COMMUTATIVE ALGEBRA AND ALGEBRAIC GEOMETRY SEMINAR

NOTES BY JOHN S. NOLAN, SPEAKERS LISTED BELOW

Abstract. These are notes from the Berkeley Commutative Algebra and Algebraic Geometry Seminar.

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1. 8/27

I missed today's talks. If you have notes and would like to share them, please let me know!

2. 9/3

I missed today's talks. If you have notes and would like to share them, please let me know!

3. 9/17a (Peter Haine) – Reconstructing Schemes from their Étale Topoi

This is based on joint work with M. Carlson and S. Wolf centered around a conjecture in Grothendieck's anabelian letter to Faltings.

3.1. **Grothendieck's conjecture.** For a scheme X, let $\mathsf{X}_{\text{ét}}$ be the étale topos of X (i.e. $\mathsf{Sh}(\text{Ét}_{\mathsf{X}}, \mathsf{Set})$). We would like to understand when X can be recovered from $\mathsf{X}_{\text{ét}}$. One motivation is to try to understand the following theorem:

Theorem 3.1 (Neukirch-Uchida 1969, Pop 1994)**.** *If K and L are infinite fields of the same characteristic that are finitely generated over their prime fields, then there is a bijection between:*

- *Isomorphisms* $\mathsf{K} \xrightarrow{\sim} \mathsf{L}$
- *Equivalences of categories* $(\operatorname{Spec} \mathsf{L})_{\text{ét}} \xrightarrow{\sim} (\operatorname{Spec} \mathsf{K})_{\text{ét}}$ *up to conjugacy.*

Classical statements of this involve the absolute Galois groups, though the statement about étale topoi is equivalent. It is necessary to assume the fields are infinite: for finite fields, the absolute Galois groups are always $\hat{\mathbb{Z}}$.

We'll work over a field k for simplicity. There are four main issues that arise when trying to reconstruct a scheme from its étale topos:

(1) If $\mathsf{L} \supset \mathsf{k}$ is an extension of separably closed fields, then $(\operatorname{Spec} \mathsf{L})_{\text{ét}} \simeq (\operatorname{Spec} \mathsf{k})_{\text{ét}}$. Thus we must restrict to finite type schemes.

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(2) If k is an algebraically closed field of characteristic zero and X and Y are smooth proper curves over k, then $\mathsf{X}_{\text{ét}} \simeq \mathsf{Y}_{\text{ét}}$ iff $\mathsf{g}(\mathsf{X}) = \mathsf{g}(\mathsf{Y})$. Thus we must restrict to "small" fields k. Grothendieck suggests that we fix this by only considering k which are finitely generated over their prime field.

(3) If $\mathsf{f} : \mathsf{X} \to \mathsf{Y}$ is a universal homeomorphism, then $\mathsf{X} \times_{\mathsf{Y}} (-) : \text{Ét}_{\mathsf{Y}} \to \text{Ét}_{\mathsf{X}}$ is an equivalence of categories. (Examples include the normalization of the cuspidal cubic, the absolute Frobenius are both universal, and any nil-immersion.) Thus we must invert universal homeomorphisms.

(4) A fourth subtle point involving constructibility.

Let's explain this fourth point. We must start with a small amount of topos theory.

Definition 3.2. Given topoi $\mathcal{X}$ and $\mathcal{Y}$, a *geometric morphism* is a functor $\mathsf{f}_* : \mathcal{X} \to \mathcal{Y}$ with a right adjoint $\mathsf{f}^* : \mathcal{Y} \to \mathcal{X}$ such that f^* preserves geometric morphisms.

Every topos $\mathcal{X}$ has an associated topological space $|\mathcal{X}|$, and $|{-}|$ is functorial in geometric morphisms. For a scheme X, we have $|\mathsf{X}_{\text{ét}}| = |\mathsf{X}|$. Furthermore, if T is a sober topological space, then $|\mathsf{Sh}(\mathsf{T})| \simeq \mathsf{T}$.

