Introduction to Mathematical Logic

Evan Russenberger-Rosica

Contents

Preface

Congratulations; If you're reading this and you're human, then hundreds of millions of years of evolution have gifted you the most advanced logical computer known to exist: the human brain. Indeed, humans have an innate logical ability passed down by our ancestors whose very survival depended on the ability to observe and understand cause and effect relations. Today, even young children are able to understand certain logical statements. They understand the conditional statement that a hot stove will burn them if and only if they touch it. They also correctly deduce that if they do not touch a hot stove, then that stove will not burn them. However, if you asked that child to precisely explain the process of steps made in that deduction, he would likely be at a loss. This not because he is unintelligent, in fact quite the opposite, merely that he uses logic implicitly. However to truly understand the subject, as well as for the purpose of conversing with computers, which do not have our innate logical abilities, we must seek to formalize our thinking process. To this end, we have constructed symbolic logic as a "mathematical model of deductive thought." [**5**, ix] Thus to study logic is to study the process of thought itself at the meeting point between math and philosophy.

There are, of course many texts available on the subject of mathematical logic. My goals in adding to this bounty are to address the following deficiencies in traditional logic textbooks:

(1) The PhD authors of logic textbooks tend to be decades removed from their introduction to the subject, making it hard to anticipate the questions of students. Subjects which seem straightforward to the professional may actually be confusing to the beginner.
(2) Many texts either are either precise or comprehensible. This book seeks to strike a balance through the use of many clear examples, diagrams, and exposition.
(3) Standard logic texts tend to take an abstract approach to the subject, making it seem somewhat removed from reality. We address this shortfall by implementing logical concepts in the Python programming language in the second half of this book.

That said, we encourage the reader to examine other excellent texts such as [**15**][1], [**8**], [**17**], or the first three chapters of [**13**]. Engaging with several different logic texts may benefit the reader as their degree and quality of exposition varies by topic. Moreover, multiple different explanations of a topic can help eliminate any misunderstandings or ambiguities that the reader may have.

[1] Professor Sundstrom has graciously made his text freely available here: http://scholarworks.gvsu.edu/books/7/

CHAPTER 1

Propositional Logic

When we first begin the study of logic, we are principally concerned with what is known as propositional logic, propositional calculus, sentential logic, or zero-th order logic. This branch of logic is concerned with the truth value of propositions or statements.

Definition 1.0.1. In logic, a **proposition** or **statement** is a "declarative sentence that is either true or false but not both. The key is that there must be no ambiguity. To be a statement, a sentence must be true or false, and it cannot be both". [**15**, 1]

Example 1.0.2. A sentence such as "The sky is beautiful" is not a statement since whether the sentence is true or not is a matter of opinion. A question such as "Is it raining?" is not a statement because it is a question and is not declaring or asserting that something is true." [**15**, 1]. The sentences $1+1=2$ and $1+1=3$ are examples of true and false propositions respectively. Note that $1+x=2$ is not a statement, since it may be either true or false depending on the value of x.

Just as letters are used to represent numerical values in algebra, (e.g. $x=2$) logicians also use (typically capital) letters to represent propositions; P, Q, R, S are especially common. When used in this manner, such letters are known as **propositional variables**.

1.1. Connectives and Compound Statements

In this section, we discuss how statements can be joined or "glued" together by **logical connectives** to form new **compound statements**.

Definition 1.1.1. A **logical operator** or **connective** is a symbol, word, or combination of words that is used together with one or more mathematical statements to create a new statement whose truth value depends only on the connective used and the truth values of the original statements. The new statements formed by the use of connectives are known as **compound statements**. [**15**, 33] To emphasize that some statement P should be viewed as part of a larger compound statement R, we shall call statement P a component or **sub-statement** of R. Statements which cannot be decomposed further are known as **atomic**.

There are four basic connectives used in mathematics. We summarize these connectives, their symbols, and the compound statements they create below.

Note that the negation connective only takes a single proposition as an argument, while the other connectives take two. Thus, the negation operator constructs a compound proposition from a single existing proposition, whereas the other operators require two existing propositions to form a compound proposition. For this

Connective	Symbol	Written	Read
Conjunction	$\wedge$	$P \wedge Q$	P and Q
Disjunction	$\vee$	$P \vee Q$	P or Q
Negation	$\neg$	$\neg P$	not P
Implication	$\rightarrow$	$P \rightarrow Q$	If P then Q, P implies Q

TABLE 1. The Basic Connectives and their Compound Statements

P	Q	$P \wedge Q$	$P \vee Q$	$P \rightarrow Q$	$\neg P$
T	T	T	T	T	F
T	F	F	T	F	F
F	T	F	T	T	T
F	F	F	F	T	T

TABLE 2. Truth tables for the four basic connectives

reason, the negation connective is said to be **unary statement**, while the others are **binary statements**.

EXAMPLE 1.1.2. The conjunction of P and Q, written $P \wedge Q$, is a compound statement containing one operator: $\wedge$, and two component statements, P, Q. Observe that the component statements of a compound statement may themselves be compound statements.

1.2. Truth Values

In 1.0.1, we defined statements to have truth values, specifying that a statement is either true or false but not both. Since compound statements are themselves statements, they too must be either true or false but not both. The truth value of a compound statement as a whole depends entirely on the truth values of its component statements and the logical connective used. A compound statement which is always true is known as a **tautology** and is written T. A proposition which is always false is known as a **contradiction** and written F.

DEFINITION 1.2.1. **Truth tables** illustrate how the truth value of a compound statement as a whole depends on the truth values of each of its component statements.

Each logical connective, when supplied with the appropriate number of propositions as arguments generates/constructs a new compound statement. For instance, the connective $\vee$ supplied with the arguments P, Q, generates the statement $P \vee Q$. We define each of the the four basic logical connectives by the relationship they create between the truth values of their arguments and the compound statements they produce. For instance, the logical Disjunction $\vee$ is defined such that the compound statement $P \vee Q$ is false precisely when P and Q are false.

DEFINITION 1.2.2. We present truth tables for the four basic connectives $\wedge, \vee, \neg \rightarrow$.

Note that since the negation connective is unary, the compound statement $\neg P$ only contains one substantiate P. In contrast, as the other 3 basic compound statements which are generated by binary connectives each contain two sub -statements, P, Q.

For instance, the first row of the truth table for $P \wedge Q$ states that when P is True and Q is True, the compound statement $P \wedge Q$ is also True. From the other rows, we see that for all other truth values of P and Q, $P \wedge Q$ is False.

From the truth table, we can see the compound statement $P \vee Q$ is false precisely when both P and Q are false, (and therefore true in all other cases). This is the **inclusive or**, so called because $A \vee B$ is true in the case that both A and B are true, that is $A \vee B$ includes the possibilty that A and B are both true. In conversation, we sometimes say "A and/or B" in order to emphasize that we are using the inclusive or. If we wish to exclude the possibility and A and B are both true, we may use the **exclusive or** operation, indicated by $\oplus$. The inclusive and exclusive or operations are identical except the exclusive or $\oplus$ is false when *both* A and B are true. The subtle distinction between the exclusive and inclusive or operations can lead to confusion. In mathematics, when we say a or b, we always mean the inclusive or $\vee$, unless we explicitly state otherwise. In general speach however, people are not always careful to make the distinction between the inclusive and exclusive or. Thus, we must use contextual clues to discern whether or not the speaker intends the inclusive or exclusive or. For example, a game show might tell a contestant that they can keep their money or try to double it at the risk of losing it all. Clearly, they do not mean to allow the possibility that a contestant can keep their money, *and* try to double it and thus they are using the exclusive or. In contrast, if a weather forecaster states that it will be cold or rainy tomorrow, we are somewhat at a loss. It is certainly meteorologically possible for it to be both cold and rainy, thus this statement is somewhat ambiguous. For this reason, we sometimes say A and/or B in order to make it explicit that we are using the inclusive or in conversation.

Logical connectives and truth tables enable us to abstract verbal and mathematical statements down to their logical structure, which simplifies analyzing such statements. Indeed, the logical variables P, Q can stand for any statement, we shall see in the example below:

EXAMPLE 1.2.3. The truth table for $P \vee Q$ could represent the statement that "I have studied calculus or I have studied algebra". This compound statement consists of two sub-statements, "I have studied algebra" which we shall call P, and "I have studied calculus", which we shall call Q.

1.3. Conditional Statements

Conditional statements are a family of compound statements which involve the implication connective. The most common type of conditional statements are of the form "if P then Q," written $P \to Q$. Statement P is known as the **anteceedent** or **hypothesis** and Q is known as the **consequent** or **conclusion** of the conditional statement. Observe from table 2 on the facing page that the compound statement $P \to Q$ is false precisely when P is true and Q is false, and true otherwise.

EXAMPLE 1.3.1. The truth table for $P \to Q$ could represent the compound statement that "if it is raining tomorrow then I will bring an umbrella". This statement consists of two sub-statements, so that it has the form if (sub-statement one), then (substantiate two). We see that the two sub-statements are related to one another by the implication connective. From the truth table for $P \to Q$, we see that this statement is true for all values of sub-statements P, Q, except when the antecedent or hypothesis "it is raining tomorrow" is true, and the consequent or conclusion, "I will bring an umbrella" is false.

A	B	$A \to B$	$B \to A$	$A \leftrightarrow B$
T	T	T	T	T
F	T	T	F	F
T	F	F	T	F
F	F	T	T	T

TABLE 3. The Biconditional Connective

We shall now see how we can use all the connectives to create a more complex compound statement.

EXAMPLE 1.3.2. Consider the statement "if it is raining or snowing then I will get wet and I will be unhappy." By replacing each statement with a logical variable, we can more easily understand its meaning. Let R be the statement "it is raining", S be the statement "it is snowing", W be the statement "I will get wet", and H be the statement "I will be happy," (and thus $\neg H$ means I will be unhappy). Then we can rewrite the statement above as

$$(R \vee S) \to (W \wedge \neg H)$$

DEFINITION 1.3.3. Vacuous Truth

We saw earlier that $P \to Q$ is false precisely when P is true and Q is false, and true in all other cases. If P is false, the statement $P \to Q$ is said to be **vacuously true**. We use the modifier vacuous meaning: "emptied of or lacking content" or "marked by lack of ideas or intelligence" because although the statement is defined to be true in this case, the fact that the antecedent is false prevents us from using the conditional statement to infer anything about the truth value of the consequent.[**12**] For example, in later chapters, the logical rule known as modus ponens allows us to argue $((A \to B) \wedge A) \to B$. However, if $\neg A$, we cannot make any claims about B, that is $((A \to B) \wedge \neg A) \not\to B$. The important thing to remember is that in logic, statements of the form $A \to B$ do not imply and causal relationship between the two propositions. Later, in example 3.1.6 we will see an excellent example of vacuous truth in our proof that the empty set is a subset of every set.

EXAMPLE 1.3.4. The statement "if the author is a helicopter, then he will be given \$1,000,000 is a (vacuously) true statement, because the author is not a helicopter. The notion of vacuous truth often comes into play in statements that asset that all members of the empty set have a certain property. For example, we can use vacuous truth to prove that for any set A, $\emptyset \subseteq A$. By the definition of a subset *LINK* this means we must show every element of $\emptyset$, is an element of A. Since $\emptyset$ has no elements, this is vacuously true. it is vacuously true that they are all in A. Thus $\emptyset \subseteq A$.

DEFINITION 1.3.5. The Biconditional Statement

In mathematics and real life, we often wish to express statements of the form $(A \to B) \wedge (B \to A)$. To condense this statement, we introduce the **biconditional** connective $\leftrightarrow$, so that $A \leftrightarrow B$ has the same meaning as $(A \to B) \wedge (B \to A)$. We show in the truth table below that $A \leftrightarrow B$ means A always has the same truth value as B; if A is true, B is true, if A is false, B is false, and vice versa.

It is important to note that in natural language, the biconditional is often implicit. For instance, a parent might tell a child that if they do their chores, then

Statement	Form
Implication	$P \to Q$
Contrapositive	$\neg Q \to \neg P$
Converse	$Q \to P$
Inverse	$\neg P \to \neg Q$
Biconditional	$P \leftrightarrow Q, P \to Q \wedge Q \to P$

Table 4. Important Conditional Statements

they will get their allowance, which we can represent by $C \to A$. By the definition of the conditional statement, (see the truth table), $C \to A$ is also (vacuously) true when C is false. A precious child might then (correctly) argue that they can skip their chores and still receive allowance. To make it explicit that a child who does not do chores will not get allowance, a parent should use "if and only." For this reason, lawyers are careful to distinguish between a conditional and a biconditional statement. Note that when defining some term, is is customary for mathematicians to write if, when they really mean if and only if.

Definition 1.3.6. Converse, Contrapositive and Inverse
Given the conditional statement $P \to Q$, the statements $Q \to P$, $\neg Q \to \neg P$, and $\neg P \to \neg Q$ are known as the **converse**, **contrapositive**, and **inverse** of $P \to Q$ respectively.

Example 1.3.7. Suppose we have the conditional statement "If my pet is Golden Retriever, then it is a dog." Then:

(1) The contrapositive says, if my pet is not a dog, then it is not a Golden Retriever
(2) The converse says if my pet is a dog, then it is a Golden Retriever
(3) The inverse says if my pet is not a golden retriever then it is not a dog
(4) The biconditional says "if that animal is a golden retriever then it is a dog, and if that animal is a dog then it is its golden" retriever. We abbreviate this by saying that animal is a Golden Retriever if and only if it is a dog. The phrase "if and only if" is itself often abbreviated by "iff."

You may notice some of these statements seem to express the same idea. For instance the original statement, and its contrapositive seem to be saying the same thing. In the next section, we will make this idea of similar statements precise by defining logical equivalence.

1.4. Logical Equivalence

One of the most fundamental of all reasoning skills is the ability to recognize the property of sameness. An infant recognizes that the round peg goes in the round hole because they have the same shape. In algebra, we recognize that for any numbers a, b if $a = b$, then $f(a) = f(b)$ for *any* function f. In logic then, it is important to have an expression for denoting when two compound statements are the same.

Definition 1.4.1. A Tautology, Logical Equivalence

A statement which is true under all possible truth values of its sub-statements is a **tautology** and is represented by T. Propositions p and q are called **logically**

A	$\neg A$	$A \vee \neg A$
T	F	T
F	T	T

TABLE 5. The Law of the Excluded Middle

equivalent if $p \leftrightarrow q$ is a tautology. We express this by writing $p \equiv q$ or $p \iff q$. [**13**, 26]

It is important to note that the symbol $\equiv$ is not a logical connective. It is merely shorthand for the statement that $P \leftrightarrow Q$ is a tautology.

THEOREM 1.4.2. *P and Q are equivalent iff their columns in a truth table are identical.*

PROOF. Suppose that P and Q are logically equivalent. Then by definition it is the case that $P \leftrightarrow Q$ is a tautology. By the definition of the biconditional connective here 3, this means that P and Q always have the same truth value. Therefore their columns in a truth table must be identical.

Now suppose that P and Q have identical columns in a truth table, this means that their rows are identical. Recall that the material conditional is defined so that $P \to Q$ is false iff P is true and Q is false. So, suppose $P \to Q$ is false. Then there is some row where P is true and Q is false. Since the columns of P and Q are identical, this is a contradiction (i.e. an impossibility). Thus, if their columns are identical it must be the case that $P \to Q$ is always true, and by symmetry, $Q \to P$ is always true. Therefore $P \leftrightarrow Q$ is always true, i.e. a tautology, and therefore $P \equiv Q$. □

EXAMPLE 1.4.3. The statement $A \vee \neg A$, known as the **law of the excluded middle** is a simple but important tautology. In words, $A \vee \neg A$ tells us that for any proposition, either that proposition is true, or its negation is true. We can verify that $A \vee \neg A$ is a tautology by contructing a truth table and showing that its column is filled entirley with Ts.

Logical equivalencies are important in mathematical proofs and logical arguments as they allow us to substitute long, complex, or counterrinuitive statements with shorter, simpler, or more intuitive statements which are easier to prove. One of the more important equivalencies is that of a conditional statement with its contrapositive, see exercise 1.4.5. For example of a how a contrapositive can simply a statement, consider the statement "If an animal is not a dog, then the animal is not a golden retriever." The contrapositive of this statement is that if an animal is golden retriever then it is a dog, or more naturally, golden retrievers are dogs. The simplest way to verify whether or not two statements are equivalent is by comparing their truth tables.

