Chapter 2

LTI SYSTEM

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Linear Time Invariant System.

input → LTI → output

A system which satisfies property of linearity and time invariance is known as linear time invariant system.

LTI system studied mainly for two reason

(1). most of the physical process possesses these properties.
(2). An LTI system can be represented in impulse response.

Continuous Time LTI System

$x(t)$ → CT LTI System → $y(t)$

$h(t)$: Impulse Response (or) System Response

Impulse → Impulse response
Unit step → Unit step response
Unit ramp → Unit ramp response
Unit parabolic → Unit parabolic response

☆ Important Laplace Transform

$$x(t) \xleftrightarrow{LT} X[s]$$

$$\delta(t) \xleftrightarrow{LT} 1$$

$$u(t) \xleftrightarrow{L.T.} 1/s$$

$$e^{-at}u(t) \xleftrightarrow{LT} \frac{1}{s+a}$$

Impulse Response

$$\underset{X[s]}{x(t)} \longrightarrow \boxed{H[s] = \frac{1}{s+3}} \longrightarrow \underset{Y[s]}{y(t)}$$

$h(t)$: Impulse Response

$\Rightarrow Y[s] = X[s] \cdot H[s]$

For impulse response

$x(t) = \delta(t) \Rightarrow \underbrace{y(t)}_{\text{Impulse response}}$

$\updownarrow L.T.$

$X[s] = 1$

$\Rightarrow Y[s] = 1 \cdot H[s] = H[s]$

$\Rightarrow Y[s] = H[s]$

$\downarrow L^{-1}$

$\Rightarrow y(t) = \underbrace{h(t)}_{\text{Impulse Response}}$

$h(t)$: Impulse Response

$y(t) = T\{x(t)\}$

$x(t) = \delta(t) \Rightarrow y(t) = T\{\delta(t)\} = h(t)$

Unit Step Response

$$x(t) \longrightarrow \boxed{H[s] = \frac{1}{s+3}} \longrightarrow y(t)$$

$\Rightarrow$ For unit step Response : $x(t) = u(t) \Rightarrow y(t)$: Unit Step Response

$\Rightarrow X[s] = \frac{1}{s}$

$\Rightarrow x(t) = u(t)$

$\Rightarrow Y[s] = X[s] \cdot H[s]$

$\Rightarrow y(t) = T\{u(t)\}$

$\Rightarrow Y[s] = \frac{1}{s} \cdot H[s] \Rightarrow y(t) \neq h(t)$

$\Rightarrow y(t) \neq h(t)$

Continuous Time L.T.I. System

$$x(t) \longrightarrow \boxed{\begin{array}{c} \text{C T} \\ \text{LTI System} \end{array}} \longrightarrow y(t)$$

$h(t)$: Impulse Response

$\Rightarrow y(t) = T\{x(t)\}$

$\Rightarrow x(t) = \delta(t)$

$\Rightarrow y(t) = T\{\delta(t)\} = h(t)$

$\Rightarrow h(t) = T\{\delta(t)\}$

$\Rightarrow h(t-\tau) = T\{\delta(t-\tau)\}$

$$\Rightarrow x(t) = \int_{\tau=-\infty}^{\tau=+\infty} x(\tau)\cdot\delta(t-\tau)\cdot d\tau$$

$$\Rightarrow y(t) = T\{x(t)\}$$

$$\Rightarrow y(t) = T\left\{\int_{\tau=-\infty}^{\tau=+\infty} x(\tau)\cdot\delta(t-\tau)\cdot d\tau\right\}$$

$$\Rightarrow y(t) = \int_{\tau=-\infty}^{\tau=+\infty} x(\tau)\cdot T\{\delta(t-\tau)\}\cdot d\tau$$

$$\Rightarrow y(t) = \int_{\tau=-\infty}^{\tau=+\infty} x(\tau)\cdot h(t-\tau)\cdot d\tau$$

$$\Rightarrow y(t) = x(t) * h(t) \quad = \int_{\tau=-\infty}^{\tau=+\infty} x(\tau)\cdot h(t-\tau)\cdot d\tau = y(t)$$

CT Convolution between $x(t)$ & $h(t)$

$$y(t) = x(t) * h(t)$$

If $x(t) = \delta(t) \Rightarrow y(t) = \delta(t) * h(t) = h(t)$

LTI System

$$\Rightarrow \delta(t) * h(t) = h(t)$$

$$\Rightarrow \delta(t) * x(t) = x(t)$$

$$\Rightarrow x(t) * \delta(t) = x(t)$$

$$\Rightarrow x(t) * \delta(t-t_0) = x(t-t_0)$$

$$\Rightarrow x(t-t_0) * \delta(t) = x(t-t_0)$$

$$\Rightarrow x(t-t_1) * \delta(t-t_2) = x(t-t_1-t_2)$$

$$x(t) * h(t) = y(t)$$

$$x(t \pm t_1) * h(t \pm t_2) = y(t \pm t_1 \pm t_2)$$

$$\boxed{\begin{aligned} x(t)\cdot\delta(t-t_0) &= x(t_0)\cdot\delta(t-t_0) \\ x(t) * \delta(t-t_0) &= x(t-t_0) \end{aligned}}$$

QUESTIONS

Q-1 $e^{-2t}u(t) * \delta(t-3) = \_\_\_\_$

$$y(t) = e^{-2(t-3)}\cdot u(t-3)\cdot$$

Q-2 $\delta(t-3) * \delta(3-2t) = \_\_\_\_$

$$y(t) = \delta(t-3) * \frac{1}{|-2|}\delta\left(t-\frac{3}{2}\right)$$

$$\Rightarrow y(t) = \frac{1}{2}\delta\left(t-\frac{3}{2}-3\right)$$

$$\Rightarrow y(t) = \frac{1}{2}\delta(t-4.5)$$

$$\Rightarrow y(t) = \delta(3-2(t-3))$$

$$\Rightarrow y(t) = \delta(3-2t+6)$$

$$\Rightarrow y(t) = \delta(-2t+9)$$

$$\Rightarrow y(t) = \frac{1}{2}\delta(t-4.5)$$

Q-3 $u(t-3) * \delta(3-t) = \_\_\_\_$

$$y(t) = u(t-3) * \delta(t-3)$$

$$y(t) = u(t-3-3) = u(t-6)$$

Q-4 $e^{-2t}u(t-2) * \delta(1.5-t) = \_\_\_\_$

$$y(t) = e^{-2t}\cdot u(t-2) * \delta(t-1.5)$$

$$\Rightarrow y(t) = e^{-2(t-1.5)}\cdot u(t-1.5-2)$$

$$\Rightarrow y(t) = e^{-2t+3}\cdot u(t-3\ 5)$$

Q-5 $\delta(-t+2) * \delta(t-5) = \_\_\_\_$

$$\Rightarrow y(t) = \delta(t-2) * \delta(t-5)$$

$$\Rightarrow y(t) = \delta(t-7)$$

Q-6 $\delta(4t+2) * u(t+4) = \_\_\_\_$

$$\Rightarrow y(t) = \frac{1}{4}\delta\left(t+\frac{1}{2}\right) * u(t+4)$$

$$\Rightarrow y(t) = \frac{1}{4}u\left(t+\frac{1}{2}+4\right) = \frac{1}{4}u(t+4.5)$$

➢ Properties of CT Convolution

(1) Commutative Property

If $x(t) * h(t) = y(t) = \int_{-\infty}^{+\infty} x(\tau) \cdot h(t-\tau)\, d\tau$

Then, $h(t) * x(t) = y(t) = \int_{-\infty}^{+\infty} h(\tau) \cdot x(t-\tau)\, d\tau$

(2) Distributive Property

If $x(t) * [h_1(t) + h_2(t)] = y(t)$

Then $x(t) * h_1(t) + x(t) * h_2(t) = y(t)$

(3) Associative Property

If $x(t) * [h_1(t) * h_2(t)] = y(t)$

Then $[x(t) * h_1(t)] * h_2(t) = y(t)$

(4) Time Shifting Property

If $x(t) * h(t) = y(t)$

Then.

- $x(t) * h(t-t_1) = y(t-t_1)$
- $x(t-t_1) * h(t) = y(t-t_1)$
- $x(t-t_1) * h(t-t_2) = y(t-t_1-t_2)$

(5) Differentiation Property

If $x(t) * h(t) = y(t)$

Then

- $\dfrac{dx(t)}{dt} * h(t) = \dfrac{dy(t)}{dt}$

- $x(t) * \dfrac{dh(t)}{dt} = \dfrac{dy(t}{dt}$

- $\dfrac{dx(t)}{dt} * \dfrac{dh(t)}{dt} = \dfrac{d^2y(t)}{dt^2}$

(6) Time Scaling Property

If $x(t) * h(t) = y(t)$

Then, $x(at) * h(at) = \dfrac{1}{|a|} y(at)$

$$\Rightarrow \quad |a| \cdot \left[x(at) * h(at)\right] = y(at)$$

QUESTIONS

Q-1 $u(t) * u(t) = \underline{r(t)}$

Solⁿ:-

$$y(t) = u(t) * u(t)$$

$$\Rightarrow \frac{dy(t)}{dt} = \frac{du(t)}{dt} * u(t)$$

$$\Rightarrow \frac{dy(t)}{dt} = \delta(t) * u(t) = u(t)$$

$$\Rightarrow \frac{dy(t)}{dt} = u(t)$$

$$\Rightarrow y(t) = \int_{\tau=-\infty}^{\tau=t} u(\tau)\cdot d\tau$$

$u(\tau) = 1 \; ; \; \tau > 0$

$u(\tau) = 0 \; ; \; \tau < 0$

$$\Rightarrow y(t) = \int_{\tau=0}^{\tau=t} 1 \cdot d\tau$$

$$\Rightarrow y(t) = (\tau)_0^t \; ; \; t > 0$$

$$\Rightarrow y(t) = t \; ; \; t > 0$$

$$\Rightarrow y(t) = t \cdot u(t) = r(t).$$

Q-2 $e^{-2t} u(t) * u(t)$

Solⁿ:-

$$y(t) = e^{-2t} \cdot u(t) * u(t)$$

$$\Rightarrow \frac{dy(t)}{dt} = e^{-2t} \cdot u(t) * \frac{d}{dt}(u(t)) \cdot$$

$$\Rightarrow \frac{dy(t)}{dt} = e^{-2t} \cdot u(t) * \delta(t) = e^{-2} \cdot u(t)$$

$$\Rightarrow y(t) = \int_{-\infty}^{t} e^{-2\tau} \cdot u(\tau) \cdot d\tau$$

$u(\tau) = 1 \; ; \; \tau > 0$

$u(\tau) = 0 \; ; \; \tau < 0$

$$\Rightarrow y(t) = \int_0^t e^{-2\tau} \cdot d\tau$$

$$\Rightarrow y(t) = \left(\frac{e^{-2\tau}}{-2} \right)_0^t \; ; \; t > 0$$

$$\Rightarrow y(t) = \frac{1 - e^{-2t}}{2} \; ; \; t > 0$$

$$\Rightarrow y(t) = \frac{1 - e^{-2t}}{2} \cdot u(t)$$

Q-3 $r(t) * u(t) = \underline{\qquad}$

$$\Rightarrow y(t) = \int_{\tau=-\infty}^{\tau=t} r(\tau) \cdot d\tau$$

$$\Rightarrow y(t) = \int_{-\infty}^{t} \tau \cdot u(\tau) \cdot d\tau$$

$$\Rightarrow y(t) = \int_0^t \tau \cdot d\tau = \left(\frac{\tau^2}{2} \right)_0^t = \frac{t^2}{2} \; ; \; t > 0$$

$$\therefore \; y(t) = \frac{t^2}{2} \cdot u(t)$$

Q-4 $e^{-2t} u(t) * u(t-3) = \underline{\qquad}$

$$e^{-2t} \cdot u(t) * u(t) = \frac{1 - e^{-2t}}{2} \cdot u(t)$$

$$\Rightarrow e^{-2t} \cdot u(t) * u(t-3) = \frac{1 - e^{-2(t-3)}}{2} \cdot u(t-3)$$

Q-5 $3u(t-2) * 5u(t+5) = \underline{\qquad}$

$$\Rightarrow u(t) * u(t) = r(t)$$

$$\Rightarrow 3 \cdot u(t-2) * 5 \cdot u(t+5)$$

$$= 3 \times 5 \times r(t-2+5) = 15 \cdot r(t+3)$$

Q-6 $e^{-2(t-1)} u(t-1) * u(t+2) = \underline{\qquad}$

$$e^{-2t} u(t) * u(t) = \frac{1 - e^{-2t}}{2} \cdot u(t)$$

$$e^{-2(t-1)} \cdot u(t-1) * u(t+2)$$

$$= \frac{1 - e^{-2(t-1+2)}}{2} \cdot u(t-1+2)$$

Important Result

$$y(t) = x(t) * h(t)$$

$$\Rightarrow \quad x(t) = u(t)$$

$$\Rightarrow \quad y(t) = u(t) * h(t)$$

$$\Rightarrow \quad \frac{dy(t)}{dt} = \frac{du(t)}{dt} * h(t)$$

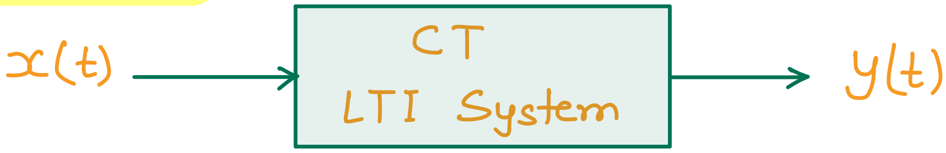
$$\Rightarrow \quad \frac{dy(t)}{dt} = \delta(t) * h(t) = h(t)$$

$$\Rightarrow \quad y(t) = \int_{-\infty}^{t} h(\tau)\, d\tau$$

$$\Rightarrow \quad h(t) = u(t)$$

$$\Rightarrow \quad y(t) = \int_{-\infty}^{t} x(\tau) \cdot d\tau$$

Concept



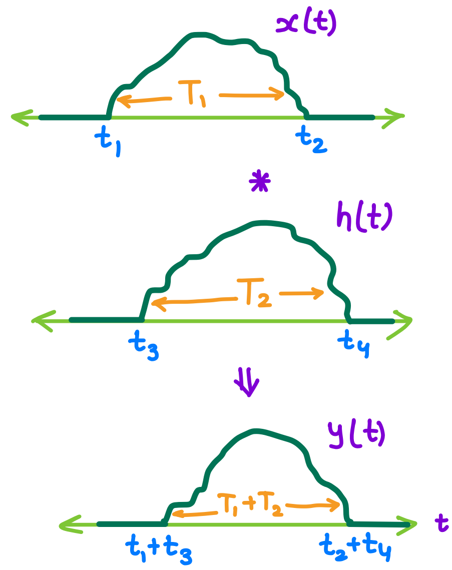
h(t) : Impulse Response

$$x(t) * h(t) = y(t) = \int_{-\infty}^{+\infty} x(\tau) \cdot h(t-\tau)\, d\tau$$

There are two methods to perform Convolution

1) Graphical Method 2) Analytical Method

1. Graphical Method :-



$$y(t) = x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau) \cdot h(t-\tau) \cdot d\tau$$

$$y(t) = h(t) * x(t) = \int_{-\infty}^{+\infty} h(\tau) \cdot x(t-\tau)\, d\tau$$

Step-I. Finding limits of y(t)

⇒ Sum of lower limits of $x(t)$ & $h(t)$ $\leq t \leq$ Sum of upper limits of $x(t)$ & $h(t)$.