Knowing this, we can now state our final condition. If $\mathsf{f} : \mathsf{X} \to \mathsf{Y}$ is a morphism of schemes locally of finite type over a field, then f must send closed points to closed points. This is not true for general geometric morphisms, so we must require it as an extra condition.

Definition 3.3. A geometric morphism $\mathsf{f}_* : \mathcal{X} \to \mathcal{Y}$ is *pinned* if $|\mathsf{f}_*| : |\mathcal{X}| \to |\mathcal{Y}|$ sends closed points to closed points.

One last thing: in the Neukirch-Uchida theorem, we quotiented out by conjugation. We'd like to know that this doesn't really affect anything (i.e. that we shouldn't really be doing something "stacky"). Thankfully, this is true:

Proposition 3.4. *Let k be a field, and let X and Y be finite type k-schemes. Then the groupoid $\mathrm{Hom}_{\mathsf{k}}^{\mathrm{pin}}(\mathsf{X}, \mathsf{Y})$ of pinned geometric morphisms $\mathsf{X}_{\text{ét}} \to \mathsf{Y}_{\text{ét}}$ over $(\operatorname{Spec} \mathsf{k})_{\text{ét}}$ is equivalent to a set.*

Now we can state Grothendieck's conjecture:

Conjecture 3.5 (Grothendieck 1983)**.** *If k is a finitely generated field, then taking étale topoi gives an equivalence between:*

- $\mathsf{Sch}_{\mathsf{k}}^{\mathsf{ft}}[\mathsf{UH}^{-1}]$*, the category of finite type k-schemes with universal homeomorphisms inverted.*
- *The category of topoi over $(\operatorname{Spec} \mathsf{k})_{\text{ét}}$ and pinned geometric morphisms.*

Theorem 3.6 (CHW)**.** *The conjecture is true if k is infinite.*

At present, there is not a complete characterization of the image of $(-)_{\text{ét}}$.

3.2. **Inverting universal homeomorphisms.** We ran into a few issues with universal homeomorphisms. One was the existence of resolutions of cuspidal cubics. Another was the existence of the absolute Frobenius.

Definition 3.7. A ring A is:

(1) *seminormal* if, whenever $\mathsf{x}^2 = \mathsf{y}^3$, there exists $\mathsf{a} \in \mathsf{A}$ such that $\mathsf{x} = \mathsf{a}^2$ and $\mathsf{y} = \mathsf{a}^2$.

(2) *absolutely weakly normal* (or *awn*) if A is seminormal and, for all primes ℓ and equations $\ell^\ell \mathsf{x} = \mathsf{y}^\ell$, there exists $\mathsf{a} \in \mathsf{A}$ such that $\mathsf{x} = \mathsf{a}^\ell$ and $\mathsf{y} = \ell \mathsf{a}$.

These properties can be extended to general schemes (where we require that they hold affine-locally).

Theorem 3.8 (Traverso, Swan)**.** *A ring A is seminormal if and only if the inclusion $\mathsf{A} \hookrightarrow \mathsf{A}[\mathsf{t}]$ induces an isomorphism $\mathsf{Pic}\mathsf{A} \xrightarrow{\sim} \mathsf{Pic}\mathsf{A}[\mathsf{t}]$.*

Theorem 3.9. *(1) The inclusion $\mathsf{Sch}^{\mathrm{awn}} \hookrightarrow \mathsf{Sch}$ admits a right adjoint $(-)^{\mathrm{awn}}$ (known as* absolute weak normalization*). Moreover, $(-)^{\mathrm{awn}}$ induces $\mathsf{Sch}[\mathsf{UH}^{-1}] \xrightarrow{\sim} \mathsf{Sch}^{\mathrm{awn}}$.*

(2) The inclusion $\mathsf{Sch}^{\mathrm{sn}} \hookrightarrow \mathsf{Sch}$ admits a right adjoint $(-)^{\mathrm{sn}}$ (known as seminormalization*). This identifies $\mathsf{Sch}^{\mathrm{sn}}$ with the category obtained from Sch by inverting universal homeomorphisms that induce isomorphisms on residue fields.*

(3) A $\mathbb{Q}$-scheme is awn if and only if it is seminormal.

