Completing the Square

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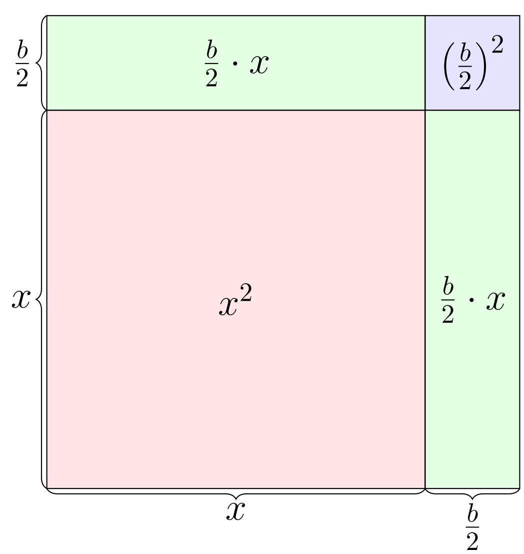
June 10, 2023

Abstract

Completing the square is a method for converting a quadratic polynomial of the form $ax^2 + bx + c$ to the form $a\left(x+h\right)^2 + k$. It is useful is both elementary algebra and higher mathematics. In this article, we attempt to create a deep understanding of completing the square starting from first principles. We begin with a brief introduction to the concept of factoring and review the grouping method and its shortcomings. We then discuss how to complete the square and illustrate how it can be used to derive the quadratic formula. We conclude with a geometric visualization of completing the square as well as a brief comparison to other factoring methods. Several exercises are then provided for the reader.



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1 Introduction to Factoring

Factoring or **factorization** is the process of writing a number or another mathematical object as a product of several **factors**, usually smaller or simpler objects of the same kind. In other words, factoring a mathematical object is the process of decomposing it into smaller parts (the factors), which when multiplied together, result in the original object. For example, $2 \times 2 \times 5$, -4×-5, 10×2 and 20×1 are some factorization of the integer 20.

We factor numbers every day without even realizing it. For example, if you walk into a bank and ask to withdraw \$120 from your account in denominations of \$20, the teller should hand you 6, \$20 bills. A prudent person might then double check that $6 \times 20 = 120$, i.e. check that 6×20 is a factorization of 120.

Just as we can factor numbers, we can also factor polynomials, i.e. we can break them up into simpler factors, which when multiplied together give us back the original expression. In this document, we will focus on factoring quadratic polynomials. Recall that a **quadratic polynomial** has the form

$$ax^2 + bx + c$$

Some quadratic polynomials are easy to factor. For example $x^2 - 4$ is a **difference of two squares**, and so it can be factored as $(x+2)(x-2)$ (see exercise 1). However, not all polynomials are so easy to factor. In the following sections will will discuss and compare three methods for factoring polynomials.

2 Factoring via the Grouping Method

We often try to factor an expression $x^2 + bx + c$ by finding two numbers α, β such that $\alpha\beta = c$ and $\alpha + \beta = b$. In other words, we're trying to find two numbers α, β which multiply to give c and add to give b. We can then factor $x^2 + bx + c = (x+\alpha)(x+\beta)$. This is known as the **grouping method**. But why does the grouping method work? Working backwards from the factored expression we have:

$$\begin{aligned}
(x+\alpha)(x+\beta) &= x(x+\beta) + a(x+\beta) \\
&= x^2 + \beta x + \alpha x + \alpha\beta \\
&= x^2 + (\alpha+\beta)x + \alpha\beta \\
&= x^2 + bx + c
\end{aligned}$$

Thus we see how by making the substitutions $b = \alpha + \beta$ and $c = \alpha\beta$, we can write $x^2 + bx + c = (x+\alpha)(x+\beta)$. One advantage of such a factorization is it allows us to easily find all solutions (zeros) of this polynomial. Unfortunately, factoring in this manner is not always possible as we will see in 3.

2.1 Examples

Example 1. Factor $x^2 - 8x + 16$ using the grouping method.

We need to find two numbers that multiply to 16 and add to -8. We note that $(-4)^2 = 16$ and $-4 + -4 = -8$. Thus we can factor $x^2 - 8x + 16 = (x-4)(x-4) = (x-4)^2$.

