Area of a Circular Segment

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Abstract

We examine several different problems relating to sectors and segments of the unit circle. We examine solutions using geometry and calculus and carefully illustrate the underlying geometry.

Definition 1. Circular Sector

A **circular sector**, also known as circle sector or disk sector (symbol: $\triangledown$), is the portion of a disk (a closed region bounded by a circle) enclosed by two radii and an arc, where the smaller area is known as the minor sector and the larger being the major sector. In the diagram, θ is the **central angle**, r the radius of the circle, and L is the arc length of the minor sector.

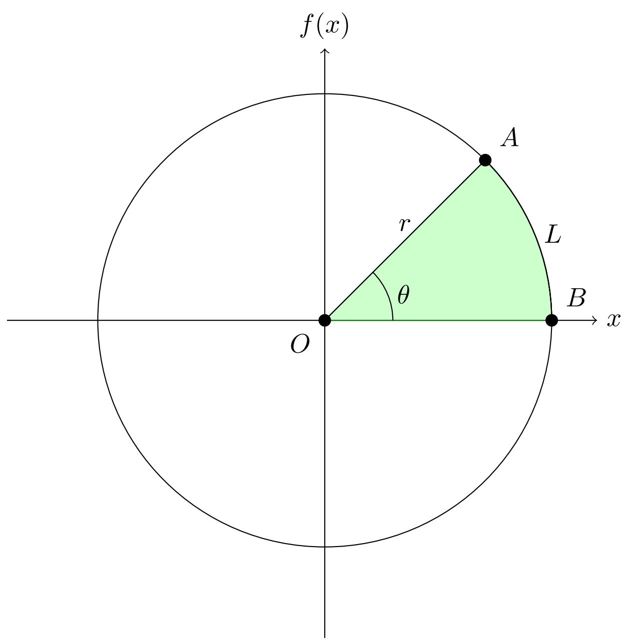

Figure 1: Circular Sector $\triangledown AOB$

Problem 2. Calculate the arc length L of sector $\triangledown AOB$. I.e. calculate the distance one would have to travel along the circle to get from A to B.

Solution. The arc length of the whole circle (i.e. the circumference) is $2\pi r$, which corresponds to a sector with central angle $\theta = 2\pi$. The arc length L corresponds to angle θ representing $\frac{\theta}{2\pi}$ of the total radians of the circle. Setting up the proportion:

$$\frac{2\pi r}{2\pi} = \frac{L}{\theta}$$
$$r\theta = L$$

Equivalently, we might observe that if the whole circle has arc length $2\pi r$, and the sector $\triangledown AOB$ represents $\frac{\theta}{2\pi}$ of the circle, then the length of $\triangledown AOB$ is $\frac{\theta}{2\pi}$ of the perimeter of the whole circle. Thus we compute

$$2\pi r \cdot \frac{\theta}{2\pi} = r\theta = L$$

Problem 3. Calculate the area of sector $\triangledown AOB$

Solution. The full circle has area πr^2 and corresponds to a sector with central angle $\theta = 2\pi$. Our sector $\triangledown AOB$ has central angle θ and unknown area x. Setting up the proportion we have

$$\frac{\pi r^2}{2\pi} = \frac{x}{\theta}$$
$$\frac{\theta r^2}{2} = x$$

Equivalently, we might observe that if the whole disk has area πr^2, and the sector represents $\frac{\theta}{2\pi}$ of the disk, then its area is $\frac{\theta}{2\pi}$ of the area of the whole disk. Thus we compute

$$\begin{aligned} x =& \pi r^2 \cdot \frac{\theta}{2\pi} \\ =& \frac{\theta r^2}{2} \end{aligned}$$

Definition 4. Circular Segment

In geometry, a **circular segment**, also known as a **disk segment**, is a region of a disk which is "cut off" from the rest of the disk by a secant or a chord. More formally, a circular segment is a region of two-dimensional space that is bounded by a circular arc (of less than π radians by convention) and by the circular chord connecting the endpoints of the arc.

