Triangular Numbers

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1 Introduction

One beautiful aspect of mathematics is that it allows us to see the similarities between situations that may appear quite different. In this article we examine several seemingly different problems:

1. What is the nth triangular number T_n?
2. Given T_n, how do we find T_{n+1}? That is, if we know some triangular number T_n, how do we find the next one?
3. What is the sum of the first n integers $(1 + 2 + 3 + 4 + \cdots + n = ?)$
4. An $n \times n$ matrix is said to be **upper triangular** if all entries below the main diagonal are equal to 0. How many entries of an $n \times n$ upper triangular matrix must be equal to 0?
5. There are n students in a science class. How many pairs can be formed?
6. Show that $\binom{n}{2} = \frac{n(n-1)}{2}$

2 Definitions

We begin by definiing a triangular number as numbers which can be respresented in the following form:

$$\begin{array}{ccccccccccccccc}
 & & & & & & & & & & \square & & & & \\
 & & & & & & \square & & & & \square & \square & & & \\
 & & & \square & & & \square & \square & & & \square & \square & \square & & \\
 & \square & & \square & \square & & \square & \square & \square & & \square & \square & \square & \square & \\
\square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\
T_1 & T_2 & & T_3 & & & T_4 & & & & T_5 & & & &
\end{array} \tag{1}$$

Each box represents "1". Thus, counting the boxes, we see the first five triangular numbers are $T_1 = 1$, $T_2 = 3$, $T_3 = 6$, $T_4 = 10$, $T_5 = 15$.

3 Solutions

We now answer the questions in order:

1. Given some triangular number T_n represented by a triangular array, we can find the value of T_n using a geometric argument. For a concreate example, we will demonstrate using T_5. We first create a copy of T_5, and attach it back to itself. We name the result $R_{5\times 6}$ because it is a 5×6 rectangle (i.e. it has 5 rows and 6 columns). This is shown below:

$$\underset{T_5}{\begin{array}{ccccc} \square & & & & \\ \square & \square & & & \\ \square & \square & \square & & \\ \square & \square & \square & \square & \\ \square & \square & \square & \square & \square \end{array}} \longrightarrow \underset{R^{5\times 6}}{\begin{array}{cccccc} \square & \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare \\ \square & \square & \blacksquare & \blacksquare & \blacksquare & \blacksquare \\ \square & \square & \square & \blacksquare & \blacksquare & \blacksquare \\ \square & \square & \square & \square & \blacksquare & \blacksquare \\ \square & \square & \square & \square & \square & \blacksquare \end{array}} \tag{2}$$

   How many items are in T_5? Clearly, there are $5 \times 6 = 30$ entries in $R_{5\times 6}$ and $R_{5\times 6}$ is twice the size of T_5. Therefore $T_5 = \frac{5\cdot 6}{2} = 15$. In general:

$$T_n = \frac{n\,(n+1)}{2} \tag{3}$$

   This shows that we can interpret T_n as half of the elements in a $n \times (n+1)$ shaped matrix.

2. Given the T_n, how do we find T_{n+1}? Well, we showed that $T_n = \frac{n(n+1)}{2}$. In the figures above, we can see that the difference between T_n and T_{n+1} is a row with $n+1$ elements. E.g. to go from T_4, we simply add a row with 5 elements. Therefore, we might guess that

$$\begin{aligned} T_{n+1} &= T_n + (n+1) \\ &= \frac{n\,(n+1)}{2} + (n+1) \end{aligned}$$

   We also know (by plugging $n+1$ into 3) that:

$$T_{n+1} = \frac{(n+1)\,((n+1)+1)}{2}$$

   If we are correct, both of these must be the equal:

$$\frac{n\,(n+1)}{2} + (n+1) = \frac{(n+1)\,((n+1)+1)}{2}$$

   And one can show via some basic algebra that this is indeed the case.

3. Now, we will use our previous work to find the sum of the first n integers ($\sum_{i=1}^{n} = 1+2+3+4+\cdots+n$). We begin by observing that the number

of entries in the nth row of T_n, is equal to n, as shown below:

$$\begin{array}{llllll} 1 \rightarrow & \square & & & & \\ 2 \rightarrow & \square & \square & & & \\ 3 \rightarrow & \square & \square & \square & & \\ 4 \rightarrow & \square & \square & \square & \square & \\ 5 \rightarrow & \square & \square & \square & \square & \square \\ & T_5 & & & & \end{array}$$

Thus counting the elements in T_n is the same as computing $\sum_{i=1}^{n} i$. But we already know $T_n = \frac{n(n+1)}{2}$, and therefore:

$$T_n = \sum_{i=1}^{n} i = \frac{n\,(n+1)}{2}$$

4. The next two questions use similar thinking but with a couple twists. Here, we must figure out how many entries of an $n \times n$ upper triangular matrix must be equal to 0. To approach this question, let's first fix $n = 4$. For example, below is a 4×4 upper triangular matrix. The 0 entries represent spots which must be 0 in an upper triangular matrix.