EXAMPLE 1.4.4. $\neg(P \vee Q)$ and $\neg P \wedge \neg Q$ are logically equivalent

EXERCISE 1.4.5. Prove that $P \to Q$ is equivalent with its contrapositive $\neg Q \to \neg P$ with a truth table.

EXERCISE 1.4.6. Verify $P \to Q \equiv \neg P \vee Q$ with a truth table. This equivalence is known as the **material implication.**

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

TABLE 6. $\neg(P \vee Q) \equiv \neg P \wedge \neg Q$

EXERCISE 1.4.7. Verify that $P \oplus Q$ where $\oplus$ represents the exclusive or, is equivalent to $(P \vee Q) \wedge \neg(P \wedge Q)$

1.5. Logical Inference

DEFINITION 1.5.1. Argument, premises, conclusion, valid, rule of inference

An **argument** in propositional logic is a sequence of propositions that ends with a conclusion. A **rule of inference** is a simple argumemt which is frequently used to construct more complex arguments. In a logical argument, all but the final proposition are called **premises**, and the final proposition is called the **conclusion**. An argument is **valid** iff the truth of all its premises implies that the conclusion is true.[**13**, 70] That is, an argument with premises, $P_1, P_2, \ldots, P_n$ and conclusion Q is valid iff, $(P_1 \wedge P_2 \wedge \cdots \wedge P_n) \to Q$

Negating the definition, we see that an argument is invalid if $\neg[(P_1 \wedge P_2 \wedge \cdots \wedge P_n) \to Q]$ which is equivalent to $(P_1 \wedge P_2 \wedge \cdots \wedge P_n) \wedge \neg Q$. In words, this tells us that is an argument is valid iff it is impossible for all the premises to be true and the conclusion to be false.

Observe that conditional statement are defined so that if the hypothesis is false, then the conclusion is vacuously true, i.e., for any statement A, the conditional statement $\mathrm{F} \to A$ is always true. By De Morgan's laws, the statement negation of $(P_1 \wedge P_2 \wedge \cdots \wedge P_n)$ given by $\neg(P_1 \wedge P_2 \wedge \cdots \wedge P_n)$ is equivalent to $\neg P_1 \vee \neg P_2 \vee \cdots \vee \neg P_2$. This tells us that if any premise P_i is false, $(P_1 \wedge P_2 \wedge \cdots \wedge P_n) \to Q$ is automatically (vacuously) true, and therefore, we need only concentrate on the the case when $(P_1 \wedge P_2 \wedge \cdots \wedge P_n)$.

THEOREM 1.5.2. *Every valid argument is a tautology*

PROOF. Suppose that an argument has the form $P_1 \wedge P_2 \wedge \cdots \wedge P_n \to Q$. Let $P = (P_1 \wedge P_2 \wedge \cdots \wedge P_n)$ be the conjunction of all our premises. Thus our argument simplifies to $P \to Q$. Consider the case that P is true. Since the argument is valid, if P is true, then Q is true, so by modus ponens, we have Q. Thus $P \to Q$ is true. Now consider the case that P is false. Then $P \to Q$ is vacuously true. Since these cases are exhaustive, the theorem is proved. $\square$

Statements which are always false are known as **contradictions**, and are typically represented by F. If F is a contradiction then $(\neg P \to F) \to P$ is valid as it is a tautology. This particular tautology forms the basis for the method of proof by contradiction. This method states that if assuming P is false (i.e. $\neg P$) leads to a contradiction F, then P must be true.

EXAMPLE 1.5.3. Show that the compound statement known as **modus ponens** $[(P \to Q) \wedge P] \to Q$ is valid.

Suppose the premises are true, that is, suppose $(P \to Q)$ and P are true. Since P is true, since for all truth values of statements P, Q, the compound statement is true (see Exercise 1.8.5 on page 16).

We can prove this by contradiction. Suppose the premises are true and the conclusion is not, that is suppose $P \to Q$ and P but $\neg Q$. Since $\neg Q$, Q is false. Thus $P \to Q \equiv T \to F \equiv F$, which is a contradiction to our premise that $P \to Q$ is true.

Exercise 1.5.4. Show $(\neg P \to F) \to P$ is a tautology
By definition of $\to$, $(\neg P \to F) \to P$ is false iff $(\neg P \to F)$ is true, and P is false. But if P is false, $\neg P$ is true, and thus by definition $(\neg P \to F)$ is false. Thus $(\neg P \to F) \to P$ is (vacuously) true by definition.

Exercise 1.5.5. Show that $P \to Q \equiv (P \wedge \neg Q) \to F$. This logical equivalence is the basis of the proof strategy called *proof by contradiction*.

Solution. Suppose that $P \to Q$ is true. Then Therefore $\neg(P \wedge \neg Q)$ is false. Thus $(P \wedge \neg Q) \to F$ is vacuously true.

Suppose that $(P \wedge \neg Q) \to F$ is true. Then it must be that $(P \wedge \neg Q)$ is false, *i.e.*, $\neg(P \wedge \neg Q)$. By Demorgan, we have $\neg P \vee Q$, which by the material condition is equivalent to $P \to Q$

1.6. Proof Strategies

Having studied the process of forming valid arguments, we can now examine how to use these arguments to "prove" statements. Some of these arguments are so important that they have their own names:
Proof by Contradiction: $(\neg P \to F) \to P$
Proof by Cases: $(P_1 \wedge P_2 \wedge \cdots \wedge P_n) \to Q$

Exercise 1.6.1. Prove the rule of inference known as disjunction introduction: $A \to A \vee B$

Proof. We use a proof by cases.

Case 1. Suppose A is true. Then by definition, $A \vee B$ is true.
Case 2. Suppose that A is false. Then by definition of $\to$, $A \to A \vee B$ is (vacuously) true.

□

1.6.1. Proving If-then statements. To prove an if then statement, we must show that it is true in all cases, i.e. it can never be false. By definition if-then statements are true in 3 of the 4 possible cases; they can only be false if the hypothesis is true and the conclusion is false. Thus we only need to verify that this cannot occur, i.e. that if the hypothesis is True, the conclusion must also be True. This means that we may assume or imagine the hypothesis is true, and deduce from it.

We can draw a parallel with the more familiar idea of a hypothetical question in which we are told to imagine a given situation is true and then answer the question. For instance, someone might ask, "if you had a million dollars, would you buy a boat?" The proper response is not, "but I don't have a million dollars," the listener must imagine that the hypothesis is true and then answer.

Example 1.6.2. Prove by contradiction that $\emptyset \subseteq A$ for all sets A

Suppose that the theorem is false. Then there exists A such that $\emptyset \not\subseteq A$. Then there exists and $x \in \emptyset$ which is not in A. This contradicts the fact that $\emptyset$ does not contain any elements. Therefore the theorem cannot be false.

Example 1.6.3. Prove that $a \geq 0 \to b - a \leq b$ by contradiction, and directly. (This theorem says that

(1) By contradiction assume the negation of the given theorem; that $a \geq 0 \wedge b - a > b$. Subtracting b from both sides of $b - a > b$, we get $-a > 0$. Adding a to both sides, we get $0 > a \perp$.
(2) To prove this directly, we may assume the hypothesis of the theorem: $a \geq 0$. Then $a + b \geq b$. Then $b \geq b - a$.

Example 1.6.4. Prove that the product of any positive number with any negative number is negative.

We can write this theorem logically as $\forall a > 0, b(b < 0 \to ab < 0)$ for $a, b \in \mathbb{R}$ (equivalent formulations are discussed in 2.6). Then the negation is $\exists a > 0(b < 0 \wedge ab > 0)$. Then we can divide both sides of $ab > 0$ by a (which we can do since $a > 0$, and thus $a \neq 0$), showing $b > 0$, which gives $b < 0 \wedge b > 0 \perp$. Thus the given theorem cannot be false.

Example 1.6.5. Proving $1 + 1 = 2$ by contradiction

Assume $1 + 1 \neq 2$. Then subtracting 1 from both sides, we get $1 \neq 1$, which is a contradiction (a statement which is always false). Thus $1 + 1 = 2$.

Remark 1.6.6. Be careful not to treat statements of the form $a \neq b$ like an equation; non-equality does not have the same properties as equality. For instance, for any $a, b \in \mathbb{R}$, if $a = b$, then multiplying both sides by a we get $a \cdot a = a \cdot b \overset{\star}{=} b \cdot b = b^2$, where equality $\star$ comes from the substitution $b = a$. Thus $a = b \to a^2 = b^2$. However, $a \neq b \not\to a^2 \neq b^2$, ($\not\to$ means "does not imply") . For instance $-1 \neq 1$, but $-1^2 = 1^2$. Given this information, the reader may then ask if example 2.16 is truly valid; i.e. is it valid to subtract 1 from both sides of $1 + 1 \neq 2$ to deduce $1 \neq 1$, or in other words, is the implication $1 + 1 \neq 2 \to 1 \neq 1$ valid?

Using the fact that a conditional statement is logically equivalent to its contrapositive, which the reader verified in Exercise 1.4.5 on page 10, we can prove that this this subtraction is valid. In fact, we can prove by contraposition the general case that for any $a, b, c \in \mathbb{R}$, $a \neq b \to a - c \neq b - c$. The contrapositive of $a \neq b \to a - c \neq b - c$ is $a - c = b - c \to a = b$. Proving this implication is simple, as if $a - c = b - c$, then adding c to both sides gives $a = b$. Since a conditional statement has the same truth value as its contrapositive, we have that proved that $a \neq b \to a - c \neq b - c$ as desired. Note that in Example 1.4.5 on page 10 $a = 1 + 1, b = 2, c = 1$.

Exercise 1.6.7. Prove for any $a, b, c \in \mathbb{R}$ that $a \neq b \to ca \neq cb$.

The contrapositive of $a \neq b \to ca \neq cb$. is $ca = cb \to a = b$. Since we may assume the antecedent/hypothesis $ca = cb$, simply multiplying both sides of this equation by $\frac{1}{c}$ gives $a = b$ as desired.

1.6.2. Proving Biconditionals.

1.7. Logical Fallacies

In the previous section, we studied valid rules for logical inference. In this section, we discuss some invalid "rules" known as logical fallacies, which are so common they have their own names. We do this so that you can be extra careful to avoid them yourself, and so that you may be on alert to their (mis-)use by others.

Recall that in a conditional statement $P \to Q$, the first statement, P in this case, is known as the hypothesis or antecedent, and the second, Q, is known as the conclusion or consequent. Affirming a disjuncture is the statement that, $(A \vee B) \wedge A \to \neg B$. In words this states that if A or B must occur, and A has occurred, then B cannot also occur. However, we know that the inclusive or is defined to that both A and B can occur. Thus affirming a disjuncture is the fallacy of equivocation between the logical OR and XOR operations.

Exercise 1.7.1. Show that $(A \vee B) \wedge A \not\to B$, but $(A \oplus B) \wedge A \to \neg B$. Hint: start by rewriting $A \oplus B$ in terms of basic logical operators.

TODO: List of fallacies

- Affirming the consequent: This is the logical fallacy that $[(P \to Q) \wedge Q] \to P$
- Denying the antecedent: This is the logical fallacy that $(P \to Q) \to (\neg P \to \neg Q)$

1.8. Boolean Algebra

Until now, we have been proving logical equivalences with truth tables. While these work fine for simple statements, they become onerous with more complex statements. We shall now discuss how to manipulate Boolean (i.e. true/false) expressions in an algebraic way. This method of manipulation is named "Boolean Algebra" after its inventor, George Boole. In order to accomplish this, we must we discuss the properties of logical operators. To this end, we introduce the symbols $\cdot$ and $+$ to represent abstract binary logical operators.

1.8.1. Associativity and Commutativity. Associativity tells us that the order of the logical operators can be switched, i.e., in an expression $A \cdot B \cdot C$, we can perform $A \cdot B$ or $B \cdot C$ first and achieve the same result. Thus the operation $\cdot$ is associative iff:

$$(A \cdot B) \cdot C \equiv A \cdot (B \cdot C)$$

When a binary operation is associative, it is not usual to omit the parenthesis. The reason being that because the operation is associative, it doesn't matter which operand we evaluate first. For example, the multiplication of real numbers is associative, so we would not be surprised to see one write $4 \cdot 5 \cdot 6$ instead of $(4 \cdot 5) \cdot 6$ is Recall that $\cdot$ is a binary operator, so it takes 2 arguments. Therefore unless we have agreed on an order of operations, statements such as $A \cdot B \cdot C$ are strictly speaking not well defined, however if $\cdot$ is associative, we

We say that a particular operation is **commutative** if the result of the operation does not depend on order of the operands (operands are the arguments to an operator). That is the operation $\cdot$ is associative iff

$$A \cdot B \equiv B \cdot A$$

A	B	C	$A \wedge B$	$B \wedge C$	$(A \wedge B) \wedge C$	$A \wedge (B \wedge C)$
T	T	T	T	T	T	T
T	T	F	T	F	F	F
T	F	T	F	F	F	F
T	F	F	F	F	F	F
F	T	T	F	T	F	F
F	T	F	F	F	F	F
F	F	T	F	F	F	F
F	F	F	F	F	F	F

TABLE 7. Logical conjunction $\wedge$ is Associative

THEOREM 1.8.1. *The logical conjunction operator $\wedge$ is associative*

PROOF. We can prove associativity using a truth table, although it is a bit tedious.

Alternative, we can apply the definition of equivalence and show that for any A, B, C, it is always the case (i.e., it is a tautology that) $(A \wedge B) \wedge C \leftrightarrow A \wedge (B \wedge C)$. By the definition of $\leftrightarrow$, this means we must show that $(A \wedge B) \wedge C \to A \wedge (B \wedge C)$ and $(A \wedge B) \wedge C \leftarrow A \wedge (B \wedge C)$.

Suppose $(A \wedge B) \wedge C$ is true. By definition of $\wedge$, this is true iff $(A \wedge B)$ and C are all true. Since $A \wedge B$ is true, A and B must be true. Thus we have A,B,C are all true. Since B and C are true, $B \wedge C$ is true, and since $B \wedge C$ is true and A is true, we have that $A \wedge (B \wedge C)$ is true.

Suppose that $A \wedge (B \wedge C)$ is true. Then A and $B \wedge C$ are true. Since $B \wedge C$ is true B and C are true. The A, B, and C are all true. Therefore $A \wedge B$ is true. Since we showed that C is true, we have that $(A \wedge B) \wedge C$ is true. □

EXERCISE 1.8.2. Show the logical disjunction operator $\vee$ is associative and commutative

1.8.2. Distributive Laws.

$$(A \vee B) \wedge C \equiv (A \wedge C) \vee (B \wedge C)$$

$$(A \wedge B) \vee C \equiv (A \vee C) \wedge (B \vee C)$$

These look very similar, and it can be hard to remember which way the $\wedge, \vee$ symbols go. The best way to remember is to try out sample statements. For instance, if A means "my pet is a dog", B means "my pet is a cat", and C means "my pet is black," the statement $(A \vee B) \wedge C$ means my white pet is either or a dog or a cat. Clearly the per must be either a black dog or a black cat, that is $(A \wedge C) \vee (B \wedge C)$.

1.8.3. De Morgan's Laws .

De Morgan's laws tell us how to take the negation of conjunctions and disjunctions. They are named after Augustus De Morgan, a 19th-century British mathematician and can be verified via truth table.

$$\neg(A \vee B) \equiv \neg A \wedge \neg B$$
$$\neg(A \wedge B) \equiv \neg A \vee \neg B$$

Example 1.8.3. In an episode of the Simpsons, Homer visits the website of the Springfield Police Department. The site asks him if it is true or false that he "has committed a crime and would like to confess." Homer selects false, and is told that this means he had committed a crime and does not want to confess, and that the police are on their way. But is this really the case? Let us examine the situation. Homer has been told

$$\neg(Crime \wedge Confess) \equiv Crime \wedge \neg Confess$$

However, using De Morgan's laws, we can see $\neg(Crime \wedge Confess) \equiv \neg Crime \vee \neg Confess$, not $Crime \wedge \neg Confess$

Exercise 1.8.4. Prove $(A \vee B) \wedge \neg A \to B$

By distributive laws, $(A \vee B) \wedge \neg A \equiv (A \wedge \neg A) \vee (B \wedge \neg A) \equiv F \vee (B \wedge \neg A) \equiv B \wedge \neg A \to B$. If we let A be the statement "My pet is a cat" and B be "My pet is a cat", then this could represent the logic, "my pet is either a cat or a dog, and its not a cat, therefore my pet is a dog."

