Sampling and Particle Filtering

CMPT 498/820 Machine Learning
Tutorial 7

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1 Monte Carlo Estimation

Often we need to estimate intractable expectations in Machine Learning of the form:

$$E\left[f(x)\right] = \int_p f(x)p(x)dx$$

.
The Monte Carlo estimate is given by

$$E\left[f(x)\right] \approx \frac{1}{N}\sum_{i=1}^{N} f(x_i), \quad \text{where } x_i \sim p(x_i)$$

How do we find x_i? We can use built-in functions for familliar distributions such as `numpy.random.rand()` and `numpy.random.normal()` for sampling from uniform and Gaussian $p(x)$, respectively.

1.1 The Problem of Sampling from a Distribution

Given: Arbitrary $p(x)$
Find: $X = [x_1 \cdots x_N]$ such that the samples are distributed according to $p(x)$, i.e, if we draw a histogram of X we see an approximation to $p(x)$

```
In [1]: %matplotlib inline
        import numpy as np
        from scipy.stats import beta
        from matplotlib import pyplot as plt
        np.random.seed(43)
        N=10**3   # number of samples
```

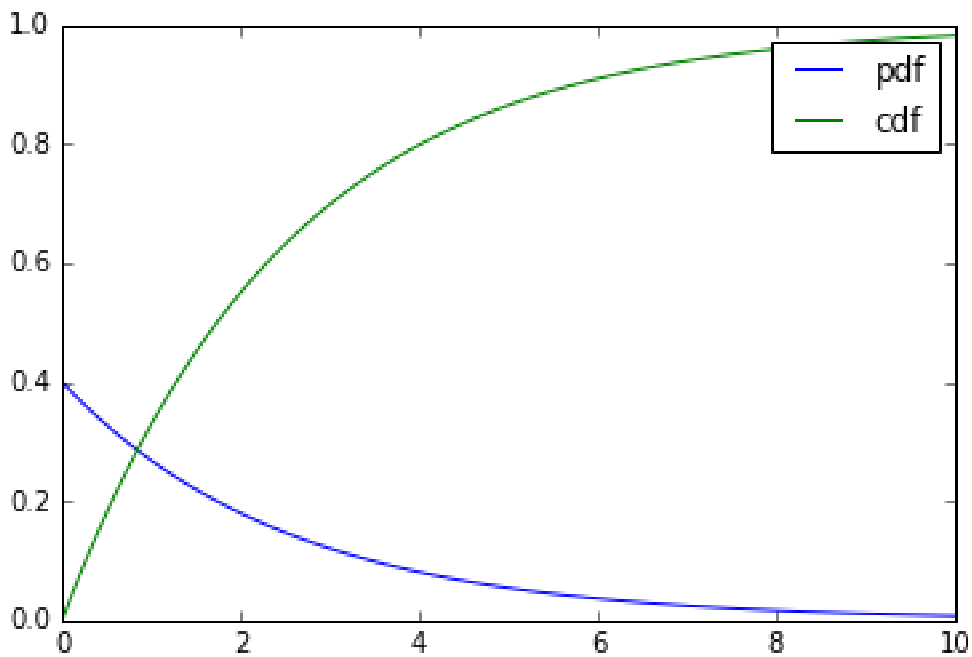
1.2 Inverse CDF

$$F(x) = Pr(X < x)$$

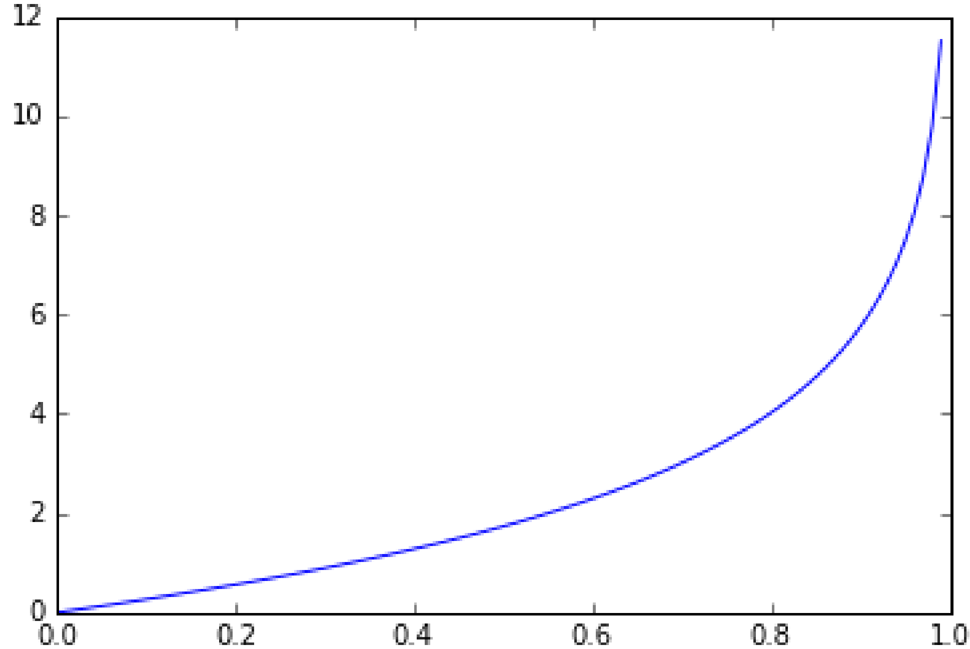
If $u \sim U(0, 1)$, then $F^{-1}(u) \sim F$

```
In [2]: lamb=0.4
        exp_pdf = lambda x: lamb*np.exp(-lamb*x)
        exp_cdf = lambda x: 1-np.exp(-lamb*x)
        exp_inv_cdf = lambda p: -(np.log(1-p))/lamb
```

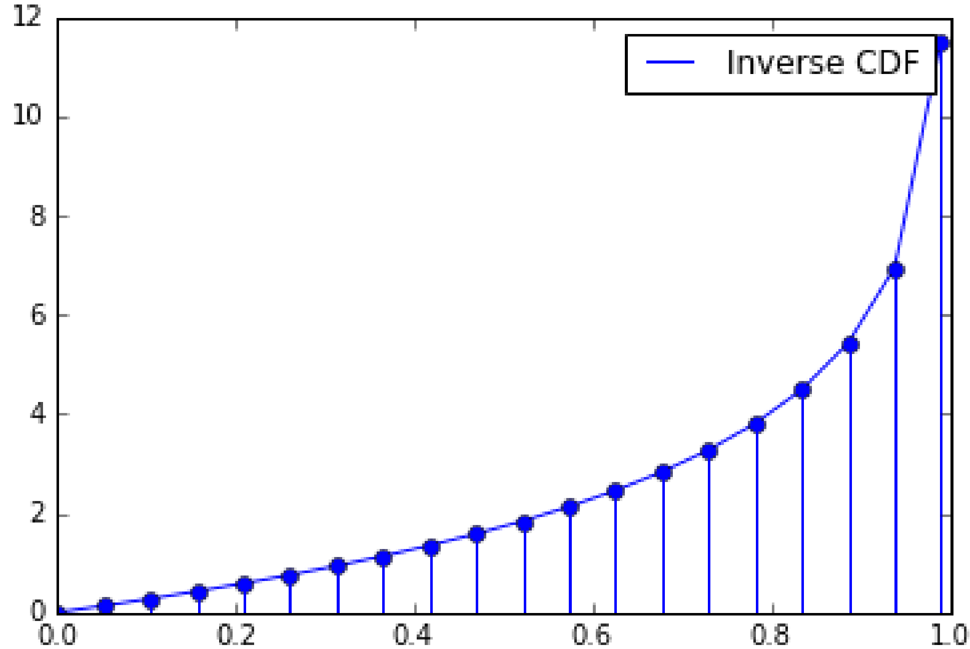
```
In [3]: x=np.linspace(0,10,100)
        p=exp_pdf(x)
        c=exp_cdf(x)
        plt.plot(x, p)
        plt.plot(x, c)
        plt.legend(labels=['pdf', 'cdf']);
```




```
In [4]: p = np.linspace(0.001,0.99,100)
        ic=exp_inv_cdf(p)
        plt.plot(p, ic);
```

