COMMUTATIVE ALGEBRA AND ALGEBRAIC GEOMETRY SEMINAR

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Abstract. These are notes from the Berkeley Commutative Algebra and Algebraic Geometry Seminar.

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1. 8/27

I missed today's talks. If you have notes and would like to share them, please let me know!

2. 9/3

I missed today's talks. If you have notes and would like to share them, please let me know!

3. 9/17a (Peter Haine) – Reconstructing Schemes from their Étale Topoi

This is based on joint work with M. Carlson and S. Wolf centered around a conjecture in Grothendieck's anabelian letter to Faltings.

Date: Fall 2024.

3.1. **Grothendieck's conjecture.** For a scheme X, let $X_{\text{ét}}$ be the étale topos of X (i.e. $\mathsf{Sh}(\text{Ét}_X, \mathsf{Set})$). We would like to understand when X can be recovered from $X_{\text{ét}}$. One motivation is to try to understand the following theorem:

Theorem 3.1 (Neukirch-Uchida 1969, Pop 1994)**.** *If K and L are infinite fields of the same characteristic that are finitely generated over their prime fields, then there is a bijection between:*

- *Isomorphisms* $K \xrightarrow{\sim} L$
- *Equivalences of categories* $(\operatorname{Spec} L)_{\text{ét}} \xrightarrow{\sim} (\operatorname{Spec} K)_{\text{ét}}$ *up to conjugacy.*

Classical statements of this involve the absolute Galois groups, though the statement about étale topoi is equivalent. It is necessary to assume the fields are infinite: for finite fields, the absolute Galois groups are always $\hat{\mathbb{Z}}$.

We'll work over a field k for simplicity. There are four main issues that arise when trying to reconstruct a scheme from its étale topos:

(1) If $L \supset k$ is an extension of separably closed fields, then $(\operatorname{Spec} L)_{\text{ét}} \simeq (\operatorname{Spec} k)_{\text{ét}}$. Thus we must restrict to finite type schemes.
(2) If k is an algebraically closed field of characteristic zero and X and Y are smooth proper curves over k, then $X_{\text{ét}} \simeq Y_{\text{ét}}$ iff $g(X) = g(Y)$. Thus we must restrict to "small" fields k. Grothendieck suggests that we fix this by only considering k which are finitely generated over their prime field.
(3) If $f : X \to Y$ is a universal homeomorphism, then $X \times_Y (-) : \text{Ét}_Y \to \text{Ét}_X$ is an equivalence of categories. (Examples include the normalization of the cuspidal cubic, the absolute Frobenius are both universal, and any nil-immersion.) Thus we must invert universal homeomorphisms.
(4) A fourth subtle point involving constructibility.

Let's explain this fourth point. We must start with a small amount of topos theory.

Definition 3.2. Given topoi $\mathcal{X}$ and $\mathcal{Y}$, a *geometric morphism* is a functor $f_* : \mathcal{X} \to \mathcal{Y}$ with a right adjoint $f^* : \mathcal{Y} \to \mathcal{X}$ such that f^* preserves geometric morphisms.

Every topos $\mathcal{X}$ has an associated topological space $|\mathcal{X}|$, and $|-|$ is functorial in geometric morphisms. For a scheme X, we have $|X_{\text{ét}}| = |X|$. Furthermore, if T is a sober topological space, then $|\mathsf{Sh}(T)| \simeq T$.

Knowing this, we can now state our final condition. If $f : X \to Y$ is a morphism of schemes locally of finite type over a field, then f must send closed points to closed points. This is not true for general geometric morphisms, so we must require it as an extra condition.

Definition 3.3. A geometric morphism $f_* : \mathcal{X} \to \mathcal{Y}$ is *pinned* if $|f_*| : |\mathcal{X}| \to |\mathcal{Y}|$ sends closed points to closed points.

One last thing: in the Neukirch-Uchida theorem, we quotiented out by conjugation. We'd like to know that this doesn't really affect anything (i.e. that we shouldn't really be doing something "stacky"). Thankfully, this is true:

Proposition 3.4. *Let k be a field, and let X and Y be finite type k-schemes. Then the groupoid $\operatorname{Hom}_k^{\text{pin}}(X, Y)$ of pinned geometric morphisms $X_{\text{ét}} \to Y_{\text{ét}}$ over $(\operatorname{Spec} k)_{\text{ét}}$ is equivalent to a set.*

Now we can state Grothendieck's conjecture:

Conjecture 3.5 (Grothendieck 1983)**.** *If k is a finitely generated field, then taking étale topoi gives an equivalence between:*

- $\mathsf{Sch}_k^{\mathsf{ft}}[\mathsf{UH}^{-1}]$*, the category of finite type k-schemes with universal homeomorphisms inverted.*
- *The category of topoi over* $(\operatorname{Spec} k)_{\text{ét}}$ *and pinned geometric morphisms.*

Theorem 3.6 (CHW)**.** *The conjecture is true if k is infinite.*

At present, there is not a complete characterization of the image of $(-)_{\text{ét}}$.

3.2. **Inverting universal homeomorphisms.** We ran into a few issues with universal homeomorphisms. One was the existence of resolutions of cuspidal cubics. Another was the existence of the absolute Frobenius.

Definition 3.7. A ring A is:

(1) *seminormal* if, whenever $x^2 = y^3$, there exists $a \in A$ such that $x = a^2$ and $y = a^2$.

(2) *absolutely weakly normal* (or *awn*) if A is seminormal and, for all primes ℓ and equations $\ell^\ell x = y^\ell$, there exists $a \in A$ such that $x = a^\ell$ and $y = \ell a$.

These properties can be extended to general schemes (where we require that they hold affine-locally).

Theorem 3.8 (Traverso, Swan)**.** *A ring A is seminormal if and only if the inclusion $A \hookrightarrow A[t]$ induces an isomorphism $\mathsf{Pic} A \xrightarrow{\sim} \mathsf{Pic} A[t]$.*

Theorem 3.9. *(1) The inclusion $\mathsf{Sch}^{\mathrm{awn}} \hookrightarrow \mathsf{Sch}$ admits a right adjoint $(-)^{\mathrm{awn}}$ (known as* absolute weak normalization*). Moreover, $(-)^{\mathrm{awn}}$ induces $\mathsf{Sch}[\mathsf{UH}^{-1}] \xrightarrow{\sim} \mathsf{Sch}^{\mathrm{awn}}$.*

(2) The inclusion $\mathsf{Sch}^{\mathrm{sn}} \hookrightarrow \mathsf{Sch}$ admits a right adjoint $(-)^{\mathrm{sn}}$ (known as seminormalization*). This identifies $\mathsf{Sch}^{\mathrm{sn}}$ with the category obtained from Sch by inverting universal homeomorphisms that induce isomorphisms on residue fields.*

(3) A $\mathbb{Q}$-scheme is awn if and only if it is seminormal.

(4) A $\mathbb{F}_p$-scheme is awn if and only if it is perfect.

In general, seminormalization / absolute weak normalization leaves the land of finite type schemes. We must allow for this!

Definition 3.10. A k-scheme X is topologically of finite type if $X \to \operatorname{Spec} k$ factors as $X \to X' \to \operatorname{Spec} k$ where $X \to X'$ is a universal homeomorphism and $X' \to \operatorname{Spec} k$ is of finite type.

Grothendieck's conjecture is then equivalent to

Conjecture 3.11. *For X and Y topologically of finite type over k with X awn, then*

$$\mathrm{Hom}_k(X, Y) \simeq \mathrm{Hom}_k^{\mathrm{pin}}(X_{\text{ét}}, Y_{\text{ét}}).$$

How do we prove this? Since we can reconstruct the underlying space from the étale topos, it suffices to reconstruct the structure sheaf. This can be rephrased as constructing the sets of morphisms to $\mathbb{A}^1$. Using the fact that both sides satisfy h-descent together with the theory of alterations, we reduce to the case where $X = (X')^{\mathrm{awn}}$ for X' regular and finite type. From here, we can reduce to the case where $Y = \mathbb{G}_m$. This lets us understand things cohomologically! We end up asking questions about Picard groups.

Theorem 3.12 (Guralnik-Jaffe-Roskind-Wiegland)**.** *If k is a finitely generated field and X is normal and of finite type over k, then $\mathsf{Pic} X$ is finitely generated.*

This is not true for seminormal schemes in general.

Example 3.13. If $X \subset \mathbb{P}^2_k$ is a nodal cubic, then $\mathsf{Pic} X \simeq k^\times \oplus \mathbb{Z}$.

Question 3.14. *For X a seminormal scheme of finite type over a finitely generated field of characteristic zero:*

(1) Does $\mathsf{Pic}(X)$ have any nontrivial infinitely divisible elements?

(2) Same as (1) but for torsion elements.

(3) Is the Tate module $\mathsf{T}(\mathsf{Pic}(X)) = \lim_n \mathsf{Pic}(X)[n]$ zero?

4. 9/17b (Hannah Larson) – Chow Rings of Moduli Spaces

4.1. Chow groups and Chow rings.

Let X be a smooth variety.

Definition 4.1. The *Chow group* in codimension i of X is the group of $\mathbb{Z}$-linear combinations of codimension i irreducible subvarieties of X modulo rational equivalence. Here *rational equivalence* is the equivalence relation generated by $Y_1 \simeq Y_2$ if there exists a family $Y \subset X \times \mathbb{P}^1$, flat over $\mathbb{P}^1$, such that Y_1 and Y_2 are fibers of this family over two points in $\mathbb{P}^1$. We write $[Y]$ for the equivalence class of the subvariety Y: this can be defined even for reducible subvarieties by $[Y \cup Z] = [Y] + [Z]$.

Definition 4.2. The *Chow ring* $A^*(X) = \oplus_i A^i(X)$ comes with an intersection product

$$A^i(X) \times A^j(X) \to A^{i+j}(X)$$

defined so that, if X and Y meet transversally, then $[Y] \cdot [Z] = [Y \cap Z]$.

4.2. **Line bundles and $A^*(\mathbb{P}^n)$.** Let $\mathcal{L}$ be a line bundle on X, and let σ be a rational section of $\mathcal{L}$.

Definition 4.3. Given σ, we can define the *divisor of zeroes* D_0 and the *divisor of poles* D_∞. The difference $c_1(\mathcal{L}) := [D_0] - [D_\infty] \in A^1(X)$ does not depend on the choice of σ. We call this the *first Chern class* of $\mathcal{L}$.

Example 4.4. Let $X = \mathbb{P}^2$ and $\mathcal{L} = \mathcal{O}_{\mathbb{P}^2}(1)$. Taking $\sigma = x_0$, we obtain $D_0 = V(x_0)$ and $D_\infty = \varnothing$, so $c_1(\mathcal{L}) = [V(x_0)]$. An alternate choice would be to take $\sigma' = x_1$, which would give $[V(x_1)]$. Well-definedness of $c_1(\mathcal{L})$ forces $[V(x_0)] = [V(x_1)]$.

We can see this directly by considering the graph of the rational map $X \to \mathbb{P}^1$, $[x_0 : x_1 : x_2] \mapsto [x_0 : x_1]$. Taking the closure of the graph gives a rational equivalence between $V(x_0)$ and $V(x_1)$. Note that this graph closure is nothing but the blowup of $\mathbb{P}^1$ at $[1 : 0 : 0]$ – this observation is more generally.

The class $\zeta = c_1(\mathcal{O}(1)) \in A^1(\mathbb{P}^n)$ is the first example of a "tautological class." In fact, $A^*(\mathbb{P}^n) \cong \mathbb{Z}[\zeta]/(\zeta^{n+1})$, and ζ^i is the class of a codimension i hyperplane. In this case, the "obvious" classes generate the Chow ring, and the "obvious" relations are all of the relations.

We can obtain Bézout's theorem from this result rather easily.

Theorem 4.5 (Bézout)**.** *If C and C' are curves in $\mathbb{P}^2$ of degrees d and d' which meet transversally, then $[C] \cdot [C'] = dd'[\mathrm{pt}]$.*

4.3. **More properties of Chow rings.** Chow rings can be defined in the non-smooth case.

Theorem 4.6 (Excision)**.** *If $Z \subset X$ is a closed subvariety of codimension c and $U = X \setminus Z$, then there is an exact sequence*

$$A^{*-c}(Z) \longrightarrow A^*(X) \longrightarrow A^*(U) \longrightarrow 0.$$

We can also pushforward cycles along proper maps. If $f : X \to Y$ is proper, then $f_* : A^i(X) \to A^{i+\dim Y - \dim X}(Y)$ is defined by

$$f_*[Z] = \begin{cases} \deg f|_Z \cdot [f(Z)] & f|_Z \text{ finite} \\ 0 & \text{otherwise} \end{cases}$$

4.4. **Moduli spaces of curves.** How can we extend things to moduli spaces of curves? Let $\mathcal{M}$ be the moduli space of curves, and let $f : \mathcal{C} \to \mathcal{M}$ be the universal curve. Consider the relative dualizing sheaf ω_f (so $\omega_f|_p = (T_p C)^\vee$). We obtain $c_1(\omega_f) \in A^1(\mathcal{C})$ and $f_* c_1(\omega_f) \in A^0(\mathcal{M}_g)$. This gives a "tautological class" on $\mathcal{M}$.

In fact, we can work a bit more generally: let

$$\kappa_i := f_* c_1(\omega_f)^{i+1} \in A^i(\mathcal{M}_g).$$

These are all "tautological classes."

Definition 4.7. The subring of $A^*(\mathcal{M}_g)$ generated by the κ_i is the *tautological ring*. We write $R^*(\mathcal{M}_g)$ for this ring.

From now on, we will work rationally, replacing A^* by $A^* \otimes \mathbb{Q}$.

Important questions:

- Does $R^*(\mathcal{M}_g) = A^*(\mathcal{M}_g)$?
- What are the relations among the κ_i? What is $R^*(\mathcal{M}_g)$?

