CENG 384 - Signals and Systems for Computer Engineers
Spring 2018-2019
Written Assignment 3

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1. (a) Let a_k denote kth Fouerier Series coefficient of x[n]
Period of the given signal is $T = 4$, so $\omega_0 = \frac{2\pi}{4}$
Using the discrete time analysis formula over the period from $n = 0$ to $n = 3$ where x[0]=0, x[1]=1, x[2]=2, x[3]=1:
$a_k = \frac{1}{4}(0.e^0 + e^{-jk\frac{\pi}{2}} + 2e^{-jk\pi} + e^{-jk\frac{3\pi}{2}})$

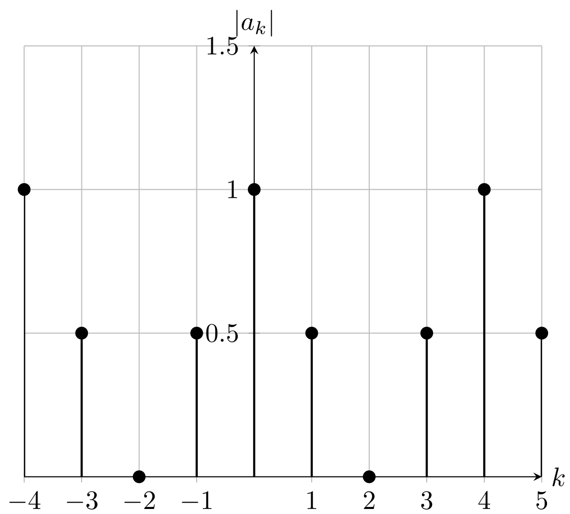
$a_0 = \frac{1}{4}4 = 1$

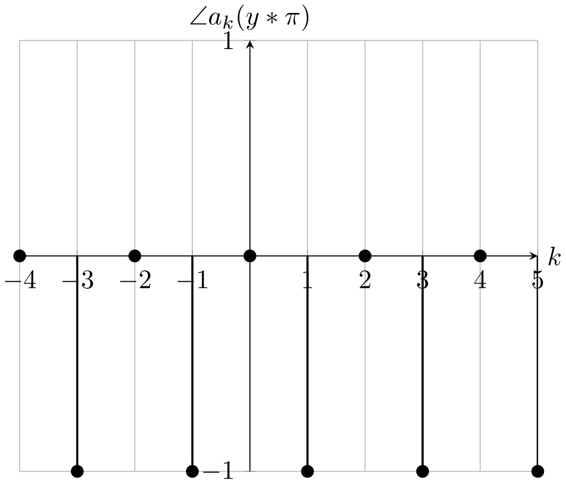
$a_1 = \frac{1}{4}(e^{-j\frac{\pi}{2}} + 2e^{-j\pi} + e^{-j\frac{3\pi}{2}})$
$a_1 = \frac{1}{4}(cos(\frac{-\pi}{2}) + j.sin(\frac{-\pi}{2}) + 2.cos(-\pi) + 2j.sin(-\pi) + cos(\frac{-3\pi}{2}) + j.sin(\frac{-3\pi}{2}))$
$a_1 = \frac{1}{4}(0 + j.(-1) + 2.(-1) + 2j.0 + 0 + j.1) = \frac{-1}{2}$

$a_2 = \frac{1}{4}(e^{-j\pi} + 2e^{-2j\pi} + e^{-3j\pi})$
$a_2 = \frac{1}{4}(cos(-\pi) + j.sin(-\pi) + 2.cos(-2\pi) + 2j.sin(-2\pi) + cos(-3\pi) + j.sin(-3\pi))$
$a_2 = \frac{1}{4}(-1 + j.0 + 2.1 + 2j.0 + -1 + j.0) = 0$

$a_3 = \frac{1}{4}(e^{-j\frac{3\pi}{2}} + 2e^{-j3\pi} + e^{-j\frac{9\pi}{2}})$
$a_3 = \frac{1}{4}(cos(\frac{-3\pi}{2}) + j.sin(\frac{-3\pi}{2}) + 2.cos(-3\pi) + 2j.sin(-3\pi) + cos(\frac{-9\pi}{2}) + j.sin(\frac{-9\pi}{2}))$
$a_3 = \frac{1}{4}(0 + j.(1) + 2.(-1) + 2j.0 + 0 + j.(-1)) = \frac{-1}{2}$