(4) A $\mathbb{F}_{\mathsf{p}}$-scheme is awn if and only if it is perfect.

In general, seminormalization / absolute weak normalization leaves the land of finite type schemes. We must allow for this!

Definition 3.10. A k-scheme X is topologically of finite type if $X \to \operatorname{Spec} k$ factors as $X \to X' \to \operatorname{Spec} k$ where $X \to X'$ is a universal homeomorphism and $X' \to \operatorname{Spec} k$ is of finite type.

Grothendieck's conjecture is then equivalent to

Conjecture 3.11. *For X and Y topologically of finite type over k with X awn, then*

$$\operatorname{Hom}_k(X, Y) \simeq \operatorname{Hom}_k^{\text{pin}}(X_{\text{ét}}, Y_{\text{ét}}).$$

How do we prove this? Since we can reconstruct the underlying space from the étale topos, it suffices to reconstruct the structure sheaf. This can be rephrased as constructing the sets of morphisms to $\mathbb{A}^1$. Using the fact that both sides satisfy h-descent together with the theory of alterations, we reduce to the case where $X = (X')^{\text{awn}}$ for X' regular and finite type. From here, we can reduce to the case where $Y = \mathbb{G}_m$. This lets us understand things cohomologically! We end up asking questions about Picard groups.

Theorem 3.12 (Guralnik-Jaffe-Roskind-Wiegland)**.** *If k is a finitely generated field and X is normal and of finite type over k, then $\mathsf{Pic} X$ is finitely generated.*

This is not true for seminormal schemes in general.

Example 3.13. If $X \subset \mathbb{P}^2_k$ is a nodal cubic, then $\mathsf{Pic} X \simeq k^\times \oplus \mathbb{Z}$.

Question 3.14. *For X a seminormal scheme of finite type over a finitely generated field of characteristic zero:*

(1) Does $\mathsf{Pic}(X)$ have any nontrivial infinitely divisible elements?
(2) Same as (1) but for torsion elements.
(3) Is the Tate module $\mathsf{T}(\mathsf{Pic}(X)) = \lim_n \mathsf{Pic}(X)[n]$ zero?

4. 9/17b (Hannah Larson) – Chow Rings of Moduli Spaces

4.1. **Chow groups and Chow rings.** Let X be a smooth variety.

Definition 4.1. The *Chow group* in codimension i of X is the group of $\mathbb{Z}$-linear combinations of codimension i irreducible subvarieties of X modulo rational equivalence. Here *rational equivalence* is the equivalence relation generated by $Y_1 \simeq Y_2$ if there exists a family $Y \subset X \times \mathbb{P}^1$, flat over $\mathbb{P}^1$, such that Y_1 and Y_2 are fibers of this family over two points in $\mathbb{P}^1$. We write $[Y]$ for the equivalence class of the subvariety Y: this can be defined even for reducible subvarieties by $[Y \cup Z] = [Y] + [Z]$.

Definition 4.2. The *Chow ring* $A^*(X) = \oplus_i A^i(X)$ comes with an intersection product

$$A^i(X) \times A^j(X) \to A^{i+j}(X)$$

defined so that, if X and Y meet transversally, then $[Y] \cdot [Z] = [Y \cap Z]$.

4.2. **Line bundles and $A^*(\mathbb{P}^n)$.** Let $\mathcal{L}$ be a line bundle on X, and let σ be a rational section of $\mathcal{L}$.

Definition 4.3. Given σ, we can define the *divisor of zeroes* D_0 and the *divisor of poles* D_∞. The difference $c_1(\mathcal{L}) := [D_0] - [D_\infty] \in A^1(X)$ does not depend on the choice of σ. We call this the *first Chern class* of $\mathcal{L}$.

