Figure 1.16 for more details about the plots.

(variation overview)

1.F.6 TSU-CD-HImp, $\rho = 0.05$

Yields move higher without any change in term premia.

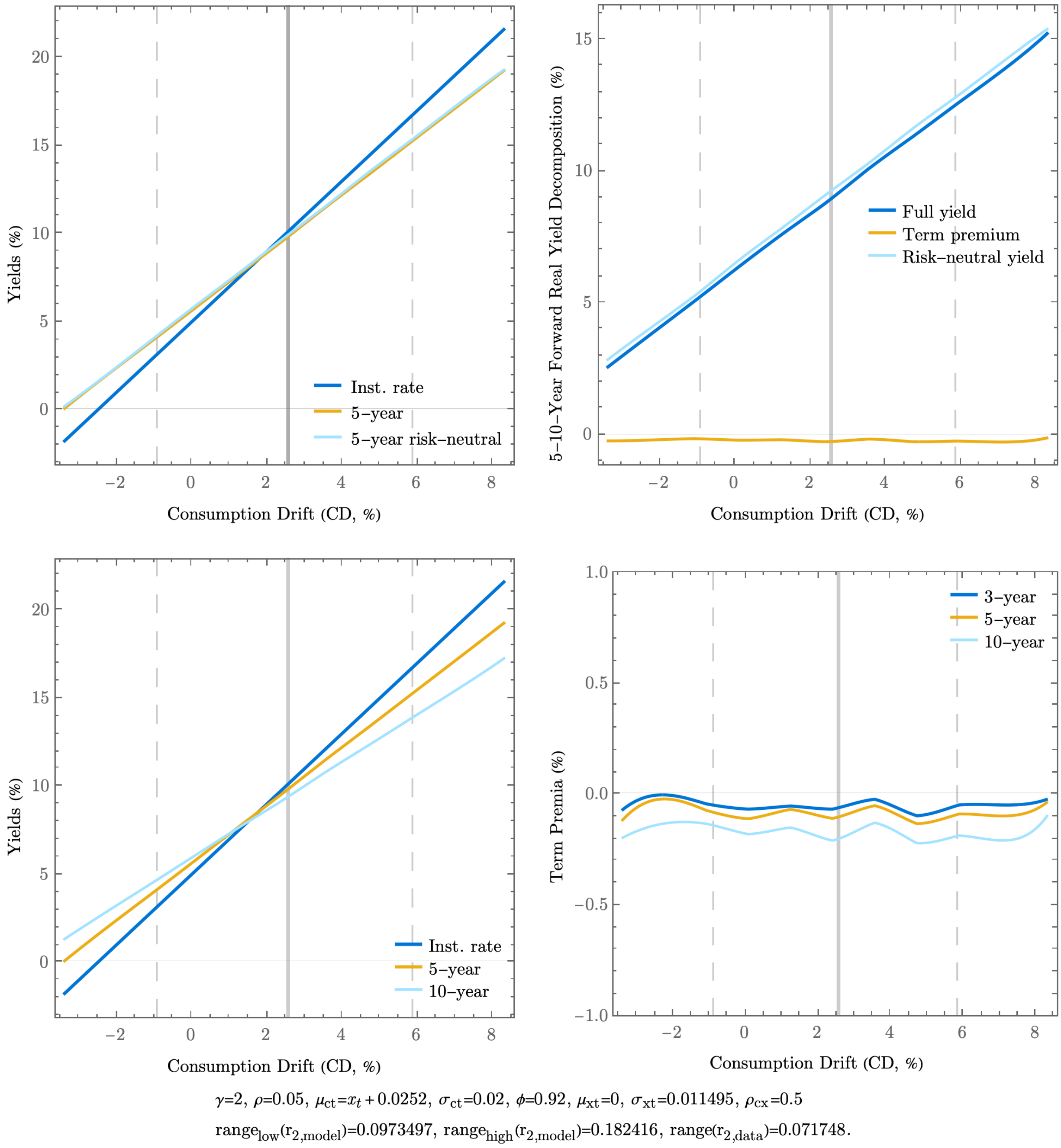


Figure 1.20: Time-varying CD with TSU and high impatience.
See Figure 1.16 for more details about the plots.

(variation overview)

1.F.7 TSU-HCD, $\mu_{c0} = 0.06$

Again, yields move higher without any change in term premia.

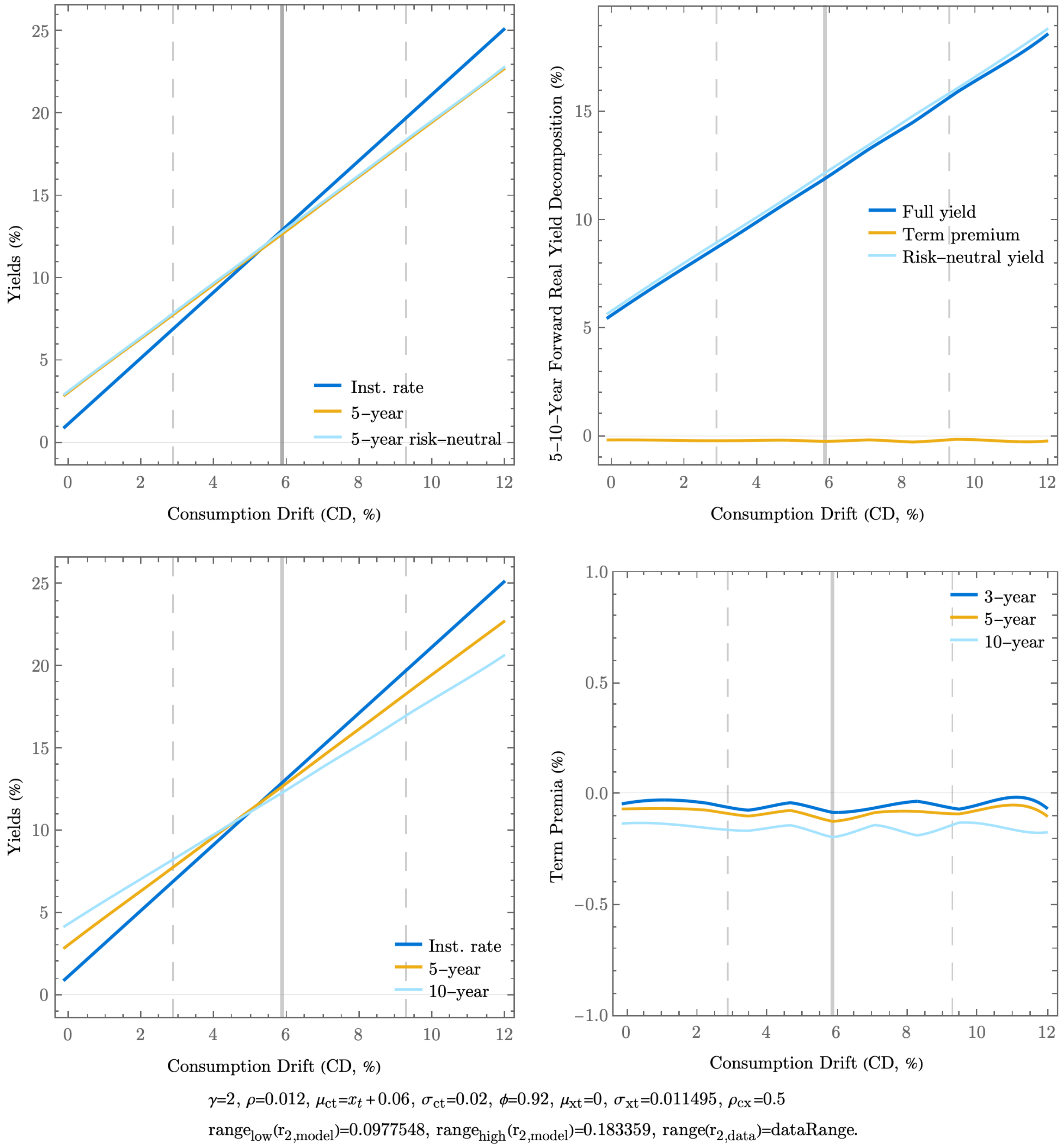


Figure 1.21: Time-varying and high CD with TSU.
See Figure 1.16 for more details about the plots.

(variation overview)

1.F.8 TSU-CD-HCV, $\sigma_{ct} = 0.16$

Yields move down and term premia increase in absolute value, but they are again constant.

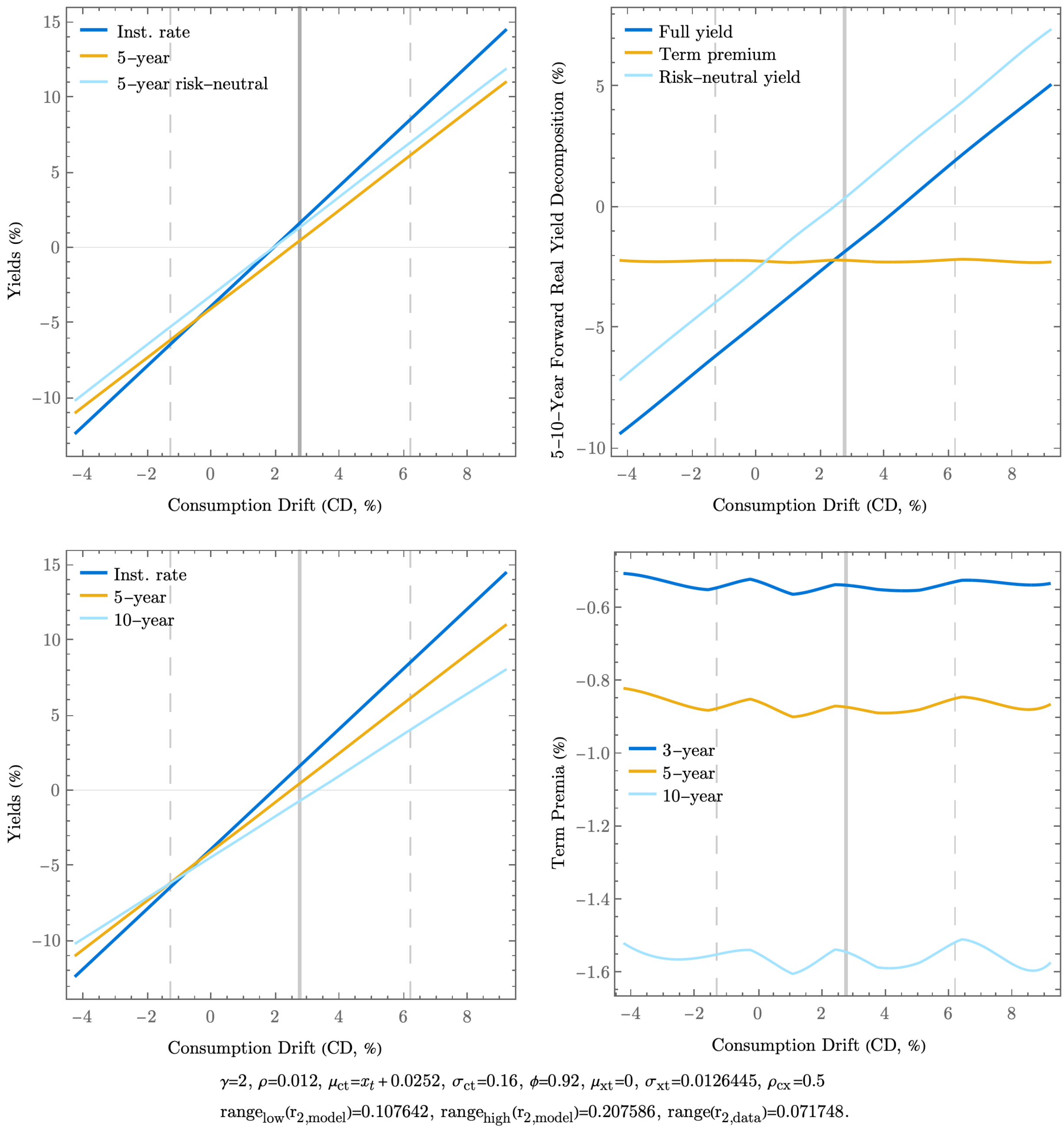
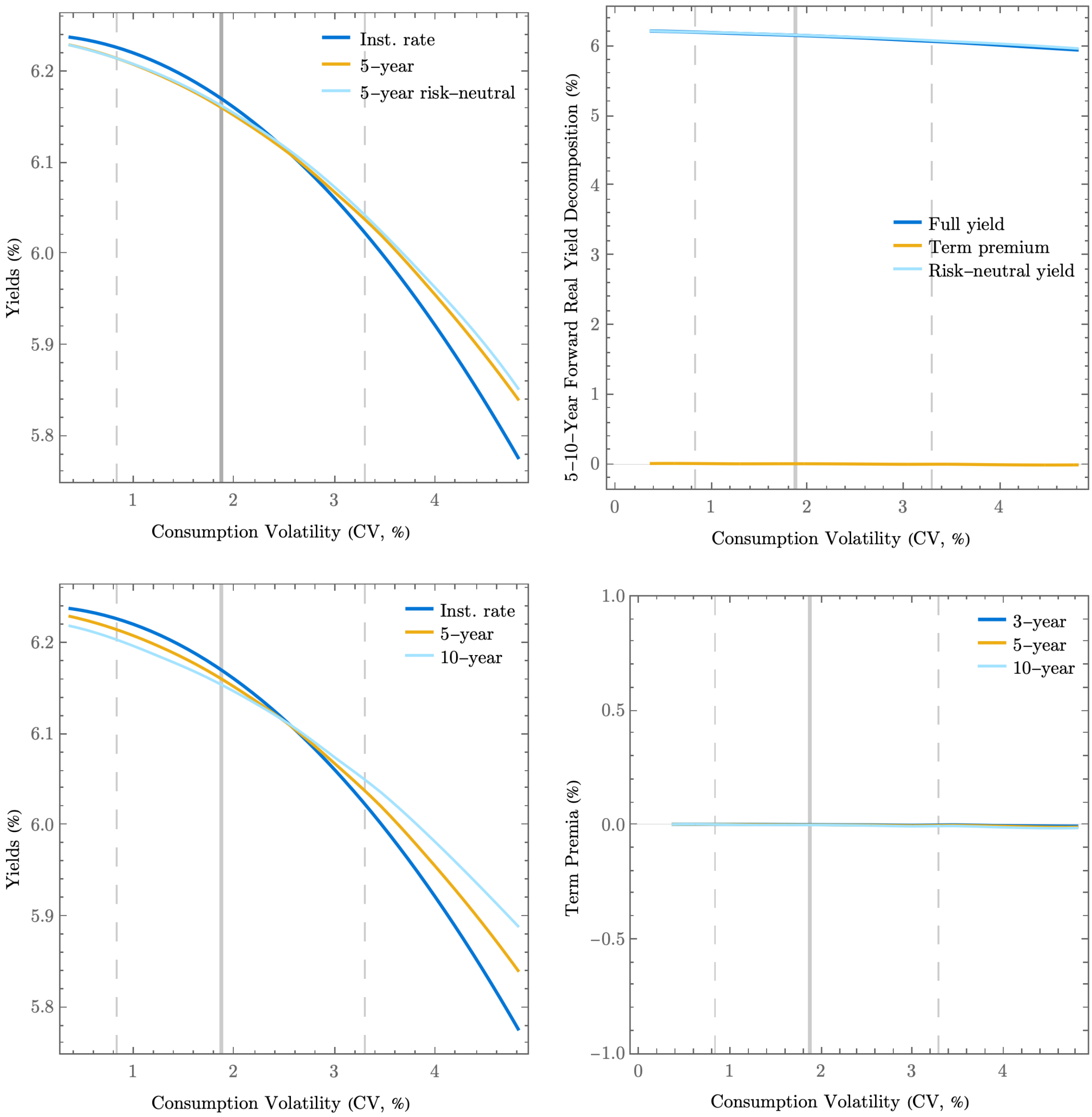


Figure 1.22: Time-varying CD with TSU and HCV
. See Figure 1.16 for more details about the plots.

(variation overview)

1.F.9 TSU-CV, Calibration used in main paper, Figure 1.3



γ=2, ρ=0.012, μ_{ct}=0.0252, σ_{ct}=0.02 x_t, ϕ=0.92, μ_{xt}=1, σ_{xt}=0.21 $\sqrt{x_t}$, ρ_{cx}=−0.5

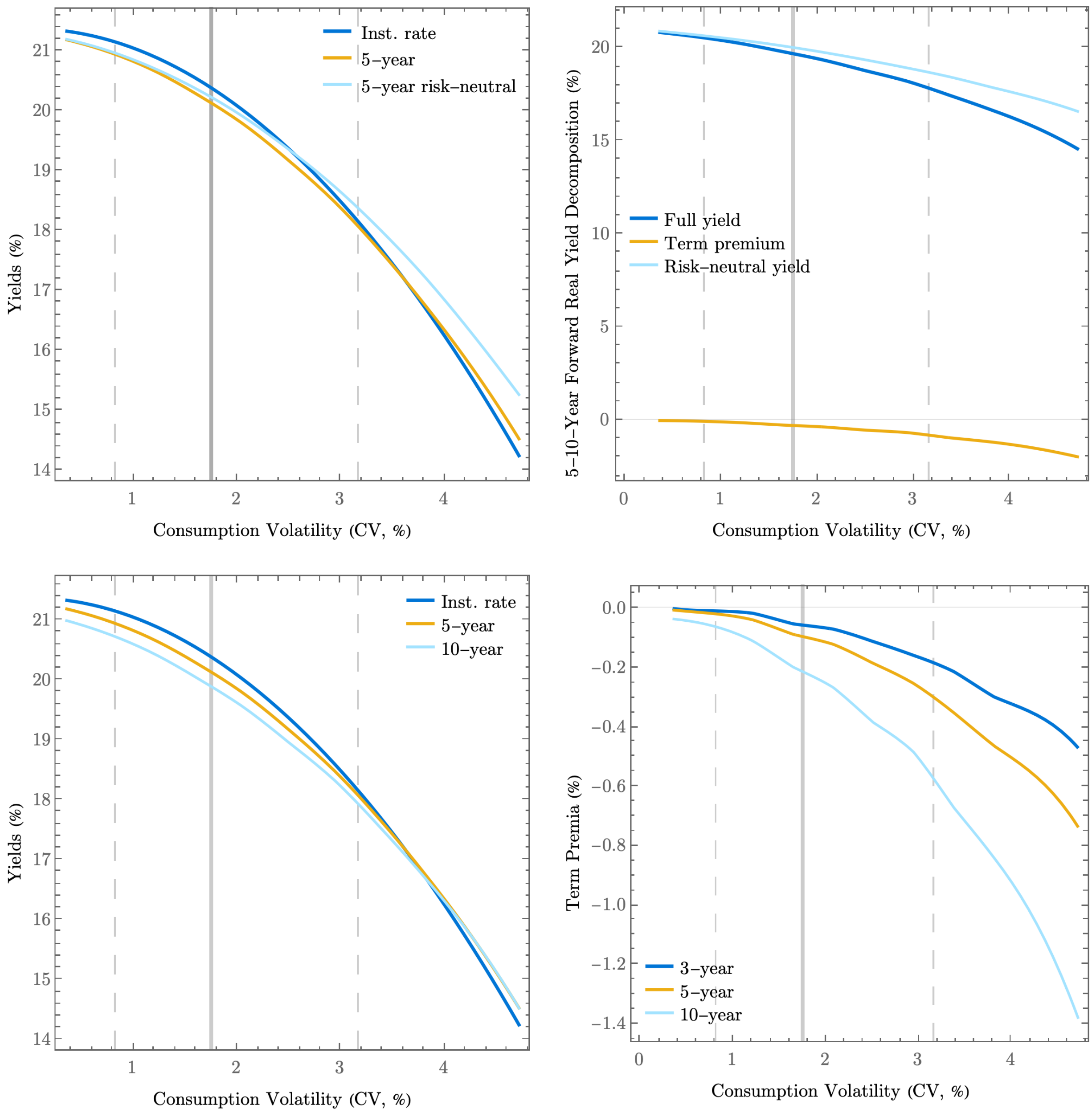
$\mathrm{range_{low}(r_{2,model})}$=0.00103855, $\mathrm{range_{high}(r_{2,model})}$=0.00408583, $\mathrm{range(r_{2,data})}$=0.071748.

Figure 1.23: Time-varying CV with TSU.
See Figure 1.16 for more details about the plots.

(variation overview)

1.F.10 TSU-CV-HRA, $\gamma = 8$

Term premia increased in absolute value but not enough and yields moved very high.



γ=8, ρ=0.012, μ_{ct}=0.0252, σ_{ct}=0.02 x_t, ϕ=0.92, μ_{xt}=1, σ_{xt}=0.21 $\sqrt{x_t}$, ρ_{cx}=−0.5

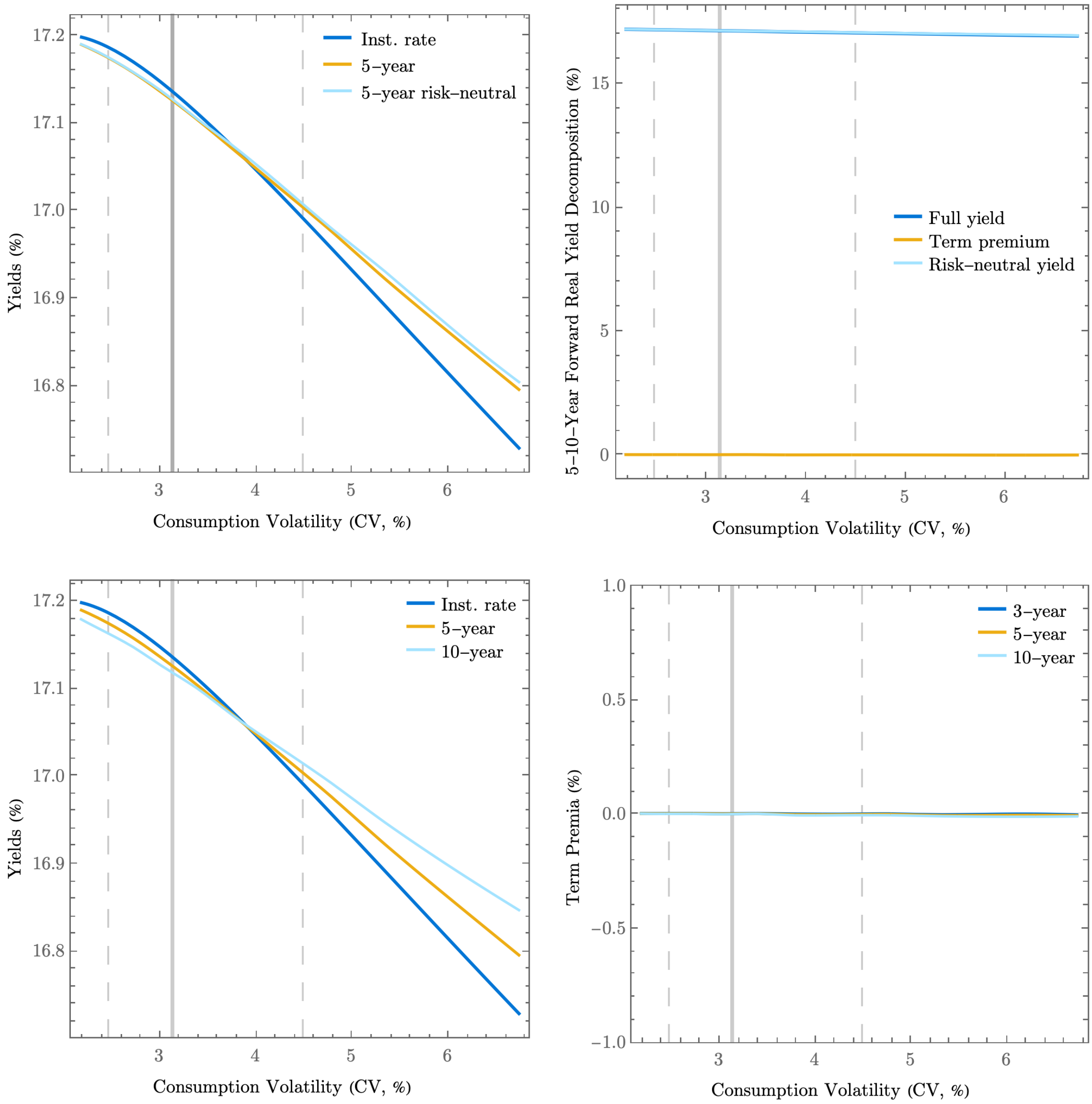
range$_{\mathrm{low}}$(r$_{2,\mathrm{model}}$)=0.0157495, range$_{\mathrm{high}}$(r$_{2,\mathrm{model}}$)=0.0684834, range(r$_{2,\mathrm{data}}$)=0.071748.

Figure 1.24: Time-varying CV with high risk aversion.
See Figure 1.16 for more details about the plots.

(variation overview)

1.F.11 TSU-CV-HCD, $\mu_c 0 = 0.08$

Term premia did not change but yields move implausibly high.



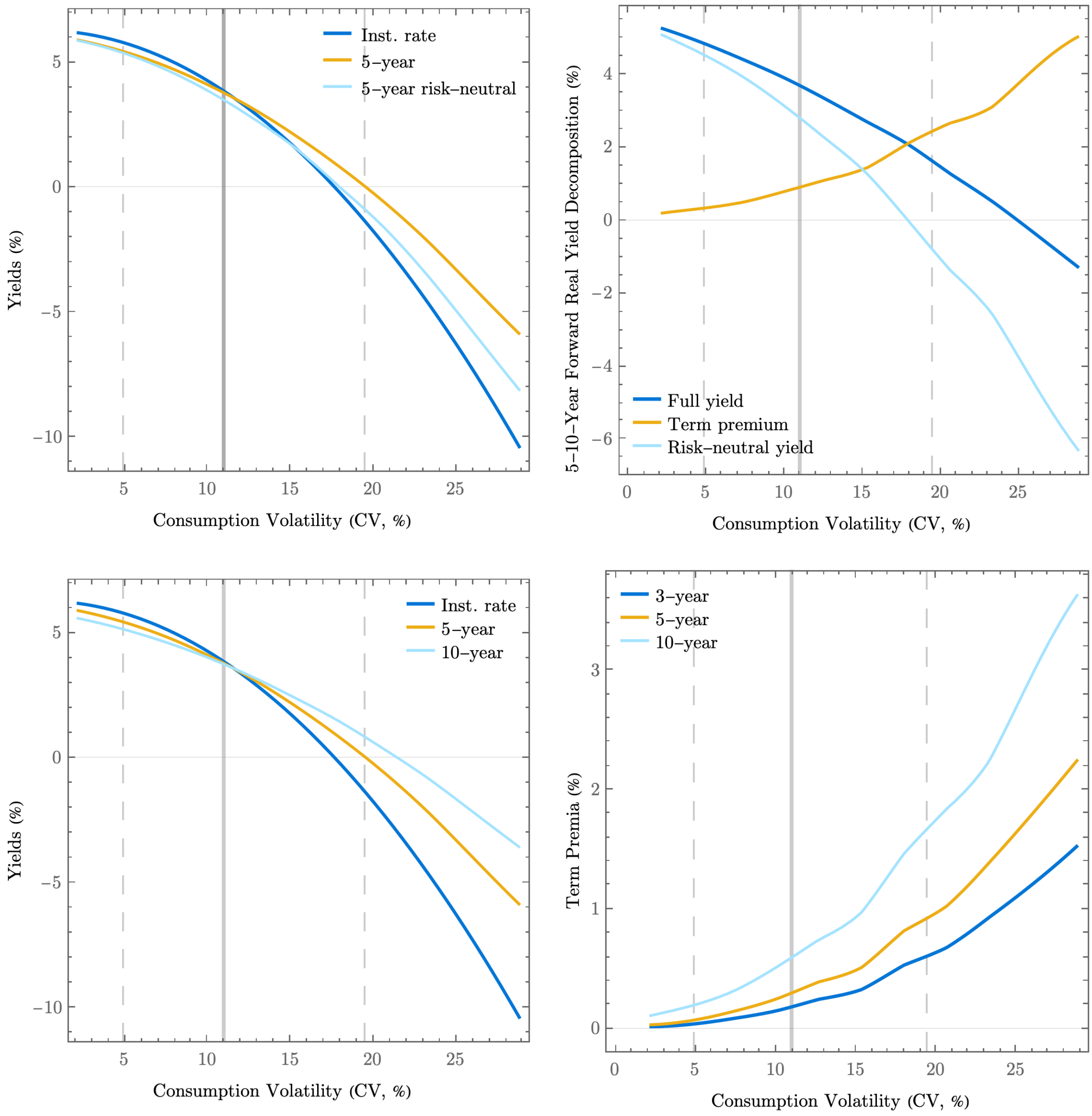
γ=2, ρ=0.012, μ_{ct}=0.08, σ_{ct}=0.02 x_t, ϕ=0.92, μ_{xt}=1, σ_{xt}=0.21 $\sqrt{x_t}$, ρ_{cx}=−0.5

range$_{low}$(r$_{2,model}$)=0.000984241, range$_{high}$(r$_{2,model}$)=0.00424656, range(r$_{2,data}$)=0.071748.

Figure 1.25: Time-varying CV with TSU and high CD
. See Figure 1.16 for more details about the plots.

(variation overview)

1.F.12 TSU-HCV, Calibration used in main paper, Figure 1.6



γ=2, ρ=0.012, μ_{ct}=0.0252, σ_{ct}=0.12 x_t, ϕ=0.92, μ_{xt}=1, σ_{xt}=0.21 $\sqrt{x_t}$, ρ_{cx}=0.5

range$_{\mathrm{low}}$($r_{2,\mathrm{model}}$)=0.0374496, range$_{\mathrm{high}}$($r_{2,\mathrm{model}}$)=0.149538, range($r_{2,\mathrm{data}}$)=0.071748.

Figure 1.26: Time-varying and HCV with TSU and positive ρ_{cx}. See Figure 1.16 for more details about the plots.

(variation overview)

1.F.13 TSU-HCV-NCor, $\rho_{cx} < 0$

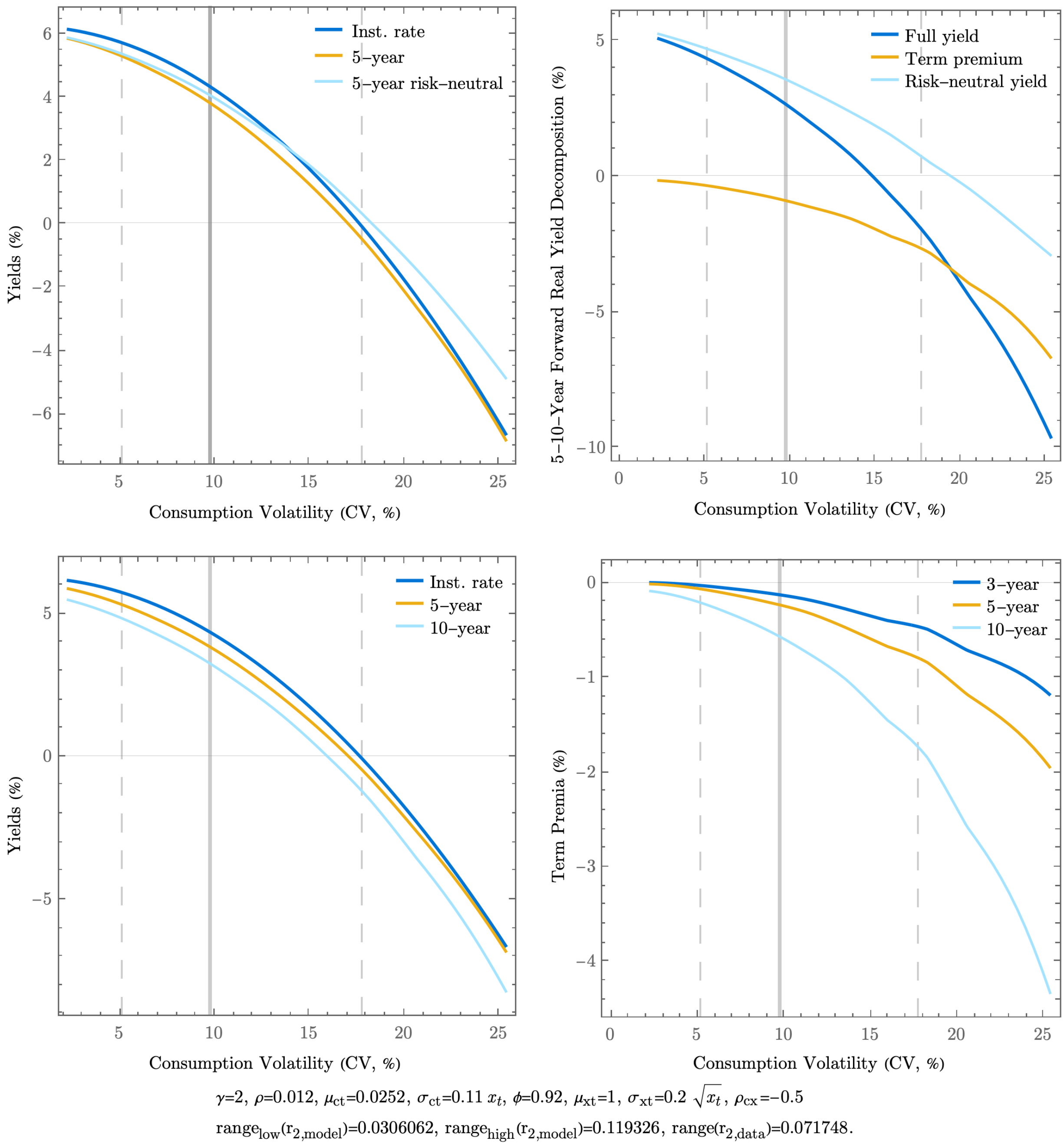
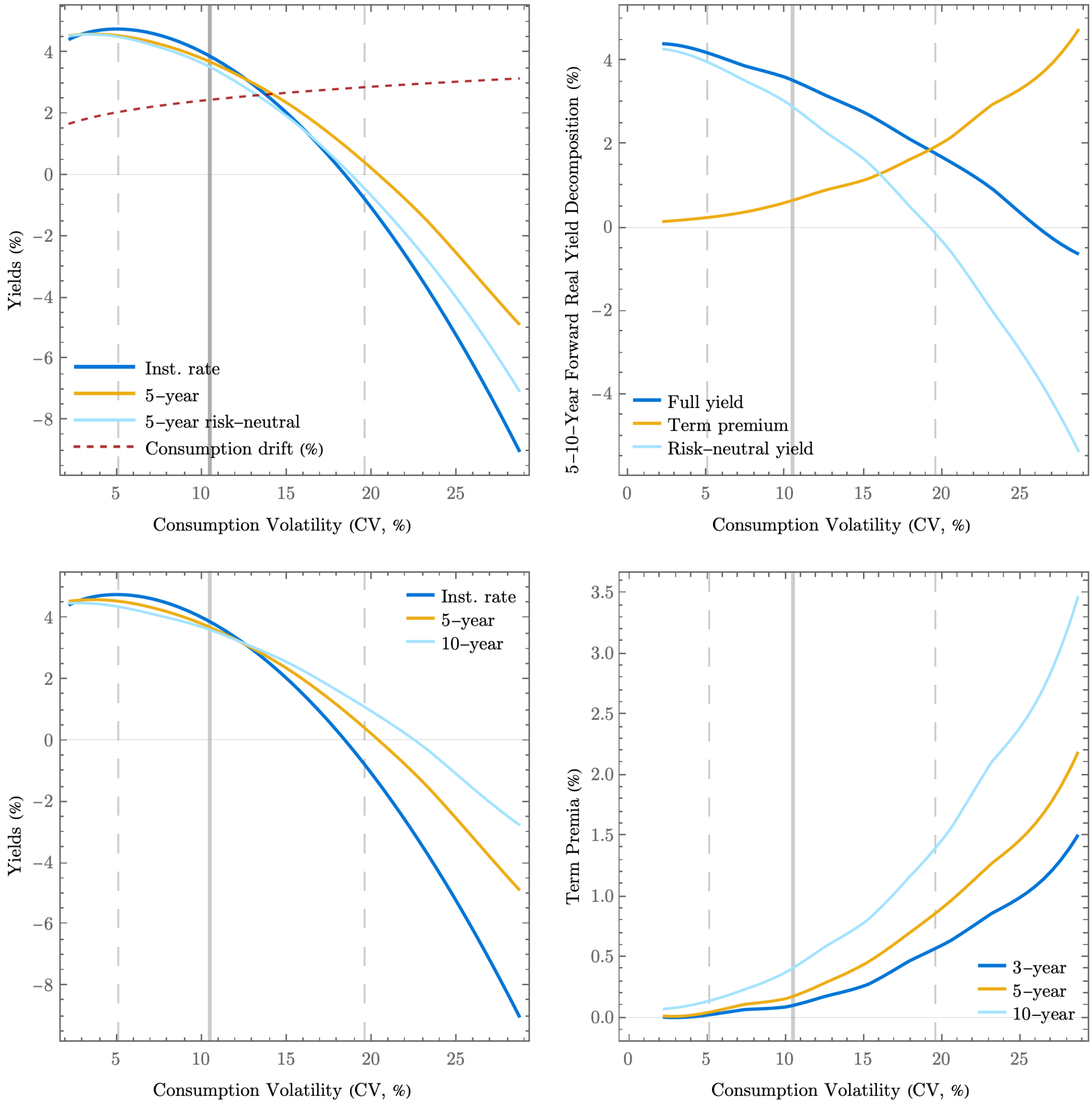


Figure 1.27: Time-varying CV with TSU and negative ρ_{cx}.
See Figure 1.16 for more details about the plots.

(variation overview)

1.F.14 Arb-DP, Calibration used in main paper, Figure 1.6



γ=2, ρ=0.012, μ_{ct}=0.0252 $\sqrt[4]{x_t}$, σ_{ct}=0.12 x_t, ϕ=0.92, μ_{xt}=1, σ_{xt}=0.21 $\sqrt{x_t}$, ρ_{cx}=0.5

range$_{\text{low}}$(r$_{2,\text{model}}$)=0.0222875, range$_{\text{high}}$(r$_{2,\text{model}}$)=0.132065, range(r$_{2,\text{data}}$)=0.071748.

Figure 1.28: Both time-varying CD and CV with TSU, short-term rate decreasing in CV and positive ρ_{cx}.
See Figure 1.6 for more details about the plots.

(variation overview)

1.F.15 Arb-IN, Calibration used in main paper, Figure 1.6

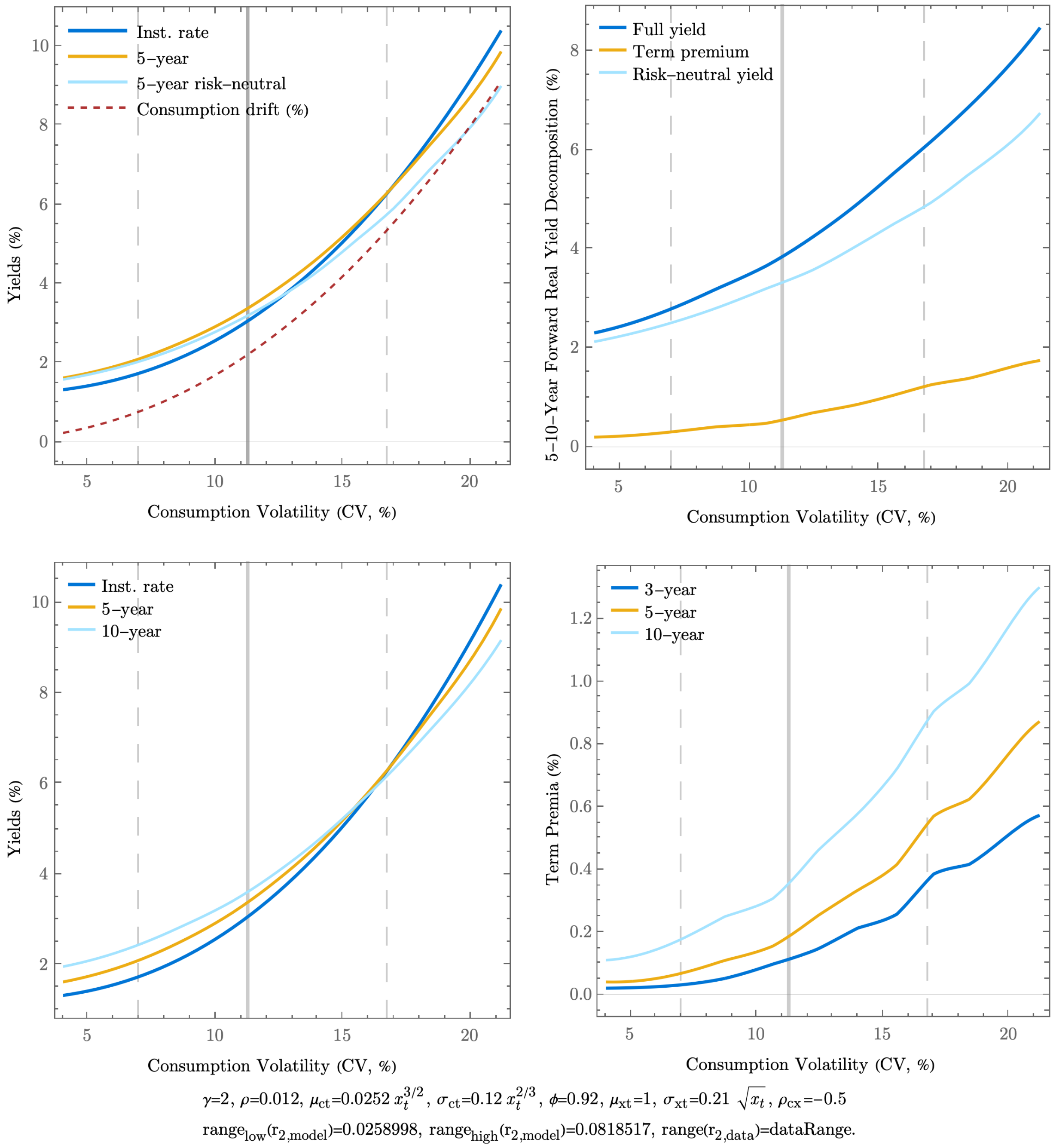
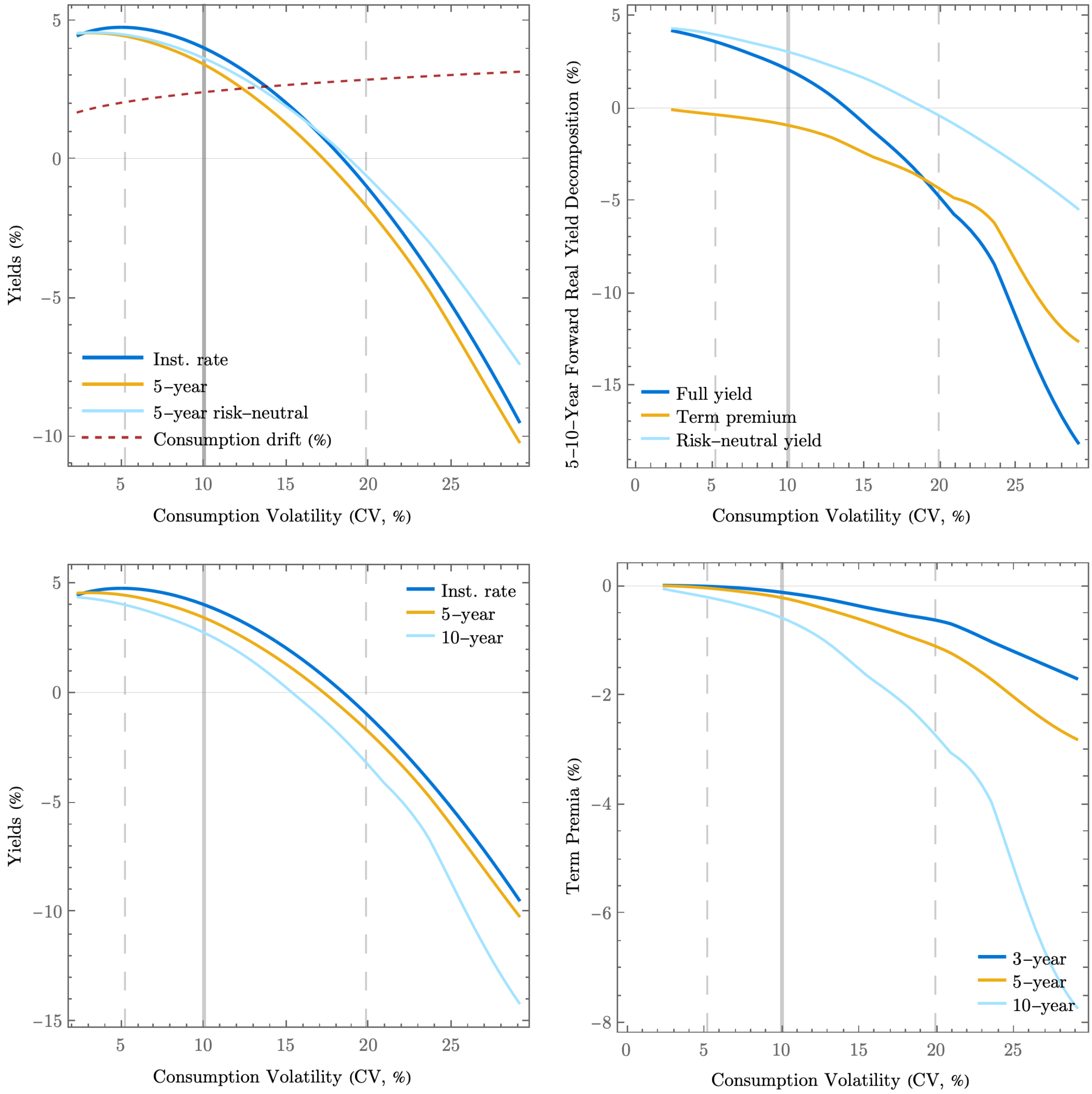


Figure 1.29: Both time-varying CD and CV with TSU, short-term rate increasing in CV and negative ρ_{cx}.
See Figure 1.6 for more details about the plots.

(variation overview)

1.F.16 Arb-DN



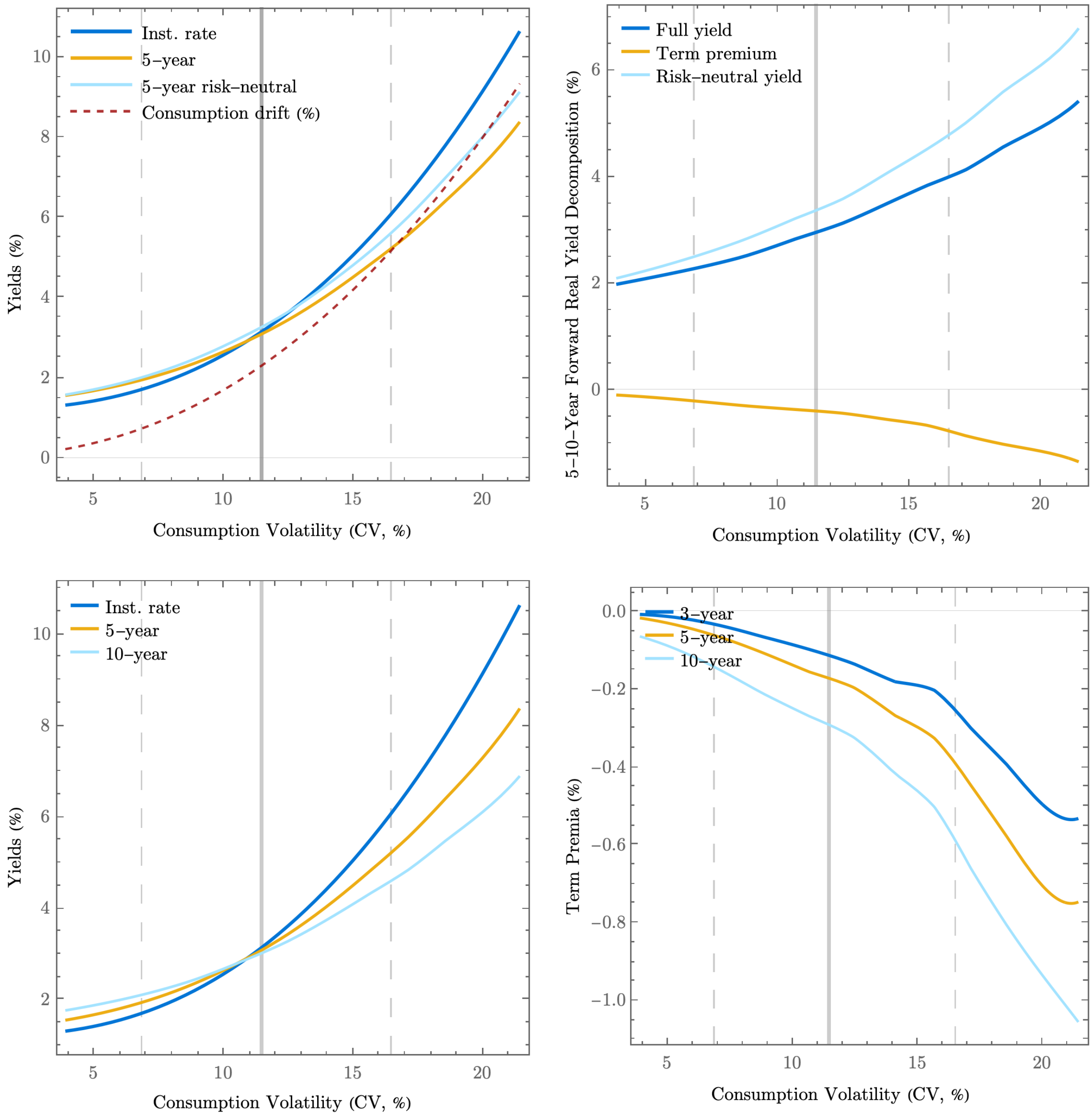
γ=2, ρ=0.012, μ_{ct}=0.0252 $\sqrt[4]{x_t}$, σ_{ct}=0.12 x_t, ϕ=0.92, μ_{xt}=1, σ_{xt}=0.21 $\sqrt{x_t}$, ρ_{cx}=−0.5

$\mathrm{range}_{\mathrm{low}}(\mathrm{r}_{2,\mathrm{model}})$=0.0224998, $\mathrm{range}_{\mathrm{high}}(\mathrm{r}_{2,\mathrm{model}})$=0.134553, $\mathrm{range}(\mathrm{r}_{2,\mathrm{data}})$=0.071748.

Figure 1.30: Both time-varying CD and CV with TSU, short-term rate decreasing in CV and negative ρ_{cx}.

(variation overview)

1.F.17 Arb-IP



γ=2, ρ=0.012, μ_{ct}=0.0252 $x_t^{3/2}$, σ_{ct}=0.12 $x_t^{2/3}$, ϕ=0.92, μ_{xt}=1, σ_{xt}=0.21 $\sqrt{x_t}$, ρ_{cx}=0.5

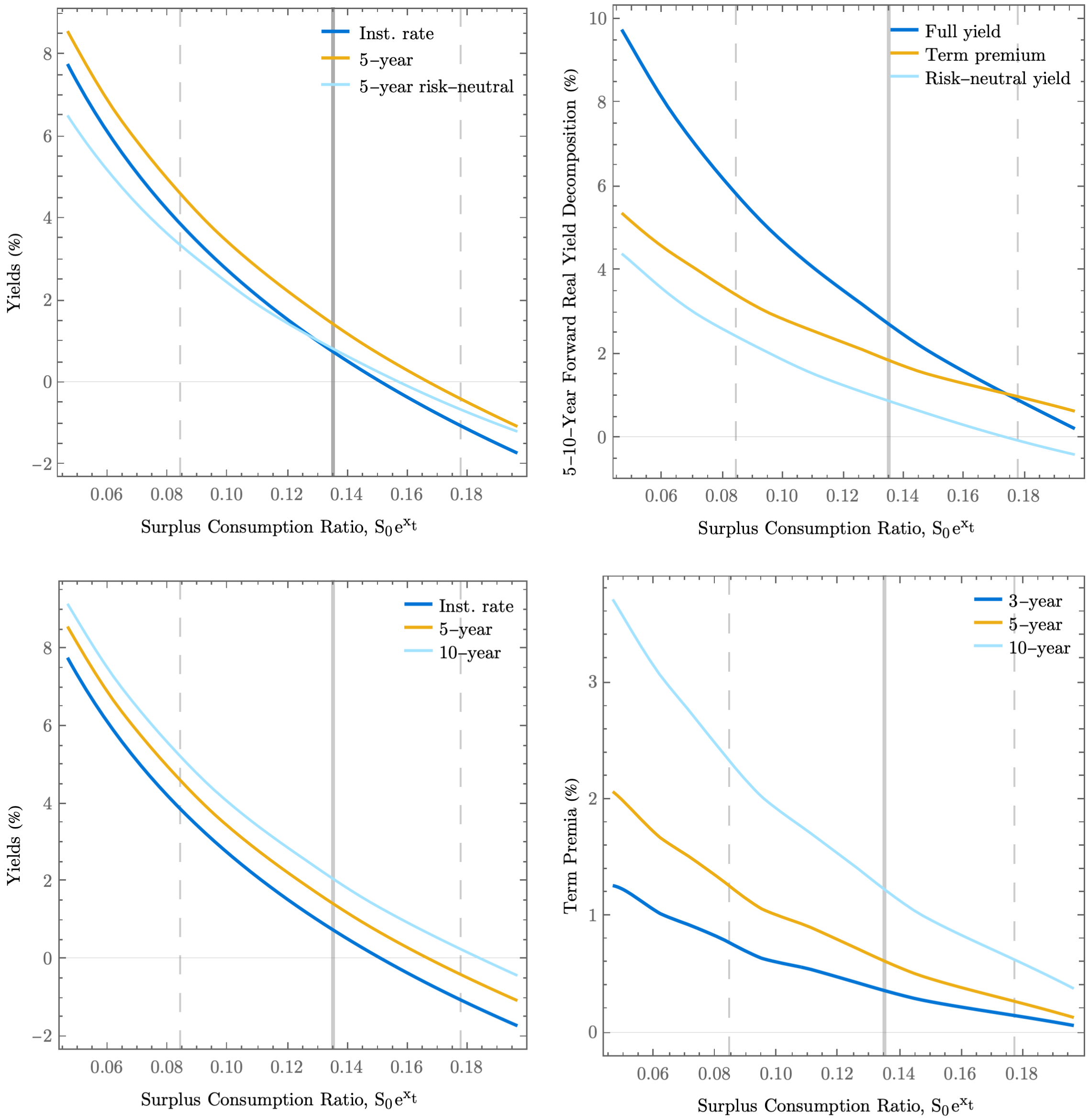
$\mathrm{range}_{\mathrm{low}}(\mathrm{r}_{2,\mathrm{model}})$=0.0263095, $\mathrm{range}_{\mathrm{high}}(\mathrm{r}_{2,\mathrm{model}})$=0.0794248, $\mathrm{range}(\mathrm{r}_{2,\mathrm{data}})$=0.071748.