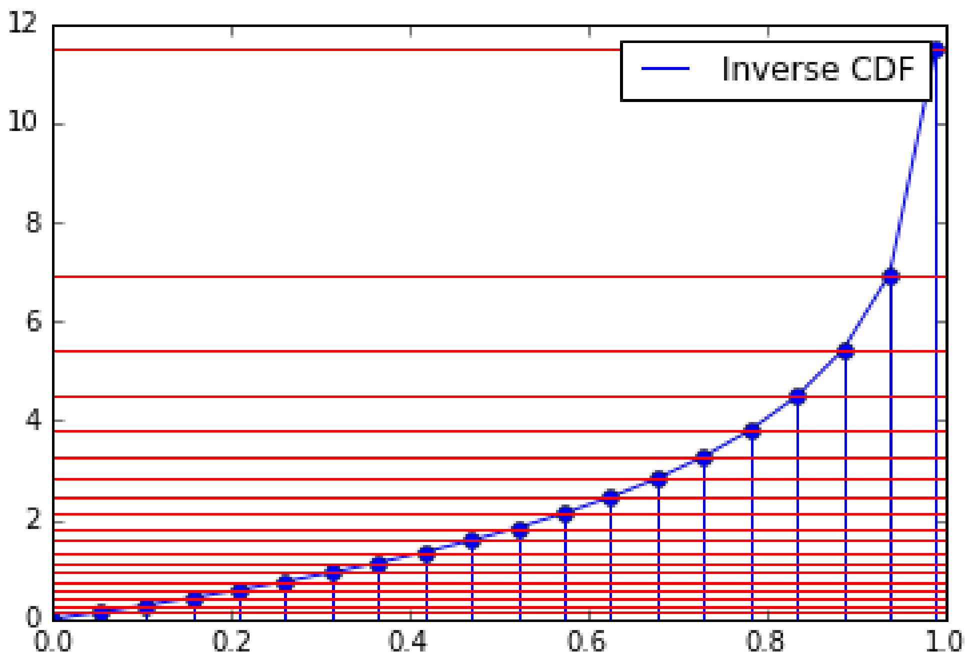



```
In [5]: p = np.linspace(0.001,0.99,20)
        ic=exp_inv_cdf(p)
        fig, ax =plt.subplots()
        ax.plot(p, ic);
        ax.stem(p, ic)
        ax.legend(['Inverse CDF']);
```

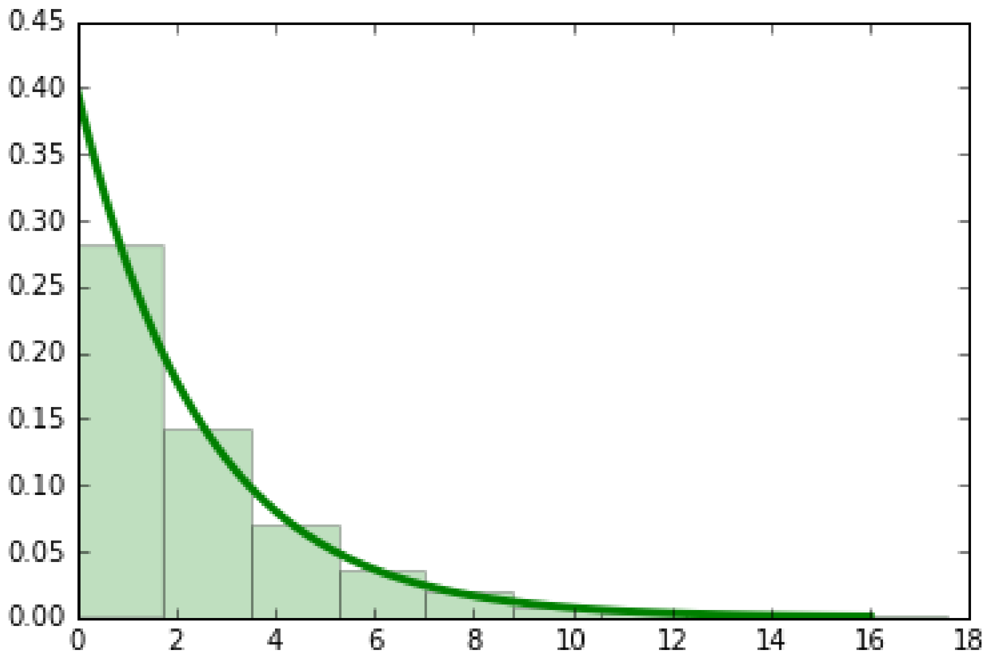


In [6]:
```
[ax.axhline(y=l, color='red') for l in ic]
fig
```

Out[6]:



