Randomized Algorithms: Assignment 1

October 14, 2024

1 Galton Board

In this assignment, we will simulate a Galton board to demonstrate the central limit theorem, in particular, the relation between binomial distribution is approximated by the normal distribution in large samples. We will simulate results of the Galton board by dropping of a large number of balls through a series of steps.

Each ball will follow a random path, with the same probability, going either left or right at each step. The probability $p_{i,n}$ of a ball ending at cell $(i, n-i)$ is given by the binomial distribution:

$$p_{i,n} = \binom{n}{i}\left(\frac{1}{2}\right)^n$$

By analyzing the final positions on the outcome of balls, with a sufficiently large sample, we will check how the distribution of the balls approximates the normal distribution, which is characterized by the mean μ and variance σ^2:

$$\mu = \frac{n}{2}, \quad \sigma^2 = \frac{n}{4}$$

1.1 Implementation

We can simulate the ball drops in Galton board by keeping track of the path taken by the balls. Balls drop from cell $(0, 0)$ up to n steps. With probability $\frac{1}{2}$, the ball moves to the left $(i+1, j)$, and also it moves to the right $(i, j+1)$. After n steps, the ball will be at a cell $(i, n-i)$.

Algorithm 1 Galton Board

```
function drop_ball(n)
    pos ← [0, 0]
    for i = 0 to n − 1 do
        if random(0, 1) = 1 then
            pos[0] ← pos[0] + 1 {Move to the left}
        else
            pos[1] ← pos[1] + 1 {Move to the right}
        end if
    end for
    return pos
end function
```

1.2 Binomial distribution

The total number of ways to arrange i moves to the left and $n-i$ moves to the right over n steps can be represented by the binomial:

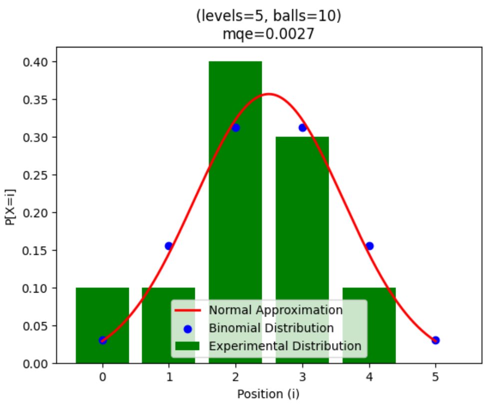
$$\binom{n}{i}$$

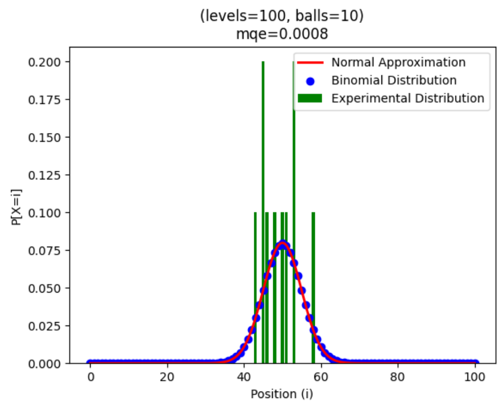
This counts the number of ways to choose i left moves out of n total moves. Since each path has a probability of $\left(\frac{1}{2}\right)^n$, the probability of landing at cell $(i, n-i)$ is given by:

$$p_{i,n} = \binom{n}{i}\left(\frac{1}{2}\right)^n$$

This is a direct application of the binomial distribution.

1.3 Experiments


(levels=5, balls=10)
mqe=0.0027
Normal Approximation
Binomial Distribution
Experimental Distribution
P[X=i]
Position (i)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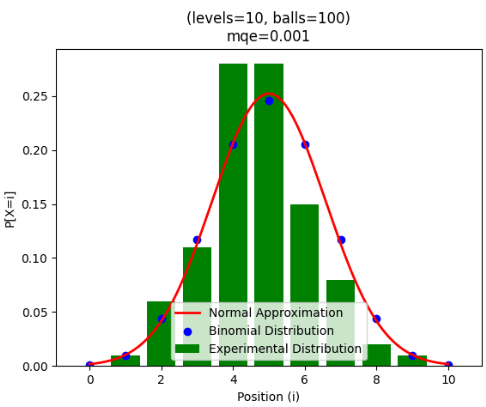
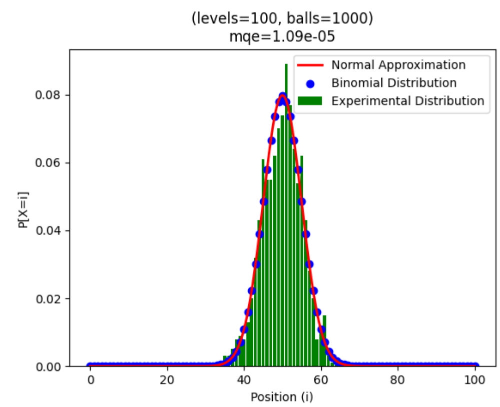
Normal Approximation
Binomial Distribution
Experimental Distribution
P[X=i]
Position (i)