$$A = \begin{bmatrix} a & b & c & d \\ 0 & e & f & g \\ 0 & 0 & h & i \\ 0 & 0 & 0 & j \end{bmatrix}$$

We will call the entry in row i and column j of matrix A the i,jth entry of A, and write $a_{i,j}$. E.g:

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1.3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{44} \end{bmatrix}$$

There are serveral ways to approach this problem. One way would be to overserve that the number of entries equal to 0 must be less than the total number of entries. Since it is a 4×4 matrix, there are $4^2 = 16$ entries. Thus, the total number of entries that must be zero must be less than 16. The simpliest solution is to then observe that the diagonal entries, i.e. those of the form $a_{i,i}$, are not among those which must be equal to 0. In the 4×4 case, we can count that there are 4 diagonal entries. When we 'delete' the diagonal entries (which we represent with a Δ), we obtain a matrix that looks like this:

$$A = \begin{bmatrix} \Delta & b & c & d \\ 0 & \Delta & f & g \\ 0 & 0 & \Delta & i \\ 0 & 0 & 0 & \Delta \end{bmatrix}$$

Observe that by deleting the diagonal entries, we have split the matrix into two equal size halves, the lower 6 elements, which must be equal to 0, and the upper 6 elements. I.e. half of the 12 entries must be equal to 0. Therefore $\frac{16-4}{2} = 6$ entries must be equal to 0. Now let's examine the general case of an $n \times n$ matrix. We follow the same reasoning as before. There are n^2 total entries, so the number of entires which must be zero must be less than that. We subtract out the diagonal elements, to create two symmetric halves. But how many diagonal elements are there in an $n \times n$ matrix? Well, each row (or equivalently each column) has exactly one diagonal element in it. E.g. in the kth row , the diagonal element is $a_{k,k}$. Since there are n rows, and each row has exactly one diagonal element, we have that there are n diagonal elements. So of the $n^2 - n$ remaining elements, exactly half (those below the main diagonal) must be equal to 0. I.e.

$$\frac{n^2 - n}{2} = \frac{n\,(n-1)}{2}$$

elements must be equal to 0.

5. Consider an $n \times n$ upper triangular matrix as before. We write the element at row i and column j as $a_{i,j}$. We intepret $a_{i,j}$ as the pairing of students i and j. All possible pairs $a_{i,j}$ of the n students are represented in matrix below

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} & \cdots & a_{2,n} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & \cdots & a_{3,n} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} & \cdots & a_{4,n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & a_{n,3} & a_{n,4} & \cdots & a_{n,n} \end{bmatrix}$$

However, some of the pairs are nonsensical. E.g. $a_{1,1,}$ represents pairing student 1 with themselves. Additionally, $a_{i,j}$ and $a_{j,i}$ represent the same pairing (since ordering doesnt matter pairing student i with j is the same as pairing student j with i). Thus we need to eliminate some entries from A to avoid doublecounting, and singleton pairs (e.g. $a_{1,1}$). Which entries to we eliminate? Well all diagonal entries are invalid since they are of the form $a_{i,i}$ and thus represent one person "pairs". Additionally entries $a_{i,j}$ and $a_{j,i}$ (which represent doublecounting) are across from each other diagonally. Thus, we only need to count the entries marked as 0 in the matrix below:

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & \cdots & a_{1,n} \\ 0 & a_{2,2} & a_{2,3} & a_{2,4} & \cdots & a_{2,n} \\ 0 & 0 & a_{3,3} & a_{3,4} & \cdots & a_{3,n} \\ 0 & 0 & 0 & a_{4,4} & \cdots & a_{4,n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_{n,n} \end{bmatrix}$$

How many zeros are there in matrix A above? Exactly the same number of entries which must be zero in an $n \times n$ upper triangular matrix. We showed in the last exercise that this number was $\frac{n^2-n}{2} = \frac{n(n-1)}{2}$. Again, we have simply taken all n^2 elements, removed the n diagonal elements, creating two equal halves, from which we select only the lower half in order to avoid double counting.

6. If you've taken a basic course in statistics, you may recall that the answer to the previous question can also be expressed as a number written $\binom{n}{2}$, and read "n choose 2". But what does $\binom{n}{2}$ mean? One interpretation is that $\binom{n}{2}$ is the number of 2 element subsets of an n-set (a set with n elements). However, we will focus on the algebraic definition which is that

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

where $n! = n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 3 \cdot 2 \cdot 1$. For example, $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$. So in the special case of $\binom{n}{2}$, we have

$$\begin{aligned}
\binom{n}{2} &= \frac{n!}{2!\,(n-2)!} \\
&= \frac{n!}{2\,(n-2)!} \\
&= \frac{n\,(n-1)\,(n-2)\,(n-3) \cdot \cdots \cdot 3 \cdot 2 \cdot 1}{2\,(n-2)\,(n-3) \cdot \cdots \cdot 3 \cdot 2 \cdot 1} \\
&= \frac{n\,(n-1)}{2}
\end{aligned}$$

which is the same as our result from the previous problem.