It is known (by work of several authors) that $R^*(\mathcal{M}_g) = A^*(\mathcal{M}_g)$ for $g \leqslant 9$. However, it is known that $R^*(\mathcal{M}_g) \neq A^*(\mathcal{M}_g)$ for $g = 12$ or $g \geqslant 16$. A heuristic explanation for this is that the lower genus moduli spaces are "more rational" than the higher genus moduli spaces.

The non-tautological class in $\mathcal{M}_{12}$ is the bielliptic locus $[B_g]$, i.e. the locus of curves which admit a degree 2 map to an elliptic curve. One can show that $[\overline{B}_g]$ is not tautological in $A^*(\overline{\mathcal{M}}_g)$ for $g \geqslant 12$. The non-tautological classes in $\mathcal{M}_{16}$ are constructed as similar "Hurwitz loci."

It is not known whether $A^*(\mathcal{M}_g)$ is finitely generated in general. The speaker expects the answer is "no" in high genus.

5. 9/24 (Christian Gaetz) – Combinatorics of Singularities of Schubert Varieties and Torus Orbit Closures Therein

Let G be a semisimple connected algebraic group over $\mathbb{C}$. Most of the setup will work for arbitrary type, but the results will be for groups with simply-laced / ADE Dynkin diagram.

5.1. Definitions and basic theory.

Let B be a Borel subgroup of G, i.e. a maximal connected closed solvable subgroup of G. We write $\mathsf{T} \subset \mathsf{B}$ for the maximal torus and $W = \mathsf{N}_\mathsf{G}(\mathsf{T})/\mathsf{T}$ for the Weyl group. For the matrix groups, we may take B to consist of the upper triangular matrices in G and T to consist of the diagonal matrices in G.

The Weyl group indexes the *Bruhat decomposition* $\mathsf{G} = \sqcup_{w \in W} \mathsf{B}w\mathsf{B}$. In the flag variety G/B, the subspaces $\mathsf{B}w\mathsf{B}/\mathsf{B} \cong \mathbb{C}^{\ell(w)}$ are the *Schubert cells*. The closures $\mathsf{X}_w = \overline{\mathsf{B}w\mathsf{B}/\mathsf{B}}$ are *Schubert varieties*.

These are classical, hard, and very useful. For example, they can be used to understand the cohomology of flag varieties: this is the subject of *Schubert calculus*. We can also relate these to problems in Grassmannians by projecting from flag varieties to Grassmannians. Schubert calculus in Grassmannians is well-understood (via e.g. Littlewood-Richardson rules), but Schubert calculus in Grassmannians is still an active area of research. We won't focus on Schubert calculus in this lecture.

Note that $\mathsf{X}_w = \sqcup_{v \leqslant w} \mathsf{B}w\mathsf{B}/\mathsf{B}$, where $\leqslant$ denotes the Bruhat order.

Example 5.1. For $\mathsf{G} = \mathrm{SL}_n$, we have $W \cong \mathsf{S}_n$. We can explicitly describe the Bruhat order on S_3: writing permutations in one-line notation, we say that $v \leqslant w$ if the one-line notation for v has more numbers located in the usual order than w.

5.2. Kazhdan-Lusztig polynomials.

The *Kazhdan-Lusztig polynomials* are $\mathsf{P}_{vw}(q) \in \mathbb{N}[q]$ which admit the following descriptions:

- Generating functions of intersection cohomology: $\mathsf{P}_{vw}(q) = \sum_i q^i \dim \mathrm{IH}_v^{2i}(\mathsf{X}_w)$ (where IH_v measures singularities at v). In the simply laced case, $\mathsf{P}_{vw}(q) = 1$ if X_w is smooth at v. In the non-simply laced case, "smooth" is replaced by "rationally smooth."
- Generating functions of Lie algebra Ext groups: $\sum_i q^i \operatorname{Ext}_{\mathfrak{g}}^{\ell(w)-\ell(v)-i}(\mathsf{M}_v, \mathsf{L}_w)$ (where M_v is a Verma module and L_w is an irrep).

Proving the equivalence between these started the field of modern geometric representation theory. The equivalence is a deep theorem.

We have $\deg \mathsf{P}_{vw} \leqslant \frac{1}{2}(\ell(w) - \ell(v) - 1)$.

The Kazhdan-Lusztig polynomials can be computed via a complicated recurrence relation. This recurrence relation contains many signs, and we don't know how to prove the nonnegativity of the polynomials from this recurrence relation.

It's impossible to control the behavior of Kazhdan-Lusztig polynomials in general (cf. a theorem of Polo ?? saying that every polynomial with constant term 1 and $\mathbb{N}$ coefficients arises as a Kazhdan-Lusztig polynomial). As a result, we'll restrict ourselves to special cases.

5.3. A conjecture of Billey and Postnikov.

Conjecture 5.2 (Billey-Postnikov). *Suppose X_w is singular, and write $\mathsf{P}_{ew}(q) = 1 + cq^{h(w)} + \dots$. If G is simply laced of rank r, then $h(w) \leqslant r - 2$.*[1]

This conjecture is somewhat surprising, as in general we only have $\deg \mathsf{P}_{ew} \leqslant O(r^2)$. The conjecture forces the first nonzero term that appears to have rank growing linearly in $\operatorname{rk} \mathsf{G}$.

Theorem 5.3 (Björner-Ekedahl). *With the above notation, $h(w)$ is the minimal i such that $h^{2i}(\mathsf{X}_w) \neq h^{2(\ell(w)-i)}(\mathsf{X}_w)$, i.e. the first dimension in which Poincaré duality fails.*

We can understand this using "patterns."

Definition 5.4. Say a permutation $w \in \mathsf{S}_n$ contains $\sigma \in \mathsf{S}_k$ as a *pattern* if there exist $1 \leqslant i_1 < \cdots < i_k < n$ such that $w(i_a) < w(i_b)$ if and only if $\sigma(a) < \sigma(b)$. Otherwise we say w *avoids* σ.

Example 5.5. The permutation $w = 45312 \in \mathsf{S}_5$ contains $\sigma = 3412 \in \mathsf{S}_4$ as a pattern (via the subpermutation 4512).

[1] The original statement is a bit weaker.

Theorem 5.6 (Lakshmibai-Sandhya)**.** *The Schubert variety* X_w *is smooth if and only if* w *avoids* 3412 *and* 4231*.*

Avoiding patterns is hard to do in high rank! In particular, X_w is almost always singular.

5.4. The theorem.

Theorem 5.7 (Gaetz-Gao)**.** *Suppose* X_w *is singular. Then*

$$h(w) = \begin{cases} 1 & w \textit{ contains } 4231 \\ \textit{minHeight}(w) & \textit{otherwise} \end{cases}$$

where

$$\text{minHeight}(w) = \min\{w(i_1) - w(i_4) \,|\, w \textit{ has } 3412 \textit{ in positions } i_1, i_2, i_3, i_4\}.$$

For any w containing 3412, we have $\text{minHeight}(w) \leqslant n-3 = (n-1)-2$. The Billey-Postnikov conjecture follows from this.

The proof is easier in the case where w contains 4231. Otherwise, w avoids 4231, and one considers projections $G/B \to G/P_J$. The images of X_w give Schubert varieties $X^J_{W^J} \subset G/P_j$. The projection maps $\pi : X_w \to X^J_{W^J}$ can be rather nasty, but by studying w carefully, one finds J such that π is a fiber bundle with fiber X_{W_J} (i.e. a smaller Schubert variety!). This allows one to understand Schubert varieties inductively.

6. 10/1a (Daigo Ito) – A New Proof of the Bondal-Orlov Reconstruction Theorem

This is based on joint work with Hiroki Matsui.

6.1. **Background.** Let X be a projective variety over $\mathbb{C}$ (though this should mostly work over any algebraically closed field). We can construct a triangulated category of perfect complexes $\mathsf{Perf}\, X$ consisting of bounded complexes of vector bundles (after inverting quasi-isomorphisms).

Question 6.1. *How much information about* X *is contained in* $\mathsf{Perf}\, X$*? In particular, in which cases can we reconstruct* X *fully from* $\mathsf{Perf}\, X$*?*

Theorem 6.2 (Bondal-Orlov, Ballard)**.** *Let* X *be Gorenstein, so the dualizing complex* ω_X *is a line bundle. If* ω_X *is ample or anti-ample, then:*

1. *We can reconstruct* X *from the triangulated category* $\mathsf{Perf}\, X$.
2. *If* $\mathsf{Perf}\, X \simeq \mathsf{Perf}\, Y$ *for* Y *projective and Gorenstein, then* $X \cong Y$.[2]

The Fano / anti-Fano assumptions here are rather strong. Another method of reconstructing X is as follows.

6.2. Balmer spectra.

Theorem 6.3 (Balmer)**.** *Let* X *be a variety (more generally, a qcqs scheme). Then we can reconstruct* X *from* $(\mathsf{Perf}\, X, \otimes^{\mathsf{L}}_{\mathcal{O}_X})$.

More precisely, for any tensor-triangulated (tt) category $(\mathcal{T}, \otimes)$*, we can construct a ringed space* $\operatorname{Spec}_\otimes \mathcal{T}$*. Taking* $(\mathcal{T}, \otimes) \cong (\mathsf{Perf}\, X, \otimes_X)$*, we obtain* $\operatorname{Spec}_\otimes \mathcal{T} \cong X$.

It is natural to ask if we can perform Balmer's construction without having the tensor structure when ω_X is anti-ample.

Definition 6.4 (Balmer)**.** Let $(\mathcal{T}, \otimes)$ be a tt-category. The *Balmer spectrum* $\operatorname{Spec}_\otimes \mathcal{T}$ is defined as follows. As a set, $\operatorname{Spec}_\otimes \mathcal{T}$ consists of *prime thick* $\otimes$*-ideals* $\mathcal{P} \subset \mathcal{T}$, meaning that:

1. (Thick subcategory) $\mathcal{P}$ is a full triangulated subcategory of $\mathcal{T}$, such that if $X \oplus Y \in \mathcal{P}$, then $X \in \mathcal{P}$ or $Y \in \mathcal{P}$.
2. ($\otimes$-ideal) If $X \in \mathcal{P}$ and $Y \in \mathcal{T}$, then $X \otimes Y \in \mathcal{P}$.
3. (Prime) $\mathcal{P}$ is a proper $\otimes$-ideal such that, if $X \otimes Y \in \mathcal{P}$, then $X \in \mathcal{P}$ or $Y \in \mathcal{P}$.

We can equip this with a natural ringed space structure.

Example 6.5. If X is a qcqs scheme, then $X \cong \operatorname{Spec}_{\otimes_{\mathcal{O}_X}} \mathsf{Perf}\, X$, where a (not necessarily closed) point $x \in X$ corresponds to the prime

$$S_X(x) = \left\{ \mathcal{F} \in \mathsf{Perf}\, X \middle| \mathcal{F}_x \simeq 0 \text{ in } \mathsf{Perf}(\mathcal{O}_{X,x}) \right\}$$

[2]This is needed to ensure that our reconstruction gives the correct result! Note that we do not assume Y is Fano / anti-Fano.

6.3. **Matsui spectra.**

Definition 6.6 (Matsui)**.** Let $\mathcal{T}$ be a triangulated category. The *Matsui spectrum* $\operatorname{Spec}_\Delta \mathcal{T}$ is defined as the set of prime thick subcategories $\mathcal{P}$, meaning that:

(1) $\mathcal{P}$ is a thick subcategory of $\mathcal{T}$.
(2) The collection $\{\mathcal{Q} \supsetneq \mathcal{P} \mid \mathcal{Q} \text{ thick}\}$ has a unique smallest element.

Example 6.7. If X is a curve and $x \in X$ is a closed point, then $S_X(x) = \langle \kappa(y) \mid y \neq x \rangle$ is prime. The smallest thick subcategory containing $S_X(x)$ is obtained by adjoining $\kappa(x)$ to $S_X(x)$.

Proposition 6.8 (Matsui)**.** *If X is a noetherian scheme, then $\operatorname{Spec}_{\otimes_{\mathcal{O}_X}} \mathsf{Perf}\, X \subset \operatorname{Spec}_\Delta \mathsf{Perf}\, X$ as sets.*

Remark 6.9. It is not known yet whether this extends to arbitrary qcqs schemes.

Theorem 6.10 (Ito-Matsui)**.** *If X is a quasiprojective scheme, then $(\operatorname{Spec}_{\otimes_{\mathcal{O}_X}} X)_{\mathrm{red}} \subset \operatorname{Spec}_\Delta \mathsf{Perf}\, X$ is an open immersion of ringed spaces.*

Remark 6.11. It is not known yet whether we can attach a natural structure sheaf to $\operatorname{Spec}_\Delta \mathsf{Perf}\, X$ that "sees the nilpotents of X."

6.4. **Proof of Bondal-Orlov.** Assume ω_X is (anti)ample. By the theorem on open immersions above, it suffices to specify the correct open subspace of the Matsui spectrum.

Let $\mathbb{S}(-) = - \otimes \omega_X[\dim X]$ be the Serre functor of $\mathsf{Perf}\, X$. This gives an automorphism of $\mathsf{Perf}\, X$.

Lemma 6.12. *The Serre functor can be uniquely determined by the triangulated category structure of $\mathsf{Perf}\, X$.*

With this lemma, we can define the *Serre-invariant locus* $\operatorname{Spec}^{\mathrm{Ser}} \mathsf{Perf}\, X \subset \operatorname{Spec}_\Delta \mathsf{Perf}\, X$ as

$$\operatorname{Spec}^{\mathrm{Ser}} \mathsf{Perf}\, X = \left\{ \mathcal{P} \in \operatorname{Spec}_\Delta \mathsf{Perf}\, X \,\middle|\, \mathbb{S}(\mathcal{P}) = \mathcal{P} \right\}$$

We claim $\operatorname{Spec}_{\otimes_{\mathcal{O}_X}} \mathsf{Perf}\, X = \operatorname{Spec}^{\mathrm{Ser}} \mathsf{Perf}\, X$.