(1). If two equal width rectangle convolve then result is a triangle.

(2). If two unequal width rectangle is convolve then result is a trapezium.

2. **Analytical Method** :-

$$y(t) = x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau)\cdot h(t-\tau)\cdot d\tau$$

(i) Both $x(t)$ & $h(t)$ are non causal signals

$\Rightarrow$ There is no change in limits

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)\cdot h(t-\tau)\cdot d\tau$$

(ii) $x(t)$ is causal & $h(t)$ is non causal signals.

$\Rightarrow$ $x(t)$: Causal Signal

$x(t) = 0;\ t<0$

$x(\tau) = 0;\ \tau<0$

$$y(t) = \int_{\tau=-\infty}^{\tau=+\infty} \underbrace{x(\tau)}\cdot h(t-\tau)\, d\tau$$

$\hookrightarrow$ C $\quad x(\tau)=0\ ;\ \tau>0$

$$y(t) = \int_{\tau=0}^{\tau=\infty} x(\tau)\cdot h(t-\tau)\cdot d\tau$$

$\leftarrow$ Change in lower limit

(iii) $x(t)$ is non-causal & $h(t)$ is causal signal

$\Rightarrow$ $h(t)$: causal signal

$\Rightarrow$ $h(t) = 0\ ;\ t<0$

$\Rightarrow$ $h(t-\tau) = 0\ ;\ t-\tau<0$

$\Rightarrow$ $h(t-\tau) = 0\ ;\ t<\tau$

$$y(t) = \int_{\tau=-\infty}^{\tau=+\infty} x(\tau)\cdot \underbrace{h(t-\tau)}\cdot d\tau$$

$\hookrightarrow$ C $\quad h(t-\tau)=0\ ;\ \tau>t$

$\leftarrow$ Change in upper limit

$$y(t) = \int_{\tau=-\infty}^{\tau=t} x(\tau)\cdot h(t-\tau)\cdot d\tau$$

(iv) Both $x(t)$ & $h(t)$ are causal signal.

$\Rightarrow x(t)$: Causal Signal

$\Rightarrow x(\tau) = 0 \; ; \; \tau < 0$

$\Rightarrow h(t)$: Causal Signal

$\Rightarrow h(t-\tau) = 0 \; ; \; \tau > t$

$$y(t) = \int_{\tau=-\infty}^{\tau=+\infty} \underbrace{x(\tau)}_{\text{C } x(\tau)=0;\ \tau<0} \cdot \overbrace{h(t-\tau)}^{\text{C } h(t-\tau)=0;\ \tau>t} \cdot d\tau$$

$$y(t) = \int_{\tau=0}^{\tau=t} x(\tau) \cdot h(t-\tau) \cdot d\tau$$

Change in Both limit.

Convolution of step function with impulse function.

$$y(t) = u(t) * \delta(t)$$

$$y(t) = u(t) * \delta(t) = u(t)$$

$$\Rightarrow K_1 u(t-t_1) * K_2 \delta(t-t_2) = K_1 \cdot K_2 \cdot u(t-t_1-t_2)$$

Convolution of step function with step function.

$$y(t) = \underset{\text{C}}{u(t)} * \underset{\text{C}}{u(t)} = r(t)$$

$$\Rightarrow y(t) = x(t) * h(t)$$

$$\Rightarrow y(t) = \int_{-\infty}^{+\infty} u(\tau) \cdot h(t-\tau) \, d\tau$$

$u(\tau)$: $1 ; \tau > 0$

$h(t-\tau)$: $1 ; t-\tau > 0$

$1 ; \tau < t$

$u(\tau) \cdot u(t-\tau) = 1 \times 1 \; ; \; 0 < \tau < t$

$= 0 \; ; \;$ otherwise

$$\Rightarrow \quad y(t) = \int_0^t 1 \times 1 \times d\tau \quad = \quad (\tau)_0^t \quad ; \; t > 0$$

$$= t \quad ; \quad t > 0$$

$$= t \cdot u(t) \quad = \quad r(t)$$

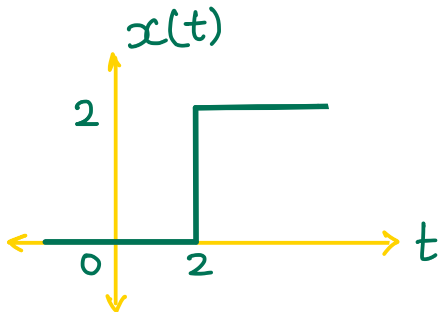
$$\Rightarrow \quad u(t) * u(t) = r(t)$$

$$\Rightarrow \quad K_1 \cdot u(t-t_1) * K_2 \cdot u(t-t_2)$$

$$= K_1 \cdot K_2 \cdot r(t - t_1 - t_2)$$

QUESTIONS

Q-1) $x(t) * h(t) = y(t)$



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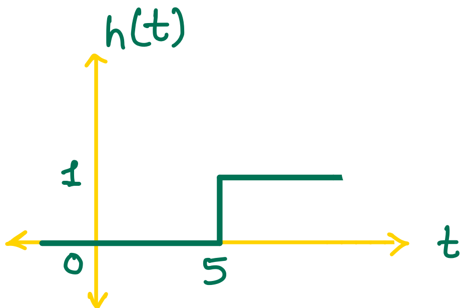


Sol^n:-

$x(t) = 2 \cdot u(t-2)$

$h(t) = 1 \cdot u(t-5)$

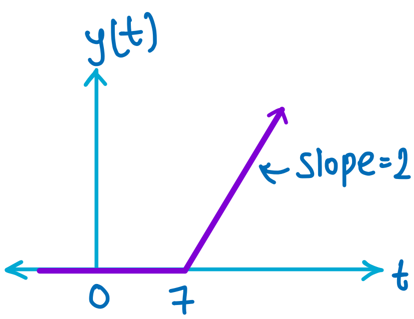
$$\Rightarrow \quad y(t) = x(t) * h(t)$$

$$\Rightarrow \quad y(t) = 2 \cdot u(t-2) * 1 \cdot u(t-5)$$

$$\Rightarrow \quad y(t) = 2 \cdot r(t-2-5)$$

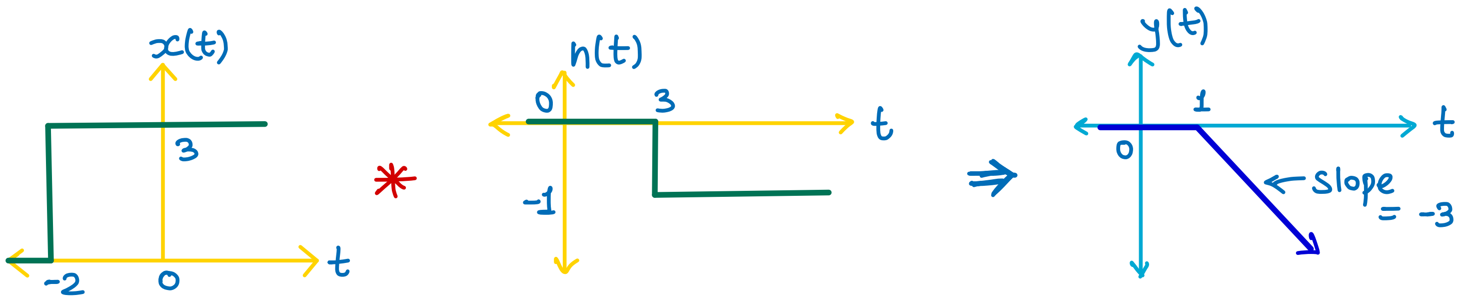
$$\Rightarrow \quad y(t) = 2 \cdot r(t-7)$$

$\Rightarrow$



Q-2

$$x(t) * h(t) = y(t)$$

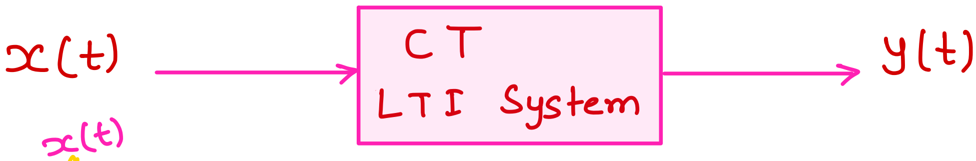


Sol^n:-

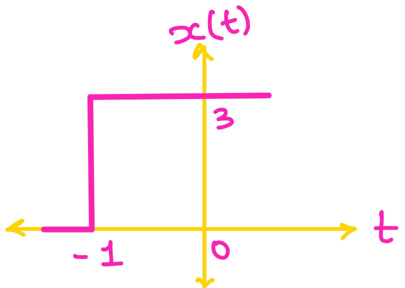
$$x(t) = 3 \cdot u(t+2)$$
$$h(t) = -1 \cdot u(t-3)$$

$$y(t) = x(t) * h(t)$$
$$\Rightarrow y(t) = 3 \cdot u(t+2) * (-u(t-3))$$
$$\Rightarrow y(t) = -3 \cdot r(t-1)$$

Qu Find output y(t) of LTI system for given input & $y(4) =$ ______ .



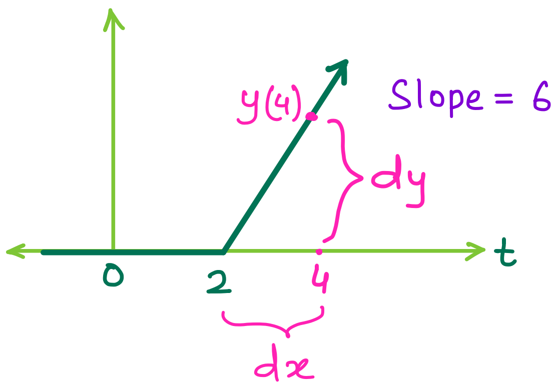
$h(t) = 2u(t-3)$



Sol^n:-

$$x(t) = 3 \cdot u(t+1)$$
$$h(t) = 2 \cdot u(t-3)$$

$$y(t) = x(t) * h(t)$$
$$\Rightarrow y(t) = 3 \cdot u(t+1) * 2 \cdot u(t-3)$$
$$\Rightarrow y(t) = 6 \cdot r(t-2)$$



$dx = 2$, $dy = ?$, slope $= 6$

$$\Rightarrow \text{Slope} = \frac{dy}{dx} \Rightarrow 6 = \frac{dy}{2}$$

$$\Rightarrow dy = 6 \times 2 = 12$$

$$\therefore y(4) = 12$$

Convolution of step function with ramp function.

$$y(t) = r(t) * u(t)$$

$$\Rightarrow y(t) = x(t) * h(t)$$

$$\Rightarrow y(t) = \int_{-\infty}^{+\infty} r(\tau) \cdot u(t-\tau) \cdot d\tau$$

$$\Rightarrow y(t) = \int_{-\infty}^{\infty} \tau \cdot \underbrace{u(\tau) \cdot u(t-\tau)}_{\hookrightarrow 1;\ 0<\tau<t} \cdot d\tau$$

$$\Rightarrow y(t) = \int_{0}^{t} \tau \cdot 1 \cdot d\tau = \left(\frac{\tau^2}{2}\right)_0^t \ ; t>0$$

$$= \frac{t^2}{2} \ ; t>0 \quad = \frac{t^2}{2} \cdot u(t)$$

$$\Rightarrow r(t) * u(t) = \frac{t^2}{2} \cdot u(t)$$

$$\Rightarrow K_1 r(t-t_1) * K_2 u(t-t_2) = K_1 \cdot K_2 \cdot \frac{(t-t_1-t_2)^2}{2} \cdot u(t-t_1-t_2)$$

Qu. $3u(t-2) * -2r(t+5) =$ ________

Sol^n:- $u(t) * r(t) = \frac{t^2}{2} \cdot u(t)$

$$\Rightarrow 3 \cdot u(t-2) * -2 \cdot r(t+5) = -6 \cdot \frac{(t-2+5)^2}{2} \cdot u(t-2+5)$$

$$= -3 \cdot (t+3)^2 \cdot u(t+3)$$

Convolution of ramp function with ramp function.