Below figures show magnitude and phase of a_k respectively





(b) i.
y[n] is a version of x[n], where different from x[n] it is 0 for $n = 3 + 4.m$ ($m \in (-\infty, \infty)$). Hence, it can stated as below:
$y[n] = (... + \delta[n+3] + \delta[n+2] + \delta[n-1] + \delta[n-2] + \delta[n-5] + \delta[n-6] + ...).x[n]$
$y[n] = x[n]\sum_{m=-\infty}^{\infty}(\delta[n-1+4m] + \delta[n-2+4m])$

ii.
Let b_k denote the Fourier Series coefficients of y[n]. Working over one period (4, same as x[n]) of y[n] where y[0]=0, y[1]=1, y[2]=2, y[3]=0:
$b_k = \frac{1}{4}(0.e^0 + e^{-jk\frac{\pi}{2}} + 2e^{-jk\pi} + 0.e^{-jk\frac{3\pi}{2}})$

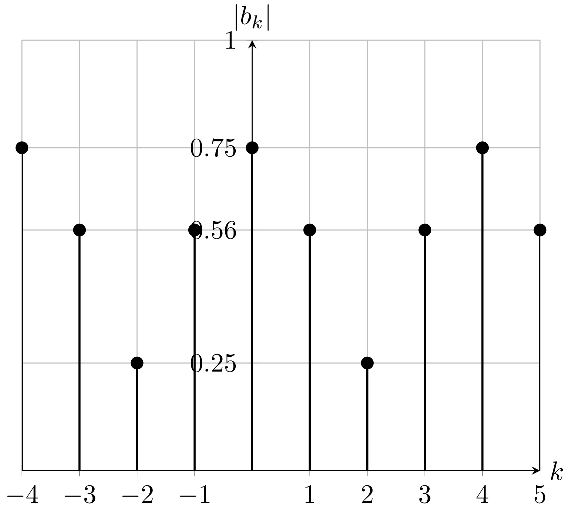
$b_0 = \frac{1}{4}(0 + 1 + 2 + 0) = \frac{3}{4}$
$|b_0| = \frac{3}{4}$, $\angle b_0 = 0$

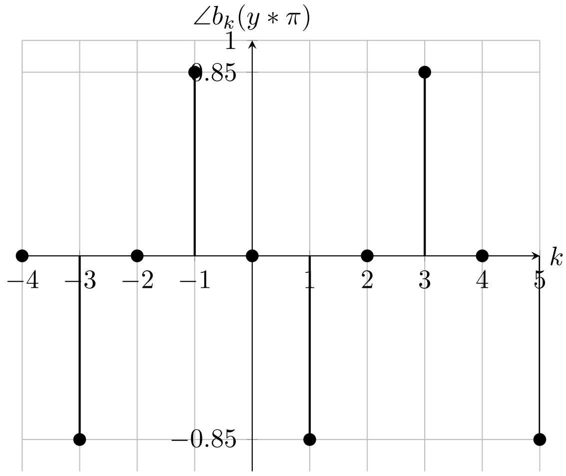
$b_1 = \frac{1}{4}(e^{-j\frac{\pi}{2}} + 2e^{-j\pi}$
$b_1 = \frac{1}{4}(cos(\frac{-\pi}{2}) + j.sin(\frac{-\pi}{2}) + 2.cos(-\pi) + 2j.sin(-\pi))$
$b_1 = \frac{1}{4}(0 + j.(-1) + 2.(-1)) = \frac{-j-2}{4}$
$|b_1| = \sqrt{(\frac{-1}{4})^2 + (\frac{-2}{4})^2} = \frac{\sqrt{5}}{4} = 0.5590$, $\angle b_1 = arctan(\frac{1}{2}) = -0.8524\pi$

$b_2 = \frac{1}{4}(e^{-j\pi} + 2e^{-2j\pi}$
$b_2 = \frac{1}{4}(cos(-\pi) + j.sin(-\pi) + 2.cos(-2\pi) + 2j.sin(-2\pi))$
$b_2 = \frac{1}{4}(-1 + j.0 + 2.1 + 2j.0)) = \frac{1}{4}$
$|b_2| = \frac{1}{4}$, $\angle b_2 = 0$

$b_3 = \frac{1}{4}(e^{-j\frac{3\pi}{2}} + 2e^{-j3\pi}$
$b_3 = \frac{1}{4}(cos(\frac{-3\pi}{2}) + j.sin(\frac{-3\pi}{2}) + 2.cos(-3\pi) + 2j.sin(-3\pi))$
$b_3 = \frac{1}{4}(0 + j.(1) + 2.(-1) + 2j.0) = \frac{j-2}{4}$
$|b_3| = \sqrt{(\frac{1}{4})^2 + (\frac{-2}{4})^2} = \frac{\sqrt{5}}{4} = 0.5590$, $\angle b_1 = arctan(\frac{-1}{2}) = 0.8524\pi$