Example 6.13. For $X = \mathbb{P}^1$, we have

$$\operatorname{Spec}_\Delta \mathsf{Perf}\, \mathbb{P}^1 = \operatorname{Spec}_{\otimes_{\mathcal{O}_{\mathbb{P}^1}}} \sqcup \coprod_{i \in \mathbb{Z}} \langle \mathcal{O}_{\mathbb{P}^1}(i) \rangle,$$

and $\mathbb{S}$ fixes $\operatorname{Spec}_{\otimes_{\mathcal{O}_{\mathbb{P}^1}}}$ while acting freely on $\coprod_{i \in \mathbb{Z}} \langle \mathcal{O}_{\mathbb{P}^1}(i) \rangle$. Thus the claim holds in this case.

In general, it is clear that $\operatorname{Spec}^{\mathrm{Ser}} \mathsf{Perf}\, X \subset \operatorname{Spec}_{\otimes_{\mathcal{O}_X}} \mathsf{Perf}\, X$. For the reverse inclusion, suppose $\mathbb{S}(\mathcal{P}) = \mathcal{P}$. Then, for all $n \in \mathbb{Z}$, we have $\mathcal{P} \otimes \omega_X^{\otimes n} = \mathcal{P}$.

Theorem 6.14 (Orlov)**.** *If ω_X is (anti)ample, then $\langle \omega_X^{\otimes n} \mid n \in \mathbb{Z} \rangle = \mathsf{Perf}\, X$.*

It follows that, for all $\mathcal{F} \in \mathsf{Perf}\, X$, we have $\mathcal{F} \otimes \mathcal{P} \subset \mathcal{P}$, i.e. $\mathcal{P}$ is a $\otimes$-ideal.

Theorem 6.15 (Matsui)**.** *A $\otimes$-ideal is a prime $\otimes$-ideal if and only if it is a prime thick subcategory.*

Thus $\mathcal{P}$ is a prime $\otimes_{\mathcal{O}_X}$-ideal of $\mathsf{Perf}\, X$, i.e. a point of $\operatorname{Spec}_{\otimes_X} \mathsf{Perf}\, X$. This concludes the reconstruction of X from $\mathsf{Perf}\, X$.

To see that $\mathsf{Perf}\, X \simeq \mathsf{Perf}\, Y$ implies $X \simeq Y$, note that, because the Serre functor commutes with any equivalence, we must have

$$X \cong \operatorname{Spec}^{\mathrm{Ser}} \mathsf{Perf}\, X \cong \operatorname{Spec}^{\mathrm{Ser}} \mathsf{Perf}\, Y.$$

We know that Y embeds as an open subspace of $\operatorname{Spec}^{\mathrm{Ser}} \mathsf{Perf}\, Y$. Thus Y is a closed (because proper) and open subspace of X. But X is connected, so $X \cong Y$.

6.5. **More applications.** Using these methods, we can also recover a theorem of Favero.

Theorem 6.16 (Favero)**.** *Let X and Y be projective varieties. Suppose there is an equivalence $\Phi : \mathsf{Perf}\, X \xrightarrow{\sim} \mathsf{Perf}\, Y$, an (anti)ample line bundle $\mathcal{A}$ on X, and a line bundle $\mathcal{L}$ on Y, such that $\Phi(- \otimes \mathcal{A}) \simeq \Phi(-) \otimes \mathcal{L}$. Then $X \cong Y$.*

Proof. By the above, we have $X \subset (\operatorname{Spec}_\Delta \mathsf{Perf}\, X)^{-\otimes \mathcal{A}} = (\operatorname{Spec}_\Delta \mathsf{Perf}\, Y)^{-\otimes \mathcal{L}}$. The same argument as above gives $X \simeq Y$. $\square$

The above proof is much shorter than Favero's.

Remark 6.17. We do not yet know how to reconstruct the tensor product $\otimes_{\mathcal{O}_X}$ directly from the Serre functor.

7. 10/1b (Noah Olander) – Fully Faithful Functors and Dimension

We'll write X and Y for smooth projective varieties over some field k. We write $\mathsf{D}^{\mathsf{b}}(\mathsf{X}) := \mathsf{D}^{\mathsf{b}}(\mathsf{Coh}(\mathsf{X})) \simeq \mathsf{Perf}\,\mathsf{X}$, viewed as a k-linear triangulated category. All functors considered will be exact and k-linear.

7.1. The main theorem.

Our goal is to prove the following:

Theorem 7.1 (Theorem 1)**.** *If there exists a fully faithful functor* $\mathsf{F} : \mathsf{D}^{\mathsf{b}}(\mathsf{X}) \to \mathsf{D}^{\mathsf{b}}(\mathsf{Y})$*, then* $\dim \mathsf{X} \leqslant \dim \mathsf{Y}$.

Let's give some examples of fully faithful functors to explain what we mean:

(1) If $\mathsf{f} : \mathsf{Y} \to \mathsf{X}$ satisfies $\mathsf{Rf}_*\mathcal{O}_{\mathsf{Y}} = \mathcal{O}_{\mathsf{X}}$, then $\mathsf{Lf}^* : \mathsf{D}^{\mathsf{b}}(\mathsf{Y}) \to \mathsf{D}^{\mathsf{b}}(\mathsf{X})$ is fully faithful. This applies for Y a projective bundle over X, a blowup of X with smooth center, and many other cases. In this case, f must be surjective, proving the theorem in this case.
(2) (Kuznetsov) There exist fully faithful functors from derived categories of K3 surfaces to derived categories of cubic fourfolds.

In these and other examples, the theorem is obvious. But the result in general wasn't known before!

We will prove the theorem by proving a weaker version of Orlov's conjecture, stated as follows. Let Rdim denote the Rouquier dimension of a triangulated category, to be defined later.

Conjecture 7.2 (Orlov)**.** *For a smooth projective variety* X*, we have* $\operatorname{Rdim} \mathsf{D}^{\mathsf{b}}(\mathsf{X}) = \dim \mathsf{X}$.

In this generality, the conjecture implies Theorem 1.

7.2. Rouquier dimension.

Let $\mathcal{T}$ be a triangulated category and $\mathsf{S} \subset \mathcal{T}$. Let $\langle \mathsf{S} \rangle_{\mathsf{d}+1} \subset \mathcal{T}$ consist of objects in $\mathcal{T}$ built by taking direct sums, shifts, passing to direct summands, and *taking at most* d *cones.*

Definition 7.3. The *Rouquier dimension* $\operatorname{Rdim} \mathcal{T}$ is the smallest integer d such that there exists a single object $\mathsf{G} \in \mathcal{T}$ with $\mathcal{T} = \langle \mathsf{G} \rangle_{\mathsf{d}+1}$.

We will make use of a variant notion.

Definition 7.4. The *countable Rouquier dimension* $\operatorname{CRdim} \mathcal{T}$ is the smallest integer d such that there exists a countable subset $\mathsf{S} \subset \operatorname{ob} \mathcal{T}$ with $\langle \mathsf{S} \rangle_{\mathsf{d}+1}$.

Example 7.5. Let R be a Dedekind domain and $\mathcal{T}$ the bounded derived category of finitely generated R-modules. Then $\operatorname{Rdim} \mathcal{T} = 1$, with $\mathcal{T} = \langle \mathsf{R} \rangle_2$. To see this, let $\mathsf{K} \in \mathcal{T}$. Then we may write $\mathsf{K} = \oplus_{\mathsf{i} \in \mathbb{Z}} \mathsf{H}^{\mathsf{i}}(\mathsf{K})[-\mathsf{i}]$, so WLOG K is a finitely generated R-module concentrated in degree zero. Then K admits a resolution

$$0 \longrightarrow \mathsf{P} \longrightarrow \mathsf{R}^{\oplus \mathsf{d}} \longrightarrow \mathsf{K} \longrightarrow 0$$

where P is projective. As P and $\mathsf{R}^{\oplus \mathsf{d}}$ both lie in $\langle \mathsf{K} \rangle_1$ (since P can be obtained as a direct summand of a free module), we have $\mathsf{K} \in \langle \mathsf{R} \rangle_2$. It follows that $\operatorname{Rdim} \mathcal{T} \leqslant 1$. The converse is left as an exercise.

Note that if R is countable, then $\operatorname{CRdim} \mathcal{T} = 0$, so CRdim and Rdim don't agree.

Example 7.6. Let X be a smooth projective curve of genus $\mathsf{g} \geqslant 1$. Let $\mathcal{L}$ be a line bundle on X of degree $\geqslant 8\mathsf{g}$. Orlov showed $\mathsf{D}^{\mathsf{b}}(\mathsf{X}) = \langle \mathcal{L}^{-1} \oplus \mathcal{O} \oplus \mathcal{L} \oplus \mathcal{L}^{\otimes 2} \rangle_2$. Thus $\operatorname{Rdim} \mathsf{X} \leqslant 1$, and equality holds by the following.

Lemma 7.7 (Rouquier)**.** *We have* $\operatorname{Rdim} \mathsf{D}^{\mathsf{b}}(\mathsf{X}) \geqslant \dim \mathsf{X}$.

The same argument also shows $\operatorname{CRdim} \mathsf{D}^{\mathsf{b}}(\mathsf{X}) \geqslant \dim \mathsf{X}$ if the ground field k is uncountable.

Lemma 7.8. *Let* $\mathsf{F} : \mathcal{T} \to \mathcal{T}'$ *be a fully faithful exact functor with an exact right adjoint. Then* $\operatorname{Rdim} \mathcal{T} \leqslant \operatorname{Rdim} \mathcal{T}'$.

Proof. If R is the right adjoint, then $\mathsf{R} \circ \mathsf{F} = \mathrm{id}$, so R is essentially surjective. If G is an optimal generator for $\mathcal{T}$, then $\mathsf{R}(\mathsf{G})$ generates $\operatorname{Rdim} \mathcal{T}$ in (at most) the same number of steps. □

This lemma (combined with a result of Bondal and van den Bergh on existence of adjoints) implies that Theorem 1 holds if Orlov's conjecture holds. The lemma also holds with Rdim replaced by CRdim.

7.3. **A weaker version of Orlov's conjecture.** Theorem 1 also follows from:[3]

Theorem 7.9 (Theorem 2). *Let the ground field* k *be uncountable. Then* $\mathrm{CRdim}\, D^b(X) \leqslant \dim X$*. More precisely,* $D^b(X) = \langle\{\mathcal{O}_X(i)\}_{i\in\mathbb{Z}}\rangle_{\dim X+1}$.

Theorem 7.10 (Theorem 3). *Given maps* $K_0 \to K_1 \to \cdots \to K_{\dim X+1}$ *in* $D^b(X)$ *such that* $H^n(K_i) \to H^n(K_{i+1})$ *is zero for all* i *and* n*, the composite map* $K_0 \to K_{\dim X+1}$ *is zero.*

Maps $K_i \to K_{i+1}$ which are nonzero despite all maps $H^n(K_i) \to H^n(K_{i+1})$ being zero are called "ghosts" in topology. They are relatively easy to find.

Example 7.11. Let $\xi \in H^1(\mathbb{P}, \mathcal{O}(-2))$ be nonzero. Then ξ gives a nonzero element of $\mathrm{Hom}(\mathcal{O}, \mathcal{O}(-2)[1])$, despite the fact that $\mathcal{O}$ and $\mathcal{O}(-2)[1]$ are concentrated in different cohomological degrees.

Proof of Theorem 3. Consider the spectral sequence $E_1^{p,q} = \prod_n \mathrm{Ext}^{2p+q}(H^n(K), H^{n-p}(L))$ which converges to $\mathrm{Ext}^{p+q}(K, L)$. This gives a filtration $F^\bullet$ on $\mathrm{Hom}(K, L)$. We can see that $F^{\dim X+1} = 0$, F^1 consists of ghost maps, and $F^r \circ F^s \subset F^{r+s}$. The result follows from these properties. □

Proof of Theorem 2. Let $K_0 = K \in D^b(X)$. Choose a finite direct sum $\oplus\mathcal{O}_X(n_i)^{\oplus d_i}[e_i]$ together with a map $\phi_0 : \oplus\mathcal{O}_X(n_i)^{\oplus d_i}[e_i] \to K_0$ which is surjective on cohomology. Let $K_1 = \mathrm{cone}\,\phi_0$, so $K_0 \to K_1$ is a ghost. Repeat this inductively to get $K_0 \to \cdots \to K_{\dim X+1}$.

We claim that $\mathrm{cone}(K_0 \to K_j) \in \langle\{\mathcal{O}_X(i)\}_{i\in\mathbb{Z}}\rangle_j$. For $j = 1$, this is true by construction. For $j > 1$, we use the octahedral axiom to construct an exact triangle

$$\mathrm{cone}(K_0 \to K_{j-1}) \to \mathrm{cone}(K_0 \to K_j) \to \mathrm{cone}(K_{j-1} \to K_j) \to$$

and use this to deduce the claim.

The theorem follows by considering $K_{\dim X+1} \oplus K_0[1] \simeq \mathrm{cone}(K_0 \xrightarrow{0} K_{\dim X+1}) \in \langle\{\mathcal{O}_X(i)\}_{i\in\mathbb{Z}}\rangle_{\dim X+1}$. □

8. 10/8 (Hannah Larson) – The Chow Rings of Moduli Spaces of Pointed Hyperelliptic Curves

This is related to the speaker's prior talk in this seminar, but knowledge of that talk is not necessary for this.

8.1. **Refresher on Chow rings.** Let X be a smooth variety. The *Chow ring* $A^*(X)$ is the ring of $\mathbb{Z}$-linear combinations of irreducible subvarieties $Y \subset X$, modulo rational equivalence. Recall that Y_1 and Y_2 are rationally equivalent (written $Y_1 \sim Y_2$) if there is a subvariety $Z \subset X \times \mathbb{P}^1$ such that the forgetful map $p : Z \to \mathbb{P}^1$ is flat and for some $t_1, t_2 \in \mathbb{P}^1$ we have $p^{-1}(t_1) = Y_1$ and $p^{-1}(t_2) = Y_2$. The Chow ring is graded by codimension. Addition in the Chow ring corresponds to taking unions, and multiplication corresponds to taking (suitable) intersections.