Problem 5. Calculate the area of the circular segment $\aleph$, $A(\aleph)$

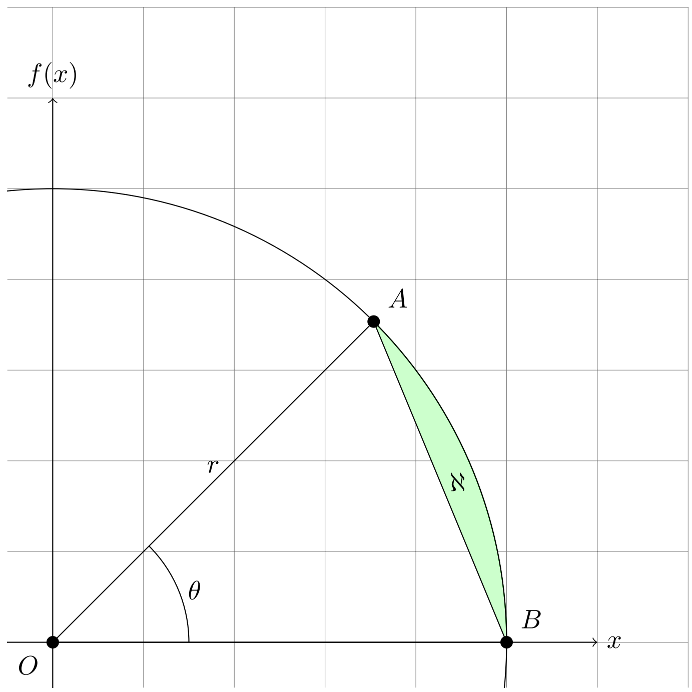

Figure 2: Circular Segment ℵ

Solution. Using our answer to problem 3, we know the area of sector $\triangledown AOB$ is $r^2\frac{\theta}{2}$. If we subtract the area of the triangle $\triangle AOB$ from $\triangledown AOB$, we obtain the area of $\aleph$. By converting to polar coordinates, we see that $\triangle AOB$ has length r and height $r\sin\theta$. Therefore the area of $\triangle AOB$ is given by $\frac{r^2\sin\theta}{2}$. Therefore:

$$\begin{aligned} A(\aleph) &= \frac{r^2\theta}{2} - \frac{r^2\sin\theta}{2} \\ &= \frac{r^2}{2}(\theta - \sin\theta) \end{aligned}$$

Problem 6.

Problem 7. Calculate the length of $\overline{AB}$, which we will denote by $|AB|$. Use the result to determine what values of θ make $|AB| < r$?

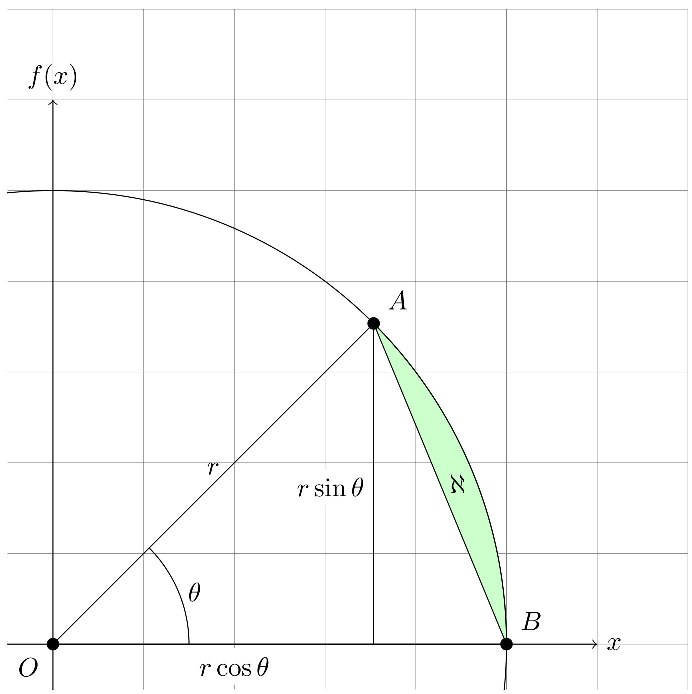

Figure 3: Using polar coordinates to solve problem 4

Solution. For simplicity, label $A = (r\cos\theta, r\sin\theta)$, $O = (0,0)$, $B = (r,0)$.