$$y(t) = r(t) * r(t)$$

$$\Rightarrow y(t) = x(t) * h(t)$$

$$\Rightarrow y(t) = \int_{-\infty}^{+\infty} x(\tau) \cdot h(t-\tau) \cdot d\tau$$

$$= \int_{-\infty}^{+\infty} \tau \cdot u(\tau) \cdot (t-\tau) \cdot u(t-\tau) \cdot d\tau$$

$$\Rightarrow y(t) = \int_{0}^{t} \tau \cdot (t-\tau)\, d\tau$$

$$= t \cdot \int_{0}^{t} \tau \cdot d\tau - \int_{0}^{t} \tau^2 \cdot d\tau$$

$$= t \cdot \left(\frac{\tau^2}{2}\right)_0^t - \left(\frac{\tau^3}{3}\right)_0^t$$

$$= \frac{t^3}{2} - \frac{t^3}{3} \quad ; \; t > 0$$

$$\Rightarrow y(t) = \frac{t^3}{6} \cdot u(t)$$

$$\Rightarrow r(t) * r(t) = \frac{t^3}{6} \cdot u(t)$$

$$\Rightarrow K_1 r(t-t_1) * K_2 \cdot r(t-t_2) = K_1 \cdot K_2 \frac{(t-t_1-t_2)^3}{6} \cdot u(t-t_1-t_2)$$

Convolution of step function with exponential function.

$$y(t) = e^{-at}u(t) * u(t)$$

$$\Rightarrow \quad y(t) = x(t) * h(t)$$

$$\Rightarrow \quad y(t) = \int_{-\infty}^{+\infty} e^{-a\tau} \cdot \underbrace{u(\tau) \cdot u(t-\tau)}_{1\,;\ 0<\tau<t} \cdot d\tau$$

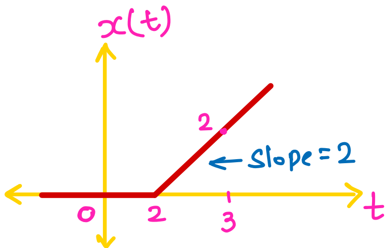
$$\Rightarrow \quad y(t) = \int_{0}^{t} e^{-a\tau} \cdot 1 \cdot d\tau \qquad = \left(\frac{e^{-a\tau}}{-a}\right)_0^t \ ; \ t>0$$

$$\Rightarrow \quad y(t) = \frac{1-e^{-at}}{a} \cdot u(t)$$

$$\Rightarrow \quad e^{-at} \cdot u(t) * u(t) = \frac{1-e^{-at}}{a} \cdot u(t)$$

$$\Rightarrow K_1 e^{-a(t-t_1)} \cdot u(t-t_1) * K_2 \cdot u(t-t_2) = K_1 K_2$$

Qu: $x(t) * h(t) = y(t)$



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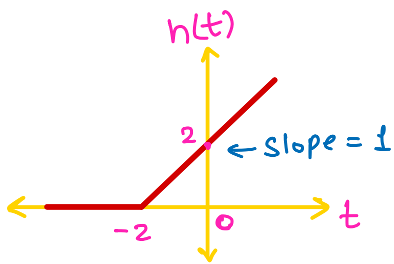


$$x(t) = 2 \cdot r(t-2)$$

$$h(t) = r(t+2)$$

$$y(t) = x(t) * h(t)$$

$$= 2 \cdot r(t-2) * r(t+2)$$

$$= 2 \cdot \frac{t^3}{6} \cdot u(t)$$

$$= \frac{t^3}{3} \cdot u(t)$$

Convolution of exponential function with exponential function.

$$y(t) = e^{-at}u(t) * e^{-bt}u(t)$$

$$\Rightarrow y(t) = x(t) * h(t)$$

$$\Rightarrow y(t) = \int_{-\infty}^{+\infty} e^{-a\tau} \cdot u(\tau) \cdot e^{-b(t-\tau)} \cdot u(t-\tau) \cdot d\tau$$

$$\Rightarrow y(t) = \int_{0}^{t} e^{-a\tau} \cdot e^{-bt} \cdot e^{b\tau} \cdot d\tau$$

$$\Rightarrow y(t) = e^{-bt} \int_{0}^{t} e^{-(a-b)\tau} \cdot d\tau$$

$$\Rightarrow y(t) = e^{-bt} \cdot \left(\frac{e^{-(a-b)\tau}}{-(a-b)} \right)_0^t$$

$$= e^{-bt} \cdot \left(\frac{e^{-at} \cdot e^{bt} - 1}{-(a-b)} \right) \; ; \; t > 0$$

$$e^{-bt}\left[\frac{e^{-at} \cdot e^{bt}}{b-a} - \frac{1}{b-a} \right]$$

$$\Rightarrow \frac{e^{-at} \cdot e^{bt} \cdot e^{-bt}}{b-a} - \frac{e^{-bt}}{b-a}$$

$$\Rightarrow \frac{e^{-at} \cdot e^{(bt-bt)} - e^{-bt}}{b-a}$$

$$\Rightarrow y(t) = \frac{e^{-at} - e^{-bt}}{b-a} \cdot u(t)$$

$$\Rightarrow e^{-at} \cdot u(t) * e^{-bt} u(t) = \frac{e^{-at} - e^{-bt}}{b-a} \cdot u(t)$$

If, $b = 0$;

$$\Rightarrow e^{-at} u(t) * u(t) = \frac{e^{-at} - 1}{-a} \cdot u(t) = \frac{1 - e^{-at}}{a} \cdot u(t)$$

If, $a = b$;

$$\Rightarrow e^{-at} u(t) * e^{-at} \cdot u(t) = \boxed{\cancel{\frac{e^{-at} - e^{-at}}{a-a}}} = e^{-at}\big(u(t) * u(t)\big)$$

$$= e^{-at} \cdot t \cdot u(t) = t \cdot e^{-at} \cdot u(t)$$

SUMMARY

- $\delta(t) * \delta(t) = \delta(t)$
- $\delta(t) * u(t) = u(t)$
- $u(t) * u(t) = r(t) = t \cdot u(t)$
- $u(t) * r(t) = \frac{t^2}{2} \cdot u(t)$
- $r(t) * r(t) = \frac{t^3}{6} \cdot u(t)$
- $e^{-at} \cdot u(t) * e^{-bt} \cdot u(t) = \frac{e^{-at} - e^{-bt}}{b-a} \cdot u(t)$
- $e^{-at} \cdot u(t) * u(t) = \frac{1 - e^{-at}}{a} \cdot u(t)$
- $e^{-at} \cdot u(t) * e^{-at} \cdot u(t) = t \cdot e^{-at} \cdot u(t)$

QUESTIONS

Q-1] $y(t) = e^{-2t} u(t) * e^{-3t} u(t) = \frac{e^{-2t} - e^{-3t}}{3-2} \cdot u(t)$

$a = 2 \quad , \quad b = 3$

Q-2] $y(t) = e^{3t} u(t) * e^{5t} u(t) = \frac{e^{3t} - e^{5t}}{-5+3} \cdot u(t)$

$a = -3 \quad , \quad b = -5$

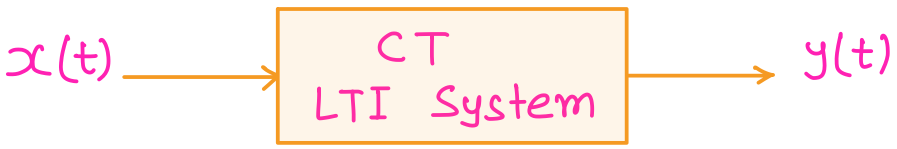
Q-3] $y(t) = e^{-4t} u(t) * e^{2t} u(t) = \frac{e^{4t} - e^{2t}}{-2-4} \cdot u(t)$

$a = 4 \quad , \quad b = -2$

Q-4] $y(t) = e^{-2t} u(t) * e^{-2t} u(t) = t \cdot e^{-2t} \cdot u(t)$

Qu. Find energy of output $y(t)$ of LTI system for given input.

$x(t) = e^{-2t} \cdot u(t)$ also find $\frac{E_{y(t)}}{E_{x(t)}}$.



$h(t) = e^{-3t} u(t)$

Sol^n:-

$x(t) = e^{-2t} \cdot u(t)$

$h(t) = e^{-3t} \cdot u(t)$

$y(t) = x(t) * h(t)$

$\Rightarrow y(t) = e^{-2t} \cdot u(t) * e^{-3t} \cdot u(t)$

$\Rightarrow y(t) = \frac{e^{-2t} - e^{-3t}}{3-2} \cdot u(t)$

$\Rightarrow y(t) = e^{-2t} \cdot u(t) - e^{-3t} \cdot u(t)$

$\Rightarrow y(t) = e^{-2t} \cdot u(t) - e^{-3t} \cdot u(t)$

$\downarrow$

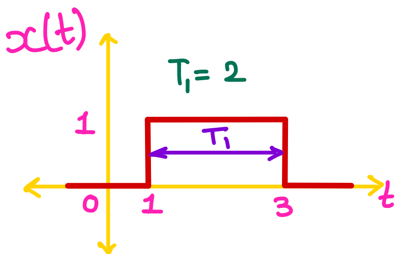
$E_{y(t)} = \frac{1}{2\times 2} + \frac{1}{2\times 3} - \frac{2}{2+3}$

$\Rightarrow E_{y(t)} = \frac{1}{60}$ units

$\Rightarrow E_{x(t)} = \frac{1}{2\times 2} = \frac{1}{4}$

$\Rightarrow \frac{E_{y(t)}}{E_{x(t)}} = \frac{\frac{1}{60}}{\frac{1}{4}} = \frac{1}{15}$

Qu. $x(t) * h(t) = y(t)$



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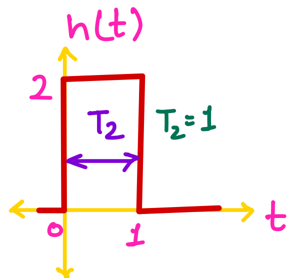
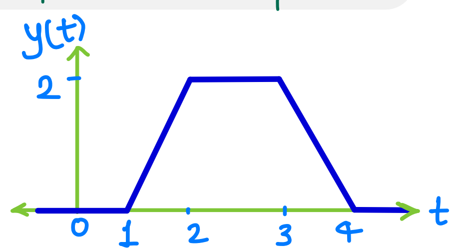


$\Rightarrow$

$\because T_1 \neq T_2 \quad \therefore$ Trapezium



Sol^n:-

$x(t) = u(t-1) - u(t-3)$ & $h(t) = 2\cdot u(t) - 2\cdot u(t-1)$

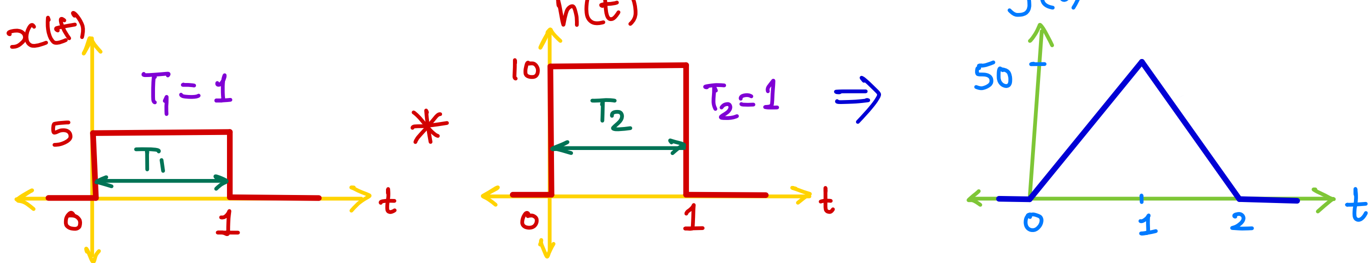
$y(t) = x(t) * h(t) = [u(t-1) - u(t-3)] * [2\cdot u(t) - 2\cdot u(t-1)]$

$\Rightarrow y(t) = u(t-1) * 2u(t) - 2\cdot u(t-1) * u(t-1) - 2\cdot u(t-3) * u(t)$
$+ 2\cdot u(t-3) * u(t-1)$

$\Rightarrow y(t) = 2\cdot r(t-1) - 2\cdot r(t-2) - 2\cdot r(t-3) + 2\cdot r(t-4)$