We will need to know a few computational facts about Chow rings:

(1) $A^*(\mathbb{A}^n) = \mathbb{Z}$, concentrated in codimension 0. For example, the subvariety $V(x^2 + y^2 - 1) \subset \mathbb{A}^2$ is rationally equivalent to $\varnothing$ via the family $V((tx^2) + (ty)^2 - 1) \subset \mathbb{A}^2 \times \mathbb{P}^1_t$.
(2) There is an *excision property*: if $Z \subset X$ is closed of codimension c, then there is an exact sequence

$$A^{*-c}(Z) \longrightarrow A^*(X) \longrightarrow A^*(U) \longrightarrow 0$$

where $U = X \setminus Z$.

These can be used to compute $A^*(\mathbb{P}^n) = \mathbb{Z}[\zeta]/(\zeta^{n+1})$, using induction and the fact that

$$\mathbb{P}^n = \mathbb{A}^n \sqcup \mathbb{P}^{n-1} = \mathbb{A}^n \sqcup \mathbb{A}^{n-1} \sqcup \cdots \sqcup \mathrm{pt}.$$

To make this calculation work, one does need to know that the class of a point in $A^*(\mathbb{P}^n)$ is nonzero, which follows from proper pushforward being well-defined. We will use similar stratification-based methods today.

[3]Some dexterity is needed to reduce the general case to the case where k is uncountable, but this is not too difficult.

8.2. **Moduli of curves.** Let $\mathcal{M}_g$ be the (coarse) moduli space of curves. One way to stratify this by gonality.

Definition 8.1. The *gonality* of a curve C is the minimal k such that there exists a degree k map $C \to \mathbb{P}^1$.

For curves of genus $g \neq 0$, the minimum gonality is 2. Curves of gonality 2 are called *hyperelliptic* and determine a closed subvariety $\mathcal{H}_g \subset \mathcal{M}_g$. Using the Riemann-Hurwitz formula, we see that if C is a hyperelliptic curve of genus g, the ramification divisor R of any degree 2 map $C \to \mathbb{P}^1$ satisfies $\deg R = 2g+2$. In particular, the coarse moduli space $\mathcal{H}_g$ corresponds to the space of collections of $2g+2$ distinct points on $\mathbb{P}^1$, all modulo automorphisms of $\mathbb{P}^1$.[4] This can be written as

$$(\mathbb{P}^{2g+2} \setminus \Delta)/\mathrm{PGL}_2$$

where Δ is the locus where the points collide. In particular, we note that the coarse moduli space of $\mathcal{H}_g$ is unirational.

We can compute the Chow ring of this space (with rational coefficients) using equivariant intersection theory. The answer is

$$A^*(\mathcal{H}_g) \otimes \mathbb{Q} = \mathbb{Q}$$

Since we are using rational coefficients, the coarse moduli space gives the same answer as the stacky computation.

8.3. **Introducing marked points.** Let $\mathcal{H}_{g,n}$ be the moduli space of smooth hyperelliptic curves with n distinct marked points. We have good reason to care about these: the boundary of the compactification $\overline{\mathcal{M}}_{g'}$ can be stratified by products of moduli spaces $\mathcal{M}_{g,n}$ of curves with marked points, and the $\mathcal{H}_{g,n}$ appear in the stratifications of these moduli spaces by gonality.

Let's consider $\mathcal{H}_{g,1}$ first. For a hyperelliptic curve C with $f : C \to \mathbb{P}^1$, if p is a marked point on C, there is an automorphism of C taking p to the other point $\overline{p} \in f^{-1}(f(p))$. Thus all we really care about is the image $f(p)$, i.e. $\mathcal{H}_{g,1}$ has coarse moduli space given by the space of collections of $2g+2$ distinct points on $\mathbb{P}^1$ together with one distinguished point on $\mathbb{P}^1$. As a variety, this is

$$((\mathbb{P}^2 \setminus \Delta) \times \mathbb{P}^1)/\mathrm{PGL}_2.$$

The forgetful map $\mathcal{H}_{g,1} \to \mathcal{H}_g$ is a $\mathbb{P}^1$-bundle, and we get

$$A^*(\mathcal{H}_{g,1}) \otimes \mathbb{Q} = \mathbb{Q}[\phi]/(\phi^2).$$

As n grows, the spaces $\mathcal{H}_{g,n}$ become more complicated. This can be made precise:

Theorem 8.2 (Barros-Mullane, Schwarz)**.** *For $n \geqslant 4g+7$, the space $\mathcal{H}_{g,n}$ is of general type.*

In particular, it is not possible to find a dominant map from a rational variety to $\mathcal{H}_{g,n}$ (i.e. a "parametrization") for n large. Nevertheless, $\mathcal{H}_{g,n}$ is easier to understand in low degrees:

Theorem 8.3 (Casnati)**.** *For $n \leqslant 2g+8$, the space $\mathcal{H}_{g,n}$ is rational.*

Theorem 8.4 (Canning-Larson)**.** *For $n \leqslant 2g+6$, we have*

$$A^*(\mathcal{H}_{g,n}) \otimes \mathbb{Q} = \frac{\mathbb{Q}[\phi_1, \dots, \phi_n]}{(\phi_1, \dots, \phi_n)^2}$$

where the classes ϕ_i all lie in codimension 1. Furthermore, $\mathcal{H}_{g,n}$ is rational for $n \leqslant 3g+6$.

The classes ψ_i here are tautological classes. Over $\mathcal{H}_{g,n}$, there is a universal hyperelliptic curve $f : \mathcal{C} \to \mathcal{H}_{g,n}$ with disjoint sections $\sigma_1, \dots, \sigma_n : \mathcal{H}_{g,n} \to \mathcal{C}$. One defines $\psi_i = c_1(\sigma_i^* \omega_f)$. This is useful for understanding $A^*(\mathcal{M}_{g,n}) \otimes \mathbb{Q}$: the only contributions to this coming from $\mathcal{H}_{g,n}$ will be tautological!

Canning and Larson also have some similar results for higher gonality.

9. 10/15a (Smita Rajan) – Kinematic Varieties for Massless Particles

This is joint work with Svala Sverrisdottir and Bernd Sturmfels. The goal is to understand scattering amplitudes in theoretical physics.

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[4]This is not quite true for the stacks, but we will not concern ourselves with this.

9.1. **Background.** One can specify a QFT by specifying a Lagrangian $\mathcal{L}$. From this Lagrangian, one can extract Feynman diagrams, which describe the interaction of particles. "Tree-level" Feynman diagrams (those without loops) are the easiest to work with and produce the largest physical effects. To compute scattering amplitudes and other physical quantities from Feynman diagrams, we need to compute Feynman integrals. These are hard in general, but if we just care about tree-level scattering amplitudes, we can use the "spinor-helicity formalism" (which we'll discuss later).

The spinor-helicity formalism is well-understood in 4 ($= 3 + 1$) dimensions. However, extending it to higher dimensions turns out to be an interesting problem because of the relations between variables.

9.2. **Particles in d-dimensional spacetime.**

Definition 9.1. By d-dimensional spacetime, we mean the real vector space $\mathbb{R}^d$ with the Lorentzian inner product

$$x \cdot y = -x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

We consider a configuration of n particles in d dimensions. Each particle has a momentum $p_i = (p_{i,1}, \ldots, p_{i,d})$. For massless particles, we obtain n quadric relations ($p_i^2 = 0$ for $1 \leqslant i \leqslant n$). We also impose momentum conservation, giving d linear relations ($\sum_{i=1}^n p_{ij} = 0$ for $1 \leqslant j \leqslant d$).

Algebraically, we focus on the polynomial ring $\mathbb{C}[p_{ij}]$ in nd variables.[5] Let $I_{d,n}$ be the ideal generated by the aforementioned n quadrics and d linear forms. Then $V(I_{d,n})$ is irreducible and of the expected dimension $(nd - n - d)$.

Theorem 9.2. *The ideal* $I_{d,n}$ *is prime and is a complete intersection if* $\max(d, n) \geqslant 4$.

Proof. First consider the case $d \geqslant 5$. By eliminating the variable p_n, we can write $I_{d,n} = J + \langle f \rangle$, where J is generated by $n - 1$ quadrics and

$$f = \left(\sum_{i=1}^{n-1} p_i\right) \cdot \left(\sum_{i=1}^{n-1} p_i\right).$$

We can write

$$\frac{\mathbb{C}[p_1, \ldots, p_{n-1}]}{J} \cong \frac{\mathbb{C}[p_1]}{p_1 \cdot p_1} \otimes \cdots \otimes \frac{\mathbb{C}[p_{n-1}]}{p_{n-1} \cdot p_{n-1}}.$$

A Hartshorne exercise shows that each of the factors has trivial divisor class group, so the ring $\mathbb{C}[p_1, \ldots, p_{n-1}]/J$ is a UFD. One can use this to show that the ideal in question is prime.

For $d = 3$ and $n \geqslant 4$, we can use Serre's criterion to show that $\mathbb{C}[p_i]/\langle p_i \cdot p_i \rangle$ is normal. This can be used to show the result.

The $d = 4$ case was handled in the paper "Spinor Helicity Varieties." □

9.3. **The Clifford algebra and spinors.**

Definition 9.3. Let V be a vector space equipped with a bilinear form B. The Clifford algebra is

$$C\ell(V) = T(V)/\langle v \otimes w + w \otimes v - 2B(v, w) \rangle.$$

We will write $C\ell(1, d-1)$ for the Clifford algebra corresponding to the Lorentzian inner product η on $\mathbb{C}^d$. This can be written explicitly as

$$C\ell(1, d-1) = \frac{\mathbb{C}\langle \gamma_1, \ldots, \gamma_d \rangle}{\langle \gamma_i \gamma_j + \gamma_j \gamma_i - 2\eta_{ij} \rangle}$$

Remark 9.4. The associated graded of the Clifford algebra is just the usual exterior algebra.

We'd like to construct a matrix representation of $C\ell(1, n-1)$ of dimension 2^k, where $k = \lfloor d/2 \rfloor$. For $d = 2$, we send γ_1 and γ_2 to the matrices

$$\Gamma_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad \Gamma_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

We can define the representations in dimensions $d > 2$ inductively.

[5] The complexification occurs here for physical reasons – certain quantities aren't well-defined if we only work with real numbers.

For $d = 2k$, we send γ_i for $1 \leqslant i \leqslant d-2$ to

$$\Gamma_{2k,i} = \Gamma_{k-1,i} \otimes \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

We can also define matrices for γ_{d-1} and γ_d. Similar arguments work for odd d.

These representations give rise to the *spinor representations* of the Lie algebra $\mathfrak{so}(1, d-1)$. Specifically, send the ijth generator of $\mathfrak{so}(1, d-1)$ to $\Sigma_{ij} = \frac{1}{4}[\Gamma_i, \Gamma_j]$.

9.4. **Back to particles.** For each particle, we construct a *momentum space Dirac matrix*

$$P_i = -p_{i1}\Gamma_1 + p_{i2}\Gamma_2 + \cdots + p_{id}\Gamma_d.$$

Definition 9.5. The *charge conjugation matrix* C is an equivariant map from the spinor representation of $\mathfrak{so}(1, d-1)$ to its dual satisfying

$$C\Gamma_i = -\Gamma_i^T C.$$

Example 9.6. For $d = 4$, we have

$$C = \begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{bmatrix}$$

We'd like to parametrize the column space of the matrix P_i by the variables

$$z_i = (z_{i,1}, \ldots, z_{i,2^{k-2}}, 0, \ldots, 0, z_{i,2^{k-1}+1}, \ldots, z_{2^k})^T.$$

We define the *spinor helicity variable* as $|i\rangle = P_i z_i$. We also let $\langle i| = (P_i z_i)^T$.

We can use these to define *spinor brackets*:

- Order 2: $\langle ij\rangle = \langle i|\, C\, |j\rangle = z_i^T P_i^T C P_j z_j$
- Order 3: $\langle ijk\rangle = \langle i|\, CP_j\, |k\rangle$
- Higher order: defined similarly.

Theorem 9.7. *These spinor brackets are invariant under the $\mathfrak{so}(1, d-1)$ action in the spinor representation (via conjugation by the P-matrices and left multiplication on the z-variables).*

Consider matrices S and T_j, where $S_{ij} = \langle ij\rangle$ and $(T_j)_{ik} = \langle ijk\rangle$.

Definition 9.8. The *kinematic variety* $K^{(2)}_{d,n}$ is the variety of possible matrices S as above.

Theorem 9.9. *For $d = 3$, the ideal of $K^{(2)}_{3,n}$ is generated by the 4×4 Pfaffians of a skew-symmetric matrix $n \times n$ matrix, so $K^{(2)}_{3,n} \cong \mathrm{Gr}(2, n)$. For $d = 4$ and $d = 5$, we have $K^{(2)}_{4,n} \cong K^{(2)}_{5,n}$, and these varieties are cut out by 6×6 Pfaffians of a skew-symmetric $n \times n$ matrix. In particular, $K^{(2)}_{4,n} \cong K^{(2)}_{5,n}$ is the first secant variety of $\mathrm{Gr}(2, n)$.*

10. 10/22 (Will Fisher) – Introduction to Hochschild Homology

10.1. **Introduction and motivation.** Let A be an associative k-algebra (where k is unital and commutative). What is the "universal" commutative algebra associated to A?

We can interpret this question categorically: what are the adjoints to $\mathsf{CAlg}_k \hookrightarrow \mathsf{Alg}_k$? One adjoint is given by $A \mapsto A/[A, A]$, but the other doesn't exist. Trying to take the center doesn't work, as it's not functorial in general. Let's fix this by considering a more general problem.