Figure 1.31: Both time-varying CD and CV with TSU, short-term rate increasing in CV and positive ρ_{cx}.
See Figure 1.6 for more details about the plots.

(variation overview)

1.F.18 TSU-Habit

1.F.19 Calibration used in main paper, Figure 1.3



γ=2, ρ=0.012, μ_{ct}=0.0252, σ_{ct}=0.02, ϕ=0.92, μ_{xt}=0, σ_{xt}=0.02 $\mathrm{If}\left[x_t > 0.492061, 0, \frac{\sqrt{1-2\,x_t}}{0.02\sqrt{\frac{2}{-\frac{0.066}{2}-\log(0.92)}}}-1\right]$, ρ_{cx}=1

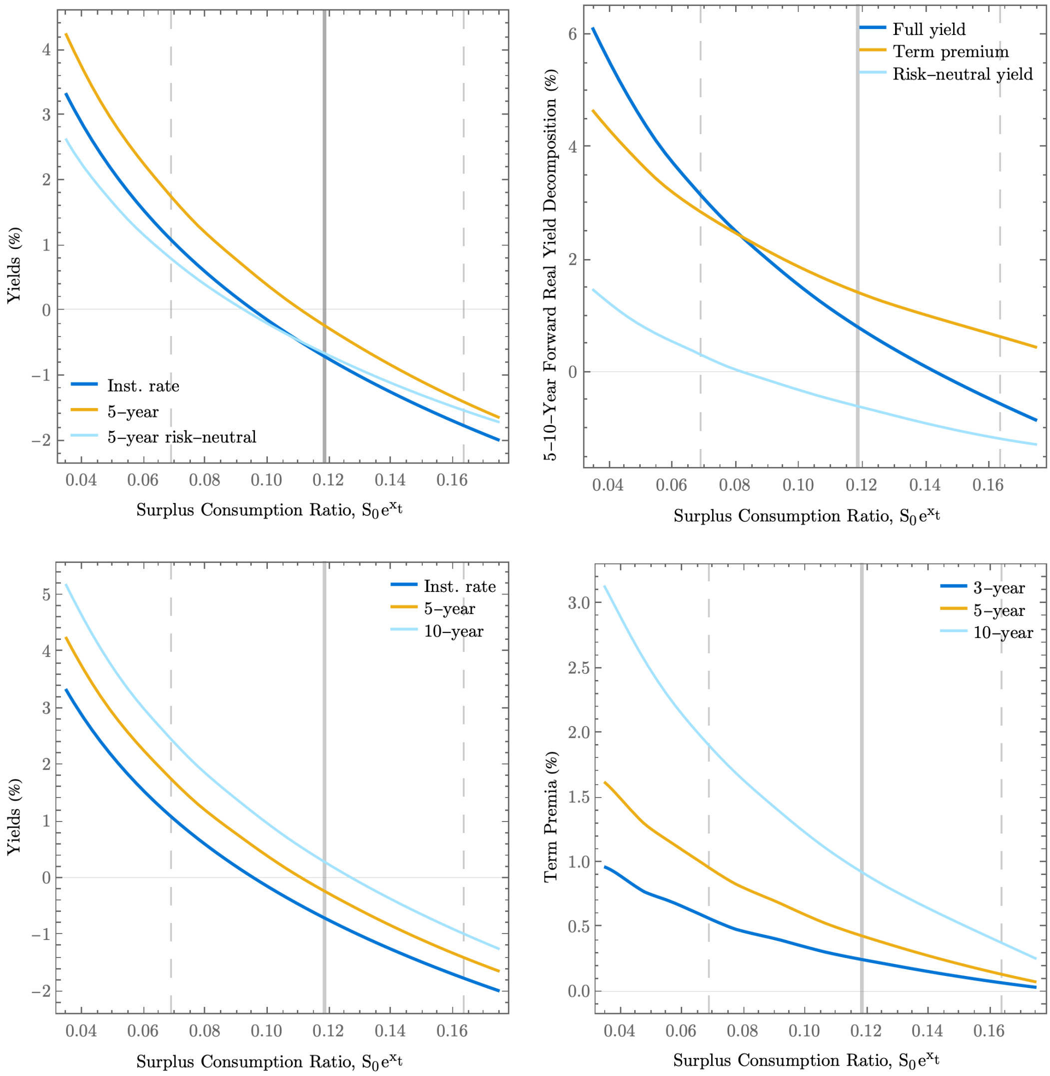
$\mathrm{range_{low}}(\mathrm{r}_{2,\mathrm{model}})$=0.0323706, $\mathrm{range_{high}}(\mathrm{r}_{2,\mathrm{model}})$=0.0831215, $\mathrm{range}(\mathrm{r}_{2,\mathrm{data}})$=dataRange.

Figure 1.32: Time-varying external habit with TSU.
See Figure 1.16 for more details about the plots.

(variation overview)

1.F.20 TSU-Habit-Low.b, $b = 0.033$

Term premia did not change but yields became flatter. This is noteworthy because in Abrahams et al. (2016) forward term premia are big while the forward risk-neutral yields are small in absolute value.



$\gamma=2$, $\rho=0.012$, $\mu_{ct}=0.0252$, $\sigma_{ct}=0.02$, $\phi=0.92$, $\mu_{xt}=0$, $\sigma_{xt}=0.02\,\mathrm{If}\left[x_t > 0.494019, 0, \frac{\sqrt{1-2\,x_t}}{0.02\sqrt{\frac{2}{-\frac{0.033}{2}-\log(0.92)}}}-1\right]$, $\rho_{cx}=1$

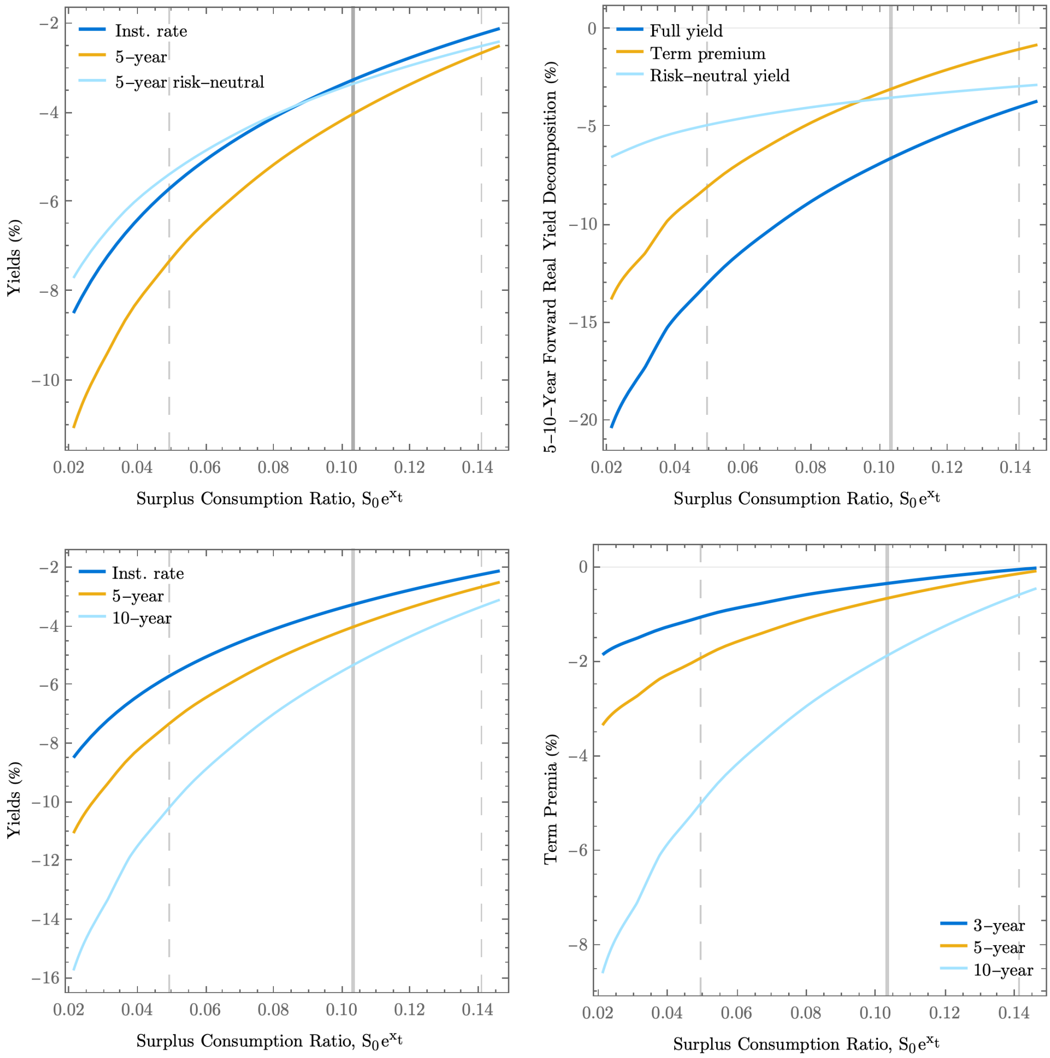
$\mathrm{range}_{\mathrm{low}}(r_{2,\mathrm{model}})=0.0181377$, $\mathrm{range}_{\mathrm{high}}(r_{2,\mathrm{model}})=0.0475884$, $\mathrm{range}(r_{2,\mathrm{data}})=\mathrm{dataRange}$.

Figure 1.33: Time-varying external habit with TSU.
See Figure 1.16 for more details about the plots.

(variation overview)

1.F.21 TSU-Habit-Neg.b, $b = -0.033$

The short-term rate is now pro-cyclical and term premia are negative.



γ=2, ρ=0.012, μ_{ct}=0.0252, σ_{ct}=0.02, ϕ=0.92, μ_{xt}=0, σ_{xt}=0.02 If$\left[x_t > 0.495995,\ 0,\ \frac{\sqrt{1-2\,x_t}}{0.02\sqrt{\frac{2}{--\frac{0.033}{2}-\log(0.92)}}} - 1\right]$, ρ_{cx}=1

$\text{range}_{\text{low}}(r_{2,\text{model}})$=0.0191972, $\text{range}_{\text{high}}(r_{2,\text{model}})$=0.0644309, $\text{range}(r_{2,\text{data}})$=dataRange.

Figure 1.34: Time-varying external habit with TSU and $b < 0$.
See Figure 1.16 for more details about the plots.

(variation overview)

1.F.22 TSU-Habit-CSV, $\sigma_{xt} = \lambda(0)\sigma_{c0}$

The term premia are now constant. This is partially contrary to the spirit of Campbell and Cochrane (1999), because the surplus consumption ratio does not get more volatile in bad states of the economy, but it illustrates how heteroskedasticity is crucial for the generation of variable term premia.

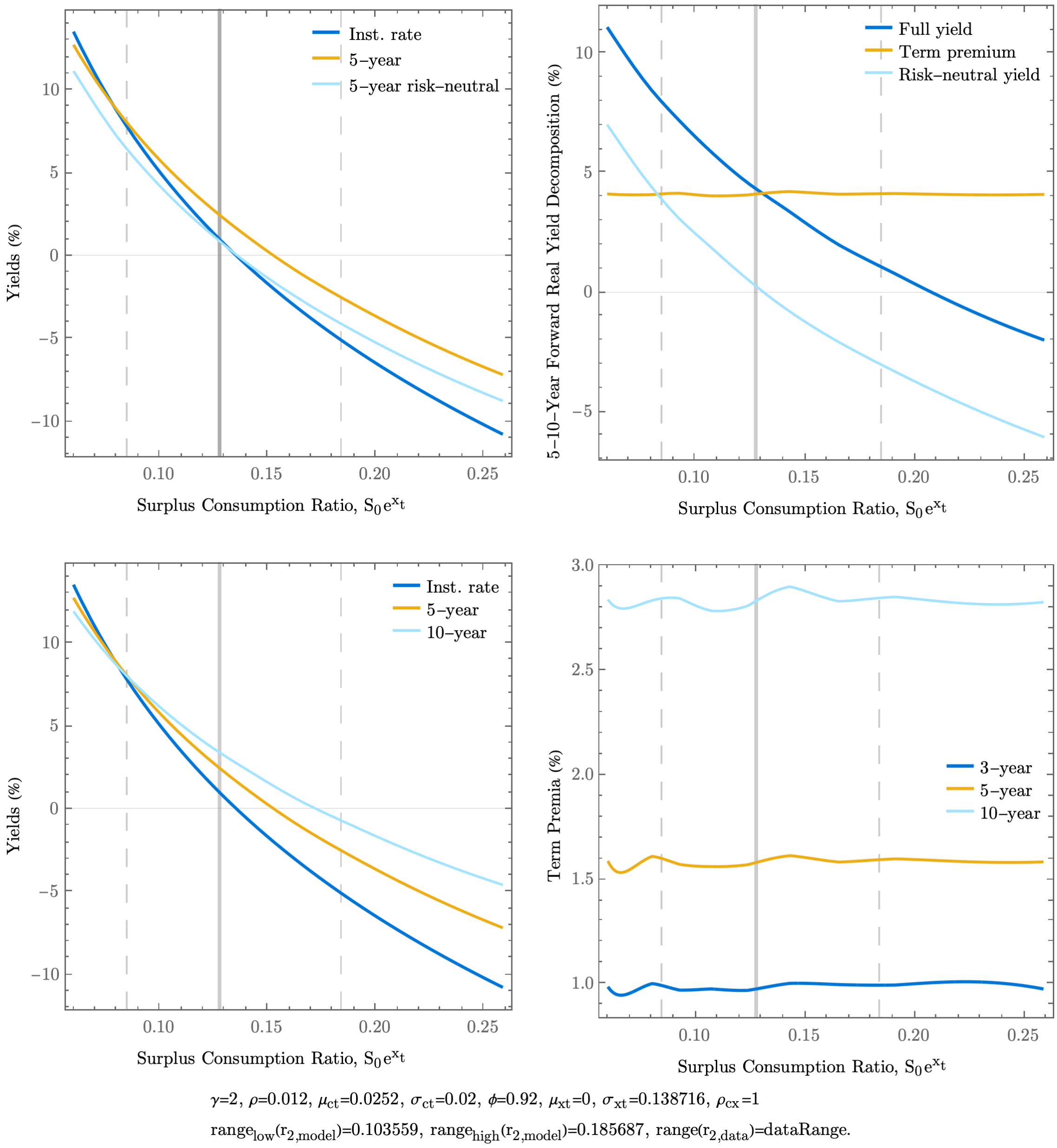
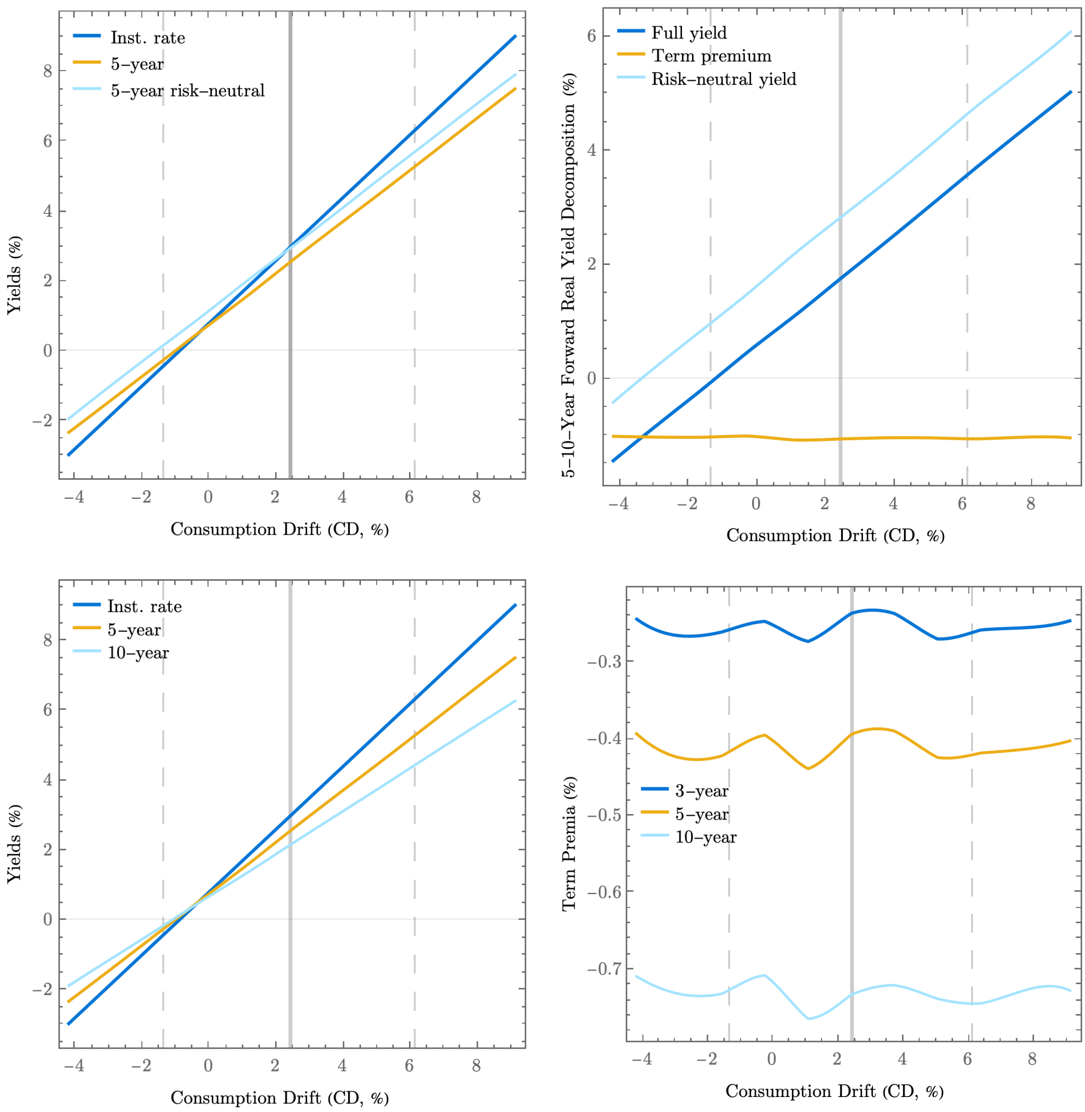


Figure 1.35: Time-varying external habit with TSU and constant state variable volatility.
See Figure 1.16 for more details about the plots.

(variation overview)

1.F.23 RU-CD, Calibration used in main paper, Figure 1.5



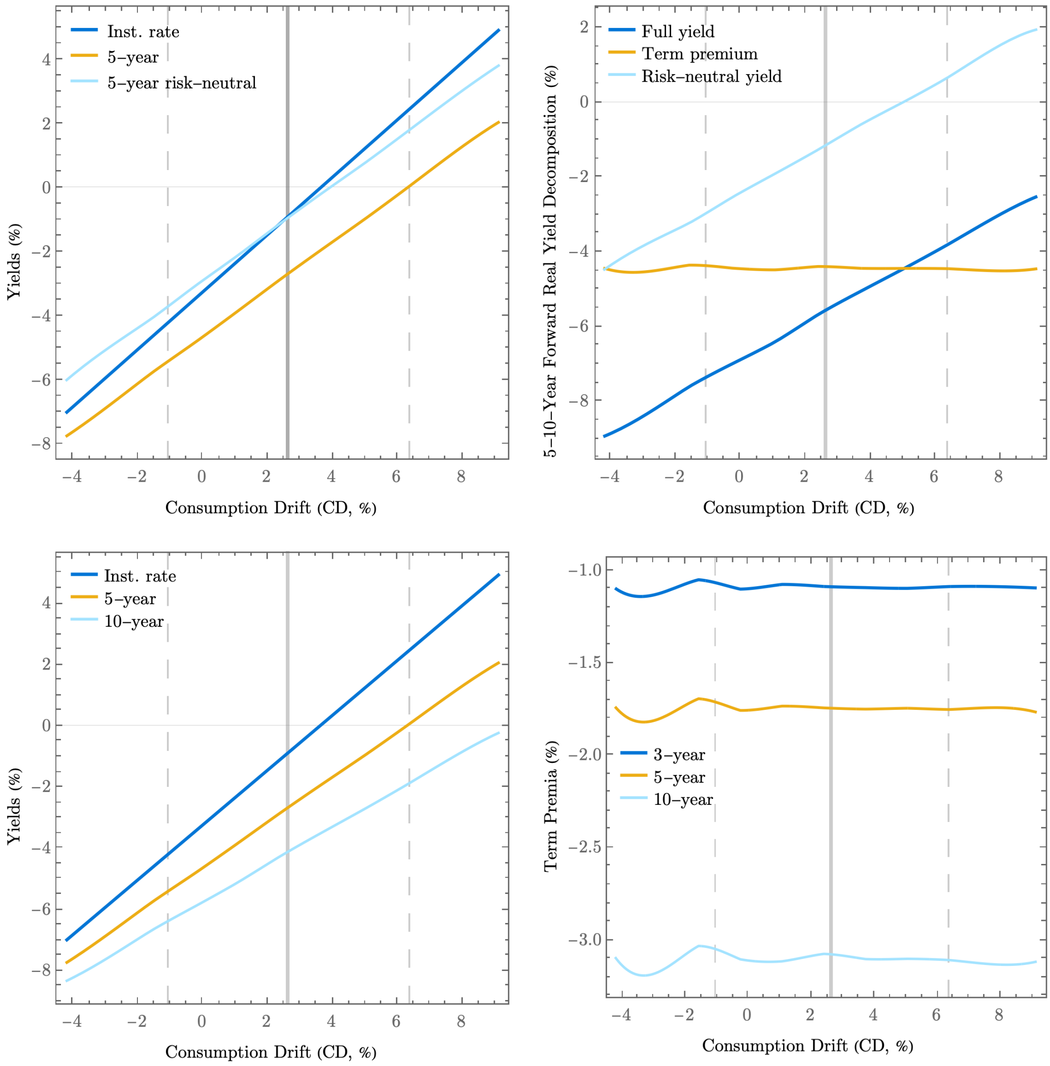
γ=2, ρ=0.012, μ_{ct}=x_t+0.0252, σ_{ct}=0.02, ϕ=0.92, μ_{xt}=0, σ_{xt}=0.0126445, ρ_{cx}=0.5, ψ=1.11, approx. order=8
$\mathrm{range_{low}}(\mathrm{r_{2,model}})$=0.050672, $\mathrm{range_{high}}(\mathrm{r_{2,model}})$=0.0927875, $\mathrm{range}(\mathrm{r_{2,data}})$=dataRange.

Figure 1.36: Time-varying CD with RU.
See Figure 1.16 for more details about the plots.

(variation overview)

1.F.24 RU-CD-HRA, $\gamma = 6$

Term premia stay constant and negative but it becomes significantly larger in absolute value. In this paper I do not consider a time-varying γ parameter, but this suggests that a time-varying risk aversion would be able to produce time-varying term premia. The habit model essentially provides a similar mechanism.



γ=6, ρ=0.012, μ_{ct}=x_t + 0.0252, σ_{ct}=0.02, ϕ=0.92, μ_{xt}=0, σ_{xt}=0.0126445, ρ_{cx}=0.5, ψ=1.11, approx. order=8

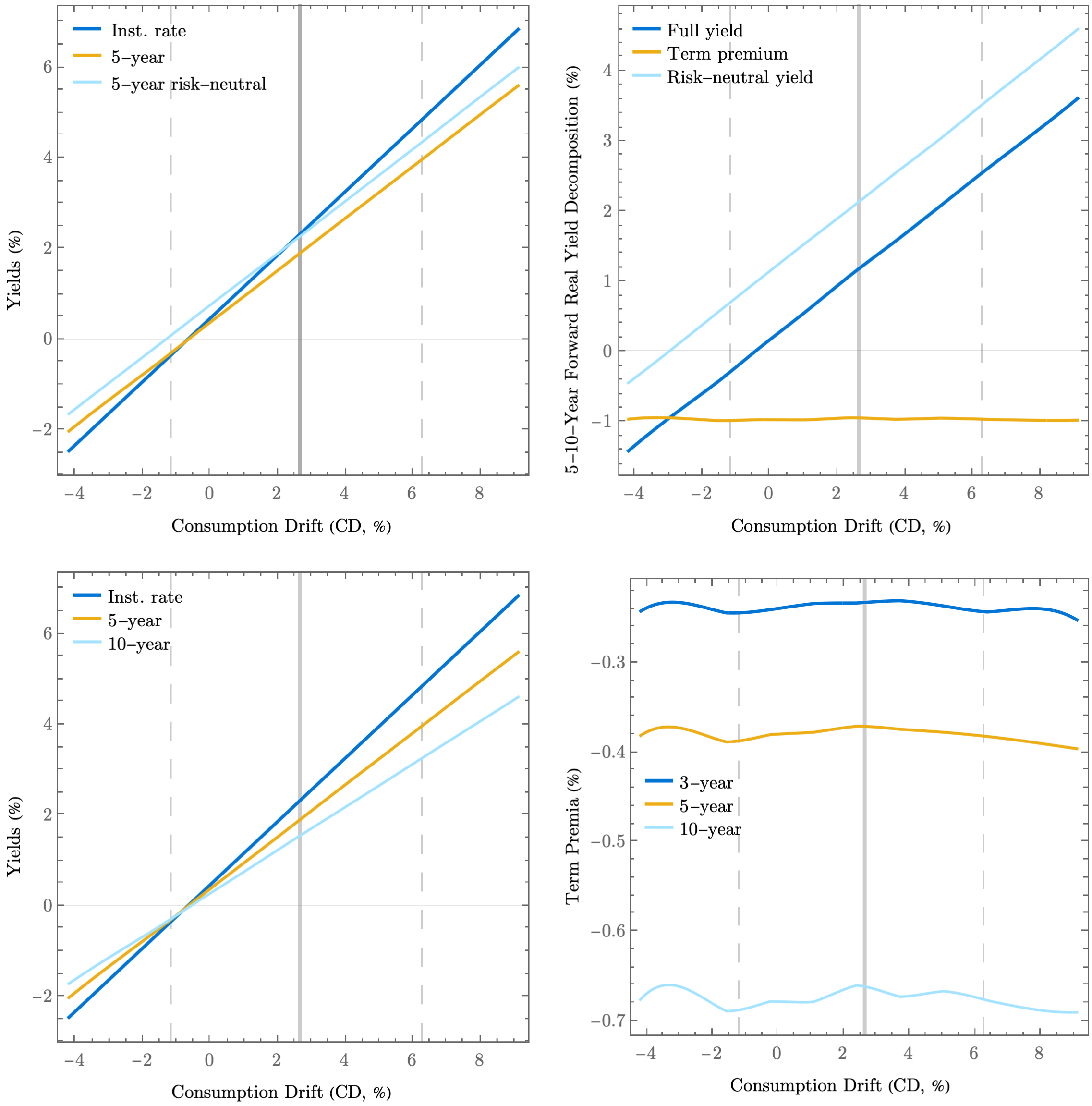
range$_{\mathrm{low}}$(r$_{2,\mathrm{model}}$)=0.0491264, range$_{\mathrm{high}}$(r$_{2,\mathrm{model}}$)=0.0893581, range(r$_{2,\mathrm{data}}$)=0.071748.

Figure 1.37: Time-varying CD with RU and high risk aversion.
See Figure 1.5 for more details about the plots.

(variation overview)

1.F.25 RU-CD-HIES, $\psi = 1.43$

Term premia do not seem to change significantly. The range of the short rate increases.



γ=2, ρ=0.012, μ_{ct}=x_t+0.0252, σ_{ct}=0.02, ϕ=0.92, μ_{xt}=0, σ_{xt}=0.0126445, ρ_{cx}=0.5, ψ=1.43, approx. order=8

$\text{range}_{\text{low}}(\text{r}_{2,\text{model}})$=0.037872, $\text{range}_{\text{high}}(\text{r}_{2,\text{model}})$=0.071687, $\text{range}(\text{r}_{2,\text{data}})$=0.071748.

Figure 1.38: Time-varying CD with HIES.
See Figure 1.16 for more details about the plots.

(variation overview)

1.F.26 RU-CD-LIES, $\psi = 0.83$

Term premia do not seem to change significantly. Curiously the range of the short rate increases again.

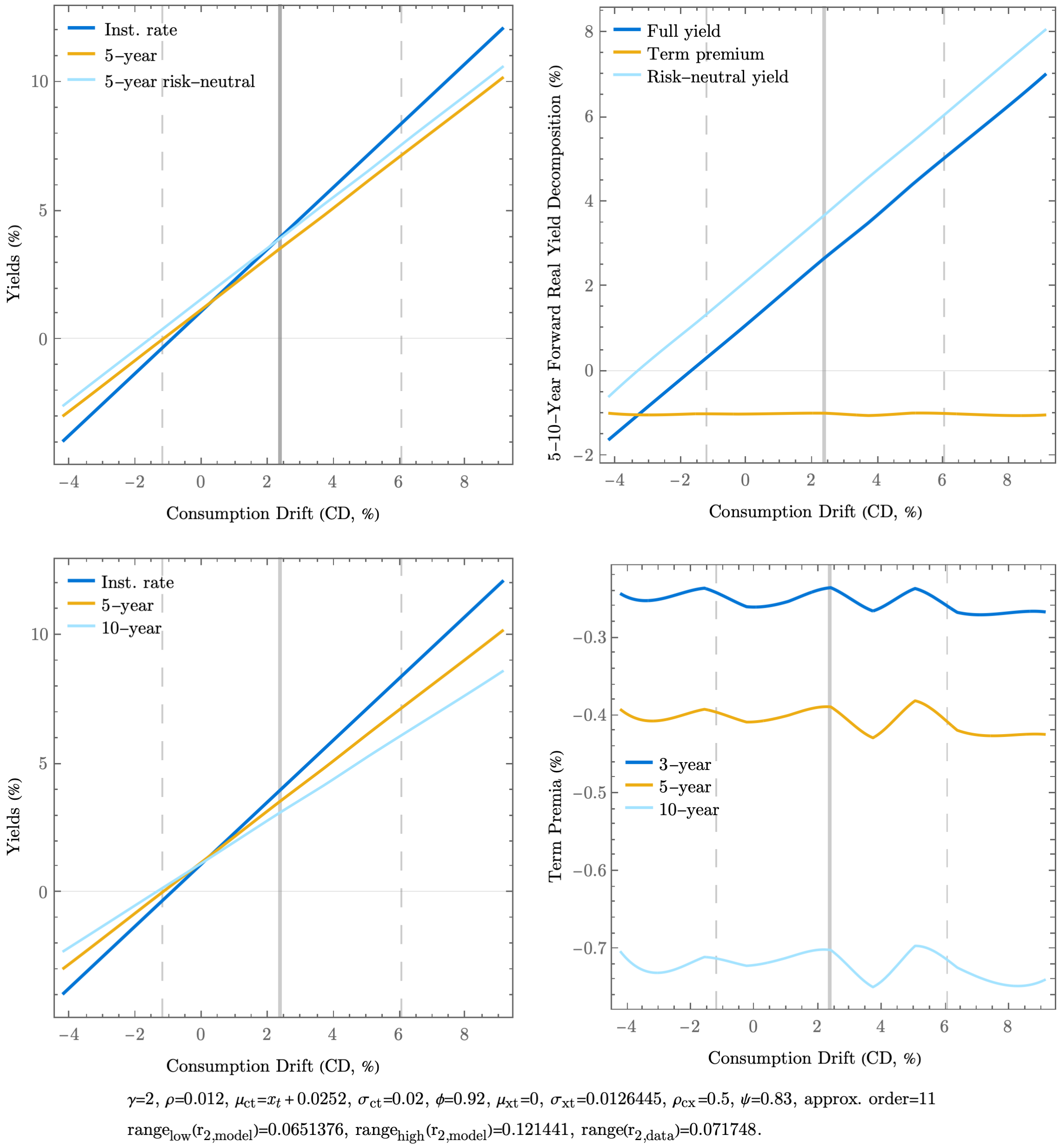
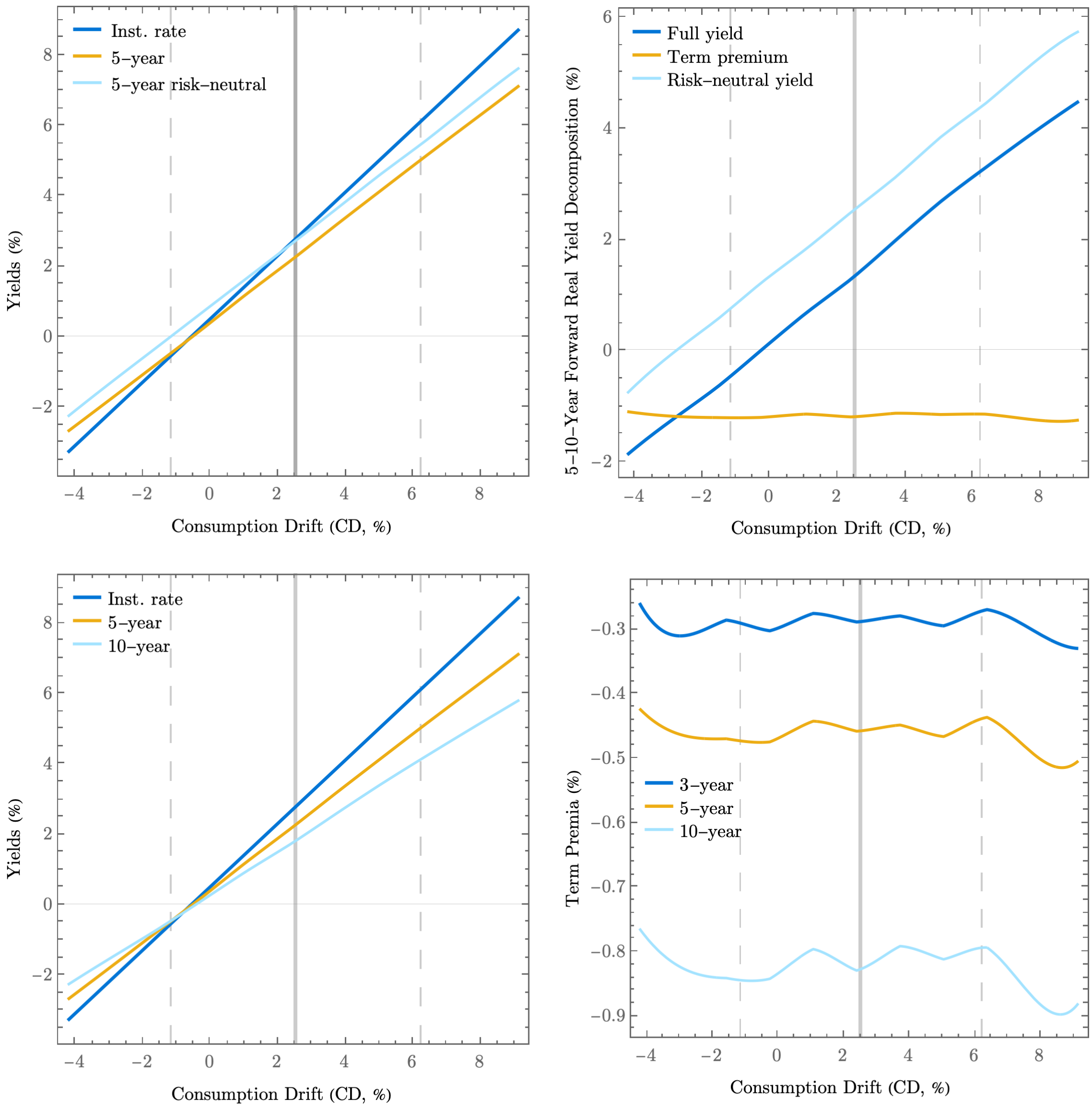


Figure 1.39: Time-varying CD with RU with LIES.
See Figure 1.16 for more details about the plots.

(variation overview)

1.F.27 RU-CD-HCor, $\rho_{cx} = 1$

Term premia increase in absolute value but do not double in size as did the correlation parameter.



γ=2, ρ=0.012, μ_{ct}=x_t + 0.0252, σ_{ct}=0.02, ϕ=0.92, μ_{xt}=0, σ_{xt}=0.0126445, ρ_{cx}=1, ψ=1.11, approx. order=8

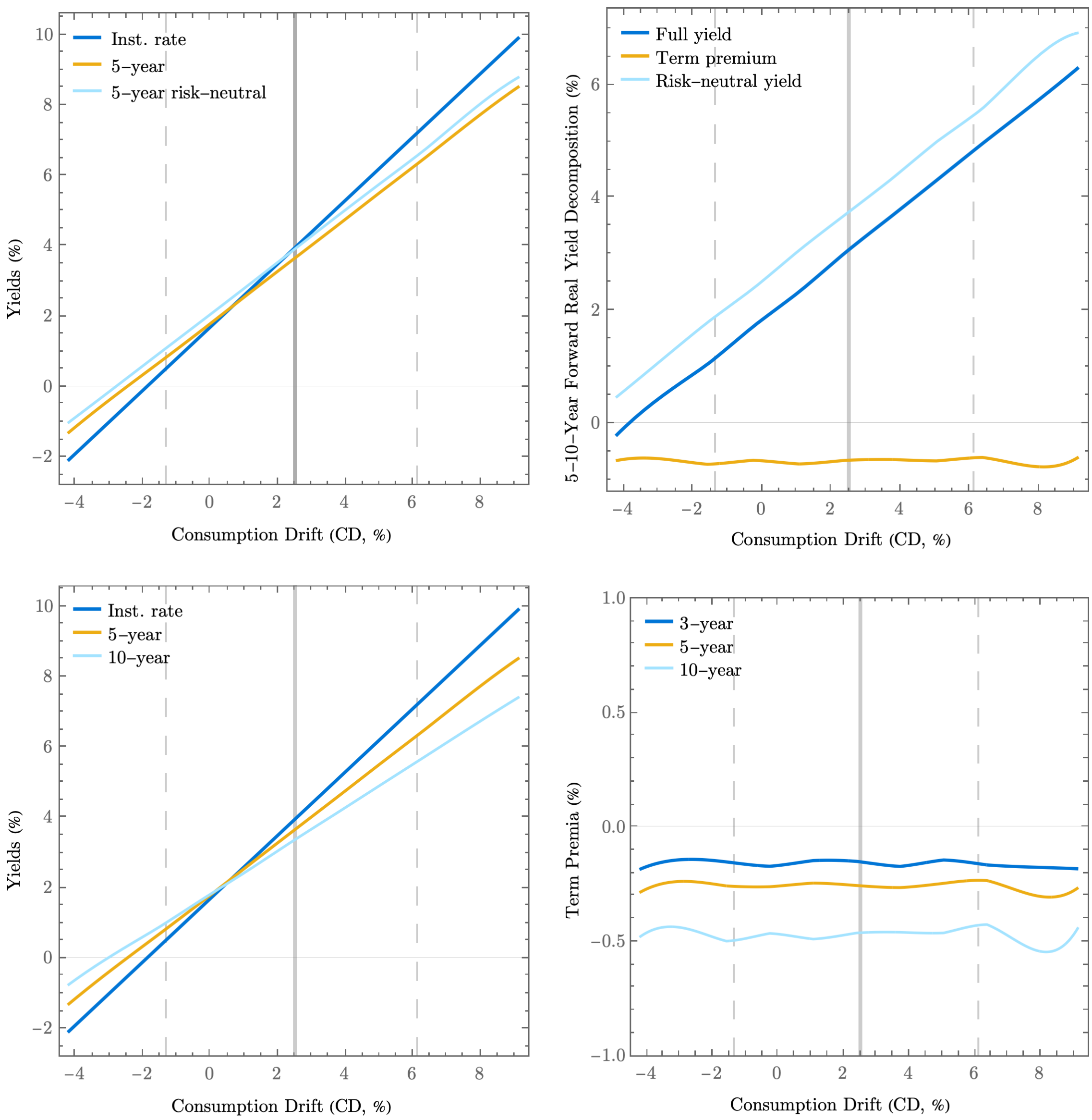
$\mathrm{range}_{\mathrm{low}}(\mathrm{r}_{2,\mathrm{model}})$=0.0506132, $\mathrm{range}_{\mathrm{high}}(\mathrm{r}_{2,\mathrm{model}})$=0.0914971, $\mathrm{range}(\mathrm{r}_{2,\mathrm{data}})$=0.071748.

Figure 1.40: Time-varying CD with RU and high ρ_{cx}.
See Figure 1.16 for more details about the plots.

(variation overview)

1.F.28 RU-CD-NCor, $\rho_{cx} < 1$

Term premia increase but they remain negative as in RU term premia are dominated by the term, not including ρ_{cx}.



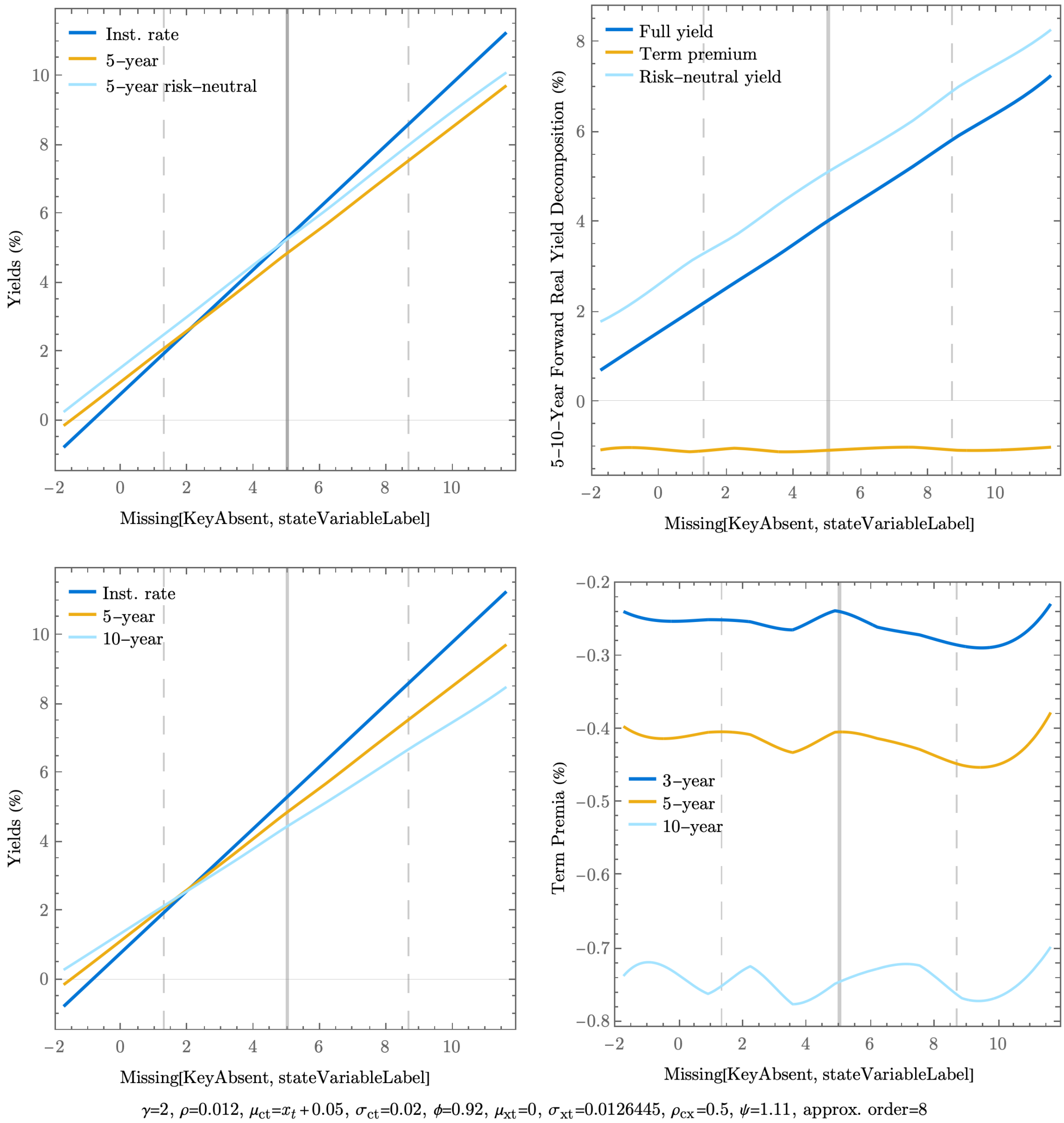
γ=2, ρ=0.012, $\mu_{\mathrm{ct}}=x_t+0.0252$, σ_{ct}=0.02, ϕ=0.92, μ_{xt}=0, σ_{xt}=0.0126445, ρ_{cx}=−1, ψ=1.11, approx. order=8
$\mathrm{range}_{\mathrm{low}}(\mathrm{r}_{2,\mathrm{model}})$=0.0484668, $\mathrm{range}_{\mathrm{high}}(\mathrm{r}_{2,\mathrm{model}})$=0.0897518, $\mathrm{range}(\mathrm{r}_{2,\mathrm{data}})$=0.071748.

Figure 1.41: Time-varying CD with RU and negative ρ_{cx}.
See Figure 1.16 for more details about the plots.

(variation overview)

1.F.29 RU-HCD, $\mu_{x0} = 0.05$

Term premia do not change, but yields increase.



γ=2, ρ=0.012, $\mu_{\mathrm{ct}}=x_t+0.05$, σ_{ct}=0.02, ϕ=0.92, μ_{xt}=0, σ_{xt}=0.0126445, ρ_{cx}=0.5, ψ=1.11, approx. order=8

$\mathrm{range_{low}}(\mathrm{r_{2,model}})$=0.0503055, $\mathrm{range_{high}}(\mathrm{r_{2,model}})$=0.0877158, $\mathrm{range}(\mathrm{r_{2,data}})$=0.071748.

Figure 1.42: Time-varying and high CD with RU.
See Figure 1.16 for more details about the plots.

(variation overview)

1.F.30 RU-CD-HCV, $\sigma_{ct} = 0.08$

Term premia do not change, but yields decrease.

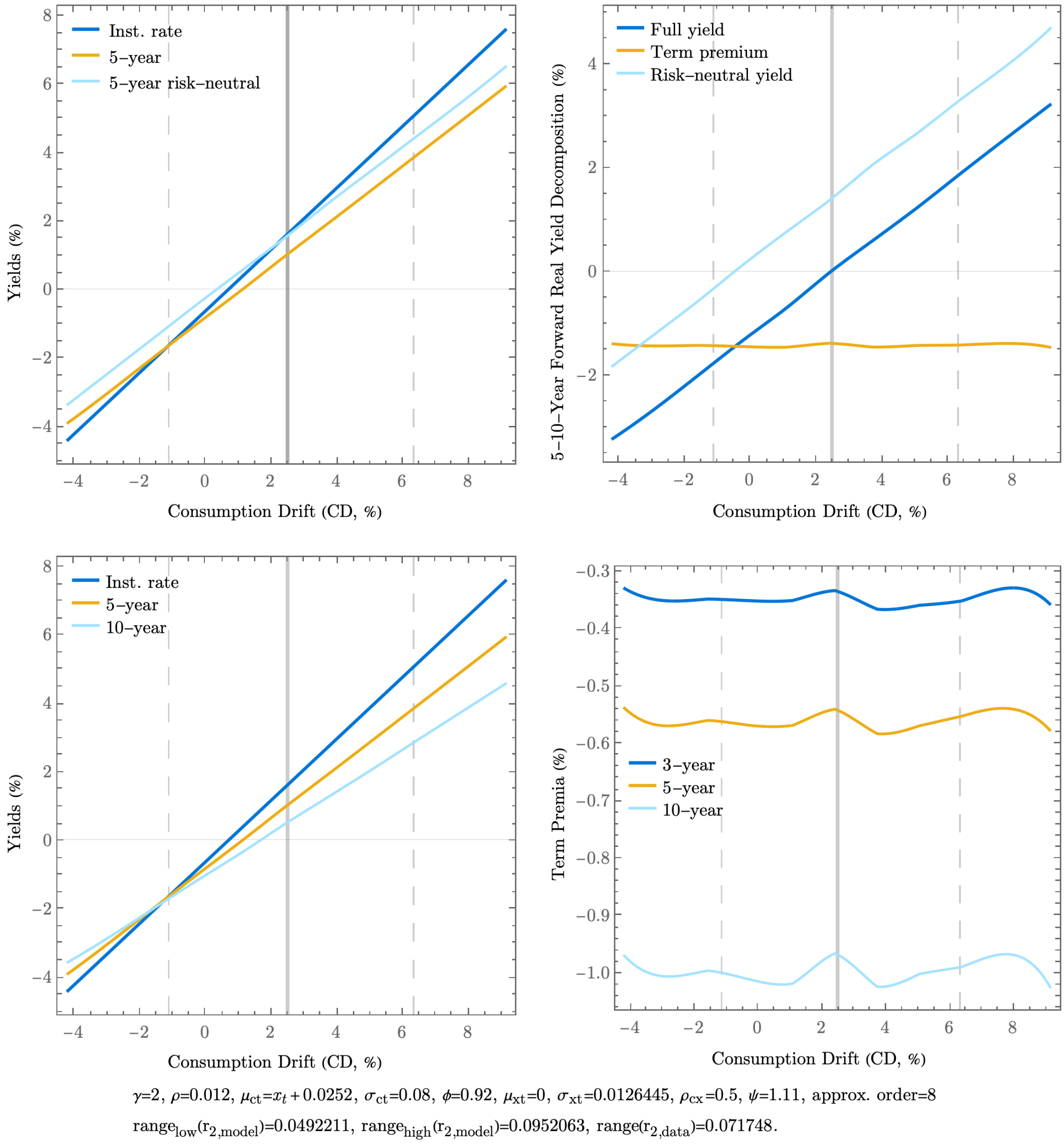
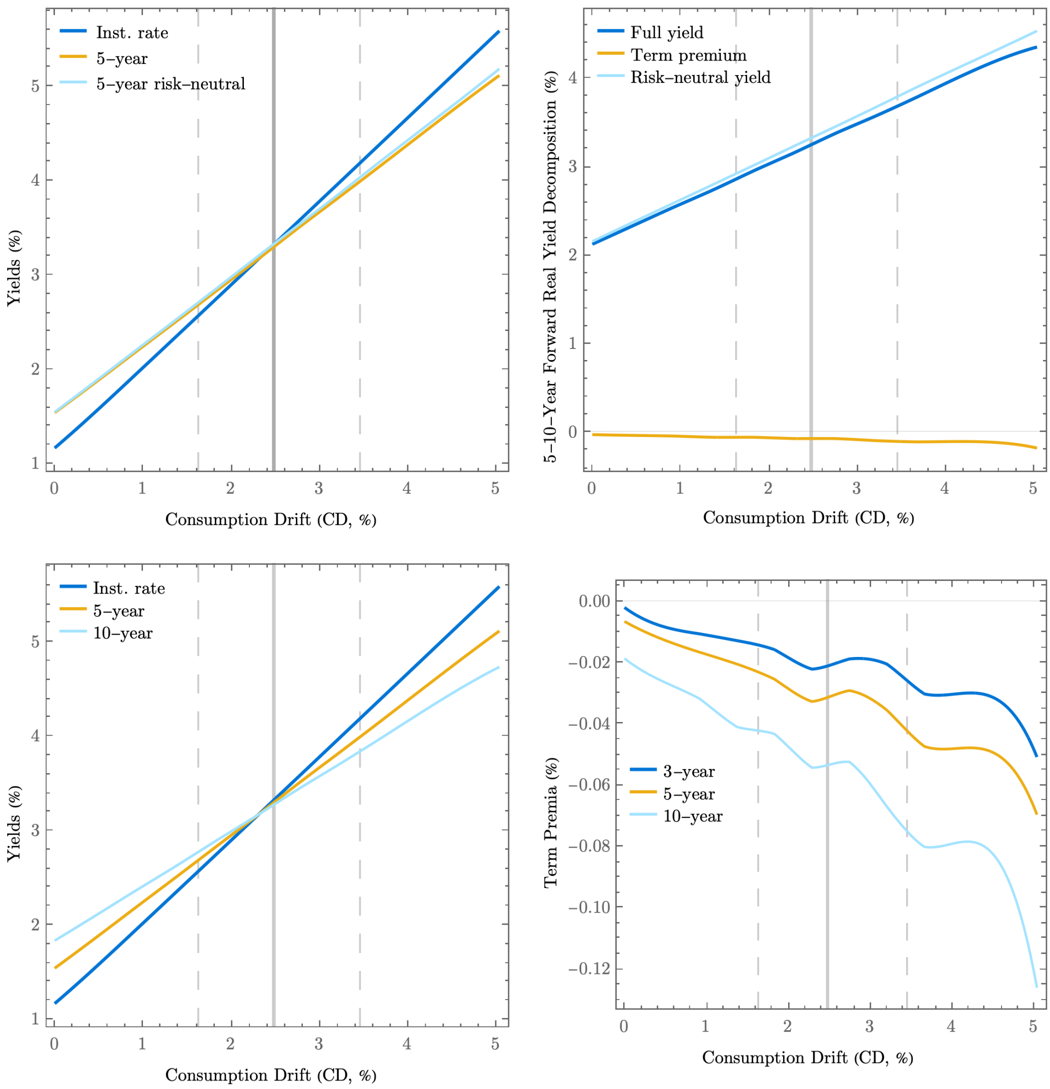


Figure 1.43: Time-varying CD with RU and HCV
. See Figure 1.16 for more details about the plots.

(variation overview)

1.F.31 RU-CD-Heterosk-PCor

When the state variable is heteroskedastic, term premia become time-varying. Here term premia are quite small, but this could change once a more volatile state variable is introduced. However, term premia are again negative.



γ=2, ρ=0.012, μ_{ct}=0.0252 x_t, σ_{ct}=0.021, ϕ=0.92, μ_{xt}=μ_x, σ_{xt}=0.12 $\sqrt{x_t}$, ρ_{cx}=0.5, ψ=1.11, approx. order=8
$\text{range}_{\text{low}}(\text{r}_{2,\text{model}})$=0.0115265, $\text{range}_{\text{high}}(\text{r}_{2,\text{model}})$=0.0221465, $\text{range}(\text{r}_{2,\text{data}})$=0.071748.

Figure 1.44: Time-varying and heteroskedastic CD with RU with positive ρ_{cx}. See Figure 1.16 for more details about the plots.

(variation overview)

1.F.32 RU-CD-Heterk-PCor

Despite changing the correlation compared to the previous case term premia are still negative given that the dominant component in function A does not contain ρ_{cx}.