The previous theorem has many neat applications. Suppose you want to prove $A \subseteq B \wedge A \neq B \to B \not\subseteq A$. Doing algebra, we get $A \neq B \iff \neg(A \subseteq B \wedge B \subseteq A) \equiv \neg(A \subseteq B) \vee \neg(B \subseteq A)$. Thus we have $[\neg(A \subseteq B) \vee \neg(B \subseteq A)] \wedge (A \subseteq B) \equiv [[\neg(A \subseteq B) \wedge (A \subseteq B)] \vee [\neg(B \subseteq A) \wedge (A \subseteq B)] \equiv F \vee [\neg(B \subseteq A) \wedge (A \subseteq B)] \implies \neg(B \subseteq A)$.

Alternatively, we could prove this by contradiction: assume $(A \subseteq B) \wedge (A \neq B) \wedge \neg(B \not\subseteq A)$. But $\neg(B \not\subseteq A) \equiv B \subseteq A$. Thus by commutative, we have $(A \subseteq B \wedge B \subseteq A) \wedge A \neq B$. But by definition, $(A \subseteq B \wedge B \subseteq A) \equiv A = B$. Thus we have a contradiction $A = B \wedge A \neq B$, and thus the given theorem cannot be false.

Example 1.8.5. Verify $[(P \to Q) \wedge P] \to Q$ is a tautology algebraically

$$\begin{aligned}
[(P \to Q) \wedge P] \to Q &\equiv ((\neg P \vee Q) \wedge P) \to Q \\
&\overset{DL}{\equiv} [(\neg P \wedge P) \vee (Q \wedge P)] \to Q \\
&\equiv [F \vee (Q \wedge P)] \to Q \\
&\equiv (Q \wedge P) \to Q \\
&\equiv \neg (Q \wedge P) \vee Q \\
&\equiv \neg Q \vee \neg P \vee Q \\
&\equiv \neg Q \vee Q \vee P \\
&\equiv T \vee P \\
&\equiv T
\end{aligned}$$

Exercise 1.8.6. Verify $(\neg P \to F) \to P$ is a tautology

$$\begin{aligned}
(\neg P \to F) \to P &\equiv (P \vee F) \to P \\
&\overset{DL}{\equiv} [(\neg P \wedge P) \vee (Q \wedge P)] \to Q \\
&\equiv [F \vee (Q \wedge P)] \to Q \\
&\equiv (Q \wedge P) \to Q \\
&\equiv \neg (Q \wedge P) \vee Q \\
&\equiv \neg Q \vee \neg P \vee Q \\
&\equiv \neg Q \vee Q \vee P \\
&\equiv T \vee P \\
&\equiv T
\end{aligned}$$

Name	Form
Modus Ponens	$[(P \implies Q) \wedge P] \implies Q$
Modus Tollens	$[(P \implies Q) \wedge \neg Q] \implies \neg P$
Hypothetical Syllogism	$[(P \implies Q) \wedge (Q \implies R)] \implies (P \implies R)$
Proof by Contradiction	$(\neg P \implies F) \implies P$ Where F is a contradiction (i.e. always False)
Material Implication	$P \to Q \equiv \neg P \vee Q$
Disjunction Introduction	$P \implies P \vee Q$
Conjunction Introduction	$P, Q \implies P \wedge Q$

Table 9. Valid Rules of Inference

1.8.4. Visualization. Mathematicians have constructed various ways to visualize logical relationships. One way to visualize and intuitively understand logical statements is by the use of Venn Diagrams. In a Venn diagram for a statement S, we represent each logical sub-statement A as a circle A'. The region inside A' is where statement A is true. The region outside A' is where statement A is false. Thus, for example, the intersection of two circles A' and B' represents that A and B are true there.

Note that 11, 13, and 15, are the negations of 10, 12, and 14 respectively. This can be observed from the diagrams from the fact that a diagram is inversely shaded from its negation.

Particularly interesting is the diagram for $A \to B$. The easiest way to understand this diagram is to remember that the proposition $A \to B$ is only false when A is true and B is false. Thus the only unshaded (i.e. false) areas, are those. The fact that the area outside of the circle A' is shaded represents the fact that when A is false, the conditional statement $A \to B$ is (vacuously) true.

Exercise 1.8.7. Construct a Venn Diagram for $A \oplus B$

Solution. Our diagram is reproduced below.

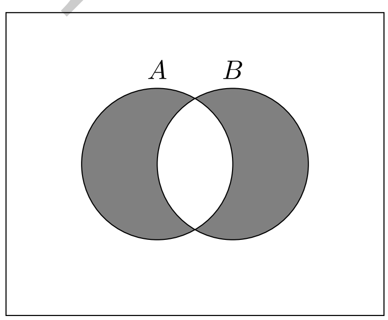

Figure 1.8.1. $A \oplus B$

1.8.5. Precedence of Logical Operators. When dealing with an expression which contains multiple operators, it is important to know which operators should be evaluated first, as this may affect the value of the expression. In general,

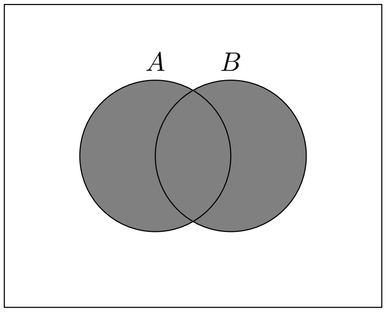

TABLE 10. $A \vee B$

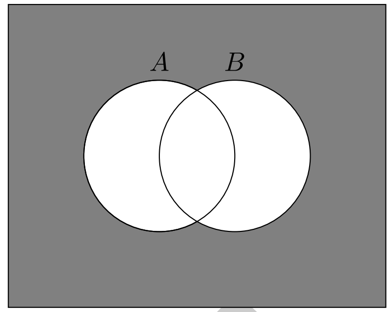

TABLE 11. $\neg A \wedge \neg B$

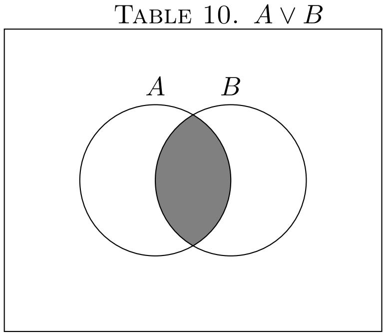

TABLE 12. $A \wedge B$

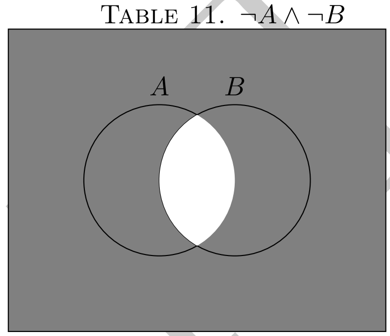

TABLE 13. $\neg A \vee \neg B$

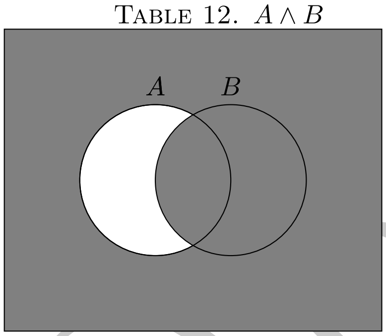

TABLE 14. $A \to B$

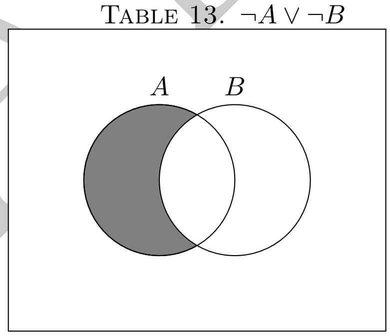

TABLE 15. $A \wedge \neg B$

TABLE 16. Statements (left column) and their Negations (right column)

when operation $\cdot$ should be evaluated before operation $+$, we say $\cdot$ takes **precedence** over $+$. For example, in algebra, the precedence of operators is **p**arentheses, **e**xponents, **m**ultiplication, **d**ivision, **a**ddition and **s**ubtraction, which we abbreviate by the acronym P.E.M.D.A.S. Such a sequence of operators in decreasing order of precedence is known as an **order of operations**. For example P.E.M.D.A.S. tells to evaluate division before subtraction, and thus

$$\begin{aligned} 2 + 4/2 &= 2 + 2 \\ &= 4 \end{aligned}$$

In contrast, if we evaluated addition first, we would have $2 + 4/2 = (2 + 4)/2 = 6/2 = 3$.

Having a fixed order of operations allows us to reduce the number of parentheses needed in a given expression. Thus we introduce the standard order of operations

Logic	NLS	NLS2	NLS3
$A \wedge B$	A and B	A but B	
$A \vee B$	A or B		
$\neg A$	not A		
$P \to Q$	P is sufficient for Q	If P then Q	P only if Q
$P \to Q$	Q is necessary for P	P implies Q	Q, if P
$P \leftrightarrow Q$	P precisely if Q	P if and only if Q	
$\neg B \to A$	A unless B	A, if not B	

TABLE 18. Natural Language and Logic Equivalents

for logical operators as negation, conjunction, disjunction, material conditional, and biconditional.[**13**, 11] To help remember this, we suggest the acronym NAOMI-B short for **n**ot, **a**nd, **o**r, **m**aterial **i**mplication[1], and **b**iconditional. This should be relatively easy to remember considering that Naomi is both a common name and a biblical figure. We summarize the logical order of operations in the following table.

Operator	Precedence
$\neg$	1
$\wedge$	2
$\vee$	3
$\to$	4
$\leftrightarrow$	5

TABLE 17. Precedence of Logical Operators

EXAMPLE 1.8.8. Precedence of Logical Operators

- $P \wedge Q \vee R \equiv (P \wedge Q) \vee R$
- $P \vee Q \wedge \neg R \equiv P \vee (Q \wedge \neg R)$
- $A \vee B \to C \equiv (A \vee B) \to C$

1.9. Natural Language and Logic

By design, the language of logic is simple and repetitive. However, the "natural language" in which we ordinarily speak is much more varied, and thus the reader may often need to translate between natural language and logic.

Here we list some common expressions in natural language and their logical equivalents.

1.9.1. Necessary and Sufficient Conditions. The terms **necessary** and **sufficient** are often used to describe conditional relationships between logical statements. Consider the statements "I am in Alabama" and "I am in the United States",

[1]The term *material implication* may be used in two different ways. Firstly, it is often treated as a synonym for the material conditional $\to$, which is how we use it here. Alternatively, it can be used to refer to the important equivalence $A \to B \equiv \neg A \vee B$. We take the latter view, but we make an exception here for purposes of creating a memorable acronymn. In any case, it should be clear from context whether by material implication an author means the connective $\to$ or the equivalence.

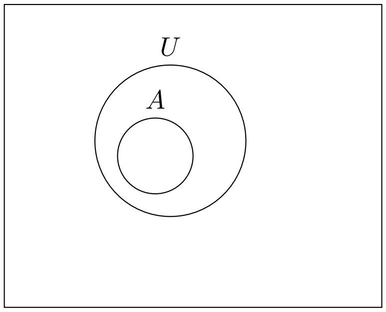

FIGURE 1.9.1. A is Sufficient for U

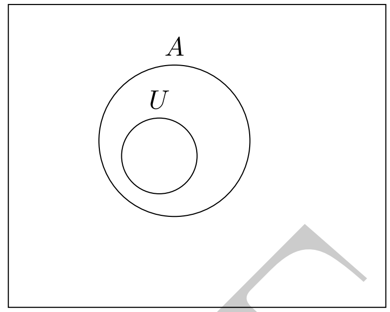

FIGURE 1.9.2. A is Necessary for U

which we'll abbreviate A and U respectively. From these two statements and the two terms, we can make 4 English sentences.

- Being in Alabama is a sufficient condition for being in the United States
- Being in the United States is a necessary condition for being in Alabama.

Although it may not initially appear so, both of these sentences express the same statement; $A \to U$. When we say "A is sufficient for U", we're trying to express the idea that knowing A is automatically enough (i.e, it is sufficient) to know U. That is, we're saying $A \to U$. When we say "U is necessary for A", we mean that if U is not true, then A cannot be true either, that is, we're saying $\neg U \to \neg A$. Note that$\neg U \to \neg A$ is the contrapositive of $A \to U$, and therefore these statements are equivalent. Thus "A is sufficient for U" and "U is necessary for A" express the same idea from different points of view.

- Being in the Unites States is a sufficient condition for being in Alabama.
- Being in Alabama is a necessary condition for being in the Unites States

Likewise, both of these sentences express the (false) statement that $U \to A$. The later statement expresses the contrapositive of the former.

We can use Euler diagrams to picture the the logical relationship between A and U.[2] In figure 1.9.1, we see A is entirely contained within (i.e., is a subset of) U. Thus if we imagine ourselves within A, we automatically have U as well since U completely surrounds A. This reflects the fact that A is sufficient for U, i.e. if we have A, we automatically have U as well. As discussed earlier, this diagram also represents the statement that "U is necessary for A", expressed as $\neg U \to \neg A$. To see this point of view, imagine that we are outside U in figure 1.9.1 (i.e. we are in $\neg U$), then clearly we cannot have A since A is entirely contained within U, thus $\neg A$. Thus we see how 1.9.1 represents both equivalent points of view.

If it is the case that A is both sufficient and necessary for U, we say that A is **necessary and sufficient** for U. Since this means $A \to U \wedge U \to A$, the statement that A is necessary and sufficient for U means that A and U are equivalent. Pictorially, this means the circles of A and U coincide (i.e., perfectly overlap) To

[2] In the language of the next chapter, the circles represent predicates $P(x)$. Under this interpretation, point x is contained within circle P if and only if $P(x)$ is true. For example, circles A and U (or more precisely $A(x)$ and $U(x)$) in figure 1.9.1 could represent the predicates "x is a power of 2" and "x is even" respectively. Thus $2, 4, 8, 16, 32, \ldots \in A$ and $2, 4, 6, 8, 10, 12, 14 \in U$, and $6, 10, 12, 14, 18, 0 \in U \backslash A$.

Sentence	Interpret As	Equivalent To
A is sufficient for U	$A \to U$	$A \to U$
U is necessary for A	$\neg U \to \neg A$	$A \to U$
U is sufficient for A	$U \to A$	$U \to A$
A is necessary for U	$\neg A \to \neg U$	$U \to A$
A is necessary and sufficient for U	$A \leftrightarrow B$	

TABLE 19. Necessary and Sufficient Conditions

avoid patronizing the reader, we omit this diagram. We summarize our discussion in 19.

1.10. Further Reading

- For a more extensive discussion of these topics including more examples, see [**13**, 1.1-1.3].
- For a more advanced discussion see
- For more information on logic circuits see: [**9**]

CHAPTER 2

First Order or Predicate Logic

There exist certain statements which cannot be represented using propositional calculus. The preceding sentence is really a statement; it is either true or false. However, it is also inexpressible using the tools of propositional calculus. To express and evaluate such statements, we come to **first order** or **predicate logic**. Predicate Logic keeps the machinery of predicate calculus, but introduces three new concepts: variables, quantifiers, and predicates. We shall begin by defining what exactly a predicate is.

DEFINITION 2.0.1. A **predicate** is a sentence $P(x_1, x_2, \ldots, x_n)$ involving variables with the property that when specific values from the universal set U are assigned to $x_1, x_2, \ldots, x_n$ then the resulting sentence is either true or false. That is, the resulting sentence is a statement. A predicate is also called an **open sentence**, or **propositional function**.[**15**, 56] Predicates involving one or two variables are said to be **unary** and **binary** respectively. For all other non-negative integers n, a predicate is said to be **n-ary**.

EXAMPLE 2.0.2. If the universal set is $\mathbb{R}$, then the sentence $4 = 2x + 2$ is a predicate in x. If we substitute $x = 1$, the predicate becomes the (true) statement $4 = 2 \cdot 1 + 2$. If we substitute any other x, we obtain a false statement.

2.1. Introduction to Quantifiers

We have seen that one way to create a statement from an open sentence is to substitute a specific element from the universal set for each variable in the open sentence. Another way is to make some claim about the truth set of the open sentence. This is often done by using a quantifier. For example, if the universal set is $\mathbb{R}$, then the following sentence is a statement.

$$\text{For each real number } x, x^2 > 0$$

The phrase "For each real number x" is said to **quantify the variable** that follows it in the sense that the sentence is claiming that something is true for all real numbers. Therefore, this sentence is a statement (which happens to be false since $0^2 = 0 \not> 0$).