In [7]:
```
u = np.random.rand(N)
exp_samples=exp_inv_cdf(u)
plt.hist(exp_samples, 10, normed=1, facecolor='green', alpha=0.25)
x=np.linspace(0,16,100)
p=exp_pdf(x)
plt.plot(x,p, linewidth=3);
```



1.3 Rejection Sampling

Let our target probability distribution be (Remember that we generally don't know how to sample from p(x)):

$$p(x) = \frac{1}{B(\alpha, \beta)} x^{\alpha-1}(1-x)^{\beta-1}$$

and let our proposal distribution be a uniform pdf.
Define the target p and proposal distributions q

```
In [8]: a=5; b=1.5
        truepdf     = lambda x: beta.pdf(x, a=a, b=b) # beta probability distribut
        proposalpdf = lambda n: np.random.rand(n)      # uniform probability distr
```

Sample from the proposal distribution

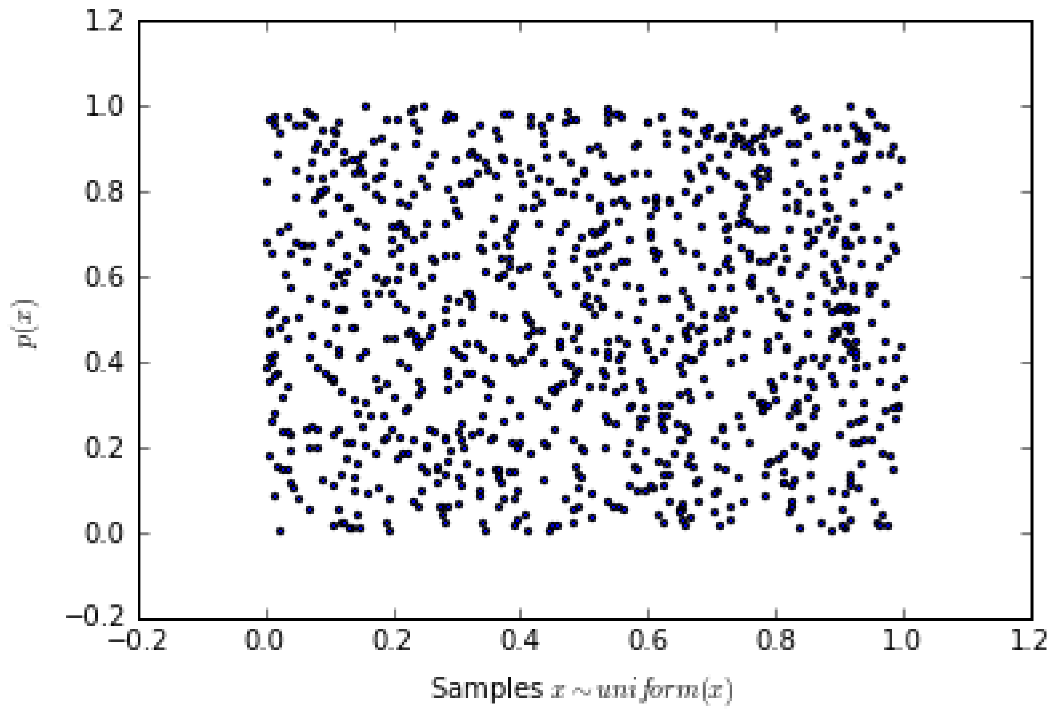
```
In [9]: q_samples = proposalpdf(N)
```

Evaluate the probability of the generated samples

```
In [10]: p_values = truepdf(q_samples)
```