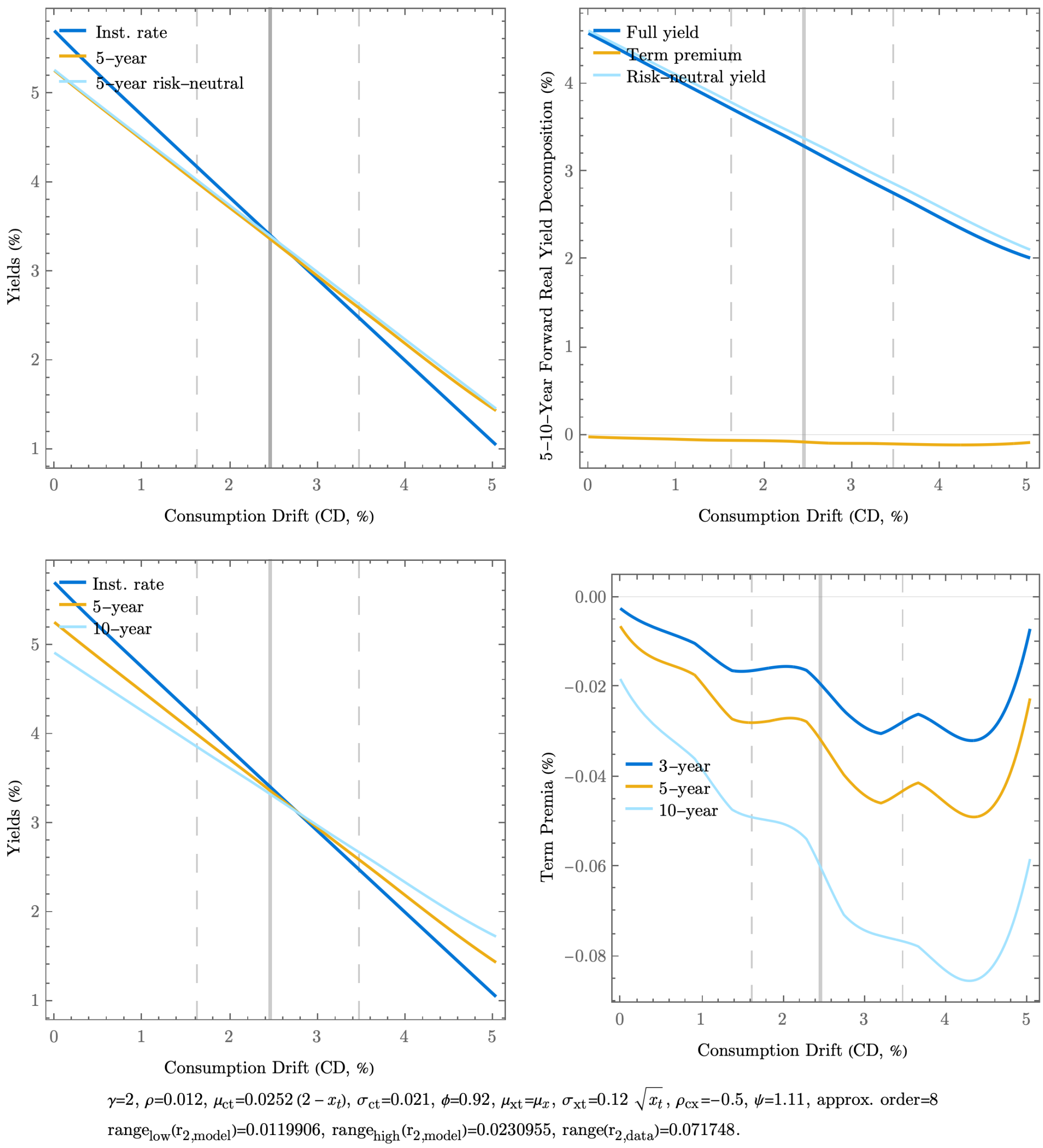
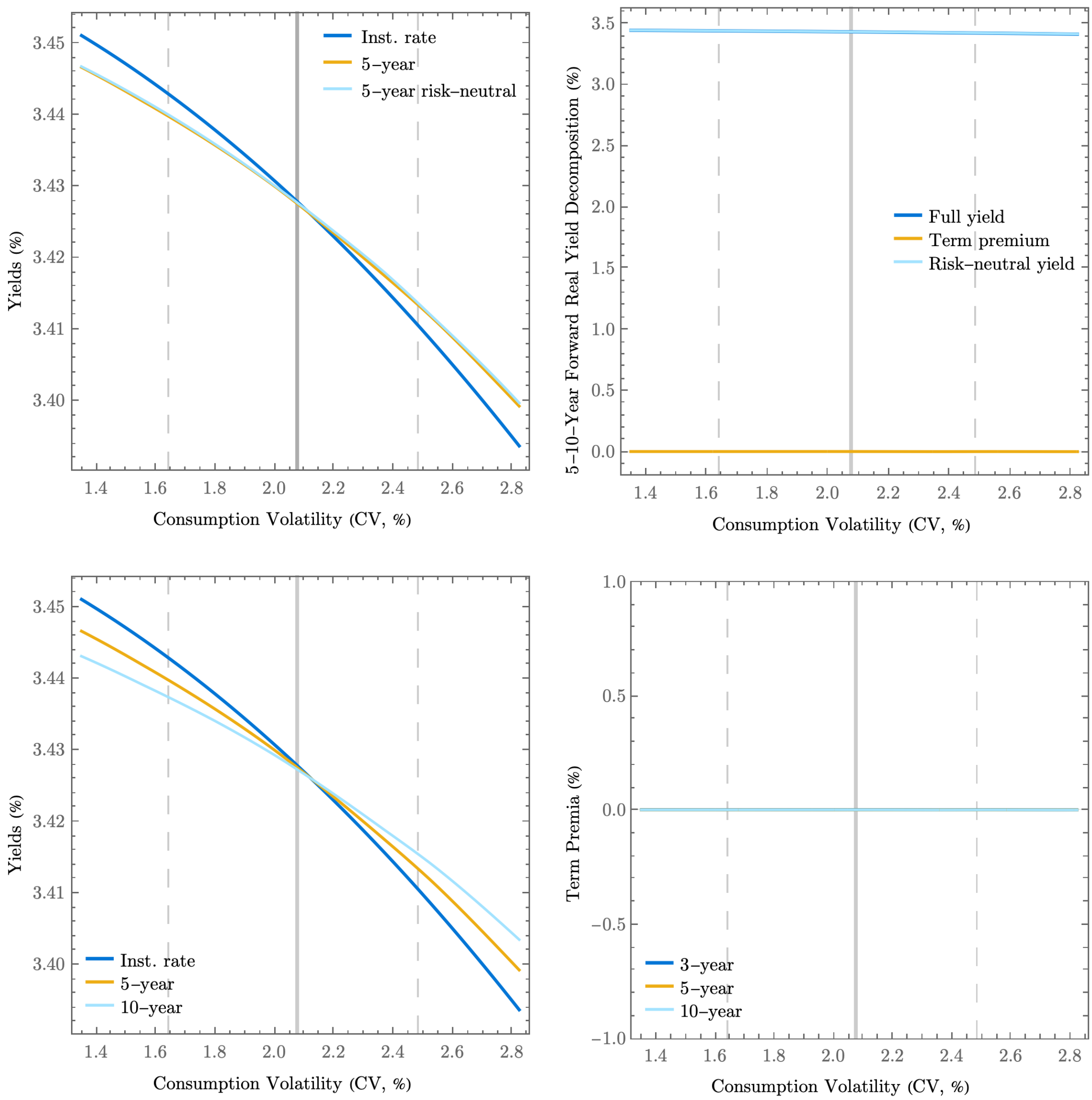


Figure 1.45: Time-varying and heteroskedastic CD with RU with negative ρ_{cx}. See Figure 1.16 for more details about the plots.

(variation overview)

1.F.33 RU-CV, Calibration used in main paper, Figure 1.5



γ=2, ρ=0.012, μ_{ct}=0.0252, σ_{ct}=0.021 $\sqrt{x_t}$, ϕ=0.92, μ_{xt}=μ_x, σ_{xt}=0.13 $\sqrt{x_t}$, ρ_{cx}=−0.5, ψ=1.11, approx. order=8

$\mathrm{range_{low}}(\mathrm{r}_{2,\mathrm{model}})$=0.000234865, $\mathrm{range_{high}}(\mathrm{r}_{2,\mathrm{model}})$=0.000434075, $\mathrm{range}(\mathrm{r}_{2,\mathrm{data}})$=dataRange.

Figure 1.46: Time-varying CV with RU.
See Figure 1.16 for more details about the plots.

(variation overview)

1.F.34 RU-CV-HRA, $\gamma = 6$

The term premia have hardly moved.

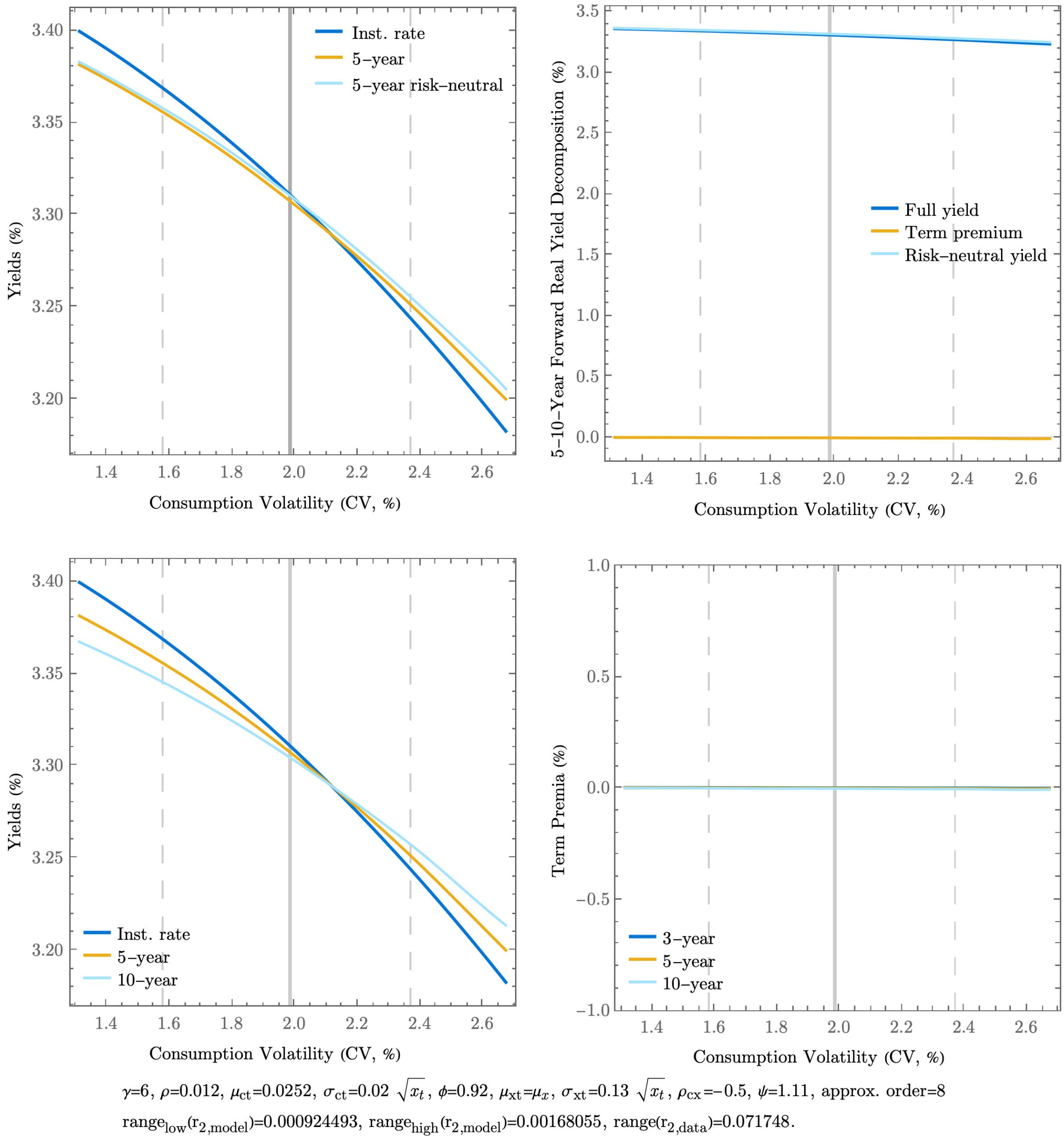


Figure 1.47: Time-varying CV with RU and high risk aversion.
See Figure 1.16 for more details about the plots.

(variation overview)

1.F.35 RU-CV-HP, $\phi = 0.96$

The term premia have hardly moved and curiously the yields have become slightly more variable again.

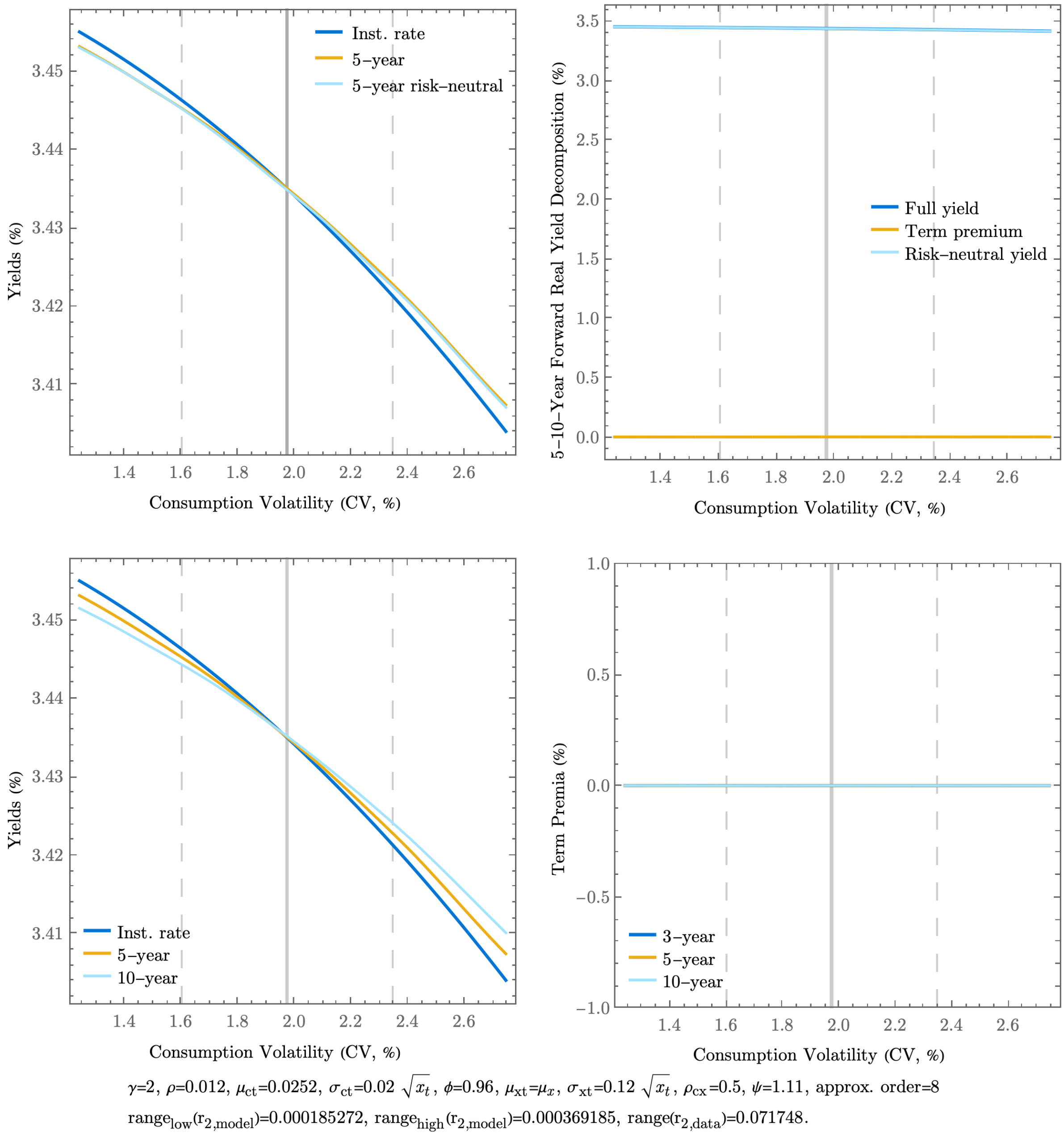


Figure 1.48: Time-varying CV with RU and high persistence.
See Figure 1.16 for more details about the plots.

(variation overview)

1.F.36 RU-CV-HIES, $\psi = 1.43$

The term premia have hardly moved and the yields have become slightly more variable.

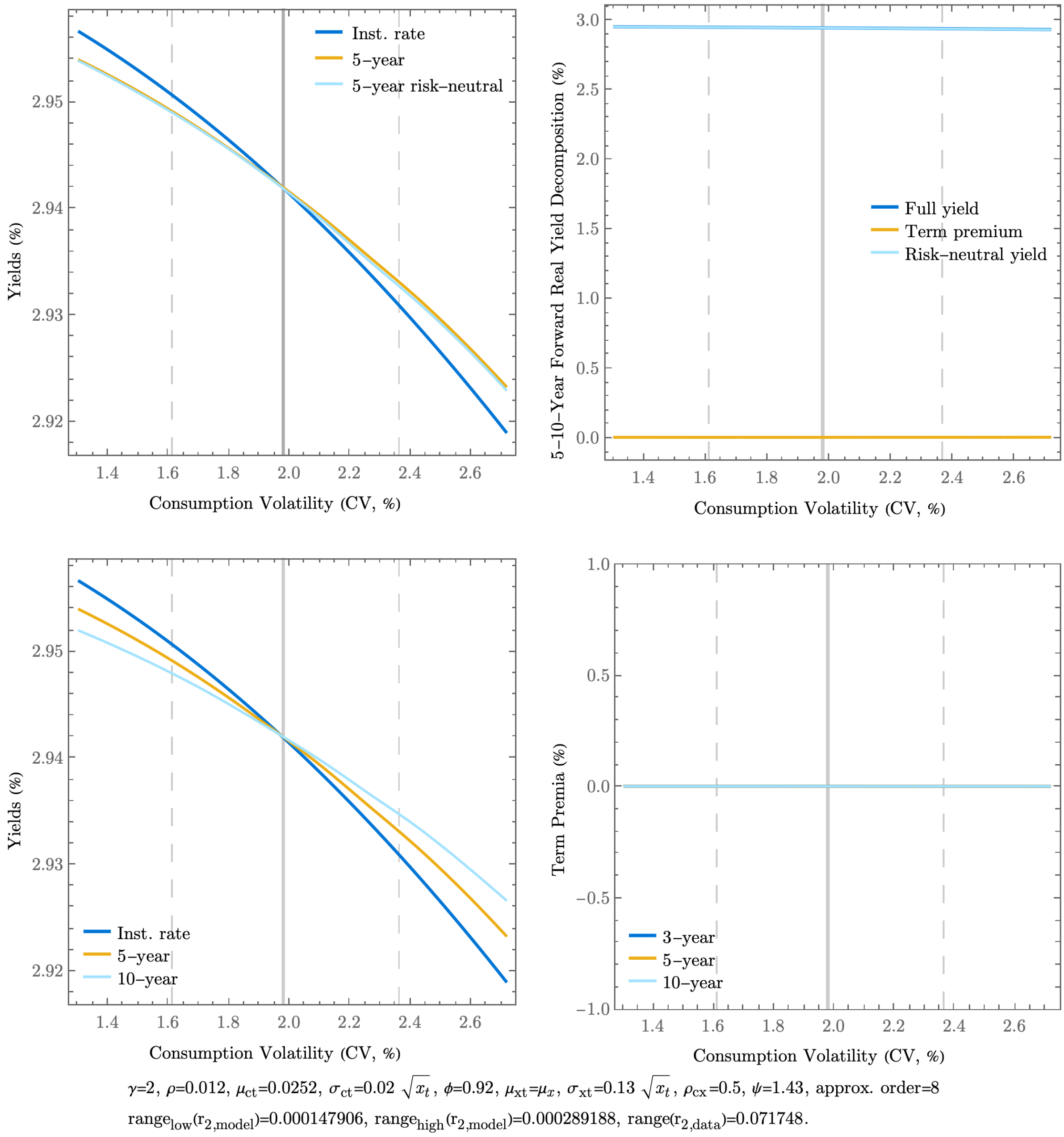


Figure 1.49: Time-varying CV with RU and HIES.
See Figure 1.16 for more details about the plots.

(variation overview)

1.F.37 RU-CV-LIES, $\psi = 0.77$

The term premia have hardly moved and curiously the yields have become slightly more variable again.

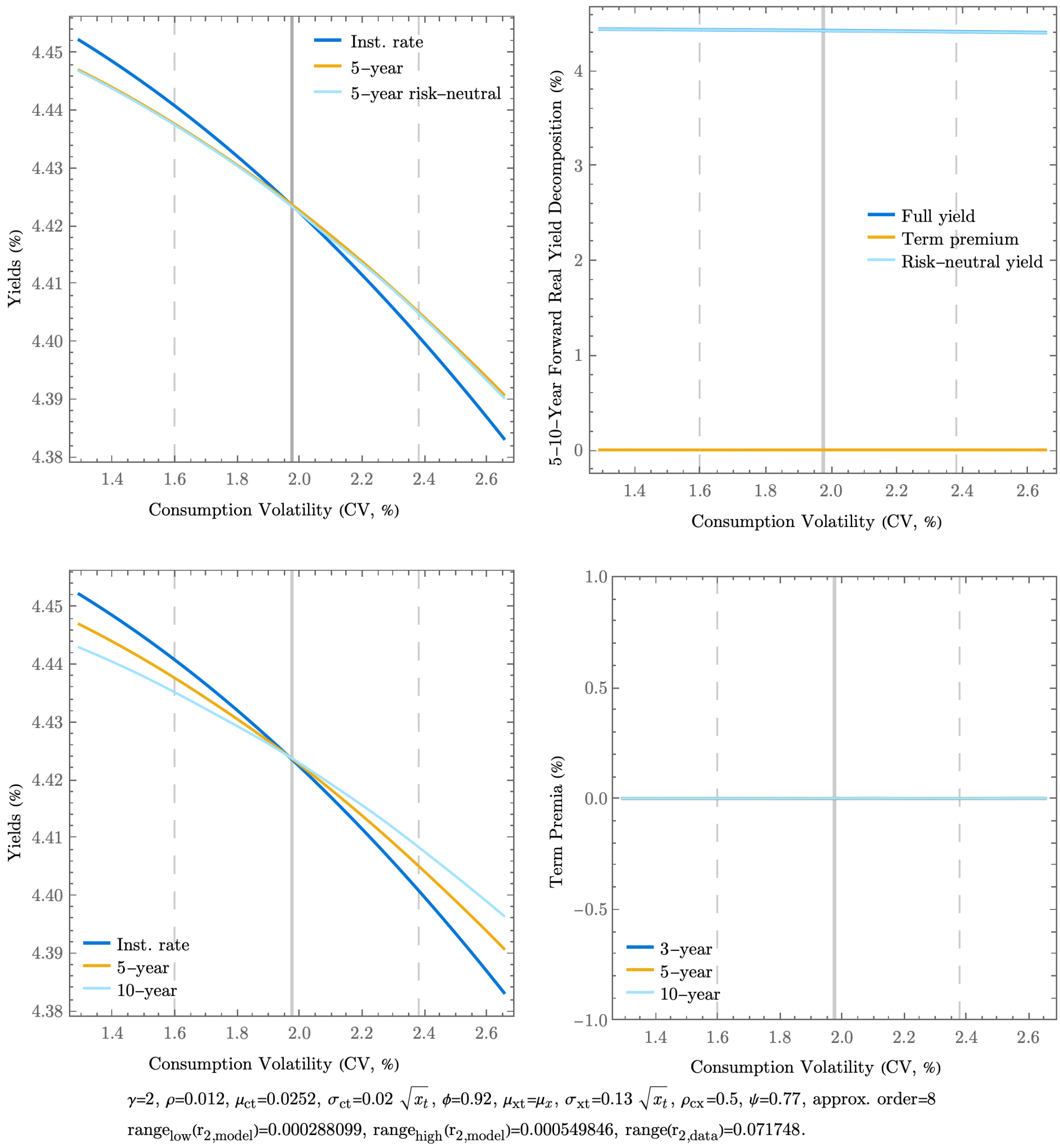
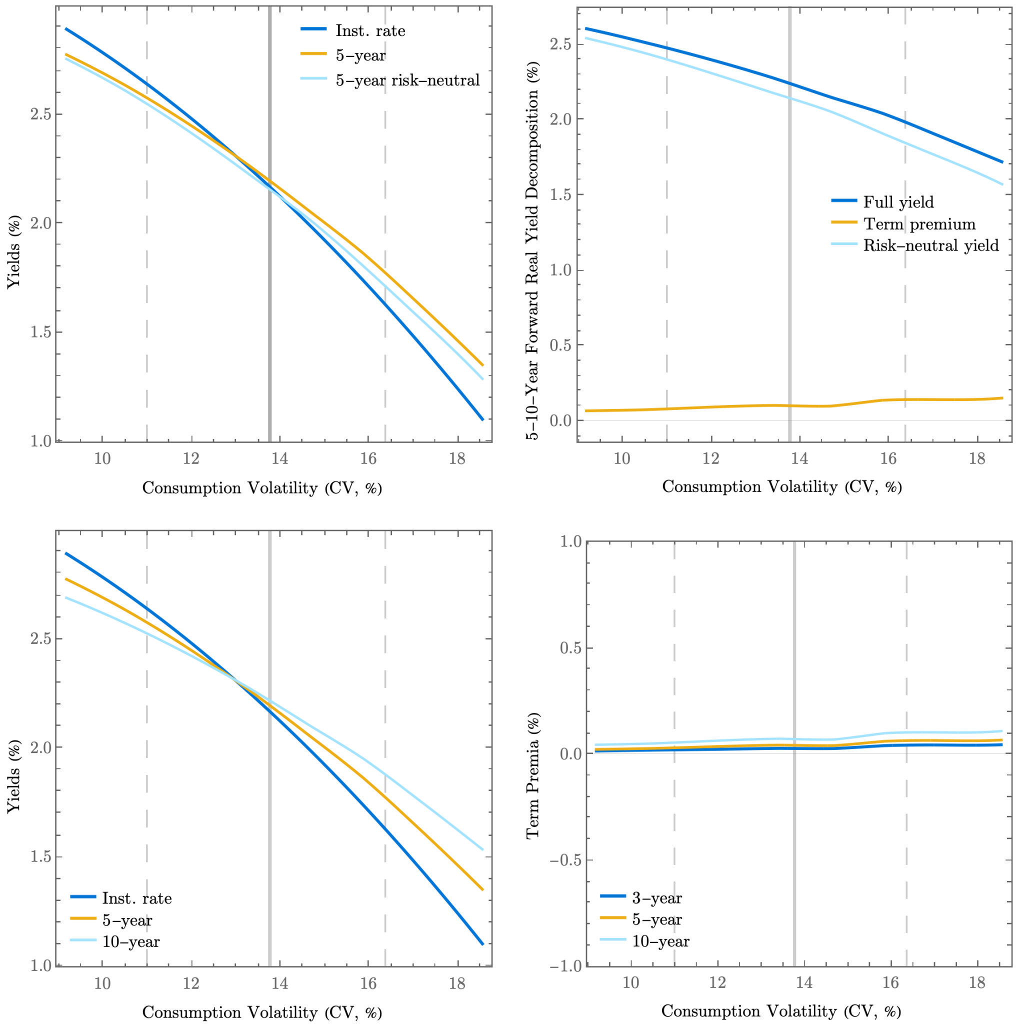


Figure 1.50: Time-varying CV with RU and LIES.
See Figure 1.16 for more details about the plots.

(variation overview)

1.F.38 RU-HCV-PCor, $\sigma_{c0} = 0.14, \rho_{cx} = 0.5$

Here term premia are positive, which means that the first component of function A that contains ρ_{cx} has become dominant due to the increase in σ_{c1}. Nevertheless, term premia are still smaller than the corresponding term premia in the TSU case, because the second term in function A is still negative.



γ=2, ρ=0.012, μ_{ct}=0.0252, σ_{ct}=0.14 $\sqrt{x_t}$, ϕ=0.92, μ_{xt}=μ_x, σ_{xt}=0.13 $\sqrt{x_t}$, ρ_{cx}=0.5, ψ=1.11, approx. order=8

range$_{\mathrm{low}}$(r$_{2,\mathrm{model}}$)=0.0074128, range$_{\mathrm{high}}$(r$_{2,\mathrm{model}}$)=0.0146724, range(r$_{2,\mathrm{data}}$)=0.071748.

Figure 1.51: Time-varying HCV with RU and PCor.
See Figure 1.16 for more details about the plots.

(variation overview)

1.F.39 RU-HCV-NCor, $\sigma_{c0} = 0.14, \rho_{cx} = 0.5$

Here both terms in function A are negative, so term premia are negative. They are also larger in absolute value than the corresponding term premia in RU-CV.

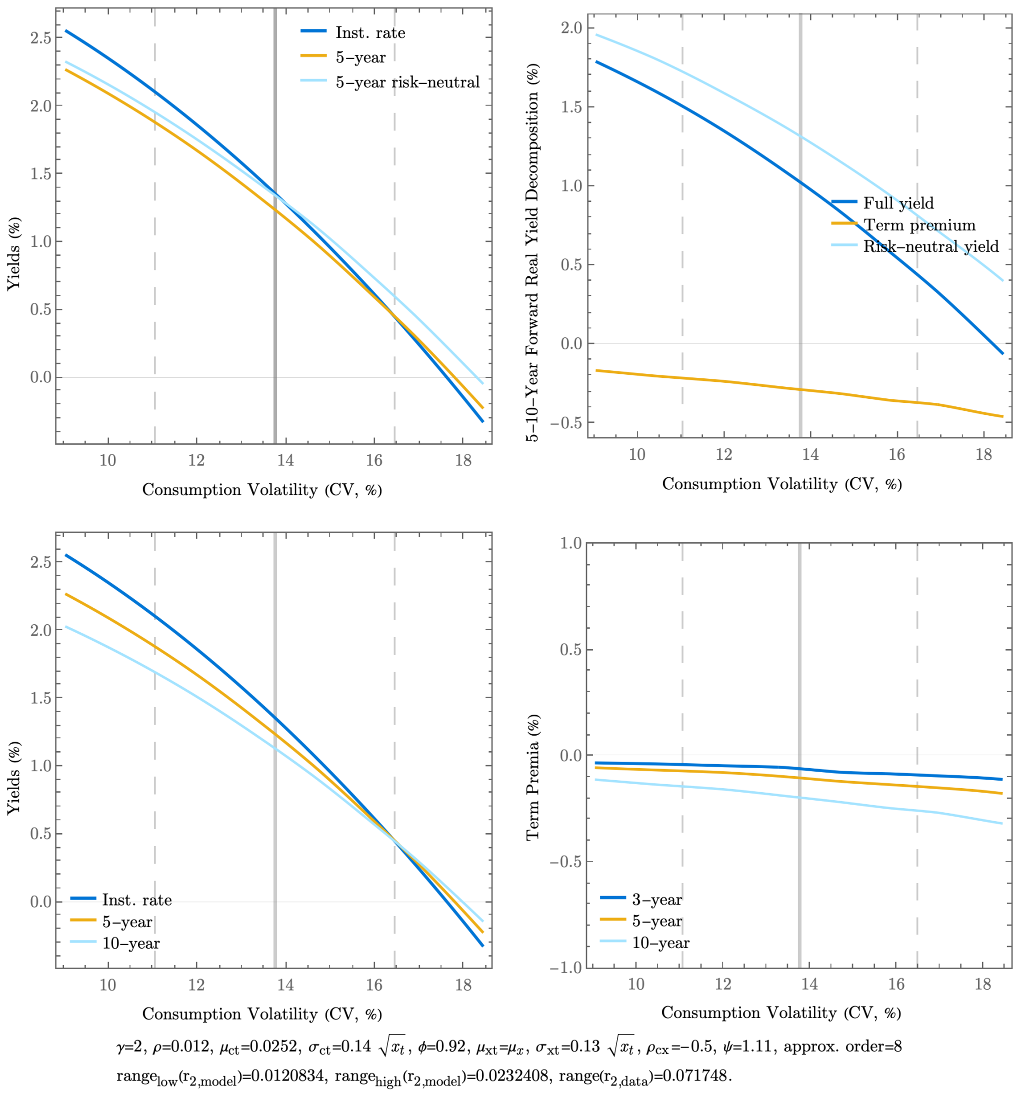


Figure 1.52: Time-varying HCV with RU and NCor.
See Figure 1.16 for more details about the plots.

(variation overview)

1.G Deriving the stochastic discount factor

1.G.1 Derivation of the SDF with TSU

Here I derive the SDE of SDF, including the case of the habit model. I present the terms that only apply to the habit model in grey colour. The following is the regular form of the SDF, in which I have substituted the state variable and log consumption:

$$\Lambda_t = e^{-\rho t}(e^{c_t}\bar{S}e^{x_t})^{-\gamma} \tag{1.25}$$

Then, in order to get the SDE form, I apply Ito's Lemma:

$$\begin{aligned}
\mathrm{d}\Lambda_t &= \frac{\partial \Lambda_t}{\partial t}\mathrm{d}t + \frac{\partial \Lambda_t}{\partial c}dc + \underbrace{\frac{\partial \Lambda_t}{\partial x_t}\mathrm{d}x_t}_{\text{habit model}} + \frac{1}{2}\Bigg(\frac{\partial^2 \Lambda_t}{\partial c^2}(\mathrm{d}c_t)^2 + \underbrace{\frac{\partial^2 \Lambda_t}{\partial x^2}(\mathrm{d}x)^2 + \frac{\partial^2 \Lambda_t}{\partial x \partial c_t}\mathrm{d}x_t\mathrm{d}c_t}_{h.m.}\Bigg) \\
&= -\rho\Lambda_t\mathrm{d}t - \gamma\Lambda_t\mathrm{d}c_t - \underbrace{\gamma\Lambda_t\mathrm{d}x_t}_{\text{h.m.}} + \frac{1}{2}\Big(\gamma^2\Lambda_t(\mathrm{d}c_t)^2 + \underbrace{\gamma^2\Lambda_t(\mathrm{d}x_t)^2 + \gamma^2\Lambda_t\mathrm{d}x_t\mathrm{d}c_t}_{h.m.}\Big)
\end{aligned}$$

$$\Rightarrow$$

$$\begin{aligned}
\frac{\mathrm{d}\Lambda_t}{\Lambda_t} &= \Big(-\rho - \gamma\mu_{ct} + \underbrace{\gamma\log(\phi)(\mu_{x0} - x_t)}_{h.m.} + \frac{\gamma^2\sigma_{ct}^2}{2} + \underbrace{\frac{\gamma^2\sigma_{xt}^2}{2} + \gamma^2\rho_{cx}\sigma_{xt}\sigma_{ct}}_{\text{h.m.}}\Big)\mathrm{d}t \\
&\quad - \gamma\sigma_{ct}\mathrm{d}W_{ct} \underbrace{-\gamma\sigma_{xt}\mathrm{d}W_{xt}}_{\text{h.m.}}
\end{aligned} \tag{1.26}$$

1.G.2 Derivation of the SDF with RU

As mentioned in the main paper the SDE of the SDF can be derived based on the following expression:

$$\frac{\mathrm{d}\Lambda_t}{\Lambda_t} = F_V(C_t, V_t)\mathrm{d}t + \frac{\mathrm{d}F_C(C_t, V_t)}{F_C(C_t, V_t)} \tag{1.27}$$

thus, flow utility is a central component of the derivation:

$$F(C_t, V_t) = \frac{\rho}{1 - 1/\psi}\Big((1-\gamma)V_t\Big)\left(\left(C_t((1-\gamma)V_t)^{-\frac{1}{1-\gamma}}\right)^{1-1/\psi} - 1\right) \tag{1.28}$$

The partial derivative of F with respect to V_t is:

$$F_V(C_t, V_t) = \frac{\rho\left((\gamma-1)\psi + (1-\gamma\psi)\left(C_t(V_t - \gamma V_t)^{\frac{1}{\gamma-1}}\right)^{\frac{\psi-1}{\psi}}\right)}{\psi - 1} \tag{1.29}$$

The partial derivative of F with respect to C_t is:

$$F_C(C_t, V_t) = -\frac{(\gamma-1)\rho V_t \left(C_t (V_t - \gamma V_t)^{\frac{1}{\gamma-1}}\right)^{\frac{\psi-1}{\psi}}}{C_t} \tag{1.30}$$

As I implement Ito's Lemma directly using c_t and x_t as independent variables, I make the following replacements in the expressions above:

$$c_t = \log(C_t), \quad V_t = \frac{C_t^{1-\gamma}}{1-\gamma} e^{(1-\gamma)K(x_t)} \Rightarrow K(x_t) = \frac{\log\left(-\frac{C_t^{1-\gamma}}{(\gamma-1)V_t}\right)}{\gamma-1} \tag{1.31}$$

And after simplification, they become:

$$F_V(C_t, V_t) \to G_1(c_t, x_t) = \frac{\rho(-(1-\gamma\psi)e^{-\frac{(\psi-1)K[x_t]}{\psi}} - \gamma\psi + \psi)}{1-\psi} \tag{1.32}$$

$$F_C(C_t, V_t) \to G_2(c_t, x_t) = \rho e^{(\frac{1}{\psi}-\gamma)K(x_t) - c_t\gamma} \tag{1.33}$$

I implement Ito's Lemma on G_2. The partial derivatives are:

$$\begin{aligned}
\frac{\partial G_2(c_t, x_t)}{\partial c_t} &= \gamma\rho\left(-e^{\left(\frac{1}{\psi}-\gamma\right)K[x_t]-\gamma c_t}\right) = -\gamma G_2(c_t, x_t) \\
\frac{\partial h(c_t, x_t)}{\partial x_t} &= \rho\left(\frac{1}{\psi}-\gamma\right) K'(x_t)\, e^{\left(\frac{1}{\psi}-\gamma\right)K[x_t]-\gamma c_t} = \left(\frac{1}{\psi}-\gamma\right) K'(x_t)\, G_2(c_t, x_t) \\
\frac{\partial^2 G_2(c_t, x_t)}{\partial c_t^2} &= \gamma^2 \rho e^{\left(\frac{1}{\psi}-\gamma\right)K[x_t]-\gamma c_t} = \gamma^2 h(c_t, x_t) \\
\frac{\partial^2 G_2(c_t, x_t)}{\partial x_t^2} &= \frac{\rho(\gamma\psi-1)\left((\gamma\psi-1)K'(x_t)^2 - \psi K''(x_t)\right) e^{\left(\frac{1}{\psi}-\gamma\right)K[x_t]-\gamma c_t}}{\psi^2} \\
&= \frac{(\gamma\psi-1)\left((\gamma\psi-1)K'(x_t)^2 - \psi K''(x_t)\right)}{\psi^2} G_2(c_t, x_t) \\
\frac{\partial G_2(c_t, x_t)}{\partial c_t \partial x_t} &= \frac{\gamma\rho(\gamma\psi-1)K'(x_t)\, e^{\left(\frac{1}{\psi}-\gamma\right)K[x_t]-\gamma c_t}}{\psi} = \frac{\gamma(\gamma\psi-1)K'(x_t)\, G_2(c_t, x_t)}{\psi}
\end{aligned} \tag{1.34}$$

The expressions above should be plugged into the expression:

$$\begin{aligned}\frac{dF_C}{F_C} =& \left(\frac{\partial G_2(c_t, x_t)}{\partial c_t}\mu_{ct} + \frac{\partial G_2(c_t, x_t)}{\partial x_t}\Big(-\log(\phi)\Big)(\mu_{x0} - x_t)\right.\\ &\left. + \frac{\sigma_{ct}^2}{2}\frac{\partial^2 G_2(c_t, x_t)}{\partial c_t^2} + \frac{\sigma_{xt}^2}{2}\frac{\partial^2 G_2(c_t, x_t)}{\partial x_t^2} + \frac{\rho_{cx}\sigma_{ct}\sigma_{xt}}{2}\frac{\partial^2 G_2(c_t, x_t)}{\partial c_t \partial x_t}\right)\mathrm{d}t\\ &+ \frac{\partial G_2(c_t, x_t)}{\partial x_t}\sigma_{xt}\mathrm{d}W_{xt} + \frac{\partial G_2(c_t, x_t)}{\partial c_t}\sigma_{ct}\mathrm{d}W_{ct}\end{aligned} \tag{1.35}$$

Then everything is plugged into Equation (1.27) to give the final result:

$$\begin{aligned}\frac{\mathrm{d}\Lambda_t}{\Lambda_t} =& \left(\frac{\gamma(\gamma\psi - 1)\rho_{cx}\sigma_{xt}\sigma_{ct}K'(x_t)}{\psi} + \frac{\gamma^2\sigma_{ct}^2}{2} - \gamma\mu_{ct}\right.\\ &+ \frac{(\gamma\psi - 1)\left(2\psi(\mu_{x0} - x_t)\log(\phi)K'(x_t) + \sigma_{xt}^2\left((\gamma\psi - 1)K'(x_t)^2 - \psi K''(x_t)\right)\right)}{2\psi^2}\\ &\left.\frac{\rho\left(-(1-\gamma\psi)e^{-\frac{(\psi-1)K[x_t]}{\psi}} - \gamma\psi + \psi\right)}{1-\psi}\right)\mathrm{d}t\\ &- \frac{(\gamma\psi - 1)\sigma_{xt}K'(x_t)}{\psi}\mathrm{d}W_{xt} - \gamma\sigma_{ct}\mathrm{d}W_{ct}\end{aligned} \tag{1.36}$$

Bibliography

Abrahams, Michael, Tobias Adrian, Richard K Crump, Emanuel Moench, and Rui Yu. 2016. "Decomposing Real and Nominal Yield Curves." *Journal of Monetary Economics*, 84: 182–200.

Ait-Sahalia, Yacine, Jonathan A Parker, and Motohiro Yogo. 2004. "Luxury Goods and the Equity Premium." *The Journal of Finance*, 59(6): 2959–3004.

Angeletos, George-Marios, Fabrice Collard, and Harris Dellas. 2018. "Quantifying Confidence." *Econometrica*, 86(5): 1689–1726.

Backus, David K, Allan W Gregory, and Stanley E Zin. 1989. "Risk Premiums in the Term Structure: Evidence from Artificial Economies." *Journal of Monetary Economics*, 24(3): 371–399.

Bansal, Ravi, and Amir Yaron. 2004. "Risks for the Long Run: A Potential Resolution of Asset Pricing Puzzles." *The Journal of Finance*, 59(4): 1481–1509.

Bansal, Ravi, and Ivan Shaliastovich. 2013. "A Long-Run Risks Explanation of Predictability Puzzles in Bond and Currency Markets." *The Review of Financial Studies*, 26(1): 1–33.

Barro, Robert J. 2006. "Rare Disasters and Asset Markets in the Twentieth Century." *The Quarterly Journal of Economics*, 121(3): 823–866.

Bauer, Michael D, and Glenn D Rudebusch. 2020. "Interest Rates under Falling Stars." *American Economic Review*, 110(5): 1316–1354.

Beechey, Meredith J, and Jonathan H Wright. 2009. "The High-Frequency Impact of News on Long-Term Yields and Forward Rates: Is it real?" *Journal of Monetary Economics*, 56(4): 535–544.

Bekaert, Geert, Eric Engstrom, and Steven R Grenadier. 2010. "Stock and Bond Returns with Moody Investors." *Journal of Empirical Finance*, 17(5): 867–894.

Benzoni, Luca, Pierre Collin-Dufresne, and Robert S Goldstein. 2011. "Explaining Asset Pricing Puzzles Associated with the 1987 Market Crash." *Journal of Financial Economics*, 101(3): 552–573.

Campbell, John Y, and John H Cochrane. 1999. "By Force of Habit: A Consumption-Based Explanation of Aggregate Stock Market Behavior." *Journal of Political Economy*, 107(2): 205–251.

Campbell, John Y, Carolin Pflueger, and Luis M Viceira. 2020. "Macroeconomic Drivers of Bond and Equity Risks." *Journal of Political Economy*, 128(8): 3148–3185.

Carpenter, Seth, Selva Demiralp, Jane Ihrig, and Elizabeth Klee. 2015. "Analyzing Federal Reserve Asset Purchases: From whom does the Fed buy?" *Journal of Banking & Finance*, 52: 230–244.

Chen, Yu, Thomas F Cosimano, Alex A Himonas, and Peter Kelly. 2009. "Asset Pricing with Long Run Risk and Stochastic Differential Utility: An Analytic Approach." *Available at SSRN 1502968.*

Chen, Yu, Thomas F Cosimano, and Alex A Himonas. 2010. "Continuous Time One-Dimensional Asset-Pricing Models with Analytic Price–Dividend Functions." *Economic Theory*, 42(3): 461–503.

Cochrane, John H. 2009. *Asset Pricing: Revised edition.* Princeton University Press.

Cochrane, John H, and Monika Piazzesi. 2005. "Bond Risk Premia." *American Economic Review*, 95(1): 138–160.

Cochrane, John H, and Monika Piazzesi. 2009. "Decomposing the Yield Curve."

Collin-Dufresne, Pierre, Michael Johannes, and Lars A Lochstoer. 2016. "Parameter Learning in General Equilibrium: The Asset Pricing Implications." *American Economic Review*, 106(3): 664–698.

Constantinides, George M, and Darrell Duffie. 1996. "Asset Pricing with Heterogeneous Consumers." *Journal of Political Economy*, 104(2): 219–240.

d 'Amico, Stefania, Don H Kim, and Min Wei. 2018. "TIPS from TIPS: The Informational Content of Treasury Inflation-Protected Security Prices." *Journal of Financial and Quantitative Analysis*, 53(1): 395–436.

Den Haan, Wouter J. 1995. "The Term Structure of Interest Rates in Real and Monetary Economies." *Journal of Economic Dynamics and Control*, 19(5-7): 909–940.

Donaldson, John B, Thore Johnsen, and Rajnish Mehra. 1990. “On the Term Structure of Interest Rates.” *Journal of Economic Dynamics and Control*, 14(3-4): 571–596.

Duffee, Gregory R. 2002. “Term Premia and Interest Rate Forecasts in Affine Models.” *The Journal of Finance*, 57(1): 405–443.

Duffee, Gregory R. 2013. “Bond pricing and the macroeconomy.” *Handbook of the Economics of Finance*, 2: 907–967.

Duffie, Darrell, and Larry G Epstein. 1992. “Stochastic Differential Utility.” *Econometrica: Journal of the Econometric Society*, 353–394.

Ellison, Martin, and Andreas Tischbirek. 2021. “Beauty Contests and the Term Structure.” *Journal of the European Economic Association*, 19(4): 2234–2282.

Fama, Eugene F, and Robert R Bliss. 1987. “The Information in Long-Maturity Forward Rates.” *The American Economic Review*, 680–692.

Gabaix, Xavier. 2012. “Variable Rare Disasters: An Exactly Solved Framework for Ten Puzzles in Macro-Finance.” *The Quarterly Journal of Economics*, 127(2): 645–700.

Gomez-Cram, Roberto, and Amir Yaron. 2021. “How important are inflation expectations for the nominal yield curve?” *The Review of Financial Studies*, 34(2): 985–1045.

Gürkaynak, Refet S, Brian Sack, and Jonathan H Wright. 2007. “The US Treasury Yield Curve: 1961 to the Present.” *Journal of Monetary Economics*, 54(8): 2291–2304.

Gürkaynak, Refet S, Brian Sack, and Jonathan H Wright. 2010. “The TIPS Yield Curve and Inflation Compensation.” *American Economic Journal: Macroeconomics*, 2(1): 70–92.

He, Zhiguo, and Arvind Krishnamurthy. 2013. “Intermediary Asset Pricing.” *American Economic Review*, 103(2): 732–770.

Hsu, Alex, Erica XN Li, and Francisco Palomino. 2021. “Real and Nominal Equilibrium Yield Curves.” *Management Science*, 67(2): 1138–1158.

Jappelli, Ruggero, Marti G Subrahmanyam, and Loriana Pelizzon. 2023. "Quantitative Easing, the Repo Market, and the Term Structure of Interest Rates."

Katagiri, Mitsuru. 2022. "Equilibrium Yield Curve, the Phillips Curve, and Monetary Policy." *Journal of Money, Credit and Banking*, 54(8): 2235–2272.

Kekre, Rohan, Moritz Lenel, and Federico Mainardi. 2022. "Monetary Policy, Segmentation, and the Term Structure."

Kliem, Martin, and Alexander Meyer-Gohde. 2022. "(Un) Expected Monetary Policy Shocks and Term Premia." *Journal of Applied Econometrics*, 37(3): 477–499.

Kraft, Holger, Thomas Seiferling, and Frank Thomas Seifried. 2017. "Optimal Consumption and Investment with Epstein–Zin Recursive Utility." *Finance and Stochastics*, 21: 187–226.

Lettau, Martin, and Jessica A Wachter. 2011. "The Term Structures of Equity and Interest Rates." *Journal of Financial Economics*, 101(1): 90–113.

Malloy, Christopher J, Tobias J Moskowitz, and Annette Vissing-Jørgensen. 2009. "Long-Run Stockholder Consumption Risk and Asset Returns." *The Journal of Finance*, 64(6): 2427–2479.

Mehra, Rajnish, et al. 2007. "The Equity Premium Puzzle: A Review." *Foundations and Trends® in Finance*, 2(1): 1–81.

Melissinos, Errikos. 2023. "A Perturbation Solution for the Pricing of Long-Term Bonds under Recursive Utility."

Melissinos, Errikos. 2024. "SDFPricing: A Julia Package for Asset Pricing Based on a Stochastic Discount Factor Process."

Neely, Christopher J, David E Rapach, et al. 2008. "Real Interest Rate Persistence: Evidence and Implications." *Federal Reserve Bank of St. Louis Review*, 90(6): 609–641.

Pflueger, Carolin E, and Luis M Viceira. 2016. "Return Predictability in the Treasury Market: Real Rates, Inflation, and Liquidity." *Handbook of Fixed-Income Securities*, 191–209.

Piazzesi, Monika, and Martin Schneider. 2006. "Equilibrium Yield Curves." *NBER Macroeconomics Annual*, 21: 389–442.

Rietz, Thomas A. 1988. "The Equity Risk Premium a Solution." *Journal of Monetary Economics*, 22(1): 117–131.

Rudebusch, Glenn D, and Eric T Swanson. 2008. "Examining the Bond Premium Puzzle with a DSGE Model." *Journal of Monetary Economics*, 55: S111–S126.

Rudebusch, Glenn D, and Eric T Swanson. 2012. "The Bond Premium in a DSGE Model with Long-Run Real and Nominal Risks." *American Economic Journal: Macroeconomics*, 4(1): 105–43.

Savov, Alexi. 2011. "Asset Pricing with Garbage." *The Journal of Finance*, 66(1): 177–201.

Schneider, Andrés. 2022. "Risk-Sharing and the Term Structure of Interest Rates." *The Journal of Finance*, 77(4): 2331–2374.

Shiller, Robert J. 1979. "The Volatility of Long-Term Interest Rates and Expectations Models of the Term Structure." *Journal of Political Economy*, 87(6): 1190–1219.

Singleton, Kenneth J. 1980. "Expectations Models of the Term Structure and Implied Variance Bounds." *Journal of Political Economy*, 88(6): 1159–1176.

Tsai, Jerry. 2015. "Rare Disasters and the Term Structure of Interest Rates." *Available at SSRN 2669992*.

Tsai, Jerry, and Jessica A Wachter. 2018. "Pricing Long-Lived Securities in Dynamic Endowment Economies." *Journal of Economic Theory*, 177: 848–878.

Van Binsbergen, Jules H, Jesús Fernández-Villaverde, Ralph SJ Koijen, and Juan Rubio-Ramírez. 2012. "The Term Structure of Interest Rates in a DSGE Model with Recursive Preferences." *Journal of Monetary Economics*, 59(7): 634–648.

Vayanos, Dimitri, and Jean-Luc Vila. 2021. "A Preferred-Habitat Model of the Term Structure of Interest Rates." *Econometrica*, 89(1): 77–112.

Wachter, Jessica A. 2006. "A Consumption-Based Model of the Term Structure of Interest Rates." *Journal of Financial Economics*, 79(2): 365–399.

Wachter, Jessica A. 2013. "Can time-varying risk of rare disasters explain aggregate stock market volatility?" *The Journal of Finance*, 68(3): 987–1035.

Wachter, Jessica A, and Yicheng Zhu. 2019. "Learning with Rare Disasters." *Available at SSRN 3407397.*

Chapter 2

SDFPricing: A Julia Package for Asset Pricing Based on a Stochastic Discount Factor Process

Abstract

I introduce a package in the Julia programming language to perform asset pricing based on a stochastic discount factor in continuous time. Prices are computed through Monte Carlo simulations according to a pricing partial differential equation and the corresponding Feynman-Kac formula. At this stage it is possible to compute a) prices of zero-coupon fixed-income securities and b) price-dividend ratios, which also allow the calculation of prices and returns of dividend-paying securities. The package is easy to use and is designed for applications in teaching and research. I illustrate the functionality of the package with examples and an application. In particular, I show how asset prices react to shifts in economic variables within a consumption-based model, and I discuss to what extent these shifts can be classified as monetary policy shocks or information shocks in connection to monetary policy announcements.

2.1 Introduction

The stochastic discount factor (SDF) is a fundamental concept in asset pricing. Given a joint process for the SDF and a payoff stream, the corresponding security can be priced. This paper introduces a package in the Julia programming language, that allows users to easily perform this computation. The package is called *SDFPricing* and is available at Github. While there are other asset pricing resources publicly available, they do not focus on using the SDF process as an input.

The package allows the pricing of zero-coupon bonds, which can then be combined to get the prices of coupon bonds.[1] In addition, it is possible to specify a process for a stochastic dividend stream, to calculate the price-dividend ratio of the corresponding security, and subsequently its expected return and price. The results can be made arbitrarily accurate given the processes of the SDF and the dividend stream, as long as enough samples are computed. In the future, I am planning to expand the functionality of the package, to include more capabilities and specialised functions. In particular, functionality that enables the user to a) choose the right tradeoff between accuracy and speed, and b) automatically calculate the price of special kinds of securities, such as options.

I have chosen to develop the package in Julia for two main reasons. Firstly, Julia is a high-level programming language that is powerful and easy to use. Its syntax is similar to Python and MATLAB, which are widely used languages, and it is designed for scientific programming. Secondly, Julia includes the *DifferentialEquations* package, which arguably includes the most comprehensive publicly available tools for solving differential equations, including stochastic differential equations, which are at the core of the pricing problem. Furthermore, there is particular interest in Julia from the finance and economics communities.[2] The asset pricing package that I introduce is the first within the context of the Julia programming language.