DEFINITION 2.1.1. Existential and Universal Quantifiers

The phrase "for every" (or its equivalents "for each", "for all") is called a **universal quantifier**. The phrase "there exists" (or its equivalents "for some") is called an **existential quantifier**. The symbol $\forall x$ is used to denote a universal quantifier, and the symbol $\exists x$ is used to denote an existential quantifier. Generally, where $P(x)$ is a predicate, we can write $\forall x P(x)$ to mean that "for all x, $P(x)$ is true", and $\exists x P(x)$ to mean "there exists x such that $P(x)$ is true" where x is from some universal set X.

Example 2.1.2. Using quantifier notation, the statement "the square of every real number is non-negative" or equivalently "for each real number x, $x^2 \geq 0$" could be written in symbolic form as:

$$\forall x(x^2 > 0)$$

where the universal set $U = \mathbb{R}$. Likewise the statement "there exists an integer x such that $3x - 2 = 0$", can be written in symbolic form as:

$$\exists x \in \mathbb{Z}(3x - 2 = 0)$$

where the universal set $U = \mathbb{Z}$. We note that even though these are relatively simple mathematical statements, they cannot be expressed purely in terms of the propositional logic discussed in the previous chapter. Thus we can appreciate the additional power that predicate logic gives us over propositional logic.

As a way to visualize quantified statements, we introduce what we'll call **truth matrices**, which we can loosely think of as the first order logic analogues of truth tables. Recall that the element in the ith row and jth column of a matrix A is written as $a_{i,j}$. The truth matrix A of a unary predicate $P(x)$ only needs one row, thus we can form A by letting $a_{1,j} = P(x_j)$. In other words, we form A by letting the jth column equal the value of the predicate evaluated at x_j, which we denote $P(x_j)$. Since the value of a predicate is always either true or false, which we represent by T and F, these matrices are quite easy to construct.

$$\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & \cdots & x_n & \cdots \end{array} \\ \begin{bmatrix} P(x_1) & P(x_2) & P(x_3) & \cdots & P(x_n) & \cdots \end{bmatrix} \end{array}$$

Figure 2.1.1. A Truth Matrix for Predicate $P(x)$ and Universal Set X

Since the statement $\forall x P(x)$ means that for every $x \in X$, $P(x)$ is true, $\forall x P(x)$ is true iff every column of its truth matrix has a T.

$$\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & \cdots & x_n & \cdots \end{array} \\ \begin{bmatrix} \mathrm{T} & \mathrm{T} & \mathrm{T} & \cdots & \mathrm{T} & \cdots \end{bmatrix} \end{array}$$

Figure 2.1.2. $\forall x P(x)$

Likewise, since the statement $\exists x P(x)$ means that there is some $c \in X$ such that $P(c)$ is true, $\exists x P(x)$ is true iff some column of its truth matrix has a T.

$$\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & \cdots & x_n & \cdots \end{array} \\ \begin{bmatrix} F & F & \mathrm{T} & \cdots & F & \cdots \end{bmatrix} \end{array}$$

Figure 2.1.3. $\exists x P(x)$

We must be clear to note that unlike propositional logic, most problems of first order logic cannot be solved by simply examining a truth matrix. Indeed, for some types of infinite sets Ω, it is hard or impossible to even write a truth matrix for a subset of Ω. However as an introductory conceptual tool, they may be useful. We conclude this section with three definitions we will be useful going forward.

EXERCISE 2.1.3. Construct a truth matrix for the predicate $P(x)$ "x is prime" for the universal set $X = \{2, 3, 4, \ldots, 10\}$. Use this matrix to evaluate the truth of the statements $\forall x P(x)$, $\exists x P(x)$ are true and explain your solution.

DEFINITION 2.1.4. When a quantifier is used on the variable x, we say that this occurrence of the variable is **bound**. An occurrence of a variable that is not bound by a quantifier or set equal to a particular value is said to be **free**. A predicate with no free variables is a proposition or statement (i.e. a sentence which is true or false). The part of a logical expression to which a quantifier is applied is called the **scope** of this quantifier.[**13**, 44-45]

EXAMPLE 2.1.5. TODO ^example of above

2.2. Multiple Quantifiers and Variables

Statements may have many variables and many quantifiers, and the order in which they are written is important. To see the significance of the quantifier order, consider the following, where C is the set of cars, P is the set of people, and R is a relation such that cRp means car $c \in C$ is owned by person $p \in P$. Observe that this relation is a predicate of two variables $P(c, p)$.

$$\forall c \in C, \exists p \in P(cRp)$$

$$\exists p \in P, \forall c \in C(cRp)$$

Observe that in the first statement of the example, the universal quantifier precedes the existential quantifier. This statement means that for each car c there exists an owner p, or more naturally, each car c has an owner p. Observe that the person p depends on the car. In the second statement, the universal quantifier follows the existential quantifier. This statement means there is some person p who owns **every** car. Thus this person doesn't depend on the car (since he has all of them, or in other words; given any car c, he has it). To conclude, for any variables x, y, y can depend on x if and only if the universal quantifier $\forall x$ precedes the existential quantifier $\exists y$.

For a more abstract example, consider the definitions from real analysis given below:

$f : \mathbb{R} \to \mathbb{R}$ is continuous if: $\forall x \in \mathbb{R}, \forall \epsilon > 0, \exists \delta > 0, \forall y \in \mathbb{R}(|x-y| < \delta \implies |x, y| < \epsilon)$

$f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous if: $\forall \epsilon > 0, \exists \delta > 0, \forall x, y \in \mathbb{R}(|x-y| < \delta \implies |x, y| < \epsilon)$

Observe that in the definition of continuity, the universal quantifiers $\forall x \in \mathbb{R}$ and $\forall \epsilon$ precede$\exists \delta$. Thus here δ may depend on both x, ϵ. However, in the definition of uniform continuity, the only universal quantifier that precedes δ is $\forall \epsilon$. Thus δ may only depend on ϵ and not x. In other words, one δ must work for all x. Thus uniform continuity is a stronger notion than continuity (i.e., uniform continuity implies continuity). Observe that the order of adjacent universal quantifiers does not matter; i.e. they commute. For example the definition of continuity given above is equivalent to

$$\forall \epsilon > 0, \forall x \in \mathbb{R}, \exists \delta > 0, \forall y \in \mathbb{R}(|x - y| < \delta \implies |x, y| < \epsilon)$$

as all we have done is change the order of $\forall x \in \mathbb{R}, \forall \epsilon > 0$ to $\forall \epsilon > 0, \forall x \in \mathbb{R}$.

In the above example, there are quite a few quantifiers. Writing them as a sequence $(\forall, \forall, \exists, \forall)$ we have see that the quantifier $\forall$ is repeated twice at the beginning. To simplify notation, any time we have two or more quantifiers of the same type adjacent to each other, we can abbreviate by only writing the repeated quantifier once, followed by all the variables separated by commas.

EXAMPLE 2.2.1. An useful mathematical fact is that for any real numbers x, y, their arithmetic mean $\frac{x+y}{2}$ is always greater than or equal to their geometric mean $\sqrt{xy}$. To express their logically, we could write $\forall x, \forall y(\frac{x+y}{2} \geq \sqrt{xy})$, which is equivalent to $\forall x, y(\frac{x+y}{2} \geq \sqrt{xy})$.

EXAMPLE 2.2.2. We can simplify $\forall w > 0, \exists x, \exists y, \forall z(P(w,x,y,z))$ to $\forall x > 0, \exists y, z, \forall z P(w,x,y,z)$. Note that though this statement has two existential and two universal quantifiers, only the existential quantifiers are repeated (i.e. adjacent, back to back). Thus only the variables following the two existential quantifiers are able to be consolidated under one quantifier.

Since we have two variables, our truth matrices now need two dimensions. If $P(x,y)$ is a binary predicate on universal sets X, Y, then the value of the $a_{i,j}th$ element is $P(x_i, y_j)$.

$$\begin{array}{c c} & \begin{array}{ccc} y_1 & y_2 & y_3 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} & \begin{bmatrix} P(x_1,y_1) & P(x_1,y_2) & P(x_1,y_3) \\ P(x_2,y_1) & P(x_2,y_2) & P(x_2,y_3) \\ P(x_3,y_1) & P(x_3,y_2) & P(x_2,y_3) \end{bmatrix} \end{array}$$

FIGURE 2.2.1. Truth Matrix for Predicate $P(x,y)$ on X, Y

The statements $\forall x \forall y P(x,y)$, $\exists x \forall y P(x,y)$, $\forall x \exists y P(x,y)$, $\exists x \exists y P(x,y)$ created by quantifying the binary predicate $P(x,y)$ may be visualized by the following (not necessarily unique) truth matrices.

$$\begin{array}{c c} & \begin{array}{ccc} y_1 & y_2 & y_3 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} & \begin{bmatrix} \mathrm{T} & \mathrm{T} & \mathrm{T} \\ \mathrm{T} & \mathrm{T} & \mathrm{T} \\ \mathrm{T} & \mathrm{T} & \mathrm{T} \end{bmatrix} \end{array}$$

FIGURE 2.2.2. $\forall x \forall y P(x,y)$

$$\begin{array}{c c} & \begin{array}{ccc} y_1 & y_2 & y_3 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} & \begin{bmatrix} F & F & F \\ F & \mathrm{T} & F \\ F & F & F \end{bmatrix} \end{array}$$

FIGURE 2.2.3. $\exists x \exists y P(x,y)$

$$\begin{array}{c c} & \begin{array}{ccc} y_1 & y_2 & y_3 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} & \begin{bmatrix} \mathrm{T} & F & F \\ F & \mathrm{T} & F \\ F & F & \mathrm{T} \end{bmatrix} \end{array}$$

FIGURE 2.2.4. $\forall x \exists y P(x,y)$

$$\begin{array}{c c} & \begin{array}{ccc} y_1 & y_2 & y_3 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} & \begin{bmatrix} F & F & F \\ \mathrm{T} & \mathrm{T} & \mathrm{T} \\ F & F & F \end{bmatrix} \end{array}$$

FIGURE 2.2.5. $\exists x \forall y P(x,y)$

FIGURE 2.2.6. Truth Matrices for a Binary Predicate

Note how the truth matrices for $\forall x \exists y P(x,y)$ and $\exists x \forall y P(x,y)$ differ. The statement $\forall x \exists y P(x,y)$ simply requires that every $x \in X$ have some $y \in Y$ such

Rule of Inference	Name
$\forall x P(x) \to P(c)$	Universal Instantiation
$p(c)$ for arbitrary c $\to \forall x P(x)$	Universal Generalization
$\exists x P(x) \to P(c)$ for some element c	Existential Instantiation
$P(c)$ for some element $c \to \exists x P(x)$	Existential Generalization

TABLE 1. Rules of Inference for First Order Logic

that $P(x, y)$ is true. In other words, every row must have a (i.e. at least one) T. In contrast, the statement $\exists x \forall y P(x, y)$ requires that there is some $x \in X$ such that for every $y \in Y$, $P(x, y)$ is true. In other words, some row must be all Ts.

EXERCISE 2.2.3. For each of the four statements $\forall x \forall y P(x, y)$, $\exists x \forall y P(x, y)$, $\forall x \exists y P(x, y)$, $\exists x \exists y P(x, y)$, draw an additional truth matrix which satisfies the statement. If there are no such matrices for a particular statement, explain why.

2.3. Rules of Inference

First order logic has certain rules of inference, just as propositional logic did. The rules of inference of first order logic are used extensively, and often implicitly, in mathematical logic. We examine them here.

Universal Instantiation is the rule of inference used to conclude that if it is the case that $\forall x P(x)$, then for any given c in the domain $P(c)$. For example, from the statement, "It's raining everywhere in New York," we can use universal instantiation to conclude "is raining in New York City" since New York City is a member of the domain of all places in New York. A mathematical example is that from the statement $\forall x \in \mathbb{R}, x^2 \geq 0$, we may deduce that $7^2 \geq 0$ since $7 \in \mathbb{R}$.

Universal generalization is the rule of inference that states that if $P(c)$ is true for all elements c in the domain, then $\forall x P(x)$ is true. We use universal generalization (often implicitly) when we argue that the truth of a predicate $P(x)$ for an arbitrary element c of the domain implies that $P(x)$ is satisfied by every element of the domain. In other words, universal generalization allows us to argue that the truth of $P(c)$ for an arbitrary c implies $\forall x P(x)$.

EXAMPLE 2.3.1. We universal generalization when we show that two sets A, B are equal. Recall that this means showing that A and B are subsets of one another. By universal generalization, we need only show that an arbitrary element of a is an element of B, which allows us to show that every element of A is an element of B (and vice versa).

Universal generalization is often used implicitly in mathematics. When we do so, we must be careful not to make unwarranted assumptions about the arbitrary element c. For example, it would be an error to assume that the product of any two integers $a \cdot b$ is even based on the fact that $2 \cdot 4$ is even and $6 \cdot 8$ is even since these integers are not arbitrary. However, it would be correct to argue that the product of any two integers is even. Say a, b are arbitrary even numbers. Then $\exists k, n \in \mathbb{Z}(a = 2k \wedge b = 2n)$. Then $ab = 2k \cdot 2n = 2(2kn)$ is even. Thus, universal generalization allows us to deduce that the product of any two even numbers is even.

Existential instantiation is the rule that allows us to conclude that if $\exists(x)P(x)$, then there is an element c in the domain for which $P(c)$ is true. Usually, we do not know what the value of c is, only that it exists.

EXAMPLE 2.3.2. In calculus, Rolle's theorem states that any real-valued differentiable function that attains equal values at two distinct points must have at least one stationary point somewhere between them—that is, a point where the first derivative $f'(x)$ (the slope of the tangent line to the graph of the function) is zero. If $P(x)$ is the predicate $f'(x) = 0$ and $\mathbb{R}$ is the universal set, then Rolle's theorem tells us $\exists x P(x)$. We may use existential instantiation to name the particular element of the domain for which $P(x)$ is true c and say that $f'(c) = 0$. For a concrete case, observe that $f(x) = -x^2+1$ is a real valued differentiable function and $f(x)$ attains equal values at two distinct points ($f(-1) = f(1) = 0$). Therefore we may apply Rolle's theorem to deduce $\exists x f'(x) = 0$. We then use existential instantiation to give this point a name and say that $f'(c) = 0$. It so happens in this case that there is exactly one point c such that $f'(c) = 0$; $c = 0$.

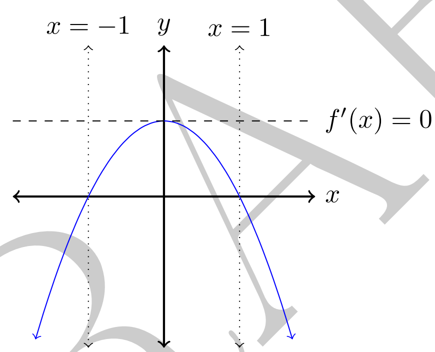

FIGURE 2.3.1. Rolle's Theorem Applied to $f(x) = -x^2 + 1$

Existential generalization is the rule of inference that states that if a for a particular element c, $P(c)$ is true then we may conclude that $\exists x P(x)$ is true. That is, if we know one element c in the domain for which $P(c)$ is true, then we know that $\exists x P(x)$ is true. For example, if $P(x)$ is the predicate that a carrot x is spoiled, then a farmer who finds one spoiled carrot c in his field might can correctly deduce that some carrot is spoiled, i.e. $\exists x P(x)$.

2.4. Logical Equivalences with Quantifiers

DEFINITION 2.4.1. Statements involving predicates and quantifiers are **logically equivalent** if and only if they have the same truth value no matter which predicates are substituted into these statements and which domain of discourse is used for the variables in these propositional functions. We use the notation $S \equiv T$ to indicate that two statements S and T involving predicates and quantifiers are logically equivalent.

EXAMPLE 2.4.2. Universal quantifiers distribute over conjunction $\forall x(P(x) \wedge Q(x)) \equiv \forall x P(x) \wedge \forall x Q(x)$

Solution. To show these statements are equivalent, we have to show that they imply each other.