Example 4.4. Let $X = \mathbb{P}^2$ and $\mathcal{L} = \mathcal{O}_{\mathbb{P}^2}(1)$. Taking $\sigma = x_0$, we obtain $D_0 = V(x_0)$ and $D_\infty = \varnothing$, so $c_1(\mathcal{L}) = [V(x_0)]$. An alternate choice would be to take $\sigma' = x_1$, which would give $[V(x_1)]$. Well-definedness of $c_1(\mathcal{L})$ forces $[V(x_0)] = [V(x_1)]$.

We can see this directly by considering the graph of the rational map $X \to \mathbb{P}^1$, $[x_0 : x_1 : x_2] \mapsto [x_0 : x_1]$. Taking the closure of the graph gives a rational equivalence between $V(x_0)$ and $V(x_1)$. Note that this graph closure is nothing but the blowup of $\mathbb{P}^1$ at $[1 : 0 : 0]$ – this observation is more generally.

The class $\zeta = c_1(\mathcal{O}(1)) \in A^1(\mathbb{P}^n)$ is the first example of a "tautological class." In fact, $A^*(\mathbb{P}^n) \cong \mathbb{Z}[\zeta]/(\zeta^{n+1})$, and ζ^i is the class of a codimension i hyperplane. In this case, the "obvious" classes generate the Chow ring, and the "obvious" relations are all of the relations.

We can obtain Bézout's theorem from this result rather easily.

Theorem 4.5 (Bézout)**.** *If C and C' are curves in $\mathbb{P}^2$ of degrees d and d' which meet transversally, then* $[C] \cdot [C'] = dd'[\text{pt}]$.

4.3. **More properties of Chow rings.**
Chow rings can be defined in the non-smooth case.

Theorem 4.6 (Excision). *If $Z \subset X$ is a closed subvariety of codimension c and $U = X \setminus Z$, then there is an exact sequence*

$$A^{*-c}(Z) \longrightarrow A^*(X) \longrightarrow A^*(U) \longrightarrow 0.$$

We can also pushforward cycles along proper maps. If $f : X \to Y$ is proper, then $f_* : A^i(X) \to A^{i+\dim Y - \dim X}(Y)$ is defined by

$$f_*[Z] = \begin{cases} \deg f|_Z \cdot [f(Z)] & f|_Z \text{ finite} \\ 0 & \text{otherwise} \end{cases}$$

4.4. **Moduli spaces of curves.**
How can we extend things to moduli spaces of curves? Let $\mathcal{M}$ be the moduli space of curves, and let $f : \mathcal{C} \to \mathcal{M}$ be the universal curve. Consider the relative dualizing sheaf ω_f (so $\omega_f|_p = (T_p C)^\vee$). We obtain $c_1(\omega_f) \in A^1(\mathcal{C})$ and $f_* c_1(\omega_f) \in A^0(\mathcal{M}_g)$. This gives a "tautological class" on $\mathcal{M}$.

In fact, we can work a bit more generally: let

$$\kappa_i := f_* c_1(\omega_f)^{i+1} \in A^i(\mathcal{M}_g).$$

These are all "tautological classes."

Definition 4.7. The subring of $A^*(\mathcal{M}_g)$ generated by the κ_i is the *tautological ring*. We write $R^*(\mathcal{M}_g)$ for this ring.

From now on, we will work rationally, replacing A^* by $A^* \otimes \mathbb{Q}$.

Important questions:

- Does $R^*(\mathcal{M}_g) = A^*(\mathcal{M}_g)$?
- What are the relations among the κ_i? What is $R^*(\mathcal{M}_g)$?

It is known (by work of several authors) that $R^*(\mathcal{M}_g) = A^*(\mathcal{M}_g)$ for $g \leqslant 9$. However, it is known that $R^*(\mathcal{M}_g) \neq A^*(\mathcal{M}_g)$ for $g = 12$ or $g \geqslant 16$. A heuristic explanation for this is that the lower genus moduli spaces are "more rational" than the higher genus moduli spaces.