$$\begin{aligned}
|AB| &= \|A - B\| \\
&= \sqrt{(r\cos\theta - r)^2 + (r\sin\theta - 0)^2} \\
&= \sqrt{(r\cos\theta - r)^2 + r^2\sin^2\theta} \\
&= \sqrt{(r\cos\theta - r)(r\cos\theta - r) + r^2\sin^2\theta} \\
&= \sqrt{r^2\cos^2\theta - 2r^2\cos\theta + r^2 + r^2\sin^2\theta} \\
&= \sqrt{r^2\cos^2\theta + r^2\sin^2\theta - 2r^2\cos\theta + r^2} \\
&= \sqrt{r^2\left(\cos^2\theta + \sin^2\theta\right) - 2r^2\cos\theta + r^2} \\
&\overset{\alpha}{=} \sqrt{r^2(1) - 2r^2\cos\theta + r^2} \\
&= \sqrt{r^2 - 2r^2\cos\theta + r^2} \\
&= \sqrt{2r^2 - 2r^2\cos\theta} \\
&= \sqrt{2r^2(1 - \cos\theta)} \\
&= r\sqrt{2(1 - \cos\theta)} \\
&\overset{\beta}{=} 2r\sin\frac{\theta}{2}
\end{aligned}$$

where equation α uses the Pythagorean identity, and equation β uses the half-angle identity for $\sin\frac{\theta}{2}$. Since $|AB| = \|A - B\| = 2r\sin\frac{\theta}{2}$, we need to find when $2r\sin\frac{\theta}{2} < r$. Solving for θ we have

$$\begin{aligned}
2r\sin\frac{\theta}{2} &< r \\
\sin\frac{\theta}{2} &< \frac{1}{2} \\
\arcsin\left(\sin\frac{\theta}{2}\right) &< \arcsin\left(\frac{1}{2}\right) \\
\frac{\theta}{2} &< \frac{\pi}{6} \\
\theta &< \frac{\pi}{3}
\end{aligned}$$

Problem 8. Use the Law of Cosines to calculate $|AB|$. Compare the result to that calculated in 7.

Solution. Recall the Law of Cosines states that if γ is the angle between sides a, b and opposite from c then

$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$

In our case $a = b = r$, which simplifies this expression to

$$\begin{aligned}
c^2 &= 2r^2 - 2r^2 \cos\gamma \\
c &= \sqrt{2r^2 \left(1 - \cos\gamma\right)} \\
&= r\sqrt{2\left(1 - \cos\gamma\right)} \\
&\overset{\beta}{=} 2r \sin\frac{\theta}{2} \\
&= |AB|
\end{aligned}$$

where equation β uses the half-angle identity for $\sin\frac{\theta}{2}$. This is the same expression as we obtained in 7. We observe that we have proved a special case of the Law of Cosines, namely that for an isosceles triangle.

Problem 9. Prove the Law of Cosines using the distance formula.

Solution. Let $A = (r\cos\theta, r\sin\theta)$, $O = (0,0)$, $B = (a,0)$. Note that by having $B = (a,0)$ instead of $B = (r,0)$ we have relaxed the assumption that B lies on the same circle as A so that we can give a general proof.

$$\begin{aligned}
|AB| &= \|A - B\| \\
&= \sqrt{\left(r\cos\theta - a\right)^2 + \left(r\sin\theta - 0\right)^2} \\
&= \sqrt{\left(r\cos\theta - a\right)^2 + r^2\sin^2\theta} \\
&= \sqrt{\left(r\cos\theta - a\right)\left(r\cos\theta - a\right) + r^2\sin^2\theta} \\
&= \sqrt{r^2\cos^2\theta - 2ar\cos\theta + a^2 + r^2\sin^2\theta} \\
&= \sqrt{r^2\cos^2\theta + r^2\sin^2\theta - 2ar\cos\theta + a^2} \\
&= \sqrt{r^2\left(\cos^2\theta + \sin^2\theta\right) - 2ar\cos\theta + a^2} \\
&\overset{\alpha}{=} \sqrt{r^2\left(1\right) - 2ar\cos\theta + a^2} \\
&= \sqrt{a^2 + r^2 - 2ar\cos\theta}
\end{aligned}$$

where equation α uses the Pythagorean identity.