$\because T_1 = T_2 \therefore$ Triangle

Qu. $x(t) * h(t) = y(t)$



Sol^n:-

$$x(t) = 5\cdot u(t) - 5\cdot u(t-1) \quad \& \quad h(t) = 10\cdot u(t) - 10\cdot u(t-1)$$

$$y(t) = \left(5\cdot u(t) - 5\cdot u(t-1) \quad * \quad 10\cdot u(t) - 10\cdot u(t-1)\right)$$

$$y(t) = 50\,r(t) - 50\,r(t-1) - 50\,r(t-1) + 50\,r(t-2)$$

$$y(t) = 50\,r(t) - 100\,r(t-1) + 50\,r(t-2)$$

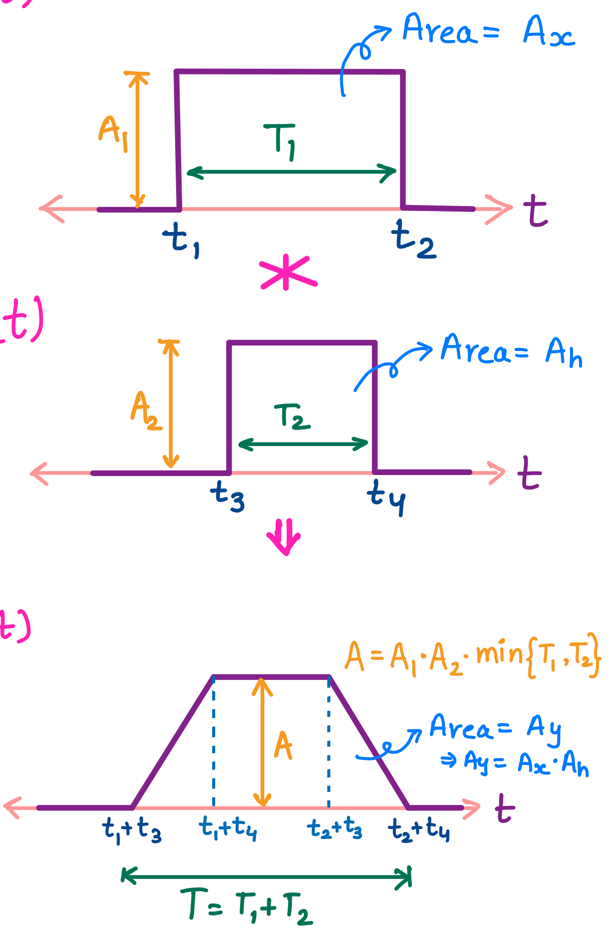
IMPORTANT SHORT TRICK

① $T_1 \neq T_2$

$x(t)$

$h(t)$

$y(t)$



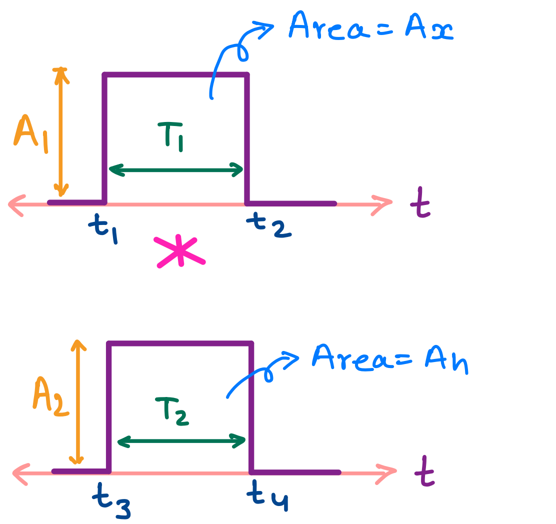
② $T_1 = T_2$

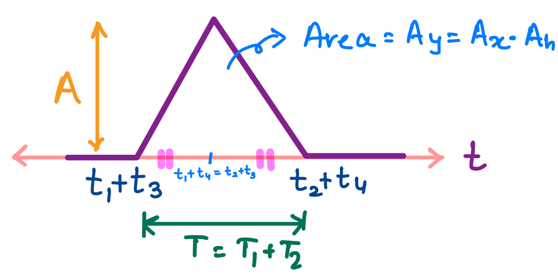
$x(t)$

$h(t)$

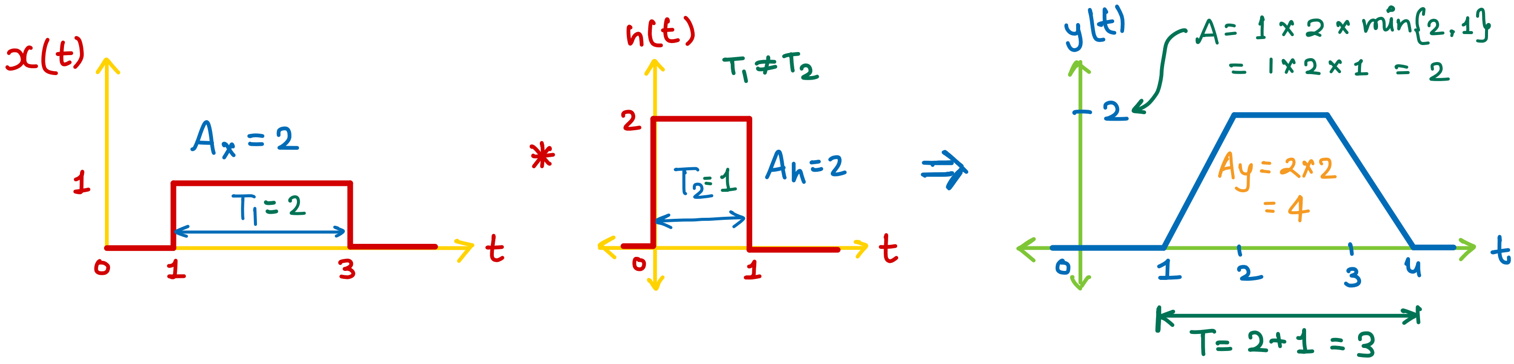
$y(t)$

$A = A_1 \cdot A_2 \cdot T$

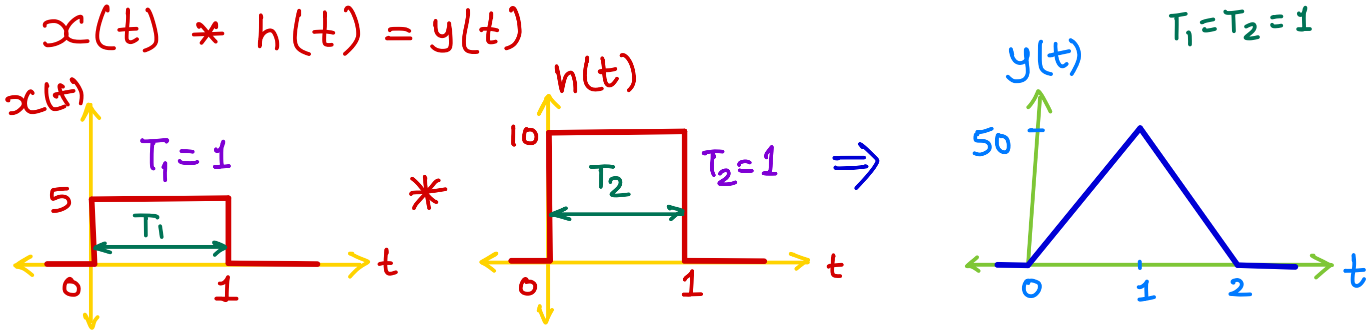




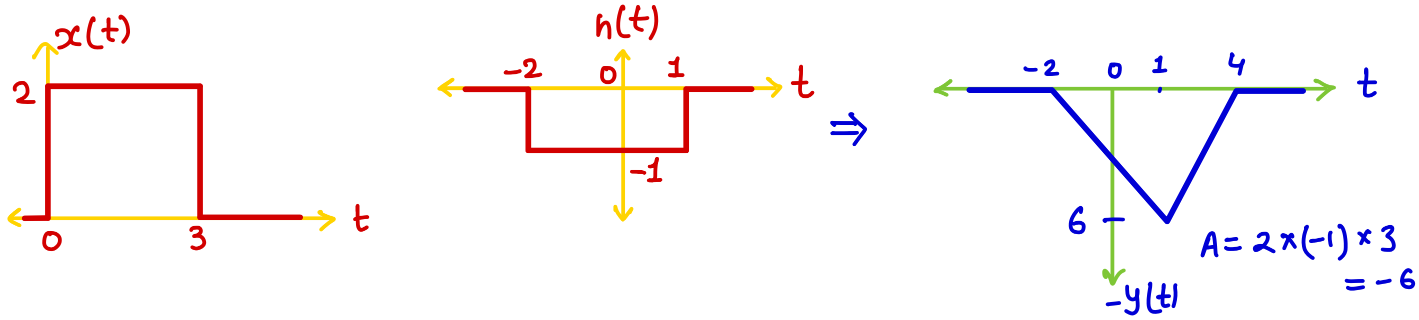
Qu. $x(t) * h(t) = y(t)$



Qu. $x(t) * h(t) = y(t)$

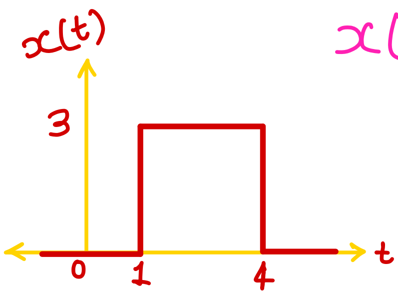


Qu.



Qu. Find energy of output $y(t)$ of LTI system for given input also find $\frac{E_{y(t)}}{E_{x(t)}}$.

$x(t)$ → CT LTI System → $y(t)$

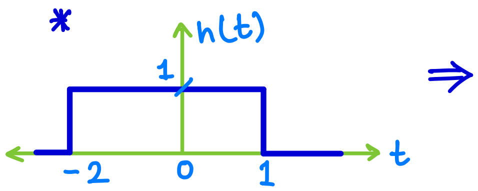


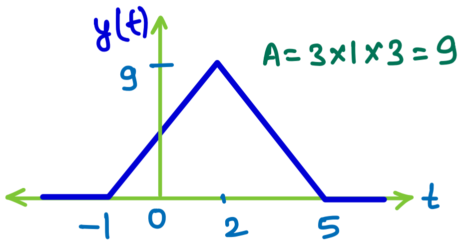
$h(t) = u(t+2) - u(t-1)$

$y(t) = x(t) * h(t)$

$E_{y(t)} = \frac{81 \times 6}{3} = 162\ J$

$E_{x(t)} = 9 \times 3 = 27\ J$





$\Rightarrow \frac{E_{y(t)}}{E_{x(t)}} = \frac{162}{27} = 6$

Discrete Time LTI System

$$x[n] \longrightarrow \boxed{\text{D.T. LTI System}} \longrightarrow y[n]$$

$h[n]$: Impulse Response

$\Rightarrow \quad y[n] = x[n] * h[n]$

$\Rightarrow \quad y[n] = \sum_{k=-\infty}^{k=+\infty} x[k] \cdot h[n-k]$

$y[n] = T\{x[n]\}$

If $x[n] = \delta[n]$

$\Rightarrow$ $y[n]$: Impulse Response $= h[n]$

$\Rightarrow \quad y[n] = \delta[n] * h[n]$

$= h[n]$

QUESTIONS

Q-1 $x[n] = 2\delta[n-2]$ & $h[n] = -3\delta[n+2]$. Find $y[n] = x[n] * h[n]$

$y[n] = 2 \cdot \delta[n-2] * -3 \cdot \delta[n+2]$

$\Rightarrow y[n] = -6 \cdot \delta[n-2+2]$

$\Rightarrow y[n] = -6 \cdot \delta[n]$

Q-2 $x[n] = \delta[n-2]$ & $h[n] = \left(\frac{1}{3}\right)^n u[n]$. Find $y[n] = x[n] * h[n]$ also calculate $y[6]$ & $y[1]$.

$\Rightarrow y[n] = \delta[n-2] * \left(\frac{1}{3}\right)^n \cdot u[n]$

$\Rightarrow y[n] = \left(\frac{1}{3}\right)^{n-2} \cdot u[n-2]$

$\Rightarrow y[n] = \left(\frac{1}{3}\right)^{n-2} \quad ; \; n \geq 2$

$= 0 \quad ; \; n < 2$

$y[6] = \left(\frac{1}{3}\right)^{6-2} = \left(\frac{1}{3}\right)^4 = \frac{1}{81}$

$y[1] = 0 \quad \because n < 2 = 0$

$1 < 2 = 0$

Convolution of step function with impulse function.

$$y[n] = u[n] * \delta[n]$$

$$\Rightarrow y[n] = u[n] * \delta[n] = u[n]$$

$$K_1 \cdot u[n-n_1] * K_2 \cdot \delta[n-n_2] = K_1 \cdot K_2 \cdot u[n-n_1-n_2]$$

Convolution of step function with step function.

$$y[n] = u[n] * u[n]$$

$$\Rightarrow y[n] = x[n] * h[n]$$

$$\Rightarrow y[n] = \sum_{K=-\infty}^{K=+\infty} \underbrace{u[K]}_{1;\ K \geq 0} \cdot \underbrace{u[n-K]}_{\substack{1;\ n-K \geq 0 \\ 1;\ K \leq n}} \longrightarrow u[K] \cdot u[n-K] = 1\ ;\ 0 \leq K \leq n$$

$$\Rightarrow y[n] = \sum_{K=0}^{K=n} 1 = (n+1)\ ;\ n \geq 0 \longrightarrow \sum_{K=0}^{K=3} 1 = \underset{K=0}{1} + 1 + 1 + \underset{K=3}{1} = 4 = 3+1$$

$$\Rightarrow y[n] = (n+1) \cdot u[n]$$

$$\Rightarrow u[n] * u[n] = (n+1) \cdot u[n]$$

$$\Rightarrow u(t) * u(t) = t \cdot u(t) = r(t)$$

Observation

$$u[n] * u[n] = \underline{(n+1) \cdot u[n]} = n \cdot u[n] + u[n] = r[n] + u[n]$$

$$\text{or, } (n+1) \cdot u[n+1] = r[n+1]$$

$$\Rightarrow (n+1) \cdot u[n] = (n+1)\ ;\ n \geq 0$$

$$= \{\underset{\uparrow}{1}, 2, 3, 4, 5, \cdots\} = r[n+1]$$

$$\Rightarrow r[n] = n\ ;\ n \geq 0$$

$$= \{\underset{\uparrow}{0}, 1, 2, 3, 4, \cdots\}$$

$$\Rightarrow r[n+1] = \{0, \underset{\uparrow}{1}, 2, 3, 4, \cdots\}$$

- $u(t) * u(t) = t \cdot u(t) = r(t)$
- $u[n] * u[n] = (n+1) \cdot u[n] = r[n] + u[n] = r[n+1]$

Qu. $x[n] = u[n-2]$ & $h[n] = -3u[n+3]$. Find $y[n] = x[n] * h[n]$ also calculate $y[2]$.