The non-tautological class in $\mathcal{M}_{12}$ is the bielliptic locus $[B_g]$, i.e. the locus of curves which admit a degree 2 map to an elliptic curve. One can show that $[\overline{B}_g]$ is not tautological in $A^*(\overline{\mathcal{M}}_g)$ for $g \geqslant 12$. The non-tautological classes in $\mathcal{M}_{16}$ are constructed as similar "Hurwitz loci."

It is not known whether $A^*(\mathcal{M}_g)$ is finitely generated in general. The speaker expects the answer is "no" in high genus.

5. 9/24 (Christian Gaetz) – Combinatorics of Singularities of Schubert Varieties and Torus Orbit Closures Therein

Let G be a semisimple connected algebraic group over $\mathbb{C}$. Most of the setup will work for arbitrary type, but the results will be for groups with simply-laced / ADE Dynkin diagram.

5.1. **Definitions and basic theory.**
Let B be a Borel subgroup of G, i.e. a maximal connected closed solvable subgroup of G. We write $T \subset B$ for the maximal torus and $W = N_G(T)/T$ for the Weyl group. For the matrix groups, we may take B to consist of the upper triangular matrices in G and T to consist of the diagonal matrices in G.

The Weyl group indexes the *Bruhat decomposition* $G = \sqcup_{w \in W} BwB$. In the flag variety G/B, the subspaces $BwB/B \cong \mathbb{C}^{\ell(w)}$ are the *Schubert cells*. The closures $X_w = \overline{BwB/B}$ are *Schubert varieties*.

These are classical, hard, and very useful. For example, they can be used to understand the cohomology of flag varieties: this is the subject of *Schubert calculus*. We can also relate these to problems in Grassmannians by projecting from flag varieties to Grassmannians. Schubert calculus in Grassmannians is well-understood (via e.g. Littlewood-Richardson rules), but Schubert calculus in Grassmannians is still an active area of research. We won't focus on Schubert calculus in this lecture.

Note that $X_w = \sqcup_{v \leqslant w} BwB/B$, where $\leqslant$ denotes the Bruhat order.

Example 5.1. For $G = SL_n$, we have $W \cong S_n$. We can explicitly describe the Bruhat order on S_3: writing permutations in one-line notation, we say that $v \leqslant w$ if the one-line notation for v has more numbers located in the usual order than w.

5.2. **Kazhdan-Lusztig polynomials.** The *Kazhdan-Lusztig polynomials* are $P_{vw}(q) \in \mathbb{N}[q]$ which admit the following descriptions:

- Generating functions of intersection cohomology: $P_{vw}(q) = \sum_i q^i \dim IH_v^{2i}(X_w)$ (where IH_v measures singularities at v). In the simply laced case, $P_{vw}(q) = 1$ if X_w is smooth at v. In the non-simply laced case, "smooth" is replaced by "rationally smooth."
- Generating functions of Lie algebra Ext groups: $\sum_i q^i \operatorname{Ext}_{\mathfrak{g}}^{\ell(w)-\ell(v)-i}(M_v, L_w)$ (where M_v is a Verma module and L_w is an irrep).

Proving the equivalence between these started the field of modern geometric representation theory. The equivalence is a deep theorem.

We have $\deg P_{vw} \leqslant \frac{1}{2}(\ell(w) - \ell(v) - 1)$.

The Kazhdan-Lusztig polynomials can be computed via a complicated recurrence relation. This recurrence relation contains many signs, and we don't know how to prove the nonnegativity of the polynomials from this recurrence relation.

It's impossible to control the behavior of Kazhdan-Lusztig polynomials in general (cf. a theorem of Polo ?? saying that every polynomial with constant term 1 and $\mathbb{N}$ coefficients arises as a Kazhdan-Lusztig polynomial). As a result, we'll restrict ourselves to special cases.

