Graph Transformations

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Abstract

In this note, we examine the four elementary transformations of a function. Visual examples are provided in addition to written discussion. We show how to build more complex transformations from compositions of these elementary transformations. We conclude with a discussion of the implications of the commutativity of the elementary transformations.

1 The Four Elementary Graph Transformations

We begin by reviewing the four elementary transformations of a function. These consist of shifting a graph horizontally or vertically, as well as horizontal and vertical scaling.

1. To shift the graph of $f(x)$ up, we add a positive number k to $f(x)$, and to shift it down we add a negative number. Thus vertically shifting $f(x)$ by k takes the form $f(x) + k$, with the direction of movement (up or down) depending on the sign of k (i.e. if k is positive or negative).

2. To stretch or shrink the graph of $f(x)$ vertically by a factor of a, we simply multiply $f(x)$ by a. If $|a| > 1$ the graph is vertically stretched, and if $1 > |a|$ the graph shrinks vertically. If $a < 0$, the graph is flipped or reflected about the x-axis and stretched by a. In particular, if we encounter $-af(x)$, we might think of it as $-1 \cdot af(x)$, so it is clear that $-af(x)$ represents both flipping and vertically stretching/shrinking of $f(x)$ by a.

3. The transformation $f(x + k)$ will move the $f(x)$ to the left or right depending if the sign of k is positive or negative.

4. We can scale f horizontally, that is make $f(x)$ narrower or wider, by multiplying x by a constant. In particular $f(ax)$ becomes narrower if $|a| > 1$, wider if $|a| > 0$, and if $a < 0$, $f(x)$ is flipped across the y-axis.

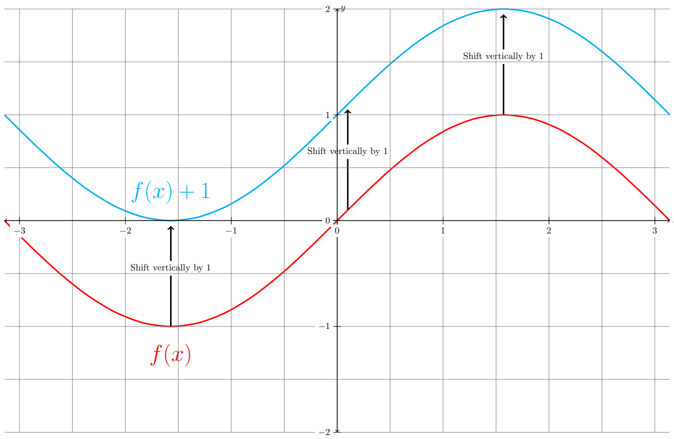
Many students often have trouble remembering which direction the graph moves, when we shift the graph horizontally so we will illustrate with an example.

Example 1. Consider the graph of $f(x) = x^2$. We know what this graph looks like; it is a parabola opening upwards with one root at $x = 0$. Now consider $f(x+3) = (x+3)^2$, where is its root? I.e, for what value of x is is $f(x+3) = 0$? Well, when $x = -3$, $f(x+3) = (-3+3)^2 = 0^2 = 0$. So now the root is at -3, so changing $f(x)$ to $f(x+3)$ has moved $f(x)$ three units to the left. If we repeat this experiment, we will see that $f(x-3)$ has a zero at $x = 3$. Thus $f(x-3)$ moves $f(x)$ three units to the right.