Sol^n:-

$$y[n] = x[n] * h[n]$$

$$u[n] * u[n] = r[n+1] = (n+1) \cdot u[n]$$

$$\Rightarrow y[n] = u[n-2] * -3u[n+3] = -3(n-2+3+1) \cdot u[n-2+3]$$

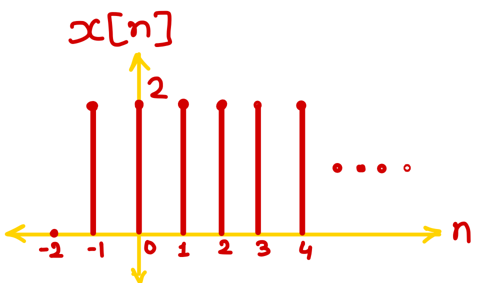
$$\Rightarrow y[n] = -3(n+2) \cdot u[n+1]$$

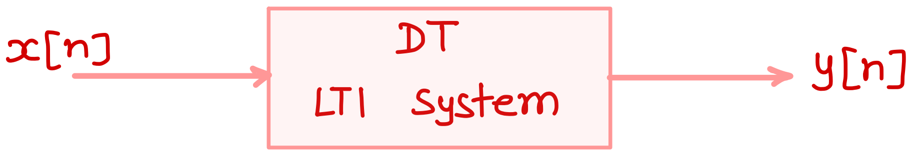
$$\Rightarrow y[n] = -3(n+2) \; ; \; n \geq -1$$

$$= 0 \; ; \; n < 1$$

$$y[2] = -3(2+2) = -12$$

Qu. For the given DT LTI system find output $y(n)$ for the given input also calculate $y[3]$.





Sol^n:-

$$x[n] = 2 \cdot u[n+1]$$

$$h[n] = u[n-2]$$

$$y[n] = x[n] * h[n]$$

$$y[n] = 2 \cdot u[n+1] * u[n-2]$$

$$\Rightarrow y[n] = 2(n+1-2+1) \cdot u[n+1-2]$$

$$\Rightarrow y[n] = 2(n) \cdot u[n-1]$$

$$\Rightarrow y[n] = 2n \; ; \; n \geq 1$$

$$= 0 \; ; \; n < 1$$

$$y[3] = 2 \times 3 = 6$$

Convolution of step function with ramp function.

$$y[n] = u[n] * r[n]$$

$\Rightarrow y[n] = x[n] * h[n]$

$\Rightarrow y[n] = \sum_{K=-\infty}^{K=+\infty} r[K] \cdot u[n-K]$

$\Rightarrow y[n] = \sum_{K=-\infty}^{K=\infty} K \cdot \underline{u[K] \cdot u[n-K]}$

$\Rightarrow y[n] = \sum_{K=0}^{K=n} K = 0+1+2+3+\cdots+n$

$\Rightarrow y[n] = \frac{n\cdot(n+1)}{2} \; ; n \geq 0$

$\Rightarrow y[n] = \frac{n\cdot(n+1)}{2} \cdot u[n]$

$u(t) * r(t) = \frac{t^2}{2} \cdot u(t)$

$u[n] * r[n] = \frac{n\cdot(n+1)}{2} \cdot u[n]$

Qu. $x[n] = 3u[n-2]$ & $h[n] = -2r[n-3]$. find $y[n] = x[n] * h[n]$ also calculate $y[2]$ & $y[0]$.

Soln:-

$y[n] = x[n] * h[n]$

$\Rightarrow u[n] * r[n] = \frac{n\cdot(n+1)}{2} \cdot u[n]$

$\Rightarrow 3\cdot u[n-2] * -2r[n-3] = -6 \cdot \frac{(n-2-3)(n-2-3+1)}{2} \cdot u[n-2-3]$

$\Rightarrow y[n] = -3(n-5)(n-4) \cdot u[n-5]$

$\Rightarrow y[n] = -3(n-5)(n-4) \; ; n \geq 5$

$= 0 \; ; n < 5$

$y[2] \Rightarrow n=2 < 5$

$\therefore y[2] = 0$

$y[0] = 0$

Convolution of ramp function with ramp function.

$$y[n] = r[n] * r[n]$$

$$\Rightarrow y[n] = x[n] * h[n]$$

$$\Rightarrow y[n] = \sum_{k=-\infty}^{k=+\infty} r[k] \cdot r[n-k]$$

$$= \sum_{k=-\infty}^{k=+\infty} k \cdot (n-k) \cdot \underbrace{u[k] \cdot u[n-k]}$$

$$\Rightarrow y[n] = n \cdot \sum_{k=0}^{k=n} k - \sum_{k=0}^{k=n} k^2$$

$$\Rightarrow y[n] = n(\underbrace{0+1+2+\cdots+n}_{\frac{n\cdot(n+1)}{2}}) - (\underbrace{0+1^2+2^2+3^2+\cdots+n^2}_{\frac{n\cdot(n+1)(2n+1)}{6}})$$

$$\Rightarrow y[n] = n\left(\frac{n\cdot(n+1)}{2}\right) - \left(\frac{n\cdot(n+1)(2n+1)}{6}\right)$$

$$\Rightarrow y[n] = \frac{n\cdot(n+1)}{2} \cdot \left(n - \frac{2n+1}{3}\right)$$

$$\Rightarrow y[n] = \left(\frac{n\cdot(n+1)}{2}\right) \cdot \left(\frac{3n-2n-1}{3}\right)$$

$$\Rightarrow y[n] = \frac{n\cdot(n+1)\cdot(n-1)}{6} \cdot u[n]$$

$$r[n] * r[n] = \frac{n\cdot(n+1)\cdot(n-1)}{6} \cdot u[n]$$

- $r(t) * r(t) = \frac{t^3}{6} \cdot u(t)$

Convolution of Increasing/Decreasing function with Increasing/Decreasing function.

$$y[n] = (\alpha)^n u[n] * (\beta)^n u[n]$$

$$\Rightarrow y[n] = x[n] * h[n]$$

$$\Rightarrow y[n] = \sum_{k=-\infty}^{k=\infty} \alpha^k \cdot u[k] \cdot \beta^{n-k} \cdot u[n-k] = \sum_{k=0}^{k=n} \beta^n \cdot \alpha^k \cdot \beta^{-k}$$

$$\Rightarrow y[n] = \beta^n \sum_{k=0}^{k=n} \left(\frac{\alpha}{\beta}\right)^k$$

$$\boxed{\sum_{k=m}^{k=n} (\gamma)^k = \frac{(\gamma)^{n+1} - (\gamma)^m}{(\gamma - 1)}}$$

$$\Rightarrow y[n] = \beta^n \left[\frac{\left(\frac{\alpha}{\beta}\right)^{n+1} - \left(\frac{\alpha}{\beta}\right)^0}{\frac{\alpha}{\beta} - 1} \right]$$

$$\Rightarrow y[n] = \cancel{\beta^{n+1}} \left[\frac{\frac{\alpha^{n+1} - \beta^{n+1}}{\cancel{\beta^{n+1}}}}{\alpha - \beta} \right] ; \quad n \geq 0$$

$$\Rightarrow \alpha^n \cdot u[n] * \beta^n \cdot u[n] \Rightarrow y[n] = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} \cdot u[n]$$

If $\beta = 1$

$$\Rightarrow \alpha^n \cdot u[n] * u[n] = \frac{1 - \alpha^{n+1}}{1 - \alpha} \cdot u[n]$$

If $\alpha = \beta$

$$\Rightarrow \alpha^n \cdot u[n] * \alpha^n \cdot u[n] = \alpha^n \left(u[n] * u[n]\right)$$

$$= (n+1) \cdot u[n] \cdot \alpha^n$$

QUESTIONS

1] $y[n] = (3)^n u[n] * (4)^n u[n]$

$\alpha = 3$

$\beta = 4$

$$y[n] = \frac{(3)^{n+1} - (4)^{n+1}}{3-4} \cdot u[n] = (4^{n+1} - 3^{n+1}) \cdot u[n]$$

2] $y[n] = \left(\frac{1}{5}\right)^n u[n] * \left(\frac{1}{7}\right)^n u[n]$

$$y[n] = \frac{\left(\frac{1}{5}\right)^{n+1} - \left(\frac{1}{7}\right)^{n+1}}{\frac{1}{5} - \frac{1}{7}} \cdot u[n]$$

3] $y[n] = \left(\frac{1}{2}\right)^n u[n] * \left(\frac{1}{2}\right)^n u[n]$

$$y[n] = \left(\frac{1}{2}\right)^n \cdot \left(u[n] * u[n]\right) = \left(\frac{1}{2}\right)^n \cdot (n+1) \cdot u[n]$$

4] $y[n] = (5)^{-n} u[n] * (4)^n u[n]$

$$\Rightarrow y[n] = \left(\frac{1}{5}\right)^n u[n] * (4)^n u[n] = \frac{\left(\frac{1}{5}\right)^{n+1} - 4^{n+1}}{\frac{1}{5} - 4} \cdot u[n]$$

Convolution of Finite duration discrete signal.

1] Linear Convolution

$$\Rightarrow y[n] = x[n] * h[n]$$

$$\Rightarrow \underset{[n_1+n_3,\ n_2+n_4]}{y[n]} = \underset{\substack{[n_1,\ n_2] \\ \downarrow \\ N_1 : \text{No. of Samples}}}{x[n]} * \underset{\substack{[n_3,\ n_4] \\ \downarrow \\ N_2 : \text{No of Samples}}}{h[n]}$$

$N = N_1 + N_2 - 1$: No of Samples

Concept

Q-1 Given $x[n] = \{1, \underset{\uparrow}{2}, 3, 4\}$ & $h[n] = \{2, \underset{\uparrow}{3}, 5\}$ · find $y[n] = x[n] * h[n]$.

Sol^n:-

$\Rightarrow y[n] = x[n] * h[n]$

$\Rightarrow x[n] = [-1, 2] \rightarrow N_1 = 4$

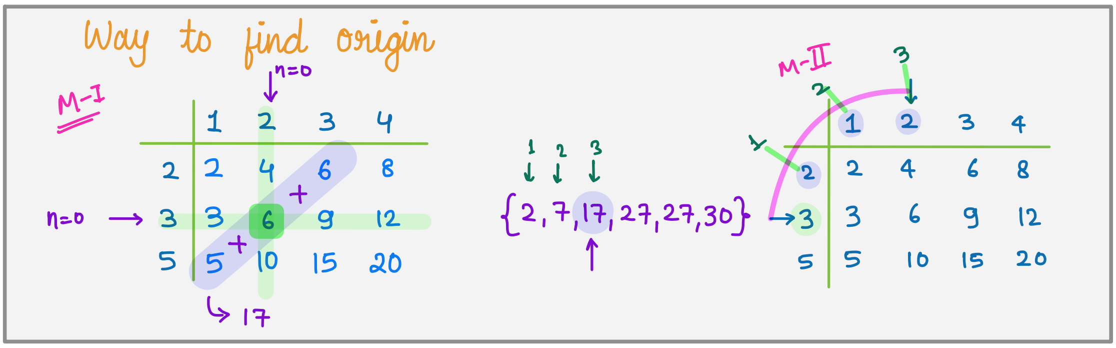
$\Rightarrow h[n] = [-1, 1] \rightarrow N_2 = 3$

$\Rightarrow y[n] = [-2, 3] \rightarrow N = 4 + 3 - 1$

$N = 6$

h[n] \ x[n]	1	2	3	4
2	2	4	6	8
3	3	6	9	12
5	5	10	15	20

$x[n] = \{\underset{n=-1}{1}, \underset{\uparrow}{2}, 3, \underset{n=2}{4}\} * h[n] = \{\underset{n=-1}{2}, \underset{\uparrow}{3}, \underset{n=1}{5}\} \Rightarrow y[n] = \{\underset{n=-2}{2}, 7, \underset{n=0}{17}, 27, 27, \underset{n=3}{20}\}$



QUESTIONS

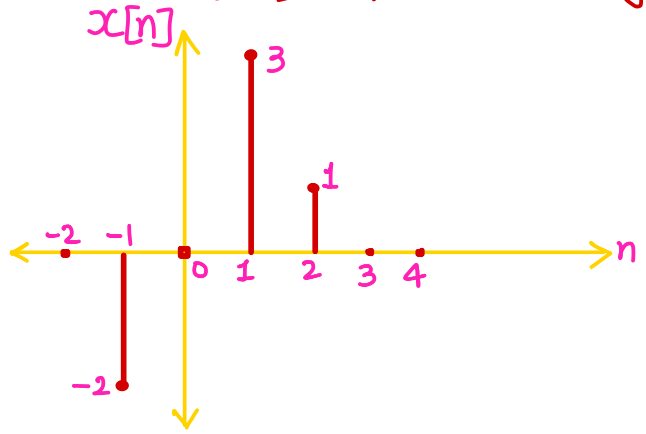
Q-1 Given $x[n] = \{\underset{n=-2}{2}, 0, \underset{\uparrow}{-1}, \underset{n=1}{3}\}$ & $h[n] = \{\underset{n=-1}{1}, \underset{\uparrow}{2}, 0, \underset{n=2}{-1}\}$.

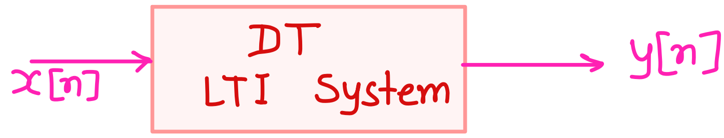
find $y[n] = x[n] * h[n]$

h[n] \ x[n]	2	0	-1	3
1	2	0	-1	3
2	4	0	-2	6
0	0	0	0	0
-1	-2	0	1	-3

$y[n] = \{\underset{n=-3}{2}, 4, -1, \underset{n=0}{-1}, 6, 1, \underset{n=3}{-3}\}$

Qu-2 For the given DT LTI system find the output $y[n]$ for the given input, also calculate $\frac{E_{y[n]}}{E_{x[n]}}$.





$h[n] = \{\underset{\uparrow}{1}, 2, 3\}$

	-2	0	3	1
1	-2	0	3	1
2	-4	0	6	2
3	-6	0	9	3

$\Rightarrow x[n] = \{\underset{n=-1}{-2}, \underset{\uparrow}{0}, 3, \underset{n=2}{1}\}$

$\Rightarrow h[n] = \{\underset{n=0}{1}, 2, \underset{n=2}{3}\}$

$\Rightarrow y[n] = \{\underset{n=-1}{-2}, \underset{\uparrow}{-4}, -3, 7, 11, \underset{n=4}{3}\}$

$E_{y[n]} = 4+16+9+49+121+9$

$= 208$ Units

$E_{x[n]} = 4+9+1 = 14$

$\frac{E_{y[n]}}{E_{x[n]}} = \frac{208}{14} = 14.85$

Qu. $x[n] = \{1, \underset{\uparrow}{2}, 3, 4\}$ & $y[n] = \{2, 7, \underset{\uparrow}{17}, 27, 27, 20\}$

find Impulse response $h[n]$.