Example 2. Factor $x^2 - 8x + 15$ using the grouping method.

We need to find two numbers that multiply to 15 and add to -8. We note that $-3 \times -5 = 15$ and $-3 + -5 = -8$. Thus we can factor $x^2 - 8x + 15 = (x-3)(x-5)$.

Example 3. Factor $x^2 - 8x + 17$ using the grouping method.

We need two numbers which multiply to 17 and add to -8. But 17 is a prime number, so the only pairs of integers that multiply to give 17 are $(17, 1)$ and $(-17, -1)$, which do not add to -8. We could try to guess two non-integers which multiply to 17 and sum to -8, however this is very difficult, so we need to find a better method.

3 Factoring via Completing the Square

Definition 4. In elementary algebra, **completing the square** is a technique for converting a quadratic polynomial of the form

$$ax^2 + bx + c$$

to

$$a(x+h)^2 + k$$

where k is known as the **constant term**. In theorem 11 we will show $k = \frac{c}{a} - \left(\frac{b}{2a}\right)^2$.

Proposition 5. *We can factor $x^2 + bx + \left(\frac{b}{2}\right)^2$ as $\left(x + \frac{b}{2}\right)^2$. That is to say:*

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$$

Proof. We can observe that $x^2 + bx + \left(\frac{b}{2}\right)^2$ does factor to $\left(x + \frac{b}{2}\right)^2$ by factoring it using the grouping method. So we want to find α, β such that $\alpha\beta = \left(\frac{b}{2}\right)^2$ and

$\alpha + \beta = b$. We observe that if $\alpha = \beta$ then we can solve explicitly for α (or β):

$$\begin{aligned}
\alpha\beta &= \left(\frac{b}{2}\right)^2 \\
\alpha^2 &= \left(\frac{b}{2}\right)^2 \\
\sqrt{\alpha^2} &= \sqrt{\left(\frac{b}{2}\right)^2} \\
\alpha &= \frac{b}{2}
\end{aligned}$$

Thus $\alpha = \beta = \frac{b}{2}$. We can also check that $\alpha = \beta = \frac{b}{2}$ satisfies the equation $\alpha + \beta = b$. Sure enough

$$\begin{aligned}
\alpha + \beta &= \alpha + \alpha \\
&= 2\alpha \\
&= 2\frac{b}{2} \\
&= b
\end{aligned}$$

□

Definition 6. The **“add-zero” trick**

The “add-zero” trick is a useful algebraic tool that involves rewriting an expression by adding b and $-b$ to the same expression simultaneously. Because $b + (-b) = 0$, using this trick does not change the value of the expression. That is, for any numbers a, b it is true that easily.

$$\begin{aligned}
a &= a + b - b \\
&= a + 0 \\
&= a
\end{aligned}$$

Example 7. Rewrite the function $ax^2 + bx + c$ using the “add-zero” trick with some number d

$$ax^2 + bx + c = ax^2 + bx + c + d - d$$

Theorem 8. *An expression of the form $x^2 + bx$, can be factored into the form $\left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2$.*

Proof. We use the “add-zero” trick (see definition 6) to simultaneously add $\left(\frac{b}{2}\right)^2$ and $-\left(\frac{b}{2}\right)^2$ to the expression $x^2 + bx$ and then apply proposition 5 to factor the resulting expression.

$$\begin{aligned}
x^2 + bx &= x^2 + bx + \left(\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 \\
&= \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2
\end{aligned}$$

$\square$

Definition 9. A **monic** quadratic polynomial is a quadratic polynomial $ax^2 + bx + c$ where $a = 1$. I.e. it has the form $x^2 + bx + c$.