Below figures show magnitude and phase of b_k respectively





2. In fact (a), we see that $N = 4$ for the signal. And in fact (b) there is a sum of the signal over two periods ($\frac{4-(-3)+1}{2} = 2$). So, a sum over one period of the signal is equal to $\frac{8}{2} = 4$. And using the fact that a_0 is equal to the mean of the signal over one period, we can conclude that $a_0 = \frac{4}{4} = 1$

   In fact (c), since Fourier coefficients of a periodic signal are also periodic, bu using the equalities ($a_{-3} = a_1, a_{11} = a_3$ and $a_{15} = a_3$), we can conclude that $a_1 = a_3^*$ and $|a_1 - a_3| = 1$

   In fact (e), by using the Discrete Time Fourier Series analysis equation, we can see that $a_1 = \frac{1}{4}\sum_{k=0}^{3} x[k]e^{-jk\pi/2}$ and $a_3 = \frac{1}{4}\sum_{k=0}^{3} x[k]e^{-jk3\pi/2}$ (Here $\omega_0 = \frac{1}{N} = \frac{1}{4}$). Hence, given summation in fact (e) is equal to $4a_1 + 4a_3 = 4$.
   So, $a_1 + a_3 = 1$

   By using the information about a_1 and a_3, we can conclude that there are two possible options for these coefficients to be:
   $\{a_1 = \frac{1}{2} + \frac{j}{2}, a_3 = \frac{1}{2} - \frac{j}{2}\}$ or $\{a_1 = \frac{1}{2} - \frac{j}{2}, a_3 = \frac{1}{2} + \frac{j}{2}\}$

   By using fact (d), we can conclude that the remaining coefficient which is a_2 is equal to 0

   Using $\{a_0 = 1, a_1 = \frac{1}{2} + \frac{j}{2}, a_2 = 0, a_3 = \frac{1}{2} - \frac{j}{2}\}$ and synthesis equation ($x[n] = \sum_{k=0}^{3} a_k e^{jk\pi n/2}$):
   $x[n] = a_0 + a_1 e^{j\pi n/2} + a_2 e^{j\pi n} + a_3 e^{j3\pi n/2}$

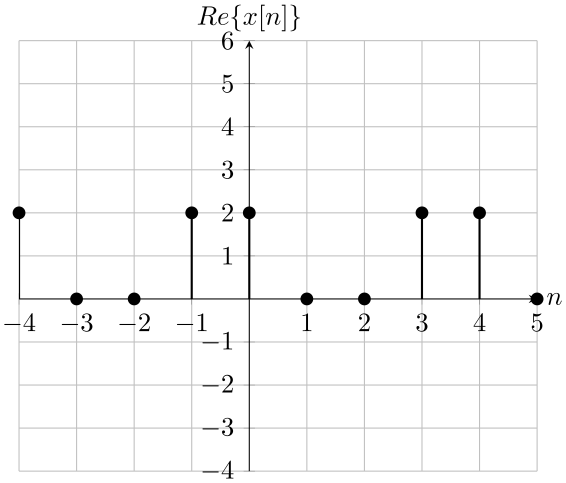
   x[0] $= 1 + \frac{1}{2}+\frac{j}{2} + 0 + \frac{1}{2}-\frac{j}{2} = 2$

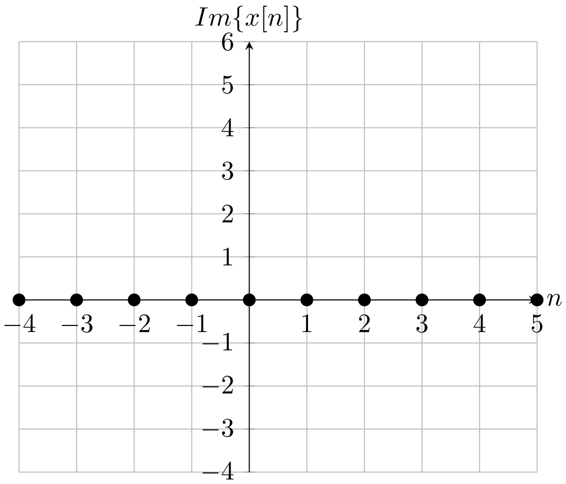
   x[1] $= 1 + (\frac{1}{2}+\frac{j}{2})[cos(\pi/2) + j.sin(\pi/2)] + (\frac{1}{2}-\frac{j}{2})[cos(3\pi/2) + j.sin(3\pi/2)]$
   x[1] $= 1 + (\frac{1}{2}+\frac{j}{2}).j + (\frac{1}{2}-\frac{j}{2}).-j = 0$