Sol^n:-

$x[n] = \{\underset{-1}{1}, \underset{\uparrow}{2}, 3, \underset{n=2}{4}\} \leftarrow N_1 = 4$

$y[n] = \{\underset{-2}{2}, 7, \underset{\uparrow}{17}, 27, 27, \underset{n=3}{20}\} \leftarrow N = 6$

$h[n] = \{\underset{n=-1}{a}, \underset{0}{b}, \underset{n=1}{c}\}$

	1	2	3	4
a	a	2a	3a	4a
b	b	2b	3b	4b
c	c	2c	3c	4c

$x[n] : N_1 = 4$

$h[n] : N_2 = ?$

$y[n] : N = 6$

$N = N_1 + N_2 - 1$

$\Rightarrow 6 = 4 + N_2 - 1$

$\therefore N_2 = 3$

$x[n] = [-1, 2]$

$h[n] = [n_3, n_4]$

$y[n] = [-2, 3]$

\# $-1 + n_2 = -2 \Rightarrow n_2 = -1$

\# $2 + n_4 = 3 \Rightarrow n_4 = 1$

$\Rightarrow y[n] = \{a, b+2a, \text{----}, 4c\}$

$a = 2$ & $4c = 20 \Rightarrow c = 5$

$b + 2a = 7 \Rightarrow b + 4 = 7$

$b = 3$

$\therefore h[n] = \{2, 3, 5\}$

Qu. $x[n] = \{3, \underset{\uparrow}{-5}, 2, 4\}$ & $y[n] = \{15, -19, \underset{\uparrow}{12}, 4, 16, 16\}$. Find $h[n]$

$x[n] : N_1 = 4 : [-1, 2]$

$y[n] : N = 6 : [-2, 3]$

$\therefore h[n] : N_2 = 3 : [-1, 1]$

Let, $h[n] = \{a, \underset{\uparrow}{b}, c\}$

$$\begin{array}{c|cccc} & 3 & -5 & 2 & 4 \\ \hline a & 3a & -5a & 2a & 4a \\ b & 3b & -5b & 2b & 4b \\ c & 3c & -5c & 2c & 4c \end{array}$$

$y[n] = \{3a, -5a+3b, ----, 4c\}$

$3a = 15 \Rightarrow a = 5$ & $4c = 16 \Rightarrow c = 4$

$3b - 5a = -19 \Rightarrow 3b = -19 + 25 \Rightarrow b = 2$

$\therefore h[n] = \{5, \underset{\uparrow}{2}, 4\}$

#

$$x[n] * h[n] = y[n]$$

$$x[n-1] * h[n+1] = y[n]$$

$$\Rightarrow x[n] * h[n] = x[n-1] * h[n+1]$$

2] Circular Convolution

→ It should always start from origin.

→ x[n] & h[n] should have same number of samples.

→ $y[n] = x[n] \; © \; h[n]$

$y[n] \downarrow N = \max(N_1, N_2)$, $x[n] \downarrow N_1$, $h[n] \downarrow N_2$

Qu. $x[n] = \{\underset{\uparrow}{2}, -3, 4, 5\}$ & $h[n] = \{\underset{\uparrow}{5}, 4, 1, 2\}$.

find $x[n] \circledcirc h[n]$

$$\begin{bmatrix} 2 & 5 & 4 & -3 \\ -3 & 2 & 5 & 4 \\ 4 & -3 & 2 & 5 \\ 5 & 4 & -3 & 2 \end{bmatrix} \times \begin{bmatrix} 5 \\ 4 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 10+20+4+(-6) \\ -15+8+5+8 \\ 20-12+2+10 \\ 25+16-3+4 \end{bmatrix}$$

$$y[n] = \{\underset{\uparrow}{28}, 6, 20, 42\}$$

Qu. $x[n] = \{1, \underset{\uparrow}{2}, 3, 4\}$ & $h[n] = \{2, \underset{\uparrow}{3}, 5\}$

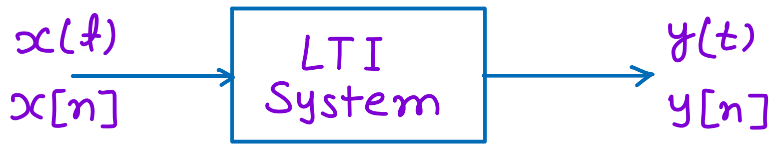
Soln :- $x[n] = \{1, \underset{\uparrow}{2}, 3, 4\}$ & $h[n] = \{2, \underset{\uparrow}{3}, 5, 0\}$ equal samples

$$\begin{bmatrix} 2 & 1 & 4 & 3 \\ 3 & 2 & 1 & 4 \\ 4 & 3 & 2 & 1 \\ 1 & 4 & 3 & 2 \end{bmatrix} \times \begin{bmatrix} 3 \\ 5 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 17 \\ 27 \\ 29 \\ 27 \end{bmatrix}$$

Start from origin

$$y[n] = \{\underset{\uparrow}{17}, 27, 29, 27\}$$

Properties of LTI Systems:



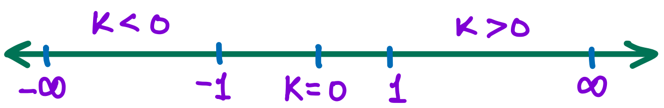
$\Rightarrow y[n] = x[n] * h[n]$

$h[n]$: Impulse Response.

$$\Rightarrow y[n] = \sum_{K=-\infty}^{K=\infty} h[K]\cdot x[n-K]$$

$y[n] = x[n-K]$

1. $K=0 \Rightarrow y[n]=x[n] \rightarrow$ Present
2. $K>0 \Rightarrow y[n]=x[n-2] \rightarrow$ Past
3. $K<0 \Rightarrow y[n]=x[n+1] \rightarrow$ future



$$\Rightarrow y[n] = \underbrace{\sum_{K=-\infty}^{K=-1} h[K]\cdot x[n-K]}_{\substack{K<0 \\ \text{future i/p}}} + \underbrace{h[0]\cdot x[n]}_{\substack{K=0 \\ \text{Present i/p}}} + \underbrace{\sum_{K=1}^{K=\infty} h[K]\cdot x[n-K]}_{\substack{K>0 \\ \text{Past i/p}}}$$

⟨1⟩ Causality:

$$\Rightarrow \text{If } \underbrace{\sum_{K=-\infty}^{K=-1} h[K]\cdot x[n-K]}_{K<0} = 0 \Rightarrow \text{LTI system is causal}$$

$\Rightarrow h[K]=0 \; ; \; K<0$

$$\boxed{\begin{array}{l} h[n]=0 \; ; \; n<0 \\ \text{LTI System} \Rightarrow \text{Causal} \\ h(t)=0 \; ; \; t<0 \end{array}}$$

⟨2⟩ Static System:

$\Rightarrow$ If $h[K]=0$ for $K>0$ & $K<0$

or

$h[K] \neq 0 \; ; \; K=0$

$\quad\;\; = 0 \; ; \; K \neq 0$

$\Bigg\}$ LTI System = Static

$h[n] = A \; ; \; n=0$

$\quad\;\; = 0 \; ; \; n \neq 0$

$\Rightarrow$

$$\boxed{\begin{array}{l} h[n] = A\cdot\delta[n] \\ h(t) = A\cdot\delta(t) \end{array}}$$

(3) Stability :

$\Rightarrow$ Bounded input $\rightarrow$ Bounded output $\Rightarrow$ Stable.

$$\Rightarrow |x(t)| \le M_x < \infty \longrightarrow |y(t)| \le M_y < \infty$$

$$\Rightarrow y(t) = \int_{-\infty}^{+\infty} h(\tau) \cdot \underbrace{x(t-\tau)}_{B.I.} \cdot d\tau$$

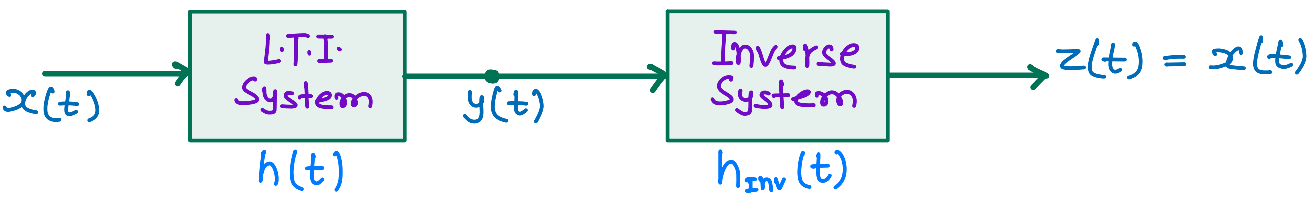
$$\Rightarrow |y(t)| < \infty \Rightarrow B.O. \rightarrow \text{Stable}$$

$$\Rightarrow \left| \int_{-\infty}^{+\infty} h(\tau) \cdot \underbrace{x(t-\tau)}_{\text{finite}} \cdot d\tau \right| < \infty$$

$$\Rightarrow \int_{-\infty}^{\infty} |h(\tau)| \cdot d\tau < \infty \Rightarrow \text{Stable}$$

$$\boxed{\begin{array}{l} \Rightarrow \displaystyle\int_{-\infty}^{+\infty} |h(t)| \cdot dt < \infty \Rightarrow \text{Stable} \\ \Rightarrow \displaystyle\sum_{n=-\infty}^{n=\infty} |h[n]| < \infty \Rightarrow \text{Stable} \end{array}}$$

(4) Invertibility :



$$\Rightarrow y(t) = \underbrace{x(t) * h(t)}$$

$$\Rightarrow z(t) = \underbrace{y(t)} * h_{Inv}(t)$$

$$\boxed{\begin{array}{l} h(t) * h_{Inv}(t) = \delta(t) \\ h[n] * h_{Inv}[n] = \delta[n] \end{array} \rightarrow \text{Invertible System}}$$

$$\Rightarrow z(t) = x(t) * \underbrace{h(t) * h_{Inv}(t)} = x(t) \longrightarrow \text{Invertible}$$

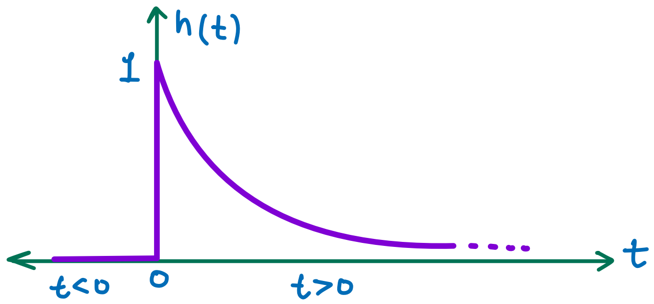
If $h(t) * h_{Inv}(t) = \delta(t) \Rightarrow z(t) = x(t)$

$$z(t) = x(t) * \delta(t) = x(t)$$

Summary Table :

Properties	C.T.S.	D.T.S.
Causality	$h(t)=0;\ t<0$	$h[n]=0\ ;\ n<0$
Statics	$h(t)=A\,\delta(t)$	$h[n]=A\cdot\delta[n]$
Stability	$\int_{-\infty}^{+\infty}\lvert h(t)\rvert\cdot dt<\infty$	$\sum_{n=-\infty}^{n=+\infty}\lvert h[n]\rvert<\infty$
Invertibility	$h(t)*h_{Inv}(t)=\delta(t)$	$h[n]*h_{Inv}[n]=\delta[n]$

Qu. $h(t)=e^{-3t}\cdot u(t)$. Check Causality & Stability.



$$\Rightarrow \int_{-\infty}^{+\infty}\lvert h(t)\rvert\cdot dt$$

$$=\int_{-\infty}^{+\infty}\lvert e^{-3t}\cdot u(t)\rvert\cdot dt$$

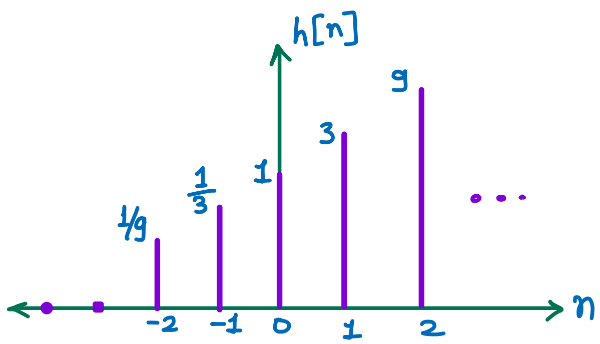
$$=\int_{0}^{\infty}\lvert e^{-3t}\rvert\cdot dt<\infty$$

$\hookrightarrow$ Stable

$\Rightarrow h(t)=0;\ t<0$ $\rightarrow$ Causal System

$\rightarrow$ Dynamic System.