Generate uniformly distributed samples

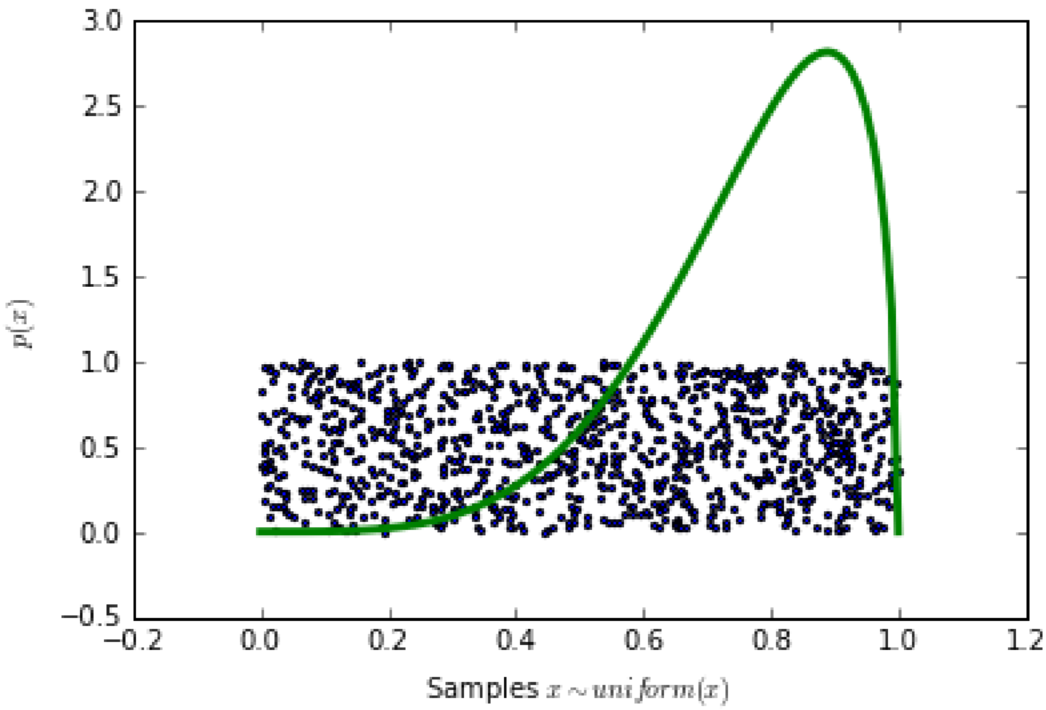
```
In [11]: rnd_sample  = np.random.rand(N)  # uniform distribution
         fig, ax = plt.subplots()
         ax.scatter(q_samples, rnd_sample, marker=".")
         ax.set_xlabel('Samples $x \sim uniform(x)$'); ax.set_ylabel('$p(x)$');
```



These samples however do not encompass our true pdf

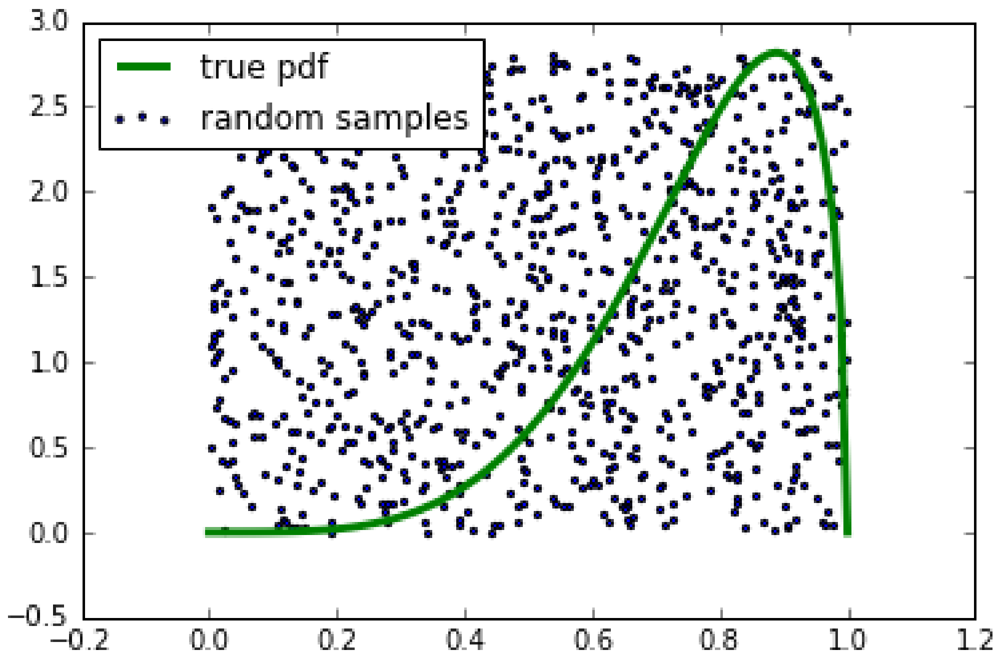
```
In [12]: x = np.linspace(0,1,100);
         ax.plot(x, truepdf(x), color='green' ,linewidth=3 )
         fig
```