   x[2] $= 1 + (\frac{1}{2}+\frac{j}{2})[cos(\pi) + j.sin(\pi)] + (\frac{1}{2}-\frac{j}{2})[cos(3\pi) + j.sin(3\pi)]$
   x[2] $= 1 + (\frac{1}{2}+\frac{j}{2}).-1 + (\frac{1}{2}-\frac{j}{2}).-1 = 0$

x[3] = $1 + (\frac{1}{2}+\frac{j}{2})[cos(3\pi/2) + j.sin(3\pi/2)] + (\frac{1}{2}$-$\frac{j}{2})[cos(9\pi/2) + j.sin(9\pi/2)]$
x[3] = $1 + (\frac{1}{2}+\frac{j}{2}).-j + (\frac{1}{2}$-$\frac{j}{2}).j = 2$

Since x[n] is a periodic signal with period 4, we can plot it using above findings:





3. This can be accomplished by using frequency selective filters. Namely, we can use lowpass filter which enables low frequencies to pass and high frequencies not to pass. By using this approach, x(t) will pass the filter due to low frequency and r(t) will not pass the filter due to high frequency.

   In order for this lowpass filter to work, we need to determine our boundaries, i.e. cutoff frequency. Call it ω_c. Now, our filter passes the signals $e^{j\omega t}$ with $-\omega_c \leq \omega \leq \omega_c$ and reject all others which have higher frequencies. We can detect from the question text that $\omega_c = K2\pi/T$. Below is the frequency response of the filter:

   $$H(j\omega) = \begin{cases} 1, & |\omega| \leq \omega_c \\ 0, & \text{otherwise} \end{cases}$$

   In order to obtain impulse response from frequency response, we can use inverse transform relation as below:

   $h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(j\omega)e^{j\omega t}d\omega$
   $h(t) = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} 1.e^{j\omega t}d\omega$
   $h(t) = \frac{1}{2\pi}\frac{1}{jt}[e^{j\omega_c t} - e^{-j\omega_c t}]$
   $h(t) = \frac{1}{2\pi}\frac{1}{jt}[e^{j\omega_c t} - e^{-j\omega_c t}]$ where $\omega_c = K2\pi/T$
   $h(t) = \frac{1}{\pi t}sin(\omega_c t)$
   $h(t) = \frac{1}{\pi t}sin(\frac{2K\pi t}{T})$ is the desired impulse response of the system.

4. (a) First of all, we should find the DE. In order to solve it easily let's make a change of variables shown on the left hand side and right hand side of the question figure, which won't affect fourier transforms.
   Let y(t) be y'(t), and let x(t) be x'(t) on the question figure.
   So , if rewrite the equation,

   y"(t) = -5y'(t) + 4x'(t) - 6y(t) + x(t)

   That is,
   y"(t) + 5y'(t) + 6y(t) = 4x'(t) + x(t)

   By equation (4.76) in the textbook,

   H(jw) $= \frac{4(jw)+1}{(jw)^2+5(jw)+6}$

   (b) We found H(jw) in part (a). Thus, by inverse transform we can find the h(t) (impulse response).

   We can split $(jw)^2 + 5(jw) + 6$ as $(jw + 3)(jw + 2)$

   $\frac{4(jw)+1}{(jw)^2+5(jw)+6} = \frac{4(jw)+1}{(jw+3)(jw+2)}$

   Let's rewrite it :

   $\frac{A}{jw+3} + \frac{B}{jw+2}$

   By equality of polinomials,

   A = 11, B = -7.
   That is, H(jw) $= \frac{11}{jw+3} + \frac{-7}{jw+2}$

   The inverse transform is:

   h(t) $= 11e^{-3t}u(t) - 7e^{-2t}u(t)$.

   (c) We know Y(jw) = H(jw)X(jw)

   So, inserting given input into the equation as X(jw):

   $\frac{4(jw)+1}{(jw+3)(jw+2)}\frac{\frac{1}{4}}{jw+\frac{1}{4}}$

   Which results in : $\frac{1}{(jw+3)(jw+2)}$

   Again, by rewriting it as : $\frac{1}{(jw+3)(jw+2)} = \frac{A}{jw+3} + \frac{B}{jw+2}$

   By equality of polinomials,

A = -1, B = 1.

The inverse transform of equation above is :

y(t) = $-e^{-3t}u(t) + e^{-2t}u(t)$