Qu. $h[n]=(3)^n\cdot u[n+2]$



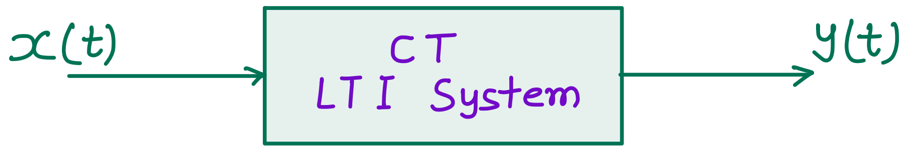
$h[n]\neq 0\ ;\ n<0$

$\hookrightarrow$ Non-Causal

$$\sum_{n=-\infty}^{n=+\infty}\lvert h[n]\rvert=\sum_{n=-\infty}^{n=\infty}\lvert 3^n\cdot u[n]\rvert=\infty$$

$\hookrightarrow$ Unstable

Impulse Response & Step Response



(1) Impulse Response

$\Rightarrow$ $x(t) = \delta(t)$

$\Rightarrow$ $y(t) = \text{I.R.}$

$\Rightarrow$ $y(t) = x(t) * h(t)$

$\Rightarrow$ $y(t) = \delta(t) * h(t)$

$\Rightarrow$ $y(t) = h(t) \rightarrow$ Impulse Response

(2) Unit Step Response

$\Rightarrow$ $x(t) = u(t)$

$\Rightarrow$ $y(t) = \text{U.S.R.} = s(t)$

$\Rightarrow$ $y(t) = x(t) * h(t)$

$\Rightarrow$ $y(t) = u(t) * h(t)$

$\Rightarrow$ $s(t) = u(t) * h(t)$

$\Rightarrow$ $\dfrac{ds(t)}{dt} = h(t)$

$$s(t) = \int_{-\infty}^{t} h(\tau)\cdot d\tau$$

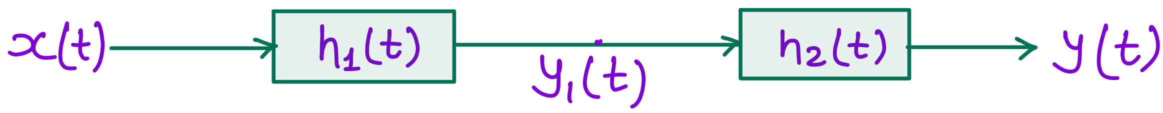
$s(t) \downarrow$ Unit Step Response

$h(\tau) \downarrow$ Impulse Response

$\Rightarrow$ $\boxed{h(t) = \dfrac{ds(t)}{dt}}$.

Interconnection of System

(1) Series Connection or Cascade Connection



$\Rightarrow$ $y_1(t) = x(t) * h_1(t)$

$\Rightarrow$ $y(t) = y_1(t) * h_2(t)$

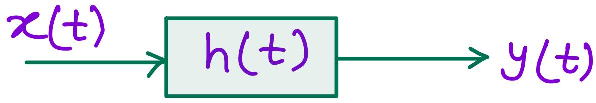
$\Rightarrow$ $y(t) = x(t) * \underbrace{h_1(t) * h_2(t)}$

Let,

$h(t) = h_1(t) * h_2(t)$

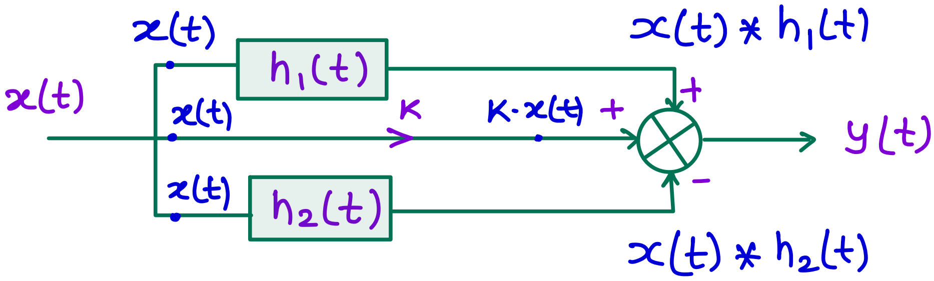
$\Rightarrow$ $y(t) = x(t) * h(t)$

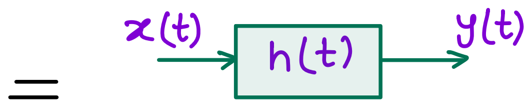
$|||$



$h(t) = h_1(t) * h_2(t)$

(2) Parallel Connection





$h(t) = h_1(t) + K\cdot\delta(t) - h_2(t)$

$$y(t) = x(t) * h_1(t) + K\cdot x(t) - x(t) * h_2(t)$$

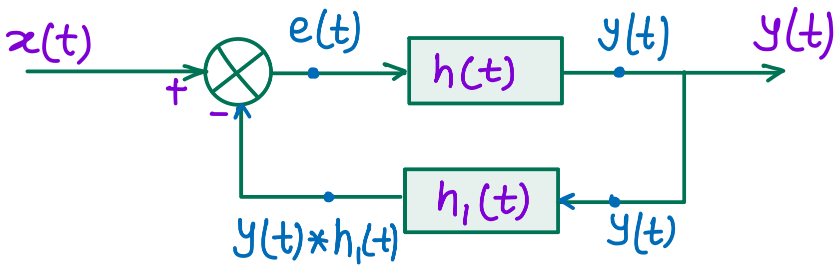
$$\Rightarrow y(t) = x(t) * h_1(t) + K\cdot x(t) * \delta(t) - x(t) * h_2(t)$$

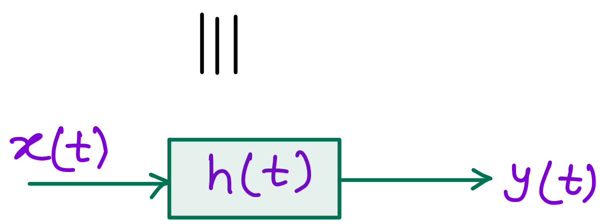
$$\Rightarrow y(t) = x(t) * \big(\underbrace{h_1(t) + K\cdot\delta(t) - h_2(t)}_{h(t)}\big)$$

Let, $h(t) = h_1(t) + K\cdot\delta(t) - h_2(t)$

$$\Rightarrow y(t) = x(t) * h(t)$$

(3) Feedback Connection





$$e(t) = x(t) - y(t) * h_1(t)$$

$$\Rightarrow y(t) = e(t) * h(t)$$

$$\Rightarrow y(t) = [x(t) - y(t) * h_1(t)] * h(t)$$

$$\Rightarrow y(t) = x(t) * h(t) - y(t) * h_1(t) * h(t)$$

$$\Rightarrow y(t) + y(t) * h(t) * h_1(t) = x(t) * h(t)$$

$$\Rightarrow y(t) * \delta(t) + y(t) * h(t) * h_1(t) = x(t) * h(t).$$

$$\Rightarrow y(t) * [\delta(t) + h(t) * h_1(t)] = x(t) * h(t)$$

Time Domain में Equivalent Block Diagram नहीं बनेगा। Laplace में बनेगा।

Qu. A continuous-time linear system with input x(t) and output y(t) yield the following input-output pairs:

$$x_a(t) = e^{j2t} \Longleftrightarrow y(t) = e^{j5t}$$
$$x_b(t) = e^{-j2t} \Longleftrightarrow y(t) = e^{-j5t}$$

If $x_1(t) = \cos(2t-1)$, the corresponding $y_1(t)$ is

(a) $\cos(5t-1)$ (c) $\cos 5(t-1)$

(b) $e^{-j}\cos(5t-1)$ (d) $e^{j}\cos(5t-1)$

Sol^n:-

$$x_1(t) = \cos(2t-1) = \frac{e^{j(2t-1)} + e^{-j(2t-1)}}{2}$$

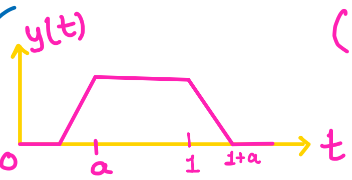
$$\Rightarrow x_1(t) = \frac{1}{2}\cdot e^{-j}\cdot \underbrace{e^{j2t}}_{x_a(t)} + \frac{1}{2}\, e^{j}\cdot \underbrace{e^{-j2t}}_{x_b(t)}$$

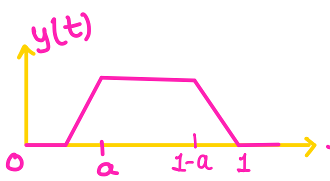
$$\Rightarrow y_1(t) = \frac{1}{2}\cdot e^{-j}\cdot e^{j5t} + \frac{1}{2}\cdot e^{j}\cdot e^{-j5t}$$

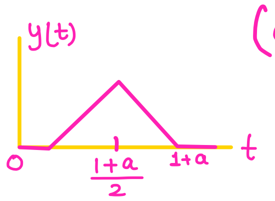
$$\Rightarrow y_1(t) = \frac{e^{j(5t-1)} + e^{-j(5t-1)}}{2} = \cos(5t-1)$$

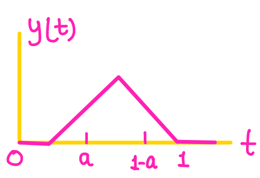
Qu. Suppose that $x(t) = \begin{cases} 1, & 0 \le t \le 1 \\ 0, & \text{elsewhere} \end{cases}$

$h(t) = x\left(\frac{t}{a}\right)$, where $0 < a \le 1$. The $y(t) = x(t) * h(t)$ is

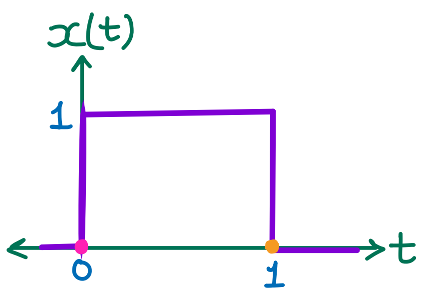
(a) 

(b) 

(c) 

(d) 

Sol^n:-



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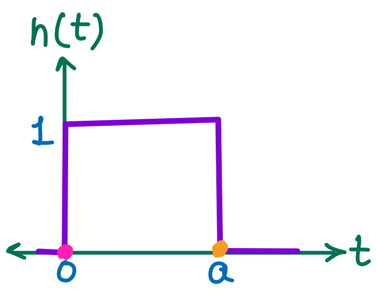

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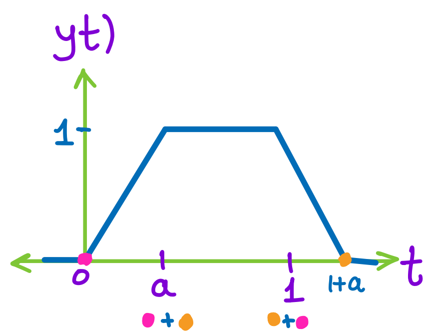
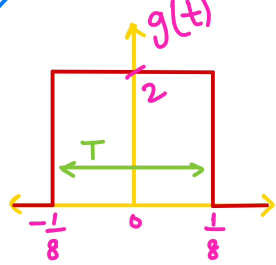


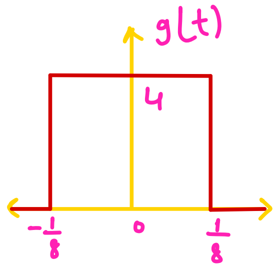
Qu. The graph for function $g(t) = \text{rect}(4t) * 4\delta(2t)$ is

(a)

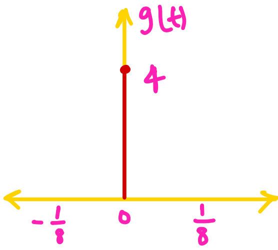
$T = 4$



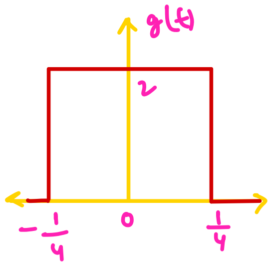
(b)



(c)



(d)



$$g(t) = \text{rect}(4t) * \frac{4}{2}\delta(t)$$

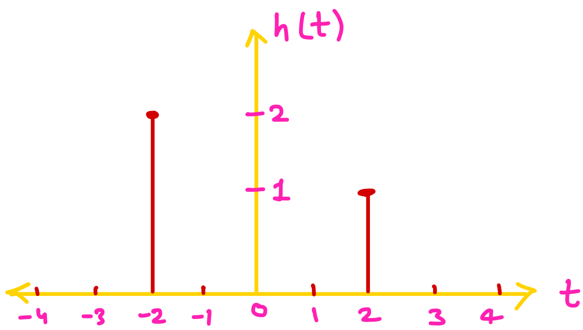
$$g(t) = \text{rect}(4t) * 2\delta(t)$$

$$g(t) = 2\cdot\text{rect}(4t)$$
$$= A\cdot\text{rect}\left(\frac{t}{T}\right)$$

$\therefore A = 2$, $T = \frac{1}{4}$

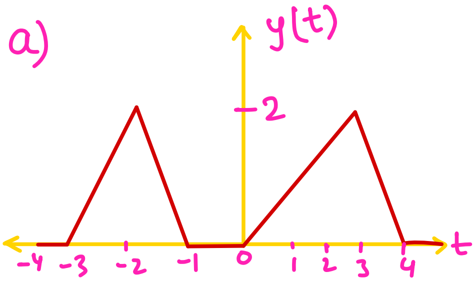
Qu. Impulse response of a CT system is shown in figure below. The system is excited by an input given as

$$x(t) = \begin{cases} 3 - 1.5|t| & ; |t| < 2 \\ 0 & ; |t| \geq 2 \end{cases}$$

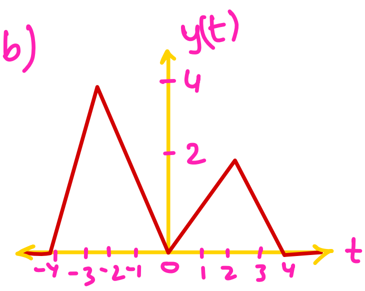


The correct graph of output $y(t)$ is –

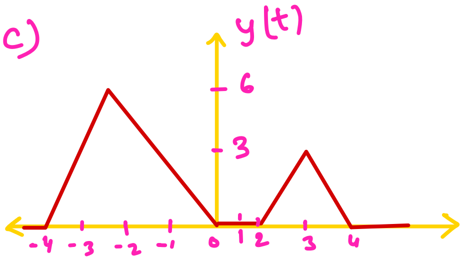
a)



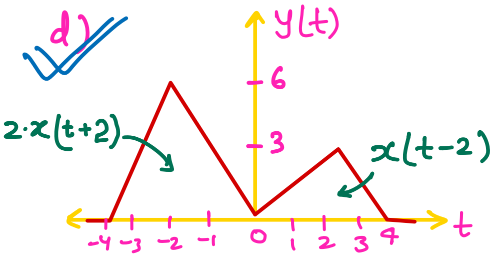
b)



c)

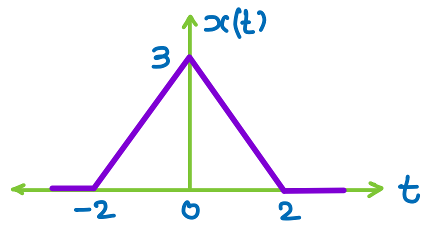


d)



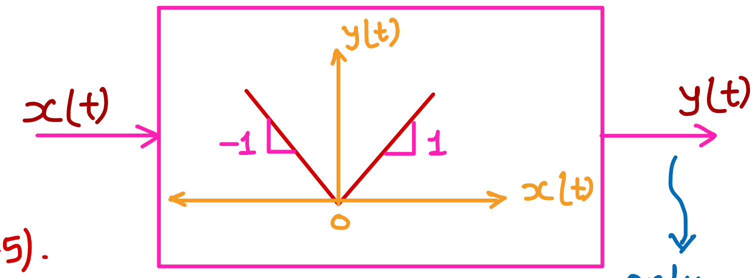
$$h(t) = \delta(t-2) + 2\cdot\delta(t+2)$$
$$y(t) = x(t) * h(t)$$
$$= x(t) * [\delta(t-2) + 2\cdot\delta(t+2)]$$
$$= x(t-2) + 2\cdot x(t+2)$$

$t = 2 \Rightarrow 0$, $t = -2 \Rightarrow 0$, $t = 0 \Rightarrow 3$



Qu. Consider a CT system shown in figure.