$\rightarrow$Suppose that $\forall x\,(P(x) \wedge Q(x))$. By universal instantiation, if a is in the domain, we have that $P(a) \wedge Q(a)$. Since a was arbitrary in the domain, by universal generalization, we have that $P(a) \wedge Q(a)$ are true for all x in the domain, i.e., $\forall x P(x) \wedge \forall x Q(x)$

$\leftarrow$Suppose that $\forall x P(x) \wedge \forall x Q(x)$. By universal instantiation, if a is in the domain, we have $P(a)$ and $Q(a)$. Thus $P(a) \wedge Q(a)$. Since a was an arbitrary element of the domain, by universal generalization, we have $\forall x\,(P(a) \wedge Q(a))$

Exercise 2.4.3. Show that existential quantifiers distribute over disjunction: $\exists x\,(P(x) \vee Q(x)) \equiv \exists x P(x) \vee \exists x Q(x)$

Solution. $\rightarrow$Suppose $\exists x\,(P(x) \vee Q(x))$. Then by existential instantiation, there is some a in the domain such that $P(a) \vee Q(a)$. By existential generalization, if $P(a)$ is true, then $\exists x P(x)$, and by disjunction introduction, we have $\exists x P(x) \vee \exists x Q(x)$. Likewise if $Q(a)$ is true, then $\exists x Q(x)$ and by disjunction introduction $\exists x Q(x) \vee \exists x P(x)$. Thus in both cases we have $\exists x P(x) \vee \exists x Q(x)$.

$\leftarrow$Suppose $\exists x P(x) \vee \exists x Q(x)$. WLOG suppose $\exists x P(x)$. Then by existential instantiation, there is some a in the domain such that $P(a)$. Then by disjunction introduction, $P(a) \vee Q(a)$. Then by existential generalization, $\exists x\,(P(a) \vee Q(a))$. By symmetry the result also holds if $Q(x)$. Thus in each case we have $\exists x\,(P(a) \vee Q(a))$.

Theorem 2.4.4. *Universal Modus Ponens*
If a is a member of the domain, then

$$[\forall x\,(P(x) \rightarrow Q(x)) \wedge P(a)] \rightarrow Q(a)$$

Proof. By universal instantiation, from $\forall x\,(P(x) \rightarrow Q(x))$, we may deduce that $P(a) \rightarrow Q(a)$. We are given that $P(a)$. Thus we have $(P(a) \rightarrow Q(a)) \wedge P(a)$. Therefore by modus ponens, $Q(a)$ must also be true. □

Theorem 2.4.5. *Universal Modus Tollens*
If a is a member of the domain, then

$$[\forall x\,(P(x) \rightarrow Q(x)) \wedge \neg Q(a)] \rightarrow \neg P(a)$$

Proof. TODO □

Exercise 2.4.6. Justify the rule of universal transitivity, which states that if $\forall x\,(P(x) \rightarrow Q(x))$ and $\forall x\,(Q(x) \rightarrow R(x))$ are true, then $\forall x\,(\neg R(x) \rightarrow P(x))$.

Solution 2.4.7. Let a be any element of the domain. Then from $\forall x\,(P(x) \rightarrow Q(x))$ by universal instantation, we have that $P(a) \rightarrow Q(a)$ are true. Likewise, from $\forall x\,(Q(x) \rightarrow R(x))$ by universal instantiation, we have that $Q(a) \rightarrow R(a)$. Observe that since a is a specific element of the domain $P(a), Q(a), R(a)$ are just three statements. Recall the rule of inference known as hypothetical syllogism in propositional logic. This stays that for any statements A, B, C, $A \rightarrow B \wedge B \rightarrow C$ implies that $A \rightarrow C$. By hypothetical syllogism, we have that $P(a) \rightarrow R(a)$. But a was an arbitrary element of the domain. Therefore, by universal generalization, we have that $\forall x\,(P(a) \rightarrow R(a))$.

2.5. Negating Quantified Statements

Much like we could take the negation of statements like $(A \vee B)$ in propositional calculus, we can also take the negations of statements such as $\forall x P(x)$ and $\exists x P(x)$ in predicate logic. We write these negations as $\neg\forall x P(x)$ and $\neg\exists x P(x)$ respectively. **De Morgan's Laws for Quantifiers** are an important equivalence which allows us to express the negation of a universally quantified in terms of an existentially quantified statement and the negation of an existentially quantified statement in terms of a universally quantified statement.[**13**, 47] They are essential to predicate logic and mathematics as a whole and should be memorized.

THEOREM 2.5.1. *De Morgan's Laws For Quantifiers*

(1) $\neg\forall x P(x) \equiv \exists x \neg P(x)$
(2) $\neg\exists x P(x) \equiv \forall x \neg P(x)$

In English, $\neg\forall x P(x) \equiv \exists x \neg P(x)$ states that the following are equivalent:

(1) It is not the case that $P(x)$ is true for every x
(2) there is some x for which $P(x)$ is false

Likewise, $\neg\exists x P(x) \equiv \forall x \neg P(x)$ states that the following are equivalent:

(1) There is no x such that $P(x)$ is true
(2) for every x $P(x)$ is false.

Put simply, De Morgan's laws for quantifiers tell us that the opposite of everything is one which is not (and vice versa), and that the opposite of nothing is something (and vice versa).

EXAMPLE 2.5.2. Let $P(x)$ be the predicate "x is red". If our universe U is the set of all dogs, then the statement $\forall x P(x)$ means "every dog is red". The negation of this statement is $\neg\exists x P(x)$; "there exists a dog which is not red." Likewise, the statement $\exists x P(x)$ means "there is a red dog." The negation of this statement is $\forall x(\neg P(x))$; "all dogs are not red".

Although a we cannot give a formal proof of De Morgan's laws at this stage, the general idea is as follows:

(1)

$$\begin{aligned}\neg\forall x P(x) &\equiv \forall x P(x) \equiv False\\ &\equiv \exists x\,(P(x) \equiv False)\\ &\equiv \exists x \neg P(x)\end{aligned}$$

(2)

$$\begin{aligned}\neg\exists x P(x) &\equiv \exists x P(x) \equiv False\\ &\equiv \neg\exists x\,(P(x) \equiv True)\\ &\equiv \forall x\,(P(x) \equiv False)\\ &\equiv \forall x \neg P(x)\end{aligned}$$

EXERCISE 2.5.3. Negate the following statement:$\forall x(x \in A \iff x \in B)$

$$\begin{aligned}\neg\forall x[x \in A \iff x \in B] &\equiv \exists x\neg[x \in A \iff x \in B]\\ &\equiv \exists x\neg[x \in A \implies x \in B \wedge x \in B \implies x \in A]\\ &\equiv \exists x[\neg(x \in A \implies x \in B) \vee \neg(x \in B \implies x \in A)]\end{aligned}$$

EXAMPLE 2.5.4. Use Proof by contradiction to show $\forall x(x \in A \iff x \in B)$ implies A and B have exactly the same elements.

We begin by supposing the negation of the thing we want to prove: that is suppose $\forall x(x \in A \iff x \in B)$ and A and B do not have the same elements. Then one set must have an element not belonging to the other. WLOG, suppose that it is A which contains an element not present in B. Symbolically, this means $\exists x(x \in A \wedge x \notin B)$. Thus, we have supposed that the following is true:

$$\forall x(x \in A \iff x \in B) \wedge \exists x(x \in A \wedge x \notin B)$$

Using our negation rules, we have $\neg\forall x(x \in A \implies x \in B)$. Thus $\forall x(x \in A \implies x \in B) \equiv F$. For brevity, we will abbreviate $\forall x(x \in A \impliedby x \in B) \equiv P$. Keep in mind that we can expand $\forall x(x \in A \iff x \in B)$ so that:

$$\begin{aligned} \forall x(x \in A \iff x \in B) &\equiv \forall x(x \in A \impliedby x \in B) \wedge \forall x(x \in A \implies x \in B) \\ &\equiv P \wedge \forall x(x \in A \implies x \in B) \end{aligned}$$

We argue as follows:

$$\begin{aligned} \forall x(x \in A \iff x \in B) \wedge \exists x(x \in A \wedge x \notin B) &\equiv \forall x(x \in A \iff x \in B) \wedge \neg\forall x(x \in A \implies x \in B) \\ &\equiv P \wedge \forall x(x \in A \implies x \in B) \wedge \neg\forall x(x \in A \implies x \in B) \\ &\equiv P \wedge F \\ &\equiv F \end{aligned}$$

Thus, despite assuming our initial conditions were true, we have deduced that they are false which is a contradiction. Thus the initial conditions must have been false.

2.6. Bounded Quantifiers

$$\forall x \in X, P(x) \equiv \forall x(x \in X \to P(x))$$

$$\exists x \in X, P(x) \equiv \exists x(x \in X \wedge P(x))$$

EXAMPLE 2.6.1. In 1.6.4, we expressed the statement that the product of any pair of positive and negative real numbers is negative with:

$$\forall a > 0, b(b < 0 \to ab < 0) \tag{2.6.1}$$

We now see that it is equivalent to writing:

$$\forall a > 0 \forall b < 0(ab < 0) \equiv \forall a > 0, b < 0(ab < 0)$$

We could even go further and eliminate the bounds on the quantifier for a by writing the expression as:

$$\forall a, b(a > 0 \to (b < 0 \to ab < 0))$$

In the example, we also found the negation of 2.6.1 to be $\exists a > 0(b < 0 \wedge ab > 0)$. We now see this is equivalent to:

$$\exists a > 0 \exists b < 0(ab > 0) \equiv \exists a > 0, b < 0(ab > 0)$$

$$\begin{array}{cc} & Y \\ X & \begin{bmatrix} \mathrm{T} & \mathrm{T} & \mathrm{T} \\ \mathrm{T} & \mathrm{T} & \mathrm{T} \\ \mathrm{T} & \mathrm{T} & \mathrm{T} \end{bmatrix} \end{array}$$

FIGURE 2.5.1. $\forall x \forall y P(x,y)$

$$\begin{array}{cc} & Y \\ X & \begin{bmatrix} \mathrm{T} & \mathrm{T} & \mathrm{T} \\ \mathrm{T} & F & \mathrm{T} \\ \mathrm{T} & \mathrm{T} & \mathrm{T} \end{bmatrix} \end{array}$$

FIGURE 2.5.2. $\exists x \exists y \neg P(x,y)$

$$\begin{array}{cc} & Y \\ X & \begin{bmatrix} \mathrm{T} & \mathrm{F} & \mathrm{F} \\ \mathrm{F} & \mathrm{T} & \mathrm{F} \\ \mathrm{F} & \mathrm{F} & \mathrm{T} \end{bmatrix} \end{array}$$

FIGURE 2.5.3. $\forall x \exists y P(x,y)$

$$\begin{array}{cc} & Y \\ X & \begin{bmatrix} \mathrm{T} & \mathrm{T} & \mathrm{T} \\ \mathrm{F} & \mathrm{F} & \mathrm{F} \\ \mathrm{T} & \mathrm{T} & \mathrm{T} \end{bmatrix} \end{array}$$

FIGURE 2.5.4. $\exists x \forall y \neg P(x,y)$

$$\begin{array}{cc} & Y \\ X & \begin{bmatrix} \mathrm{F} & \mathrm{F} & \mathrm{F} \\ \mathrm{T} & \mathrm{T} & \mathrm{T} \\ F & \mathrm{F} & \mathrm{F} \end{bmatrix} \end{array}$$

FIGURE 2.5.5. $\exists x \forall y P(x,y)$

$$\begin{array}{cc} & Y \\ X & \begin{bmatrix} \mathrm{T} & \mathrm{T} & \mathrm{F} \\ \mathrm{T} & \mathrm{F} & \mathrm{T} \\ \mathrm{F} & \mathrm{T} & \mathrm{T} \end{bmatrix} \end{array}$$

FIGURE 2.5.6. $\forall x \exists y \neg P(x,y)$

$$\begin{array}{cc} & Y \\ X & \begin{bmatrix} \mathrm{F} & \mathrm{F} & \mathrm{F} \\ \mathrm{F} & \mathrm{T} & \mathrm{F} \\ \mathrm{F} & \mathrm{F} & \mathrm{F} \end{bmatrix} \end{array}$$

FIGURE 2.5.7. $\exists x \exists y P(x,y)$

$$\begin{array}{cc} & Y \\ X & \begin{bmatrix} \mathrm{F} & \mathrm{F} & \mathrm{F} \\ \mathrm{F} & \mathrm{F} & \mathrm{F} \\ \mathrm{F} & \mathrm{F} & \mathrm{F} \end{bmatrix} \end{array}$$

FIGURE 2.5.8. $\forall x \forall y \neg P(x,y)$

FIGURE 2.5.9. Statements (left column) and their negations (right column)

2.7. Negating Multiple Quantifiers

We can negate statements with multiple quantifiers with a little extra effort. We begin with an example, and finish with a more rigorous look.

EXAMPLE 2.7.1. Let $X = \{x_1, x_2, x_3\}, Y = \{y_1, y_2, y_3\}$, and $P(x,y)$ be a predicate. Now, consider the truth value of the statement $S \equiv \forall y \in Y, \exists x \in X, P(x,y)$. In a truth matrix, the statement that for each y, there is some x such that $P(x,y)$ is true has the interpretation that for every column y, there must be some row x such that the intersection of column y and row x has value T. In simpler terms, S is the statement that each column must have a T in it.

$$\begin{array}{cc} & Y \\ X & \begin{bmatrix} \mathrm{T} & \mathrm{F} & \mathrm{T} \\ \mathrm{F} & \mathrm{T} & \mathrm{F} \\ \mathrm{T} & \mathrm{F} & \mathrm{T} \end{bmatrix} \end{array}$$

Having constructed this matrix, we see that each column does indeed contain a T, and thus S is true. In this context, it is easier to see the negation of S. Namely, if S is the statement that each column must have a T in it, then $\neg S$ is the statement that there is some column that doesn't have a T in it. That is,

$$\neg S \equiv \neg\left(\forall y \in Y, \exists x \in X, P(x,y)\right) \equiv \exists y \in Y, \forall x \in X \neg P(x,y)$$

So in words, the negation of the statement "for every $y \in Y$, there exists an $x \in X$ such that $P(x, y)$" is "there exists some $y \in Y$ such that for every $x \in X$, it is not the case that $P(x, y)$."

From the example we may extrapolate a pattern: in the negation of S, all the quantifiers swap; $\exists x$ becomes $\forall x$, and vice versa, and the predicate is negated. By extrapolation, we might (correctly) guess that this means the following equivalences hold:

$$\boxed{\begin{aligned}
\neg\left(\forall y \in Y, \exists x \in X, P(x, y)\right) &\equiv \exists y \in Y, \forall x \in X, P(x, y) \\
\neg\left(\exists y \in Y, \forall x \in X, P(x, y)\right) &\equiv \forall y \in Y, \exists x \in X, \neg P(x, y) \\
\neg\left(\exists y \in Y, \exists x \in X, P(x, y)\right) &\equiv \forall y \in Y, \forall x \in X, \neg P(x, y)
\end{aligned}}$$

These equivalences seem to be intuitively true, but how do we know for sure? What happened if there are even more quantifiers? Fortunately, in order to handle the general case, we simply must make the key realization that a statement such as $\exists x \in X, P(x, y)$ is just a predicate in y. We must know what y is in order for $\exists x \in X, P(x, y)$ to become a statement (i.e. a sentence which is true or false), or in other words, y is the only free variable in this sentence.

Following our example, we cannot determine if the predicate $P(y)$ "there exists an $x \in X$ such that $x + y$ is even" is true or false until we determine what y is. Thus $\exists x \in X, P(x, y)$ is just some predicate of y, say $\Phi(y)$. Thus we can write,

$$\forall y \in Y, \exists x \in X, P(x, y) \equiv \forall y \in Y, \Phi(y)$$

and apply De Morgan's Laws for quantifiers to obtain $\neg\left(\forall y \in Y, \Phi(y)\right) \equiv \exists y \in Y \neg\Phi(y)$. Obtaining the solution is simply a matter of substituting back in $\Phi(y) = \exists x \in X, P(x, y)$, and applying De Morgan's laws for quantifiers again.

$$\begin{aligned}
\neg\left(\forall y \in Y, \exists x \in X, P(x, y)\right) &\equiv \neg\left(\forall y \in Y, \Phi(y)\right) \\
&\equiv \exists y \in Y \neg\Phi(y) \\
&\equiv \exists x \in X, \neg\left(\exists x \in X, P(x, y)\right) \\
&\equiv \exists x \in X, \forall x \in X, \neg P(x, y)
\end{aligned}$$

This illustrates that finding the negation of a statement multiple quantifiers simply boils down to making substitutions and applying De Morgan's laws for quantifiers recursively.[**13**, 63]

Exercise 2.7.2. Negate $\forall x \in X, \exists y \in Y, \forall z \in Z, P(x, y, z)$.