(levels=10, balls=100)
mqe=0.001
Normal Approximation
Binomial Distribution
Experimental Distribution
P[X=i]
Position (i)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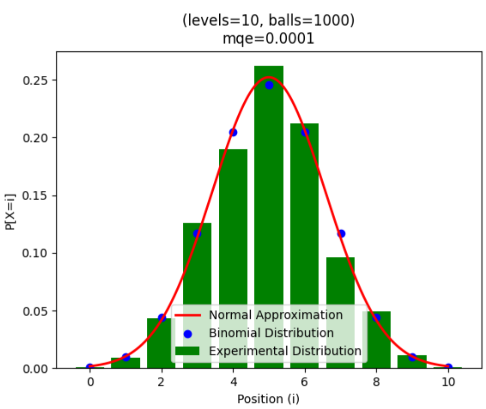
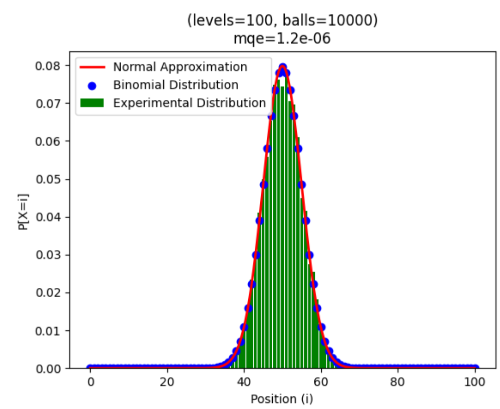
Normal Approximation
Binomial Distribution
Experimental Distribution
P[X=i]
Position (i)


(levels=10, balls=1000)
mqe=0.0001
Normal Approximation
Binomial Distribution
Experimental Distribution
P[X=i]
Position (i)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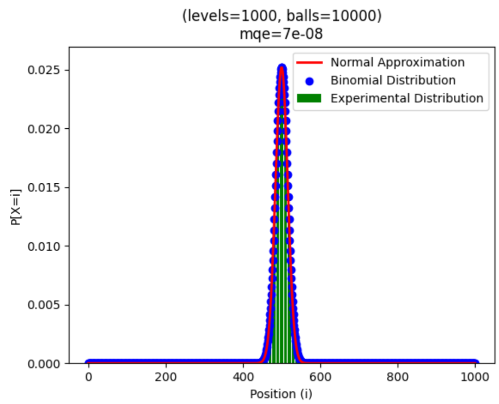
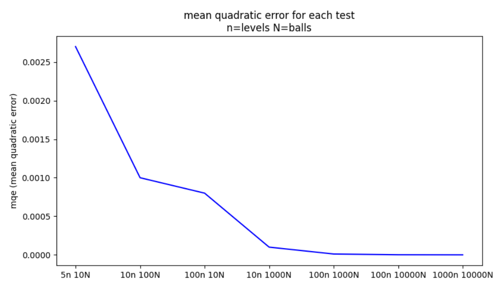
Normal Approximation
Binomial Distribution
Experimental Distribution
P[X=i]
Position (i)





Increasing N (balls dropped) helps the results match the expected predictions because of the law of large numbers. The same happens when increasing n (the number of steps) allowing to reflect better the actual distribution, making the results more accurate.

1.4 Conclusion

With the results achieved, we can confirm the central limit theorem. They show that the binomial distribution approaches the normal distribution with the increase of n and N, made clear by the mean quadratic error between the empirical and theoretical models decreasing near 0 as n and N are increased.

2 Sources

Github Repository: https://github.com/ertem0/RA-1