Given an A/k-bimodule ("A-over-k bimodule") M, we'd like to find the universal bimodule where the left and right actions agree. We recall the definition of A/k-bimodules, as this is somewhat more restrictive than the naïve notion of bimodule.

Definition 10.1. An A/k-bimodule is a k-module M with the structure of a left and right A-module such that

(1) $(a \cdot m) \cdot b = a \cdot (m \cdot b)$ for $a, b \in A$ and $m \in M$
(2) $\lambda \cdot m = m \cdot \lambda$ for $\lambda \in k$ and $m \in M$.

There are two obvious options for these "universal bimodules:"

(1) $Z(M) = \{m \in M \mid am = ma \forall a \in A\}$
(2) $M/[A, M] = M/\langle am - ma \mid m \in M, a \in A\rangle$

If M is an A/k-bimodule such that $a \cdot m = m \cdot a$ for all $a \in A$, then $[A, A]$ annihilates M. In fact, this commutativity condition is equivalent to an $A/[A, A]$-left module structure on M. Thus we have an inclusion $(A/[A, A])\text{-Mod} \hookrightarrow A/k\text{-Bimod}$. One can check that $M \mapsto Z(M)$ and $M \mapsto M/[A, M]$ are the left and right adjoints to this inclusion.

Definition 10.2. *Hochschild homology* $\mathrm{HH}_*(A/k, -)$ is the derived functor of

$$A/k\text{-Bimod} \to (A/[A, A])\text{-Mod}$$
$$M \mapsto M/[A, M].$$

If A is a *commutative* algebra, then an A/k-bimodule M is an $A \otimes_k A$-module via $(a \otimes b) \cdot m = a \cdot m \cdot b$. A similar construction lets us also view M as a right $A \otimes_k A$-module. Let $\iota : A\text{-Mod} \to (A \otimes_k A)\text{-Mod}$ be defined by $(a \otimes b) \cdot m = ab \cdot m$, so we are viewing M as an $A \otimes_k A$-module via the multiplication $m : A \otimes_k A \to A$. Geometrically, this corresponds to pushforward along the diagonal $\Delta : \operatorname{Spec} A \to \operatorname{Spec} A \times_k \operatorname{Spec} A$.

Corollary 10.3. *If A is commutative, then $M/[A, M] \cong M \otimes_{A \otimes_k A} A$, where A is an $A \otimes_k A$-module via multiplication.*

10.2. **The bar complex.** We'll take A to be commutative from now on, but note that we can make most of this work generally. For notational simplicity, we'll write $A^e = A \otimes_k A$.

Thinking of $\mathrm{HH}_*(A/k, -)$ as the derived functor of $\Delta^* : \mathsf{QCoh}(\operatorname{Spec} A \times_k \operatorname{Spec} A) \to \mathsf{QCoh}(\operatorname{Spec} A)$, we see that

$$\mathrm{HH}_*(A/k, M) \cong \operatorname{Tor}_*^{A^e}(M, A).$$

By the symmetry of Tor, we see that it suffices to take a projective resolution of A. This can be accomplished as follows.

Definition 10.4. The *bar complex* (or *standard complex*) is the complex of A^e-modules

$$\dots \xrightarrow{d_2} A \otimes_k A \otimes_k A \xrightarrow{d_1} A \otimes_k A \xrightarrow{m} A \longrightarrow 0$$

where

$$d_n(a_0 \otimes \dots a_{n+1}) = \sum_{i=0}^{n} (-1)^i a_0 \otimes \dots \otimes a_i a_{i+1} \otimes \dots \otimes a_{n+1} + (-1)^{n+1} a_{n+1} a_0 \otimes \dots \otimes a_n.$$

Here the left (resp. right) factor of $A \otimes_k A$ acts on the leftmost (resp. rightmost) factor of $A \otimes_k \dots \otimes_k A$.

Proposition 10.5. *The bar complex is exact. Furthermore, as left A^e-modules, we have $A^{\otimes n+2} \xrightarrow{\sim} A^e \otimes A^{\otimes n}$ via*

$$a_0 \otimes \dots \otimes a_{n+1} \mapsto (a_0 \otimes a_{n+1}) \otimes a_1 \otimes \dots \otimes a_n.$$

In particular, if A is free as a k-module, than $A^{\otimes n+2}$ is free as an A-module.

Theorem 10.6. *If A is a free k-module, then $\mathrm{HH}_*(A/k, M)$ is equivalent to the homology of the complex*

$$\dots \xrightarrow{d_3} M \otimes_k A \otimes_k A \xrightarrow{d_2} M \otimes_k A \xrightarrow{d_1} M \longrightarrow 0$$

where

$$d_n(m \otimes \dots a_n) = ma_1 \otimes a_2 \otimes \dots \otimes a_n + \sum_{i=1}^{n-1} (-1)^i m \otimes a_1 \otimes \dots \otimes a_i a_{i+1} \otimes \dots \otimes a_n + (-1)^n a_n m \otimes a_1 \otimes \dots \otimes a_n.$$

Proof. If A is free, use the standard resolution and $M \otimes_{A^e} (A^{\otimes n+2}) \cong M \otimes_k A^{\otimes n}$. $\square$

10.3. HKR and computations.

Theorem 10.7. *If*

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$$

is a short exact sequence of A^e-modules, we get a LES

$$\dots \longrightarrow \mathrm{HH}_1(A/k, M_3) \longrightarrow \mathrm{HH}_0(A/k, M_1) \longrightarrow \mathrm{HH}_0(A/k, M_2) \longrightarrow \mathrm{HH}_0(A/k, M_3) \longrightarrow 0.$$

where $\mathrm{HH}_0(A/k, M) = M/[A, M] = M \otimes_{A^e} A$. *Furthermore, if* M *is a free* A^e*-module, then* $\mathrm{HH}_i(A/k, M) = 0$ *for all* $i > 0$.

Example 10.8. Consider the SES

$$0 \longrightarrow I \longrightarrow A^e \longrightarrow A \longrightarrow 0$$

where $I = \ker(M)$. Because A^e is a free module over itself, we get an LES containing the terms

$$0 \longrightarrow \mathrm{HH}_1(A/k, A/k) \longrightarrow I \otimes_{A^e} A \xrightarrow{\ \phi\ } A.$$

As

$$\phi\big((a \otimes b) \otimes \alpha\big) = a\alpha b = ab\alpha = 0$$

(because $a \otimes b \in I$), we see

$$\mathrm{HH}_1(A/k, A/k) \cong I \otimes_{A^e} A \cong I \otimes_{A^e} A^e/I \cong I/I^2 \cong \Omega^1_{A/k}.$$

Thus $\mathrm{HH}_1(A/k, A/k) \cong \Omega^1_{A/k}$, regardless of A.

Theorem 10.9 (Hochschild-Kostant-Rosenberg)**.** *There exists a cup product on* $\mathrm{HH}_*(A/k, A/k)$ *and a map* $\wedge^*\Omega^1_{A/k} \to \mathrm{HH}_*(A/k, A/k)$. *If* k *is a field and* A/k *is smooth, then this map is an isomorphism.*

11. 10/29 (Catherine Cannizzo) – Homological Mirror Symmetry for Theta Divisors

I didn't take notes for this talk because the speaker already had well-prepared slides.

12. 11/5 (Martin Olsson) – Ample Vector Bundles and Projective Geometry of Stacks

This is based on joint work with Dan Bragg and Rachel Webb. For simplicity, we work over a field of characteristic zero.

12.1. Warmup.

Let $g \geqslant 2$, and consider the Deligne-Mumford moduli space of curves $\overline{\mathcal{M}}_g$. This has a coarse moduli space $\overline{M}_g$.

Recall that $\overline{M}_g$ is a projective scheme. More precisely, for all $k \geqslant 1$, we get a line bundle Λ_k on $\overline{\mathcal{M}}_g$, with sections over R-points (corresponding to curves C over R) given by $\det H^0(C, \omega_{C/R}^{\otimes k})$. For $k \gg 0$, this descends to an ample line bundle $\mathcal{L}_k$ on $\overline{M}_g$. Thus we may write

$$\overline{M}_g = \operatorname{Proj} \oplus_{m \geqslant 0} H^0(\overline{\mathcal{M}}_g, \Lambda_k^{\otimes m}) = \operatorname{Proj} \oplus_{m \geqslant 0} H^0(\overline{M}_g, \mathcal{L}_k^{\otimes m}).$$

Question 12.1. *How can we produce an analogous description of* $\overline{\mathcal{M}}_g$*?*

12.2. An example: $\overline{\mathcal{M}}_{1,1}$.

Consider $\overline{\mathcal{M}}_{1,1}$; this is a "stacky curve."[6] We define a line bundle Λ with sections over an R-point (E, e) given by $H^0(E, \omega_{E/R}) = e^*\omega_{E/R}$. The coarse moduli space here is $\mathbb{P}^1$ (with the coarse moduli map given by the j-invariant). Mumford showed that $\Lambda^{\otimes 12}$ descends to $\mathcal{O}_{\mathbb{P}^1}(1)$.

We'd like to compute the graded ring $\Gamma_*(\overline{\mathcal{M}}_{1,1}, \Lambda)$. Note that $\overline{\mathcal{M}}_{1,1}$ has two special points: $i_0 : B\mu_6 \to \overline{\mathcal{M}}_{1,1}$ and $i_{1728} : B\mu_4 \to \overline{\mathcal{M}}_{1,1}$. Viewing these as divisors $\mathcal{D}_0$ and $\mathcal{D}_{1728}$, we can show $\Lambda = \mathcal{O}_{\overline{\mathcal{M}}_{1,1}}(\mathcal{D}_{1728} - \mathcal{D}_0)$. From this, we obtain

$$H^0(\overline{\mathcal{M}}_{1,1}, \Lambda^{\otimes m}) = \begin{cases} 0 & m \text{ odd} \\ H^0\left(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}\left(\lfloor \frac{m}{4} \rfloor - \lceil \frac{m}{6} \rceil\right)\right) & m \text{ even.} \end{cases}$$

This can be used to show that $\Gamma_*(\overline{\mathcal{M}}_{1,1}, \Lambda) = k[x, y]$, where $\deg x = 4$ and $\deg y = 6$.

Observe that $\overline{\mathcal{M}}_{1,1} \simeq [(\operatorname{Spec} k[x, y] \backslash 0)/\mathbb{G}_m]$. This looks like a reasonable "projective embedding of stacks."

[6] For more on stacky curves, see the AMS Memoirs book by Voight and Zureick-Brown.

12.3. **Desiderata.** In general, we should ask:

Question 12.2. *What plays the role of projective spaces for stacks?*

Question 12.3. *What plays the role of ample line bundles for stacks?*

The *weighted projective spaces* $\mathcal{P}(a_0, \ldots, a_n) = [(\mathbb{A}^{n+1}\setminus 0)/_{\underline{a}}\mathbb{G}_m]$ should be examples of "stacky projective spaces." However, they shouldn't give all of the examples, as that would only allow cyclic stabilizers. Similarly, restricting to line bundles only gives embeddings into stacks with cyclic stabilizers. Abramovich and Hassett study a version of moduli theory for stacks with cyclic stabilizers, but we really want to allow nonabelian examples.

Example 12.4. Let's think about the case where $\mathfrak{X} = BG$ for a finite group G. Line bundles on $\mathfrak{X}$ correspond to multiplicative characters of G. Vector bundles on G correspondto representations of G – knowing these tells us a lot more about G!

12.4. **Generalizing projective spaces.** Let V be a representation of GL_r, and let $\mathbb{A}_V = \operatorname{Spec}\operatorname{Sym} V$. Define a subvariety $\mathbb{A}_V^{s,\det} \subset \mathbb{A}_V$ by the condition $x \in \mathbb{A}_V^{s,\det}$ if and only if there exists $f \in \operatorname{Sym} V$ such that

1. $g \cdot f = (\det g)^m f$ for all $g \in \mathrm{GL}_r$,
2. $f(x) \neq 0$, and
3. x has closed orbit in $D(f)$.

This should be thought of as similar to a GIT "stability condition."

We define $\mathcal{QP}(V) = [\mathbb{A}_V^{s,\det}/\mathrm{GL}_r]$. By GIT, it follows that $\mathcal{QP}(V)$ is a DM stack with finite diagonal. The coarse moduli space of $\mathcal{QP}(V)$ is $\operatorname{Proj} \oplus_{m \geqslant 0} A_m$, where A_m consists of the f above.

12.5. **Generalizing ample line bundles.** The generalization of ample line bundles used here is inspired by Robin Hartshorne's "ample vector bundles."

Let $\mathfrak{X}$ be a finite type DM stack with affine diagonal over an algebraically closed field[7] k, and let $\pi : \mathfrak{X} \to X$ be the coarse moduli space. For a point $x : \operatorname{Spec} k \to \mathfrak{X}$, we write G_x for the corresponding stabilizer group. In particular, if $\mathcal{F} \in \mathsf{Coh}(\mathfrak{X})$, then the fiber $\mathcal{F}(x) = x^*\mathcal{F}$ is naturally an object of $\mathsf{Rep}(G_x)$.

Definition 12.5. Let $\mathcal{E}$ be a vector bundle on $\mathfrak{X}$. We say that $\mathcal{E}$ is:

1. *faithful* if, for all $x \in \mathfrak{X}(k)$, the representation $\mathcal{E}(x)$ of G_x is faithful.
2. *H-ample* if $\mathcal{E}$ is faithful and, for all $\mathcal{F} \in \mathsf{Coh}(\mathfrak{X})$, there exists n_0 such that $\pi_*(\mathcal{F} \otimes \operatorname{Sym}^n \mathcal{E})$ is generated by global sections for all $n \geqslant n_0$.
3. det-*ample* if $\mathcal{E}$ is faithful and there exists $N > 0$ such that $\det(\mathcal{E})^{\otimes N}$ descends to an ample line bundle on X.