Out[12]:



So we scale it by M, the maximum value of the true pdf

```
In [13]: M = max(p_values)
         rnd_sample = M*rnd_sample
         fig, ax = plt.subplots()
         ax.scatter(q_samples, rnd_sample, marker=".", label='random samples')
         ax.plot(x, truepdf(x), color='green' ,linewidth=3, label='true pdf' )
         ax.legend(loc=2);
```



Find out acceptable samples

```
In [14]: # find out acceptable samples
         accepted = rnd_sample < p_values
         a_samples = q_samples[accepted]
         r_samples = q_samples[~accepted]
```

Estimated expected value:

```
In [15]: np.mean(a_samples)

Out[15]: 0.77244358056829099
```

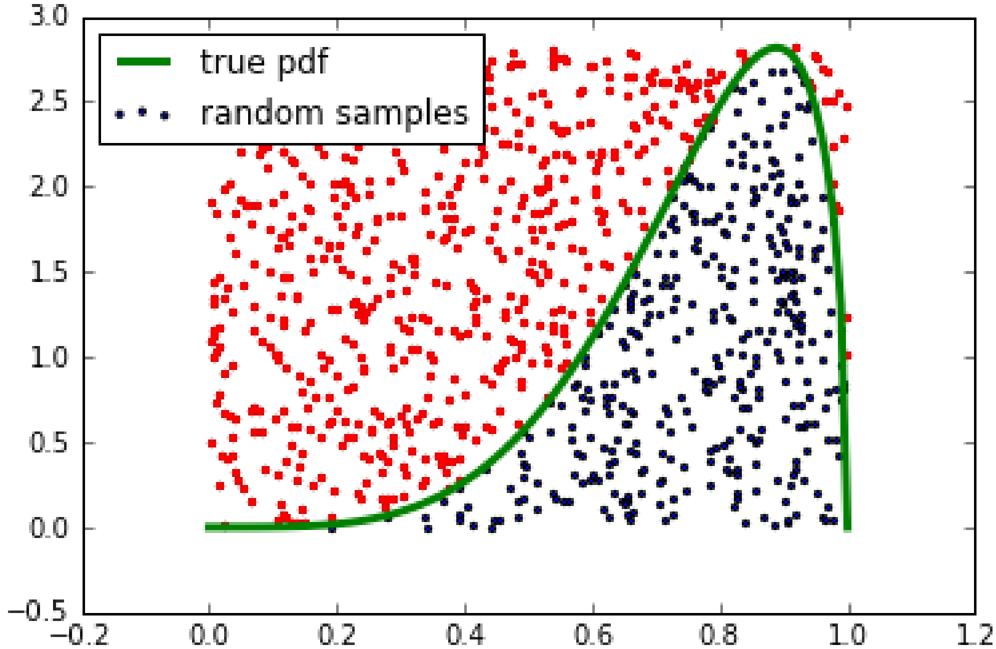
```
In [16]: a/(a+b)

Out[16]: 0.7692307692307693
```