In other programming languages, the licensed Computational Finance Suite of MATLAB contains resources that can be used for asset pricing.[3] Moreover, Quantlib, which is a free and open-source C++ library, provides asset pricing

[1] Each coupon can be thought of as a separate zero-coupon bond. Adding up the prices of all coupons (including the final payoff) gives the price of the full coupon bond.

[2] For example, the QuantEcon Organisation is actively engaged in the language.

[3] Information can be found on the official website `https://www.mathworks.com/solutions/computational-finance/computational-finance-suite.html`

tools, while it can also be used in Python via a dedicated library.[4] Both Quantlib and the resources available in Matlab are tools tailored to the industry and they do not focus on the pricing of assets starting with an SDF as an input, as is the case with my package.

In Section 2.2, I describe the package and its functionality. A central function of the package is *solve*, which takes a variable of type *Problem* and a variable of type *SolutionSettings* and returns a variable of type *Solution*, which contains the solution to the pricing problem described by the inputs. The solution is calculated using Monte Carlo simulations and the Feynman-Kac formula.

In Section 2.3, I provide concrete examples of how to use the package along with the results. I use SDFs that depend both on one state variable and two state variables, and I compute, zero-coupon bond prices, price-consumption ratios, and returns of dividend-paying securities. The examples also include a replication of the model in Campbell and Cochrane (1999), which can be easily computed with the package.

In Section 2.4, I present an application using the functionality of the package. In particular, I analyse the effect of general monetary policy shocks on interest rates and asset prices in the context of a simple consumption-based model. This analysis is motivated by the literature on Delphic and Odyssean shocks (Campbell et al. 2012; Nakamura and Steinsson 2018; Jarociński and Karadi 2020; Andrade and Ferroni 2021; Altavilla et al. 2019), which are different types of monetary policy surprises (Gürkaynak, Sack and Swanson 2005; Kuttner 2001). Both kinds of shocks are caused by actions or announcements of central banks. However, Delphic shocks reveal information about the underlying state of the economy. In contrast, Odyssean shocks reveal information about the future conduct of monetary policy.[5] In the literature, the type of shock is identified depending on how financial variables react after the monetary policy announcement. Using a simple consumption-based setup I analyse the possible reactions of interest rates and asset prices to monetary policy announcements. Importantly, monetary policy may not operate through one channel only. While a traditional monetary policy shock

[4] Information can be found on the official website https://www.quantlib.org.

[5] The term Odyssean is due to the central bank committing to a future policy beforehand, as Odysseus did when he tied himself to the mast of his ship to avoid the Sirens. Originally, in Campbell et al. (2012) the emphasis was placed on whether the central bank is committed to the future policy or not. In subsequent literature, the emphasis regarding the distinction between the two kinds of shocks was placed more on whether the shocks reveal important information about monetary policy actions as opposed to underlying macroeconomic conditions. In these papers Delphic shocks are identified with information shocks and Odyssean shocks are identified with monetary policy shocks.

can be justified within a simple consumption-based model, my analysis shows that if monetary policy can also affect consumption volatility, then the response of financial variables can be much more unpredictable and give rise to movements in the short-term rate, and in asset prices that are usually associated with an information shock. In addition, information shocks may also induce the effects that are associated with a standard monetary policy shock. In particular, this can happen if the central bank *reveals* the kind of information about the consumption process that is usually assumed to be *caused* by a standard monetary policy shock. These results suggest that accurate theoretical models may be required for the classification of monetary policy shocks.

Finally, Section 2.5 concludes.

2.2 Package Description

In the following description, I assume a single state variable and a single Wiener process for the simplicity of the formulas. However, it is also possible to have several state variables and several Wiener processes. The corresponding formulas are given in Appendix 2.A.

2.2.1 Theory

2.2.1.1 Zero Coupon Bond

Following Cochrane (2009) the pricing equation for an asset that pays no dividends in continuous time can be written as:

$$\mathrm{E}\Big[\mathrm{d}\Big(\Lambda_t Q(x_t,t,m)\Big)\Big] = 0, \quad Q(x_t,t,0) = g(t,x_t) \tag{2.1}$$

where m is the remaining maturity until the payoff is due, $g(\cdot)$ is a function for the terminal payoff for each time t, which can also depend on the state variable, but which typically is just equal to 1, Λ_t is the SDF and Q is the price function of the asset in terms of the state variable, current time and the remaining maturity of the asset. If the process for the SDF and the state variable is known, then this equation can be solved to derive the price function $Q(x_t, t, m)$ for any value of x_t, t, and m.[6] In particular, the state variable follows some process that can be

[6]Here, I show the dependence on time explicitly. Often it will be the case that there will be no dependence on time and the state variable and remaining maturity will be sufficient to determine the price of assets. The case without explicit time dependence is also easier to compute

written in stochastic differential equation (SDE) form as:

$$\mathrm{d}x_t = \mu(x_t, t)\mathrm{d}t + \sigma(x_t, t)\mathrm{d}W_t \tag{2.2}$$

where μ is the drift of the state variable, σ is the diffusion of the state variable, and W_t is the Wiener process. The process for the SDF can be written in SDE form as:

$$\frac{\mathrm{d}\Lambda_t}{\Lambda_t} = \mu_\Lambda(x_t, t)\mathrm{d}t + \sigma_\Lambda(x_t, t)\mathrm{d}W_t \tag{2.3}$$

Next, using Ito's Lemma, it is possible to express the SDE that the price function Q follows. This is given by:

$$\mathrm{d}Q = \left(\frac{\partial Q}{\partial t} - \frac{\partial Q}{\partial m} + \mu(x_t, t)\frac{\partial Q}{\partial x_t} + \frac{1}{2}\sigma(x_t, t)^2\frac{\partial^2 Q}{\partial x_t^2}\right)\mathrm{d}t + \sigma(x_t, t)\frac{\partial Q}{\partial x_t}\mathrm{d}W_t \tag{2.4}$$

where I have stopped showing the dependence of Q on the arguments for simplicity, and I have used that $\mathrm{d}m = -\mathrm{d}t$. This expression can now be inserted in Equation (2.1) to derive the pricing partial differential equation (SDE) that the price function Q follows:

$$\mathrm{E}\left[\frac{\mathrm{d}(\Lambda_t Q)}{\Lambda_t}\right] = 0 \Rightarrow \mathrm{E}\left[\frac{\mathrm{d}\Lambda_t}{\Lambda_t}Q + \mathrm{d}Q + \frac{\mathrm{d}\Lambda_t}{\Lambda_t}\mathrm{d}Q\right] = 0 \tag{2.5}$$

$$\Rightarrow -r(x_t, t)Q + \frac{\partial Q}{\partial t} - \frac{\partial Q}{\partial m} + \mu(x_t, t)\frac{\partial Q}{\partial x_t} + \frac{1}{2}\sigma(x_t, t)^2\frac{\partial^2 Q}{\partial x_t^2} + \sigma(x_t, t)\sigma_\Lambda(x_t, t)\frac{\partial Q}{\partial x_t} = 0$$

where $r(x_t, t) = -\mathrm{E}[\mathrm{d}\Lambda_t/\Lambda_t]$ is the risk-free rate. This PDE can be solved using the Feynman-Kac formula, which states that the solution to the PDE is given by the expected value of the terminal payoff of the asset under the risk-neutral measure. This is given by:

$$Q(x_t, t, m) = \mathrm{E}\left[\exp\left(-\int_t^{t+m} r(\hat{x}_s, s)\mathrm{d}s\right)g(t+m, \hat{x}_{t+m})\middle|\hat{x}_t = x_t\right] \tag{2.6}$$

where $\hat{x}$ follows the modified process:

$$\mathrm{d}\hat{x}_t = \Big(\mu(\hat{x}_t, t) + \rho_{cx}\sigma(\hat{x}_t, t)\sigma_\Lambda(\hat{x}_t, t)\Big)\mathrm{d}t + \sigma(\hat{x}_t, t)\mathrm{d}W_t \tag{2.7}$$

---

with the package.

This process has a modified process compared to the original process for the state variable. If the modification is equal to zero then the process for the state variable is the same as the original process, which implies that prices are set by risk-neutral investors (or equivalent to risk-neutral investors). If the modification is not equal to zero then there is a risk premium or a risk discount. By Monte Carlo simulations of the modified process in Equation (2.7), it is possible to compute the expectation in Equation (2.6), which gives the value of the zero-coupon bond. This allows us to also derive the instantaneous return on bonds ($\mathrm{d}Q/Q$), which follows directly from Ito's Lemma as shown in Equation (2.4).

Finally, if the zero-coupon bond price is integrated up to infinity then this gives rise to the price of a perpetuity:

$$Z(x_t, t) \equiv \int_t^{\infty} Q(x_t, t, s)\mathrm{d}s \tag{2.8}$$

2.2.1.2 Price-Dividend Ratio

Based on the zero-coupon bonds it is also possible to derive the price-dividend ratio of a dividend-paying security. In particular, if at time t the price of the security is P_t and the dividend stream is D_t, then the price-dividend ratio is defined as:[7]

$$p_t(X_t) \equiv \frac{P_t}{D_t} \tag{2.9}$$

Where D_t follows the process:

$$\frac{\mathrm{d}D_t}{D_t} = \mu_D(x_t, t)\mathrm{d}t + \sigma_D(x_t, t)\mathrm{d}W_{Dt} \tag{2.10}$$

In order, to show the connection to zero-coupon bonds, I also define the dividend-strip $P_t^{str}(T)$, which pays an amount equal to the dividend of the security only at a specific time T ($P_t^{str}(0) = D_t$). Then the dividend-strip price ratios are:

$$p_t^{str}(m) \equiv \frac{P_t^{str}(m)}{D_t}, \quad p_t^{str}(0) \equiv 1 \tag{2.11}$$

By definition the price of the dividend-paying security is the sum of the prices of the dividend strips:

$$P_t = \int_0^{\infty} P_t^{str}(s)\mathrm{d}s \tag{2.12}$$

[7] Given the homotheticity of preferences, the price-dividend and price-consumption ratios are only functions of the state variable.

The price-dividend ratio of the dividend-paying security is then:[8]

$$\frac{P_t}{D_t} \equiv p_t = \int_0^\infty \frac{P_t^{str}(s)}{D_t} \mathrm{d}s = \int_0^\infty p_t^{str}(s) \mathrm{d}s \tag{2.13}$$

Given this setup we can again apply Equation (2.1) as before:[9]

$$\mathrm{E}\Big[\mathrm{d}\Big(\Lambda_t P^{str}(x_t, t, m)\Big)\Big] = 0 \quad \forall t \in (t_0, T), \quad P^{str}(x_t, t, 0) = g(t) \quad \forall x_t \tag{2.14}$$

This can then be expressed in terms of the ratio p_t^{str} as (where I stop showing the explicit dependence on the arguments for simplicity):[10]

$$\begin{aligned} &\mathrm{E}\Big[\mathrm{d}\Big(\Lambda_t p_t^{str} D_t\Big)\Big] = 0 \\ \Rightarrow &\mathrm{E}\left[\frac{\mathrm{d}\Lambda_t}{\Lambda_t} p^{str} + \mathrm{d}p^{str} + \frac{\mathrm{d}D_t}{D_t} p^{str} + \frac{\mathrm{d}\Lambda_t}{\Lambda_t} \mathrm{d}p^{str} + \frac{\mathrm{d}\Lambda_t}{\Lambda_t}\frac{\mathrm{d}D_t}{D_t} p^{str} + \frac{\mathrm{d}D_t}{D_t} \mathrm{d}p^{str}\right] = 0 \end{aligned} \tag{2.15}$$

In addition, Ito's lemma also applies to the price-dividend ratio, giving rise to an expression similar to Equation (2.4):

$$\mathrm{d}p_t^{str} = \left(\frac{\partial p_t^{str}}{\partial t} - \frac{\partial p_t^{str}}{\partial m} + \mu(x_t, t)\frac{\partial p_t^{str}}{\partial x_t} + \frac{1}{2}\sigma(x_t, t)^2 \frac{\partial^2 p_t^{str}}{\partial x_t^2}\right)\mathrm{d}t + \sigma(x_t, t)\frac{\partial p_t^{str}}{\partial x_t}\mathrm{d}W_{xt} \tag{2.16}$$

By inserting the expressions for p_t^{str} (Equation 2.16), D_t (Equation 2.10), and Λ_t (Equation 2.3) into Equation (2.15) we get:

$$\begin{aligned} &- r(x_t, t)p_t^{str} + \frac{\partial p_t^{str}}{\partial t} - \frac{\partial p_t^{str}}{\partial m} + \mu(x_t, t)\frac{\partial p_t^{str}}{\partial x_t} + \frac{1}{2}\sigma(x_t, t)^2 \frac{\partial^2 p_t^{str}}{\partial x_t^2} + \mu_D(x_t, t) p_t^{str} \\ &+ \rho_{cx}\sigma(x_t, t)\sigma_\Lambda(x_t, t)\frac{\partial p_t^{str}}{\partial x_t} + \rho_{cD}\sigma_D(x_t, t)\sigma_\Lambda(x_t, t)p_t^{str} + \rho_{xD}\sigma_D(x_t, t)\sigma(x_t, t)\frac{\partial p_t^{str}}{\partial x_t} = 0 \end{aligned} \tag{2.17}$$

[8] A similar expression for discrete time is given in Wachter (2006).

[9] Now, I have adapted the notation as the price of the dividend-strip depends on the state variable and time explicitly.

[10] The derivation here is similar to Chen, Cosimano and Himonas (2010)

which is similar to Equation (2.5) but with additional terms that depend on the dividend process. This PDE can also be solved using the Feynman-Kac formula:

$$p_t^{str}(x_t, t, m) = \mathrm{E}\left[\exp\left(-\int_t^{t+m} \tilde{r}(\tilde{X}_s, s)\mathrm{d}s\right) g(t+m) \middle| \tilde{x}_t = x_t\right] \tag{2.18}$$

where

$$\tilde{r}(x_t, t) = r(x_t, t) - \mu_D(x_t, t) - \rho_{cD}\sigma_D(x_t, t)\sigma_\Lambda(x_t, t) \tag{2.19}$$

is adjusted due to the extra terms coming from the dividend stream. The process for the modified state variable is given by:

$$\mathrm{d}\tilde{x}_{it} = \Big(\sigma(\tilde{x}_{it}, t)(\rho_{cx}\sigma_\Lambda(\tilde{x}_{it}, t) + \rho_{xD}\sigma_D) + \mu(\tilde{x}_t, t)\Big)\mathrm{d}t + \sigma(\tilde{x}_t, t)\mathrm{d}W_{xt} \tag{2.20}$$

Using the ratios of the prices of the dividend strips over the current dividend for each maturity it is possible to integrate over all maturities to get the price-dividend ratio of the dividend-paying security as shown in Equation (2.13).

Finally, the return of the dividend-paying security can be computed:

$$\begin{aligned}
&\frac{\mathrm{d}P_t}{P_t} + \frac{D_t}{P_t}\mathrm{d}t = \frac{\mathrm{d}(p_t D_t)}{p_t D_t} + \frac{1}{p_t}\mathrm{d}t = \frac{\mathrm{d}p_t}{p_t} + \frac{\mathrm{d}D_t}{D_t} + \frac{\mathrm{d}p_t\mathrm{d}D_t}{p_t D_t} + \frac{1}{p_t}\mathrm{d}t \\
=&\frac{1}{u_t}\Bigg(\left(\frac{\partial p_t}{\partial t} + \mu(x_t, t)\frac{\partial p_t}{\mathrm{d}x_t} + \frac{1}{2}\sigma(x_t, t)^2\frac{\partial^2 p}{\partial x_t^2} + \mu_D(x_t, t)p_t + \frac{1}{2}\sigma_D(x_t, t)\sigma\frac{\partial p_t}{\partial x_t} + 1\right)\mathrm{d}t \\
&+ \left(\sigma(x_t, t)\frac{\partial p_t}{\partial x_t} + \sigma_D(x_t, t)\right)\mathrm{d}W_t\Bigg)
\end{aligned} \tag{2.21}$$

Where Ito's Lemma has been applied to p_t.

2.2.2 Implementation

The implementation of the package follows closely the logic of the Feynman-Kac formula. To get the price of a zero-coupon security, the modified state variable(s) needs to be simulated and the values of these simulations are used to simulate the stochastic integral of the corresponding short rate.[11] The user needs to specify formulas for the drift and diffusion of the state variable, while for the stochastic integral, the r function needs to be given as a drift and 0 should be given as the

[11] In the case of the continuous payoff security the modified short rate should be given by the user as defined in Equation 2.19.

diffusion. Each simulation of the stochastic integral is expressed as:[12]

$$\mathcal{I}_i \approx -\int_t^{t+m} \underbrace{r(\hat{x}_s)}_{\text{drift for simulation}} \mathrm{d}s, \quad i = 1, 2, \ldots, N_s$$

or in the case of the dividend-paying security:

$$\mathcal{I}_i \approx -\int_t^{t+m} \underbrace{\tilde{r}(\tilde{x}_s)}_{\text{drift for simulation}} \mathrm{d}s, \quad i = 1, 2, \ldots, N_s \tag{2.22}$$

where N_s is the number of simulations used. Given a large number of samples, the price of the security is approximated by:

$$Q(x_t, t, m) = \mathrm{E}\left[\exp\left(-\int_t^{t+m} r(\hat{x}_s)\mathrm{d}s\right)\right] g(t+m, x_{t+m}) \approx \frac{1}{N_s}\sum_{i=1}^{N_s} \exp\left(\mathcal{I}_i\right) g(t+m, x_{t+m}) \tag{2.23}$$

This becomes a good approximation for a large enough number of simulations.

The main function of the package, which performs the computation above is *solve* which takes as input a variable of type *Problem* and a variable of type *SolutionSettings*. It then returns a variable of type *Solution*. The *Problem* type contains the information for the drift and the diffusion of the processes to be simulated, it also contains the terminal function for the payoff (typically just equal to one at maturity). The drift and diffusion functions for the processes that are meant to be simulated are given as they would be for the standard DifferentialEquations package in Julia.[13] The *SolutionSettings* type contains information that is necessary for the solution of the problem, such as the grid for the state variable, the number of simulations to be performed, and the algorithm to be used for the simulations.[14] This type can also be given a specification that a continuous payoff variable is being simulated. In this case, prices of dividend strips are computed[15] and they are then also integrated to give the price-dividend ratio. Finally, the *Solution* type constitutes the result of the computation that can be used as a normal function. This means that it can be called with specific arguments for time and the state variable(s) to return the price of the zero-coupon security or the price-dividend ratio of the dividend-paying security if a continuous payoff specification

[12] The simulation of the following integral is handled internally by the StochasticDiffEq.jl package, as is the simulation of the state variable.

[13] Or the more specialised package StochDiffEq.jl for stochastic differential equations.

[14] The algorithm is one of the algorithms offered in the standard DifferentialEquations.jl package. More information can be found in the documentation page.

[15] From the point of view of the code these are the same as zero-coupon securities. The difference should be in the drift and the modified short rate of the given process, which should be modified as specified in 2.13.

is given.[16]

Finally, a convenience function is also given to compute the derivatives of the price-dividend ratio with respect to the state variable. This can facilitate the computation of the return of the dividend-paying security as shown in Equation (2.21).[17]

A more detailed technical description of the package is provided in Appendix 2.B, while the following examples illustrate how the package can be used.

2.3 Examples

2.3.1 One State Variable

2.3.1.1 Time-Varying Consumption Drift – Zero-Coupon Security

While the package can be used with any process for the SDF, the examples in this paper are from the context of a consumption-based model, in which the investor has CRRA utility.[18] The specific code and all the results for the examples are shown in Jupyter notebooks that are included in Appendix 2.C.[19] The consumption flow (or just consumption for simplicity) process is exogenous and given by:

$$\mathrm{d}\log C_t = \mathrm{d}c_t = \mu_c(x_t)\mathrm{d}t + \sigma_c \mathrm{d}W_t, \quad \mu_c(x_t) = \mu_{c0} + x_t \tag{2.24}$$

By Ito's Lemma and the fact that $\Lambda_t = e^{-\rho t}C^{-\gamma}$, the process for the SDF is given by:

$$\frac{\mathrm{d}\Lambda_t}{\Lambda_t} = \Big(-\rho - \gamma\mu_c(x_t) + \frac{1}{2}\gamma^2\sigma_c^2\Big)\mathrm{d}t - \gamma\sigma_c\mathrm{d}W_t \tag{2.25}$$

This also provides the function for the short rate:

$$r(x_t) = -\mathrm{E}\left[\frac{\mathrm{d}\Lambda_t}{\Lambda_t}\right]\frac{1}{\mathrm{d}t} = \rho + \gamma\mu_c(x_t) - \frac{1}{2}\gamma^2\sigma_c^2 \tag{2.26}$$

The process for the state variable is given by:

$$\mathrm{d}x = -\log\phi(\bar{x} - x_t)\mathrm{d}t + \sigma_x\mathrm{d}W_t \tag{2.27}$$

[16] The *Solution* type is further subdivided into *SinglePayoffSolution* and *ContinuousPayoffSolution*.

[17] At the point of writing this function can only be used when the problem has one state variable.

[18] It would also be possible to use more general utility processes, such as recursive utility as is shown in Melissinos (2023*a*).

[19] These examples are also included as part of the code of the package to facilitate users.

where $\bar{x}$ is the stochastic steady state (or steady state for simplicity), the point at which the process has a drift of zero. So, based on Equation (2.7), the process for the modified state variable is:

$$\mathrm{d}\hat{x} = \underbrace{\Big(-\log\phi(\bar{x} - \hat{x}_t) - \gamma\rho_{cx}\sigma_x\sigma_c \Big)}_{\text{drift for simulation}}\mathrm{d}t + \underbrace{\sigma_x}_{\text{diffusion for simulation}}\mathrm{d}W_t \tag{2.28}$$

Apart from computing the regular price of the zero-coupon bond, I also compute the price of the risk-neutral zero-coupon bond. In this case, I simulate the unmodified state variable of the problem. Then the term premium can be computed as the difference between the yield and the risk-neutral yield. As can be seen in the results, the term structure can be either upward or downward-sloping (positive or negative yield spread respectively), depending on the value of the state variable. This is because the state variable is expected to revert to the steady state. So, long-term yields which are in some respect combinations of expected future short-rates are higher (lower) compared to short-term yields, when short rates are expected to increase (decrease). In addition, within this stylised model, even though consumption drift is significantly variable, the term premium is negative and tiny.

2.3.1.2 Time-Varying Consumption Diffusion - Zero-Coupon Security

In this example, instead of having a time-varying consumption drift as previously, I have a time-varying consumption diffusion. The consumption process is given by:

$$\mathrm{d}\log C_t = \mathrm{d}c_t = \mu_c\mathrm{d}t + \sigma_c(x_t)\mathrm{d}W_t, \quad \sigma_c(x_t) = \begin{cases} \frac{2\sigma_{c0}}{1+\exp(-2x)} & \text{if } x < 0 \\ \frac{4\sigma_{c0}}{1+\exp(-x)} - 1 & \text{otherwise} \end{cases} \tag{2.29}$$

while the state variable follows the same process as in the previous example. The process could have also followed a CIR process (Cox, Ingersoll Jr and Ross 1985) and then be used as the consumption diffusion. This would also ensure that the consumption diffusion is positive. However, I use this different process to show that the package can handle processes that are non-affine in terms of the state variable. In addition, I avoid using a simple exponential that would ensure the positivity of the consumption diffusion, because the exponential can increase too fast and make some paths unstable. The functional form above ensures that the consumption diffusion is bounded between 0 and $3\sigma_{c0}$, where σ_{c0} is the value at the steady state. The modified state is then given by:

$$\mathrm{d}\hat{x} = \Big(-\log\phi(\bar{x} - \hat{x}_t) - \gamma\rho_{cx}\sigma_x\sigma_c(\hat{x}_t)\Big)\mathrm{d}t + \sigma_x \mathrm{d}W_{ct} \tag{2.30}$$

As can be seen in the results, the short-term rate is slightly decreasing with consumption volatility due to the precautionary savings motive of agents. The term premium is negative and also very small.

2.3.1.3 Time-Varying Consumption Drift - Dividend-Paying Security

In this example, I show how the package can be used to compute the price-dividend ratio of a dividend-paying security for the same consumption process as in the first example. In general, the dividend can follow any process. However, here I assume that the dividend process follows the same process as the consumption process. In this case, the price-dividend ratio is called price-consumption ratio, and dividend strips are called consumption strips. Here, the modified process is not the same as in the case with a zero-coupon bond. Following Equation (2.20) the modified process is given by:

$$\mathrm{d}\tilde{x} = \underbrace{\Big(-\log\phi(\bar{x} - \tilde{x}_t) - (\gamma - 1)\rho_{cx}\sigma_x\sigma_c\Big)}_{\text{drift for simulation}}\mathrm{d}t + \underbrace{\sigma_x}_{\text{diffusion for simulation}} \mathrm{d}W_{ct} \tag{2.31}$$

Unlike the case of the zero-coupon bond for the consumption strip the short rate function that is used in the simulation is also modified. So, following Equation (2.19) the modified short rate is given by:

$$\tilde{r}(\tilde{x}_t, t) = \rho + \gamma\mu_c(\tilde{x}_t) - \frac{1}{2}\gamma^2\sigma_c^2 - \mu_c(\tilde{x}_t) + \gamma\sigma_c^2 \tag{2.32}$$

The results show that when consumption drift is significantly varying the price of the consumption perpetuity is significantly volatile. In addition, it is possible to use the resulting price-consumption ratio to calculate its derivatives and then use the derivatives to get the return of the security as a function of the state variable.[20] The return of the security is very close to the short-term rate verifying the idea behind the equity premium puzzle. In particular, this model would predict a very small equity premium, which is not consistent with the data.

[20] Here, the derivatives work best when the solution is computed based on an interpolation function that is calculated based on the DataInterpolations package.

2.3.1.4 Time-Varying Consumption Diffusion - Dividend-Paying Security

The dividend-paying security can also be computed when consumption diffusion is time-varying. Again I assume that the dividend is following the same process as consumption. So, the modified process for the state variable is given by:

$$\tilde{r}(\tilde{x}_t, t) = \rho + \gamma\mu_c - \frac{1}{2}\gamma^2\sigma_c(\tilde{x}_t)^2 - \mu_c + \gamma\sigma_c(\tilde{x}_t)^2 \tag{2.33}$$

Interestingly for $\gamma = 2$ the function above becomes a constant which makes the price-consumption ratio also a constant. This implies that holding consumption constant and changing consumption volatility does not affect prices and returns. However, the short rate is still a decreasing function of consumption volatility so the equity premium is increasing in the consumption diffusion.

2.3.2 Two State Variables

The package also works with more than one state variable. In this example, I compute prices of zero-coupon bonds, while also allowing consumption drift and consumption diffusion to vary independently. So, the consumption process is given by:

$$\mathrm{d}c_t = \mu_c(x_{1t})\mathrm{d}t + \sigma_c(x_{2t})\Big(1 - |\rho_{cx1}| - |\rho_{cx2}|\Big)\mathrm{d}W_{c1t} + \rho_{cx1}\sigma_c(x_{1t})\mathrm{d}W_{x1t} + \rho_{cx2}\sigma_c(x_{2t})\mathrm{d}W_{x2t} \tag{2.34}$$

Where $\mu_c(\cdot)$ and $\sigma_c(\cdot)$ are the same as in Subsections 2.3.1.1 and 2.3.1.2 respectively. The processes of the state variables are given by:

$$\begin{aligned}
\mathrm{d}x_{1t} &= -\log\phi_1(\bar{x}_1 - x_{1t})\mathrm{d}t + \sigma_{x1}\frac{1}{1+\rho_{12}}\mathrm{d}W_{x1t} + \sigma_{x1}\frac{\rho_{12}}{1+\rho_{12}}\mathrm{d}W_{x2t} \\
\mathrm{d}x_{2t} &= -\log\phi_2(\bar{x}_2 - x_{2t})\mathrm{d}t + \sigma_{x2}\frac{\rho_{21}}{1+\rho_{21}}\mathrm{d}W_{x1t} + \sigma_{x1}\frac{\rho_{21}}{1+\rho_{21}}\mathrm{d}W_{x2t}
\end{aligned} \tag{2.35}$$

W_{ct}, W_{x1t}, and W_{x2t} are independent Wiener processes, but based on the structure above the correlations between the various components are:

$$\begin{aligned}
\mathrm{E}[\mathrm{d}c_t \mathrm{d}x_{1t}] &= \left(\rho_{cx1}\frac{1}{1+\rho_{12}} + \rho_{cx2}\frac{\rho_{12}}{1+\rho_{12}}\right)\sigma_c(x_{2t})\sigma_{x1}\mathrm{d}t \approxeq \rho_{cx1}\sigma_c\sigma_{x1}\mathrm{d}t \\
\mathrm{E}[\mathrm{d}c_t \mathrm{d}x_{2t}] &= \left(\rho_{cx2}\frac{1}{1+\rho_{21}} + \rho_{cx1}\frac{\rho_{21}}{1+\rho_{21}}\right)\sigma_c(x_{2t})\sigma_{x2}\mathrm{d}t \approxeq \rho_{cx2}\sigma_c(x_{2t})\sigma_{x_2}\mathrm{d}t \\
\mathrm{E}[\mathrm{d}x_{1t}\mathrm{d}x_{2t}] &= \sigma_{x1}\sigma_{x2}\frac{\rho_{21}+\rho_{12}}{1+\rho_{12}+\rho_{21}+\rho_{12}\rho_{21}}\mathrm{d}t \approxeq (\rho_{12}+\rho_{21})\sigma_{x1}\sigma_{x2}\mathrm{d}t \qquad (2.36)
\end{aligned}$$

with the approximate equalities being valid when ρ_{12} and ρ_{21} are small. The benefit of this setup is that consumption diffusion is equal to $\sigma_c(\hat{x}_{2t})$, the diffusion of the state variables is close to σ_{x1} and σ_{x2} when ρ_{12} and ρ_{21} are small, and a correlation structure can still be maintained between the consumption process and the state variables by an appropriate choice of parameters ρ_{cx1}, ρ_{cx2}, ρ_{12}, and ρ_{21}. For example, a negative correlation between consumption and consumption diffusion can be specified by letting ρ_{cx2} be negative, or a correlation between the two state variables can be specified without introducing correlation with consumption by letting ρ_{12} and ρ_{21} be different from zero. Similar to the above the modified process is given by:

$$\begin{aligned}
\mathrm{d}\hat{x}_{1t} &= \Big(-\log\phi_1\cdot(\bar{x}_1-\hat{x}_{1t}) + \rho_{cx1}\sigma_c(x_t)\sigma_{x1}\Big)\mathrm{d}t + \sigma_{x1}\frac{1}{1+\rho_{12}}\mathrm{d}W_{x1t} + \sigma_{x1}\frac{\rho_{12}}{1+\rho_{12}}\mathrm{d}W_{x2t} \\
\mathrm{d}\hat{x}_{2t} &= \Big(-\log\phi_2\cdot(\bar{x}_2-\hat{x}_{2t}) + \rho_{cx2}\sigma_c(x_t)\sigma_{x2}\Big)\mathrm{d}t + \sigma_{x2}\frac{\rho_{21}}{1+\rho_{21}}\mathrm{d}W_{x1t} + \sigma_{x2}\frac{1}{1+\rho_{21}}\mathrm{d}W_{x2t}
\end{aligned} \qquad (2.37)$$

Finally, for the computation of the prices of zero-coupon bonds, the short rate is unmodified and a function of two variables:

$$r(x_{1t},x_{2t}) = -\mathrm{E}\left[\frac{\mathrm{d}\Lambda_t}{\Lambda_t}\right]\frac{1}{\mathrm{d}t} = \rho + \gamma\mu_c(x_{1t}) - \frac{1}{2}\gamma^2\sigma_c(x_{2t})^2 \qquad (2.38)$$

Seen as a function of one state variable at a time the results are not significantly different compared to the examples before. However, the two-variable model can be used to show further moments including cross moments between different financial variables.

2.3.3 Replication of Habit Model in Campbell and Cochrane (2006)

Here I show how the package can be used to replicate the computation of the price-consumption ratio in Campbell and Cochrane (1999). As before consumption evolves according to the following equation:

$$\mathrm{d}\log C_t = \mathrm{d}c_t = \mu_c \mathrm{d}t + \sigma_c \mathrm{d}W_{ct} \tag{2.39}$$

where μ_c and σ_c are constants. Flow utility is given by:

$$u(C_t, S_t) = \frac{(C_t S_t)^{1-\gamma} - 1}{1-\gamma} \tag{2.40}$$

where S_t, the surplus consumption ratio, captures the level of the habit, and it evolves according to:[21]

$$\mathrm{d}\log x_t = -\log(\phi)(\bar{x} - x_t)\mathrm{d}t + \sigma_c \lambda(x_t)\mathrm{d}W_{xt}, \quad S_t = \bar{S}\exp(x_t) \tag{2.42}$$

where $\bar{S}$ is the steady state level of the surplus consumption ratio. The process for the SDF can be derived using Ito's Lemma:

$$\frac{\mathrm{d}\Lambda_t}{\Lambda_t} = \Big(-\rho - \gamma\mu_c + \frac{1}{2}\gamma^2\sigma_c^2\Big)\mathrm{d}t \tag{2.43}$$

The price-consumption ratio corresponds to the security whose dividends follow the same process as consumption. Based on the above, the package can be used to compute the price-consumption ratio. The code and the results are shown in Example 6 in Appendix 2.C. The results are somewhat different compared to the original paper, which shows the importance of using appropriate numerical methods for each problem. The results are more in line with Wachter (2005), where the issue of numerical methods in solving habit models is discussed. In future development of the package, I plan to provide a transparent treatment of these issues along with the possibility of choosing the most appropriate algorithm and settings for each problem.

[21] The exact form of $\lambda(\cdot)$ is:

$$\sigma_{xt} = \sigma_{ct}\lambda(x_t) = \begin{cases} \sigma_{ct}\left(\frac{\sqrt{1-2x_t}}{\bar{S}} - 1\right) & \text{if } x_t < \frac{1-\bar{S}^2}{2} \\ 0 & \text{if } x \geq \frac{1-\bar{S}^2}{2} \end{cases}, \quad \bar{S} = \sqrt{\frac{\gamma}{-\log(\phi) - b/\gamma}} \tag{2.41}$$

2.4 Application

The package allows the computation of asset prices in response to changes in the state variable and/or consumption. As also mentioned in the introduction, the literature has focused on two kinds of shocks, Delphic and Odyssean. Delphic shocks reveal information about the underlying state of the economy, and Odyssean shocks introduce an unexpected monetary policy. Performing an analysis in the context of a consumption-based model highlights the fact that interest rates and asset prices ultimately only move when some component of the SDF changes (or is perceived to change). This holds regardless of whether the central bank is revealing information or whether it is committing to a different monetary policy. In addition, the analysis highlights the importance of the channel through which monetary policy is conducted. For example, a standard monetary tightening is supposed to decrease output and this increases expected output growth as the economy is expected to revert to the steady state. This implies that the real short-term rate increases to counteract the increased consumption smoothing motive, and asset prices fall due to the higher discount rate. This literature assumes that on the day of a monetary policy announcement, the announcement itself is causing the changes in asset prices and not vice versa. While this is reasonable and in most cases should be true, the observation of an increase in interest rates and a decrease in asset prices does not necessarily imply that the central bank has *caused* this effect by tightening monetary policy if the central bank could just be revealing information. Indeed the central bank could be directly revealing that output growth has increased for other reasons (in the same way that it would have increased had the actual monetary policy changed), and this by definition should produce the same response of interest rates and asset prices. So, even when we observe the “correct” pattern for monetary policy we cannot exclude the possibility of a Delphic/information shock.

Inversely, one could still claim that when the “wrong” pattern occurs, then it is due to an information shock and not due to monetary policy affecting the economy. However, even in this case, monetary policy could be affecting the economy through different channels (even before the episodes of explicitly unconventional monetary policy after the global financial crisis). In this section, I explore the behaviour of asset prices when different components of the SDF change, and my results suggest that, whatever the pattern, it is not trivial to classify them as Odyssean/monetary shocks and Delphic/information shocks. I analyse two main cases, in the first the state variable is consumption drift and in the second it is consumption diffusion. One could ask whether monetary policy can affect consumption diffusion at all. And while indeed this channel is less standard, such a relationship can find support in at least two different strands of literature. Firstly, there is literature suggesting the importance of the "risk-taking channel" of monetary policy (Borio and Zhu 2012; Adrian and Shin 2010), and such a channel could be modelled as affecting consumption diffusion in the context of a consumption-based model.[22] Secondly, Vayanos and Vila (2021) also suggested that the term structure of interest rates is driven by arbitrageurs taking on more or less risk. In their model, this would directly translate to more or less wealth volatility, which can be naturally modelled as consumption volatility within a model that has exogenous consumption.[23]

In the following cases that I examine, I will compare my results to the classification of monetary policy announcements in Jarociński and Karadi (2020) (JK) and Cieslak and Schrimpf (2019) (CS), which I summarize in Table 2.1. In JK the co-movement of stock prices with the short-term rate is exclusively considered as a criterion to classify shocks into pure monetary policy shocks and information shocks. In the case of CS the yields of long-term bonds are also considered, and essentially the difference is that information shocks are sub-categorised into "growth" and "risk premium" shocks, in the former (latter) short-term (long-term) yields react more aggressively than long-term (short-term) yields. The calibration for both cases is shown in Table 2.2.

2.4.1 Case 1: Time-Varying Consumption Drift

In the first case that I analyse, the underlying model has a time-varying consumption drift, while the central bank is also able to "externally" affect the variables in the model. I assume that this takes place without adding an extra state variable.[24]

[22] A different approach within a consumption-based model would be to assume that the risk aversion parameter itself can stochastically change as in Lettau and Wachter (2011).

[23] In the original model there is no consumption, and arbitrageurs are just optimising the mean and variance of their portfolio value. In particular, arbitrageurs are rather not associated with real consumers but with financial institutions. Here, I use a model with consumption without any explicit wealth, but this could be thought of as modelling either the behaviour of real consumers or the behaviour of financial institutions that use a consumption-based SDF. While Vayanos and Vila (2021) is mostly associated with the conduct of unconventional monetary policy, it suggests a more general explanation of the term structure. So, assuming that monetary policy has played an important role in shaping the term structure of interest rates even before the introduction of unconventional monetary policies, it is arguable that arbitrageurs would be adjusting their levels of risk even before the introduction of unconventional monetary policies.

[24] Technically, the monetary policy variable should also have a distribution. However, for the stylised model I use here, I just assume that monetary policy can affect the economy in an

	Shock	Yields		Stocks	Stock-Yield
		short	long		Co-movement
Jarociński and Karadi (2020)	Monetary policy:	$\uparrow$	-	$\downarrow$	-
	Information:	$\uparrow$	-	$\uparrow$	+
Cieslak and Schrimpf (2019)	Monetary policy:	$\uparrow$	$\uparrow$	$\downarrow$	-
	Growth:	$\uparrow$	$\uparrow$	$\uparrow$	+
	Risk premium:	$\downarrow$	$\downarrow$	$\downarrow$	+

Table 2.1: This table is partly taken from a corresponding table in Cieslak and Schrimpf (2019), to which I have added the classification in Jarociński and Karadi (2020). In the latter paper, the authors do not use long-term yields in the classification and they do not compare the size of the movements. The former paper uses both kinds of yields and compares the size of the movements. This is expressed through the size of the arrows in the table.

So, the processes evolve according to:

$$\begin{aligned} \mathrm{d}x_t &= -\log \phi(x_t - x_t)\mathrm{d}t + \sigma_x \mathrm{d}W_{xt} + \mathcal{M}_x \mathrm{d}q_t \\ \mathrm{d}c_t &= (\mu_{c0} + x_t)\mathrm{d}t + \sigma_c \mathrm{d}W_t + \mathcal{M}_c \mathrm{d}q_t \end{aligned} \tag{2.44}$$

Where $\mathrm{d}q_t$ is a Poisson shock assumed to be activated when the monetary policy announcement occurs. $\mathcal{M}_x$ and $\mathcal{M}_c$ express the size of the effect on the state variable and consumption respectively.[25] I focus on the effect of the shock on yields and on an asset that pays dividends equal to consumption. I call this asset *consumption perpetuity.*

If the announcement only affects the current level of consumption ($\mathcal{M}_x = 0, \mathcal{M}_c \neq 0$), then the short-term rate is unaffected, the price-consumption ratio is unaffected, but the price of the consumption perpetuity undergoes a level shift that persists in time.[26] This is because there is only one state variable in the

unanticipated way. A different approach would be to introduce monetary policy as a separate state variable or assume that the state variable is already equivalent to monetary policy. The results would not be significantly different.

[25] For mathematical consistency with the rest of the model, as also mentioned in the previous footnote, $\mathrm{d}q_t$ should have 0 intensity, which implies that the probability of such a change is equal to 0.

[26] When discussing how monetary policy announcements affect variables, this could be taking place either by monetary policy literally affecting the variable or by revealing information and making investors aware that a variable has some value. For simplicity, I do not explicitly and separately model perceived and real variables.

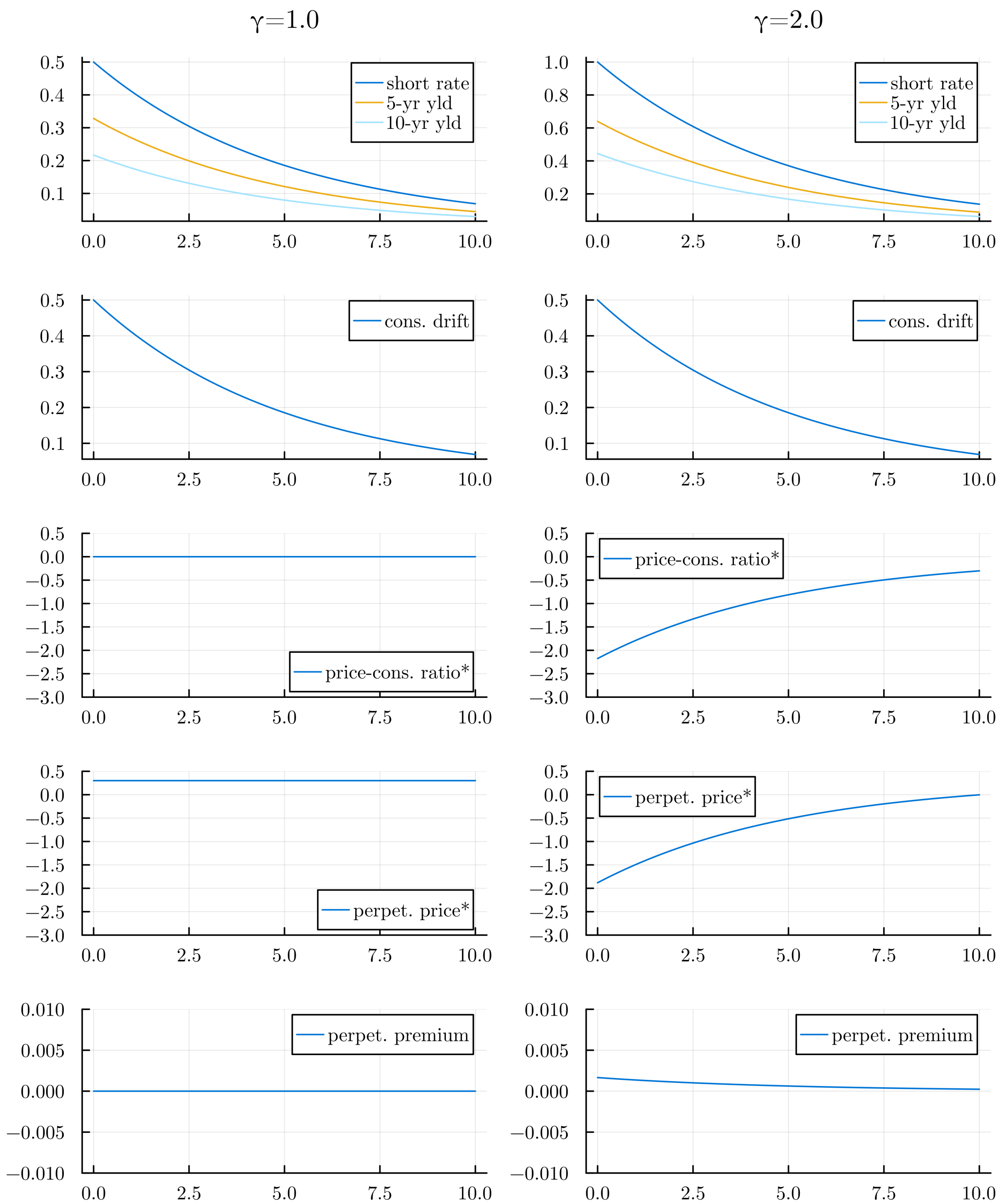


Figure 2.1: **Impulse Responses to Consumption Drift Shock**
These are responses after a consumption drift change. The period shown is ten years. The size of the shock corresponds to one standard deviation. I interpret these responses as taking place after a monetary policy announcement, and I assess to which type of shock they correspond. The short-term rate increases, but the response of the asset price is different depending on the risk aversion parameter. *Plots normally show the per cent deviation from the steady state. Plots with starred variables show per cent relative deviations from the steady state.

Parameter	Model: Time-varying consumption drift	Model: Time-varying consumption diffusion
γ	1/2	1/2/2.5
ϕ	0.82	0.82
$\bar{x}$	0.0	0.0
ρ	0.02	0.02
μ_{c0}	0.01	-
μ_c	state variable	0.01
σ_{c0}	-	0.06
σ_c	0.01	state variable
σ_x	0.005	0.5
ρ_{cx}	0.3	0.3

Table 2.2: **Calibration for the two cases**

model other than consumption, which by itself does not affect prices and the price-consumption ratios. In addition, the jump in consumption does not dissipate given the process chosen for consumption, which explains the persistence of the effect on the price of the consumption perpetuity. In a more realistic case in which the announcement affects both variables ($\mathcal{M}_x > 0, \mathcal{M}_c < 0$) the results are shown in Figure 2.1. For both risk aversion values the short-term rate increases, while longer-term yields increase also but less with maturity due to the mean reversion of the steady state. In addition, in both cases, there is no significant change in the equity premium. Interestingly, this case does not turn out to exactly fit the pattern of a traditional monetary policy shock, instead for the special case of $\gamma = 1$ the price-consumption ratio is constant as the higher discounting exactly offsets the higher expected dividends. The price of the asset still undergoes a level increase due to the increase in the dividends, which do not revert to the previous level. When $\gamma = 2.0$ the traditional pattern of monetary policy arises, in which the short rate goes up and the asset price goes down, with both effects subsequently dying out.[27] This is consistent with the monetary policy classification in both JK

[27] Even in this case though there is a lasting residual effect on the price of the dividend-paying security that is due to the increase of the dividend which in this model does not revert

and CS. Nevertheless, these results highlight that ultimately the effects occur in connection to a shift in the consumption drift. So, if the central bank can affect the economy through an information channel, then the shift could be *caused* or alternatively it could be *revealed* by the central bank. Therefore, even in this conventional monetary policy case, a method is required to distinguish between the two alternatives.

2.4.2 Case 2: Time-Varying Consumption Diffusion

For the second case, I work on the model with time-varying consumption diffusion. Again I focus on the yields and the consumption perpetuity. Here the processes evolve according to:

$$\begin{aligned} \mathrm{d}x_t &= -\log\phi(x_t - x_t)\mathrm{d}t + \sigma_x \mathrm{d}W_{xt} + \mathcal{M}_x \mathrm{d}q_t \\ \mathrm{d}c_t &= \mu_c \mathrm{d}t + \sigma_c(x_t)\mathrm{d}W_t + \mathcal{M}_c \mathrm{d}q_t \end{aligned} \tag{2.45}$$

with σ_c defined in Equation (2.29). The cases where only consumption increases is the same as was discussed before. The effect of the monetary policy announcement affecting an increase in consumption volatility ($\mathcal{M}_x > 0, \mathcal{M}_c = 0$) is shown in Figure 2.2. Here, I use a higher average consumption volatility compared to the previous case, so that consumption volatility changes have a significant effect on the short rate and the returns. I also use a positive correlation between consumption volatility changes and consumption changes as was primarily done in Melissinos (2023*b*).[28] For all three values of the risk aversion parameter ($\gamma = 1.0, 2.0.2.5$), the short-term rate decreases albeit with different intensities. This would be associated with an easing of monetary policy. In addition, despite consumption volatility and the equity premium rising considerably in all three cases, the shock is never classified as a risk premium shock of CS because long-term yields react less than short-term yields. This is due to the reversion of the state variable to the steady state, which implies that over the long run, the state variable should revert to its previous value. In addition, consistent with an information shock in JK and a growth shock in CS the price of the asset decreases when $\gamma = 1.0$ when the short-term rate also decreases.[29] Interestingly, this has nothing to do with a

to the previous level. There is also a tiny increase in the equity premium, significantly less than 0.01%, as the equity premium depends on the state variable, but only very slightly given that consumption volatility is constant.

[28] For this analysis using negative correlation would produce practically the same impulse response functions.

[29] This is also consistent with a corresponding result in Bansal and Yaron (2004).

growth shock conceptually as by construction only consumption volatility is affected. On the contrary, the case of a growth shock (shift in consumption drift) was analysed in the previous subsection, and it had the effect of a monetary policy shock according to the classifications. Furthermore, there are two more cases for the risk aversion parameter that would imply a different classification. For $\gamma = 2.0$ the price of the asset remains constant, which is an alternative not considered by JK or CS. For $\gamma = 2.5$ the price of the asset increases, when the short-term rate decreases. So, this would be classified as a normal monetary policy shock both by JK and CS. The model variations analysed here are highly stylised, but the results are nevertheless suggestive. If monetary policy is inducing investors to take on more risk, by moving the interest rate, then the effect on asset prices may depend on the value of the risk aversion parameter of the investors who are responsible for pricing the assets. And there are some cases, in which, unlike what is assumed in most classifications of monetary policy shocks, the central bank *causes* the short-term rate and asset prices to move in the same direction.

2.5 Conclusion

The primary goal of this paper is to introduce and explain a Julia package that can facilitate asset pricing in settings in which the process of the SDF is known. The package takes advantage of the already available DifferentialEquations.jl package, that is used to simulate SDEs. The package can then generate price functions for fixed-income securities. In addition, the price-dividend ratio of dividend-paying securities can also be computed, as long as the joint processes of the SDF and the dividend are given. The package can handle single-variable and multi-variable problems for both fixed-income and dividend-paying securities. In the case of single-variable problems, the package also facilitates the calculation of the expected return. I illustrate the use of the package in fully worked out examples that have been explained in the paper and whose code is included in the Appendix.[30] The examples include consumption-based models in which the consumption process is exogenous and generates an SDF process. In the examples, both one and two state variables are used.

The package is suited for academic research, and I illustrate this with an application, in which I study the possible effects of monetary policy announcements on interest rates and asset prices. While several papers have performed classification

[30] The examples are also included in the Github page for the package.

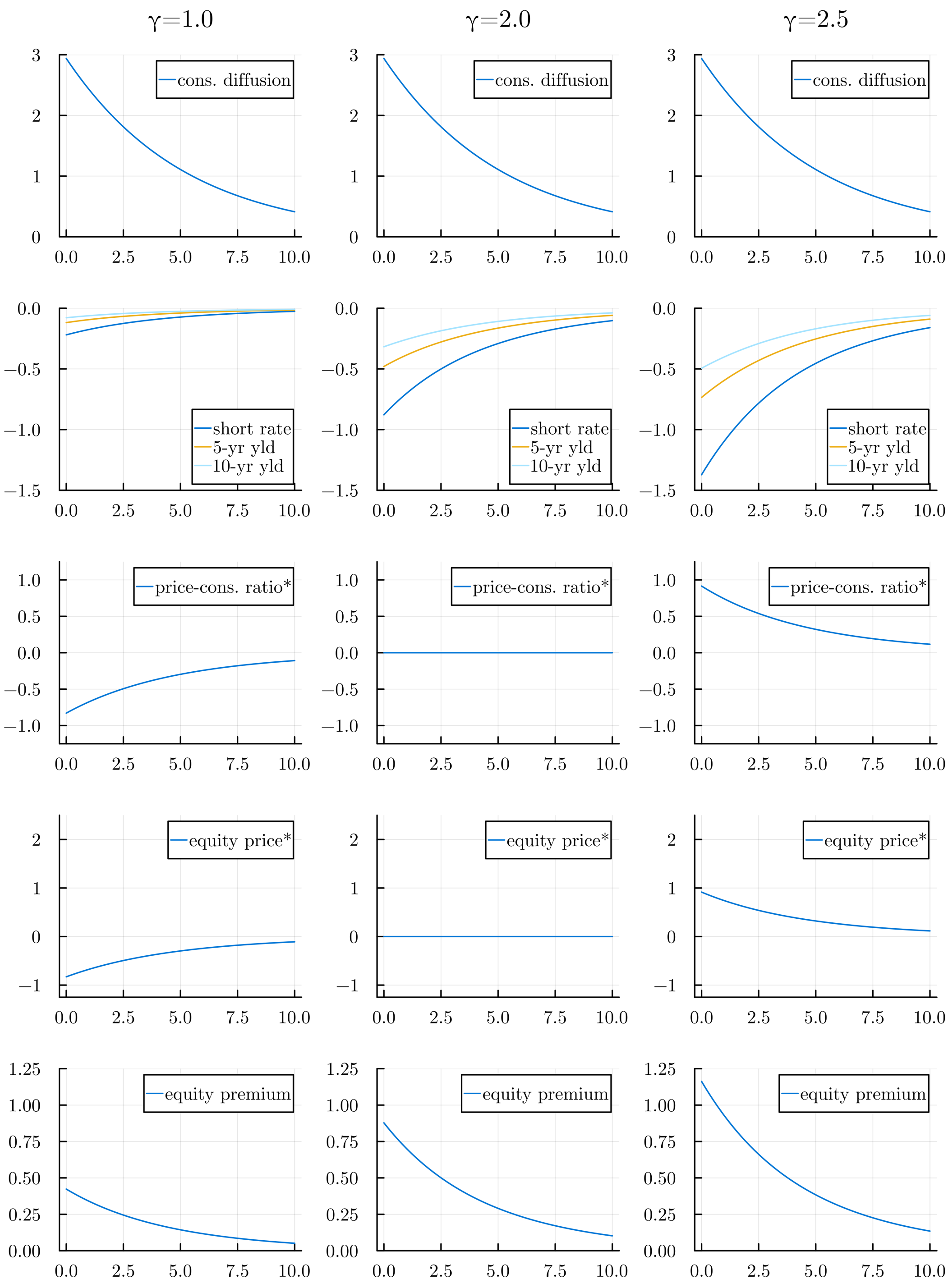


Figure 2.2: **Impulse Responses to Consumption Diffusion Shock**
These are responses after a consumption diffusion change. The period shown is ten years. The size of the shock corresponds to one standard deviation. I interpret these responses as taking place after a monetary policy announcement, and I assess to which type of shock they correspond. The short-term rate and the expected excess return (equity premium) always fall, but the response of the asset price is different depending on the risk aversion parameter.
*Plots normally show a per cent deviation from the steady state. Plots with starred variables show per cent relative deviations from the steady state.

of monetary policy announcements, based on responses of financial markets, few provide an explicit theory that shows how these effects are channelled.[31] In general higher interest rates are associated with higher discounting and hence lower asset prices, but if monetary policy increases interest rates via a channel that also increases dividend flows, it becomes non-trivial whether asset prices should increase or decrease. In addition, if monetary policy runs through a risk-taking channel the effects on asset prices can again be ambiguous. My results highlight two things. Firstly, once it is accepted that the central bank can affect the economy by revealing information about the state of the economy, then it becomes difficult to identify a pure monetary policy shock even if financial variables follow the anticipated pattern. Secondly, given unconventional monetary policy or to the extent that monetary policy does not operate through the traditional channel, it is possible that "tightening" ("easing") does not necessarily lead to a decrease (increase) in asset prices. This suggests that the identification of information shocks should also be accompanied by a specific theory that outlines the potential channels of monetary policy.