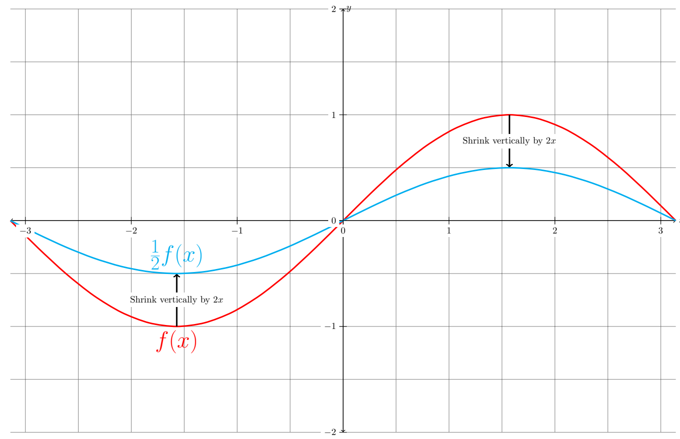
We summarize the basic graph transformations below in table 1 and illustrate them in figure 1.1 and figure 1.2.

Action	Algebra	Note
Vertical Shift	$f(x) + k$	Direction of shift $(\uparrow, \downarrow)$ if k $(+, -)$
Vertical Scaling (Stretch/Shrink)	$af(x)$	Stretch if $a > 1$, shrink if $0 < a < 1$, flip over x-axis when $a < 0$
Horizontal Shift	$f(x+k)$	Shift $\leftarrow$ if k positive, shift $\rightarrow$ if k is negative
Horizontal Scaling	$f(ax)$	Narrower if $a > 1$, wider if $a < 1$, flip over y-axis when $a < 0$

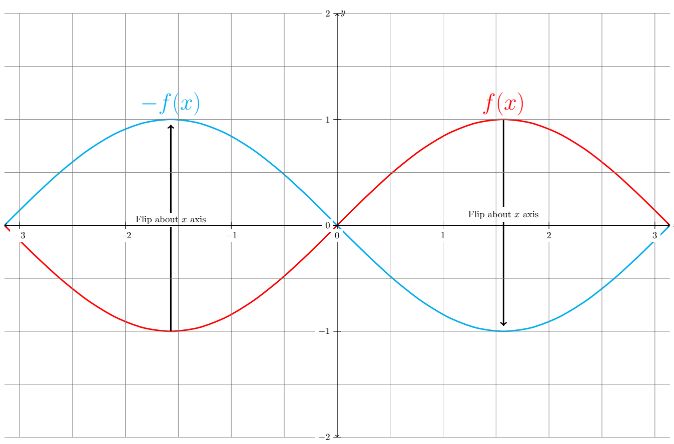
Table 1: Graph Transformations



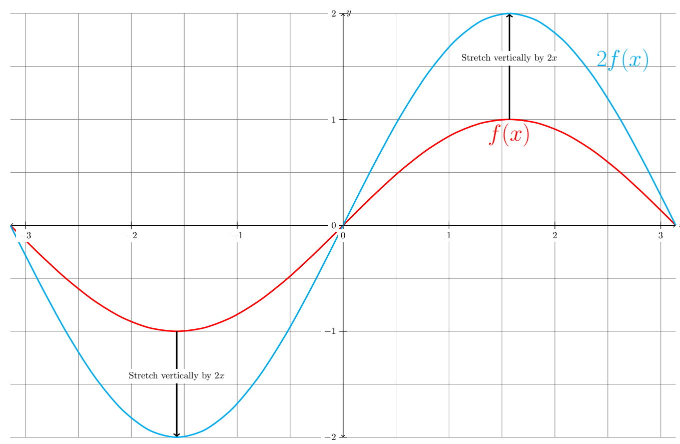
(a) Shifting $f(x)$ vertically by adding a constant outside $f(x)$



(b) Scaling $f(x)$ vertically by multiplying $f(x)$ by a constant

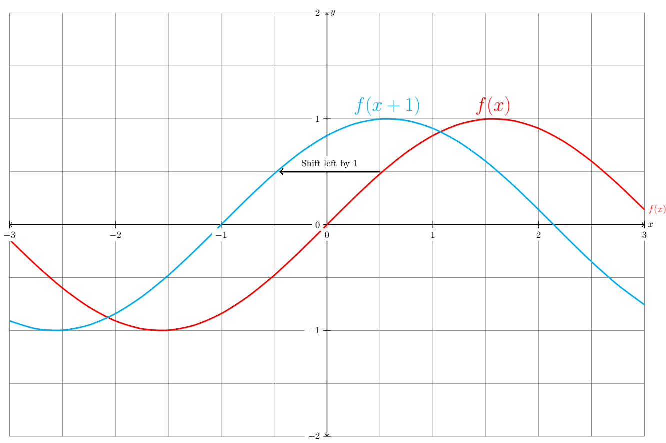


(c) Flipping $f(x)$ across the x-axis by multiplying $f(x)$ by a negative constant