Theorem 12.6. *The following are equivalent:*

(1) X *is quasiprojective and* $\mathfrak{X} = [U/\mathrm{GL}_r]$ *for some scheme* U *with action of* GL_r *(for some* $r \geqslant 1$*)*
(2) $\mathfrak{X}$ *admits a vector bundle that is both H-ample and* det-*ample.*
(3) $\mathfrak{X}$ *admits a* det-*ample vector bundle.*
(4) $\mathfrak{X}$ *admits an immersion into* $\mathcal{QP}(V)$ *for some* V.

Given a det-ample vector bundle, how do we obtain a map $\mathfrak{X} \to \mathcal{QP}(V)$? Let's forget about stability conditions for the moment and just consider maps from $\mathfrak{X}$ to $[\mathbb{A}_V/\mathrm{GL}_r]$. These correspond to GL_r-torsors $P \to \mathfrak{X}$ together with GL_r-equivariant maps $P \to \mathbb{A}_V$. If P corresponds to the rank r vector bundle $\mathcal{E}$, then P embeds as an open substack of $\operatorname{Spec}\operatorname{Sym} \mathcal{E}^{\oplus r}$. Requiring that $\mathcal{E}$ is faithful implies that P is a *scheme* open in $\operatorname{Spec}_X(\pi_* \operatorname{Sym} \mathcal{E}^{\oplus r})$. This can be used to understand $H^0(P, \mathcal{O}_P)$, and the map $P \to \mathbb{A}_V$ can be constructed as a map $V \to H^0(P, \mathcal{O}_P)$.

13. 11/12 (Nathaniel Gallup) – The Grothendieck Ring of Certain Non-Noetherian Multigraded Algebras via Hilbert Series

This talk is motivated by Schubert calculus.

[7] We're only assuming this to clean up some of the statements we'll make.

13.1. **Basic background.** Some history:

(1) In 1982, Lascoux and Schützenberger introduced Grothendieck polynomials.
(2) In 2001, Knutson and Miller gave these a geometric interpretation in terms of *matrix Schubert varieties.*
(3) In 2005, Miller and Sturmfels wrote a book on the subject.

The Grothendieck polynomial admits an explicit combinatorial formula in terms of "pipe dreams."

Let $R_n = k[x_{11}, x_{12}, \dots, x_{nn}]$. For $\sigma \in S_n$, we define $r_{a,b}(\sigma) \in \mathbb{N}$ by counting the number of 1's in the submatrix with a rows and b columns (?) in the lower left corner of the permutation matrix of σ. We consider the determinantal ideal $I_\sigma \subset R_n$ generated by minors of size $r_{a,b}(\sigma)+1$ in the variables $x_{\geqslant a, \leqslant b}$.

Example 13.1. For the permutation $213 \in S_3$, we have $I_\sigma = \langle x_{31}, x_{32}, x_2, x_{32} - x_{22}x_{31}\rangle$.

We consider R_n/I_σ, which is $\mathbb{Z}^n$-graded where each column of variables $x_{\bullet,i}$ corresponds to one of the $\mathbb{Z}$ factors in $\mathbb{Z}^n$.

13.2. **Grothendieck rings.** Let Γ be a finitely generated abelian group and R a Γ-graded K-algebra. We define $K_0(R)$ to be the group completion of the monoid of isomorphism classes of finitely generated Γ-graded *projective* R-modules. The group $G_0(R)$ is defined similarly, except we use *all* finitely generated Γ-graded R-modules (not just projective modules). There is always a map $K_0(R) \to G_0(R)$.

Proposition 13.2. *If* $R = k[x_1, \dots, x_n]$*, then* $K_0(R) \cong G_0(R)$.

Proof. Let M be a finitely generated Γ-graded R-module. By the Hilbert syzygy theorem, there exists a Γ-graded finite free resolution

$$0 \longrightarrow F_n \longrightarrow \dots \longrightarrow F_0 \longrightarrow M \longrightarrow 0.$$

This implies $[M] = [F_0] - [F_1] + \cdots + (-1)^n [F_n]$, and we can use this to define an inverse of $K_0(R) \to G_0(R)$. □

How do we compute $G_0(R)$ in general? For nice R and finitely generated R-modules M, we want to define a *multigraded Hilbert series*

$$H_M(t) = \sum_{g \in \Gamma} \dim_k M_g t^g.$$

Proposition 13.3. *Suppose* R *satisfies the following properties:*

(1) $\Lambda := \operatorname{supp}_\Gamma(R) \subset \Gamma$ *is* pointed *(i.e. if* $p, q \in \Lambda$ *and* $p+q=0$*, then* $p=q=0$*),*
(2) R *is connected (i.e.* $R_0 = k$*),*
(3) M *is finitely generated.*

Then $\dim_k M_g < \infty$ *for all* g.

It follows that, under these hypotheses, the multigraded Hilbert series of M is well-defined.

Proposition 13.4. *If* $R = k[x_1, \dots, x_n]$ *is pointed and connected with respect to a certain* Γ*-grading, and* M *is a finitely generated* R*-module then:*

(1) $H_M(t) = \frac{K_M(t)}{\prod_i (1 - t^{\deg x_i})}$*, where the* K-polynomial $K_M(t)$ *is in* $\mathbb{Z}[\Gamma]$.
(2) The K*-polynomial of* $\oplus_i R(-g_i)$ *is* $\sum_i t^{g_i}$.
(3) The K*-polynomial of* R *is* 1.

Theorem 13.5. *If* $R = k[x_1, \dots, x_n]$ *is pointed and connected, then taking* K*-polynomials induces equivalences*

$$K_0(R) \cong \mathbb{Z}[\Gamma] \cong G_0(R).$$

Proof. We use that finitely generated projectives over R are free together with a Γ-graded Nakayama lemma: $R_+M = M$ implies $M = 0$. □

In particular, the K-polynomial of $[R_n/I_\sigma] \in G_0(R)$ is the Grothendieck polynomial.

13.3. **Infinite Matrix Schubert Varieties.** We'd like to mimic the construction of Grothendieck polynomials for $\sigma \in S_\infty$. Let $R_{\mathbb{N}} = k[x_{ij}]_{i,j \in \mathbb{N}}$. The ideals $I_\sigma \subset R_{\mathbb{N}}$ are defined as before (using matrices of size $\mathbb{N} \times \mathbb{N}$).

We'd like to define a suitable category C of "small enough" graded $R_{\mathbb{N}}$-modules in which:

(1) Objects of C have Hilbert series.
(2) Projectives in C are free.
(3) $K_0(R) \xrightarrow{\sim} G_0(R)$
(4) C contains syzygies of all I_σ.

The first condition is handled by the following.

Proposition 13.6. *Let Γ be an abelian group grading $R = k[x_1, x_2, \dots]$, and suppose R satisfies conditions (PDCF):*

(1) R is pointed.
(2) R is connected.
(3) R is downward finite (using the order on Γ induced by declaring elements of Λ to be nonnegative, for all $p \in \Lambda$, the sets $\{q < p \mid q \in \Lambda\}$ are finite).
(4) Only finitely many x_i are in any R_g.

Then $\dim_k R_g < \infty$ for all $g \in \Gamma$. Suppose furthermore a Γ-graded R-module M satisfies conditions (BDF):

(1) M is supported on finitely many translates of Γ.
(2) $\dim_k M_g < \infty$ for all $g \in \Gamma$.

Then M admits a Hilbert series.

Letting C be the category of BDF modules, we obtain a version of Nakayama's lemma that implies projectives in C are free. Furthermore, defining $K_0(R_{\mathbb{N}})$ using finitely generated BDF modules and $G_0(R_{\mathbb{N}})$ using finitely generated modules, we get an isomorphism $K_0(R) \to G_0(R)$. This follows by taking free resolutions $F_\bullet \to M$ and noticing that $[M] = [\oplus_{n \text{ odd}} F_n] - [\oplus_{n \text{ even}} F_n]$. One can now generalize the previous theorem:

Theorem 13.7. *If $R = k[x_1, x_2, \dots]$ satisfies conditions PDCF, then taking Hilbert series induces isomorphisms*

$$K_0(R) \cong \mathbb{Z}[[\Lambda]][\Gamma] \cong G_0(R).$$

From this we obtain the desired notion of Grothendieck series $K_M(t) \in \mathbb{Z}[[\Lambda]][\Gamma]$. Current work in progress (joint with Anna Chlopecki) focuses on producing an "infinite pipe dream" formula for $K_M(t)$. The ultimate goal is to be able to do Schubert calculus in infinite dimensions.

14. 11/19a (David Eisenbud) – Some One-Dimensional Local Rings

We work throughout over $k = \overline{k}$.

14.1. **Review of multiplicity.** Suppose X is a d-dimensional subvariety of $\mathbb{P}^n$. We'd like to have a good notion of the multiplicity of X at a point p. The "correct answer" was found by Samuel in the 1950's: consider the maximal ideal $\mathfrak{m} \subset \mathcal{O}_{X,p}$. The dimension of the quotients $\mathfrak{m}^{q-1}/\mathfrak{m}^q$ is an eventually polynomial function in q (the "Hilbert-Samuel function"), and we define the *multiplicity* $\operatorname{mult}(X,p)$ of X at p to be the leading coefficient of this polynomial.

In the Cohen-Macaulay case, things become simpler. In this case, we can take a regular sequence $x_1, \dots, x_d$ generating $\mathfrak{m}_{X,p}$. The multiplicity is equal to the dimension of $\mathcal{O}_{X,p}/(x_1 \dots x_d)$. In particular, $\operatorname{mult}(X,p) \geqslant \operatorname{codim}_{\mathbb{P}^n} X + 1$.

The degree of a projective variety X is the multiplicity of the vertex of the cone over the variety. A similar argument shows that $\deg X \geqslant \operatorname{codim}_{\mathbb{P}^n} X + 1$.

Suppose $(R, \mathfrak{m})$ is a one-dimensional local domain. Then one can show that the multiplicity of $\mathfrak{m}$ is at least the number of generators of $\mathfrak{m}$.

14.2. **Resolution of singularities for curves.** Let $(R, \mathfrak{m})$ be a one-dimensional local domain. Resolution of singularities is easy in this case: we just take the normalization $\overline{R}$. However, it might be preferable to resolve singularities via iterated blowups. This is also doable and gives an explicit algorithmic procedure. Things are particularly nice for unibranch singularities, and the general case can be reduced to these.

For curves in $\mathbb{C}^2$, we can draw small spheres S^3 around the singularities. The intersections of the curves with S^3 give links, and it turns out that these singularities are intricately related to links and 3-manifold topology! We won't discuss this in more detail, but it's worth noting.

14.3. **Unibranch singularities and Arf rings.** Let's move back to unibranch singularities. These can be defined in various ways: one condition is that $\overline{R}$ is still local. We could also require that $(\hat{R}, \hat{\mathfrak{m}}) \subset k[[t]]$. In particular, $\hat{R}$ cuts out a "semigroup of values" in $\mathbb{N}$.

Example 14.1. The local ring R of $k[x, y]/(y^2 - x^3)$ at the origin is unibranch, with $\hat{R} = k[[t^2, t^3]]$. In particular, the semigroup is $\langle 2, 3\rangle \subset \mathbb{N}$.

More generally, we can construct semigroup rings S giving semigroups $G = \langle g_1, \dots, g_s\rangle \subset \mathbb{N}$, corresponding to $k[[t^{g_1}, \dots, t^{g_s}]] \subset k[[t]]$. Let m be the minimal element of G. There's a story about these in terms of "Apéry sets," but I didn't really follow it. The *Kunz cone* of m is the rational cone in $\mathbb{Q}^{m-1}$ generated by Apéry sets of S with $\operatorname{mult} S = m$. This can be understood explicitly.

An *Arf ring* is a one-dimensional local domain of minimal multiplicity such that each infinitely near point is also of minimal multiplicity.

Theorem 14.2 (Dao, Isobe-Kumashiro)**.** *A one-dimensional local domain R is Arf if and only if every reflexive module over R is a direct sum of the ith blowups of R (or localizations thereof).*

In particular, every ith syzygy for $i \geqslant 2$ is a direct sum of such components.

An open question remains: what is the first syzygy of R_i? Computer evidence suggests it is a direct sum of opies of R_j for $j \leqslant i$. This appears to be difficult to prove.

15. 11/19b (Cameron Chang) – Applications of Molien's Formula

This is an expository talk on the paper "Invariants of Finite Groups" by Stanley.

15.1. **Molien's formula.** Let V be a finite-dimensional complex vector space and $G \subset \mathrm{GL}(V)$ a finite subgroup. Let $R = \operatorname{Sym} V$. We are interested in understanding the ring of invariants R^G and, in particular, the *Molien series*

$$F_G(\lambda) = \sum_{k=0}^{\infty} \dim_{\mathbb{C}}(\operatorname{Sym}^k(V))^G \lambda^k.$$

More generally, let W be an irrep of G, and let d_k be the number of copies of W in $\operatorname{Sym}^k V$.

$$F_{G,W}(\lambda) = \sum_{k=0}^{\infty} d_k \lambda^k$$

In particular, $F_G = F_{G,\mathrm{triv}}$.