In the future, I am planning to extend the functionality of the package. In particular, as first steps I am planning to a) add the possibility of discrete jumps of the relevant variables,[32] b) include specialised functions to compute the prices of special payoff structures that correspond to different known financial securities, and c) facilitate the choice of underlying solution algorithms, to strike the desired balance between speed and accuracy for each kind of problem. Next, functions can be added for the calibration and/or estimation of models to macroeconomic and/or financial data.

[31] An exception is Cieslak and Schrimpf (2019), who included a non-consumption-based macro-finance model.

[32] Assuming that an equilibrium exists, then SDFs should be able to price the relevant assets (Duffie 2010).

Appendix

2.A Multivariable formulas

- Equation 2.2:

$$\mathrm{d}x_t i = \mu_i(X_t, t)\mathrm{d}t + \sum_{j=1}^{M} \sigma_{i,j}(X_t, t)\mathrm{d}W_{jt}, \quad i = 1, 2, \ldots, N, \quad \mathrm{E}[\mathrm{d}W_{jt}\mathrm{d}W_k(t)] = \rho_{j,k}\mathrm{d}t$$

  where $X_t = (x_1, x_2, \ldots, x_N)$, μ_i is the drift of state variable i, W_{jt} is the j-th Wiener process, M is the number of Wiener processes, $\sigma_{i,j}$ is the diffusion of state variable i with respect to W_{jt}, and $\rho_{j,k}$ is the correlation between the jth and kth Wiener processes.

- Equation 2.3:

$$\frac{\mathrm{d}\Lambda_t}{\Lambda_t} = \mu_\Lambda(X_t, t)\mathrm{d}t + \sum_{j=1}^{M} \sigma_{\Lambda,j}(X_t, t)\mathrm{d}W_{jt}$$

- Equation 2.4:

$$\begin{aligned}\mathrm{d}Q =& \Bigg(\frac{\partial Q}{\partial t} - \frac{\partial Q}{\partial m} + \sum_{i=1}^{N} \mu_i(X_t, t)\frac{\partial Q}{\partial x_i} \\ &+ \frac{1}{2}\sum_{a=1}^{N}\sum_{b=1}^{N}\sum_{c=1}^{M}\sum_{d=1}^{M} \rho_{c,d}\sigma_{a,c}(X_t, t)\sigma_{b,d}(X_t, t)\frac{\partial^2 Q}{\partial x_a \partial x_b}\Bigg)\mathrm{d}t \\ &+ \sum_{i=1}^{N}\sum_{j=1}^{M} \sigma_{i,j}(X_t, t)\frac{\partial Q}{\partial x_i}\mathrm{d}W_{jt}\end{aligned}$$

- Equation 2.5:

$$\begin{aligned}&\mathrm{E}\left[\frac{\mathrm{d}(\Lambda_t Q)}{\Lambda_t}\right] = 0 \Rightarrow \mathrm{E}\left[\frac{\mathrm{d}\Lambda_t}{\Lambda_t}Q + \mathrm{d}Q + \frac{\mathrm{d}\Lambda_t}{\Lambda_t}\mathrm{d}Q\right] = 0 \\ \Rightarrow& - r(X, t)Q + \frac{\partial Q}{\partial t} - \frac{\partial Q}{\partial m} + \sum_{i=1}^{N} \mu_i(X_t, t)\frac{\partial Q}{\partial x_i} \\ &+ \frac{1}{2}\sum_{a=1}^{N}\sum_{b=1}^{N}\sum_{c=1}^{M}\sum_{d=1}^{M} \rho_{c,d}\sigma_{a,c}(X_t, t)\sigma_{b,d}(X_t, t)\frac{\partial^2 Q}{\partial x_a \partial x_b} \\ &+ \sum_{i=1}^{N}\sum_{j=1}^{M}\sum_{k=1}^{M} \rho_{j,k}\sigma_{i,j}(X_t, t)\sigma_{\Lambda,k}(X_t, t)\frac{\partial Q}{\partial x_t i} = 0\end{aligned}$$

- Equation 2.6:

$$Q(X_t, t, m) = \mathrm{E}\left[\exp\left(-\int_t^{t+m} r(\hat{X}_s, s)\mathrm{d}s\right)g(t+m)\middle|\hat{X}_t = X_t\right]$$

where $\hat{X} = (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n)$.

- Equation 2.7:

$$\mathrm{d}\hat{x}_{it} = \left(\sum_{j=1}^{M}\sum_{k=1}^{M}\rho_{j,k}\sigma_{i,j}(\hat{X}_t, t)\sigma_{\Lambda,k} + \mu_i(\hat{X}_t, t)\right)\mathrm{d}t + \sum_{j=1}^{M}\sigma_{i,j}(\hat{X}_t, t)\mathrm{d}W_{jt}$$

- Equation 2.8:

$$Z(X_t, t) \equiv \int_t^{\infty} Q(X_t, t, s)\mathrm{d}s$$

- Equation 2.9:

$$u_t(X_t) \equiv \frac{U_t}{D_t(X_t)}$$

- Equation 2.10:

$$\frac{\mathrm{d}D_t}{D_t} = \mu_D(X_t, t)\mathrm{d}t + \sum_{j=1}^{M}\sigma_{D,j}(X_t, t)\mathrm{d}W_{jt}$$

- Equation 2.14:

$$\mathrm{E}\Big[\mathrm{d}\Big(\Lambda_t Y(X_t, t, m)\Big)\Big] = 0 \quad \forall t \in (t_0, T), \quad Y(X, t, 0) = g(t) \quad \forall X$$

- Equation 2.16:

$$\begin{aligned}\mathrm{d}y =& \left(\frac{\partial y}{\partial t} - \frac{\partial y}{\partial m} + \sum_{i=1}^{N}\mu_i(X_t, t)\frac{\partial y}{\partial x_i} + \frac{1}{2}\sum_{a=1}^{N}\sum_{b=1}^{N}\sum_{c=1}^{M}\sum_{d=1}^{M}\rho_{c,d}\sigma_{a,c}(X_t, t)\sigma_{b,d}(X_t, t)\frac{\partial^2 y}{\partial x_a \partial x_b}\right)\mathrm{d}t \\ &+ \sum_{i=1}^{N}\sum_{j=1}^{M}\sigma_{i,j}(X_t, t)\frac{\partial y}{\partial x_i}\mathrm{d}W_{jt}\end{aligned}$$

- Equation 2.17:

$$
\begin{aligned}
&-r(X_t,t)p_t^{str}+\frac{\partial p_t^{str}}{\partial t}-\frac{\partial p_t^{str}}{\partial m}+\sum_{i=1}^{N}\mu_i(X_t,t)\frac{\partial p_t^{str}}{\partial x_i}+\frac{1}{2}\sum_{a=1}^{N}\sum_{b=1}^{N}\sum_{c=1}^{M}\sum_{d=1}^{M}\rho_{c,d}\sigma_{a,c}(X_t,t)\sigma_{b,d}(X_t,t)\frac{\partial^2 p_t^{str}}{\partial x_a\partial x_b}\\
&+\mu_D(X_t,t)p_t^{str}+\sum_{i=1}^{N}\sum_{j=1}^{M}\sum_{k=1}^{M}\rho_{j,k}\sigma_{i,j}(X_t,t)\sigma_{\Lambda,k}(X_t,t)\frac{\partial p_t^{str}}{\partial x_i}+\sum_{j=1}^{M}\sum_{k=1}^{M}\rho_{j,k}\sigma_{D,j}(X_t,t)\sigma_{\Lambda,k}(X_t,t)p_t^{str}\\
&+\sum_{i=1}^{N}\sum_{j=1}^{M}\sum_{k=1}^{M}\rho_{j,k}\sigma_{D,j}(X_t,t)\sigma_{i,j}(X_t,t)\frac{\partial p_t^{str}}{\partial x_i}=0
\end{aligned}
$$

- Equation 2.18:

$$
p_t^{str}(X_t,t,m)=\mathrm{E}\left[\exp\left(-\int_t^{t+m}\tilde{r}(\tilde{X}_s,s)\mathrm{d}s\right)g(t+m)\middle|\tilde{X}_t=X_t\right]
$$

- Equation 2.19:

$$
\tilde{r}(X_t,t)=r(X_t,t)+\mu_D(X_t,t)+\sum_{j=1}^{M}\sum_{k=1}^{M}\rho_{j,k}\sigma_{D,j}(X_t,t)\sigma_{\Lambda,k}
$$

- Equation 2.20:

$$
\mathrm{d}\tilde{x}=\left(\sum_{j=1}^{M}\sum_{k=1}^{M}\rho_{j,k}\sigma_{i,j}(X_t,t)(\sigma_{\Lambda,k}(X_t,t)+\sigma_{D,k}(X_t,t))+\mu_i(\tilde{X}_t,t)\right)\mathrm{d}t+\sum_{j=1}^{M}\sigma_{i,j}(\tilde{X}_t,t)\mathrm{d}W_{jt}
$$

- Equation 2.21:

$$
\begin{aligned}
\frac{\mathrm{d}P_t}{P_t}+\frac{D_t}{P_t}\mathrm{d}t=&\frac{\mathrm{d}(p_tD_t)}{p_tD_t}+\frac{1}{p_t}\mathrm{d}t=\frac{\mathrm{d}p_t}{p_t}+\frac{\mathrm{d}D_t}{D_t}+\frac{\mathrm{d}p_t\mathrm{d}D_t}{p_tD_t}+\frac{1}{p_t}\mathrm{d}t\\
=&\frac{1}{u_t}\Bigg(\bigg(\frac{\partial p_t}{\partial t}+\sum_{i=1}^{N}\mu_i(X_t,t)\frac{\partial p_t}{\mathrm{d}x_i}+\frac{1}{2}\sum_{a=1}^{N}\sum_{b=1}^{N}\sum_{c=1}^{M}\sum_{d=1}^{M}\rho_{c,d}\sigma_{a,c}(X_t,t)\sigma_{b,d}(X_t,t)\frac{\partial^2 p}{\partial x_a\partial x_b}\\
&+\frac{\mu_D(X_t,t)}{D_t}p_t+\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{M}\sum_{k=1}^{M}\rho_{j,k}\sigma_{D,j}(X_t,t)\sigma_{i,k}\frac{\partial p_t}{\partial x_i}\bigg)\mathrm{d}t\\
&+\bigg(\sum_{i=1}^{N}\sum_{j=1}^{M}\sigma_{i,j}(X_t,t)\frac{\partial p_t}{\partial x_i}+\sum_{j=1}^{M}\sigma_{D,j}(X_t,t)\bigg)\mathrm{d}W_{jt}\Bigg)
\end{aligned}
$$

2.B Documentation

The following is the documentation that is also available within the package.

```
/////////////////////////////////////////////////////////////
SDFPricing.Problem struct
Variable that holds the information of the problem.
=============================================================

Fields
======

drift            :: Function
  --> drift function of the SDE, consistent with
  DifferentialEquations package.
diffusion        :: Function
  --> diffusion function of the SDE, consistent with
  DifferentialEquations package.
numNoiseVariables :: Int64
  --> number of noise variables in the SDE
outVariables     :: Vector{Int64}
  --> vector of indices of the output variables
  (corresponds to the simulation of the discounting
  integral, in the drift and diffusion functions
  of the problem struct).
terminalFunction  :: Function
  --> terminal function, which includes disounting.
  In most cases default should be used unless, there
  is a more exotic payoff at maturity (for example
  depending on the state variable).
diagonalNoise    :: Bool
  --> true if the noise is diagonal (implies one noise
  variable), should be false if more than one noise
  variables are used.

* All arguments are given as keyword arguments. If no
arguments are given a default problem is created.
```

```
==============================================================
==============================================================



//////////////////////////////////////////////////////////////
struct SDFPricing.SolutionSettings{alg} struct
Variable that holds the information required for the solution
that are not strictly part of the problem.
==============================================================
Fields
======

xRanges                  :: Vector
        --> vector of ranges for the state variables. Just
        one range in a vector, when there is only one state
        variable.
initialValues            :: Vector{Vector{Float64}}
        --> vector of initial values for the state variables.
        There is a vector for each set of initial values,
        which contains the values, just one value if there is
        just one state variable.
tRange                   :: AbstractRange{Float64}
        --> time range from 0 up to the highest maturity of
        the bond for the SinglePayoffSolution. For the
        ContinuousPayoffSolution, the time range should be
        long so that the dividend strips reach a value of 0.
        And the price-dividend ratio is converged.
dt                       :: Float64
        --> time step for the simulation, it makes a
        difference only when the algorithm requires a fixed
        time step. Otherwise, the algorithm adaptively picks
        a time step.
pathsPerInitialValue     :: Int64
        --> number of paths to simulate for each initial
        value.
algorithm                :: alg
        --> algorithm to use for the simulation according
```

```
        to the DifferentialEquations package.
continuousPayoffRanges   :: Vector
        --> xRanges that will only apply to the continuous
        payoff solution, by default equal to XRanges.
continuousPayoffVars     :: Vector
        --> specifies the variable for which the
        price-dividend ratio is going to be computed. For
        this variable, the SinglePayoffSolution is computed
        as well, and then integrated to get the price-
        dividend ratio.
continuousPayoffDuration :: Float64
        --> specifies the period for which the dividend-
        strips are integrated. By default, it is equal to
        the end of the tRange. It can be specified to be
        smaller if the user wants to calculate the price-
        dividend ratio for a security that pays for a
        limited period of time.

* All arguments are given as keyword arguments. If no
arguments are given, a default SolutionSettings variable
is created.
==============================================================
==============================================================

//////////////////////////////////////////////////////////////
SDFPricing.SinglePayoffSolution{arr, intp, sp} struct
The variable returned by the solve function.
==============================================================

  Fields
  ======

  array            :: arr
        --> array containing prices for different states
        and different maturities.
  intp             :: intp
        --> interpolation of prices as a function remaining
```

```
        maturity and state, also called automatically when
        the struct is called.
  samplePaths      :: sp
        --> a sample simulation.
  problem          :: SDFPricing.Problem
        --> the problem struct that was solved.
  solutionSettings :: SDFPricing.SolutionSettings
        --> the solutionSettings struct used to solve the
        problem.
  outVariable      :: Int64
        --> the index of the output variable (corresponds
        to the simulation of the discounting integral, in
        the drift and diffusion functions of the problem
        struct).

Supertype Hierarchy
===================

SDFPricing.SinglePayoffSolution{arr, intp, sp}
<: SDFPricing.Solution <: Any
==============================================================
==============================================================

//////////////////////////////////////////////////////////////
SDFPricing.ContinuousPayoffSolution{arr, intp} struct
The variable returned by the solve function.
==============================================================

  Fields
  ======

  array                :: arr
        --> array containing prices for different states and
        different maturities
  intp                 :: intp
        --> interpolation of prices as a function remaining
        maturity and state,
```

```
  singlePayoffSolution :: SDFPricing.SinglePayoffSolution
        --> contains the singlePayoffSolution struct that
        is also computed during the solution
  problem              :: SDFPricing.Problem
        --> the problem struct that was solved
  solutionSettings     :: SDFPricing.SolutionSettings
        --> the solutionSettings struct used to solve the
        problem
  outVariable          :: Int64
        --> the index of the output variable (corresponds to
        the simulation of the discounting integral, in the
        drift and diffusion functions of the problem struct)

Supertype Hierarchy
===================

SDFPricing.ContinuousPayoffSolution{arr, intp}
<: SDFPricing.Solution <: Any
==============================================================
==============================================================

//////////////////////////////////////////////////////////////
SDFPricing.solve function
Main function that solves the pricing problem.
==============================================================

Input
=====
1. Problem struct
2. SolutionSettings struct.

Output
======
Tuple of size two containing a:
1. tuple for each SinglePayoffSolution struct
2. tuple for each ContinuousPayoffSolution struct
* examples:
```

```
- if there is only one bond computed:
((bondPrice,),) = sdf.solve(problem, solutionSettings)
- if there is one bond and one price-dividend ratio:
((bondPrice,dividendStrip),(pdratio,)) = sdf.solve(problem, solutionSettings)
#* the dividend-strip needs to be computed as
#* SinglePayoffSolution for the price-dividend ratio to also
#* get computed.

* For convenience it is also possible to call solve without
any arguments, in which case a default solution is computed.
==============================================================
==============================================================

//////////////////////////////////////////////////////////////
SDFPricing.derivatives function
Computes derivatives of price-dividend interpolation.
==============================================================

The input is the ContinuousPayoffSolution struct.
The output is a tuple of two functions,
the first and second derivatives.

The first derivative is computed with the built-in function
from the DataInterpolations package.
The second derivative is then computed ``manually'' by
computing a ratio of small differences.

The epsilon value can be chosen by the user as a keyword
argument.
By default: epsilon=1e-5.
==============================================================
==============================================================
```

2.C Examples

Example 1 – One state variable

Time-Varying Consumption Drift – Zero-Coupon Bond

The state variable is associated with the consumption drift. Given a CRRA utility function the SDF process can be computed, inserted in the pricing equation and then solved using a Feynman-Kac formula. The modified state variable follows the process:

$$\mathrm{d}\hat{x}_t = \big(-\log\phi(\bar{x} - \hat{x}_t) + \rho_{cx}\sigma_c\sigma_x \big)\mathrm{d}t + \sigma_x \mathrm{d}W_{xt}$$

While the state variable is not modified when there is no correlation between the process for consumption and the process for the state variable:

$$\mathrm{d}x_t = -\log\phi(\bar{x} - x_t)\mathrm{d}t + \sigma_x \mathrm{d}W_{xt}$$

In order to get the price of the zero-coupon security a process for the integral of the short-term rate will also be needed:

$$\mathrm{d}\mathcal{I} = r(\bar{x}_t)\mathrm{d}t$$

Import the packages

[1]:
```
import SDFPricing as sdf
import StochasticDiffEq as sde # this is needed in order to specify the
↪algorithm
```

Define the parameters

[2]:
```
cs = (
    phi = 0.92, # mean reversion
    xbar = 0.0, # steady state
    rho = 0.01, # time preference parameter
    gamma = 2, # risk aversion
    muc0 = 0.005, # mean of consumption drift
    sigmac = 0.01, # consumption diffusion
    sigmax = 0.005, # state variable diffusion
    rhocx = 0.3 # correlation between consumption and state variable
);
```

Drift and Diffusion of the processes I also include the unmodified process which will correspond to "risk-neutral pricing". By comparing normal pricing with risk-neutral pricing it is possible to compute excess returns.

[3]:
```
mu0(x,c) = -log(c.phi)*(c.xbar-x) # drift of unmodified state
sigma(x,c) = c.sigmax; # diffusion of both modified and unmodified state
mu(x,c) = mu0(x,c)-c.rhocx*c.gamma*c.sigmac*sigma(x,c) # drift of modified state
```

[3]:
```
mu (generic function with 1 method)
```

Short-term rate function

[4]:
```
r(x,c) = c.rho+c.gamma*(c.muc0+x)-c.gamma^2*c.sigmac^2/2;
r(x) = r(x,cs); # define with one argument for convenience
```

Define setup consistent with SDE package in Julia

[5]:
```
function drift(du,u,p,t,c)
    du[1] = mu0(u[1],c)
    du[2] = mu(u[2],c)
    du[3] = r(u[1],c)
    du[4] = r(u[2],c)
end
drift(du,u,p,t) = drift(du,u,p,t,cs);
function diffusion(du,u,p,t,c)
    du[1] = sigma(u[1],c)
    du[2] = sigma(u[2],c)
    du[3] = 0.0
    du[4] = 0.0
end
diffusion(du,u,p,t) = diffusion(du,u,p,t,cs);
```

Define the Problem and SolutionSettings variables

[6]:
```
prob = sdf.
 ↪Problem(drift=drift,diffusion=diffusion,numNoiseVariables=1,outVariables=[3,4],
terminalFunction=(ik, x, y, z) -> exp(-x));
xRange = -0.05:0.01:0.05;
sett = sdf.SolutionSettings(xRanges=[xRange,], initialValues=[[x, x, 0.0, 0.0]␣
 ↪for x in xRange],
algorithm=sde.LambaEM(), pathsPerInitialValue=10000, tRange=0.0:1.0:10.0);
```

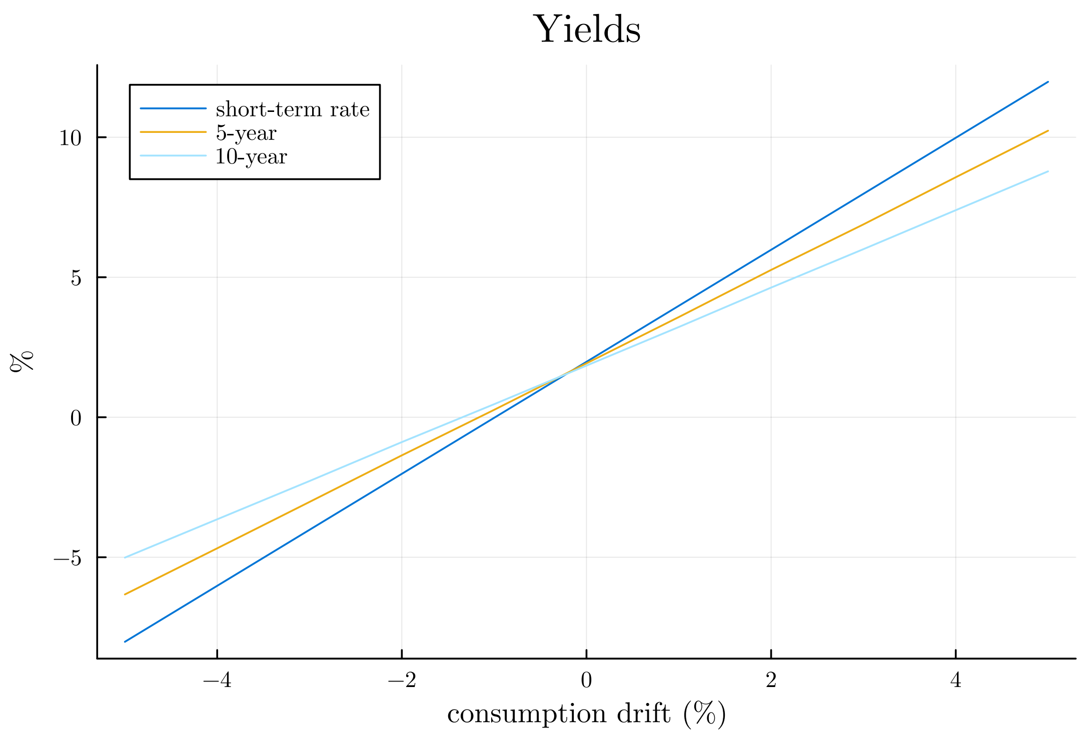
Solve Problem and Define Yield

[7]:
```
((bondPriceRiskNeutral,bondPrice),) = sdf.solve(prob, sett);
yld(t,x) = -log(bondPrice(t,x))/t;
yldRiskNeutral(t,x) = -log(bondPriceRiskNeutral(t,x))/t;
```

Plot the yield

[8]:
```
# colors: "#0075d6", "#edad14", "#a3e3ff", "#9c0000"
import Plots as plt
plt.default(titlefont= (14,"Computer Modern"),legendfont=(8,"Computer Modern"),
    tickfont=(8,"Computer Modern"),guidefont=(10,"Computer Modern"))
plt.plot(100*xRange, xRange .|> x->100*r(x), title="Yields",
    xlabel="consumption drift (%)",label="short-term␣
 ↪rate",color="#0075d6",ylabel="%")
plt.plot!(100*xRange, 100*yld.(5.0, xRange), label="5-year",color= "#edad14")
plt.plot!(100*xRange, 100*yld.(10.0, xRange), label="10-year",color="#a3e3ff")
```

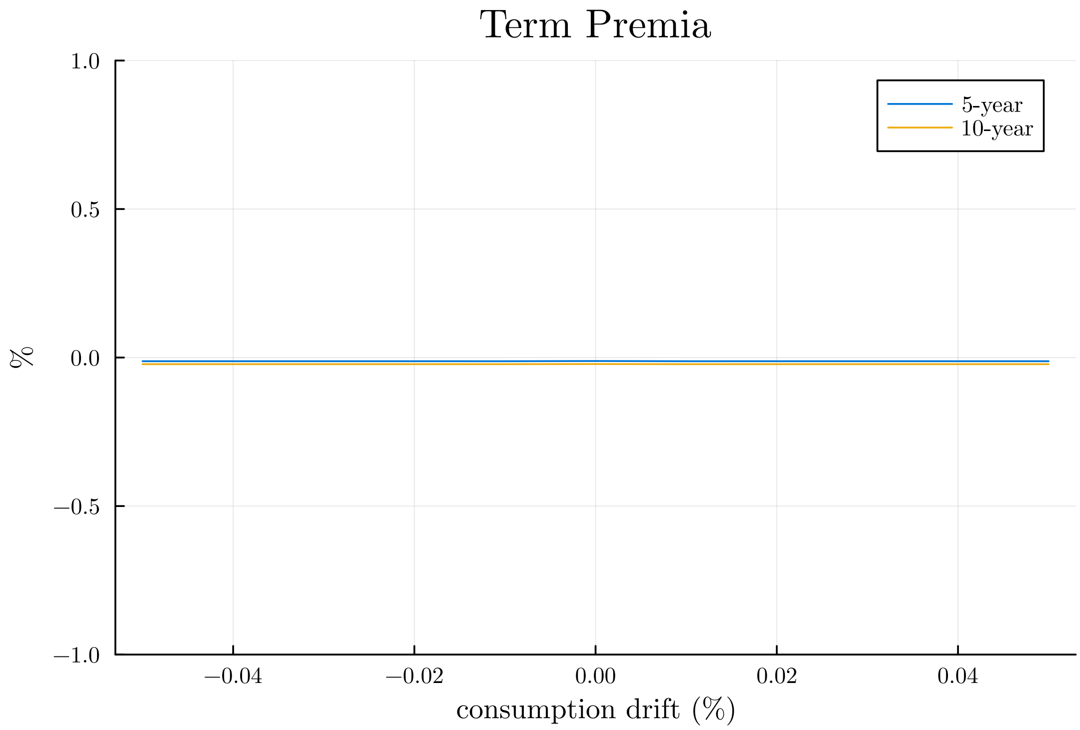
[8]:



Plot the term premium

[9]:
```
plt.plot(xRange, 100*(yld.(5.0, xRange) .- yldRiskNeutral.(5.0,␣
↪xRange)),title="Term Premia",
    xlabel="consumption drift (%)",label="5-year",ylims=(-0.005,0.
↪005),color="#0075d6",ylabel="%")
plt.plot!(xRange, 100*(yld.(10.0, xRange) .- yldRiskNeutral.(10.0, xRange)),
    label="10-year",ylims=(-1,1),color="#edad14")
```

[9]:



This shows that term premia in such a model with a time-varying consumption drift are negative, very small, and constant.

Example 2 – One state variable

Time-Varying Consumption Diffusion – Zero-Coupon Bond

The state variable is now associated with the consumption diffusion unlike example in which it was associated with consumption drift. Given a CRRA utility function the SDF process can be computed, inserted in the pricing equation and then solved using a Feynman-Kac formula. The modified state variable follows the process:

$$\mathrm{d}\hat{x}_t = \big(-\log\phi(\bar{x}-\hat{x}_t)+\rho_{cx}\sigma_{ct}\sigma_x\big)\mathrm{d}t+\sigma_x\mathrm{d}W_{xt}$$

While the state variable is not modified when there is no correlation between the process for consumption and the process for the state variable:

$$\mathrm{d}x_t = -\log\phi(\bar{x}-x_t)\mathrm{d}t+\sigma_x\mathrm{d}W_{xt}$$

In order to get the price of the zero-coupon security a process for the integral of the short-term rate will also be needed:

$$\mathrm{d}\mathcal{I} = r(\bar{x}_t)\mathrm{d}t$$

Import the packages

[1]:
```
import SDFPricing as sdf
import StochasticDiffEq as sde # this is needed in order to specify the␣
 ↪algorithm
```

Define the parameters

[2]:
```
cs = (
    phi = 0.92, # steady state reversion
    xbar = 0.0, # steady state
    rho = 0.01, # time preference parameter
    gamma = 2, # risk aversion
    muc0 = 0.005, # mean of consumption drift
    sigmac0 = 0.04, # consumption diffusion ###- higher compared to example 1
    sigmax = 0.5, # state variable diffusion ###- higher compared to example 1
    rhocx = -0.3 # correlation between consumption and state variable
);
```

Drift and Diffusion of the processes I also include the unmodified process which will correspond to "risk-neutral pricing". By comparing normal pricing with risk-neutral pricing it is possible to compute excess returns.

[3]:
```
###- now consumption diffusion is a non-linear function of the state,
###- given that it needs to be positive.
###- I use this function because the simple exponential can get too high for␣
 ↪some samples.
sigmac(x,c) = c.sigmac0*(x<0 ? 2/(1+exp(-2x)) : 4/(1+exp(-x))-1)
```

```
mu0(x,c) = -log(c.phi)*(c.xbar-x) # drift of unmodified state
sigma(x,c) = c.sigmax; # diffusion of modified and unmodified state
mu(x,c) = mu0(x,c)-c.rhocx*c.gamma*sigmac(x,c)*sigma(x,c) # drift of modified␣
↪state
```

[3]: mu (generic function with 1 method)

Short-term rate function

[4]:
```
r(x,c) = c.rho+c.gamma*c.muc0-c.gamma^2*sigmac(x,c)^2/2;
r(x) = r(x,cs);
```

Define setup consistent with SDE solution in Julia

[5]:
```
function drift(du,u,p,t,c)
    du[1] = mu0(u[1],c)
    du[2] = mu(u[2],c)
    du[3] = r(u[1],c)
    du[4] = r(u[2],c)
end
drift(du,u,p,t) = drift(du,u,p,t,cs);
function diffusion(du,u,p,t,c)
    du[1] = sigma(u[1],c)
    du[2] = sigma(u[2],c)
    du[3] = 0.0
    du[4] = 0.0
end
diffusion(du,u,p,t) = diffusion(du,u,p,t,cs);
```

Define the Problem and SolutionSettings variables

[6]:
```
prob = sdf.
 ↪Problem(drift=drift,diffusion=diffusion,numNoiseVariables=1,outVariables=[3,4],
terminalFunction=(ik, x, y, z) -> exp(-x));
xRange = -2.0:0.25:2.0;
sett = sdf.SolutionSettings(xRanges=[xRange,], initialValues=[[x, x, 0.0, 0.0]␣
 ↪for x in xRange],
algorithm=sde.LambaEM(), pathsPerInitialValue=20000, tRange=0.0:1.0:10.0);
```

Solve Problem and Define Yield

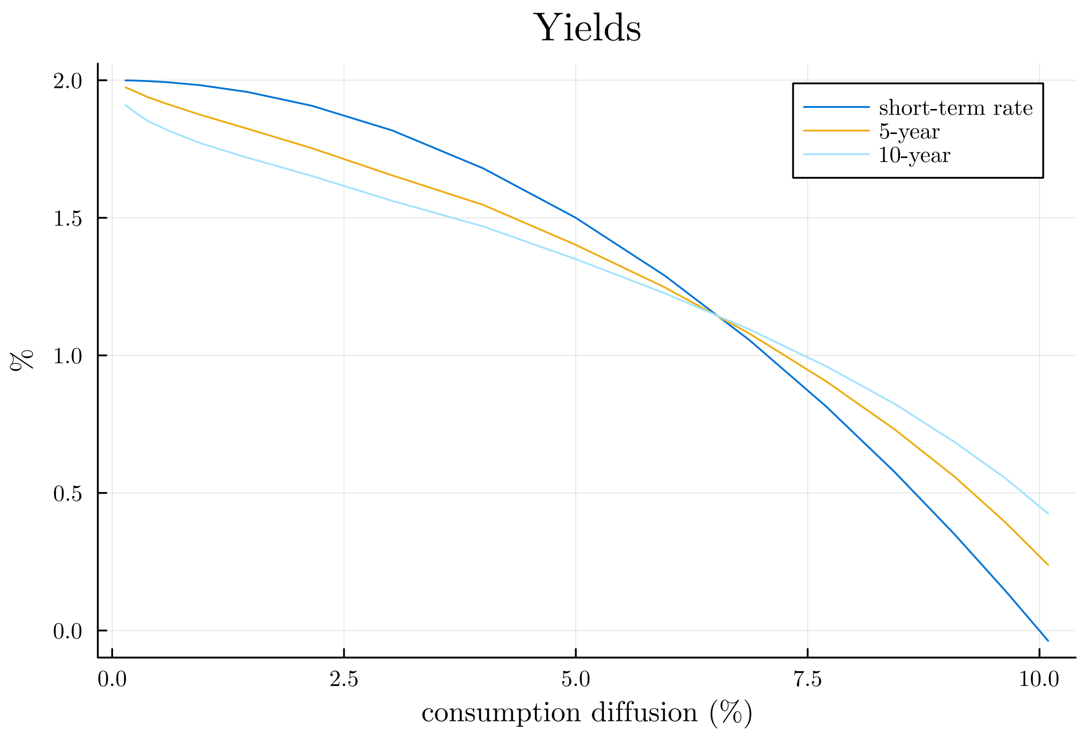
[7]:
```
((bondPriceRiskNeutral,bondPrice),) = sdf.solve(prob, sett);
yld(t,x) = -log(bondPrice(t,x))/t;
yldRiskNeutral(t,x) = -log(bondPriceRiskNeutral(t,x))/t;
```

Plot the yield

[8]:
```
# colors: "#0075d6", "#edad14", "#a3e3ff", "#9c0000"
import Plots as plt
plt.default(titlefont= (14,"Computer Modern"),legendfont=(8,"Computer Modern"),
    tickfont=(8,"Computer Modern"),guidefont=(10,"Computer Modern"))
```

```
plt.plot(xRange .|>x->100*sigmac(x,cs), xRange .|> x->100*r(x), title="Yields",
    xlabel="consumption diffusion (%)",label="short-term␣
↪rate",color="#0075d6",ylabel="%")
plt.plot!(xRange .|>x->100*sigmac(x,cs), 100*yld.(5.0, xRange),␣
↪label="5-year",color= "#edad14")
plt.plot!(xRange .|>x->100*sigmac(x,cs), 100*yld.(10.0, xRange),␣
↪label="10-year",color="#a3e3ff")
```

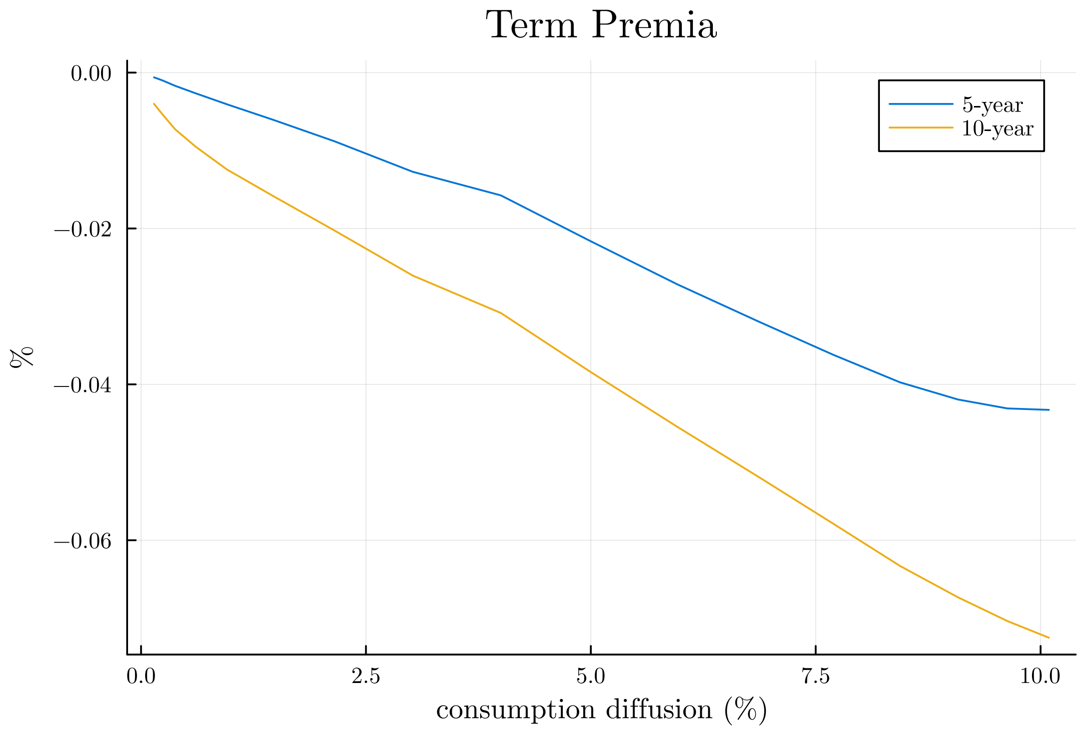
[8]:



Plot the term premium

[9]:
```
plt.plot(xRange .|>x->100*sigmac(x,cs), 100*(yld.(5.0, xRange) .-␣
↪yldRiskNeutral.(5.0, xRange)),title="Term Premia",
    xlabel="consumption diffusion␣
↪(%)",label="5-year",color="#0075d6",ylabel="%")
plt.plot!(xRange .|>x->100*sigmac(x,cs), 100*(yld.(10.0, xRange) .-␣
↪yldRiskNeutral.(10.0, xRange)),
    label="10-year",color="#edad14")
```

[9]:



This shows that term premia are state-dependent when consumption diffusion is time-varying. They can also get larger in absolute value when consumption volatility is relatively high.

Example 3 – One state variable

Time-Varying Drift – Price Consumption Ratio

Here the setup is exactly the same as in example 1, but now I calculate the price consumption ratio instead of the price of zero coupon bond. By changing the values of the parameters it is also possible to compute a more general price-dividend ratio, for an asset that does not have the same dividend process as consumption. The modified process in this case is:

$$\mathrm{d}\tilde{x}_t = \big(-\log\phi(\bar{x} - \tilde{x}_t) + \rho_{cx}\sigma_c\sigma_x + \rho_{xD}\sigma_x\sigma_D \big)\mathrm{d}t + \sigma_x \mathrm{d}W_{xt}$$

In order to get the price of the zero-coupon security a process for the integral of the short-term rate will also be needed:

$$\mathrm{d}\mathcal{J} = \tilde{r}(\tilde{x}_t)\mathrm{d}t$$

Import the packages

[1]:
```
import SDFPricing as sdf
import StochasticDiffEq as sde # this is needed in order to specify the␣
↪algorithm
```

Define the parameters

[2]:
```
cs = (
    phi = 0.92, # steady state reversion
    xbar = 0.0, # steady state
    rho = 0.02, # time preference parameter
    gamma = 2, # risk aversion
    muc0 = 0.01, # steady state consumption drift
    sigmac = 0.01, # consumption diffusion
    sigmax = 0.005, # state variable diffusion
    rhocx = 0.3, # correlation between consumption and state variable
    # sigmaD = 0.02, # dividend diffusion ###- added compared to example 1
    # muD = 0.02, # dividend drift ###- added compared to example 1
    # rhoxD = 0.5, # correlation between dividends and state variable ###-␣
↪added compared to example 1
    # rhocD = 0.4 # correlation between dividends and consumption ###- added␣
↪compared to example 1
    sigmaD = 0.01, # dividend diffusion ###- case of consumption perpetuity
    muD = 0.01, # dividend drift ###- case of consumption perpetuity
    rhoxD = 0.3, # correlation between dividends and state variable ###- case␣
↪of consumption perpetuity
    rhocD = 1.0 # correlation between dividends and consumption ###- case of␣
↪consumption perpetuity
);
```

Drift and Diffusion of the processes I also include the unmodified process which will correspond to "risk-neutral pricing". By comparing normal pricing with risk-neutral pricing it is possible to compute excess returns.

[3]:
```
# diffusion of modified state
sigma(x,c) = c.sigmax;
# drift of modified state
mu(x,c) = -log(c.phi)*(c.xbar-x)-c.rhocx*c.gamma*c.sigmac*sigma(x,c)+c.
  ↪rhoxD*sigma(x,c)*c.sigmaD
```

[3]:
```
mu (generic function with 1 method)
```

Short-term rate function

[4]:
```
r(x,c) = c.rho+c.gamma*(c.muc0+x)-c.gamma^2*c.sigmac^2/2;
r(x) = r(x,cs);
muD(x) = cs.muD+x; #- case of consumption perpetuity
# muD(x) = cs.muD; #- perpetuity with constant dividend drift
rmod(x,c) = r(x,c)-(muD(x)-c.gamma*c.rhocD*c.sigmac*c.sigmaD);
rmod(x) = rmod(x,cs);
```

Define setup consistent with SDE solution in Julia

[5]:
```
function drift(du,u,p,t,c)
    du[1] = mu(u[1],c)
    du[2] = rmod(u[1],c)
end
drift(du,u,p,t) = drift(du,u,p,t,cs);
function diffusion(du,u,p,t,c)
    du[1] = sigma(u[1],c)
    du[2] = 0.0
end
diffusion(du,u,p,t) = diffusion(du,u,p,t,cs);
```

Define the Problem and SolutionSettings variables In the theory I state that the price consumption ratio is computed from the integral over all consumption strip maturities. In practice it is not possible to integrate to infinity. So, I compute consumption strips up to a maturity of 300 years.

[6]:
```
prob = sdf.
  ↪Problem(drift=drift,diffusion=diffusion,numNoiseVariables=1,outVariables=[2],
terminalFunction=(ik, x, y, z) -> exp(-x));
xRange = -0.05:0.006:0.05;
tRange = 0.0:5.0:300.0;
sett = sdf.SolutionSettings(xRanges=[xRange,], initialValues=[[x, 0.0] for x in␣
  ↪xRange],
algorithm=sde.LambaEM(), pathsPerInitialValue=5000, tRange=tRange);
# add the settings in order to compmute price-dividend ration of the continuous␣
  ↪payoff security
```

```
sett2 = sdf.SolutionSettings(sett; continuousPayoffVars=[2]);
```

Solve Problem, Get Yield and Price Consumption Ratio

[7]:
```
((consumptionStrip,),(priceConsumptionRatio,)) = sdf.solve(prob, sett2);
```

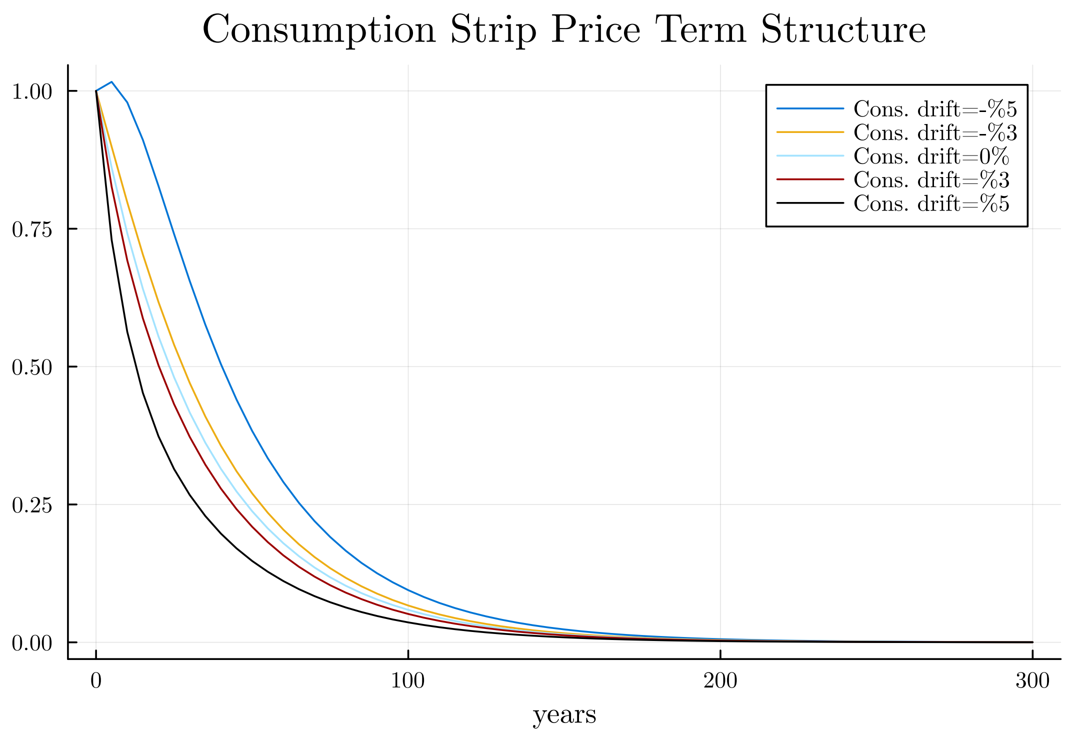
Get the Return of the Consumption Perpetuity The calculation requires the computation of the first and second derivatives of the price consumption ratio with respect to the state of the economy.

[8]:
```
(DPC,D2PC) = sdf.derivatives(priceConsumptionRatio);
ret(x) = (DPC(x)*(mu(x, cs) + sigma(x, cs) * cs.sigmac * cs.rhocx) +
    D2PC(x)* sigma(x, cs)^2/2.0 + 1.0)/priceConsumptionRatio(x) + muD(x);
```

Plot the Consumption Strip Term Structure

[9]:
```
import Plots as plt
# # colors: "#0075d6", "#edad14", "#a3e3ff", "#9c0000", "#000000"
plt.default(titlefont= (14,"Computer Modern"),legendfont=(8,"Computer Modern"),
    tickfont=(8,"Computer Modern"),guidefont=(10,"Computer Modern"))
plt.plot(tRange, consumptionStrip.(tRange, -0.04),color="#0075d6",
    title="Consumption Strip Price Term Structure", xlabel="years",label="Cons.␣
 ↪drift=-%5")
plt.plot!(tRange, consumptionStrip.(tRange, -0.01),label="Cons.␣
 ↪drift=-%3",color="#edad14")
plt.plot!(tRange, consumptionStrip.(tRange, 0.0),label="Cons.␣
 ↪drift=0%",color="#a3e3ff")
plt.plot!(tRange, consumptionStrip.(tRange, 0.01),label="Cons.␣
 ↪drift=%3",color="#9c0000")
plt.plot!(tRange, consumptionStrip.(tRange, 0.04),label="Cons.␣
 ↪drift=%5",color="#000000")
```

[9]:

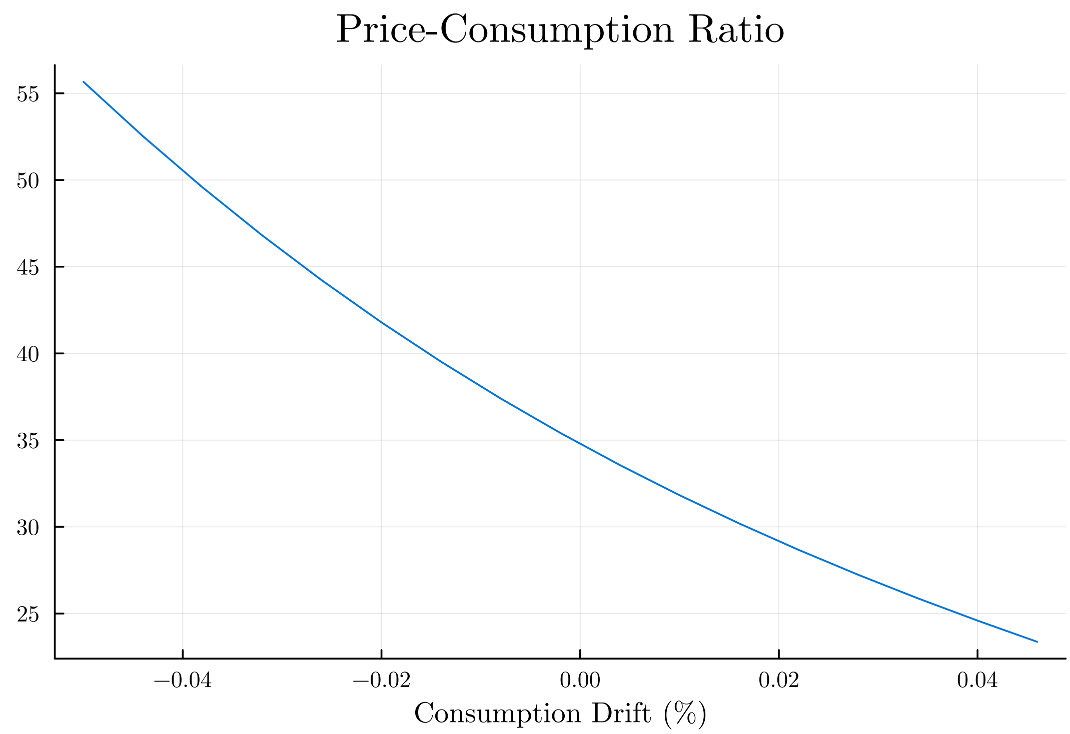


It can be seen that for all values of the state variable the price of the securities comes close to 0 for maturities as long as 300 years.

Plot the Price Consumption Ratio

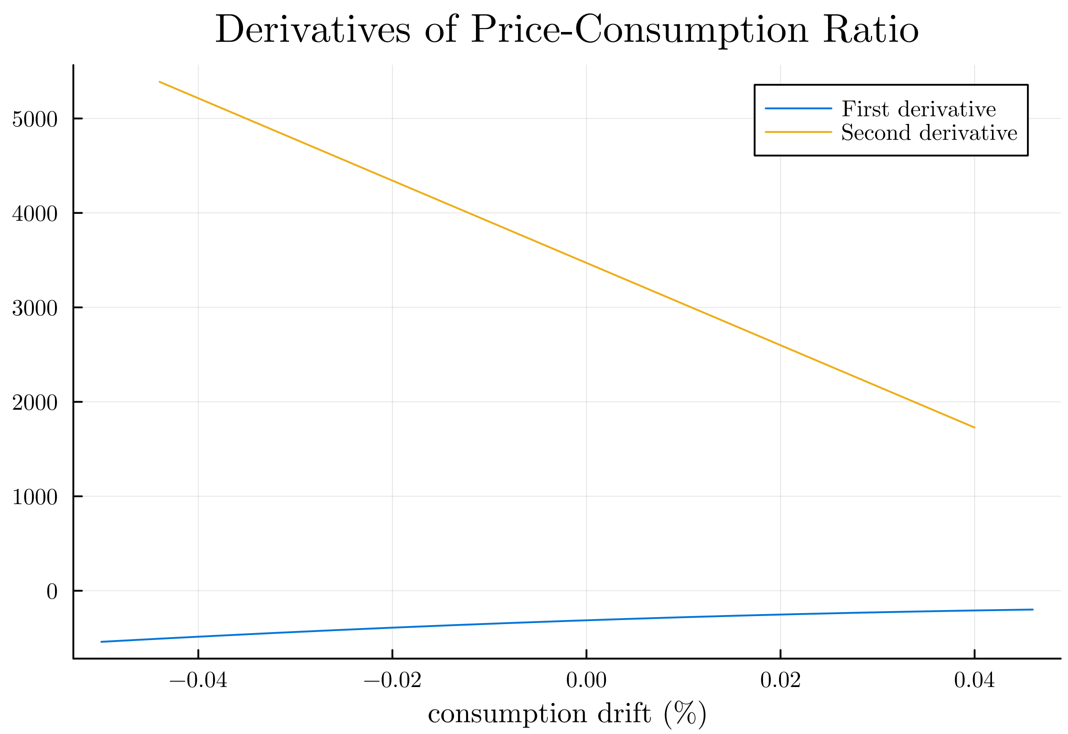
[10]:
```
plt.plot(xRange, priceConsumptionRatio(xRange), legend=false,
    title="Price-Consumption Ratio",color="#0075d6", xlabel="Consumption Drift␣
↪(%)")
```

[10]:




```
[11]: plt.plot(xRange, DPC.(xRange),
          title="Derivatives of Price-Consumption Ratio",label="First␣
      ↪derivative",color="#0075d6")
      plt.plot!(xRange[2:end-1], D2PC.(xRange[2:end-1]),label="Second␣
      ↪derivative",color="#edad14",xlabel="consumption drift (%)")
```

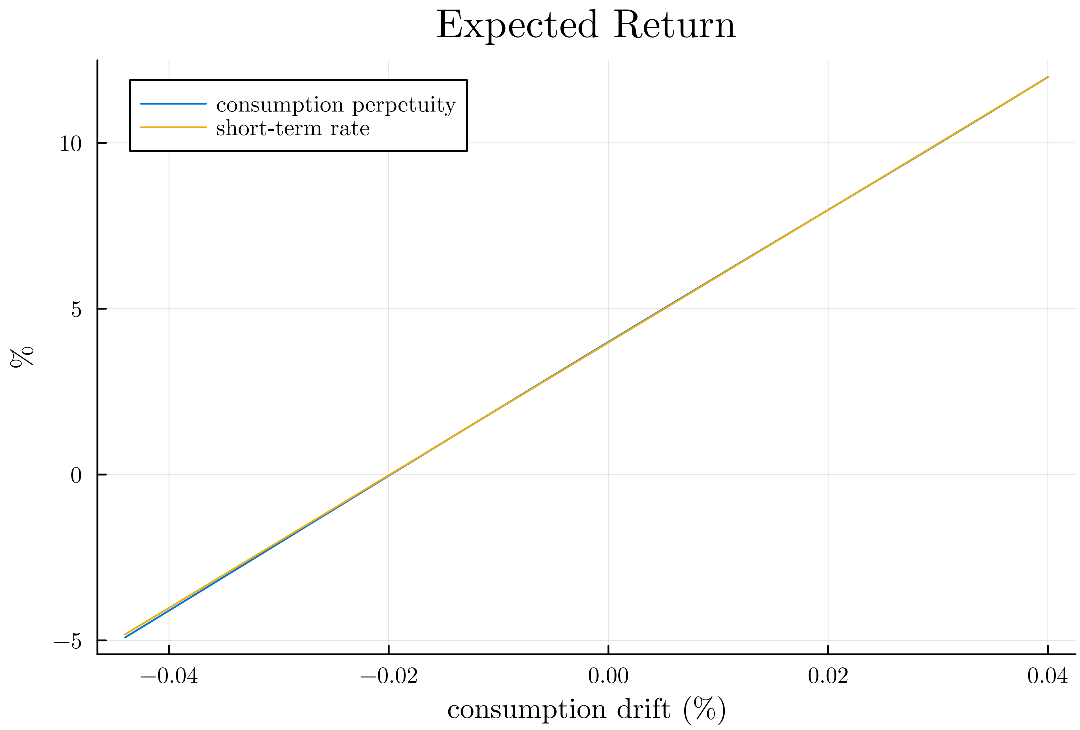
[11]:



Plot the Return

[12]:
```
plt.plot(xRange[2:end-1], 100*ret.(xRange[2:end-1]), label="consumption␣
↪perpetuity",
    title="Expected Return", color="#0075d6",xlabel="consumption drift␣
↪(%)",ylabel="%")
plt.plot!(xRange[2:end-1], 100*r.(xRange[2:end-1]),
    label="short-term rate",color="#edad14")
```

[12]:



As expected in the standard model with time-varying consumption drift the premium compared to the short-term rate is almost zero.