Problem 10. Use elementary geometry to show that $|AB| = r$ when $\theta = \frac{\pi}{3}$.

Solution. Suppose $|AB| = r$. Then since $|OB| = |OA| = r$, all three sides of the triangle $\triangle OAB$ are equal to r, that is to say $\triangle OAB$ is an equilateral triangle. By applying the isosceles triangle theorem twice, we can show that an equilateral triangle must also be equiangular (i.e. have 3 equal angles). Thus all three interior angles of $\triangle OAB$ are equal. Since the angles of a triangle sum to π, each angle must be $\frac{\pi}{3}$. So we know $|AB| = r$ when $\theta = \frac{\pi}{3}$.

Problem 11. Given an arbitrary line in slope-intercept form $y = mx + b$, re-express the equation in polar coordinates with $r = f\left(\theta\right)$

Solution. Changing to polar, we have:

$$\begin{aligned} y &= r\sin\theta \\ x &= r\cos\theta \end{aligned}$$

then

$$\begin{aligned} y &= mx + b \\ r\sin\theta &= mr\cos\theta + b \\ r\sin\theta - mr\cos\theta &= b \\ r\left(\sin\theta - m\cos\theta\right) &= b \\ r &= \frac{b}{\sin\theta - m\cos\theta} \end{aligned}$$

Problem 12. Find the slope of the line through $\overline{AB}$

Solution. The slope m of $\overline{AB}$ is given by

$$\begin{aligned} m &= \frac{r\sin\theta - 0}{r\cos\theta - r} \\ &= \frac{r\sin\theta}{r\left(\cos\theta - 1\right)} \\ &= \frac{\sin\theta}{\cos\theta - 1} \\ &\overset{\gamma}{=} \frac{-1}{\tan\frac{\theta}{2}} \\ &\overset{\delta}{=} -\cot\frac{\theta}{2} \end{aligned}$$

where equation γ follows from the half-angle formula for $\tan\frac{\theta}{2}$, and δ by the definition of cot.

Problem 13. Find the y intercept of the line through $\overline{AB}$.

Solution. To find the y-intercept of the line with slope $m = -\cot\frac{\theta}{2}$ passing through the point $(r, 0)$, we use the result of problems 11 and 12

$$\begin{aligned}
\frac{b}{\sin\theta - m\cos\theta} &= r \\
b &= r\left(\sin\theta - m\cos\theta\right) \\
&= r\left(\sin\theta + \cot\frac{\theta}{2}\cos\theta\right) \\
&= r\left(\sin\theta + \frac{\sin\theta}{1-\cos\theta}\cos\theta\right) \\
&= r\sin\theta\left(1 + \frac{\cos\theta}{1-\cos\theta}\right) \\
&\overset{\alpha}{=} r\sin\theta\left(\frac{1-\cos\theta}{1-\cos\theta} + \frac{\cos\theta}{1-\cos\theta}\right) \\
&= r\sin\theta\left(\frac{1-\cos\theta+\cos\theta}{1-\cos\theta}\right) \\
&= r\frac{\sin\theta}{1-\cos\theta} \\
&= r\cot\left(\frac{\theta}{2}\right)
\end{aligned}$$

Note that in to obtain equation α, we multiply 1 by $\frac{1-\cos\theta}{1-\cos\theta}$ to convert to a common denominator.

Problem 14. In this problem, we will repeat our previous work but using the point-slope equation of a line instead of the slope-intercept form.