Solution 2.7.3. Let $\Phi(x) = \exists y \in Y, \forall z \in Z, P(x, y, z)$ and $\zeta(x, y) = \forall z \in Z, P(x, y, z)$. Thus $\Phi(x) = \exists y \in Y, \zeta(x)$

$$\begin{aligned}
\forall x \in X, \exists y \in Y, \forall z \in Z, P(x, y, z) &\equiv \forall x \in X, \Phi(x) \\
\neg\left(\forall x \in X, \exists y \in Y, \forall z \in Z, P(x, y, z)\right) &\equiv \neg\left(\forall x \in X, \Phi(x)\right) \\
&\equiv \exists x \in X \neg\Phi(x) \\
&\equiv \exists x \in X \neg\left(\exists y \in Y, \zeta(x, y)\right) \\
&\equiv \exists x \in X, \forall y \in Y, \neg\zeta(x, y) \\
&\equiv \exists x \in X, \forall y \in Y, \neg\left(\forall z \in Z, P(x, y, z)\right) \\
&\equiv \exists x \in X, \forall y \in Y, \exists z \in Z, \neg P(x, y, z)
\end{aligned}$$

Thus:

$$\neg\left(\forall x \in X, \exists y \in Y, \forall z \in Z, P(x,y,z)\right) \equiv \exists x \in X, \forall y \in Y, \exists z \in Z, \neg P(x,y,z)$$

EXERCISE 2.7.4. The defining property of a function $f : A \to B$ states that states that "if any element a of domain A has two images b_1, b_2 under f, then $b_1 = b_2$", i.e.

$$\forall a \in A, \forall b_1, b_2 \in B\left[(f(a) = b_1 \wedge f(a) = b_2) \to (b_1 = b_2)\right]$$

Show that this is equivalent to the statement that "no element of the domain has two distinct images under f", i.e.

$$\neg\exists a \in A, \exists b_1, b_2 \in B\left[(f(a) = b_1 \wedge f(a) = b_2) \wedge b_1 \neq b_2\right]$$

PROOF. Recall De Morgan's laws for quantifiers, namely that $\neg\exists x(P(x)) \equiv \forall x(\neg P(x))$. Let $Q(x) = \neg P(x)$. Then we have $\forall x(Q(x)) \equiv \neg\exists x(\neg Q(x))$. To simplify notation, also let $\Phi(a,b)$ be the predicate $f(a) = b_1 \wedge f(a) = b_2$. Then

$$\begin{aligned}
\forall a \in A, \forall b_1, b_2 \in B\left[\Phi(a,b) \to b_1 = b_2\right] &\equiv \forall a \in A, \forall b_1, b_2 \in B\left[\Phi(a,b) \to b_1 = b_2\right] \\
&\equiv \neg\exists a \in A \neg\left(\forall b_1, b_2 \in B\left[\Phi(a,b) \to (b_1 = b_2)\right]\right) \\
&\equiv \neg\exists a \in A, \exists b_1, b_2 \in B \neg\left[\Phi(a,b) \to (b_1 = b_2)\right] \\
&\equiv \neg\exists a \in A, \exists b_1, b_2 \in B\left[\Phi(a,b) \wedge \neg(b_1 = b_2)\right] \\
&\equiv \neg\exists a \in A, \exists b_1, b_2 \in B\left[\Phi(a,b) \wedge b_1 \neq b_2\right]
\end{aligned}$$

□

2.8. Further Reading

There are a great number of books dealing with first order logic. We reccomend: [**14, 16, 5, 11**]. There are in fact many different kinds of logic beyond those which we have mentioned. For a survey of these, see: [**18**].

CHAPTER 3

Sets and Functions

3.1. Introduction to Sets

A **set** is a collection of objects, which we call its **members**, **elements**, or **points**. These objects could be almost anything, numbers, people, planets, etc. We denote that a is a member of set A by $a \in A$, and likewise that a is not a member by $a \notin A$.

There are two methods for writing out sets. The first is known as the **roster method**, and specifies a set as a list of comma separated elements enclosed in curly brackets. For example, the set containing the numbers $1, 2, 3$ is written $\{1, 2, 3\}$. We can also define a set A in terms of a given property $P(a)$ that all elements a of A must have. We can think of this property as an entrance test; $a \in A$ if and only if a has the given property and thus passes the entrance test. In other words, $a \in A$ if and only if $P(a)$ is true. This method of set construction is known as **set comprehension** or **set abstraction** [**6**, 4]. Sets defined in this way are said to be written in **set builder notation**

Set-builder notation has three parts: a variable x, a colon ":" or vertical bar "|" separator (read as "such that"), and a property $P(x)$, which the variable x must satisfy to be a member of the set. To be precise, when we say $P(x)$ is some property, we really mean $P(x)$ is a predicate. Recall that predicates are Boolean functions[1], they take in an element, and return either True or False. Thus when we say "x must satisfy the predicate", we mean x must make the predicate $P(x)$ True.

Thus, the set of all objects x such that satisfy the predicate $P(x)$ is written

$$\{x : P(x)\}$$

Example 3.1.1. The set of all even numbers can be written in set builder notation as $\{x : x \text{ is even}\}$. By definition, a number x is even if there is some integer k such that $x = 2k$. Thus we have

$$\{x : x \text{ is even}\} = \{x : \exists k \in \mathbb{Z}(x = 2k)\} = \{x : \exists k(k \in \mathbb{Z} \wedge x = 2k)\}$$

[1]We will define functions in 3.4 in terms of set builder notation. This gives the appearance that our definitions for functions and set builder notation are circular, however this is not true. The canonical Zermelo–Fraenkel (ZF) set theory (which is one of many set theories), is governed by a series of axioms; statements accepted without proof. Roughly speaking, an axiom schema is a template for creating a (possibly countably infinite) number axioms.

In ZF, there is a set existence axiom scheme, which states that if E is a set and $\Phi(x)$ is a formula in the language of set theory, then there is a set Y whose members are exactly the elements of E that satisfy Φ:

$(\forall E)(\exists Y)(\forall x)[x \in Y \Leftrightarrow x \in E \wedge \Phi(x)].(\forall E)(\exists Y)(\forall x)[x \in Y \Leftrightarrow x \in E \wedge \Phi(x)]$. The set Y obtained from this axiom is exactly the set described in set builder notation as$\{x \in E \mid \Phi(x)\}\{x \in E \mid \Phi(x)\}$.

Thus $P(x)$ is the statement $\exists k(k \in \mathbb{Z} \wedge x = 2k)$, and $P(x)$ is true for $0, 2, -2, 4, -, 4, \ldots$. Thus $\{x : \exists k(k \in \mathbb{Z} \wedge x = 2k)\} = \{\ldots, -4, 2, 0, 2, 4, \ldots\}$.

Definition 3.1.2. Subset

Suppose A and B are sets. If every object that belongs to A also belongs to B then we say A is a **subset** of B, written $A \subseteq B$. Symbolically, $A \subseteq B \iff \forall a(a \in A \to a \in B)$. Likewise we write $A \nsubseteq B$ to mean $\neg(A \subseteq B)$, i.e. A is not a subset of B.

Exercise 3.1.3. * Negate the statement $A \subseteq B \iff \forall a(a \in A \to a \in B)$ to find an equivalent expression for $A \nsubseteq B$.

$$\begin{aligned} A \subseteq B &\iff \forall a\,(a \in A \to a \in B) \\ \neg(A \subseteq B) &\iff \neg\forall a\,(a \in A \to a \in B) \\ A \nsubseteq B &\iff \exists a \neg\,(a \in A \to a \in B) \\ A \nsubseteq B &\iff \exists a\,(a \in A \wedge \neg(a \in B)) \\ A \nsubseteq B &\iff \exists a\,(a \in A \wedge a \notin B) \end{aligned}$$

Definition 3.1.4. Set Equality

Two sets X, Y are equal iff they have the same elements. This means for any sets X, Y, $X = Y$ iff $X \subseteq Y$ and $Y \subseteq X$. This definition comes from the axiom of extensionality which states $\forall x \forall y(\forall z(z \in x \leftrightarrow z \in y) \to x = y)$. The converse of this $\forall x \forall y(x = y \to \forall z(z \in x \leftrightarrow z \in y))$ comes from TODO, giving the iff statement.

From 3.1.4, we can make two important deductions: sets are unordered and that repeat elements may be ignored. Thus for instance $\{1, 2, 3\} = \{3, 1, 2\} = \{1, 1, 3, 2\}$ because all of these sets have the same elements. The set with no elements is called the **empty set**, and is denoted $\emptyset$, or $\{\}$.

Example 3.1.5. Prove that $\{a\} = \{a, b\} \iff a = b$

$\implies$ Suppose $\{a\} = \{a, b\}$. Then $\{a, b\} \subseteq \{a\}$. Thus every element of $\{a, b\}$ must be an element of $\{a\}$. Since there is only one element of $\{a\}$, we have that $b = a$. Formally, $\{a, b\} \subseteq \{a\} \implies \forall x \in \{a, b\}, (x \in \{a\})$. By universal instantiation, we have:

$$\forall x \in \{a, b\}(x \in \{a\}) \wedge b \in \{a, b\} \implies b \in \{a\} \implies b = a$$

$\impliedby$ Suppose $a = b$. Then $\{a, b\} = \{a, a\}$. Its easy to show $\{a, a\} \subseteq \{a\}$ and $\{a\} \subseteq \{a, a\}$, and therefore $\{a, b\} = \{a, a\} = \{a\}$.

Example 3.1.6. Prove that the Empty Set $\emptyset$ is a subset of every set Y.

By the definition of 3.1.2, to show that a given set X is a subset of a set Y, we must show that every element of X is also an element of Y. Since the empty set has no elements, it is vacuously true that every element of $\emptyset$ is contained in Y.

Alternatively, we could see from 3.1.3 that $A \nsubseteq B \iff \exists a(a \in A \wedge a \notin B)$ and use proof by contradiction. We begin by assuming that the theorem is not true, i.e. that $\emptyset \nsubseteq Y$. Then $\exists a(a \in \emptyset \wedge a \notin Y)$. But the statement that $\exists a \in \emptyset$ is a contradiction. Thus the theorem must be true.

Definition 3.1.7. For any set A, the **powerset** (or **power set**) of A, denoted $\mathcal{P}(A)$ is the set whose members are precisely the subsets of A. Thus $\mathcal{P}(A) = \{x : x \subseteq A\}$.

Definition 3.1.8. Collection

It is very common to have a set whose elements are themselves sets. For example $\{\{0,1\},\{1,2\}\}$ is a set with two elements $\{0,1\},\{1,2\}$ both of which are sets. We call such sets **collections** and denote them as bolded capital letters, $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \ldots, \boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}$. Observe that all collections are sets.

Example 3.1.9. Since the empty set is a subset of every set, we have that $\mathcal{P}(\emptyset) = \emptyset$. Similarly, $\mathcal{P}(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}$. Finally,

$$\mathcal{P}(\{1,2,3\}) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,3\}, \{1,2,3\}\}$$

. $\mathcal{P}(\{1,2,3\})$ is an example of a collection.

3.2. Set Operations

In arithmetic, we manipulate numbers via the operations of addition, subtraction, multiplication, and division. In this section we will introduce the operations with which we can manipulate sets.

Definition 3.2.1. Set Union and Intersection: $A \cup B$, $A \cap B$

The **union** of A and B written $A \cup B$ is the set of all objects which are members of A or B. The **intersection** of A and B written $A \cap B$ is the set of all things that are members of both A and B. Sets A and B are called **disjoint** iff their intersection is empty, i.e. they have no members in common. A collection of sets is **pairwise disjoint** iff any any two elements of the collection are disjoint. Formally, we can define the union and intersection of arbitrary amounts of sets as follows; Let $\boldsymbol{A}$ be a collection. Then the union of $\boldsymbol{A}$, $\bigcup \boldsymbol{A}$ is the set consisting of every x such that x is belongs to some member of $\boldsymbol{A}$. Similarly, $\bigcap \boldsymbol{A}$ is the set consisting of every x such that x belongs to all members of A. Formally, we have:

$$\bigcup \boldsymbol{A} = \{x : \exists A \in \boldsymbol{A}(x \in A)\}$$
$$\bigcap \boldsymbol{A} = \{x : \forall A \in \boldsymbol{A}(x \in A)\}$$

Example 3.2.2. If $\boldsymbol{A} = \{\{0,1\},\{1,2\}\}$, $\bigcup A = \{0,1,2\}$, $\bigcap A = \{1\}$. If the collection $\boldsymbol{A}$ only contains a few elements, say A, B, C, we can use the infix notation to denote the union and intersection of these elements as $A \cup B \cup C$ and $A \cap B \cap C$ respectively.

Example 3.2.3. Let $\boldsymbol{A} = \{\{0,1\},\{1,2\}\{3\}\}$. Then $\bigcup \boldsymbol{A} = \{0,1,2,3\}$.

Example 3.2.4. Suppose $\boldsymbol{A} = \emptyset$. What are $\bigcup \emptyset$ and $\bigcap \emptyset$?

$\bigcup \emptyset = \{x \in X : \exists A \in \emptyset(x \in A)$. Likewise, $\bigcap \emptyset = \{x \in X : \forall A \in \emptyset(x \in A)\}$. Let $x \in X$. Since there are no $A \in \emptyset$, it is vacuously true that $\forall A \in \emptyset(x \in A)$. Since x was arbitrary in X, by universal generalization, we have $X \subseteq \bigcap \emptyset$, and since X is the universal set, we have $\bigcap \emptyset \subseteq X$, and thus $X = \bigcap \emptyset$.

If two sets intersect (i.e. have common elements), we may wish to discuss the elements which are contained in one set and not in the other. For example, the odd numbers are the elements of $\mathbb{Z}$, which are not even, i.e: $\{x : x \in \mathbb{Z} \wedge x \text{ is not even}\} = \{x : x \in \mathbb{Z} \wedge \neg(\exists n \in \mathbb{Z}(x = 2n))\}$. To express this notion more succinctly, we have the following

Definition 3.2.5. Set Difference, Relative Compliment, Absolute Compliment: $B \backslash A$

The **set-theoretic difference** of B and A (also known as the **relative complement of A in B**) is the set of elements in B but not in A, written

$$B \backslash A = \{x : x \in B \wedge x \notin A\}$$

Although this notation is standardized in [**10**, 5], some authors write $B - A.$, which may also be refereed to as "B delete A".

If U is the universal set, then we refer to $U \backslash A$ as the **absolute complement**of A (or simply the **complement** of A). I.e, it is the set of all elements in the universe U which are not in A. We simplify our notation by writing $A^{\complement} := U \backslash A$.

Example 3.2.6. Let $U = \{1, 2, 3, 4, 5, 6\}$, $A = \{2, 4, 6\}$, $B = \{1, 3, 5\}$. Then $U \backslash A = A^{\complement} = B$, $B \backslash A = B$, $A \backslash \{2\} = \{4, 6\}$.

Theorem 3.2.7. *Let A, B be subsets of a universal set U, and $B^c = U \backslash B$. Then $A \backslash B = A \cap B^c$.*

Proof. Let $x \in A \backslash B$. Then $x \in A \wedge x \notin B$, this implies $x \in A \wedge x \in U \backslash B \equiv x \in A \cap B^{\complement}$.