Example 4 – One state variable

Time-Varying Consumption Drift – Price Consumption Ratio

Here the setup is exactly the same as in example 2, but now I calculate the price consumption ratio instead of the price of zero coupon bond. The modified process in this case is:

$$\mathrm{d}\tilde{x}_t = \big(- \log\phi(\bar{x}_0 - \tilde{x}_t) + \rho_{cx}\sigma_{ct}\sigma_x + \rho_{xD}\sigma_x\sigma_D \big)\mathrm{d}t + \sigma_x \mathrm{d}W_{xt}$$

In order to get the price of the zero-coupon security a process for the integral of the short-term rate will also be needed:

$$\mathrm{d}\mathcal{I} = \tilde{r}(\tilde{x}_t)\mathrm{d}t$$

Import the packages

[1]:
```
import SDFPricing as sdf
import StochasticDiffEq as sde # this is needed in order to specify the␣
↪algorithm
```

Define the parameters

[2]:
```
cs = (
    phi = 0.92, # steady state reversion
    xbar = 0.0, # steady state
    rho = 0.02, # time preference parameter
    gamma = 2.0, # risk aversion
    muc0 = 0.005, # consumption drift
    sigmac = 0.08, # steady state consumption diffusion
    sigmax = 0.5, # state variable diffusion
    rhocx = -0.3, # correlation between consumption and state variable
    # sigmaD = 0.10, # dividend diffusion ###- added compared to example 1
    # muD = 0.01, # dividend drift ###- added compared to example 1
    # rhoxD = -0.5, # correlation between dividends and state variable ###-␣
↪added compared to example 1
    # rhocD = 0.4 # correlation between dividends and consumption ###- added␣
↪compared to example 1
    sigmaD = 0.08, # dividend diffusion ###- case of consumption perpetuity
    muD = 0.005, # dividend drift ###- case of consumption perpetuity
    rhoxD = -0.3, # correlation between dividends and state variable ###- case␣
↪of consumption perpetuity
    rhocD = 1.0 # correlation between dividends and consumption ###- case of␣
↪consumption perpetuity
);
```

Drift and Diffusion of the processes I also include the unmodified process which will correspond to "risk-neutral pricing". By comparing normal pricing with risk-neutral pricing it is possible to compute excess returns.

```
[3]: ###- now consumption diffusion is a non-linear function of the state,
###- given that it needs to be positive.
###- I use this function because the simple exponential can get too high for␣
 ↪some samples.
sigmac(x,c) = c.sigmac*(x<0 ? 2/(1+exp(-2x)) : 4/(1+exp(-x))-1);
sigmac(x) = sigmac(x,cs);
# sigmaD(x,c) = defineSomeFunctionOf(x,c); #- general dividend diffusion
sigmaD(x,c) = sigmac(x,c); #- case of consumption perpetuity
sigma(x,c) = c.sigmax; # diffusion of modified state
mu(x,c) = -log(c.phi)*(c.xbar-x)-c.rhocx*c.gamma*sigmac(x,c)*sigma(x,c)+c.
 ↪rhoxD*sigma(x,c)*sigmaD(x,c) ; # drift of modified state
```

Short-term rate function

```
[4]: r(x,c) = c.rho+c.gamma*c.muc0-c.gamma^2*sigmac(x,c)^2/2;
r(x) = r(x,cs);
muD(x) = cs.muD; # perpetuity with constant dividend drift
rmod(x,c) = r(x,c)-(muD(x)-c.gamma*c.rhocD*sigmac(x,c)*sigmaD(x,c));
rmod(x) = rmod(x,cs);
```

Define setup consistent with SDE solution in Julia

```
[5]: function drift(du,u,p,t,c)
    du[1] = mu(u[1],c)
    du[2] = rmod(u[1],c)
end
drift(du,u,p,t) = drift(du,u,p,t,cs);
function diffusion(du,u,p,t,c)
    du[1] = sigma(u[1],c)
    du[2] = 0.0
end
diffusion(du,u,p,t) = diffusion(du,u,p,t,cs);
```

Define the Problem and SolutionSettings variables

```
[6]: prob = sdf.
 ↪Problem(drift=drift,diffusion=diffusion,numNoiseVariables=1,outVariables=[2],
terminalFunction=(ik, x, y, z) -> exp(-x));
xRange = -2.0:0.4:2.0;
tRange = 0.0:5.0:300.0;
sett = sdf.SolutionSettings(xRanges=[xRange,], initialValues=[[x, 0.0] for x in␣
 ↪xRange],
algorithm=sde.LambaEM(), pathsPerInitialValue=5000, tRange=tRange);
# add the settings in order to compmute price-dividend ration of the continuous␣
 ↪payoff security
sett2 = sdf.SolutionSettings(sett; continuousPayoffVars=[2]);
```

Solve Problem and Define Yield

```
[7]: ((consumptionStrip,),(priceConsumptionRatio,)) = sdf.solve(prob, sett2);
```

Get the Return of the Consumption Perpetuity The calculation requires the computation of the first and second derivatives of the price consumption ratio with respect to the state of the economy.

[8]:
```
(DPC,D2PC) = sdf.derivatives(priceConsumptionRatio);
ret(x) = (DPC(x)*(mu(x, cs) + sigma(x, cs) * sigmac(x,cs) * cs.rhocx) +
    D2PC(x)* sigma(x, cs)^2/2.0 + 1.0)/priceConsumptionRatio(x) + muD(x);
```

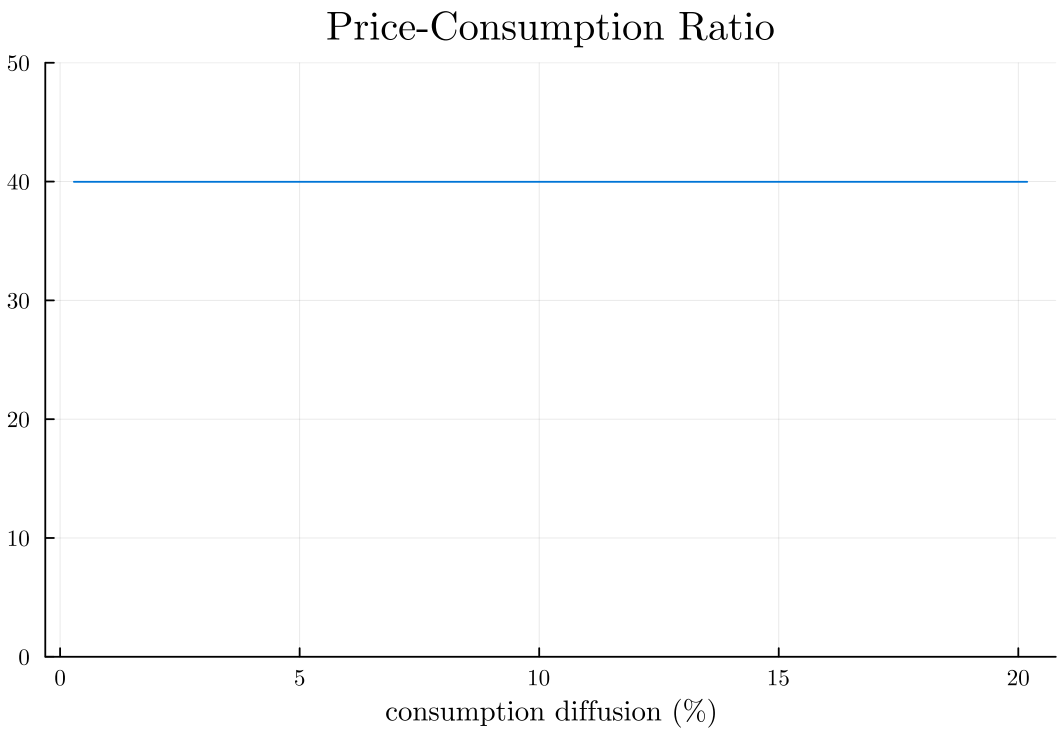
In the theory I state that the price consumption ratio is computed from the integral over all consumption strip maturities. In practice it is not possible to integrate to infinity. So, I compute consumption strips up to a maturity of 300 years.

It can be seen that for all values of the state variable the price of the securities comes close to 0 for maturities as long as 300 years.

Plot the Price Consumption Ratio

[9]:
```
import Plots as plt
plt.default(titlefont= (14,"Computer Modern"),legendfont=(8,"Computer Modern"),
    tickfont=(8,"Computer Modern"),guidefont=(10,"Computer Modern"))
plt.plot(100*sigmac.(xRange), priceConsumptionRatio(xRange), legend=false,
    title="Price-Consumption Ratio",color="#0075d6",ylims = (0.0, 50.0),␣
 ↪xlabel="consumption diffusion (%)")
```

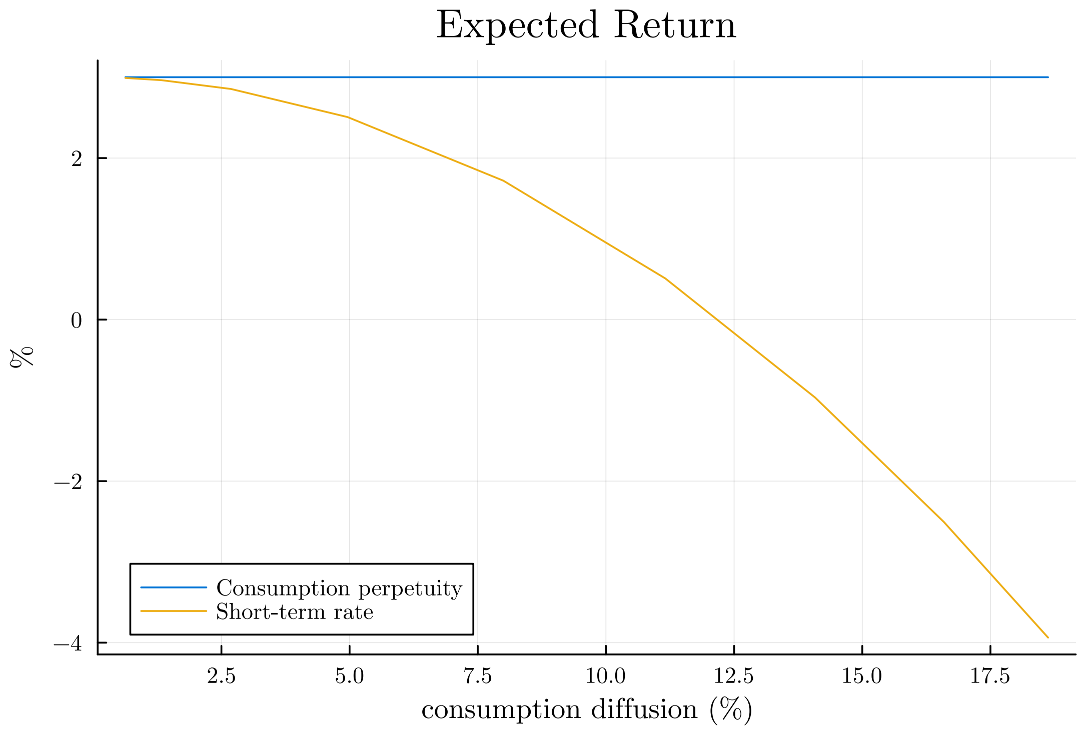
[9]:



Plot the Return

[10]:
```julia
plt.plot(100*sigmac.(xRange[2:end-1]), 100*ret.(xRange[2:end-1]),␣
↪label="Consumption perpetuity",
    title="Expected Return", color="#0075d6", xlabel="consumption diffusion␣
↪(%)",ylabel="%")
plt.plot!(100*sigmac.(xRange[2:end-1]), 100*r.(xRange[2:end-1]),
    label="Short-term rate",color="#edad14")
```

[10]:



It turns out that for a value of gamma=2 the effects cancel out and the price consumption ratio constant. However, there is still an expected return due to the dividend and the equity premium is large and increasing as volatility increases, due to the falling risk-free rate.

Example 5 – Two state variables

Time-Varying Drift and Diffusion – Zero-Coupon Bond

Here the setup is more complicated, having two state variables. Both consumption drift and diffusion are simultaneously time-varying. The modified processes that will give the price of the zero coupon security are the following (where $\hat{x}_1$ and $\hat{x}_2$ revert to $\bar{x}_1$ and $\bar{x}_2$ respectively):

$$\mathrm{d}c_t = \mu_{ct}\mathrm{d}t+\sigma_{ct}(1-|\rho_{c1}|-|\rho_{c2}|)\mathrm{d}W_{ct}+\sigma_{ct}\rho_{cx1}\mathrm{d}W_{x1t}+\sigma_{ct}\rho_{cx1}\mathrm{d}W_{x2t}\mathrm{d}\hat{x}_{1t} = (-\log\phi\cdot(\bar{x}_1-\hat{x}_{1t})+\rho_{cx1}\sigma_{ct}\sigma_{1x})\mathrm{d}t+\sigma_{x1}$$

where W_{c1}, W_{x1} and W_{x2} are independent and:

$$\mathrm{E}[\mathrm{d}c_t\mathrm{d}\bar{x}_{1t}] = \left(\rho_{cx1}\frac{1}{1+\rho_{12}}+\rho_{cx2}\frac{\rho_{12}}{1+\rho_{12}}\right)\sigma_{ct}\sigma_{x1}\mathrm{d}t \approxeq \rho_{cx1}\sigma_{ct}\sigma_{x1}\mathrm{d}t$$

$$\mathrm{E}[\mathrm{d}c_t\mathrm{d}\bar{x}_{2t}] = \left(\rho_{cx2}\frac{1}{1+\rho_{21}}+\rho_{cx1}\frac{\rho_{21}}{1+\rho_{21}}\right)\sigma_{ct}\sigma_{x2}\mathrm{d}t \approxeq \rho_{cx2}\sigma_{ct}\sigma_{x2}\mathrm{d}t$$

$$\mathrm{E}[\mathrm{d}\bar{x}_{1t}\mathrm{d}\bar{x}_{2t}] = \sigma_{x1}\sigma_{x2}\frac{\rho_{21}+\rho_{12}}{1+\rho_{12}+\rho_{21}+\rho_{12}\rho_{21}}\mathrm{d}t \approxeq (\rho_{12}+\rho_{21})\sigma_{x1}\sigma_{x2}\mathrm{d}t$$

and the approximate equations are valid if ρ_{12} and ρ_{21} are small.

In order to get the price of the zero-coupon security a process for the integral of the short-term rate will also be needed:

$$\mathrm{d}\mathcal{I} = r(\hat{x}_{1t},\hat{x}_{2t})\mathrm{d}t$$

Import the packages

[1]:
```
import SDFPricing as sdf
import StochasticDiffEq as sde # this is needed in order to specify the␣
 ↪algorithm
```

Define the parameters

[2]:
```
cs = (
    phi1 = 0.91, # steady state reversion
    phi2 = 0.96, # steady state reversion
    xbar1 = 0.0, # steady state
    xbar2 = 0.0, # steady state
    rho = 0.02, # time preference parameter
    gamma = 2.0, # risk aversion
    muc = 0.005, # mean of consumption drift
    sigmac = 0.08, # mean of consumption diffusion
    sigmax1 = 0.005, # volatility
    sigmax2 = 0.2, # volatility
    rhocx1 = 0.0, # correlation parameter
    rhocx2 = -0.6, # correlation parameter
    rho12 = 0.1, # correlation parameter
```

```
    rho21 = 0.1 # correlation parameter
    );
```

Drift and Diffusion of the processes I also include the unmodified process which will correspond to "risk-neutral pricing". By comparing normal pricing with risk-neutral pricing it is possible to compute excess returns.

[3]:
```
###- now consumption diffusion is a non-linear function of the state,
###- given that it needs to be positive.
###- I use this function because the simple exponential can get too high for␣
 ↪some samples.
sigmac(x,c) = c.sigmac*(x<0 ? 2/(1+exp(-2x)) : 4/(1+exp(-x))-1);
muc(x,c) = c.muc + x;
sigmac(x) = sigmac(x,cs);
sigma1_1(x,c) = c.sigmax1/(1+c.rho12);
sigma1_2(x,c) = c.sigmax1*c.rho12/(1+c.rho12);
sigma1(x,c) = sigma1_1(x,c)+sigma1_2(x,c);
sigma2_1(x,c) = c.sigmax2*c.rho21/(1+c.rho21);
sigma2_2(x,c) = c.sigmax2/(1+c.rho21);
sigma2(x,c) = sigma2_1(x,c)+sigma2_2(x,c);
mu1(x,c) = -log(c.phi1)*(c.xbar1-x)-c.gamma*sigmac(x,c)*(sigma1_1(x,c)*c.
 ↪rhocx1+sigma1_2(x,c)*c.rhocx2); # drift of modified state
mu2(x,c) = -log(c.phi2)*(c.xbar2-x)-c.gamma*sigmac(x,c)*(sigma2_1(x,c)*c.
 ↪rhocx1+sigma2_2(x,c)*c.rhocx2); # drift of modified state
mu10(x,c) = -log(c.phi1)*(c.xbar1-x); # drift of unmodified state
mu20(x,c) = -log(c.phi2)*(c.xbar2-x); # drift of unmodified state
```

Short-term rate function

[4]:
```
r(x1,x2,c) = c.rho+c.gamma*muc(x1,c)-c.gamma^2*sigmac(x2,c)^2/2;
r(x1,x2) = r(x1,x2,cs);
```

Define setup consistent with SDE solution in Julia

[5]:
```
function drift(du,u,p,t,c)
    du[1] = mu1(u[1],c)
    du[2] = mu2(u[2],c)
    du[3] = mu10(u[3],c)
    du[4] = mu20(u[4],c)
    du[5] = r(u[1],u[2],c)
    du[6] = r(u[3],u[4],c)
end
drift(du,u,p,t) = drift(du,u,p,t,cs);
function diffusion(du,u,p,t,c)
    du[1, 1] = sigma1_1(u[1], c)
    du[1, 2] = sigma1_2(u[1], c)
    du[2, 1] = sigma2_1(u[2], c)
    du[2, 2] = sigma2_2(u[2], c)
```

```
    du[3, 1] = sigma1_1(u[3], c)
    du[3, 2] = sigma1_2(u[3], c)
    du[4, 1] = sigma2_1(u[4], c)
    du[4, 2] = sigma2_2(u[4], c)
    du[5, 1] = 0.0
    du[5, 2] = 0.0
    du[6, 1] = 0.0
    du[6, 2] = 0.0
end
diffusion(du,u,p,t) = diffusion(du,u,p,t,cs);
```

Define the Problem and SolutionSettings variables

[6]:
```
prob = sdf.
 ↪Problem(drift=drift,diffusion=diffusion,numNoiseVariables=2,outVariables=[5,6],
terminalFunction=(ik, x, y, z) -> exp(-x),diagonalNoise=false);
xRanges = [-0.03:0.005:0.03,-2.0:0.4:2.0];
tRange = 0.0:1.0:20.0;
sett = sdf.SolutionSettings(xRanges=xRanges, initialValues=vcat([[x, y,x,y, 0.
 ↪0,0.0] for y in xRanges[2] for x in xRanges[1]]),
algorithm=sde.LambaEM(), pathsPerInitialValue=5000, tRange=tRange);
```

Solve Problem and Define Yield

[7]:
```
((bondPrice,riskNeutralPrice),) = sdf.solve(prob, sett);
yld(t,x1,x2) = -log(bondPrice(t,x1,x2))/t;
yldRiskNeutral(t,x1,x2) = -log(riskNeutralPrice(t,x1,x2))/t;
```

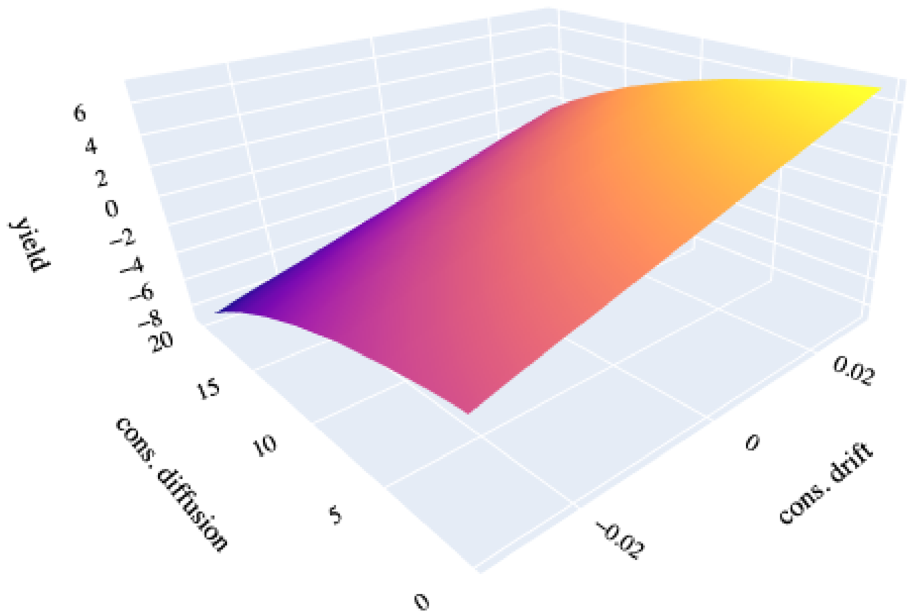
Plot the Yield in 3D

[10]:
```
import Plots as plt
import PlotlyJS as pltjs
coordinates = pltjs.surface(
    z=[100*yld(10.0, x1, x2) for x1 in xRanges[1], x2 in xRanges[2]],␣
 ↪x=xRanges[1],
    y=100*sigmac.(xRanges[2]),
    showscale=false)

layout = pltjs.Layout(
    width=800, height=350,
    title_x=0.5,
    titlefont_size="16",
    scene_aspectratio=pltjs.attr(x=1, y=1, z=0.5),
    scene=pltjs.attr(
        xaxis=pltjs.attr(title="cons. drift"),
        yaxis=pltjs.attr(title="cons. diffusion"),
        zaxis=pltjs.attr(title="yield"),
        camera=pltjs.attr(
            center=pltjs.attr(x=0.3, y=0, z=-0.40),
```

```
            eye=pltjs.attr(x=-.95, y=-1.25, z=0.65)
        )
    ),
    font=pltjs.attr(family="Computer Modern", size=12, color="black"),
    margin=pltjs.attr(l=0, r=0, b=0, t=0, pad=0))
  pltjs.plot([coordinates], layout)
```

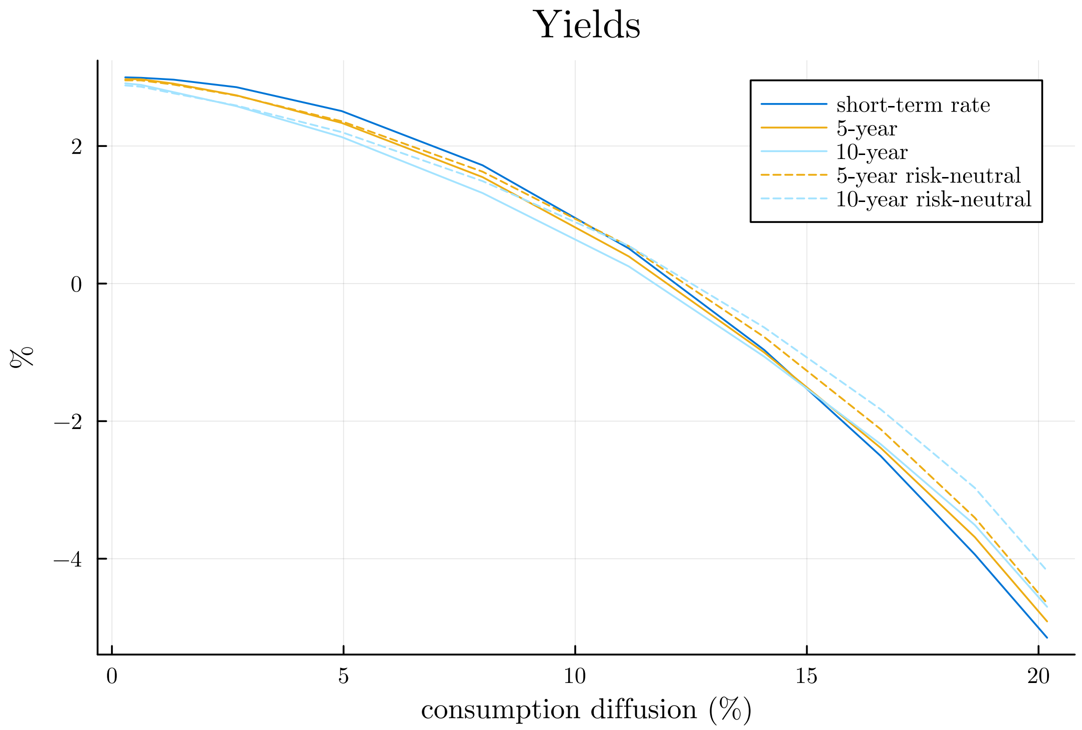
[10]:



[9]:

```
# colors: "#0075d6", "#edad14", "#a3e3ff", "#9c0000"
import Plots as plt
x1 = 0.0
plt.default(titlefont= (14,"Computer Modern"),legendfont=(8,"Computer Modern"),
    tickfont=(8,"Computer Modern"),guidefont=(10,"Computer Modern"))
plt.plot(100*sigmac.(xRanges[2]), xRanges[2] .|> x2->100*r(x1,x2),␣
  ↪title="Yields",
    xlabel="consumption diffusion (%)",ylabel="%",label="short-term␣
  ↪rate",color="#0075d6")
plt.plot!(100*sigmac.(xRanges[2]), 100*yld.(5.0, x1,xRanges[2]),␣
  ↪label="5-year",color= "#edad14")
plt.plot!(100*sigmac.(xRanges[2]), 100*yld.(10.0, x1,xRanges[2]),␣
  ↪label="10-year",color="#a3e3ff")
plt.plot!(100*sigmac.(xRanges[2]), 100*yldRiskNeutral.(5.0, x1,xRanges[2]),␣
  ↪label="5-year risk-neutral",color= "#edad14",style=:dash)
plt.plot!(100*sigmac.(xRanges[2]), 100*yldRiskNeutral.(10.0, x1,xRanges[2]),␣
  ↪label="10-year risk-neutral",color="#a3e3ff",style=:dash)
```

[9]:



Focusing on one state variable shows that the results are similar to the single variable case.

Example 6 – Replication of Habit Model in Wachter (2006) in Continuous Time

Time-Varying Surplus Consumption Ratio

This example replicates some main results from Wachter (2006) as adapted to continuous time. The results replicated are the yields of real bonds and the price-consumption ratio. The state variable in the model is the surplus consumption ratio, which is part of the utility function. It evolves according to:

$$\mathrm{d}s_t = -\log\phi(\bar{s} - s_t)\mathrm{d}t + \lambda(s_t)\sigma_c \mathrm{d}W_t$$

The modified state variable and short rate for the calculation of the price-consumption ratio evolves according to:

$$\mathrm{d}\tilde{s}_t = \Big(-\log\phi(\bar{s} - \gamma\tilde{s}_t) + \lambda(\tilde{s}_t)\sigma_c + \Big)\mathrm{d}t + \lambda(\tilde{s}_t)\sigma_c \mathrm{d}W_t$$

$$\tilde{r}(\tilde{s}_t) = \rho + \gamma\log(\phi)\tilde{s}_t + \gamma\mu_c - \frac{\gamma^2\sigma_c^2}{2}\big(1+\lambda(s)\big)^2 - \mu_c - \sigma_c^2\lambda(\tilde{s}_t)$$

In order to get the price of the consumption strip, the following stochastic integral needs to be simulated:

$$\mathrm{d}\mathcal{J}_2 = \tilde{r}(\tilde{x}_t)\mathrm{d}t$$

Import the packages

[50]:
```
import SDFPricing as sdf
import StochasticDiffEq as sde # this is needed in order to specify the␣
↪algorithm
```

Define the parameters

[51]:
```
#? parameters from Campbell, Cochrane (1999) (also Wachter (2005) Solving␣
↪Models with External Habit)
cs = (
    phi = 0.87, # mean reversion
    gamma = 2.0, # risk aversion
    b = 0.0,
    xbar = 0.0, # long-run mean
    rho = -log(0.895), # time preference parameter
    muc0 = 0.0189, # consumption drift
    sigmac = 0.015, # consumption diffusion
    rhocx = 1.0, # correlation between consumption and state variable
    sigmaD = 0.015, # dividend diffusion ###- case of consumption perpetuity
    muD = 0.0189, # dividend drift ###- case of consumption perpetuity
    rhoxD = 1.0, # correlation between dividends and state variable ###- case␣
↪of consumption perpetuity
    rhocD = 1.0 # correlation between dividends and consumption ###- case of␣
↪consumption perpetuity
```

```
);
```

Drift and Diffusion of the processes I also include the unmodified process which will correspond to "risk-neutral pricing". By comparing normal pricing with risk-neutral pricing it is possible to compute excess returns.

[52]:
```
###- now consumption diffusion is a non-linear function of the state,
###- given that it needs to be positive.
###- I use this function because the simple exponential can get too high for␣
 ↪some samples.
Sbar = cs.sigmac*(cs.gamma/(-log(cs.phi)-cs.b/cs.gamma))^(1/2);
# Sbar = cs.sigmac*(cs.gamma/(1-cs.phi-cs.b/cs.gamma))^(1/2);
lambda(x) = x<(1-Sbar^2)/2 ? (1-2*x)^(1/2)/Sbar-1 : 0;
sigmac(x,c) = c.sigmac;
sigmac(x) = sigmac(x,cs);
sigmaL(x) = -cs.gamma*sigmac(x)*(1+lambda(x))
# sigmaD(x,c) = defineSomeFunctionOf(x,c); #- general dividend diffusion
sigmaD(x,c) = sigmac(x,c); #- case of consumption perpetuity
sigma(x,c) = sigmac(x)*lambda(x); # diffusion of modified state
mu00(x,c) = -log(c.phi)*(c.xbar-x); # drift of unmodified state
mu0(x,c) = mu00(x,c)+c.rhocx*sigma(x,c)*sigmaL(x); # drift of modified state␣
 ↪for yield
# drift of modified state for price consumption ratio
mu(x,c) = mu0(x,c)+c.rhoxD*sigma(x,c)*sigmaD(x,c);
```

Short-term rate function

[54]:
```
r(x,c) = c.rho+c.gamma*log(c.phi)*x+c.gamma*c.muc0-sigmaL(x)^2/2;
r(x) = r(x,cs);
muD(x) = cs.muD; # perpetuity with constant dividend drift
rmod(x,c) = r(x,c)-muD(x)-c.rhocD*sigmaD(x,c)*sigmaL(x);
rmod(x) = rmod(x,cs);
```

Define setup consistent with SDE solution in Julia

[56]:
```
function drift(du,u,p,t,c)
    du[1] = mu(u[1],c)
    du[2] = rmod(u[1],c)
end
drift(du,u,p,t) = drift(du,u,p,t,cs);
function diffusion(du,u,p,t,c)
    du[1] = sigma(u[1],c)
    du[2] = 0.0
end
diffusion(du,u,p,t) = diffusion(du,u,p,t,cs);
```

Define the Problem and SolutionSettings variables

[57]:
```
prob = sdf.
  ↪Problem(drift=drift,diffusion=diffusion,numNoiseVariables=1,outVariables=[2],
terminalFunction=(ik, x, y, z) -> exp(-x));
xRange = -2.5:0.05:0.5;
tRange = 0.0:5.:120.0;

sett = sdf.SolutionSettings(xRanges=[xRange,], initialValues=[[x,0.0] for x in␣
  ↪xRange],
algorithm=sde.LambaEM(), pathsPerInitialValue=1000, tRange=tRange,dt=0.005);
# add the settings in order to compmute price-dividend ration of the continuous␣
  ↪payoff security
sett2 = sdf.SolutionSettings(sett; continuousPayoffVars=[2]);
```

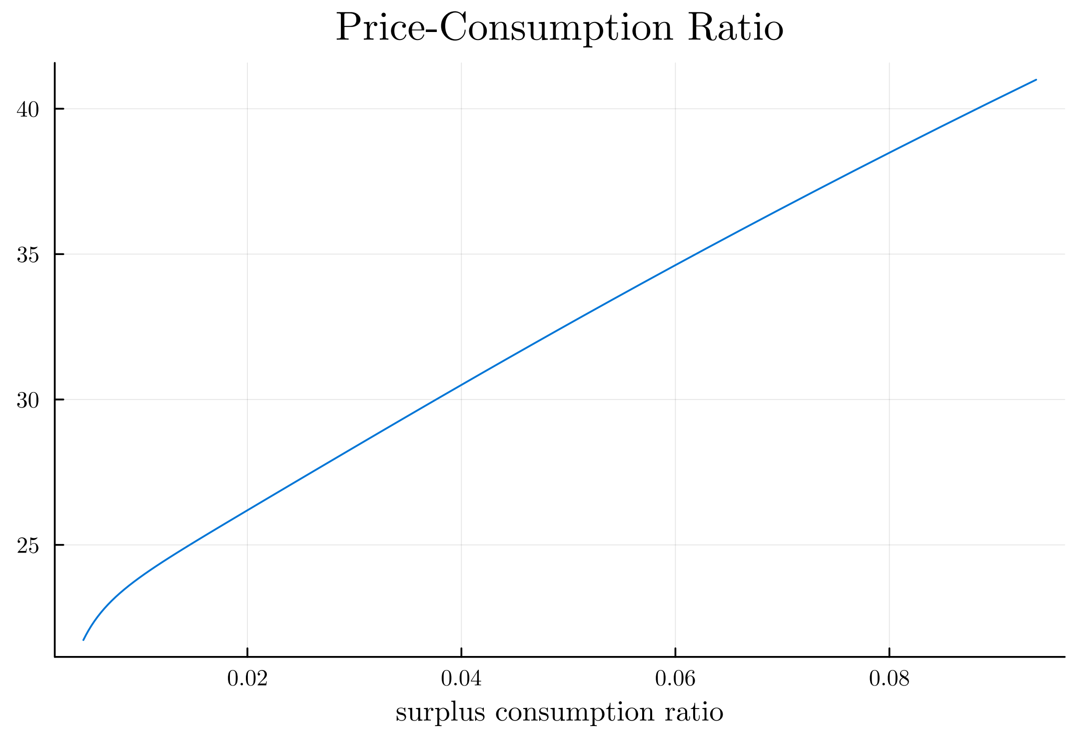
Solve Problem and Define Yield

[58]:
```
((singlePayoff),continuousPayoff) = sdf.solve(prob, sett2);
yld1(t,x) = -log(singlePayoff[1](t,x))/(t);
```

Plot the price-consumption ratio

[60]:
```
import Plots as plt
plt.default(titlefont= (14,"Computer Modern"),legendfont=(8,"Computer Modern"),
    tickfont=(8,"Computer Modern"),guidefont=(10,"Computer Modern"))
plt.plot(Sbar.*exp.(xRange), continuousPayoff[1](xRange), legend=false,
    title="Price-Consumption Ratio",color="#0075d6", xlabel="surplus␣
  ↪consumption ratio")
```

[60]:



Here the result is significantly different compared to the original paper. This shows that the result can vary significantly based on the numerical methods used. See Wachter (2005) Solving Models with External Habit for a discussion of the numerical issues involved. With further development of the package I am planning to make these issues more transparent and these problems better manageable with the appropriate methods.

Bibliography

Adrian, Tobias, and Hyun Song Shin. 2010. “Financial Intermediaries and Monetary Economics.” In *Handbook of Monetary Economics*. Vol. 3, 601–650. Elsevier.

Altavilla, Carlo, Luca Brugnolini, Refet S Gürkaynak, Roberto Motto, and Giuseppe Ragusa. 2019. “Measuring Euro Area Monetary Policy.” *Journal of Monetary Economics*, 108: 162–179.

Andrade, Philippe, and Filippo Ferroni. 2021. “Delphic and Odyssean Monetary Policy Shocks: Evidence from the Euro Area.” *Journal of Monetary Economics*, 117: 816–832.

Bansal, Ravi, and Amir Yaron. 2004. “Risks for the Long Run: A Potential Resolution of Asset Pricing Puzzles.” *The Journal of Finance*, 59(4): 1481–1509.

Borio, Claudio, and Haibin Zhu. 2012. “Capital Regulation, Risk-Taking and Monetary Policy: A Missing Link in the Transmission Mechanism?” *Journal of Financial Stability*, 8(4): 236–251.

Campbell, Jeffrey R, Charles L Evans, Lonas DM Fisher, and Alejandro Lustiniano. 2012. “Macroeconomic Effects of Federal Reserve Forward Guidance.” *Brookings Papers on Economic Activity.*

Campbell, John Y, and John H Cochrane. 1999. “By Force of Habit: A Consumption-Based Explanation of Aggregate Stock Market Behavior.” *Journal of Political Economy*, 107(2): 205–251.

Chen, Yu, Thomas F Cosimano, and Alex A Himonas. 2010. “Continuous Time One-Dimensional Asset-Pricing Models with Analytic Price–Dividend Functions.” *Economic Theory*, 42(3): 461–503.

Cieslak, Anna, and Andreas Schrimpf. 2019. “Non-Monetary News in Central Bank Communication.” *Journal of International Economics*, 118: 293–315.

Cochrane, John H. 2009. *Asset Pricing: Revised edition.* Princeton University Press.

Cox, John C, Jonathan E Ingersoll Jr, and Stephen A Ross. 1985. “An Intertemporal General Equilibrium Model of Asset Prices.” *Econometrica: Journal of the Econometric Society*, 363–384.

Duffie, Darrell. 2010. *Dynamic Asset Pricing Theory.* Princeton University Press.

Gürkaynak, Refet S, Brian Sack, and Eric T Swanson. 2005. "Do Actions Speak Louder Than Words? The Response of Asset Prices to Monetary Policy Actions and Statements." *International Journal of Central Banking.*

Jarociński, Marek, and Peter Karadi. 2020. "Deconstructing Monetary Policy Surprises—The Role of Information Shocks." *American Economic Journal: Macroeconomics*, 12(2): 1–43.

Kuttner, Kenneth N. 2001. "Monetary Policy Surprises and Interest Rates: Evidence from the Fed Funds Futures Market." *Journal of Monetary Economics*, 47(3): 523–544.

Lettau, Martin, and Jessica A Wachter. 2011. "The Term Structures of Equity and Interest Rates." *Journal of Financial Economics*, 101(1): 90–113.

Melissinos, Errikos. 2023*a*. "A Perturbation Solution for the Pricing of Long-Term Bonds under Recursive Utility."

Melissinos, Errikos. 2023*b*. "Real Term Premia in Consumption-Based Models." *Available at SSRN 4582708.*

Nakamura, Emi, and Jón Steinsson. 2018. "High-Frequency Identification of Monetary Non-Neutrality: The Information Effect." *The Quarterly Journal of Economics*, 133(3): 1283–1330.

Vayanos, Dimitri, and Jean-Luc Vila. 2021. "A Preferred-Habitat Model of the Term Structure of Interest Rates." *Econometrica*, 89(1): 77–112.

Wachter, Jessica A. 2005. "Solving Models With External Habit." *Finance Research Letters*, 2(4): 210–226.

Wachter, Jessica A. 2006. "A Consumption-Based Model of the Term Structure of Interest Rates." *Journal of Financial Economics*, 79(2): 365–399.

Chapter 3

A Perturbation Solution Method for Models with Recursive Utility

Abstract

I illustrate a novel method for pricing assets within recursive utility models in continuous time, that has first been used in Melissinos (2023). My method builds on the analytic solution of the problem for a value of the intertemporal elasticity of substitution, ψ, equal to 1. I provide the full perturbation series for the value function, when consumption is exogenous, in terms of ψ, which gives rise to a global perturbation approximation. This allows the pricing of assets for a much larger range of values for ψ, which are economically meaningful. I comment on the convergence properties of the perturbation series, and I show that the method provides a straightforward and reliable approach to asset pricing. I employ my method to derive the prices of long-term bonds, the price-consumption ratio and the instantaneous return of the consumption perpetuity. Finally, I extend the method, and I show that it can also be used to solve the consumption-investment problem.

3.1 Introduction

Recursive utility (RU) (Epstein and Zin 1989) is extensively used in the literature, to express more general preferences than can be expressed by time-separable utility (TSU). Indeed some models, such as in Bansal and Yaron (2004) and Wachter (2013), require the use of recursive utility, to produce important features of the data. RU models in continuous time were introduced by Duffie and Epstein (1992*b*) and an important literature describing their properties has since followed, for example Duffie and Epstein 1992*a*; Duffie, Schroder and Skiadas 1996; Schroder and Skiadas 1999 and many others. Even though solving these models in continuous time remains challenging, for some special cases the problem does have a closed-form solution. Chacko and Viceira (2005) for example presented a closed-form solution to the consumption-investment problem with recursive utility when the intertemporal elasticity of substitution (IES) is equal to one. Tsai and Wachter (2018) provided a similar solution for the value function when the IES is equal to one, while the stochastic processes can also undergo discrete jumps. In this paper, I introduce a perturbation method that is based on this closed-form result. The method provides a solution in terms of an expansion of the IES parameter.

In particular, in this paper, I follow Tsai and Wachter (2018), who claimed that their solution can also be used for other IES values. However, a priori it is not obvious under which conditions this is true. My more general solution is valid for a large range of IES values and this also allows the evaluation of the accuracy of the approximation offered by Tsai and Wachter (2018). My approach is based on the perturbation theory described in Bender, Orszag and Orszag (1999), which has also been advanced in economics and finance by Judd (1996). Apart from Melissinos (2023), where the method was used to solve long-run risk type models, to the best of my knowledge, this is the first paper to use a perturbation method with respect to the IES parameter. Chacko and Viceira (2005) and Leisen (2016) also provided two separate perturbation solutions for the consumption-investment problem.[1] However, these perturbations are not with respect to the IES parameter, as is done in this paper.

Once the value function is derived based on the perturbation approximation, I can derive the stochastic differential equation (SDE) form of the stochastic discount factor (SDF), following the derivation of Chen et al. (2009). Then I can

[1] Caldara et al. (2012) also uses perturbation methods to solve a DSGE model with recursive utility.

proceed to solve the partial differential equation (PDE) that is associated with the pricing of the long-term zero-coupon bond. In addition, quantities like the price-dividend ratio, the wealth-consumption ratio and the price of dividend-paying securities can also be computed. I solve the pricing equation for the long-term bond by using the Feynman-Kac formula, which I implement through Monte Carlo simulations. The method is not restricted to solving the problem with an exogenous consumption process. Indeed the same method can also be extended to solve the consumption-investment problem with RU by using the results in Kraft, Seiferling and Seifried (2017).

The rest of the paper is organised as follows. Section 2 introduces the general framework including the RU component, Section 3 introduces the perturbation expansion, Section 4 performs the pricing of the long-term bond based on the previous results, Section 5 extends the method to the consumption-investment problem, and Section 6 concludes.

3.2 Recursive Utility Framework

This section closely follows the framework introduced by Tsai and Wachter (2018). For simplicity, I present my method using only one state variable. Introducing multiple state variables is straightforward based on the single-variable case.

3.2.1 Consumption process

The consumption process has two components: a deterministic trend and a Brownian motion component:[2]

$$\frac{\mathrm{d}C_t}{C_t} = \mathrm{d}\log(C_t) = \mathrm{d}c_t = \mu_{ct}\mathrm{d}t + \sigma_{ct}W_{ct} \tag{3.1}$$

where the t subscript denotes variables at time t,[3] C_t is consumption flow (or just consumption for simplicity), c_t is log consumption, x_t is the state variable characterising the economy, μ_{ct} is the deterministic consumption trend, σ_{ct} determines consumption volatility and W_{ct} is the Brownian motion component. μ_{ct} and σ_{ct} are either parameters or they depend on the state variable.

[2]Here, for simplicity, I assume that consumption does not undergo discontinuous jumps (with probability 1), but my solution method can be extended to the case, in which the consumption process includes Poisson jumps.

[3]For generality I use a subscript t for all symbols that can correspond to variables. However, in some variations, these symbols may also correspond to parameters.

3.2.2 State variable

Similar to consumption, the state variable also has two components:

$$\mathrm{d}x_t = \mu_{xt}\mathrm{d}t + \sigma_{xt}\mathrm{d}W_{xt} \tag{3.2}$$

Here, x_t denotes the state variable, and the functions and parameters are analogous to the case of consumption.

3.2.3 Utility

Lifetime utility is:[4]

$$V_0 = \mathrm{E}_t\left[\int_0^\infty F(C_t, V_t)\mathrm{d}t\right] \tag{3.3}$$

This equation highlights why utility is referred to as "recursive", as the integrand depends on the current value of the agent at each point in time. The combination of the consumption with the concurrent lifetime utility takes place via the so-called aggregator function:[5]

$$F(C_t, V_t) = \frac{(1-\gamma)\rho V_t(\Big(C_t\big((1-\gamma)V_t\big)^{-\frac{1}{1-\gamma}}\Big)^{1-\frac{1}{\psi}} - 1)}{1-\frac{1}{\psi}} \tag{3.4}$$

This function represents a flow that depends both on current consumption, C_t, and on the current level of the value function, V_t. ρ denotes a time preference parameter, γ denotes the risk aversion parameter and ψ denotes the IES. RU is a useful modelling tool because it allows the separation of the risk aversion parameter from the IES parameter. This utility specification reduces to the more familiar time-additive case when $\gamma = 1/\psi$. In this case, the agent is indifferent about when uncertainty is resolved. For more general parameter specifications the agent may exhibit a preference for late or early resolution of uncertainty. In particular, for $\gamma > 1/\psi$ ($\gamma < 1/\psi$) there is a preference for early (late) resolution of uncertainty. The intuition for this mechanism can also be explained differently.

[4]Following Tsai and Wachter (2018) I do not prove the existence and uniqueness of my solution. Hence, I use the infinite horizon case for simplicity. When considering a proof of existence and/or uniqueness, a finite horizon may be easier to deal with. In general, it would be important to derive results that would guarantee the existence of a solution in general to this problem, as also discussed in Tsai and Wachter (2018). However, there are existence results for similar problems in special cases. Seiferling and Seifried (2016) for instance showed that a solution exists for a problem with a finite horizon when $\gamma\psi, \psi \geq 1$ or $\gamma\psi, \psi \leq 1$. In the cases displayed in this paper, many fulfil the first condition $\gamma\psi, \psi \geq 1$.

[5]This is the normalised form of the aggregator in Duffie and Epstein (1992*b*)

In particular, $\gamma > 1/\psi$ implies that:

$$\frac{\partial^2 F(C_t, V_t)}{\partial C_t \partial V_t} < 0 \text{ and } \frac{\partial^3 F(C_t, V_t)}{\partial C_t^2 \partial V_t} > 0$$

$$\Rightarrow \mathrm{E}_t\left[\frac{\partial F(C_t, V_t)}{\partial C_t}\right] < \mathrm{E}_t\left[\frac{\partial F(C_t, \mathrm{E}_{t+1}[V_t])}{\partial C_t}\right] \tag{3.5}$$

On the right-hand side, the notation means that the agent has been given early information about the state of the world in $t+1$, while on the left-hand side, this is not the case. It follows that the ex-ante expectation of these two situations leads to a preference for early resolution of uncertainty, as the utility flow is expected to be higher, in a state where the agent has early knowledge. The opposite is true in the case of a preference for late resolution of uncertainty. The mathematics of the situation is similar to the case, in which a risk-averse agent prefers a safe sum of money to a risky lottery with the same expected value. So, it is crucial for the preference of early resolution of uncertainty that consumption becomes less enjoyable as the value of the agent increases. Notice that this is not the familiar diminishing marginal utility of consumption. Instead, this implies that the same quantity of consumption is less enjoyable when the agent becomes happier for reasons that are not related to current consumption, for example, she may have learned that expected consumption in the future has increased and this makes her current consumption less enjoyable.

3.2.4 Decomposition of the value function

In the RU framework, there exists a scale invariance property (Duffie and Epstein 1992*b*), which allows us to express the value of the agent in a way that separates the dependence on consumption from the dependence on the state variable. As shown in Tsai and Wachter (2018), the value function can be written as:[6]

$$V_t = \frac{C_t^{1-\gamma} e^{K(x_t)(1-\gamma)}}{1-\gamma} \tag{3.6}$$

[6]Similar results are common in the literature, see for example Benzoni, Collin-Dufresne and Goldstein 2011; Kraft, Seiferling and Seifried 2017.

Where $K(x_t)$ satisfies the following differential equation:[7]

$$\rho\frac{\psi}{1-\psi}\left(1-e^{(1/\psi-1)K(x_t)}\right)-\frac{\gamma\sigma_{ct}^2}{2}+\mu_{ct}+\frac{\sigma_{xt}^2}{2}K''(x_t)+\mu_{xt}K'(x_t)+\frac{(1-\gamma)\sigma_{xt}^2}{2}K'(x_t)^2=0 \tag{3.7}$$

A proof of this result is included in Appendix 3.A.[8]

3.2.5 Functional forms for consumption and state variable processes

The asset pricing problem based on the above setup is generally not easy to solve. However, Tsai and Wachter (2018) showed that significant progress can be made under the following specification for the consumption process and the process of the state variable:

$$\begin{aligned}
\mu_{ct} &= \mu_{c0} + \mu_{c1}x_t \\
\sigma_{ct} &= \sqrt{\sigma_{c0} + \sigma_{c1}x_t} \\
\mu_{xt} &= -\log(\phi)(\mu_{x0} - x_t) \\
\sigma_{xt} &= \sqrt{\sigma_{x0} + \sigma_{x1}x_t}
\end{aligned} \tag{3.8}$$

The parameters on the right-hand side can be chosen. Notably, this specification is particularly useful because plugging these expressions into Equation (3.7) produces linear terms in x_t.

3.3 Method Description

3.3.1 Exact Solution for ψ=1

As shown Tsai and Wachter (2018), Equation (3.7) has an exact solution for $\psi = 1$. In particular, after using the expressions in Equation (3.8), the differential

[7] This equation is also valid for $\psi = 1$, in which case the expressions are replaced by their limits.

[8] A very similar result is also proven by Tsai and Wachter (2018).

equation becomes:

$$\begin{aligned}-\rho K(x_t) - \frac{1}{2}\gamma(\sigma_{c0} + x_t\sigma_{c1}) + \mu_{c0} + x_t\mu_{c1} + \frac{1}{2}K''(x_t)(x\sigma_{x1} + \sigma_{x0})\\ - \log(\phi)K'(x_t)(\mu_{x0} - x_t) - \frac{1}{2}\gamma K'(x_t)^2(x_t\sigma_{x1} + \sigma_{x0}) + \frac{1}{2}K'(x_t)^2(x_t\sigma_{x1} + \sigma_{x0}) = 0\end{aligned} \tag{3.9}$$

and the solution takes the form, $K(x_t) = a_{0,0} + a_{0,1}x$. The coefficients, $a_{0,0}$ and $a_{0,1}$ can be solved by sequentially solving the following equations:

$$\begin{aligned}0 =& -\rho a_{0,0} - \frac{1}{2}\gamma a_{0,1}^2\sigma_{x0} - a_{0,1}\mu_{x0}\log(\phi) + \frac{1}{2}a_{0,1}^2\sigma_{x0} - \frac{\gamma\sigma_{c0}}{2} + \mu_{c0}\\ 0 =& -\rho a_{0,1} - \frac{1}{2}\gamma a_{0,1}^2\sigma_{x1} + \frac{1}{2}a_{0,1}^2\sigma_{x1} + a_{0,1}\log(\phi) - \frac{\gamma\sigma_{c1}}{2} + \mu_{c1}\end{aligned} \tag{3.10}$$

The second equation is solved first as it only includes parameter $a_{0,1}$. Then, using this solution the first equation can also be solved.[9]

$$\begin{aligned}a_{0,1} &= \begin{cases} -\frac{\rho-\log(\phi)\pm\sqrt{2(\gamma-1)\mu_{c1}\sigma_{x1}-(\gamma-1)\gamma\sigma_{c1}\sigma_{x1}+(\rho-\log(\phi))^2}}{(\gamma-1)\sigma_{x1}} & \text{if } \sigma_{x1} \neq 0\\ \frac{2\mu_{c1}-\gamma\sigma_{c1}}{2\rho-2\log(\phi)} & \text{if } \sigma_{x1} = 0\end{cases}\\ a_{0,0} &= -\frac{\gamma a_{0,1}^2\sigma_{x0}}{2\rho} - \frac{a_{0,1}\mu_{x0}\log(\phi)}{\rho} + \frac{a_{0,1}^2\sigma_{x0}}{2\rho} - \frac{\gamma\sigma_{c0}}{2\rho} + \frac{\mu_{c0}}{\rho}\end{aligned} \tag{3.11}$$

Tsai and Wachter (2018) used this solution to derive analytical expressions for the pricing of long-term assets, when $\psi = 1$. They also use these results to derive approximate expressions for the case when $\psi \neq 1$.