1. Write the point slope equation of a line
2. Given the point-slope equation of a line, identify an expression for the y-intercept. Call this expression b.
3. Compute the y-intercept of the line through $\overline{AB}$, in polar form:
4. Write the point slope equation of a line in polar form
5. Write the point slope equation of the line through $\overline{AB}$ in polar form
6. Write the point slope equation of the line through $\overline{AB}$ in slope-intercept form
7. Write the point slope equation of the line through $\overline{AB}$ in slope-intercept form, assuming $\theta = \frac{\pi}{3}$ and $r = 2$
8. What are the slope and intercept of the line through $\overline{AB}$ when $\theta = \frac{\pi}{3}$ and $r = 2$?
9. Make a conjecture about the relationship between the slope of a line in polar form and the intercept

10. Write an expression for the set of all points in the circular segment $\aleph$ using polar coordinates

11. Write an expression for the set of all points in the circular segment $\aleph$ using Cartesian coordinates

Solution. Several parts:

1. Recall point-slope form is given by:

$$y - y_1 = m\left(x - x_1\right)$$

2. We identify for the intercept as follows

$$\begin{aligned} y - y_1 &= m\left(x - x_1\right) \\ y &= mx - mx_1 + y_1 \\ y - mx &= -mx_1 + y_1 \\ &= b \end{aligned}$$

3. Since $b = -mx_1 + y_1$, using the result from problem 12that $m = -\cot\frac{\theta}{2}$ and that the line passes through $(x_1, y_1) = (r, 0)$ we have

$$\begin{aligned} b &= -mx_1 + y_1 \\ &= \cot\frac{\theta}{2} r \end{aligned}$$

4. To convert the point slope form of a line to polar form, we make the usual substitutions. One wrinkle is that we must use notation that distinguishes the point $(x_1, y_1) = (r\cos\theta, r\sin\theta)$ from the variables $x = r\cos\theta$ and $y = r\sin\theta$. Thus we refer to $(x_1, y_1) = (r'\cos\theta', r''\sin\theta'')$.

$$r\sin\theta - r'\sin\theta' = m\left(r\cos\theta - r''\cos\theta''\right)$$

5. We simply plug in for m and $(r'\cos\theta', r''\sin\theta'')$ to the previous equation

$$\begin{aligned} r\sin\theta - r'\sin\theta' &= m\left(r\cos\theta - r''\cos\theta''\right) \\ r\sin\theta - 0 &= -\cot\frac{\theta'}{2}\left(r\cos\theta - r''\right) \\ r\sin\theta &= -\cot\frac{\theta'}{2}\left(r\cos\theta - r''\right) \end{aligned}$$

6. From problem 11, we know the polar form of a line in slope intercept form is:

$$r = \frac{b}{\sin\theta - m\cos\theta}$$

We also previously showed that $b = \cot\frac{\theta}{2} r$ and $m = -\cot\frac{\theta}{2}$. Thus

$$r = \frac{b}{\sin\theta - m\cos\theta} = \frac{\cot\frac{\theta}{2} r}{\sin\theta + \cot\frac{\theta}{2}\cos\theta}$$

7. When $\theta = \frac{\pi}{3}$, we have $\cot\frac{\theta}{2} = \frac{\pi}{6} = \sqrt{3}$. Since $r = 2$, we have:

$$r = \frac{\cot\frac{\theta}{2} r}{\sin\theta + \cot\frac{\theta}{2}\cos\theta} = \frac{\sqrt{3}\cdot 2}{\sin\theta + \sqrt{3}\cos\theta}$$

8. Since $b = \cot\frac{\theta}{2} r$ is the intercept and $m = -\cot\frac{\theta}{2}$ is the slope, when $r = 2$ and $\theta = \frac{\pi}{3}$ we have:

$$b = \sqrt{3}\cdot 2, m = \sqrt{3}$$

9. In this problem, we found that the intercept b was equal to the slope m multiplied by a factor of $-r$. That is

$$\begin{aligned} m &= -\cot\frac{\theta}{2} \\ -mr &= \cot\frac{\theta}{2} r \\ &= b \end{aligned}$$

we conjecture that it is always true that $b = -mr$.