Theorem 10. *Let $p(x)$ be a monic quadratic polynomial. Then*

$$x^2 + bx + c = (x - h)^2 + k$$

where $k = c - \left(\frac{b}{2}\right)^2$

Proof. This is very similar to theorem 8, except that we now have a constant term c in the initial equation. However, everything else is the same and we see that c is eventually just absorbed into the constant term k.

$$\begin{aligned} x^2 + bx + c &= x^2 + bx + \left(\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c \\ &= \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c \\ &= \left(x + \frac{b}{2}\right)^2 + k^2 \end{aligned}$$

Letting $k = c - \left(\frac{b}{2}\right)^2$, the result is proved. $\square$

Now, we relax (i.e. eliminate) the assumption that $a = 1$, so that ax^2+bx+c may represent an arbitrary quadratic polynomial. In the next theorem, we show how to rewrite $ax^2 + bx + c = 0$ by completing the square.

Theorem 11. *By completing the square, we can factor an arbitrary quadratic polynomial $ax^2 + bx + c$ as follows*

$$ax^2 + bx + c = \left(x + \frac{b}{2a}\right)^2 - k$$

with constant term $k = \frac{c}{a} - \left(\frac{b}{2a}\right)^2$

Proof. We begin by factoring out $a$from the first two terms. This simplifies the algebra somewhat vs if we had factored a out of all three terms. We then use the "add-zero" trick to add $\left(\frac{b}{2a}\right)^2$ and $-\left(\frac{b}{2a}\right)^2$ to the expression. We then then apply proposition 5 to factor the resulting expression, thus "completing the square".

$$
\begin{aligned}
ax^2 + bx + c &= a\left[x^2 + \frac{b}{a}x\right] + c \\
&= a\left[x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2\right] + c \\
&= a\left(x + \frac{b}{2a}\right)^2 - a\left(\frac{b}{2a}\right)^2 + c \\
&= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c \\
&= a\left(x + \frac{b}{2a}\right)^2 + k
\end{aligned}
$$

where the constant term $k = -\frac{b^2}{4a} + c$ □

Now that we have completed the square on an arbitrary quadratic polynomial $ax^2 + bx + c$, we can solve this expression for x, thereby obtaining the a formula for the solutions of an arbitrary quadratic polynomial. This is known as the **quadratic formula**.

Theorem 12. *Quadratic Formula*

Let $ax^2 + bx + c$ be an arbitrary quadratic formula. Then the two solutions (or "roots") are given by

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Proof. We take the factorization of an arbitrary quadratic polynomial $ax^2 + bx + c$ obtained in theorem 11 and solve for x.

$$
\begin{aligned}
a\left(x+\frac{b}{2a}\right)^2-\frac{b^2}{4a}+c&=0\\
a\left(x+\frac{b}{2a}\right)^2&=\frac{b^2}{4a}-c\\
\left(x+\frac{b}{2a}\right)^2&=\frac{b^2}{4a^2}-\frac{c}{a}\\
\sqrt{\left(x+\frac{b}{2a}\right)^2}&=\sqrt{\frac{b^2}{4a^2}-\frac{c}{a}}\\
x+\frac{b}{2a}&=\sqrt{\frac{b^2}{4a^2}-\frac{c}{a}}\\
x&=\sqrt{\frac{b^2}{4a^2}-\frac{c}{a}}-\frac{b}{2a}\\
&=\sqrt{\frac{b^2-4ac}{4a^2}}-\frac{b}{2a}\\
&=\frac{\sqrt{b^2-4ac}}{2a}-\frac{b}{2a}\\
&=\frac{-b\pm\sqrt{b^2-4ac}}{2a}
\end{aligned}
$$

□

3.1 Examples

Example 13. Rewrite the function $x^2+6x+87$ by completing the square and find all solutions.

$$
\begin{aligned}
x^2+6x+87&=x^2+6x+\left(\frac{6}{2}\right)^2-\left(\frac{6}{2}\right)^2+87\\
&=x^2+6x+9-9+87\\
&=(x+3)^2+78
\end{aligned}
$$

To find the solutions set this expression equal to zero.

$$
\begin{aligned}
(x+3)^2+78&=0\\
x+3&=\sqrt{-78}\\
x&=-3\pm\sqrt{-78}\\
&=-3\pm i\sqrt{78}
\end{aligned}
$$

Note that this function has two complex roots.