Many samples are rejected

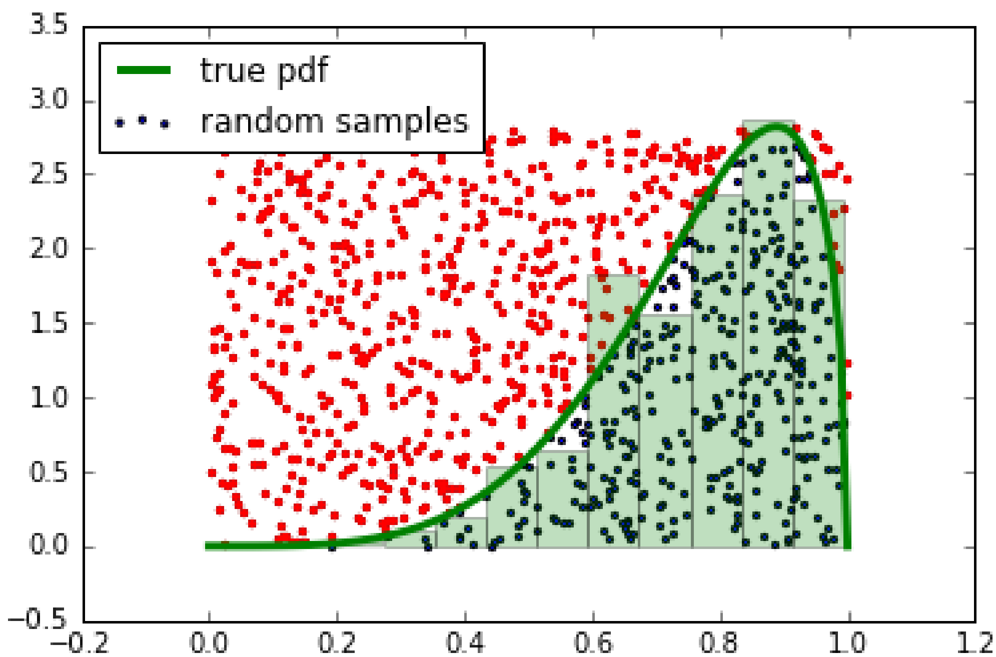
```
In [17]: ax.scatter(r_samples, rnd_sample[~accepted], marker=".", color='red')
         fig

Out[17]:
```




```
In [18]: ax.hist(a_samples, 10, normed=1, facecolor='green', alpha=0.25)
         fig
```

Out[18]:



In [19]:
```
rejection_ratio = 100.0*sum(~accepted)/N
rejection_ratio
```

Out[19]: 63.100000000000001

1.4 Importance Sampling

$$E\left[f(x)\right] = \int_p f(x)p(x)dx = \int_p f(x)\left[\frac{p(x)}{q(x)}\right]q(x)dx \quad x \sim q(x)$$

In [20]:
```
def estimate(f, p, q, qSampler, N):
    # sample from proposal
    x=qSampler(N)

    # compute weights
    w = p(x)/q(x)

    # normalize weights
    w = w/sum(w)

    # compute expectation
    fest = sum(w*f(x))
    return fest
```

In [21]:
```
# function of x
f = lambda x: x

# target distribution p(x)
p = lambda x: beta.pdf(x, a=a, b=b)

c=(a-1)/(a+b-2) # maximum value of the beta distribution
q = lambda x: c
qSampler = lambda n: np.random.rand(n)       # uniform probability distribu
```

In [22]:
```
estimate(f, p, q, qSampler, N)
```

Out[22]: 0.76464389544564559

In [23]:
```
# Analytical expectation
a/(a+b)
```

Out[23]: 0.7692307692307693

2 Particle Filters

Robot localization using Particle Filter in Python
Robot localization using Particle Filter in Javascript

In [ ]:

Algorithm 23.1: One step of a generic particle filter

for $s = 1 : S$ **do**
  Draw $\mathbf{z}_t^s \sim q(\mathbf{z}_t | \mathbf{z}_{t-1}^s, \mathbf{y}_t)$;
  Compute weight $w_t^s \propto w_{t-1}^s \frac{p(\mathbf{y}_t | \mathbf{z}_t^s) p(\mathbf{z}_t^s | \mathbf{z}_{t-1}^s)}{q(\mathbf{z}_t^s | \mathbf{z}_{t-1}^s, \mathbf{y}_t)}$;
Normalize weights: $w_t^s = \frac{w_t^s}{\sum_{s'} w_t^{s'}}$;
Compute $\hat{S}_{\text{eff}} = \frac{1}{\sum_{s=1}^{S} (w_t^s)^2}$;
if $\hat{S}_{eff} < S_{min}$ **then**
  Resample S indices $\boldsymbol{\pi} \sim \mathbf{w}_t$;
  $\mathbf{z}_t^{\cdot} = \mathbf{z}_t^{\boldsymbol{\pi}}$;
  $w_t^s = 1/S$;

Machine Learning: A Probabilistic Perspective, p826