Let $x \in A \cap B^{\complement}$. Then $x \in A \wedge x \in B^{\complement} \equiv x \in A \wedge x \in U \backslash B$, $\square$

Recall that De Morgan's Laws of propositional logic tell us that for any propositions A, B,

$$\neg(A \vee B) \equiv \neg A \wedge \neg B,$$
$$\neg(A \wedge B) \equiv \neg A \vee \neg B$$

An important consequence of this logical equivalence is the following theorem, sometimes known=p as De Morgan's Laws for sets. Whereas the logical De Morgan's laws describe how negations operate on conjunctions and disjunctions of propositions, the De Morgan's Laws for sets describe how set compliments operate on intersections and unions of sets. The simplest form of the De Morgan's Laws for sets relates absolute complements of sets:

Theorem 3.2.8. *De Morgan's Laws for Sets*
Let A, B, C be sets. Then:
(1) $A \backslash (B \cup C) = (A \backslash B) \cap (A \backslash C)$
(2) $A \backslash (B \cap C) = (A \backslash B) \cup (A \backslash C)$

Proof. Let $x \in A - (B \cup C)$. Then $x \in A \wedge x \notin (B \cup C)$. Recall $x \in B \cup C \leftrightarrow x \in B \vee x \in C$. Therefore $x \notin (B \cup C) \rightarrow \neg(x \in B \vee x \in C)$. By the Logical De Morgans Laws, we have $\neg(x \in B \vee x \in C) \equiv x \notin B \wedge x \notin C$. Thus we know $x \in A \wedge x \notin B \wedge x \notin C$. Of course for any statements $A \leftrightarrow A \wedge A$. Thus we have $x \in A \wedge x \in A \wedge x \notin B \wedge x \notin C$. By applying associativity and commutativity of the logical operator $\wedge$, we have:

$$\begin{aligned} x \in A \wedge x \in A \wedge x \notin B \wedge x \notin C &\equiv x \in A \wedge x \notin B \wedge x \in A \wedge x \notin C \\ &\equiv (x \in A \wedge x \notin B) \wedge (x \in A \wedge x \notin C) \\ &\equiv x \in A - C \wedge x \in A - B \\ &\equiv x \in (A - B) \cap (A - C) \end{aligned}$$

Conversely, let $x \in (A - B) \cap (A - C)$. Then $x \in (A - B) \wedge x \in (A - C)$. Then $x \in A \wedge x \notin B$ and $x \in A \wedge x \notin C$. Then $x \notin B \wedge x \notin C$. Thus $\neg(x \in B \vee x \in C) \equiv \neg(x \in B \cup C)$. Thus $x \notin B \cup C$. Since we showed $x \in A$, we have $x \in A \wedge x \notin (B \cup C)$. Therefore $x \in A - (B \cup C)$. $\square$

As a special case of this theorem, when A is the universal set U, $A \backslash (B \cup C) = (A \backslash B) \cap (A \backslash C)$ becomes $(B \cup C)^{\complement} = B^{\complement} \cap C^{\complement}$ and $A \backslash (B \cap C) = (A \backslash B) \cup (A \backslash C)$ becomes $(B \cap C)^{\complement} = B^{\complement} \cup C^{\complement}$. Then we have the simple and memorable form of De Morgan's laws, below:

$$(A \cup B)^{\complement} = A^{\complement} \cap B^{\complement},$$
$$(A \cap B)^{\complement} = A^{\complement} \cup B^{\complement}$$

We may read these equations as stating that the compliment of the union is the intersection of the compliments, and the compliment of the intersection is the union of the compliments.

Exercise 3.2.9. Prove that if $A \cap B = \emptyset$, then $B \backslash A = B$.

Proof. Let $x \in B \backslash A$. By definition, $x \in B \wedge x \notin A$. Thus $x \in B$. Since x was arbitrary in $B \backslash A$, by Universal Generalization, we have that $\forall x \in B \backslash A (x \in B)$. Therefore $B \backslash A \subseteq B$

Let $x \in B$. Since $A \cap B = \emptyset$, $x \notin A$. Since $x \in B \wedge x \notin A$. Thus $x \in B \backslash A$. Since x was arbitrary in B, by Universal Generalization, we have that $\forall x \in B (x \in B \backslash A)$. Therefore $B \subseteq B \backslash A$. $\square$

One question which naturally arises is, "is everything a set", or more precisely, given some set X and a predicate $P(x)$, is $\{x : P(x)\}$ a set?

Russel's Paradox arises when one considers the set B of all sets X in some universe A such that X does not contain itself. We can write this as

$$B := \{X \in A : X \notin X\}$$

Is B a member of itself? Note

$$y \in B \iff y \in A \wedge y \notin X$$

Suppose $B \in A$. Then there are two cases, $B \in B$ or $B \notin B$. If $B \notin B$, we have $B \in B$, and if $B \in B$, we have $B \notin B$. Since both cases lead to a contradiction, we have that $B \notin A$. Since the universal set A was arbitrary, we conclude there does not exist a set which contains every set. This is equivalent to stating that all sets don't include some set. In better English, nothing contains everything, and there is no universe (Halmos, 6,7).

Russel's paradox was so profound that it has made its way into popular culture as the Barber's Paradox: "In a certain town, the barber, who is a man, shaves exactly those men who don't shave themselves. Who shaves the barber?" Consider th e two cases, and the consequences.

stuff

3.3. Relations

Definition 3.3.1. We saw earlier that sets are unordered, meaning for example $\{1,2,3\} = \{3,1,2\}$. However, this is not always desirable. Thus we introduce the idea of an **ordered pair** (x, y) of objects defined so that

$$(x, y) = (u, v) \iff x = u \wedge y = v$$

One way to do this, known as the Kuratowski Formalization, is to define ordered pairs in terms of sets as follows:

$$(x, y) = \{\{x\}, \{x, y\}\}$$

More generally, ordered n-tuples are defined recursive for $n > 1$ by:

$$(x_1, \ldots, x_{n+1}) = ((x_1, \ldots, x_n), x_{n+1})$$

Exercise 3.3.2. Using the Kuratowski definition of an ordered pair, define an ordered triple (x, y, z) as a set containing two elements.

Solution. Ordered triples are defined recursively, so that $(x, y) = \{\{x\}, \{x, y\}\}$ and $(x, y, z) = ((x, y), z)$. Observe that $((x, y), z)$ only has two elements, (x, y) and z, so we can just apply the definition. To make our lives easier, let $q = (x, y) = \{\{x\}, \{x, y\}\}$. Then the substitution is simple:

$$\begin{aligned}
(x, y, z) &= ((x, y), z) \\
&= (q, z) \\
&= \{\{q\}, \{q, z\}\} \\
&= \{\{\{\{x\}, \{x, y\}\}\}, \{\{\{x\}, \{x, y\}\}, z\}\}
\end{aligned}$$

Likewise, $(x, y, z, w) = ((x, y, z), w)$, so letting $q = (x, y, z)$, we can write $(x, y, z, w) = (q, w) = \{\{q\}, \{q, w\}\}$, which we can expand as shown above.

Exercise 3.3.3. Prove the Kuratowski Formalization of an ordered pair as $(a, b) = \{\{a\}, \{a, b\}\}$ guarantees $(a, b) = (c, d) \iff a = c \wedge b = d$. (Hint, consider the cases $a = b = c = d$, $a = b, c \neq d$, and $a \neq b, c \neq d$).

Proof. We will give 2 proofs

First proof: By definition $(a, b) = \{\{a\}, \{a, b\}\}$. We WTP $(a, b) = (c, d) \iff a = c \wedge b = d$.

$\Longleftarrow$ Suppose that $a = c \wedge b = d$. Then $(a, b) = \{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\} = (c, d)$, where the first and third equalities are by definition, and the second is by assumption.

$\Longrightarrow$ Suppose that $(a, b) = (c, d)$. By definition, we have $\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\}$. Both of these sets has at least one element and at most two. In the case when both sets have one element, the theorem is trivial (as $a = b = c = d$). Thus we will consider the cases when precisely one set has one element, and when both have two elements.

Case 1. WLOG Suppose $a = b$ and $c \neq d$. Then $(a, b) = \{\{a\}, \{a, b\}\} = \{\{a\}, \{a, a\}\} = \{\{a\}\}$. By assumption, $(a, b) = (c, d)$, so $\{\{a\}\} = \{\{c\}, \{c, d\}\}$, which contradicts the fact that $c \neq d$. Therefore $(a, b) \neq (c, d)$ so the theorem is vacuously true in this case.

Case 2. Suppose $a \neq b$ and $c \neq d$.

Suppose $\{a\} = \{c, d\}$. Then, $a = c = d \perp$. Thus $\{a\} \neq \{c, d\}$

$$\{a\} \neq \{c, d\} \tag{3.3.1}$$

Since $\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\}$, we have $\{\{a\}, \{a, b\}\} \subseteq \{\{c\}, \{c, d\}\}$. Thus $\forall x \in \{\{a\}, \{a, b\}\}(x \in \{\{c\}, \{c, d\}\})$ and $x \in \{\{c\}, \{c, d\}\}$ means $x = \{c\} \vee \{c, d\}$. Thus we have:

$$\begin{aligned} &\forall x \in \{\{a\}, \{a, b\}\}(x = \{c\} \vee \{c, d\}) \\ &\{a\} \in \{\{a\}, \{a, b\}\} \\ \therefore\ &\{a\} = \{c\} \vee \{c, d\} \end{aligned}$$

And since $\{a\} \neq \{c, d\}$, we have $\{a\} = \{c\}$, and thus $a = c$.

By the same logic, $\forall x \in \{\{c\}, \{c, d\}\}(x = \{a\} \vee \{a, b\})$, and we showed in 3.3.1 $\{c, d\} \neq \{a\}$. Thus $\{c, d\} = \{a, b\}$. Thus $d = a \vee b$. If $d = a$, then $\{\{c\}, \{c, d\}\} = \{\{a\}, \{a, a\}\} = \{\{a\}\} = \{\{a\}, \{a, b\}\}$, which contradicts the assumption that $a \neq b$. Therefore $d = b$.

□

Proof. A Different Proof of $\implies$:
Suppose $(a, b) = (c, d)$, then:

$$\begin{aligned} \bigcap(a, b) = &\bigcap(c, d) \\ \{a\} = &\{c\} \end{aligned} \tag{3.3.2}$$

Therefore we have $\{a\} = \{c\}$ and we have $a = c$. Now consider the union $\bigcup(a, b) = \{a, b\}$, so we have:

$$\begin{aligned} \bigcup(a, b) = &\bigcup(c, d) \\ \{a, b\} = &\{c, d\} \end{aligned}$$

Case 1. Suppose $a = b$: then $\{a\} = \{a, b\} = \{c, d\} = \{a, d\}$, since 3.3.2 shows $a = c$. Thus, $\{a\} = \{a, d\}$, so $a = d$. So $b = a = d$. Thus $b = d$.

Case 2. Suppose $a \neq b$. Then $\{a, b\} = \{c, d\} = \{a, d\}$. Thus

$$\begin{aligned} \{a, b\} = &\{a, d\} \\ \{a, b\} - \{a\} = &\{a, d\} - \{a\} \\ \{b\} = &\{d\} \end{aligned}$$

Thus, $b = d$.

□

Definition 3.3.4. Given two sets A, B, their **Cartesian Product** $A \times B$ is given by $\{(a, b) : a \in A \wedge b \in B\}$. Observe that the notation (a, b) denotes an *ordered* pair, and is thus distinct from the set $\{a, b\}$ which is unordered. For example, $\{a, b\} = \{b, a\}$, but $(a, b) \neq (b, a)$. The Cartesian product of a set A with itself is written A^2.

DEFINITION 3.3.5. If A and B are sets, a subset R of the Cartesian product $A \times B$ is called a **relation between A and B**. The statement $(x, y) \in R$ is read "x is R-related to y, and is denoted xRy, $R(x, y)$, or $(x, y) \in R$. A relation $R \subseteq A \times A = A^2$ is called a **relation on A**. In general A^n is the set of all n-tuples of members of A.

It is important to note that order matters, if $a \neq b$, then $aRb \not\to bRa$.

EXAMPLE 3.3.6. Let

$$A = \{\text{Chevrolet, Honda, Toyota, Ford}\}$$
$$B = \{\text{Taurus, Corolla, Civic, Corvette}\}$$

Define $R = \{(a, b) \in A \times B : a \text{ manufactures } b\}$. Then $R \subsetneq A \times B$ is a relation and

$$R = \{(\text{Ford, Taurus}), (\text{Toyota, Corolla}), (\text{Honda, Civic}), (\text{Chevrolet, Corvette})\}$$

DEFINITION 3.3.7. The **domain** of a relation R, written dom$R$ is the set of all objects x such that $(x, y) \in R$ for some y. The **image**(or **range**) of R, written im$R$ (or ran$R$) is the set of all objects y such that $(x, y) \in R$ for some x. Symbolically:

$$\text{dom}R = \{x : (x, y) \in R\}$$
$$\text{im}R = \{y : (x, y) \in R\}$$

DEFINITION 3.3.8. Mathematicians have some ways of classifying relations based on their properties. A relation R is said to be:

- Reflexive iff $\forall x \in X, xRx$
- Symmetric iff $\forall x, y \in X, xRy \iff yRx$
- Transitive iff $\forall x, y, z \in X, xRy \wedge yRz \implies xRz$
- Anti-symmetric iff $\forall x, y \in X, (xRy \wedge yRx) \implies x = y$

A binary relation which is reflexive, symmetric, and transitive is known as an **equivalence relation**. A binary relation which is reflexive and transitive is known as a preorder. Given $x, y \in X$ and equivalence relation R, the **equivalence class of x under R**, denoted $[x]_R$, is defined as the set of all elements in X which are related to x by R. That is:

$$[x]_R = \{y \in X : xRy\}$$

DEFINITION 3.3.9. For a given positive integer n, two integers a, b are called **congruent modulo m** iff m divides $a - b$. We write this:

$$a \equiv b \pmod{n}$$

(Recall that for $m, n \in \mathbb{Z}$ it is said that m **divides** n, m is a **divisor** of n, or n is an **integer multiple** of m, written $m|n$ iff $\exists k \in \mathbb{Z}(mk = n)$. For instance, $37 \equiv 57 \pmod{10}$ since $37 - 57 = -20$ which equals $-2 \cdot 10$.

DEFINITION 3.3.10. Partition

A partition of a set S is a collection of subsets $\boldsymbol{S}$ of S such that:

(1) $\boldsymbol{S}$ is pairwise disjoint. I.e., there are no two distinct sets S_1, S_2 in $\boldsymbol{S}$ which contain a common element. Symbolically: $\forall S_1, S_2 \in \boldsymbol{S} : S_1 \neq S_2 \to S_1 \cap S_2 = \emptyset$, which is equivalent to saying $\neg\exists S_1, S_2 \in \boldsymbol{S} : S_1 \neq S_2 \wedge S_1 \cap S_2 \neq \emptyset$
(2) The union of $\boldsymbol{S}$ is equal to the whole set S: $\bigcup \boldsymbol{S} = S$
(3) None of the elements of $\boldsymbol{S}$ is empty: $\forall S_1 \in \boldsymbol{S} : S_1 \neq \emptyset$

THEOREM 3.3.11. *The Equivalence Classes of an Equivalence Relation on a set S constitute a partition of S*

PROOF. Let R be an equivalence relation on set X. We need to show $\bigcup_{x \in X}[x] = S$ and $\bigcap_{x \in X}[x] = \emptyset$ ☐

3.4. Functions

In elementary school, we are told to think of a function f from a set A into a set B as a rule which transforms a given input x in A into a unique output $f(x)$ in B. The notation $f(x)$ (read f of x) is intended to show that the element $f(x)$ is the result or image of function f acting on element x. The key property of functions is that they are **single-valued**; each input x has a unique output $f(x)$. For instance, figure **??**, we see a relation which is not a function because the input x_2 has two outputs.

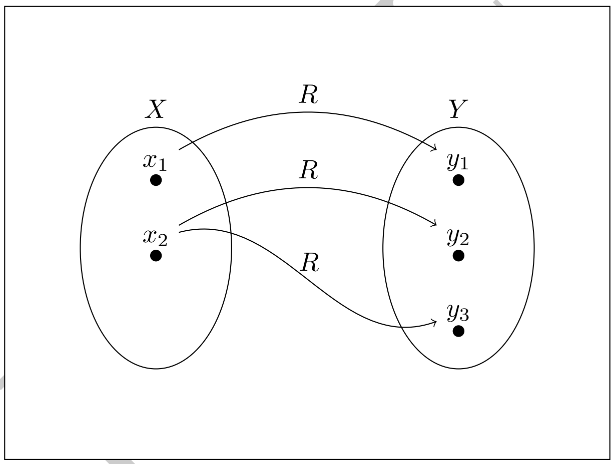

FIGURE 3.4.1. The relation R is not a function

In order to progress to more abstract mathematics, we we will need to formalize this intuitive definition. The mathematically rigorous definition of a function follows.

DEFINITION 3.4.1. Let A, B be sets. A relation $R \subseteq A \times B$ is a **function from A into B** iff:

$$\forall a \in A, \forall b_1, b_2 \in B\,(aRb_1 \wedge aRb_2 \in R \to b_1 = b_2)$$

We say that R **maps A into B** and write $R : A \to B$.

Some functions have important properties which we will discuss below. If $\forall b \in B, \exists a \in A(b = f(a))$, then f is said to be a **surjection** or **onto**. If $\forall a_{1,} a_2 \in A\,(f(a_1) = f(a_2) \to a_1 = a_2))$, then f is said to be an **injection** or **one-to-one**. A function which is both onto and one-to-one is said to be **bijective** or a **bijection**.

