Mapping the toric code to the rotated toric code

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Abstract

In this short note, we describe how the transformation from the hypergraph product code of two classical Tanner codes to a quantum Tanner code specializes to the toric code. In this case, the resulting code is the rotated toric code. This observation was first made by N. Breuckmann.

Let us consider a repetition code $[n, 1, n]$ of even length $n = 2m$. It is a Tanner code defined on the cycle graph (V, E) of length n, with local parity codes. Both sets V and E are indexed by the integers $0, 1, \dots, n-1$. Bits are associated with edges E, and the constraint on vertex $i \in [n]$ enforces the values of the bits $i-1$ and i to sum to zero: $x_{i-1} + x_i = 0$. Note that the indices are understood modulo n.

Because the graph has even length, it is bipartite and it is convenient to consider the partition $V = V_e \cup V_o$ of the vertices where V_e (resp. V_o) consists of vertices with an even (resp. odd) index.

The toric code is obtained by applying the hypergraph product construction to two copies of the repetition code [TZ14]. It is a CSS code with parameters $[\![2n^2, 2, n]\!]$ and

- $2n^2$ qubits indexed by $E \times E \cup V \times V = E \times E \cup (V_e \cup V_o) \times (V_e \cup V_o)$,
- n^2 σ_X-type generators $g^X_{p,q}$ indexed by $(V_e \cup V_o) \times E$,
- n^2 σ_Z-type generators $g^Z_{p,q}$ indexed by $E \times (V_e \cup V_o)$.

The support of the various generators is given by:

- Supp $g^X_{2i,2k} = \{(2i-1, 2k), (2i, 2k)\} \in E \times E, (2i, 2k) \in V_e \times V_e, (2i, 2k+1) \in V_e \times V_o\}$
- Supp $g^X_{2i,2k+1} = \{(2i-1, 2k+1), (2i, 2k+1) \in E \times E, (2i, 2k+2) \in V_e \times V_e, (2i, 2k+1) \in V_e \times V_o\}$
- Supp $g^X_{2i+1,2k} = \{(2i, 2k), (2i+1, 2k) \in E \times E, (2i+1, 2k) \in V_o \times V_e, (2i+1, 2k+1) \in V_o \times V_o\}$
- Supp $g^X_{2i+1,2k+1} = \{(2i, 2k+1), (2i+1, 2k+1) \in E \times E, (2i+1, 2k+2) \in V_o \times V_e, (2i+1, 2k+1) \in V_o \times V_o\}$
- Supp $g^Z_{2i,2k} = \{(2i, 2k-1), (2i, 2k)\} \in E \times E, (2i, 2k) \in V_e \times V_e, (2i+1, 2k) \in V_o \times V_e\}$

- Supp $g^Z_{2i,2k+1} = \{(2i,2k),(2i,2k+1) \in E \times E, (2i,2k+1) \in V_e \times V_o, (2i+1,2k+1) \in V_o \times V_o\}$
- Supp $g^Z_{2i+1,2k} = \{(2i+1,2k-1),(2i+1,2k) \in E \times E, (2i+1,2k) \in V_o \times V_e, (2i+2,2k) \in V_e \times V_e\}$
- Supp $g^Z_{2i+1,2k+1} = \{(2i+1,2k),(2i+1,2k+1) \in E \times E, (2i+1,2k+1) \in V_o \times V_o, (2i+2,2k+1) \in V_e \times V_o\}$.