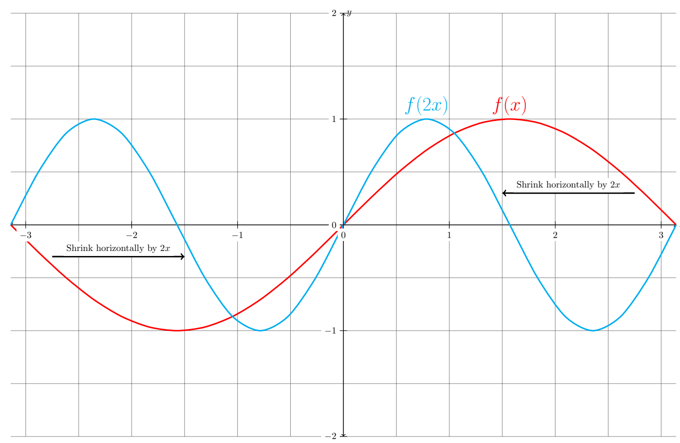


(d) Scaling $f(x)$ vertically by multiplying $f(x)$ by a constant

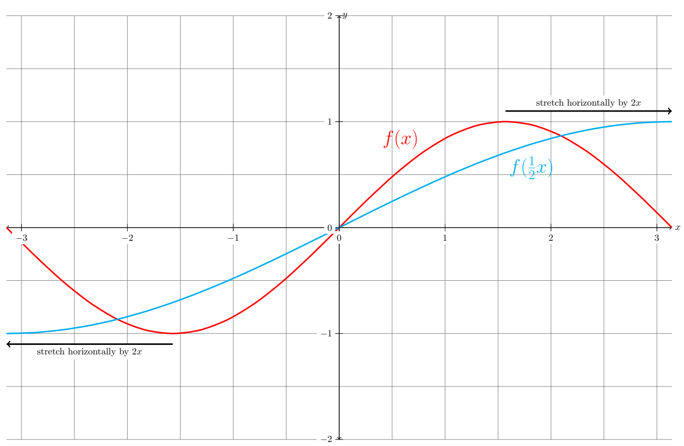
Figure 1.1: Vertical Transformations



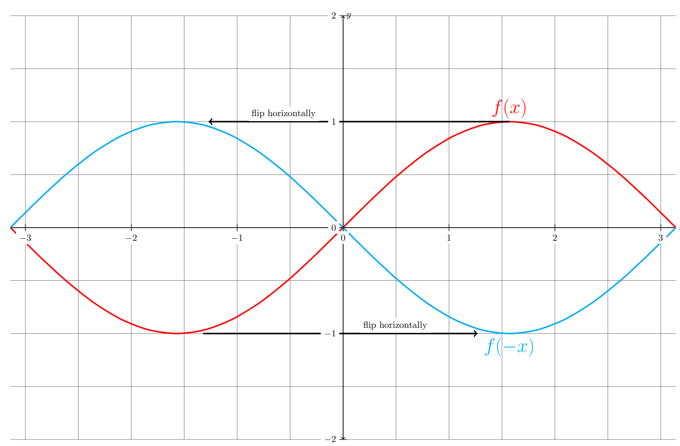
(a) Shifting $f(x)$ horizontally



(b) Shrinking $f(x)$ horizontally



(c) Stretching $f(x)$ horizontally



(d) Flipping $f(x)$ horizontally (across the y-axis)