3.3.2 Applying the method – The $\psi \neq 1$ case

3.3.2.1 General description

In this paper, I extend the above solution method, to allow the IES parameter to take a large range of values. As is common with perturbation solutions, instead of solving the problem for a specific value of ψ for which there is no analytic solution, the problem is redefined and solved for arbitrary ψ. This may seem more difficult, but since the solution for $\psi = 1$ is already known, the perturbation solution

[9] When $\sigma_{x1} \neq 0$, then $a_{0,1}$ has a double solution. However, only one of the two solutions is economically meaningful. This duplicity is explained by the existence of the square root and by the fact that the state variable can be defined to be an increasing or decreasing function of the state variable. Chacko and Viceira (2005) showed that it is possible to distinguish between the two solutions that arise when $b_1 \neq 0$, by picking the solution that leads to the correct known solution when $\gamma = 1$, which also corresponds to the case of logarithmic flow utility.

provides a way to start from the solution that is known, and then gradually move towards a solution that is valid for any ψ. I achieve this by re-expressing ψ in terms of ϵ and expanding the problem in a series of ϵ. In particular, ψ is replaced by $\frac{1}{1-\epsilon}$ and the expansion of ψ in terms of ϵ is:

$$\psi = \frac{1}{1-\epsilon} = 1 + \epsilon + \epsilon^2 + \epsilon^3 + \ldots \tag{3.12}$$

The redefinition in terms of ϵ is convenient because the analytic solution occurs for $\epsilon = 0$, and this makes the power series of $K(\cdot)$ considerably simpler. As above, I proceed by guessing the series solution of the differential equation (3.9):

$$\begin{aligned} K(x_t, \epsilon) = & \sum_{n=0}^{\infty} \epsilon^n \left(\sum_{m=0}^{n+1} a_{n,m} x_t^m \right) \\ = & (a_{0,0} + a_{0,1} x_t) \\ & + \epsilon (a_{1,0} + a_{1,1} x_t + a_{1,2} x_t^2) \\ & + \epsilon^2 (a_{2,0} + a_{2,1} x_t + a_{2,2} x_t^2 + a_{2,3} x_t^3) \\ & + \epsilon^3 (a_{3,0} + a_{3,1} x_t + a_{3,2} x_t^2 + a_{3,3} x_t^3 + a_{3,4} x_t^4) \\ & \quad \ldots \\ = & K_0(x_t) + K_1(x_t)\epsilon + K_2(x_t)\epsilon^2 + \ldots \end{aligned} \tag{3.13}$$

In the remainder of the paper, I refer to the approximations according to the highest power of ϵ. For example the approximation that only maintains $(a_{0,0} + a_{0,1} x)$ is the "zeroth" order approximation. The approximation that also maintains the next line in Equation (3.13) is the "first" order approximation and so on. The structure of the solution is a polynomial both in terms of x_t and in terms of ϵ. In particular, for each successive order of ϵ, the order of the polynomial in terms of x_t that is multiplying it increases by one. As it turns out, the solution of Tsai and Wachter (2018) corresponds to the zeroth order of the series. This makes sense given that $\epsilon = 0$ simplifies to the previous case, namely $\psi = 1$. The other higher polynomials in x_t are new and they show what the effect is from moving away from IES equal to 1. Thus, by replacing ψ with $\frac{1}{1-\epsilon}$, then plugging in the guess and expanding in terms of ϵ, Equation (3.9) becomes:

$$
\begin{aligned}
0 =& -\frac{1}{2}\gamma(\sigma_{c0} + x_t\sigma_{c1}) + \mu_{c0} + x_t\mu_{c1} - \rho K_0(x_t) - \log(\phi)K_0'(x_t)(\mu_{x0} - x_t) \\
&\quad - \frac{1}{2}\gamma K_0'(x_t)^2(x_t\sigma_{x1} + \sigma_{x0}) + \frac{1}{2}K_0'(x_t)^2(x_t\sigma_{x1} + \sigma_{x0}) + \frac{1}{2}K_0''(x_t)(x_t\sigma_{x1} + \sigma_{x0})\Big) \\
&\epsilon\Big(- \rho K_1(x_t) - \log(\phi)K_1'(x_t)(\mu_{x0} - x_t) - \gamma K_0'(x_t)K_1'(x_t)(x_t\sigma_{x1} + \sigma_{x0}) \\
&\quad + K_0'(x_t)K_1'(x_t)(x_t\sigma_{x1} + \sigma_{x0}) + \frac{1}{2}K_1''(x_t)(x_t\sigma_{x1} + \sigma_{x0})\Big) \\
&\epsilon^2\Big(- \rho K_2(x_t) - \log(\phi)K_2'(x_t)(\mu_{x0} - x_t) - \frac{1}{2}\gamma K_1'(x_t)^2(x_t\sigma_{x1} + \sigma_{x0}) \\
&\quad - \gamma K_0'(x_t)K_2'(x_t)(x\sigma_{x1} + \sigma_{x0}) + \frac{1}{2}K_1'(x_t)^2(x_t\sigma_{x1} + \sigma_{x0}) \\
&\quad + K_0'(x_t)K_2'(x_t)(x_t\sigma_{x1} + \sigma_{x0}) + \frac{1}{2}K_2''(x_t)(x_t\sigma_{x1} + \sigma_{x0})\Big)
\end{aligned} \tag{3.14}
$$

In the expression above I have still not inserted the $a_{\cdot,\cdot}$ parameters in detail, in order to not clutter the overall expression too much. Nevertheless, it can be seen that for this equation to hold for all values of ϵ, we need the coefficient for each power of ϵ to be equal to 0. Subsequently, each of these coefficients includes the K_n's and their derivatives, which contain polynomials of x_t. Following the same strategy, for these polynomials to be equal to 0 for all values of x_t, the corresponding coefficients also need to equal 0. Combining the two stages implies that for each pair of powers for ϵ and x_t, that show up in Equation (3.13), there is a corresponding equation that allows us to compute the respective coefficient.[10] In addition, each of these equations turns out to be linear and sequentially solvable given the solutions of the previous equations.[11] The order in which the corresponding coefficients are solved, increases with the powers of ϵ and decreases with the powers of x_t. Namely, the parameters are found in the following order:

$$
a_{0,1}, a_{0,0}, a_{1,2}, a_{1,1}, a_{1,0}, a_{2,3}, a_{2,2}, a_{2,1}, a_{2,0}, \ldots \tag{3.15}
$$

Given that each parameter requires the solution of a linear equation, it is easy to solve the model for high orders of approximation. However, for each order of ϵ, the number of parameters increases by one and the equations become increasingly complicated. As a result, roughly fifteen orders of approximation in terms of ϵ can

[10] Another way to think of this is the following: For each nth power of ϵ, there is a linear second order differential equation for $K_n(\cdot)$ which can be solved sequentially using the solutions of the previous differential equations.

[11] The only exception is parameter $a_{0,1}$ which was already mentioned above and might require the solution of a second order equation.

be found relatively quickly (this corresponds to 135 distinct parameter values).

3.3.2.2 First order approximation

Finding the first-order approximation requires the solution of the following equations:

$$\begin{aligned}
0 &= -\rho a_{1,2} - 2\gamma a_{0,1} a_{1,2} \sigma_{x1} + 2a_{0,1} a_{1,2} \sigma_{x1} + 2a_{1,2} \log(\phi) \\
0 &= -\rho a_{1,1} - 2\gamma a_{0,1} a_{1,2} \sigma_{x0} - 2a_{1,2} \mu_{x0} \log(\phi) + 2a_{0,1} a_{1,2} \sigma_{x0} - \gamma a_{0,1} a_{1,1} \sigma_{x1} \\
&\qquad + a_{0,1} a_{1,1} \sigma_{x1} + a_{1,2} \sigma_{x1} + a_{1,1} \log(\phi) \\
0 &= -\rho a_{1,0} - \gamma a_{0,1} a_{1,1} \sigma_{x0} - a_{1,1} \mu_{x0} \log(\phi) + a_{0,1} a_{1,1} \sigma_{x0} + a_{1,2} \sigma_{x0}
\end{aligned} \tag{3.16}$$

whose solutions are:

$$\begin{aligned}
a_{1,2} =& \frac{\rho a_{0,1}^2}{2(2\gamma a_{0,1} \sigma_{x1} - 2a_{0,1} \sigma_{x1} + \rho - 2\log(\phi))} \\
a_{1,1} =& \frac{\rho a_{0,0} a_{0,1} - 2\gamma a_{1,2} a_{0,1} \sigma_{x0} - 2a_{1,2} \mu_{x0} \log(\phi) + 2a_{1,2} a_{0,1} \sigma_{x0} + a_{1,2} \sigma_{x1}}{\gamma a_{0,1} \sigma_{x1} - a_{0,1} \sigma_{x1} + \rho - \log(\phi)} \\
a_{1,0} =& \frac{\rho a_{0,0}^2 - 2\gamma a_{0,1} a_{1,1} \sigma_{x0} - 2a_{1,1} \mu_{x0} \log(\phi) + 2a_{0,1} a_{1,1} \sigma_{x0} + 2a_{1,2} \sigma_{x0}}{2\rho}
\end{aligned} \tag{3.17}$$

As can be seen above, the values of all parameters can be found by plugging in the previous solutions.

3.3.2.3 Second order approximation

The solution of the second order proceeds similarly. The equations to be solved are the following:

$$
\begin{aligned}
0 &= \frac{1}{6}\gamma\rho a_{0,0}^3 - \frac{1}{6}\rho a_{0,0}^3 - \gamma\rho a_{1,0}a_{0,0} + \rho a_{1,0}a_{0,0} + \frac{1}{2}\gamma^2\sigma_{x0}a_{1,1}^2 - \gamma\sigma_{x0}a_{1,1}^2 + \frac{1}{2}\sigma_{x0}a_{1,1}^2 + \gamma\rho a_{2,0} \\
&\quad - \rho a_{2,0} + \gamma\log(\phi)\mu_{x0}a_{2,1} - \log(\phi)\mu_{x0}a_{2,1} + \gamma^2\sigma_{x0}a_{0,1}a_{2,1} - 2\gamma\sigma_{x0}a_{0,1}a_{2,1} + \sigma_{x0}a_{0,1}a_{2,1} \\
&\quad - \gamma\sigma_{x0}a_{2,2} + \sigma_{x0}a_{2,2} \\
0 &= \frac{1}{2}\sigma_{x1}a_{1,1}^2\gamma^2 + 2\sigma_{x0}a_{1,1}a_{1,2}\gamma^2 + \sigma_{x1}a_{0,1}a_{2,1}\gamma^2 + 2\sigma_{x0}a_{0,1}a_{2,2}\gamma^2 - \sigma_{x1}a_{1,1}^2\gamma + \frac{1}{2}\rho a_{0,0}^2a_{0,1}\gamma \\
&\quad - \rho a_{0,1}a_{1,0}\gamma - \rho a_{0,0}a_{1,1}\gamma - 4\sigma_{x0}a_{1,1}a_{1,2}\gamma + \rho a_{2,1}\gamma - \log(\phi)a_{2,1}\gamma - 2\sigma_{x1}a_{0,1}a_{2,1}\gamma \\
&\quad + 2\log(\phi)\mu_{x0}a_{2,2}\gamma - \sigma_{x1}a_{2,2}\gamma - 4\sigma_{x0}a_{0,1}a_{2,2}\gamma - 3\sigma_{x0}a_{2,3}\gamma + \frac{1}{2}\sigma_{x1}a_{1,1}^2 - \frac{1}{2}\rho a_{0,0}^2a_{0,1} \\
&\quad + \rho a_{0,1}a_{1,0} + \rho a_{0,0}a_{1,1} + 2\sigma_{x0}a_{1,1}a_{1,2} - \rho a_{2,1} + \log(\phi)a_{2,1} + \sigma_{x1}a_{0,1}a_{2,1} - 2\log(\phi)\mu_{x0}a_{2,2} \\
&\quad + \sigma_{x1}a_{2,2} + 2\sigma_{x0}a_{0,1}a_{2,2} + 3\sigma_{x0}a_{2,3} \\
0 &= 2\sigma_{x0}a_{1,2}^2\gamma^2 + 2\sigma_{x1}a_{1,1}a_{1,2}\gamma^2 + 2\sigma_{x1}a_{0,1}a_{2,2}\gamma^2 + 3\sigma_{x0}a_{0,1}a_{2,3}\gamma^2 + \frac{1}{2}\rho a_{0,0}a_{0,1}^2\gamma \\
&\quad - 4\sigma_{x0}a_{1,2}^2\gamma - \rho a_{0,1}a_{1,1}\gamma - \rho a_{0,0}a_{1,2}\gamma - 4\sigma_{x1}a_{1,1}a_{1,2}\gamma + \rho a_{2,2}\gamma - 2\log(\phi)a_{2,2}\gamma \\
&\quad - 4\sigma_{x1}a_{0,1}a_{2,2}\gamma + 3\log(\phi)\mu_{x0}a_{2,3}\gamma - 3\sigma_{x1}a_{2,3}\gamma - 6\sigma_{x0}a_{0,1}a_{2,3}\gamma - \frac{1}{2}\rho a_{0,0}a_{0,1}^2 \\
&\quad + 2\sigma_{x0}a_{1,2}^2 + \rho a_{0,1}a_{1,1} + \rho a_{0,0}a_{1,2} + 2\sigma_{x1}a_{1,1}a_{1,2} - \rho a_{2,2} + 2\log(\phi)a_{2,2} + 2\sigma_{x1}a_{0,1}a_{2,2} \\
&\quad - 3\log(\phi)\mu_{x0}a_{2,3} + 3\sigma_{x1}a_{2,3} + 3\sigma_{x0}a_{0,1}a_{2,3} \\
0 &= \frac{1}{6}\gamma\rho a_{0,1}^3 - \frac{1}{6}\rho a_{0,1}^3 - \gamma\rho a_{1,2}a_{0,1} + \rho a_{1,2}a_{0,1} + 3\gamma^2\sigma_{x1}a_{2,3}a_{0,1} - 6\gamma\sigma_{x1}a_{2,3}a_{0,1} + 3\sigma_{x1}a_{2,3}a_{0,1} \\
&\quad + 2\gamma^2\sigma_{x1}a_{1,2}^2 - 4\gamma\sigma_{x1}a_{1,2}^2 + 2\sigma_{x1}a_{1,2}^2 + \gamma\rho a_{2,3} - \rho a_{2,3} - 3\gamma\log(\phi)a_{2,3} + 3\log(\phi)a_{2,3}
\end{aligned}
\tag{3.18}
$$

The solutions are:

$$
\begin{aligned}
a_{2,3} =& -\frac{\rho a_{0,1}^3}{6(3\gamma a_{0,1}\sigma_{x1} - 3a_{0,1}\sigma_{x1} + \rho - 3\log(\phi))} + \frac{\rho a_{1,2} a_{0,1}}{3\gamma a_{0,1}\sigma_{x1} - 3a_{0,1}\sigma_{x1} + \rho - 3\log(\phi)} \\
& - \frac{2\gamma a_{1,2}^2 \sigma_{x1}}{3\gamma a_{0,1}\sigma_{x1} - 3a_{0,1}\sigma_{x1} + \rho - 3\log(\phi)} + \frac{2a_{1,2}^2\sigma_{x1}}{3\gamma a_{0,1}\sigma_{x1} - 3a_{0,1}\sigma_{x1} + \rho - 3\log(\phi)} \\
a_{2,2} =& -\frac{3a_{2,3}\mu_{x0}\log(\phi)}{2\gamma a_{0,1}\sigma_{x1} - 2a_{0,1}\sigma_{x1} + \rho - 2\log(\phi)} + \frac{3(1-\gamma)a_{2,3}a_{0,1}\sigma_{x0}}{2\gamma a_{0,1}\sigma_{x1} - 2a_{0,1}\sigma_{x1} + \rho - 2\log(\phi)} \\
& + \frac{2(1-\gamma)a_{1,2}^2\sigma_{x0}}{2\gamma a_{0,1}\sigma_{x1} - 2a_{0,1}\sigma_{x1} + \rho - 2\log(\phi)} - \frac{\rho a_{0,0}a_{0,1}^2}{2(2\gamma a_{0,1}\sigma_{x1} - 2a_{0,1}\sigma_{x1} + \rho - 2\log(\phi))} \\
& + \frac{\rho a_{1,1}a_{0,1}}{2\gamma a_{0,1}\sigma_{x1} - 2a_{0,1}\sigma_{x1} + \rho - 2\log(\phi)} + \frac{2(1-\gamma)a_{1,1}a_{1,2}\sigma_{x1}}{2\gamma a_{0,1}\sigma_{x1} - 2a_{0,1}\sigma_{x1} + \rho - 2\log(\phi)} \\
& + \frac{\rho a_{0,0}a_{1,2}}{2\gamma a_{0,1}\sigma_{x1} - 2a_{0,1}\sigma_{x1} + \rho - 2\log(\phi)} + \frac{3a_{2,3}\sigma_{x1}}{2\gamma a_{0,1}\sigma_{x1} - 2a_{0,1}\sigma_{x1} + \rho - 2\log(\phi)} \\
a_{2,1} =& -\frac{2a_{2,2}\mu_{x0}\log(\phi)}{\gamma a_{0,1}\sigma_{x1} - a_{0,1}\sigma_{x1} + \rho - \log(\phi)} + \frac{a_{2,2}\sigma_{x1}}{\gamma a_{0,1}\sigma_{x1} - a_{0,1}\sigma_{x1} + \rho - \log(\phi)} \\
& + \frac{2(1-\gamma)a_{1,1}a_{1,2}\sigma_{x0}}{\gamma a_{0,1}\sigma_{x1} - a_{0,1}\sigma_{x1} + \rho - \log(\phi)} + \frac{2(1-\gamma)a_{0,1}a_{2,2}\sigma_{x0}}{\gamma a_{0,1}\sigma_{x1} - a_{0,1}\sigma_{x1} + \rho - \log(\phi)} \\
& + \frac{3a_{2,3}\sigma_{x0}}{\gamma a_{0,1}\sigma_{x1} - a_{0,1}\sigma_{x1} + \rho - \log(\phi)} + \frac{(1-\gamma)a_{1,1}^2\sigma_{x1}}{2(\gamma a_{0,1}\sigma_{x1} - a_{0,1}\sigma_{x1} + \rho - \log(\phi))} \\
& - \frac{\rho a_{0,1}a_{0,0}^2}{2(\gamma a_{0,1}\sigma_{x1} - a_{0,1}\sigma_{x1} + \rho - \log(\phi))} + \frac{\rho a_{1,1}a_{0,0}}{\gamma a_{0,1}\sigma_{x1} - a_{0,1}\sigma_{x1} + \rho - \log(\phi)} \\
& + \frac{\rho a_{0,1}a_{1,0}}{\gamma a_{0,1}\sigma_{x1} - a_{0,1}\sigma_{x1} + \rho - \log(\phi)} \\
a_{2,0} =& -\frac{a_{2,1}\mu_{x0}\log(\phi)}{\rho} + \frac{(1-\gamma)a_{1,1}^2\sigma_{x0}}{2\rho} + \frac{(1-\gamma)a_{0,1}a_{2,1}\sigma_{x0}}{\rho} + \frac{a_{2,2}\sigma_{x0}}{\rho} - \frac{1}{6}a_{0,0}^3 + a_{1,0}a_{0,0}
\end{aligned}
\tag{3.19}
$$

As can be seen above, the expressions become complicated fast. However, it is straightforward to use a computer to derive higher orders of approximation, at least up to the point that it is also too much for the computer.

3.3.3 Convergence

Regarding convergence, the problem constitutes a regular perturbation problem (Bender, Orszag and Orszag 1999). So, the series has a non-vanishing radius of converges around $\epsilon = 0$ for each x_t. This is known rigorously but it is not clear exactly what the radius of convergence is. Based on the definition of K in Equation (3.13) some conclusions can be drawn. Firstly, a finite order of approximation will never work for all x_t. As x_t goes to $\pm\infty$, the highest power of x_t will necessarily blow up, in a way that does not correspond to the approximated function, as the highest power of x_t changes for each order of approximation. Furthermore, something can also be said regarding the convergence of the series. On the one hand, for small x_t, only parameters of the form $a_{n,0}$, $n = 0, 1, 2, 3, \ldots$ matter for the approximation. So, if their growth rate is slower than the decay rate of ϵ^n, then the approximation will converge. On the other hand, if x_t is not very small, then the parameters of the form $a_{n,n+1}$, $n = 0, 1, 2, 3, \ldots$ will matter for convergence, and the approximation converges if the growth rate of these parameters is slower than the decay rate of $(\epsilon \times x)^n$. Thus, the question is whether the growth rate of these parameters can be deduced.

Indeed the parameters follow some patterns, that can already be seen in the expression that I provided in the previous subsection. Firstly, the parameters of the form $a_{n,n+1}$, with $n = 0, 1, 2, \ldots$ are determined recursively based on other parameters of the same form, $a_{n',n'+1}$, with $n = 0, 1, 2, \ldots$ and $n' < n$. Thus, these "diagonal" parameters can be computed independently. In addition, the pattern of products implies that any parameter of the form $a_{n,n+1}$ for each n includes terms containing $a_{0,1}$ raised at most to the power $n+1$. In fact, something similar holds for all parameters of the nth order approximation. In particular, when a parameter $a_{n,m}$ is written in terms of $a_{0,0}$ and $a_{0,1}$, each term in the corresponding expression contains combinations of powers of these two initial parameters, and the sum of the powers is always less or equal to $n + 1$. This can be seen, for instance, in Figure 3.1.[12]

This indicates that as long as the following conditions jointly hold the series will converge:

- ϵ^n decays faster than the growth of $a_{0,1}^{(n+1)}$
- ϵ^n decays faster than the growth of $a_{0,0}^n$
- $(\epsilon x)^n$ decays faster than the growth of $a_{0,1}^{n+1}$

Even though these conditions are mostly reliable, they are not exact, as the series can converge when the conditions are violated, and it can also diverge when the conditions hold. To prove the exact conditions for convergence, a more extensive analysis needs to be made of the combinatorial structure of the problem, so that

[12] Here I set $\gamma = 1$. This facilitates some simplifications. In the general case and as long as the exact growth properties are considered an extra factor would arise, as in each step going from $m + 1$ to 0 there are terms that contain an extra power of $a_{0,1}$ both in the numerator and in the denominator. This would affect the third case that I show below.

	1	x	x^2	x^3	x^4	x^5
1	$a_{0,0}$	$a_{0,1}$	0	0	0	0
ϵ	$0. + 0.5\, a_{0,0}^2 + 0.806542\, a_{0,0}\, a_{0,1} + 0.360086\, a_{0,1}^2$	$0. + 0.193458\, a_{0,0}\, a_{0,1} + 0.0863705\, a_{0,1}^2$	$0. + 0.0535437\, a_{0,1}^2$	0	0	0
ϵ^2	$0. + 0.333333\, a_{0,0}^3 + 0.962574\, a_{0,0}^2\, a_{0,1} + 0.898055\, a_{0,0}\, a_{0,1}^2 + 0.271482\, a_{0,1}^3$	$0. + 0.037426\, a_{0,0}^2\, a_{0,1} + 0.129038\, a_{0,0}\, a_{0,1}^2 + 0.0651179\, a_{0,1}^3$	$0. - 0.0270929\, a_{0,0}\, a_{0,1}^2 - 0.00196799\, a_{0,1}^3$	$0. - 0.00837498\, a_{0,1}^3$	0	0
ϵ^3	$0. + 0.25\, a_{0,0}^4 + 0.99276\, a_{0,0}^3\, a_{0,1} + 1.46067\, a_{0,0}^2\, a_{0,1}^2 + 0.936459\, a_{0,0}\, a_{0,1}^3 + 0.219814\, a_{0,1}^4$	$0. + 0.00724036\, a_{0,0}^3\, a_{0,1} + 0.056934\, a_{0,0}^2\, a_{0,1}^2 + 0.0898408\, a_{0,0}\, a_{0,1}^3 + 0.0371744\, a_{0,1}^4$	$0. - 0.0176065\, a_{0,0}^2\, a_{0,1}^2 - 0.0332772\, a_{0,0}\, a_{0,1}^3 - 0.0129955\, a_{0,1}^4$	$0. - 0.000645249\, a_{0,0}\, a_{0,1}^3 - 0.00244501\, a_{0,1}^4$	$0. + 0.00044994\, a_{0,1}^4$	0
ϵ^4	$0. + 0.2\, a_{0,0}^5 + 0.998599\, a_{0,0}^4\, a_{0,1} + 1.98805\, a_{0,0}^3\, a_{0,1}^2 + 1.96372\, a_{0,0}^2\, a_{0,1}^3 + 0.958123\, a_{0,0}\, a_{0,1}^4 + 0.184172\, a_{0,1}^5$	$0. + 0.00140071\, a_{0,0}^4\, a_{0,1} + 0.0182958\, a_{0,0}^3\, a_{0,1}^2 + 0.0591872\, a_{0,0}^2\, a_{0,1}^3 + 0.0625563\, a_{0,0}\, a_{0,1}^4 + 0.0207276\, a_{0,1}^5$	$0. - 0.00634651\, a_{0,0}^3\, a_{0,1}^2 - 0.0311549\, a_{0,0}^2\, a_{0,1}^3 - 0.0346041\, a_{0,0}\, a_{0,1}^4 - 0.0113813\, a_{0,1}^5$	$0. + 0.00403845\, a_{0,0}^2\, a_{0,1}^3 + 0.00259804\, a_{0,0}\, a_{0,1}^4 + 0.000192267\, a_{0,1}^5$	$0. + 0.00136972\, a_{0,0}\, a_{0,1}^4 + 0.000603844\, a_{0,1}^5$	$0. + 0.000153175\, a_{0,1}^5$

$\{\gamma=1,\ \rho=0.02,\ \mu_{c0}=0.0252,\ \sigma_{c0}=0,\ \sigma_{c1}=0.0004,\ \phi=0.92,\ \sigma_{ct}=\sqrt{\sigma_{c0}+\sigma_{c1}\, x_t},\ \sigma_{x1}=0.0169,\ \mu_{x0}=1,\ \rho cx=\rho\rho_{cx}\ \&,\ \rho_{cx}=-0.5\}$

Figure 3.1: **Series Coefficients – Variation: Time-varying consumption volatility.**
This shows the value of the parameters in terms of $a_{0,0}$ and $a_{0,1}$. The first row and first column show the corresponding power of x_t and ϵ respectively. The nth power of ϵ and mth power of x_t correspond to $a_{n,m}$. It can be seen that the highest power of $a_{0,0}$, $a_{0,1}$ or the higher sum of their powers for the parameters in the nth order approximation is $n+1$. The calibration used is also labeled.

the growth rate of the coefficients can be exactly determined. However, in practice, these conditions are good indications regarding convergence, which can practically be checked by looking at the first partial sums of the series. Figure 3.2 shows these partial sums for the same calibration as in Figure 3.1. The top plot shows the convergence of the series for $K(x_t)$ for different values of x_t according to the approximate conditions expressed previously, the series should converge as long as $|x \times \epsilon| < 1/a_{0,1}$, that is less than 238 approximately. Actually, the series seems to converge for much larger values also, and it starts diverging when $|x \times \epsilon|$ is about 1200. These numbers are huge, as in this calibration the state variable would practically never take values larger than 10, which would mean that consumption volatility is ten times larger compared to the stochastic steady state. The bottom plot shows the convergence of the series for the derivative of $K(x_t)$ for different values of x_t, and it is clear that the convergence of the derivative follows the same pattern. This is reasonable given that in this case convergence is regulated by the terms that have a high power in terms of x_t, and these terms appear in both K and its derivative.

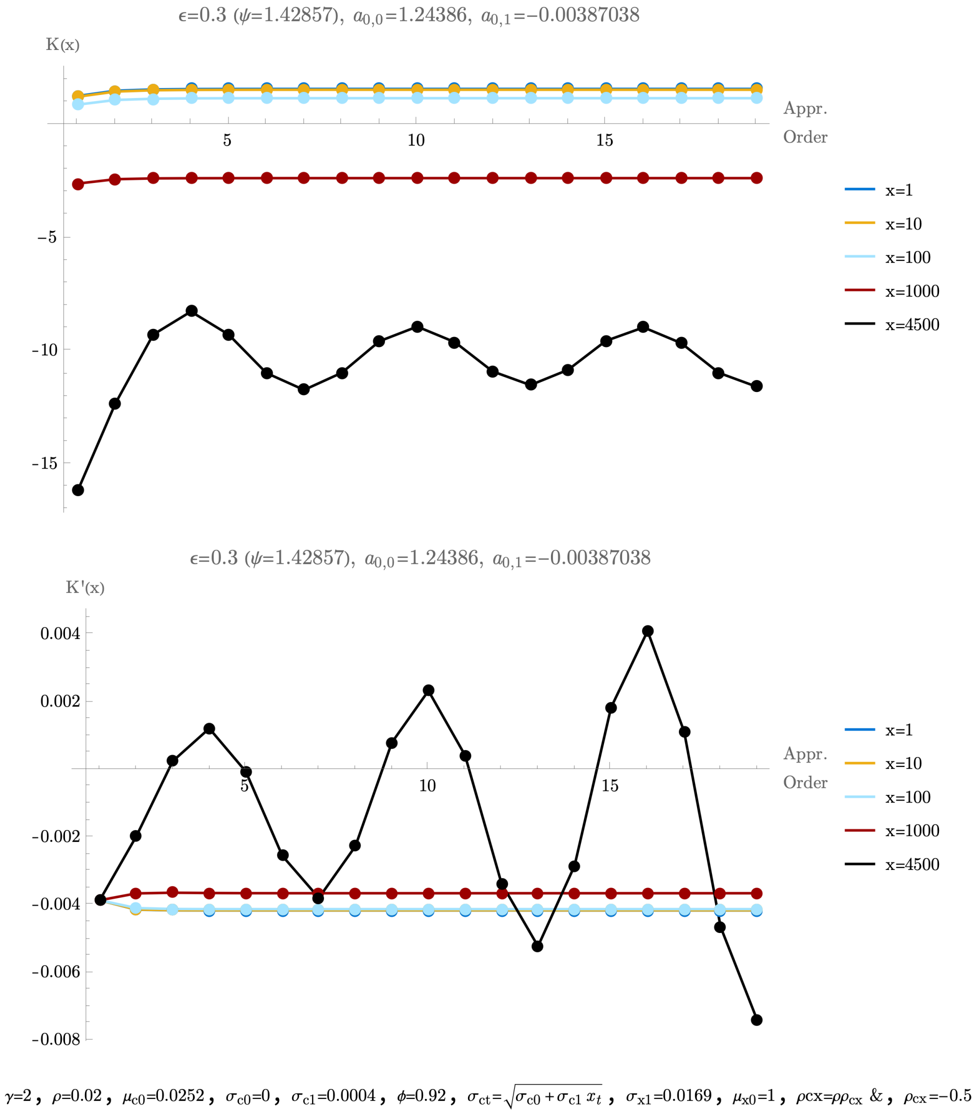


Figure 3.2: **Variation: Time-varying consumption volatility.** This shows the convergence of the problem for different values of x_t.

Figure 3.3 shows convergence for different values of ϵ using the same calibration.[13] According to the approximate conditions, the series should converge, if the ab-

[13] Apart from γ which is now equal to 2.

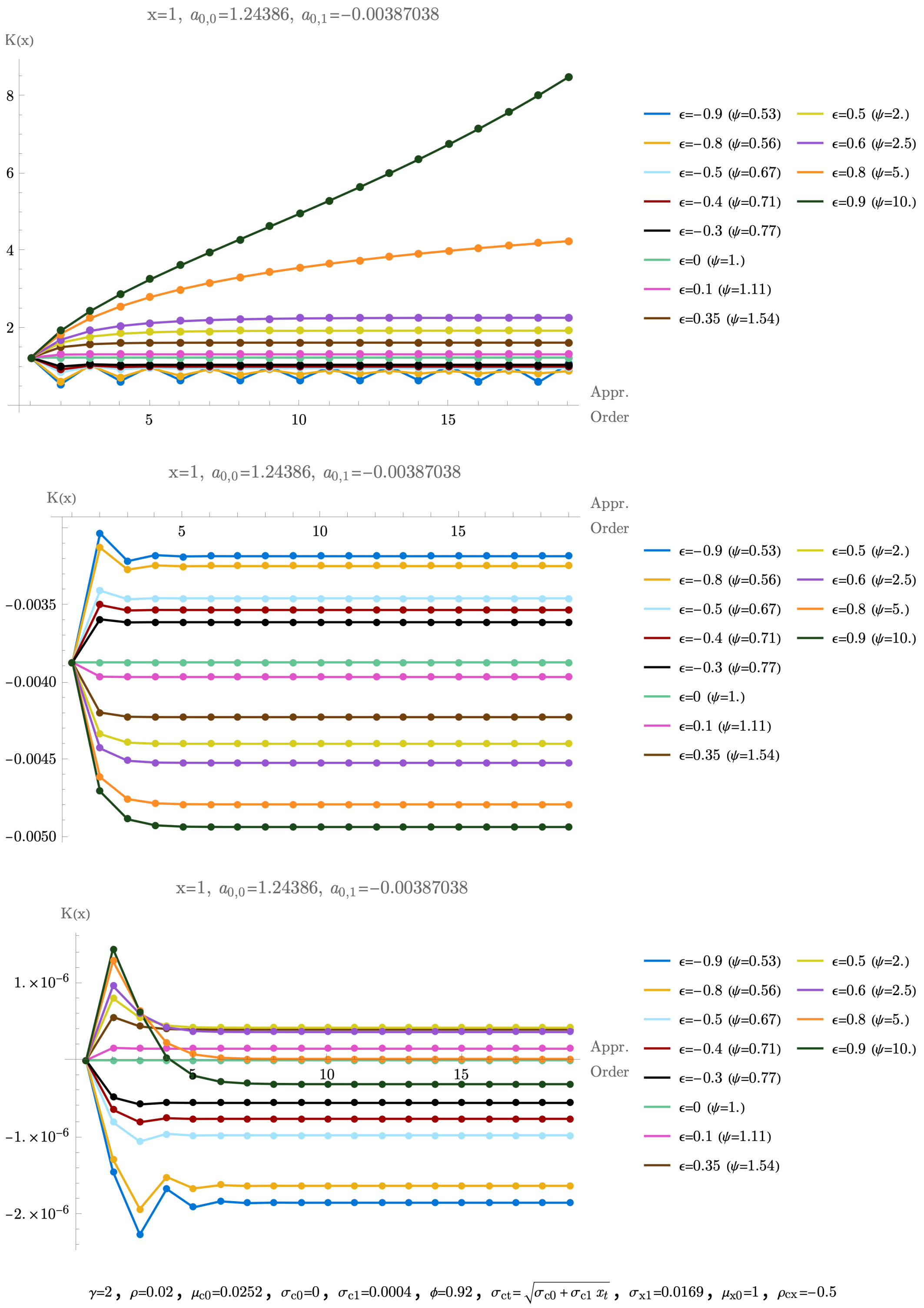


Figure 3.3: **Variation: Time-varying consumption volatility.**
This shows the convergence of the problem for different values of ϵ. The plots correspond to the series of K and its two derivatives.

solute value of ϵ is less than 0.8. Indeed, as can be seen in the figure the series begins to diverge both for $\epsilon = 0.8$ and for $\epsilon = -0.8$. The figure also shows the corresponding values of ψ for each value of ϵ, and in this example, the figure indicates that the zeroth order approximation used by Tsai and Wachter (2018) can indeed be used for a significant range of ψ values. However, using higher orders of approximation leads to a much larger range of ψ values becoming usable. The second and third plots of Figure 3.3 show respectively that the first and second derivative of K converges for all the values of ϵ, that I have chosen, including the values of ϵ for which K itself diverged. This can be explained, because the convergence of the series depends on the terms containing the lowest powers of x_t, as terms with higher powers of x_t will almost certainly be smaller. Hence, the derivatives are much more likely to converge compared to the original function. This means that for some quantities that only rely on the derivatives, the approximation may be used even when K itself diverges.

3.4 Pricing

3.4.1 Process of the stochastic discount factor

As has been shown already the method can provide reliable approximations for a large range of ψ values. The next step is to use the approximation to perform the pricing of securities. This requires the derivation of the process of the SDF. In particular, given the expression for the value function, Ito's Lemma can be implemented to get to the SDE that governs the SDF. The calculation here follows Chen et al. (2009). In particular, the fundamental relationship is:

$$\frac{\mathrm{d}\Lambda_t}{\Lambda_t} = f_V(C, V)\mathrm{d}t + \frac{df_C(C, V)}{f_C(C, V)} \tag{3.20}$$

This can be computed relatively easily. The first term is just the derivative of the flow utility with respect to the value function. The second term can be computed with an application of Ito's lemma on the derivative of flow utility with respect to consumption.[14] The result is the following:

$$
\begin{aligned}
\frac{\mathrm{d}\Lambda_t}{\Lambda_t} =& \Bigg(\frac{\rho(-(1-\gamma\psi)e^{-\frac{(\psi-1)K[x_t]}{\psi}} - \gamma\psi + \psi)}{1-\psi} - \gamma\mu_{ct} + \frac{\gamma^2\sigma_{ct}^2}{2} + \frac{\gamma(\gamma\psi-1)\rho_{cx}\sigma_{xt}\sigma_{ct}K'(x_t)}{\psi} \\
&+ \frac{(\gamma\psi-1)\Big(2\psi(\mu_{x0}-x_t)\log(\phi)K'(x_t) + \sigma_{xt}^2((\gamma\psi-1)K'(x_t)^2 - \psi K''(x_t))\Big)}{2\psi^2} \Bigg)\mathrm{d}t \\
&+ \frac{(1-\gamma\psi)\sigma_{xt}K'(x_t)}{\psi}\mathrm{d}W_{xt} - \gamma\sigma_{ct}W_{ct}
\end{aligned} \tag{3.21}
$$

where ρ_{cx} is the correlation between consumption and the state variable. TSU arises when $\gamma = 1/\psi$. As can be seen by the expression above the dependence of the SDF on $K(x_t)$ and on x_t disappears in this case. The stochastic component that is related to consumption $(-\gamma\sigma_{ct}W_{ct})$ is the same as in TSU. On the contrary, the stochastic component that is related to the state variable $((1-\gamma\psi)\sigma_{xt}K'(x_t)/\psi\mathrm{d}W_{xt})$ does not appear in TSU. So, given that these stochastic components are ultimately responsible for the generation of risk premia, RU introduces an extra mechanism by which premia can be generated. The sign of this mechanism depends on the sign of $(1-\gamma\psi)$, which corresponds to a preference for late or early resolution of uncertainty.

Furthermore, based on the SDF expression the risk-free rate can also be deduced:

$$
\begin{aligned}
r(x_t) =& -E\frac{d\Lambda_t}{\Lambda_t}\frac{1}{\mathrm{d}t} = \\
&\gamma\mu_{ct} - \frac{\gamma^2\sigma_{ct}^2}{2} + \frac{\rho((1-\gamma\psi)e^{-\frac{(\psi-1)K[x_t]}{\psi}} + \gamma\psi - \psi)}{1-\psi} + \frac{\gamma(1-\gamma\psi)\rho_{cx}\sigma_{xt}\sigma_{ct}K'(x_t)}{\psi} \\
&+ \frac{(1-\gamma\psi)\Big(2\psi(\mu_{x0}-x_t)\log(\phi)K'(x_t) + \sigma_{xt}^2((1-\gamma\psi)K'(x_t)^2 - \psi K''(x_t))\Big)}{2\psi^2}
\end{aligned} \tag{3.22}
$$

The short rate is also affected by recursive utility. While the consumption smoothing motive $(\gamma\mu_{ct})$ and the precautionary savings motive $(-\frac{\gamma^2\sigma_{ct}^2}{2})$ is the same as in TSU, the time preference parameter is multiplied by a new factor, and the remaining terms are all new.

[14]It is possible to do this operation after substituting the value function using Equation (3.6) and applying Ito's lemma based on consumption and the state variable as independent variables.

3.4.2 Long-term bonds

The process for the SDF can be inserted in the pricing differential equation as in Cochrane (2009) and Chen et al. (2009):

$$E\Big[\mathrm{d}(\Lambda_t Q)\Big] = 0 \Rightarrow E\left[\frac{\mathrm{d}\Lambda_t}{\Lambda_t} + \frac{\mathrm{d}Q}{Q} + \frac{\mathrm{d}\Lambda_t \mathrm{d}Q}{\Lambda_t Q} = 0\right] = 0 \tag{3.23}$$

Here $Q(m, x_t)$ is the price of the zero-coupon bond with maturity m when the state of the economy is x_t.[15] By Ito's Lemma:

$$\mathrm{d}Q(x,m) = \Big(-\log(\phi)(\mu_{x0} - x_t)Q_x - Q_m + \frac{1}{2}\sigma_x^2 Q_{xx}\Big)\mathrm{d}t + \sigma_{xt} Q_x \mathrm{d}W_{xt} \tag{3.24}$$

This can be directly plugged in Equation (3.23) and the result is:

$$\begin{aligned} 0 =& - Q_m + r(x_t)Q + \Big(-\log(\phi)(\mu_{x0} - x_t) + A(x_t)\Big)Q_x + \frac{1}{2}Q_{xx}\sigma_{xt}^2 \\ A(x_t) =& (\gamma + \epsilon - 1)\sigma_{xt}^2 K^{(1,0)}(x_t, \epsilon) + \gamma\rho_{cx}\sigma_{ct}\sigma_{xt} \end{aligned} \tag{3.25}$$

The subscripts $\cdot_m$ and $\cdot_x$ denote partial derivatives of maturity, m, and of the state variable, x_t, respectively. In the above equation $K(x_t)$ appears in $r(x_t)$, but the coefficients of Q_x and Q_{xx} only contain $K^{(1,0)}(x_t, \epsilon)$. This is noteworthy because Equations (3.24) and (3.22) imply that the expected instantaneous excess return obeys the following relationship:

$$E\left[\frac{\mathrm{d}Q}{Q}\right] - r(x_t)\mathrm{d}t = -E\left[\frac{\mathrm{d}\Lambda_t \mathrm{d}Q}{\Lambda_t Q}\right] = A(x_t)\mathrm{d}t = \Big((\gamma+\epsilon-1)\sigma_{xt}^2 K^{(1,0)}(x_t,\epsilon) + \gamma\rho_{cx}\sigma_{ct}\sigma_{xt}\Big)\mathrm{d}t \tag{3.26}$$

So, the term premium also primarily depends on $K'(x_t)$ and not $K(x_t)$ itself. Thus, the approximation may provide useful information about term premia even when it diverges, given the result in Section 3.3.3, that the derivative of K can converge even when K diverges.[16]

Continuing with the pricing of the long-term bond, according to the Feynman-Kac method Equation (3.25) can be solved by Monte Carlo simulations. In par-

[15] In the formulas I use Q instead of $Q(m, x_t)$ to avoid cluttering.

[16] For brave researchers this could suggest the use of this approximation even when the original series diverges when the item of interest is the risk premium, which is determined by the derivative of $K(x_t)$.

ticular:[17]

$$Q(m, x_t) = E\left[\exp\left\{\int_m^0 r(\hat{x}_{t+s})\mathrm{d}s\right\}\right] = E\left[\exp\left\{-\int_0^m r(\hat{x}_{t+s})\mathrm{d}t\right\}\right] \tag{3.27}$$

where $\hat{x}_0 = x$ and $\hat{x}_t$ follows the modified process:

$$\mathrm{d}\hat{x} = (-\log(\phi)(\mu_{x0} - \hat{x}) + (\gamma + \epsilon - 1)\sigma_x(\hat{x})^2 K'(\hat{x}) + \gamma\rho_{cx}\sigma_c(\hat{x})\sigma_x(\hat{x}))\mathrm{d}t + \sigma_x(\hat{x}_t)\mathrm{d}W_{xt} \tag{3.28}$$

This is a modified process because, while it is similar to the regular state variable of the model, the trend component of the modified process has extra terms coming from the interaction of the SDF with the stochastic components of the state variable process.[18] Based on function Q and Equation (3.24), it is also easy to derive the instantaneous expected return of long-term bonds.

In addition, term premia can also be computed. If, instead of using the modified process, the original state variable is used:

$$H(m, x_t) = \mathrm{E}\left[\exp\left\{\int_m^0 r(x_{t+s})\mathrm{d}s\right\}\right] = \mathrm{E}\left[\exp\left\{-\int_0^m r(x_{t+s})\mathrm{d}t\right\}\right] \tag{3.29}$$

The result is the price of the *risk-neutral bond*, namely a bond priced by a risk-neutral investor with the same consumption process and utility function as in the original model. The difference between the yields corresponding to Q and H is the term premium for the corresponding maturity:

$$TP(m, x_t) = -\frac{\log\left(Q(m, x_t)\right)}{m} - \left(-\frac{\log\left(H(m, x_t)\right)}{m}\right) \tag{3.30}$$

3.4.3 Price-consumption ratio

The price-consumption ratio is a concept similar to the price-dividend ratio. It is a ratio, whose numerator is the price of the consumption perpetuity, a security that continuously pays the consumption for an infinite horizon, and its denominator is the concurrent consumption. Wachter (2006) derived the price-consumption ratio in discrete time for the habit model of Campbell and Cochrane (1999). Here I use the same approach adapted for continuous time. So, I build up the price-consumption ratio from *consumption strips*, securities that pay dividends equal to consumption just once after m periods. These securities have a price

[17] This is the step that can be performed using the package introduced in Melissinos (2024).

[18] In this expression, instead of σ_{ct} and σ_{xt}, I am using $\sigma_c(\hat{x})$ and $\sigma_x(\hat{x})$ in order to make explicit that these quantities can now be functions of the modified process $\hat{x}$.

$P^{str}(m, x_t, C_t)$ at time t. The value of these securities depends on the current value of the state variable and the current value of consumption. To avoid dependence on consumption, I divide these securities by current consumption. This leads to a *consumption strip price-consumption ratio*, $p^{str}(m, x_t) = P^{str}(m, x_t, C_t)/C_t$. Then the integration of the consumption strips leads to the full price-consumption ratio:

$$p(x_t) = \int_0^\infty p^{str}(m, x_t)\mathrm{d}m = \int_0^\infty \frac{P^{str}(m, x_t, C_t)}{C_t}\mathrm{d}m \tag{3.31}$$

Furthermore, one can also define the price-consumption *annuity* ratio. The consumption annuity is similar to the consumption perpetuity, but it only pays coupons for a finite period T. For example:

$$p_T(x_t) = \int_0^T P^{str}(m, x_t)\mathrm{d}m = \int_0^T \frac{P^{str}(m, x_t, C_t)}{C_t}\mathrm{d}m \tag{3.32}$$

If M is large, this quantity likely behaves similarly to the price-consumption ratio, but in practice, this may be easier to compute as it does not require the calculation of the integral for an infinite horizon.

In order to derive $p^{str}(m, x_t)$, I follow an approach similar to Chen, Cosimano and Himonas (2010), who used the pricing equation to calculate the price-consumption ratio directly. Unlike them, I first calculate the p^{str}'s and I then build up the price-consumption ratio. This is arguably more complicated as it involves the solution of a PDE and the computation of an integral, instead of the solution of an ordinary differential equation only. However, my approach does not require the specification of initial conditions and it determines the price-consumption ratio uniquely. Thus, the pricing equation can be re-written:

$$\begin{aligned} E\Big[\mathrm{d}(\Lambda_t P^{str}(m, x_t, C_t))\Big] = 0 &\Rightarrow E\left[\frac{\mathrm{d}\Lambda_t}{\Lambda_t} + \frac{\mathrm{d}P^{str}(m, x_t, C_t)}{P^{str}(m, x_t, C_t)} + \frac{\mathrm{d}\Lambda_t \mathrm{d}P^{str}(m, x_t, C_t)}{\Lambda_t P^{str}(m, x_t, C_t)}\right] = 0 \\ &\Rightarrow E\left[\frac{\mathrm{d}\Lambda_t}{\Lambda_t} + \frac{\mathrm{d}\Big(p^{str}(m, x_t)C_t\Big)}{p^{str}(m, x_t)C_t} + \frac{\mathrm{d}\Lambda_t \mathrm{d}\Big(p^{str}(m, x_t)C\Big)}{\Lambda_t p^{str}(m, x_t)C_t}\right] = 0 \\ &\Rightarrow E\left[\frac{\mathrm{d}\Lambda_t}{\Lambda_t} + \frac{\mathrm{d}p^{str}}{p^{str}} + \frac{\mathrm{d}C_t}{C_t} + \frac{\mathrm{d}\Lambda_t \mathrm{d}p^{str}}{\Lambda_t p^{str}} + \frac{\mathrm{d}\Lambda_t \mathrm{d}C_t}{\Lambda_t C_t} + \frac{\mathrm{d}p^{str}\mathrm{d}C_t}{p^{str}C_t}\right] = 0 \end{aligned} \tag{3.33}$$

In the final line, I do not show the dependence of p^{str} for simplicity. Similar to above, by Ito's Lemma:

$$\mathrm{d}p^{str} = \Big(- \log(\phi)(\mu_{x0} - x_t)p_x^{str} - p_m^{str} + \frac{1}{2}\sigma_x^2 p_x^{str}\Big)\mathrm{d}t + \sigma_{xt} p_x^{str} \mathrm{d}W_{xt} \tag{3.34}$$

So the processes for the SDF, for the consumption strip and for consumption can all be substituted in Equation (3.33) above and this will again generate a PDE that can be solved, by computing the Feynman-Kac formula through Monte Carlo simulations. The pricing equation is:

$$\begin{aligned}
0 = & \underbrace{-r(x_t)}_{\mathrm{d}\Lambda_t/\Lambda_t} + \underbrace{\Big(- \log(\phi)(\mu_{x0} - x_t)\frac{p_x^{str}}{p^{str}} - \frac{p_m^{str}}{p^{str}} + \frac{1}{2}\frac{p_{xx}^{str}}{p^{str}}\sigma_{xt}^2\Big)}_{\mathrm{d}p^{str}/q} + \underbrace{\mu_{ct}}_{\mathrm{d}C_t/C_t} \\
& + \underbrace{\frac{(1-\gamma\psi)\rho_{cx}\sigma_{xt}\sigma_{ct}K'(x_t)}{\psi} - \gamma\sigma_{ct}^2}_{\mathrm{d}\Lambda_t\mathrm{d}C_t/(\Lambda_t C_t)} + B(x_t)\frac{p_x^{str}}{p^{str}} \\
B(x_t) = & \underbrace{\frac{(1-\gamma\psi)\sigma_{xt}^2 K'(x_t)}{\psi} - \gamma\rho_{cx}\sigma_{xt}\sigma_{ct}}_{\mathrm{d}\Lambda_t\mathrm{d}p^{str}/(\Lambda_t p^{str})} + \underbrace{\rho_{cx}\sigma_{xt}\sigma_{ct}}_{\mathrm{d}p^{str}\mathrm{d}C_t/(p^{str}C_t)}
\end{aligned} \tag{3.35}$$

The brackets show where the expressions in the equations come from. The equation can be rewritten as:

$$\begin{aligned}
0 = & -\left(r(x_t) - \mu_{ct} - \frac{(1-\gamma\psi)\rho_{cx}\sigma_{xt}\sigma_{ct}K'(x_t)}{\psi} + \gamma\sigma_{ct}^2\right)p^{str} - p_m^{str} + \frac{\sigma_{xt}^2}{2}p_{xx}^{str} \\
& + \Big(- \log(\phi)(\mu_{x0} - x_t) + \frac{(1-\gamma\psi)\sigma_{xt}^2 K'(x_t)}{\psi} - \gamma\rho_{cx}\sigma_{xt}\sigma_{ct} + \rho_{cx}\sigma_{xt}\sigma_{ct}\Big)p_x^{str}
\end{aligned} \tag{3.36}$$

The corresponding Feynman-Kac formula is:

$$p^{str}(m, x_t) = E\left[\exp\left\{\int_m^0 \tilde{r}(\tilde{x}_{t+s})\mathrm{d}s\right\}\right] = \mathrm{E}\left[\exp\left\{-\int_0^m \tilde{r}(\tilde{x}_{t+s})\mathrm{d}t\right\}\right] \tag{3.37}$$

where

$$\tilde{r}(\tilde{x}_t) = \tilde{r}(\tilde{x}_t) - \mu_c(\tilde{x}_t) - \frac{(1-\gamma\psi)\rho_{cx}\sigma_x(\tilde{x}_t)\sigma_c(\tilde{x}_t)K'(\tilde{x}_t)}{\psi} + \gamma\sigma_c(\tilde{x}_t)^2 \tag{3.38}$$

$\tilde{x}_0 = x_0$ and $\tilde{x}_t$ follows another modified process:[19]

$$\mathrm{d}\tilde{x}_t = (-\log(\phi)(\mu_{x0}-\tilde{x}_t)+\frac{(1-\gamma\psi)\sigma_x(\tilde{x}_t)^2K'(x_t)}{\psi}+(1-\gamma)\rho_{cx}\sigma_x(\tilde{x}_t)\sigma_c(\tilde{x}_t))\mathrm{d}t+\sigma_x(\tilde{x}_t)\mathrm{d}W_{xt} \tag{3.39}$$

Given the price-consumption ratio as a function of the state variable and the given stochastic process of the price-consumption ratio that can be written as follows:

$$\mathrm{d}p = \Big(-\log(\phi)(\mu_{x0}-x_t)p_x + \frac{1}{2}\sigma_x^2 p_{xx}\Big)\mathrm{d}t + \sigma_{xt}p_x\mathrm{d}W_{xt} \tag{3.40}$$

The return of the consumption perpetuity can be derived:[20]

$$\begin{aligned}
\frac{\mathrm{d}P}{P} + \frac{C_t}{P}\mathrm{d}t =& \frac{\mathrm{d}(C_tp^{str})}{C_tp^{str}} + \frac{1}{p^{str}} = \frac{\mathrm{d}C_t}{C_t} + \frac{\mathrm{d}p}{p} + \frac{\mathrm{d}C_t\mathrm{d}p}{C_tp} + \frac{1}{p}\mathrm{d}t \\
=& \mu_{ct}\mathrm{d}t + \sigma_{ct}W_{ct} - \log(\phi)(\mu_{x0}-x_t)\frac{p_x}{p}\mathrm{d}t + \frac{\sigma_{xt}^2}{2}\frac{p_{xx}}{p}\mathrm{d}t \\
&+ \sigma_{xt}\frac{p_x}{p}\mathrm{d}W_{xt} + \rho_{cx}\sigma_{ct}\sigma_{xt}\frac{p_x}{p}\mathrm{d}t + \frac{1}{p}\mathrm{d}t \\
=& \Big(\mu_{ct} - \log(\phi)(\mu_{x0}-x_t)\frac{p_x}{p} + \frac{\sigma_{xt}^2}{2}\frac{p_{xx}}{p} + \rho_{cx}\sigma_{ct}\sigma_{xt}\frac{p_x}{p} + \frac{1}{p}\Big)\mathrm{d}t + \sigma_{ct}W_{ct} + \sigma_{xt}\frac{p_x}{p}\mathrm{d}W_{xt}
\end{aligned} \tag{3.41}$$

Finally, the expected return is:

$$E\Big[\frac{\mathrm{d}P}{P}\Big] + \frac{C_t}{P}\mathrm{d}t = \Big(\mu_{ct} - \log(\phi)(\mu_{x0}-x_t)\frac{p_x}{p} + \frac{\sigma_{xt}^2}{2}\frac{p_{xx}}{p} + \rho_{cx}\sigma_{ct}\sigma_{xt}\frac{p_x}{p} + \frac{1}{p}\Big)\mathrm{d}t \tag{3.42}$$

3.5 Applications

3.5.1 Time-varying consumption drift

Given the setup introduced in the previous section, real interest rates and the price-consumption ratio can be determined. Here, I show the results for the case when consumption drift is time-varying. This is the result of setting $\mu_{c1} = 1$, which means that the consumption drift is proportional to the state variable. In this variation consumption volatility is constant, $\sigma_{c1} = 0$, and the process is ho-

[19] Similar to above, in this expression, instead of σ_{ct} and σ_{xt}, I am using $\sigma_c(\tilde{x}_t)$ and $\sigma_x(\tilde{x}_t)$ to make explicit that these quantities can now be functions of the modified process $\tilde{x}_t$.