Example 14. Rewrite $x^2 - 10x + 17$ by completing the square and find the solutions:

$$\begin{aligned} x^2 - 10x + 17 &= x^2 - 10x + 25 - 25 + 17 \\ &= (x-5)^2 - 8 \end{aligned}$$

We find the solutions by setting this expression equal to zero

$$\begin{aligned} (x-5)^2 - 8 &= 0 \\ (x-5)^2 &= 8 \\ \sqrt{(x-5)^2} &= \sqrt{8} \\ x - 5 &= \sqrt{8} \\ &= \sqrt{4 \cdot 2} \\ &= \pm 2\sqrt{2} \\ x &= 5 \pm 2\sqrt{2} \end{aligned}$$

Example 15. Rewrite the function by completing the square and find all solutions.

Since $a = 10$, this is not a monic polynomial, however we can convert it to a monic polynomial by factoring out 10 from all terms containing x.

$$10x^2 - 30x - 8 = 10\left(x^2 - 3x\right) - 8$$

We can now complete the square on the monic polynomial $x^2 - 3x$.

$$\begin{aligned} 10\left[x^2 - 3x\right] - 8 &= 10\left[x^2 - 3x + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2\right] - 8 \\ &= 10\left[\left(x - \frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2\right] - 8 \\ &= 10\left(x - \frac{3}{2}\right)^2 - 10\left(\frac{3}{2}\right)^2 - 8 \\ &= 10\left(x - \frac{3}{2}\right)^2 - \frac{45}{2} - \frac{16}{2} \\ &= 10\left(x - \frac{3}{2}\right)^2 - \frac{61}{2} \end{aligned}$$

To find the solutions set this expression equal to zero:

$$10\left[\left(x-\frac{3}{2}\right)^2-\frac{61}{20}\right]=0$$

$$\left(x-\frac{3}{2}\right)^2=\frac{61}{20}$$

$$x=\frac{3}{2}\pm\sqrt{\frac{61}{20}}$$

4 Completing the Square as a Graph Translation

4.1 Graph Translations Review

In this section we briefly review the translations of graph of a function $f(x)$ so that we may interpret the expression which results of completing the square.

1. To shift the graph of $f(x)$ up, we add a positive number k to $f(x)$, and to shift it down we add a negative number. Thus vertically shifting $f(x)$ by k takes the form $f(x)+k$, with the direction of movement (up or down) depending on the sign of k (i.e. if k is positive or negative).

2. To stretch or shrink the graph of $f(x)$ vertically by a factor of a, we simply multiply $f(x)$ by a. If $a>1$ the graph is vertically stretched, and if $1>a>0$ the graph shrinks vertically.

3. If $a<0$, the graph is flipped or reflected about the x-axis and stretched by a. In particular, if we encounter $-af(x)$, we might think of it as $-1{\cdot}af(x)$, so it is clear that we are both flipping it and vertically stretching/shrinking it by a.

4. Lastly $f(x+k)$ will move the $f(x)$ to the left or right depending if the sign of k is positive or negative.

Many students often have trouble remembering which direction the graph moves, so we will illustrate with an example. Consider the graph of $f(x)=x^2$. We know what this graph looks like; it has one root at $x=0$. Now consider $f(x+3)=(x+3)^2$, where is its root? I.e, for what value of x is is $f(x+3)=0$? Well, when $x=-3$, $f(x+3)=(-3+3)^2=0^2=0$. So now the root is at -3, so changing $f(x)$ to $f(x+3)$ has moved $f(x)$ three units to the left. If we repeat this experiment, we will see that $f(x-3)$ has a zero at $x=3$. Thus $f(x-3)$ moves $f(x)$ three units to the right. We summarize these results below in table 1:

4.2 Application to Completing the Square

Keeping in mind what we have learned about graph translations, what does $a(x+h)^2+k$ look like relative to x^2? In other words, if $f(x)=x^2$, how does $af(x+h)+k$ transform it? It is:

Action	Algebra	Note
Vertical Shift	$f(x)+k$	Direction of shift $(\uparrow,\downarrow)$ if k $(+,-)$
Vertical Stretch/Shrink	$af(x)$	Stretch if $a>1$, shrink if $0<a<1$
Vertical Flip/Reflection	$af(x)$	when $a<0$
Horizontal Shift	$f(x+k)$	Shift $\leftarrow$ if k positive, shift $\rightarrow$ if k is negative

Table 1: Graph Transformations

1. Vertically stretched/shrunk by a
2. Horizontally shifted to the left by h
3. Vertically shifted by k

Specifically, since $f(x)=x^2$ has a vertex of $(0,0)$, $af(x+h)+k$ has its vertex at $(-h,k)$.