Theorem 15.1 (Molien's Formula)**.** *Let χ_W be the character of W. Then*

$$F_{G,W}(\lambda) = \frac{1}{G} \sum_{M \in G} \frac{\overline{\chi_W(M)}}{\det(I - \lambda M)}.$$

In particular,

$$F_G(\lambda) = \frac{1}{G} \sum_{M \in G} \frac{1}{\det(I - \lambda M)}.$$

Proof. We have

$$d_k = \left\langle \chi_{\operatorname{Sym}^k V}, \chi_W \right\rangle = \frac{1}{|G|} \sum_{M \in G} \chi_{\operatorname{Sym}^k(V)}(M) \overline{\chi_W(M)}.$$

The values $\chi_{\operatorname{Sym}^k(V)}(M)$ are equal to the sums $\sum_{i_1 + \cdots + i_n = k} \lambda_1^{i_1} \dots \lambda_n^{i_n}$, where the λ_i's are the eigenvalues of M. $\square$

If $\lambda_1, \dots, \lambda_n$ are the eigenvalues of M, then a direct computation shows

$$\det(I - \lambda M) = (1 - \lambda\lambda_1)\dots(1 - \lambda\lambda_n).$$

Taking the Laurent expansion of $F_{G,W}(\lambda)$ at $\lambda = 1$, we see that the coefficient of $1/(1-\lambda)^n$ is

$$\frac{\overline{\chi_W(I)}}{|G|} = \frac{\dim W}{|G|} \neq 0.$$

Thus $F_{G,W}(\lambda) \neq 0$, and in particular every irrep of G appears in $\operatorname{Sym} V$ (under our assumption that V is a faithful G-representation)! This is a classical theorem, but it's a neat proof.

15.2. **Application to roots of unity.** We can sometimes compute $F_G(\lambda)$ by inspection. Comparing this with Molien's formula can produce beautiful theorems.

Example 15.2. Let $G \cong \mathbb{Z}/n\mathbb{Z}$ act on $\mathbb{C}^2$ by

$$[i] \mapsto \begin{bmatrix} \xi^i & 0 \\ 0 & \xi^{-i} \end{bmatrix}$$

for ξ a primitive nth root of unity. By inspection, we see that $x^i y^j \in \mathbb{C}[x,y]^G$ if and only if $i \equiv j \pmod{n}$. Thus $\mathbb{C}[x,y]^G = \oplus_{i=0}^{n-1} x^i y^i \mathbb{C}[x^n, y^n]$, and

$$F_G(\lambda) = \frac{1 + \lambda^2 + \dots + \lambda^{2(n-1)}}{(1-\lambda^n)^2}.$$

Molien's formula gives

$$F_G(\lambda) = \frac{1}{n} \sum_{\omega} \frac{1}{|1 - \lambda\omega|^2}$$

where ω ranges over the nth roots of unity. Thus

$$\frac{1 + \lambda^2 + \dots + \lambda^{2(n-1)}}{(1-\lambda^n)^2} = \frac{1}{n} \sum_{\omega} \frac{1}{|1 - \lambda\omega|^2},$$

and taking limits as $\lambda \to 1$ gives

$$\sum_{\omega \neq 1} \frac{1}{|1-\omega|^2} = \frac{n^2 - 1}{12}.$$

15.3. **Applications to elementary symmetric polynomials.** Molien's formula can also be used to compute rings of invariants: if one has a subring of R^G which one conjectures to be the whole thing, then one can use Molien's formula to check that the dimensions match.

Theorem 15.3 (Fundamental theorem of symmetric polynomials)**.** *Let $S_n \curvearrowright \mathbb{C}[x_1, \dots, x_n]$ by the usual permutation matrices. Then $\mathbb{C}[x_1, \dots, x_n]^{S_n} = \mathbb{C}[\sigma_1, \dots, \sigma_n]$, where the $\sigma_i = \sum_{t_1 < \dots < t_i} x_{t_1} \dots x_{t_i}$ are the elementary symmetric polynomials. Furthermore, the σ_i are algebraically independent.*

Sketch of proof, assuming algebraic independence. Equality holds if and only if

$$F_{S_n}(\lambda) = \frac{1}{(1-\lambda)\dots(1-\lambda^n)},$$

so let's check that these agree. Molien's formula implies that

$$F_{S_n}(\lambda) = \frac{1}{n!} \sum_{\sigma \in S_n} \frac{1}{(1-\lambda^{c_1})\dots(1-\lambda^{c_k})},$$

where σ has cycle type $(c_1, \dots, c_k)$. To compare the two, one uses

$$\sum_n F_{S_n}(\lambda) t^n = \exp\left(\sum_k \frac{t^k}{k(1-\lambda^k)}\right) = \dots = \prod_{i=0}^{\infty} \frac{1}{(1-\lambda^i t)}$$

as well as some partition identities. $\square$

15.4. **Application to Chevalley-Shephard-Todd.** When is $\mathbb{C}[x_1, \dots, x_n]^G$ a polynomial ring? Let's first consider the result over $\mathbb{R}$.

Theorem 15.4 (Chevalley-Shephard-Todd over $\mathbb{R}$)**.** *The ring $\mathbb{R}[x_1, \dots, x_n]^G$ is a polynomial ring if and only if G is generated by reflections.*

In the complex setting, we need a generalization of reflections.

Definition 15.5. An element $M \in G$ is a *pseudoreflection* if M is not the identity and $n-1$ eigenvalues of M are equalt ot 1.

Theorem 15.6 (Chevalley-Shephard-Todd over $\mathbb{C}$)**.** *The ring $\mathbb{C}[x_1, \dots, x_n]^G$ is a polynomial ring if and only if G is generated by pseudoreflections.*

Sketch of proof of $\Leftarrow$. Suppose $\theta_1, \dots, \theta_n$ are algebraically independent generators of $\mathbb{C}[x_1, \dots, x_n]^G$. We may assume these are homogeneous, say of degrees $d_1, \dots, d_n$. The Poincaré series of $\mathbb{C}[x_1, \dots, x_n]^G$ is then

$$\frac{1}{(1-\lambda^{d_1})\dots(1-\lambda^{d_n})},$$

while the Molien series is

$$\frac{1}{|G|}\sum_{M \in G} \frac{1}{\det(I - \lambda M)}.$$

To compare these, we take the Laurent series at $\lambda = 1$. The coefficient of $(1-\lambda)^{-n}$ in the Poincaré series is $1/(d_1 \dots d_n) = 1/|G|$ (in particular, $|G| = d_1 \dots d_n$). The same coefficient in the Molien series is $(\sum_i (d_i - 1))/(2d_1 \dots d_n)$, and one can use this to see that the only contributions to this term come from pseudoreflections. $\square$

16. 11/26a (Serkan Hoşten) – ML Degree Stratification of $\mathbb{P}^{m-1} \times \mathbb{P}^{n-1}$

This is based on joint work (arXiv:2312.10010) with O. Clarke, N. Kushnerchuk, and J. Oldekop.

16.1. **Segre embeddings and independence loci.** Recall that the product $\mathbb{P}^{m-1} \times \mathbb{P}^{n-1}$ admits a Segre embedding into $\mathbb{P}^{mn-1}$, given by

$$[\alpha_1 : \dots : \alpha_m] \times [\beta_1 : \dots : \beta_n] \mapsto [\alpha_i\beta_j]_{i \in \{1,\dots,m\}, j \in \{1,\dots,n\}}.$$

We think of the image $\mathrm{Seg}(m,n)$ as being described by $m \times n$ matrices of rank 1.

There is a corresponding map of standard simplices $\Delta_{m-1} \times \Delta_{n-1} \to \Delta_{mn-1}$ given by essentially the same formula

$$(\alpha_1, \dots, \alpha_m) \times (\beta_1, \dots, \beta_n) \mapsto (\alpha_i\beta_j)_{i \in \{1,\dots,m\}, j \in \{1,\dots,n\}}.$$

Thinking of points of Δ_{m-1} as probability distributions on $\{1, \dots, m\}$ (and likewise for Δ_{n-1}), this takes $P(X = i)$ and $P(Y = j)$ and returns $P(X = i, Y = j)$. The image is called the *independence locus* $\mathrm{Ind}(m, n)$.

16.2. **Maximum likelihood.** In statistics, given data $u = (u_{ij}) \in \mathbb{N}^{m \times n}$, where u_{ij} is the number of observed samples with $X = i$ and $Y = j$. We seek to maximize

$$\ell_u(p) = \sum_{j=1}^{n}\sum_{i=1}^{m} u_{ij}\log(p_{ij})$$

subject to the constraint $(p_{ij}) \in \mathrm{Ind}(m, n)$. The maximizer $(p_{ij}^{\times})$ is known as the *maximum likelihood estimator* for $\mathrm{Ind}(m, n)$ given u.

A related algebraic problem is to compute the complex critical points of $\ell_u(p)$ on

$$\mathrm{Seg}(m,n) \setminus \{(\prod_{ij} p_{ij})(\sum_{ij} p_{ij}) = 0\}.$$

The number of complex critical points for generic u is the ML degree $\mathrm{MLdeg}(\mathrm{Seg}(m,n)$. Note that the complex critical points are determined by the *derivatives* of $\ell_u(p)$, which are algebraic.

More generally, given $X \subset \mathbb{P}^{r-1}$, we view $\mathrm{Cone}(X) \cap \Delta_{r-1}$ as a probability model. The ML degree $\mathrm{MLdeg}(X)$ counts the number of complex critical points of $\ell_u(p)$ on $X \setminus \{(\prod_i p_i)(\sum_i p_i) = 0\}$. This can help estimate the complexity of a maximum likelihood estimation problem. Furthermore, if we know the ML degree, we can use methods of numerical algebraic geometry to find critical points.

16.3. **The problem.** Let $W = (w_{ij}) \in (\mathbb{C}^times)^{mn}$, and define a modified Segre embedding

$$[\alpha_1 : \cdots : \alpha_m] \times [\beta_1 : \cdots : \beta_n] \mapsto [w_{ij}\alpha_i\beta_j]_{i\in\{1,\ldots,m\}, j\in\{1,\ldots,n\}}.$$

The image $\mathrm{Seg}_W(m,n)$ is abstractly isomorphic to $\mathrm{Seg}(m,n)$ but may have a different ML degree.

Question 16.1. *What is* $\mathrm{MLdeg}(\mathrm{Seg}_W(m,n))$ *as* w *varies?*

Analogous toy examples show that the results can be interesting:

Example 16.2. For the twisted cubic

$$[s : t] \mapsto [s^3 : s^2t : st^2 : t^3]$$

we can compute that the ML degree is 3. However, if we vary the coefficients to

$$[s : t] \mapsto [s^3 : 3s^2t : 3st^2 : t^3]$$

we get ML degree 1.

Definition 16.3. Let $\hat{W} = [I_m|W] \in \mathbb{C}^{m(m+n)}$, and let M_W be the matroid associated to $\hat{W}$.

Example 16.4. For $m = n = 3$, we look at the matroid consisting of bases which consist of columns of

$$\hat{W} = \begin{bmatrix} 1 & 0 & 0 & w_{11} & w_{12} & w_{13} \\ 0 & 1 & 0 & w_{21} & w_{22} & w_{23} \\ 0 & 0 & 1 & w_{31} & w_{32} & w_{33} \end{bmatrix}$$

Up to isomorphism, there are 38 matroids of rank 3 on a 6-element set. Only 8 of these arise as M_W.

Theorem 16.5. *The integer* $\mathrm{MLdeg}(\mathrm{Seg}_W(m,n))$ *is a matroid invariant: if* $M_V \cong M_W$*, then* $\mathrm{MLdeg}(\mathrm{Seg}_V(m,n)) = \mathrm{MLdeg}(\mathrm{Seg}_W(m,n))$.

One can also ask what values of MLdeg are achievable. It is possible to show that $\binom{n+m-2}{m-1} = \deg(\mathrm{Seg}(m,n))$

Theorem 16.6. *Let* $m \in \{2,3,4\}$*. Then, for all* $1 \leqslant k \leqslant \binom{n+m-2}{m-1} = \deg(\mathrm{Seg}(m,n))$*, there exists* $W \in (\mathbb{R}^\times)^{mn}$ *such that* $\mathrm{MLdeg}(\mathrm{Seg}_W(m,n)) = k$.

This is expected to be true for more general m, but the speaker does not have a proof at the moment.

16.4. **Ideas of the proofs.** Let $f_W = \sum_{j=1}^n \sum_{i=1}^m w_{ij}x_iy_j$, and let ∇_A be the closure of the set of $W \in (\mathbb{C}^\times)^{mn}$ such that there exists $(x,y) \in (\mathbb{C}^\times)^m \times (\mathbb{C}^\times)^n$ with $f_W(x,y) = 0$ and

$$\frac{\partial f_W}{\partial x_i} = \frac{\partial f_W}{\partial y_j} = 0$$

for all i,j. If ∇_A has codimension one, the irreducible polynomial $\Delta(f_W)$ is the A-*discriminant* of Gelfand-Kapranov-Zelevinsky.

The *principal* A-*determinant* is the product $E(W) = \prod_\Gamma \Delta_{\Gamma\cap(\Delta_{m-1}\times\Delta_{n-1})}$, where the product runs over faces of $\Delta_{m-1} \times \Delta_{n-1}$. This can also be computed as a product over all minors of all sizes of W. The proofs use a recent theorem relating this principal A-determinant to the problem at hand (but I missed the statement).

Another important idea is the following:

Theorem 16.7 (Huh)**.** *If a* d*-dimensional variety* $X \setminus H$ *is smooth (for* $H = \{(\prod p_{ij})(\sum p_{ij}) = 0\}$ *is smooth, then* $\mathrm{MLdeg}(X) = (-1)^d\chi(X \setminus H)$.

Using this, we can reduce the problems at hand to clever Euler characteristic computations. This turns the ML degrees appearing here into matroid invariants ("β-invariants").

17. 11/26b (Dustin Ross) – Ehrhart Fans

This is based on ongoing work with Melody Chan, Emily Clader, and Carly Kilvans.

17.1. **Motivation.** Let X be a smooth complex algebraic variety. The K-ring $K(X)$ is the free abelian group generated by vector bundles $\mathcal{E}$ on X, modulo relations $[\mathcal{F}] = [\mathcal{E}] + [\mathcal{G}]$ whenever we have a short exact sequence

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow 0 \ ,$$

together with multiplication via tensor product.

If X is complete, there exists an Euler characteristic $\chi : K(X) \to \mathbb{Z}$ given by $[\mathcal{E}] \mapsto \sum_i (-1)^i h^i(X, \mathcal{E})$. When X is not complete, this usually doesn't exist (dimensions are typically infinite-dimensional).