Figure 1.2: Horizontal Transformations

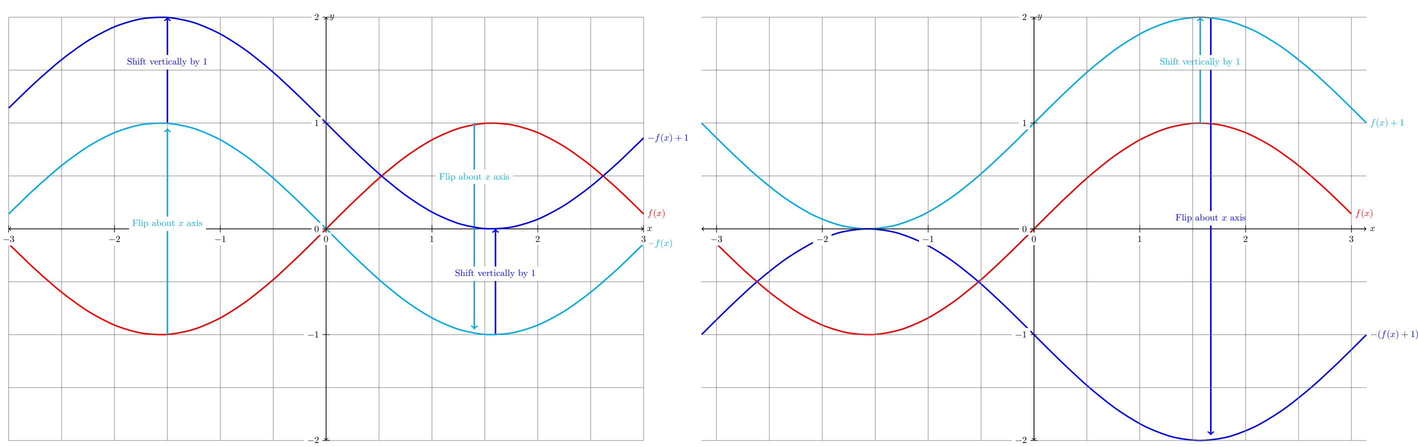
2 Building Complex Graph Transformations

We often apply multiple transformations to a single function. For example, we might scale $f(x)$ vertically and horizontally, or we might flip it vertically and shift it horizontally. We might even apply all four transformations to a single graph. How do we approach finding the shape of the translated graph?

The key insight is that each complex transformation, by which mean a transformation created as the composition of several elementary transformations, can be built as a sequence of elementary transformations. However, the order in which these elementary operations are performed matters.

Example 2. Order of Operations Matters

Consider the transformation $af(x)+b$. By following the order of operations (PEMDAS), we see that $af(x)+b$ represents scaling $f(x)$ by a then then shifting it by b. How would this transformation change if we applied the shift first? In figure 2.1, we depict each transformation as a sequence of the same two elementary transformations, but changing the order in which they are applied. We can see that changing the order of operations causes the end results to be quite different. Specifically, shifting then flipping $f(x)$ results in $-(f(x)+1) = -f(x)-1$ while flipping then shifting results in $-f(x)+1$. Thus flipping first results in a graph which is 2 units higher than the once produced by shifting first.



(a) Flipping vertically then shifting $f(x)$ vertically

(b) Shifting vertically then flipping $f(x)$ vertically

Figure 2.1: Order Matters - Shifting then Scaling and vice versa

We now discuss the general case

Theorem 3. *Any combination of the four basic transformations can be represented as*

$$af(b(x-c))+d \tag{2.1}$$

By following the order of operations in equation (2.1), we can interpret equation (2.1) as transforming $f(x)$ to $af(b(x-c))+d$ by the following sequence of elementary transformations

1. Horizontally shift $f(x)$ by c
2. Horizontal scale $f(x)$ by a factor of b
3. Vertically scaling $f(x)$ a factor of a
4. Vertically shift $f(x)$ by d