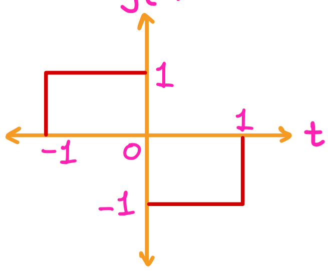
Input to this system is
$x(t) = \text{rect}(t-0.5) - \text{rect}(t+0.5)$.
The output $y(t)$ is —



only Positive Amplitude

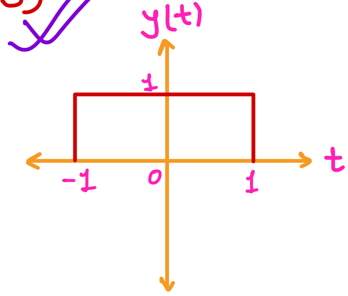
$y(t) = |x(t)|$

a)



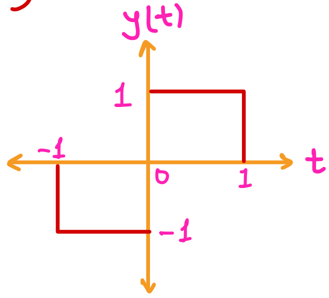
↳ (−)ve Amplitude

b)



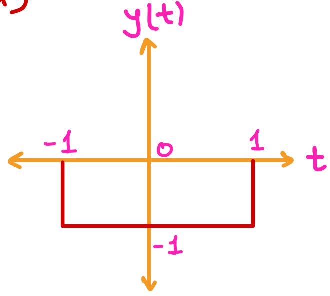
↳ only (+)ve Amplitude

c)



↳ (−) ve

d)



↳ (−) ve

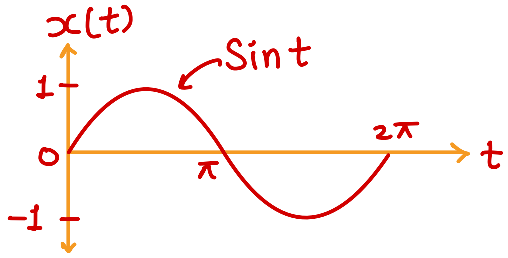
Qu. The convolution of two signals $x(t)$ and $h(t)$ is $y(t) = x(t) * h(t)$. If the signal $z(t) = x(2t) * h(2t)$, then $\dfrac{z(t)}{y(2t)} =$ ________

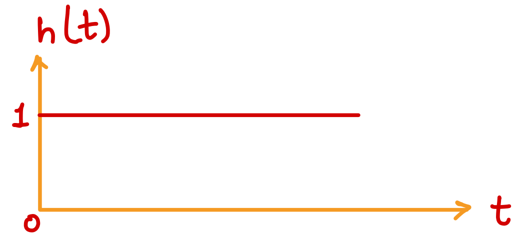
$$x(at) * h(at) = \frac{1}{|a|}\, y(at)$$

$$x(2t) * h(2t) = \frac{1}{2}\, y(2t) = z(t)$$

$$\Rightarrow \frac{z(t)}{y(2t)} = \frac{1}{2}$$

Q. Consider two signal $x(t)$ and $h(t)$ as shown in figure. What is the value of convolution $y(t) = x(t) * h(t)$ at $t = \pi$?



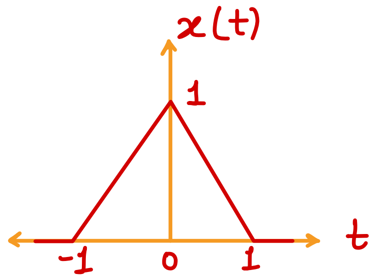


$$y(t) = x(t) * h(t)$$

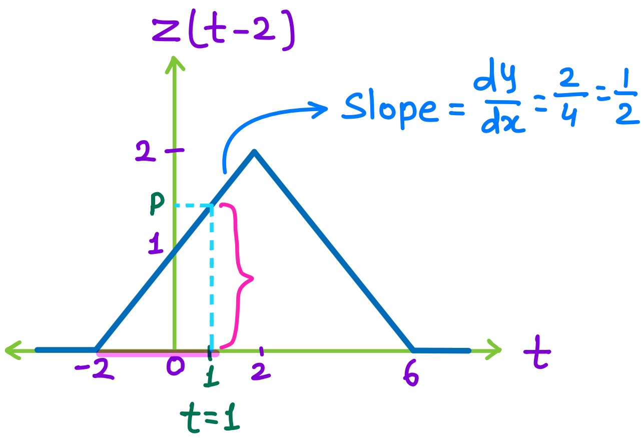
$$= \int_0^t (\sin\tau)\cdot(1)\cdot d\tau = \left(-\cos\tau\right)_0^t = 1 - \cos t \;;\; 0 \le t \le 2\pi$$

$$y(t)\Big|_{t=\pi} = 1 - \cos\pi = 2$$

Q Consider a signal $x(t)$ shown in figure below



If $y(t) = 2x(t/4) * \delta(t-2)$, where $\delta(t)$ is an unit impulse function, then what will be the value of $y(1)$?



$$z(t) = 2\cdot x\left(\frac{t}{4}\right)$$

$$y(t) = z(t) * \delta(t-2)$$

$$y(t) = z(t-2)$$

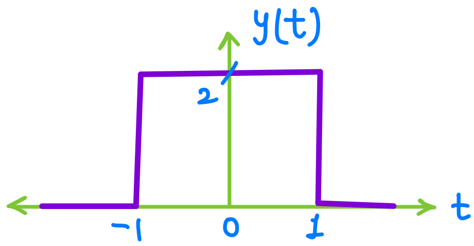
$$P = y(t)\Big|_{t=1} = \frac{1}{2}\times 3$$

$$= \frac{3}{2} = 1.5$$

Q A function $g(t)$ is given as $g(t) = 6\,\text{rect}\left(\frac{t+4}{2}\right) * \delta(3t)$ then value of $g(-4)$ is ______.

$$g(t) = 6\ \text{rect}\left(\frac{t+4}{2}\right) * \frac{1}{3}\delta(t)$$

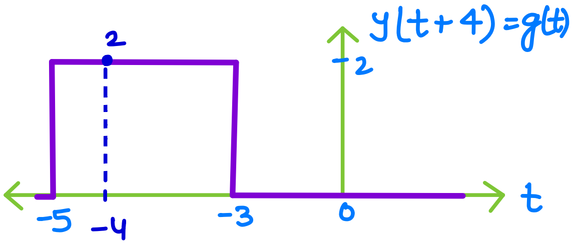
$$\Rightarrow g(t) = 2\cdot\text{rect}\left(\frac{t+4}{2}\right)$$



Let,

$$y(t) = 2\cdot\ \text{rect}\left(\frac{t}{2}\right)$$

$$\Rightarrow y(t+4) = 2\cdot\ \text{rect}\left(\frac{t+4}{2}\right) = g(t)$$



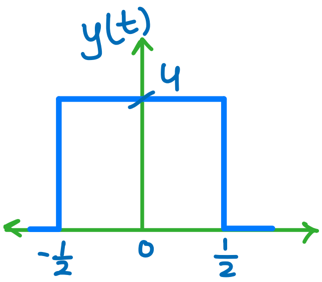
$$g(t)\Big|_{t=-4} = 2$$

Q If $\delta_T(t) = \sum_{n=-\infty}^{\infty}\delta(t-nT)$, then power of the signal $x(t) = 4\text{rect}(t) * \delta_4(t)$ is ______ unit.

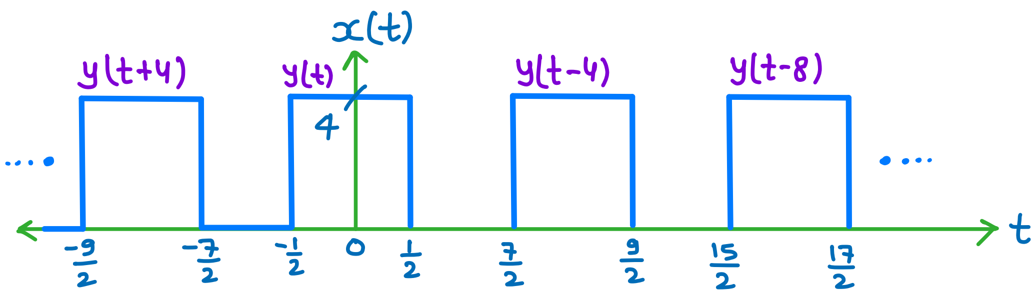
$$\delta_T(t) = \cdots + \delta(t+T) + \delta(t) + \delta(t-T) + \delta(t-2T) + \cdots$$

$$x(t) = 4\text{rect}(t) * \delta_4(t)$$

$$x(t) = \underbrace{4\ \text{rect}(t)}_{y(t)} * (\cdots + \delta(t+4) + \delta(t) + \delta(t-4) + \delta(t-8) + \cdots)$$

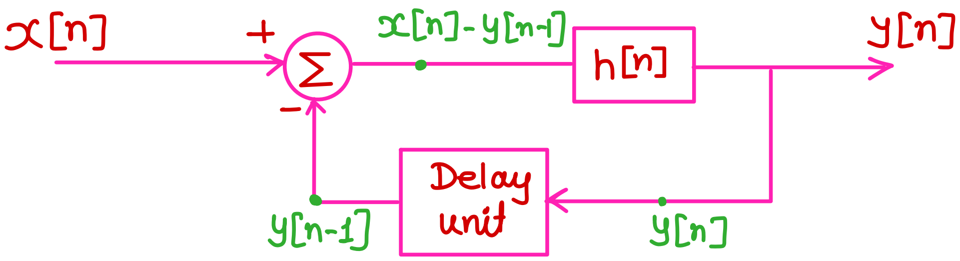


$$x(t) = \cdots + y(t+4) + y(t) + y(t-4) + y(t-8) - \cdots$$



$$P_{avg.} = \frac{E_{x(t)} \text{ over a f.T.}}{T} = \frac{16\times 1}{4} = 4$$

Q The block diagram representation of a D.T system is shown in the figure, where $h[n] = \delta[n] - \delta[n-1]$.



The difference equation that describes the complete system is —

a) $y[n] + y[n-1] = x[n] - x[n-1]$

b) $y[n] = x[n] - x[n-1]$

c) $y[n] + y[n-1] = x[n]$

d) $y[n] + y[n-1] - y[n-2] = x[n] - x[n-1]$ ✓

$\Rightarrow \quad y[n] = h[n] * (x[n] - y[n-1]) = (\delta[n] - \delta[n-1]) * (x[n] - y[n-1])$

$\Rightarrow \quad y[n] = x[n] - y[n-1] - x[n-1] + y[n-2]$

$\Rightarrow \quad y[n] + y[n-1] - y[n-2] = x[n] - x[n-1]$

Q- The impulse response of a LTI system is given as $h[n] = \left(-\frac{1}{2}\right)^n u[n]$. The step response is -

Q An input $x[n] = \alpha^n u[n]$ is applied to a discrete LTI system having the following impulse response $h[n] = \beta^n u[n]$.
If $\alpha = \beta = 2$, then for $n=1$ output $y[n]$ of the system will be ______.

Q An input $x[n] = \alpha^n u[n]$ is applied to a discrete LTI system having the following impulse response $h[n] = \beta^n u[n]$.
If $\alpha = 1$, $\beta = 2$, then what will be the output $y[n]$ at $n=2$?

$$y[n] = \alpha^n . u[n] * \beta^n . u[n] = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} . u[n]$$

$$y[n] = u[n] * 2^n . u[n] = \frac{1 - 2^{n+1}}{1-2} . u[n]$$

$$y[n]\Big|_{n=2} = \frac{1-2^{2+1}}{1-2} = 7$$

Q Consider two DT signals given as $x[n] = (0.8)^n u[n]$ and $h[n] = (0.4)^n u[n]$ if $y[n] = x[n] * h[n]$, then what will be the value of sample $y[1]$?

Q An inter-connection of LTI systems is shown in the figure. The impulse responses are $h_1[n] = 0.5\,\delta[n-1] + 0.7\,\delta[n]$ and $h_2[n] = \delta[n-1]$. If the response of overall system be $A\delta[n] - B\delta[n-1] - C\delta[n-2]$, then $A+B+C =$ ______ ?