Example 16. Consider $f(x)=8x^2-64x+16$

1. Rewrite $f(x)$ by completing the square

$$\begin{aligned}-8x^2-64x+16 &= -8\left[x^2+8x\right]+16\\ &= -8\left[x^2+8x+16-16\right]+16\\ &= -8\left[(x+4)^2-16\right]+16\\ &= -8(x+4)^2+8\cdot 16+16\\ &= -8(x+4)^2+9\cdot 16\\ &= -8(x+4)^2+144\end{aligned}$$

2. Describe $f(x)$ in terms of graph translations from $g(x)=x^2$

   This is the graph of x^2, but flipped over the x-axis (by the -1 in the -8), stretched vertically by 8, vertically shifted up by 144, and moved to the left by 4

3. Find the vertex of $f(x)$

   The vertex is at $(-4,144)$

4. Find the roots of $f(x)$

$$\begin{aligned}-8(x+4)^2+144 &= 0\\ (x+4)^2 &= \frac{-144}{-8}\\ x &= -4\pm\sqrt{18}\\ &= -4\pm\sqrt{2\cdot 3^2}\\ &= -4\pm 3\sqrt{2}\end{aligned}$$

5 Comparing Various Factoring Methods

As we have just seen, completing the square has the advantage of allowing us to directly read off the coordinates of the vertex of a quadratic polynomial. These coordinates represent the maximum or minimum (**extremum**) of the quadratic polynomial, and the vertical line through the x-coordinate represents the axis of symmetry. The downside is that if we want to find the roots of the polynomial, we must solve for them. This is in contrast with the grouping method and quadratic formula which provide the roots of the graph, but require us to solve for the extrema of the polynomial.

While the grouping method is probably the simplest and most familiar factoring method, it is important to remember that there are many quadratic polynomials which cannot be factored using the grouping method. In contrast, both completing the square and using the quadratic formula will always work. Of course, in order to use the quadratic formula, one must first memorize it, which can difficult for some students.

We summarize our comparison of the three factoring methods in table 2. Each has its advantages and disadvantages, and students should be proficient in using all three factoring methods.

Method	Shows	Must Solve For	Always Works	Difficulty
Completing the Square	Extremum	Roots	Yes	Medium
Grouping Method	Roots	Extremum	No	Easy
Quadratic Formula	Roots	Extremum	Yes	Medium

Table 2: Advantages of Various Factoring Methods

6 Geometric Visualization

Completing the square has a nice geometric visualization shown in figure 1. Notice how

$$x^2 + \frac{b}{2}x + \frac{b}{2}x = x^2 + bx$$

which represents represents the sum of the red and green rectangles, almost forms a perfect square with sides $x + \frac{b}{2}$. The missing part of the perfect square, shown in purple, has an area of $\left(\frac{b}{2}\right)^2$. By adding the missing $\left(\frac{b}{2}\right)^2$ to $x^2 + bx$ we obtain a perfect square with sides $x + \frac{b}{2}$ and area

$$\left(x + \frac{b}{2}\right)^2 = x^2 + bx + \left(\frac{b}{2}\right)^2$$

Keep in mind that when completing the square, we are able add this extra $\left(\frac{b}{2}\right)^2$ to the "almost square" $x^2 + bx$ because we simultaneously subtract $\left(\frac{b}{2}\right)^2$ as per the "add-zero" trick (see definition 6 and theorem 8).