The idea behind the construction of quantum Tanner codes (detailed in Section 7 of [LZ22a]) is to modify the set of generators so as to keep only qubits in $E \times E$. To achieve this, the generators are modified in a way that avoids the supports of σ_X-type and σ_Z-type generators to overlap on the qubits in $(V_e \cup V_o) \times (V_e \cup V_o)$. One possibility is to choose new σ_X-type generators that will only have support on $E \times E \cup V_e \times V_e \cup V_o \times V_o$ and new σ_Z-type generators that will only have support on $E \times E \cup V_e \times V_o \cup V_e \times V_o$. If this is the case, since the generators still commute pairwise, it means that they commute when restricted to $E \times E$ since they do not overlap elsewhere. In particular, by discarding the qubits in $(V_e \cup V_o) \times (V_e \cup V_o)$, one obtains a new CSS code with qubits on $E \times E$ and generators given by the new ones. We now explain how to do this transformation when starting from the toric code. Applying this transformation to the expander lifted-product codes of Panteleev and Kalachev [PK21] gives the family of quantum Tanner codes [LZ22b]. The same transformation can also be applied to any hypergraph product code of two classical Tanner codes.

We now define our new generators, which will belong to 4 families:

- for $(2i,2k) \in V_e \times V_e$, define $\tilde{g}^X_{2i,2k} := g^X_{2i,2k} g^X_{2i,2k+1}$. Its support is

  Supp $\tilde{g}^X_{2i,2k} =$
  $\{(2i-1,2k),(2i,2k),(2i-1,2k+1),(2i,2k+1) \in E \times E, (2i,2k),(2i,2k+2) \in V_e \times V_e\}$.

- for $(2i+1,2k+1) \in V_o \times V_o$, define $\tilde{g}^X_{2i+1,2k+1} := g^X_{2i+1,2k} g^X_{2i+1,2k+1}$. Its support is

  Supp $\tilde{g}^X_{2i+1,2k-1} =$
  $\{(2i,2k),(2i+1,2k),(2i,2k-1),(2i+1,2k-1) \in E \times E, (2i+1,2k-1),(2i+1,2k+1) \in V_o \times V_o\}$.

- for $(2i,2k+1) \in V_e \times V_o$, define $\tilde{g}^Z_{2i,2k+1} := g^Z_{2i,2k+1} g^Z_{2i+1,2k+1}$. Its support is

  Supp $\tilde{g}^Z_{2i,2k-1} =$
  $\{(2i,2k),(2i,2k+1),(2i+1,2k),(2i+1,2k+1) \in E \times E, (2i,2k+1),(2i+2,2k+1) \in V_e \times V_o\}$.

- for $(2i+1,2k) \in V_o \times V_e$, define $\tilde{g}^Z_{2i+1,2k} := g^Z_{2i,2k} g^Z_{2i-1,2k}$. Its support is

  Supp $\tilde{g}^Z_{2i+1,2k} =$
  $\{(2i-1,2k-1),(2i-1,2k),(2i,2k-1),(2i,2k) \in E \times E, (2i+1,2k),(2i-1,2k) \in V_o \times V_e\}$.

One can observe as promised that the σ_X-type generators don't have support on $V_e \times V_o$ or $V_o \times V_e$ and that the σ_Z-type generators don't have support on $V_e \times V_e$ or $V_o \times V_o$.

We can therefore define generators with a support restricted to $E \times E$:

- for $(2i, 2k) \in V_e \times V_e$, define $G^X_{2i,2k}$ with support

$$\{(2i-1, 2k), (2i, 2k), (2i-1, 2k+1), (2i, 2k+1)\} \subset E \times E.$$

- for $(2i+1, 2k+1) \in V_o \times V_o$, define $G^X_{2i+1,2k+1}$ with support

$$\{(2i, 2k), (2i+1, 2k), (2i, 2k-1), (2i+1, 2k-1)\} \subset E \times E.$$

- for $(2i, 2k+1) \in V_e \times V_o$, define $G^Z_{2i,2k+1}$ with support

$$\{(2i, 2k), (2i, 2k+1), (2i+1, 2k), (2i+1, 2k+1)\} \subset E \times E.$$

- for $(2i+1, 2k) \in V_o \times V_e$, define $G^Z_{2i+1,2k}$ with support

$$\{(2i-1, 2k-1), (2i-1, 2k), (2i, 2k-1), (2i, 2k)\} \subset E \times E.$$

These generators correspond to those of the rotated toric code of length n^2.

References

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