Since R is a function if and only if 3.4.1 holds, we have that R is not a function if and only if the negation of the definition holds (recall $A \leftrightarrow B \equiv \neg A \leftrightarrow \neg B$). To

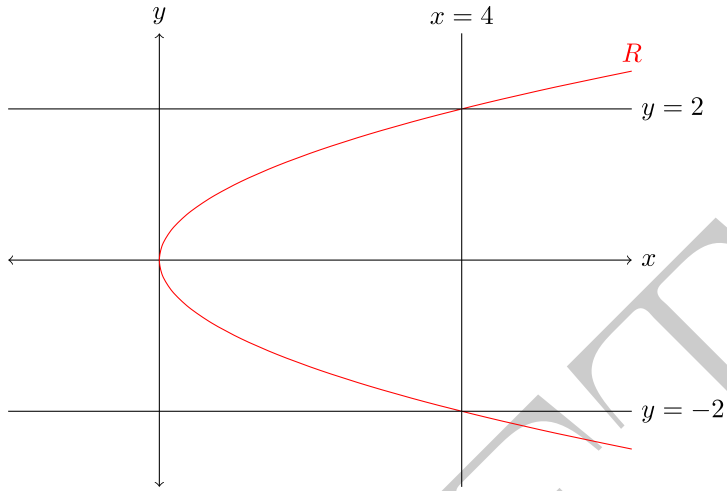

FIGURE 3.4.2. The Vertical Line Test

simplify the algebra, we define the statement $S \equiv (aRb_1 \wedge aRb_2 \in R)$. Thus the statement that a relation $R \subseteq A \times B$ is not a function iff:

$$\begin{aligned}\neg\forall a \in A, \forall b_1, b_2 \in B(S \to b_1 = b_2) &\equiv \exists a \in A, \exists b_1, b_2 \in B\neg(S \to b_1 = b_2)\\ &\equiv \exists a \in A, \exists b_1, b_2 \in B\,(S \wedge \neg(b_1 = b_2))\\ &\equiv \exists a \in A, \exists b_1, b_2 \in B\,(S \wedge b_1 \neq b_2)\\ &\equiv \exists a \in A, \exists b_1, b_2 \in B\,(S \wedge b_1 \neq b_2)\end{aligned}$$

Thus a relation is not a function iff there exists an $a \in A$ and $b_1, b_2 \in B$ such that (a, b_1) and (a, b_2) are in R and $b_1 \neq b_2$. Now, we can see precisely why, the relation $R \subseteq X \times Y$ pictured in figure **??** is not a function of x. Observe that $(x_2, y_2), (x_2, y_3) \in R$ but $y_2 \neq y_3$. For a more specific example, consider the relation

One familiar consequence of this definition is the **vertical line test**. This says that for any relation R in the XY-plane (a visual representation of $X \times Y$), R is not a function of x if a vertical line intersects R at more than one point. Consider the relation $R = \{(x^2, x) : x \in \mathbb{R})$ shown in figure 3.4.2. Then $(4, 2), (4-2) \in R$, but $2 \neq -2$, therefore R is not a function.

We use doubleheaded arrows ($\leftrightarrow$) in figure **??** to indicate the unique property of bijections that every element $y \in Y$ has a preimage in x, and that preimage is unique (i.e. no two distinct $x \in X$ map to the same y in Y).

EXAMPLE 3.4.2. In combinatorics, a bijection from a set to itself is sometimes called a **permutation**. Informally, a permutation of a set of objects is just a rearrangement, for instance, 01,2,3 is just a rearrangement of 1,2,3,0. We see how this is accomplished. Let $X = \{1, 2, 3, 4\}$, and let $f : X \to X$, $x \mapsto x + 1 \bmod 4$.

$$f : \{0, 1, 2, 3\} \to \{0, 1, 2, 3\}$$

EXERCISE 3.4.3. Prove the the identity function $i : X \to X$ defines a bijection between every set X and itself.

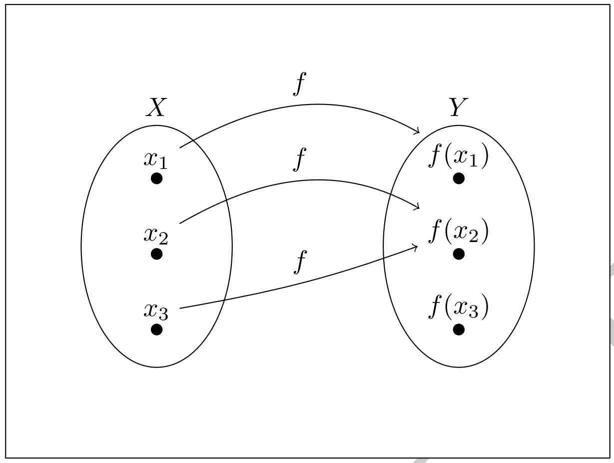

FIGURE 3.4.3. A non-surjective and non-injective function f from X into Y

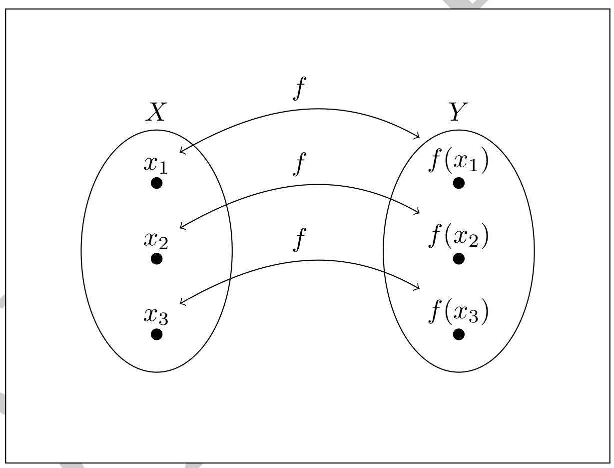

FIGURE 3.4.4. A bijection f from X onto Y

3.5. Sequences

DEFINITION 3.5.1. A function $a : N \to X$ from a subset of the integers N into a set X is called a **sequence in X**. Although sequences are simply a type of function, there is some special terminology and notation which is unique to sequences. The domain of a sequence is known as its **index set**, and an element of the index set is known as an **index**. The notation a_n is used instead of $a(n)$ to denote the image or value of n under a. We then call a_n a **term** or **element**. We use the notation $(a_n)_{n \in N}$ to refer to the sequence itself. If the domain is clear, we may simplify and write (a_n). In summary, if $a : N \to X$ is a sequence, then for each $n \in N$, $a(n)$ is denoted a_n and a itself is denoted $(a_n)_{n \in N}$ or (a_n) for short. [**13**, 156]

The most common domains for finite sequences are sets of the form $\mathbb{N}_N = \{n \in \mathbb{N} : n < N\} = \{0, 1, 2, 3, \ldots, N-1\}$, sometimes called **initial segments** or **sets of**

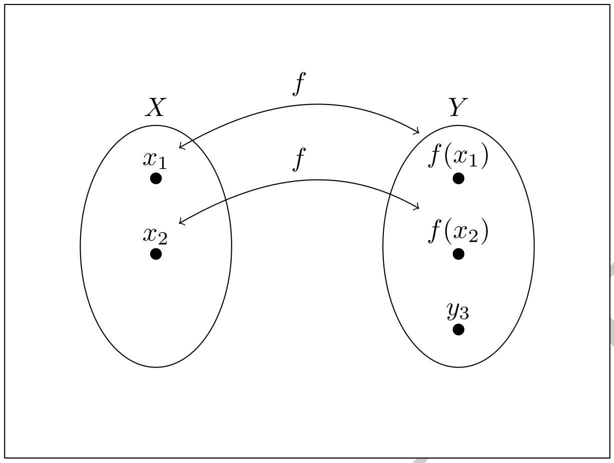

FIGURE 3.4.5. An Injection f from X into Y

strictly preceeding elements of the natural numbers.[**3**, 16] When the domain of a sequence is an initial segment, we can refer to a_n as the nth term of the sequence

EXAMPLE 3.5.2. We can define a particular sequence by giving a formula for the nth term a_n. Let (a_n) be the sequence with domain $N = \{1, 2, 3, 4, 5\}$ and whose nth term is $a_n := \frac{1}{n^2}$. Then $(a_n)_{n \in N} = (\frac{1}{1^2}, \frac{1}{2^2}, \frac{1}{3^2}, \frac{1}{4^2}, \frac{1}{5^2}) = (1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25})$, and we would say 1 is the 1st term or element of the sequence, $\frac{1}{4}$ is the 2nd term, $\frac{1}{9}$ is the third and so on. [**1**, 55]

3.5.1 shows how a sequence (a_n) maps element n of the index set $\{1, 2, \ldots\}$ to the element a_n, which we call the nth term of the sequence.

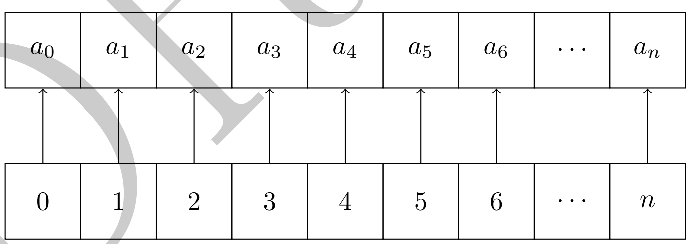

FIGURE 3.5.1. The terms a_n of the sequence (a_n)

3.6. Counting and the Cardinality of Sets

How would you go about counting the number of distinguishable objects lying on a table? One scheme is to go through each object, and label them with a natural number such that the first object is labeled 1, the second 2, and so on being sure to avoid repeats. This process is shown in the picture below in the case of three objects.

We can see that this labeling has some unique properties. Firstly, each element in $\{1, 2, 3\}$ is associated with exactly one object. This means that this labeling is a

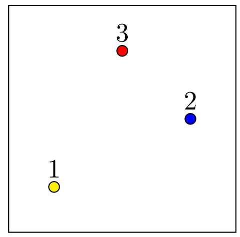

FIGURE 3.6.1. Counting Objects on a Table

function from $\{1, 2, 3\}$ into the set of objects. Moreover, each object is associated with a number from 1 to 3, (i.e. an element of $\{1, 2, 3\}$ so the function is onto. Finally we see that if two objects are the same, then they have the same label. Thus the function is 1-1. Therefore the labeling is really a bijection.

This example provides the intuition for our formal definition of the size or cardinality of a set.

DEFINITION 3.6.1. Cardinality of Sets [**1**, 16]

(1) The empty set $\emptyset$ has 0 elements
(2) If $n \in \mathbb{N}$, a set S is said to have n elements if there exists a bijection from the set $\mathbb{N}_n := \{1, 2, \ldots, n\}$ onto S.
(3) A set S is said to be **finite** if it is either empty or it has n elements for some $n \in \mathbb{N}$.
(4) A set S is said to be **infinite** if it is not finite.

Denumerable, Countable, and Uncountable Sets [**1**, 17]

- A set S is **denumberable** (or **countably infinite**) if there exists a bijection of $\mathbb{N}$ onto S

(1) A set S is **countable** if it is either finite or denumerable
(2) A set S is said to be **uncountable** if it is not countable.

The intuition behind this definition is as follows. Because the identity function is Plugging $\mathbb{N} = S$ in definition 3.26#1, we see that $\mathbb{N}$

EXAMPLE 3.6.2. We will show

$$A = \{a, b, c, d, e, f, g, h\}$$
$$B = \{1, 2, 3, 4\}$$

It may seem that set A is in some way "bigger" than set B. We can check this by "counting" the elements.

EXERCISE 3.6.3. Show that the set of integers $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$

3.7. Further Reading:

For more on modular arithmetic, see [**7**]. For more on counting and combinatorics see [**2, 4**].

Appendix

3.8. Summary Tables

For easy reference, we include the following tables:

Table 1. The basic connectives and their compound statements

Connective	Compound Statement	Written
Conjunction of P and Q	P and Q	$P \wedge Q$
Disjunction of P and Q	P or Q	$P \vee Q$
Negation of P	not P	$\neg P$
Implication	If P then Q, P implies Q	$P \to Q$

Table 2. Truth tables for the four basic connectives

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

P	Q	$P \to Q$
T	T	T
T	F	F
F	T	T
F	F	T

P	$\neg P$
T	F
F	T

Table 3. Important Logical Equivalencies

Name	Statement
De Morgan's Laws	$\neg(P \wedge Q) \equiv \neg P \vee \neg Q$
De Morgan's Laws	$\neg(P \vee Q) = \neg P \wedge \neg Q$
Contrapositive	$P \to Q \equiv \neg Q \to \neg P$
Material Conditional	$A \to B \equiv \neg A \vee B$
Negation of Conditional	$\neg(A \to B) = A \wedge \neg B$
Biconditional Statement	$P \leftrightarrow Q \equiv (P \to Q) \wedge (Q \to P)$
Distributive Laws	$P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$
Distributive Laws	$P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$
Conditionals with Disjunctions	$P \to (Q \vee R) \equiv (P \wedge \neg Q) \to R$
Conditionals with Disjunctions	$(P \vee Q) \to R \equiv (P \to R) \wedge (Q \to R)$
Negation of Conditional	$\neg(A \to B) = A \wedge \neg B$
Material Conditional	$A \to B \equiv \neg A \vee B$
Exportation	$A \to (B \to C) \equiv (A \wedge B) \to C$
Identity Laws	
Domination Laws	$T \wedge P = P$
Domination Laws	$T \vee P = T$
Domination Laws	$F \wedge P = F$
Domination Laws	$F \vee P = P$

Symbol Dictionary

Symbol	Meaning
$:=$	defined to be equal to (assignment)
$\therefore$	Therefore
$\square$	end of proofs
$\iff$	If and only if, iff

The Greek Alphabet

Capital	Lowercase	Name	Capital	Lowercase	Name
A	α	Alpha	N	ν	Nu
B	β	Beta	Ξ	ξ	Xi
Γ	γ	Gamma	O	o	Omicron
Δ	δ	Delta	Π	π	Pi
E	ϵ	Epsilon	P	ρ	Rho
Z	ζ	Zeta	Σ	σ	Sigma
H	η	Eta	T	τ	Tau
Θ	θ	Theta	Υ	υ	Upsilon
I	ι	Iota	Φ	ϕ	Phi
K	κ	Kappa	X	χ	Chi
Λ	λ	Lambda	Ψ	ψ	Psi
M	μ	Mu	Ω	ω	Omega

Index

Bibliography

[1] Robert G. Bartle and Donald R. Sherbert. *Introduction to Real Analysis*. Wiley, 2011.

[2] Norman L. Biggs. *Discrete Mathematics*. Oxford Univerity Press, New York, 2 edition, 2009.

[3] George L. Cain. *Introduction to General Topology*. Addison-Wesley, New York, 1994.

[4] Peter J. Cameron. *Combinatorics: Topics, Techniques, Algorithms*. Cambridge University Press, New York, 1994.

[5] Herbert Enderton. *A Mathematical Introduction to Logic*. Elsevier Science Publishing Co Inc, 2001.

[6] Herbert B. Enderton. *Elements of Set Theory*. Academic Press, 1977.

[7] Joseph A Gallian. *Contemporary Abstract Algebra*. Brooks/Cole, Boston, 8 edition, 2013.

[8] Richard Hammack. *Book of Proof*. Richard Hammack, Richmond, VA, 2.2 edition, 2013.

[9] Paul Horowitz and Winfield Hill. *The Art of Electronics*. Cambridge University Press, 2015.

[10] International Organization for Standardization. Quantities and units–part 2: Mathematical signs and symbols to be used in the natural sciences and technology. techreport ISO 80000-2:2009, International Organization for Standardization, Geneva, Switzerland, December 2009.

[11] Elliott Mendelson. *Introduction to Mathematical Logic*. Chapman & Hall, New York, 4 edition, 1997.

[12] Merriam-Webster. vacuous, 2018.

[13] Kenneth H. Rosen. *Discrete Mathematics and Its Applicaitons*. McGraw Hill, New York, 7 edition, 2012.

[14] Peter Smith. *An Introduction to Formal Logic*. CAMBRIDGE UNIV PR, 2003.

[15] Ted Sundstrom. *Mathematical Reasoning Writing and Proof*. Pearson, New York, 1.1 edition, 2013.

[16] Dirk van Dalen. *Logic and Structure*. Springer, Berlin, 4 edition, 2008.

[17] Daniel J. Velleman. *How To Prove It*. Cambridge, New York, 2 edition, 2006.

[18] Robert S. Wolf. *A Tour Through Mathematical Logic (Carus Mathematical Monographs)*. The Mathematical Association of America, 2005.