Let's consider some particularly nice cases.

(1) Suppose $X = X_\Sigma$ is a projective toric variety. Then $K(X)$ is spanned by ample line bundles $\mathcal{L}$, each corresponding to a lattice polytope $P_\mathcal{L}$. The Euler characteristic $\chi([\mathcal{L}])$ is equal to the number of lattice points of $P_\mathcal{L}$, relating algebraic geometry and Ehrhart theory.
(2) Suppose X is a wonderful compactification. Recent work of Larson-Li-Payne-Proudfoot computes the K-rings and Euler characteristics in terms of matroids associated to the hyperplane arrangement determining X. This can be used to construct K-rings and Euler characteristics of abstract matroids. One notes that $K(M) = K(X_{\Sigma_M})$, where M is the (typically incomplete) *Bergman fan* of the matroid.

We'd like to unify these and explain why there's an "Euler characteristic" on X_{Σ_M} even when Σ_M is not complete. Ultimately, we will define "Ehrhart fans," a class of unimodular fans containing both smooth projective fans and matroid fans. One can define a suitable "Euler characteristic" for the K-rings of the corresponding toric varieties.

17.2. **Definition of Ehrhart fans.** Let $\Sigma \subset \mathbb{R}^n$ be a pure unimodular fan. Here "pure" means that the maximal cones in Σ all dimension equal to $\dim(\Sigma)$ (which is the dimension of the support of Σ, not necessarily n). "Unimodular" means that the minimal generators of each cone can be extended to a basis of $\mathbb{Z}^n$, or equivalently that the corresponding toric variety is smooth.

Definition 17.1. A function $f : |\Sigma| \to \mathbb{R}$ is *integral piecewise-linear* if, for all $\sigma \in \Sigma$, there exists a linear function $\ell_\sigma : \mathbb{R}^n \to \mathbb{R}$, taking integer values on ray generators of σ, such that $f|_\sigma = \ell_\sigma|_\sigma$. These form a group $\mathrm{PL}(\Sigma) \cong \mathbb{Z}^{\Sigma(1)}$. We let $\underline{\mathrm{PL}}(\Sigma)$ be the quotient of $\mathrm{PL}(\Sigma)$ by the linear functions.

There are two useful ways to think of PL functions:

(1) There is an equivalence $\underline{\mathrm{PL}}(\Sigma) \cong \mathrm{Pic}(X_\Sigma) \subset K(X_\Sigma)$.
(2) If Σ is complete and f is convex, then f determines a polytope

$$P_f = \{v \in \mathbb{R}^n \,|\, u_\rho \cdot v \leqslant f(u_\rho) \text{ for all } \rho \in \Sigma(1)\}.$$

Counting the lattice points of P_f gives the Euler characteristic of the corresponding line bundle. Using elements of the quotient $\underline{\mathrm{PL}}(\Sigma)$ instead of $\mathrm{PL}(\Sigma)$ just corresponds to fixing the polytope modulo translation – in particular, the number of lattice points is left unchanged.

We'd like to adapt the latter picture to work (and give the correct answer!) for incomplete fans. One can think of lattice point counting as a recursive process: by truncating a polytope along one face, we get

$$\chi_\Sigma([f]) = \chi_\Sigma([f - \delta_\rho]) + \chi_{\Sigma^\rho}([f]^\rho)$$

Here Σ^ρ is the *star fan.* We'll use this as a definition.

Definition 17.2. A fan Σ is *Ehrhart* if all star fans Σ^ρ are Ehrhart and there exists a well-defined $\chi_\Sigma : \underline{\mathrm{PL}}(\Sigma) \to \mathbb{Z}$ such that:

(1) $\chi_\Sigma(0) = 1$
(2) $\chi_\Sigma([f]) = \chi_\Sigma([f - \delta_\rho]) + \chi_{\Sigma^\rho}([f]^\rho)$ for all $f \in \mathrm{PL}(\Sigma)$ and all $\rho \in \Sigma(1)$.

17.3. **Results.** Ehrhart fans are easy to understand in low dimensions.

Example 17.3. If $\dim \Sigma = 0$, then $\underline{\mathrm{PL}(\Sigma)} = 0$ and Σ is Ehrhart with $\chi_\Sigma = 1$.

Example 17.4. If $\dim \Sigma = 1$ and Σ is Ehrhart, then $\chi_\Sigma([f])$ must be $1 + \sum_\rho f(u_\rho)$. In order for χ_Σ to descend to $\underline{\mathrm{PL}}(\Sigma)$, we must have $\sum_\rho u_\rho = 0$.

In higher dimensions, we can say that all Ehrhart fans are tropical with uniform weights. Much as with tropical geometry, we can say that the Ehrhart condition does not depend on the fan structure:

Theorem 17.5 (CCKR)**.** *If $|\Sigma_1| = |\Sigma_2|$, then Σ_1 is Ehrhart if and only if Σ_2 is Ehrhart. Furthermore, if f is PL on both Σ_1 and Σ_2, then $\chi_{\Sigma_1}([f]) = \chi_{\Sigma_2}([f])$.*

Corollary 17.6. *If Σ is Ehrhart, there exists a canonical "Euler characteristic" $\chi_\Sigma : K(X_\Sigma) \to \mathbb{Z}$.*

Theorem 17.7 (CCKR/LLKP)**.** *Bergman fans of matroids are Ehrhart.*

Corollary 17.8. *All fans supported on Bergman fans are Ehrhart. In particular, complete fans are Ehrhart.*

The speaker expects that the Euler characteristics mentioned here can be interpreted in terms of "tropical ranks," but this is still work in progress.

18. 12/3 (Dawei Chen) – Geometry of Subcanonical Points

This was a slide talk, so I did not take notes.

19. 12/10a (Adam Boocher) – From Classical Commutative Algebra to Some Diophantine Equations

This is based on joint work with Noah Huang and Harrison Wolf.

Throughout the talk, we will let $R = k[x_1, \dots, x_n]$ and M be a finitely generated graded R-module of codimension c. (We are most interested in the case $M = R/I$.) If

$$0 \longrightarrow F_n \longrightarrow \dots \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

is a minimal free resolution of M, we define the ith Betti number of M as $\beta_i(M) = \operatorname{rk} F_i$. There are two main questions that interest us:

Conjecture 19.1.

$$\beta_i(M) \geqslant \binom{c}{i}.$$

Conjecture 19.2. *Either* $\sum_i \beta_i(M) = 2^c$ *or* $\sum_i \beta_i(M) \geqslant \left(\frac{3}{2}\right) 2^c$.

19.1. **Status of the conjectures.** The first conjecture is open for $c \geqslant 5$. It is known to be true when I is generated by a regular sequence: the Koszul complex gives

$$\beta_i(R/I) = \binom{c}{i}.$$

For a general ideal I, it was conjectured by Buchsbaum and Eisenbud that all resolutions of R/I have dga structures. This would imply the first conjecture for $M = R/I$ (known in this case as the Buchsbaum-Eisenbud-Horrocks conjecture). Unfortunately, the Buchsbaum-Eisenbud conjecture turned out to be false.

The first conjecture also refines a known recent result of Walker and VandeBogert:

$$\sum_i \beta_i(R/I) \geqslant 2^c.$$

The second conjecture is known for:

1. R/I with I monomial,
2. $c \geqslant 4$,
3. I monomial, and
4. M with "low" regularity.

19.2. **The project.** What happens for pure modules?

Definition 19.3. A Cohen-Macaulay module M of codimension c is *pure* of type $D = \{0 < d_1 < \dots < d_c\} \subset \mathbb{Z}$ if, for all i, the ith syzygy module of M is generated in a single degree d_i.

Note that, even if I is generated in some fixed degree, the same need not be true for the syzygies of R/I. That is, R/I need not be pure.

Here are some facts about pure modules:

1. (Eisenbud-Schreyer) Pure modules exist for any D.
2. Pure modules correspond to extremal rays in the semigroup / polyhedral cone of Betti tables.

(3) (Herzog-Kühl) If M is pure of type D, then for $i \geqslant 1$,

$$\beta_i(M) = \beta_0(M) \prod_{i \neq j} \frac{d_j}{|d_i - d_j|}.$$

Example 19.4. Let $D = \{0, 2, 5, 6\}$. Suppose $\beta_0 = 2N$. Then $\beta_1 = 5N$, $\beta_2 = 8N$, and $\beta_3 = 5N$. Thus, regardless of N, the first conjecture must be true for pure modules of type D.

We can ask whether this argument works more generally. Given D, let $B(D)$ be the list of numbers obtained by clearing denominators across all $\prod_{i \neq j} \frac{d_j}{|d_i - d_j|}$ (so in the above example, $B(D) = (2, 5, 8, 5)$). The list $B(D)$ can be used to look for potential counterexamples to the conjecture. However, note that the actual Betti numbers β_i may have to be scalar multiples of the elements B_i of $B(D)$. Certain results in commutative algebra (the Krull altitude theorem, Buchsbaum-Rim syzygies, the Walker-VandeBogert sum theorem, etc.) give constraints on the β_i and control which multiples we may have to take.

We can also understand $B(D)$ using Diophantine equations.

Theorem 19.5 (Boocher-Huang-Wolf). *For $c = 3$, there are infinitely many nondegenerate D such that $B_1 < \binom{3}{1}$. These are all of the form $(0, a_n, b_n, b_n + 1)$ where the values of a_n and b_n are constructed by a certain recursion relation.*

Theorem 19.6 (Boocher-Huang-Wolf). *For $c = 3$, there are three nondegenerate D such that $2^3 < \sum_i B_i < 2^3 + 2^2$.*

The behavior in higher codimension is still under investigation.

20. 12/10b (Tejas Rao) – Primality Testing and Elliptic Curves

Suppose we are given an integer N. How do we tell whether N is prime?

We'll start by discussing the classical approach to primality testing. Then we'll move on to talk about elliptic curves and complex multiplication. Finally, we'll discuss Atkin-Morain's ECPP approach.

20.1. **The classical story.** The most naïve approach is as follows: take all integers between 1 and N (exclusive) and see if any divide N. This method takes time $O(N)$. Actually, we only need to check integers up to $\sqrt{N}$, so the runtime can be reduced to $O(\sqrt{N})$. This is still quite large.

Lucas had the idea to verify primality by "constructing a group G so large that N must be prime." This principle is illustrated by the following.

Theorem 20.1 (Proth's test). *Suppose $N - 1 = FM$ where $F > \sqrt{N}$. For every prime factor q of F, suppose we can find a_q such that $a_q^{N-1} \equiv 1 \mod N$ and $\gcd(a_q^{(N-1)/q} - 1, N) = 1$. Then N is prime.*

Proof. Suppose a prime $p \leqslant \sqrt{N}$ divides N. Lagrange's theorem applied to the cyclic subgroup of $(\mathbb{Z}/p\mathbb{Z})^\times$ generated by $[a_q^{(N-1)/q}]$ (for varying q) shows that $F | p - 1$. Thus $p > \sqrt{N}$, giving a contradiction. □

Unfortunately, Proth's test almost never works in practice.

20.2. **Elliptic curves.** Recall that an elliptic curve E (over a field k) is a smooth projective curve of genus 1 together with a distinguished point 0. One can make E into an abelian group scheme – this is most easily shown (at the level of k-points) by using the equivalence $E(k) \cong \mathrm{Div}^0 E(k)$ given by $P \mapsto [P] - [0]$.

We'd like to understand the structure of $\mathrm{End}(E)$. There is always a map $\mathbb{Z} \to \mathrm{End}(E)$ given by sending n to the nth power map $[n]$ on E. If $k = \mathbb{F}_p$, there is also an endomorphism $\pi \in \mathrm{End}(E)$ given by the Frobenius. We'd like to understand the relationship between π and this copy of $\mathbb{Z}$.

For an ordinary (i.e. with nontrivial p-torsion) elliptic curve, note that $E[\ell] \cong \mathbb{Z}/\ell Z \times \mathbb{Z}/\ell\mathbb{Z}$. Given $A \in \mathrm{End}(E)$, we may consider the action $A \curvearrowright E[\ell]$. Using the Cayley-Hamilton theorem for this action, we see that

$$\pi = \frac{t \pm \sqrt{t^2 - 4p}}{2} \in \mathcal{O}_K \text{ for } K = \mathbb{Q}[\sqrt{-D}]$$

where $t = \operatorname{tr} \pi$. Thus we get $\mathbb{Z} \subset \mathcal{O}_K \subset \mathrm{End}(E(\mathbb{F}_p))$. In particular, we get a map $\mathcal{O}_K \to E(\mathbb{F}_p)$ via evaluation at 0. The behavior of this map reflects the properties of the elliptic curve.

We may use this to implement Lucas's idea above by constructing various elliptic curves $E_i(p)$ and using these to test primality.

20.3. **Primality testing.** Let's start with an input number N. To check that N is prime, suppose not – then there exists $\mathsf{p} \leqslant \sqrt{\mathsf{N}}$ dividing N. We construct an elliptic curve E with Frobenius π satisfying $\pi\overline{\pi} = \mathsf{N}$ and $(\pi - 1)(\overline{\pi} - 1) = 2\mathsf{N}_2$ (this step is nontrivial). Assume N_2 is prime. If $\mathsf{Q} \neq 0 \in \mathbb{E}(\mathbb{F}_\mathsf{p})$ and $[\mathsf{N}_2]\mathsf{Q} = 0$ as endomorphisms of $\mathsf{E}(\mathbb{F}_\mathsf{p})$, we see that N_2 divides the order of $\mathsf{E}(\mathbb{F}_\mathsf{p})$. But this order is $(\pi_\mathsf{p} - 1)(\overline{\pi}_\mathsf{p} - 1)$, where $\mathsf{p} = \pi_\mathsf{p}\overline{\pi}_\mathsf{p}$. This produces a contradiction.