[20] This calculation only applies to the consumption perpetuity. For the case of the consumption annuity, the calculation would require an extra component that accounts for the fact that the annuity at time t has infinitesimally lower duration compared to the annuity at time $t + \mathrm{d}t$. Numerically, I only calculate annuities. So, I only apply this formula for long-lived annuities, for which the security's price should not change significantly with duration.

moskedastic, $\sigma_{x1} = 0$. Figure 3.4 shows the results, while comparing the eighth order approximation, using the method introduced in this paper, to the zeroth order approximation, which is equivalent to the method of Tsai and Wachter (2018). In addition, here $\epsilon = 0.1$ ($\psi = 1.11$) which is close to the analytic solution for $\epsilon = 0$ ($\psi = 1$). The results verify that the basic approximation can be accurate for $\psi \neq 1$. The first row shows the instantaneous rate and the ten-year yield as a function of consumption drift. The results for the two approximations are very similar. The second row shows the inverse price-consumption ratio as a function of the consumption drift. The inverse price-consumption ratio is equivalent to the dividend yield of the security. As I have already mentioned, I calculate the price-consumption ratio numerically by integrating the zero coupon price-consumption ratios. However, I cannot numerically integrate to infinity. So, I put the cutoff at 200 years. This means that technically I am calculating the price-consumption ratio for the 200-year consumption annuity. The second plot of the second row shows the value of the inverse price-consumption ratio for different cutoff points and it can be seen that at 200 hundred years it is relatively close to being converged. Similar to the above, the price-consumption ratio for the two approximations is quite close, even though in this case the price-consumption ratio is not very sensitive to the value of consumption drift. So, the difference appears larger in the figure. Finally, in the third row, I also show the instantaneous expected return of the consumption perpetuity, which is very similar to the instantaneous short-term rate in this variation.

In Figure 3.5, I show the results for $\epsilon = 0.7$, which is not so close to the analytic solution of $\epsilon = 0$, and as can be seen in Figure 3.9, the value function varies significantly for different orders of approximation. This example illustrates the value of my method, as it shows that for some interesting values for the IES ($\psi = 3.3$ in this case, but in other examples it can also be lower), the value function deviates significantly between the different orders of approximation. This has consequences for the implied value of the short-term rate, the yield and the price-consumption ratio. The short-term rate and the ten-year yield appear linear as functions of the consumption drift for both approximations, but the slope is different and at the stochastic steady state there is a difference of roughly 1% for the short-term rate and a bit lower for the ten-year yield. The difference between the two approximations is also higher for the price-consumption ratio, which is now also more sensitive to the consumption drift.

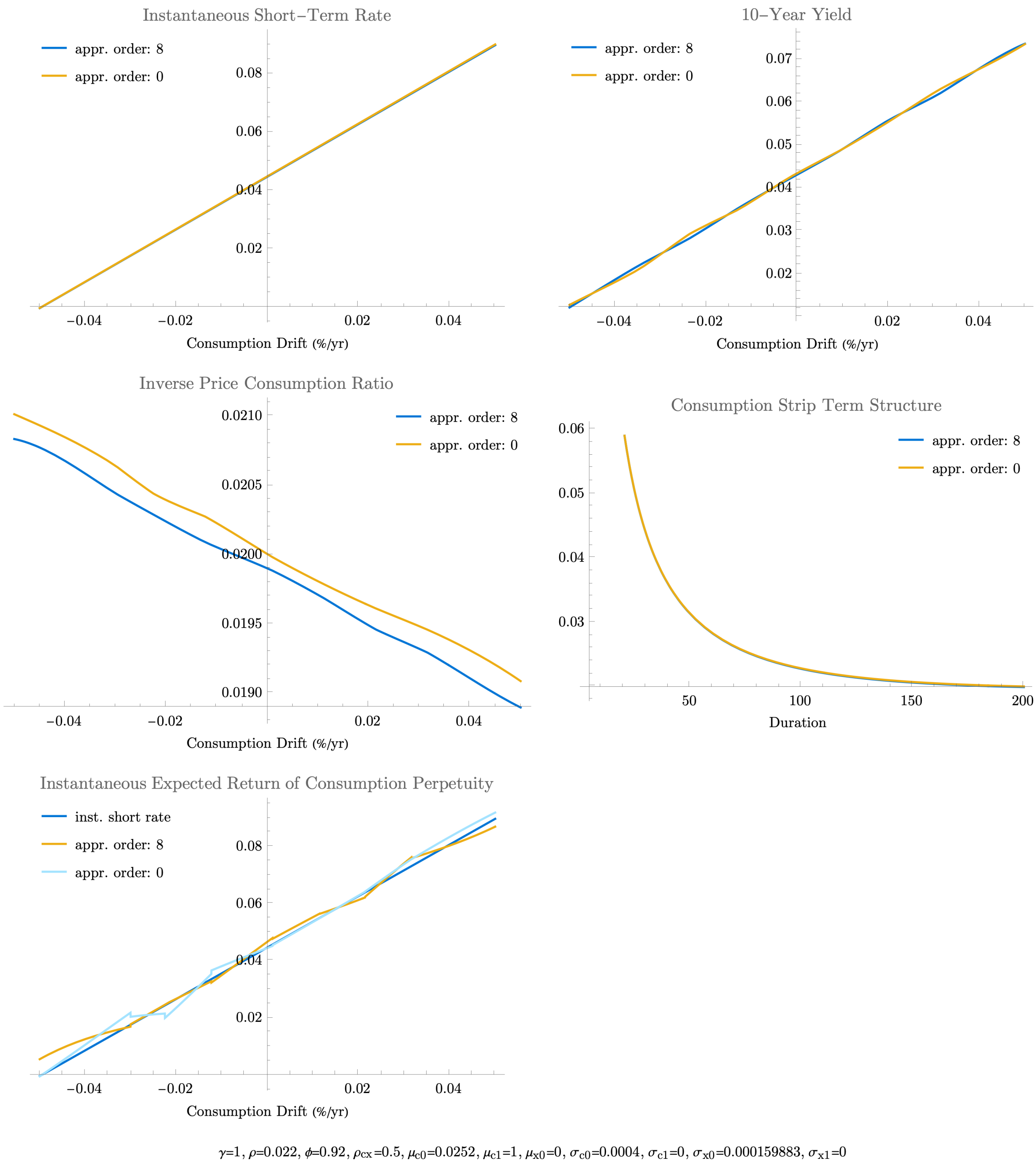


Figure 3.4: **Time-varying consumption drift** $-\ \epsilon = 0.1$
The plots show various quantities of interest for the two different approximation orders.

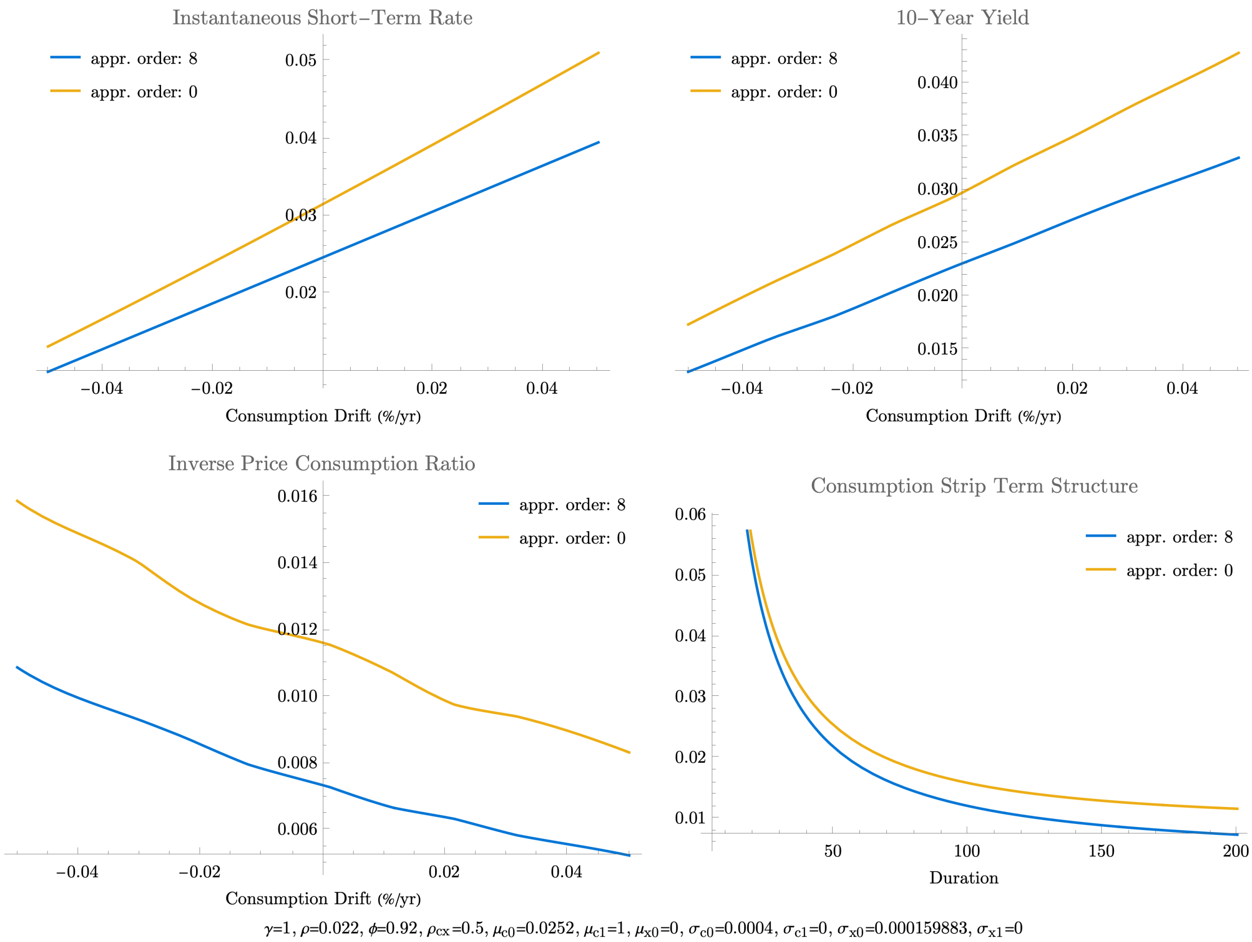


Figure 3.5: **Time-varying consumption drift** $- \epsilon = 0.7$
The first row shows the .

3.5.2 Time-varying consumption volatility

As a second application, I introduce the case when consumption volatility is time-varying. This is the result of setting $\sqrt{\sigma_{c1}} \neq 0$, while consumption drift is constant because $\mu_{x1} = 0$. The state variable is also heteroskedastic, and it is guaranteed to be non-negative with $\sigma_{x0} = 0$ and $\sqrt{\sigma_{x1}} \neq 0$. Figure 3.6 shows the case where $\epsilon = 0.1$, and all rates are not very sensitive to consumption volatility. Nevertheless, the figure demonstrates that the zeroth order approximation is roughly within 15 basis points compared to the higher approximation. Depending on the application, this difference could be considered negligible. In addition, while there is a difference in the level of the rates, the difference in the slope of the rates with respect to consumption volatility is not noticeable.

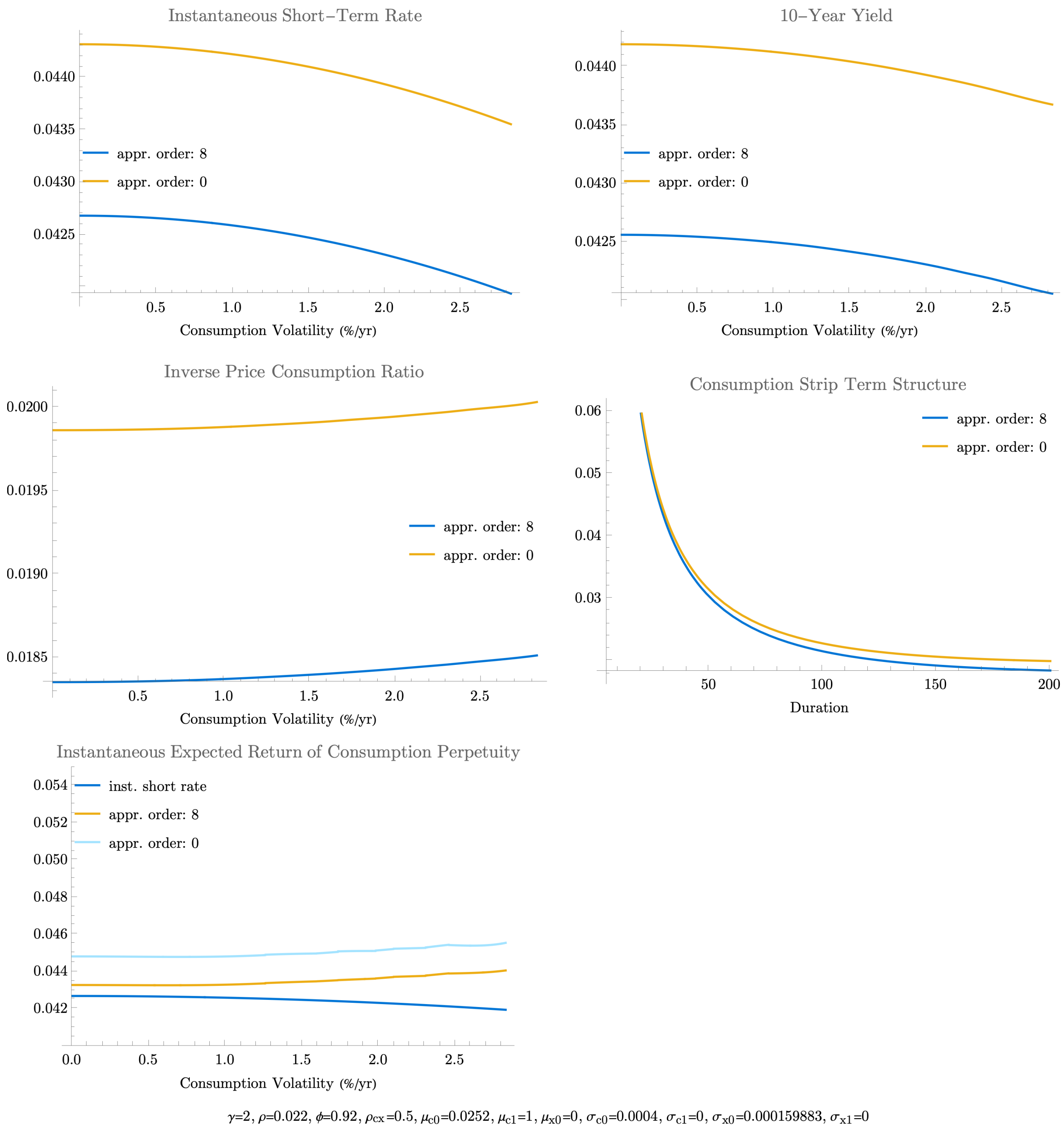
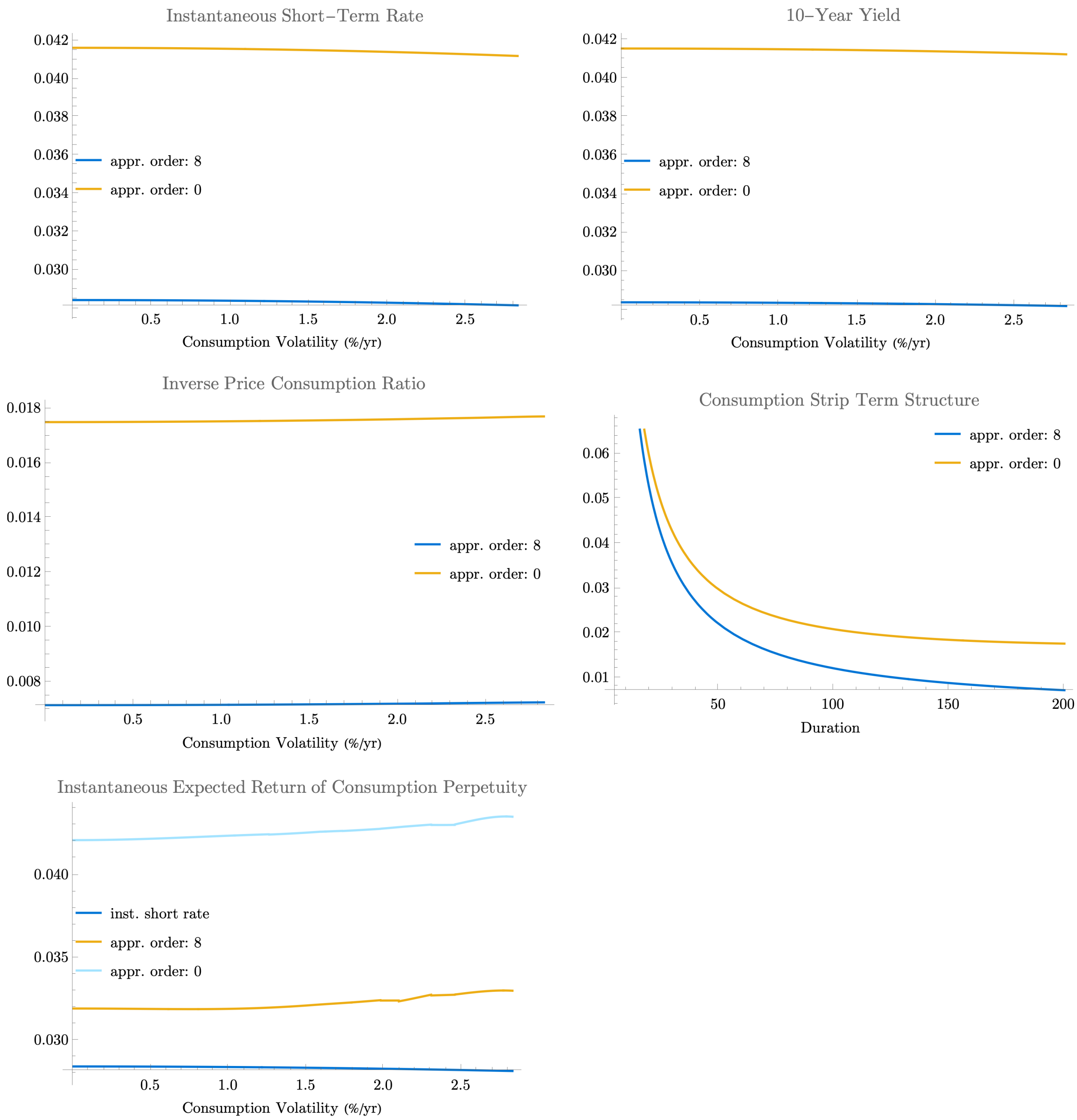


Figure 3.6: **Time-varying consumption volatility** $-\ \epsilon = 0.1$
The first row shows the .

Figure 3.7 shows the case where $\epsilon = 0.7$. Now, the difference in the rates is certainly not negligible, as it ranges around 100 basis points. Nevertheless, the slopes still appear the same. This is the result of the derivatives of K being very well approximated by the zeroth order approximation (Figure 3.9).



γ=2, ρ=0.022, ϕ=0.92, $\rho_{\rm cx}$=0.5, $\mu_{\rm c0}$=0.0252, $\mu_{\rm c1}$=1, $\mu_{\rm x0}$=0, $\sigma_{\rm c0}$=0.0004, $\sigma_{\rm c1}$=0, $\sigma_{\rm x0}$=0.000159883, $\sigma_{\rm x1}$=0

Figure 3.7: **Time-varying consumption volatility** $- \ \epsilon = 0.7$
The first row shows the.

3.6 Extension to the Consumption-Investment Problem

In the previous sections, I have shown how to derive the value function when the consumption process is exogenous. However, the same method can also be used for the case when the price of a risky asset is exogenous and the agent chooses the amount invested and consumed. Kraft, Seiferling and Seifried (2017) solved this problem and provided closed-form expressions up to the solution of a semi-linear PDE, which in their paper was solved numerically. It turns out that this equation is similar to Equation (3.7), and it can also be solved by the perturbation approach introduced in this paper.[21]

Following Kraft, Seiferling and Seifried (2017) the problem is as follows. S_t is the price of the risky asset which evolves according to:

$$\mathrm{d}S_t = S_t\left[\left(r(Y_t) + \lambda(Y_t)\right)\mathrm{d}t + \sigma(Y_t)\mathrm{d}W_t\right] \tag{3.43}$$

where $r(y)$ is the short-term rate, $\lambda(Y_t)$ is the equity premium, $\sigma(Y_t)$ is the volatility of the risky asset and Y_t is the state variable, which in turns evolves according to:

$$\mathrm{d}Y_t = \tilde{\alpha}(Y_t)\mathrm{d}t + \beta(Y_t)\left(\rho\mathrm{d}W_t + \sqrt{1-\rho^2}\mathrm{d}\bar{W}_t\right) \tag{3.44}$$

where $\beta(Y_t)$ is the volatility of the state variable, $\bar{W}$ is the Wiener process that is independent compared to the Wiener process of the risky asset, and ρ expresses the correlation between the state variable and the price of the risky asset. Based on the processes above, the wealth of the agent evolves according to:

$$\mathrm{d}X_t^{\pi,c} = X_t^{\pi,c}\left(\left(r(Y_t) + \pi_t\lambda(Y_t)\right)\mathrm{d}t + \pi_t\sigma(Y_t)\mathrm{d}W_t\right) - C_t\mathrm{d}t \tag{3.45}$$

where π is the share invested in the risky asset. In this setup, Kraft, Seiferling and Seifried (2017) showed that the optimal consumption and investment policy corresponds to:

$$\pi_t(Y_t) = \frac{\lambda(Y_t)}{\gamma\sigma(Y_t)^2} + \frac{k}{\gamma}\frac{\beta(Y_t)\rho}{\sigma(Y_t)}\frac{h_y(Y_t)}{h(Y_t)}, \quad C_t(Y_t) = \delta^{\psi}h(Y_t)^{\frac{-k(1-1/\psi)\psi}{1-\gamma}} \tag{3.46}$$

where k is a simplifying parameter, and h is the solution to the following ordinary

[21] I solve the problem here after adapting it to an infinite horizon problem, which fits my setup.

differential equation:

$$0 = -\tilde{r}(Y_t)h + \tilde{\alpha}(Y_t)h_y + \frac{1}{2}\beta(Y_t)^2 h_{yy} + \delta^{\psi} h^{1-\frac{-k(1-1/\psi)\psi}{1-\gamma}} \tag{3.47}$$

As shown in Appendix 3.D the equation can be rewritten as:

$$0 = -\tilde{R}(Y_t)L + (1 + \tilde{\alpha}(Y_t))\, L_y + \frac{1}{2}\beta(Y_t)^2(L_y^2 + L_{yy}) + \frac{(1-\gamma)\delta\left(e^{\frac{k(1-\psi)L(Y_t)}{1-\gamma}}\delta^{\psi-1} - \psi\right)}{1-\psi} \tag{3.48}$$

where $\tilde{R}(Y_t)$ is a simple transformation of $\tilde{r}(Y_t)$, and I have made the substitution, $L(Y_t) = \log(h(Y_t))$. L is similar to K in the rest of the paper. If $\tilde{R}$, $\tilde{\alpha}$, and β^2 are linear in Y_t, then Equation (3.48) can be solved with the perturbation approach, after ψ is replaced by $1/(1-\epsilon)$. The solution follows a functional form similar to Equation (3.13):

$$\begin{aligned}
L(Y_t, \epsilon) = &\sum_{n=0}^{\infty} \epsilon^n \Big(\sum_{m=0}^{n+1} b_{n,m} Y_t^m\Big) \\
=&(b_{0,0} + b_{0,1}Y_t) \\
&+ \epsilon(b_{1,0} + b_{1,1}Y_t + b_{1,2}Y_t^2) \\
&+ \epsilon^2(b_{2,0} + b_{2,1}Y_t + b_{2,2}Y_t^2 + b_{2,3}Y_t^3) \\
&+ \epsilon^3(b_{3,0} + b_{3,1}Y_t + b_{3,2}Y_t^2 + b_{3,3}Y_t^3 + b_{3,4}Y_t^4) \\
&\quad \ldots \\
=&L_0(Y_t) + L_1(Y_t)\epsilon + L_2(Y_t)\epsilon^2 + \ldots
\end{aligned} \tag{3.49}$$

The coefficients of this expression can be derived in the same way as was described in the previous sections. After transforming this solution back to h, optimal consumption and investment can be derived and expressed as an expansion in terms of the IES according to the formulas in Equation (3.46).

If $\tilde{r}$, $\tilde{\alpha}$, and β^2 have the following form:

$$\begin{aligned}
\tilde{r}(Y_t) =&\tilde{R}_0 + \tilde{R}_1 Y_t \\
\tilde{\alpha}(Y_t) =&\tilde{\alpha}_0 + \tilde{\alpha}_1 Y_t \\
\beta^2(Y_t) =&\beta_0 + \beta_1 Y_t
\end{aligned} \tag{3.50}$$

The first coefficients of the solution that correspond to the case $\epsilon = 0$ ($\psi = 1$) are:

$$b_{0,1} = \begin{cases} \frac{\tilde{R}_2}{k\delta+\tilde{\alpha}_2} & \text{if } b_1 = 0 \\ \frac{-k\delta-\tilde{\alpha}_2-\sqrt{2\beta_2\tilde{R}_2+(k\delta+\tilde{\alpha}_2)^2}}{\beta_2} \text{ or } \frac{-k\delta-\tilde{\alpha}_2+\sqrt{2\beta_2\tilde{R}_2+(k\delta+\tilde{\alpha}_2)^2}}{\beta_2} & \text{otherwise} \end{cases}$$
$$b_{0,0} = \frac{(1-\gamma)\delta(\log(\delta)-1)+\tilde{R}_1-(1+\tilde{\alpha}_1)b_{0,1}+\beta_1 b_{0,1}^2}{k\delta} \tag{3.51}$$

the remaining coefficients can be found by solving linear equations.

As was mentioned in the introduction, Chacko and Viceira (2005) and Leisen (2016) also used perturbation methods to solve essentially the same problem. The former paper performs the perturbation with respect to the consumption-wealth ratio. The latter paper performs a perturbation with respect to several variables including the outcome of flow utility.[22] However, in none of these papers is the perturbation performed with respect to the IES parameter itself.

3.7 Conclusion

In conclusion, I have introduced a new method, based on perturbation theory, to express the value function when the agent exhibits recursive utility. The value function is expressed as a series in terms of ϵ and it constitutes a global solution in terms of the state variable. The value of ϵ is determined by the value of ψ, which represents the IES in the problem. The first term in the series (which multiplies ϵ^0) gives the solution for $\psi = 1$. Each further order of approximation only requires the solution of linear equations. Computing the first fifteen orders of approximation is relatively easy, but higher orders are typically computationally demanding as the number of coefficients increases by one for each order of approximation and the equations become increasingly complicated.

The method is useful for a wide range of calibrations. Tsai and Wachter (2018) only used the zeroth order approximation. I have shown that this can produce accurate results for a relatively low absolute value of ϵ, but the approximation can deteriorate as the absolute value of ϵ increases. Higher-order approximations using my method can solve this issue, and this applies both for models with time-varying consumption drift and time-varying consumption volatility. The paper can also be extended to include multiple state variables and Poisson jump components in the consumption process. I have used the perturbation series to derive both the price of long-term bonds and the price-consumption ratio of consumption strips,

[22] The aggregator function is multiplied by a perturbation term.

consumption annuities and the consumption perpetuity. Furthermore, I have also derived the expected instantaneous return of the consumption perpetuity. Apart from being easy to implement, my method allows to easily check whether the results are accurate and how many orders of approximation are necessary for an acceptable solution. Despite not having derived exact convergence conditions, I have sketched the behaviour of the series for different orders of approximation.

Finally, it should be noted that the perturbation series uniquely determines the value function, even if it is a divergent asymptotic series for some combinations of parameters and values of x_t. This means that further work following this approach, using more sophisticated mathematical analysis, could provide an expression of the solution that is easily computable and uniformly convergent, possibly in terms of special functions.[23]

[23] Applying a Padé approximation to the problem did not yield converging results in the regions that were diverging under the regular approximation.

Appendix

3.A Proof of result in Equation (3.7)

The expression is derived from the Hamilton-Jacobi-Bellman equation, $\mathcal{D}V_t + F(C_t, V_t) = 0$, after the relevant quantities have been substituted. By applying Ito's Lemma to V_t, which is a function of C_t and x_t, the result is:

$$\frac{\mathcal{D}V_t}{V_t} = -\frac{1}{2}(\gamma-1)(-\gamma\sigma_{ct}^2+2\mu_{ct}+\sigma_{xt}^2K''(x_t)-\gamma\sigma_{xt}^2K'(x_t)^2+2\mu_{xt}K'(x_t)+\sigma_{xt}^2K'(x_t)^2) \tag{3.52}$$

Here, I can substitute the guessed expression for the value function, $V_t = \frac{C_t^{1-\gamma}e^{(1-\gamma)K(x_t)}}{1-\gamma}$, which I will verify later, in the previous expression and in the expression for flow utility:

$$\frac{\mathcal{D}V_t}{V_t} = (1-\gamma)(\mu_{ct} + \mu_{xt}K'(x_t) - \frac{\gamma\sigma_{ct}^2}{2} + \frac{(1-\gamma)\sigma_{xt}^2}{2}K'(x_t)^2 + \frac{\sigma_{xt}^2}{2}K''(x_t)) \tag{3.53}$$

$$F(C_t, V_t) = \frac{(1-\gamma)\rho V_t((C((1-\gamma)V_t)^{-\frac{1}{1-\gamma}})^{1-\frac{1}{\psi}} - 1)}{1-\frac{1}{\psi}} = (1-\gamma)\rho\frac{\psi(1-e^{(\frac{1}{\psi}-1)K[x_t]})}{1-\psi} \tag{3.54}$$

After plugging these two expressions in the JHB equation, the result is:

$$\rho\frac{\psi(1-e^{(\frac{1}{\psi}-1)K[x_t]})}{1-\psi} + \mu_{ct} + \mu_{xt}K'(x_t) - \frac{\gamma\sigma_{ct}^2}{2} + \frac{(1-\gamma)\sigma_{xt}^2}{2}K'(x_t)^2 + \frac{\sigma_{xt}^2}{2}K''(x_t) = 0 \tag{3.55}$$

This is Equation (3.7) in the main text. By the fact that this is the result of the HJB equation, assuming that the solution exists, the guess is verified.

3.B Deriving the stochastic discount factor

As mentioned in the main paper the SDE of the SDF can be derived based on the following expression:

$$\frac{\mathrm{d}\Lambda_t}{\Lambda_t} = F_V(C_t, V_t)\mathrm{d}t + \frac{\mathrm{d}F_C(C_t, V_t)}{F_C(C_t, V_t)} \tag{3.56}$$

thus, flow utility is a central component of the derivation:

$$F(C_t, V_t) = \frac{\rho}{1 - 1/\psi}\Big((1-\gamma)V_t\Big)\left(\left(C_t((1-\gamma)V_t)^{-\frac{1}{1-\gamma}}\right)^{1-1/\psi} - 1\right) \tag{3.57}$$

The partial derivative of F with respect to V_t is:

$$F_V(C_t, V_t) = \frac{\rho\left((\gamma-1)\psi + (1-\gamma\psi)\left(C_t(V_t - \gamma V_t)^{\frac{1}{\gamma-1}}\right)^{\frac{\psi-1}{\psi}}\right)}{\psi - 1} \tag{3.58}$$

The partial derivative of F with respect to C_t is:

$$F_C(C_t, V_t) = -\frac{(\gamma-1)\rho V_t\left(C_t(V_t - \gamma V_t)^{\frac{1}{\gamma-1}}\right)^{\frac{\psi-1}{\psi}}}{C_t} \tag{3.59}$$

As I implement Ito's Lemma directly using c_t and x_t as independent variables, I make the following replacements in the expressions above:

$$c_t = \log(C_t), \quad V_t = \frac{C_t^{1-\gamma}}{1-\gamma}e^{(1-\gamma)K(x_t)} \Rightarrow K(x_t) = \frac{\log\left(-\frac{C_t^{1-\gamma}}{(\gamma-1)V_t}\right)}{\gamma - 1} \tag{3.60}$$

And after simplification, they become:

$$F_V(C_t, V_t) \to G_1(c_t, x_t) = \frac{\rho(-(1-\gamma\psi)e^{-\frac{(\psi-1)K[x_t]}{\psi}} - \gamma\psi + \psi)}{1-\psi} \tag{3.61}$$

$$F_C(C_t, V_t) \to G_2(c_t, x_t) = \rho e^{(\frac{1}{\psi} - \gamma)K(x_t) - c_t\gamma} \tag{3.62}$$

And I implement Ito's Lemma on G_2. The partial derivatives are:

$$
\begin{aligned}
\frac{\partial G_2(c_t,x_t)}{\partial c_t} =& \gamma\rho\left(-e^{\left(\frac{1}{\psi}-\gamma\right)K[x_t]-\gamma c_t}\right) = -\gamma G_2(c_t,x_t) \\
\frac{\partial h(c_t,x_t)}{\partial x_t} =& \rho\left(\frac{1}{\psi}-\gamma\right)K'(x_t)\,e^{\left(\frac{1}{\psi}-\gamma\right)K[x_t]-\gamma c_t} = \left(\frac{1}{\psi}-\gamma\right)K'(x_t)\,G_2(c_t,x_t) \\
\frac{\partial^2 G_2(c_t,x_t)}{\partial c_t^2} =& \gamma^2\rho e^{\left(\frac{1}{\psi}-\gamma\right)K[x_t]-\gamma c_t} = \gamma^2 h(c_t,x_t) \\
\frac{\partial^2 G_2(c_t,x_t)}{\partial x_t^2} =& \frac{\rho(\gamma\psi-1)\left((\gamma\psi-1)K'(x_t)^2-\psi K''(x_t)\right)e^{\left(\frac{1}{\psi}-\gamma\right)K[x_t]-\gamma c_t}}{\psi^2} \\
=& \frac{(\gamma\psi-1)\left((\gamma\psi-1)K'(x_t)^2-\psi K''(x_t)\right)}{\psi^2}G_2(c_t,x_t) \\
\frac{\partial G_2(c_t,x_t)}{\partial c_t \partial x_t} =& \frac{\gamma\rho(\gamma\psi-1)K'(x_t)\,e^{\left(\frac{1}{\psi}-\gamma\right)K[x_t]-\gamma c_t}}{\psi} = \frac{\gamma(\gamma\psi-1)K'(x_t)\,G_2(c_t,x_t)}{\psi}
\end{aligned}
\tag{3.63}
$$

The expressions above should be plugged into the expression:

$$
\begin{aligned}
\frac{dF_C}{F_C} =& \left(\frac{\partial G_2(c_t,x_t)}{\partial c_t}\mu_{ct} + \frac{\partial G_2(c_t,x_t)}{\partial x_t}\Big(-\log(\phi)\Big)(\mu_{x0}-x_t)\right. \\
&\left. + \frac{\sigma_{ct}^2}{2}\frac{\partial^2 G_2(c_t,x_t)}{\partial c_t^2} + \frac{\sigma_{xt}^2}{2}\frac{\partial^2 G_2(c_t,x_t)}{\partial x_t^2} + \frac{\rho_{cx}\sigma_{ct}\sigma_{xt}}{2}\frac{\partial^2 G_2(c_t,x_t)}{\partial c_t \partial x_t}\right)\mathrm{d}t \\
&+ \frac{\partial G_2(c_t,x_t)}{\partial x_t}\sigma_{xt}\mathrm{d}W_{xt} + \frac{\partial G_2(c_t,x_t)}{\partial c_t}\sigma_{ct}\mathrm{d}W_{ct}
\end{aligned}
\tag{3.64}
$$

Then everything is plugged into Equation (3.56) to give the final result:

$$
\begin{aligned}
\frac{\mathrm{d}\Lambda_t}{\Lambda_t} =& \left(\frac{\gamma(\gamma\psi-1)\rho_{cx}\sigma_{xt}\sigma_{ct}K'(x_t)}{\psi} + \frac{\gamma^2\sigma_{ct}^2}{2} - \gamma\mu_{ct}\right. \\
&+ \frac{(\gamma\psi-1)\left(2\psi(\mu_{x0}-x_t)\log(\phi)K'(x_t) + \sigma_{xt}^2\left((\gamma\psi-1)K'(x_t)^2-\psi K''(x_t)\right)\right)}{2\psi^2} \\
&\left.\frac{\rho\left(-(1-\gamma\psi)e^{-\frac{(\psi-1)K[x_t]}{\psi}}-\gamma\psi+\psi\right)}{1-\psi}\right)\mathrm{d}t \\
&- \frac{(\gamma\psi-1)\sigma_{xt}K'(x_t)}{\psi}\mathrm{d}W_{xt} - \gamma\sigma_{ct}\mathrm{d}W_{ct}
\end{aligned}
\tag{3.65}
$$

3.C Convergence – Time-varying consumption drift

In the main paper, I show convergence for the case, in which consumption volatility is time-varying. Here, I show the case when the consumption drift is time-varying. The convergence properties are similar in this case.

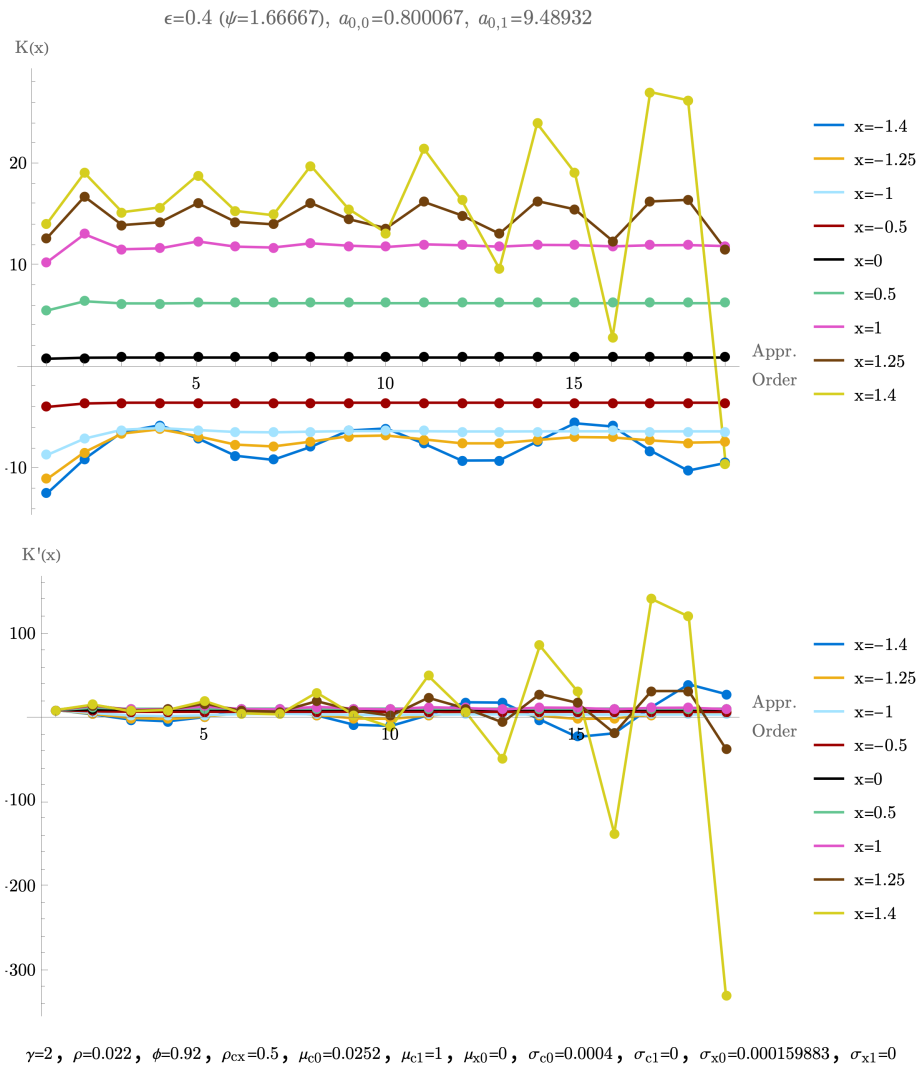
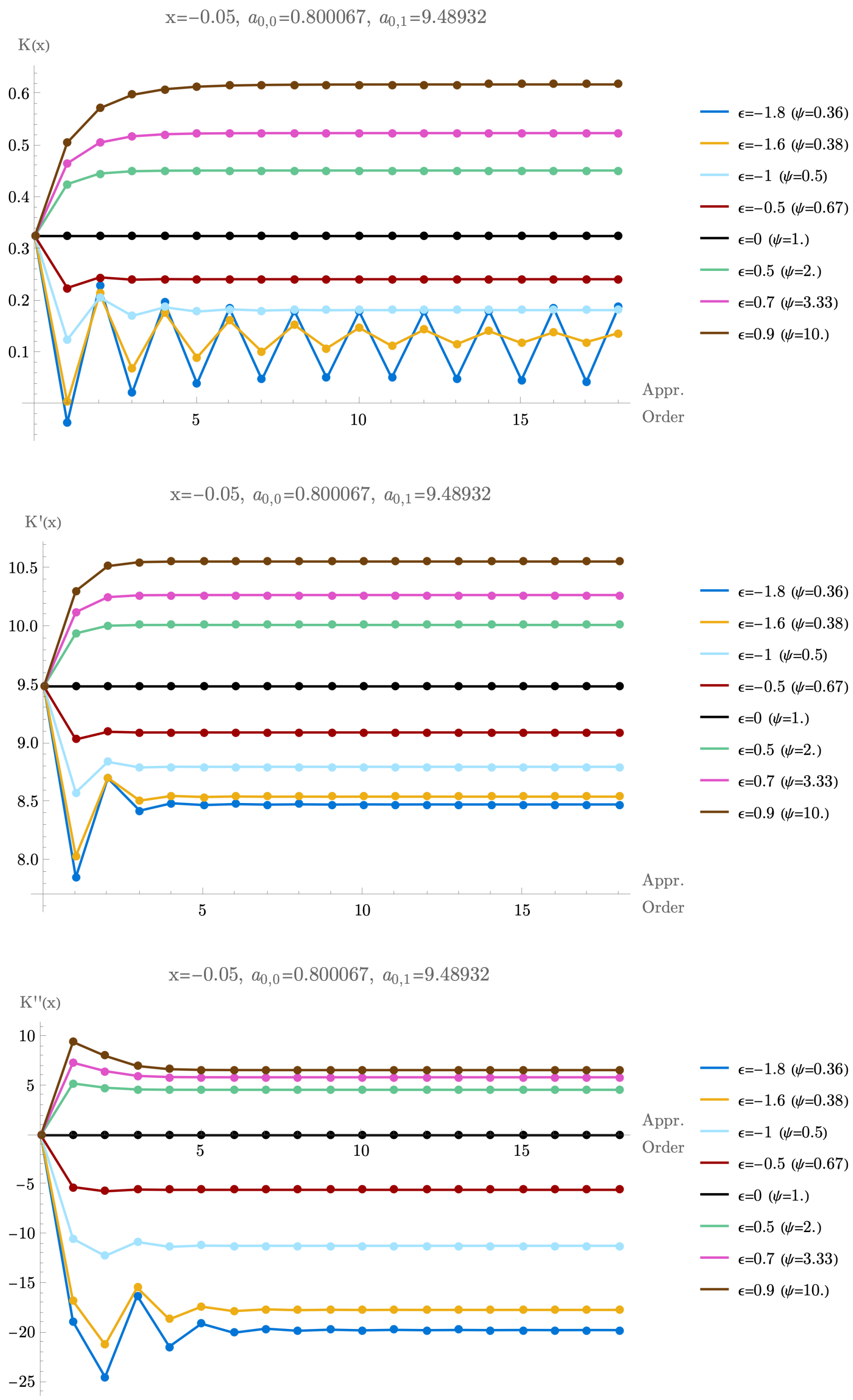


Figure 3.8: **Variation: Time-varying consumption drift.**
This shows the convergence of the problem for different values of x_t.

In this case, the series converges for all values of ϵ between 0 and 1. This means that the approximation works effectively for all values of $\psi > 1$.



γ=2, ρ=0.022, ϕ=0.92, ρ_{cx}=0.5, μ_{c0}=0.0252, μ_{c1}=1, μ_{x0}=0, σ_{c0}=0.0004, σ_{c1}=0, σ_{x0}=0.000159883, σ_{x1}=0

Figure 3.9: **Variation: Time-varying consumption drift.**
This shows the convergence of the problem for different values of ϵ. The plots correspond to the series of K and its two derivatives.

3.D Extension of Results to the Consumption-Investment Problem

The reduced form Hamilton-Jacobi-Bellman equation in Kraft, Seiferling and Seifried (2017) is a PDE. In my case, I have an infinite horizon problem. Thus, the equation does not have a dependence on time, and the equation can be reduced to an ordinary differential equation. Then Equation (3.47), is the same as the reduced form Hamilton-Jacobi-Bellman equation in Kraft, Seiferling and Seifried (2017), after q is replaced by:

$$q = 1 - \frac{k(1 - \frac{1}{\psi})\psi}{1 - \gamma} \tag{3.66}$$

Then I divide the equation by h and replace $\tilde{R}(Y_t) = \tilde{r}(Y_t) + \frac{\delta(1-\gamma)}{k(1-\frac{1}{\psi})}$ This makes it more transparent that the equation has a well-defined limit for $\psi = 1$. Finally I make the transformation $L(Y_t) = \log(h(Y_t))$ and Equation (3.48) follows.

Bibliography

Bansal, Ravi, and Amir Yaron. 2004. "Risks for the Long Run: A Potential Resolution of Asset Pricing Puzzles." *The Journal of Finance*, 59(4): 1481–1509.

Bender, Carl M, Steven Orszag, and Steven A Orszag. 1999. *Advanced Mathematical Methods for Scientists and Engineers I: Asymptotic Methods and Perturbation Theory.* Vol. 1, Springer Science & Business Media.

Benzoni, Luca, Pierre Collin-Dufresne, and Robert S Goldstein. 2011. "Explaining Asset Pricing Puzzles Associated with the 1987 Market Crash." *Journal of Financial Economics*, 101(3): 552–573.

Caldara, Dario, Jesus Fernandez-Villaverde, Juan F Rubio-Ramirez, and Wen Yao. 2012. "Computing DSGE Models with Recursive Preferences and Stochastic Volatility." *Review of Economic Dynamics*, 15(2): 188–206.

Campbell, John Y, and John H Cochrane. 1999. "By Force of Habit: A Consumption-Based Explanation of Aggregate Stock Market Behavior." *Journal of Political Economy*, 107(2): 205–251.

Chacko, George, and Luis M Viceira. 2005. "Dynamic Consumption and Portfolio Choice with Stochastic Volatility in Incomplete Markets." *The Review of Financial Studies*, 18(4): 1369–1402.

Chen, Yu, Thomas F Cosimano, Alex A Himonas, and Peter Kelly. 2009. "Asset Pricing with Long Run Risk and Stochastic Differential Utility: An Analytic Approach." *Available at SSRN 1502968.*

Chen, Yu, Thomas F Cosimano, and Alex A Himonas. 2010. "Continuous Time One-Dimensional Asset-Pricing Models with Analytic Price–Dividend Functions." *Economic Theory*, 42(3): 461–503.

Cochrane, John H. 2009. *Asset Pricing: Revised edition.* Princeton University Press.

Duffie, Darrell, and Larry G Epstein. 1992*a*. "Asset Pricing with Stochastic Differential Utility." *The Review of Financial Studies*, 5(3): 411–436.

Duffie, Darrell, and Larry G Epstein. 1992*b*. "Stochastic Differential Utility." *Econometrica: Journal of the Econometric Society*, 353–394.

Duffie, Darrell, Mark Schroder, and Costis Skiadas. 1996. "Recursive valuation of defaultable securities and the timing of resolution of uncertainty." *The Annals of Applied Probability*, 1075–1090.

Epstein, Larry G, and Stanley E Zin. 1989. "Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns: A Theoretical Framework." *Econometrica*, 57: 937–969.

Judd, Kenneth L. 1996. "Approximation, Perturbation, and Projection Methods in Economic Analysis." *Handbook of Computational Economics*, 1: 509–585.

Kraft, Holger, Thomas Seiferling, and Frank Thomas Seifried. 2017. "Optimal Consumption and Investment with Epstein–Zin Recursive Utility." *Finance and Stochastics*, 21: 187–226.

Leisen, Dietmar. 2016. "A Perturbation Approach to Continuous-Time Portfolio Selection." *Available at SSRN 2331434*.

Melissinos, Errikos. 2023. "Real Term Premia in Consumption-Based Models." *Available at SSRN 4582708*.

Melissinos, Errikos. 2024. "SDFPricing: A Julia Package for Asset Pricing Based on a Stochastic Discount Factor Process."

Schroder, Mark, and Costis Skiadas. 1999. "Optimal Consumption and Portfolio Selection with Stochastic Differential Utility." *Journal of Economic Theory*, 89(1): 68–126.

Seiferling, Thomas, and Frank Thomas Seifried. 2016. "Epstein-Zin Stochastic Differential Utility: Existence, Uniqueness, Concavity, and Utility Gradients." *Uniqueness, Concavity, and Utility Gradients (May 23, 2016)*.

Tsai, Jerry, and Jessica A Wachter. 2018. "Pricing Long-Lived Securities in Dynamic Endowment Economies." *Journal of Economic Theory*, 177: 848–878.

Wachter, Jessica A. 2006. "A Consumption-Based Model of the Term Structure of Interest Rates." *Journal of Financial Economics*, 79(2): 365–399.

Wachter, Jessica A. 2013. "Can time-varying risk of rare disasters explain aggregate stock market volatility?" *The Journal of Finance*, 68(3): 987–1035.

Nikolaos-Errikos Melissinos

Leibniz Institute for Financial Research SAFE e.V.
Data Center, Theodor-W.-Adorno-Platz 3
60323 Frankfurt am Main, Germany

Website: errikos-melissinos.com
Email: melissinos@safe-frankfurt.de

Education

2016 – 2024 (expected)	Ph.D. from the Faculty of Economics and Business Administration at Johann Wolfgang Goethe-University Frankfurt am Main First Supervisor: Prof. Volker Wieland Second Supervisor: Prof. Christian Schlag
2013 - 2014	Master of Economics and Finance, Barcelona School of Economics, Universitat Pompeu Fabra
2007 - 2012	Bachelor of Laws, University of Athens

Research Interests

I am interested in macroeconomics and finance, with a special focus on monetary economics, asset pricing and solution methods.

Research Projects

Working Papers

Real Term Premia in Consumption-Based Models (job market paper)

A Perturbation Solution Method for Models with Recursive Utility

Measuring On-the-job Learning Rates in Multidimensional Skills, *with Mariia Bondar*

Work in Progress

Jointly Explaining Stock Market Non-Participation and the Equity Premium

Monetary Policy Evaluation in Real Models

Effect of Recent Events on the Phillips Curve Relationship at the Regional Level, *with Henning Weber*

Research Experience

2018 - now	Research Assistant, Leibniz Institute - SAFE, Data Center Among other tasks, I contributed to the management of financial databases, I organized webinars, and I provided relevant support to researchers.
2020 – 2022	Researcher Assistant, Deutsche Bundesbank, Research Center Among other tasks I assisted in the paper: **The case for a positive Euro Area inflation target: Evidence from France, Germany and Italy** by *K. Adam, E. Gautier, S. Santoro and H. Weber*, Journal of Monetary Economics
2022 - 2024	Visitor Researcher, Deutsche Bundesbank, Research Center Working with German micro-price data.

Teaching

05/2022	Webinar: Pandas in Python for Economics and Finance, SAFE
11/2021	Webinar: Cryptocurrencies and Decentralised Finance, SAFE
10/2020	Webinar: Web-Scraping, SAFE
11/2019	Seminar: Python APIs for WRDS, Bloomberg and Eikon (Refinitiv), SAFE
09/2019	PhD Pre-Semester Course: Real Analysis, GSEFM

Conferences

2023	Bonn-Frankfurt-Mannheim PhD Conference
2022	15th RGS Doctoral Conference in Economics (presented by co-author)
2021	16th BiGSEM Doctoral Workshop on Economics and Management (presented by co-author)
2021	Frankfurt-Mannheim Macro Workshop

Other Experience

09/2014 – 08/2016	Lawyer, Independent Practice
09/2012 – 08/2013	Trainee, Dryllerakis and Associates Law Firm

Languages

Greek: **Native Speaker**; English: **Excellent**; German: **Good**; Spanish: **Good**

Computer Skills

Julia; Mathematica; Matlab (including Dynare); Python; R; Stata

References

Prof. Volker Wieland
IMFS Endowed Chair of Monetary Economics
Institute for Monetary and Financial Stability
Goethe University Frankfurt
Wieland@imfs-frankfurt.de

Prof. Christian Schlag
Professor of Finance
Leibniz Institute for Financial Research SAFE
Goethe University Frankfurt
Schlag@finance.uni-frankfurt.de

Dr. Henning Weber
Economist
Research Center
Deutsche Bundesbank
Henning.Weber@bundesbank.de

Feierliche Erklärung – Solemn Declaration

Frankfurt am Main, 7. Mai 2024

Ich habe die vorgelegte Dissertation selbst verfasst und dabei nur die von mir angegebenen Quellen und Hilfsmittel benutzt. Alle Textstellen, die wörtlich oder sinngemäß aus veröffentlichten oder nicht veröffentlichten Schriften entnommen sind, sowie alle Angaben, die auf mündlichen Auskünften beruhen, sind als solche kenntlich gemacht.

Nikolaos-Errikos Melissinos