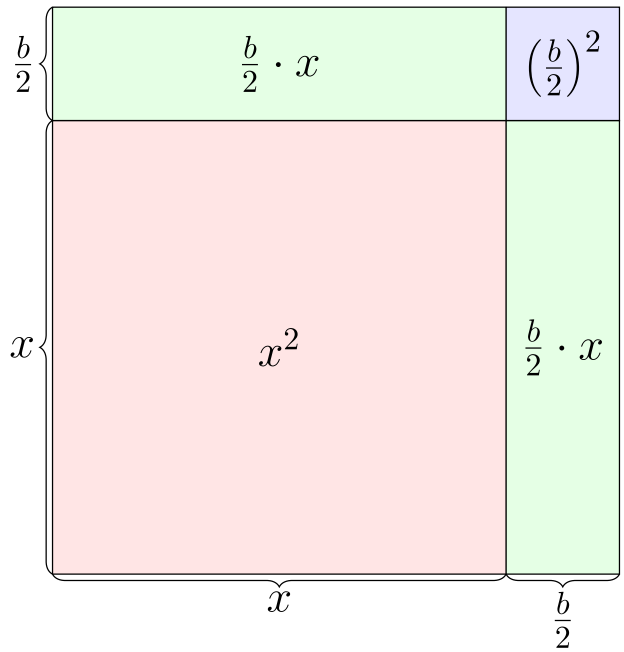

Figure 1: Adding $\left(\frac{b}{2}\right)^2$ to $x^2 + bx$ creates a perfect square $x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$

7 Exercises

Exercise 1. Let $f(x) = x^2 - a^2$ be a difference of two squares. Show that $f(x)$ can be factored as $(x+a)(x-a)$.

Exercise 2. Let $f(x) = x^2 + 13x + 42$

1. Attempt to factor $f(x)$ using the grouping method and find the roots. If the grouping method cannot be used, explain why.
2. Factor $f(x)$ by completing the square and find the roots.
3. Factor $f(x)$ using the quadratic formula and find the roots.

Exercise 3. Let $f(x) = -3x^2 - 9x - 22$

1. Attempt to factor $f(x)$ using the grouping method and find the roots. If the grouping method cannot be used, explain why.
2. Factor $f(x)$ by completing the square and find the roots.
3. Factor $f(x)$ using the quadratic formula and find the roots.

Exercise 4. Let $f(x) = 10x^2 + 64x - 16$

1. Attempt to factor $f(x)$ using the grouping method and find the roots. If the grouping method cannot be used, explain why.
2. Factor $f(x)$ by completing the square and find the roots.
3. Factor $f(x)$ using the quadratic formula and find the roots.

Exercise 5. Let $f(x) = 10x^2 + 20x + 12$

1. Attempt to factor $f(x)$ using the grouping method and find the roots. If the grouping method cannot be used, explain why.
2. Factor $f(x)$ by completing the square and find the roots.
3. Factor $f(x)$ using the quadratic formula and find the roots.

Exercise 6. Let $f(x) = 2x^2 + 8x + 15$. Complete the square in order to express $f(x)$ as a graph translation from $g(x) = x^2$.

Exercise 7. Let $f(x) = -4x^2 + 8x + 10$. Complete the square in order to express $f(x)$ as a graph translation from $g(x) = x^2$.

Exercise 8. Let $g(x) = 2f(x+3) + 7$. If $g(x)$ is a translation of $f(x)$, find a inverse translation $g^{-1}(x)$ of $g(x)$ such that $g^{-1}(g(x)) = f(x)$. If g is not a translation of f, explain why not.

Exercise 9. Let $g(x) = -6xf(x+3) + 2$. If $g(x)$ is a translation of $f(x)$, find a inverse translation $g^{-1}(x)$ of $g(x)$ such that $g^{-1}(g(x)) = f(x)$. If g is not a translation of f, explain why not.

Exercise 10. Let $f(x) = -7x^2 + 14x - 32$. Rewrite $f(x)$ by completing the square. Draw a visualization of completing the square as in figure 1.

Exercise 11. Let $f(x) = 2x^2 + 11x + 17$. Rewrite$f(x)$ by completing the square. Draw a visualization of completing the square as in figure 1.

Exercise 12. Let $f(x) = x^2 + bx + p$ be a quadratic binomial where p is a prime number. For what values of b can we factor $f(x)$ using the grouping method?