5.3. **A conjecture of Billey and Postnikov.**

Conjecture 5.2 (Billey-Postnikov)**.** *Suppose X_w is singular, and write $P_{ew}(q) = 1 + cq^{h(w)} + \dots$. If G is simply laced of rank r, then $h(w) \leqslant r - 2$.*[1]

This conjecture is somewhat surprising, as in general we only have $\deg P_{ew} \leqslant O(r^2)$. The conjecture forces the first nonzero term that appears to have rank growing linearly in $\operatorname{rk} G$.

Theorem 5.3 (Björner-Ekedahl)**.** *With the above notation, $h(w)$ is the minimal i such that $h^{2i}(X_w) \neq h^{2(\ell(w)-i)}(X_w)$, i.e. the first dimension in which Poincaré duality fails.*

We can understand this using "patterns."

Definition 5.4. Say a permutation $w \in S_n$ contains $\sigma \in S_k$ as a *pattern* if there exist $1 \leqslant i_1 < \cdots < i_k < n$ such that $w(i_a) < w(i_b)$ if and only if $\sigma(a) < \sigma(b)$. Otherwise we say w *avoids* σ.

Example 5.5. The permutation $w = 45312 \in S_5$ contains $\sigma = 3412 \in S_4$ as a pattern (via the subpermutation 4512).

Theorem 5.6 (Lakshmibai-Sandhya)**.** *The Schubert variety X_w is smooth if and only if w avoids 3412 and 4231.*

Avoiding patterns is hard to do in high rank! In particular, X_w is almost always singular.

5.4. **The theorem.**

Theorem 5.7 (Gaetz-Gao)**.** *Suppose X_w is singular. Then*

$$h(w) = \begin{cases} 1 & w \text{ contains } 4231 \\ \textit{minHeight}(w) & \textit{otherwise} \end{cases}$$

where

$$\operatorname{minHeight}(w) = \min\{w(i_1) - w(i_4) \mid w \text{ has } 3412 \text{ in positions } i_1, i_2, i_3, i_4\}.$$

For any w containing 3412, we have $\operatorname{minHeight}(w) \leqslant n - 3 = (n-1) - 2$. The Billey-Postnikov conjecture follows from this.

The proof is easier in the case where w contains 4231. Otherwise, w avoids 4231, and one considers projections $G/B \to G/P_J$. The images of X_w give Schubert varieties $X^J_{W^J} \subset G/P_j$. The projection maps $\pi : X_w \to X^J_{W^J}$ can be rather nasty, but by studying w carefully, one finds J such that π is a fiber bundle with fiber X_{W_J} (i.e. a smaller Schubert variety!). This allows one to understand Schubert varieties inductively.

[1]The original statement is a bit weaker.

6. 10/1a (Daigo Ito) – A New Proof of the Bondal-Orlov Reconstruction Theorem

This is based on joint work with Hiroki Matsui.

6.1. Background.
Let X be a projective variety over $\mathbb{C}$ (though this should mostly work over any algebraically closed field). We can construct a triangulated category of perfect complexes $\mathsf{Perf}\,\mathsf{X}$ consisting of bounded complexes of vector bundles (after inverting quasi-isomorphisms).

Question 6.1. *How much information about X is contained in $\mathsf{Perf}\,\mathsf{X}$? In particular, in which cases can we reconstruct X fully from $\mathsf{Perf}\,\mathsf{X}$?*

Theorem 6.2 (Bondal-Orlov, Ballard)**.** *Let X be Gorenstein, so the dualizing complex ω_{X} is a line bundle. If ω_{X} is ample or anti-ample, then:*
(1) We can reconstruct X from the triangulated category $\mathsf{Perf}\,\mathsf{X}$.
(2) If $\mathsf{Perf}\,\mathsf{X} \simeq \mathsf{Perf}\,\mathsf{Y}$ for Y projective and Gorenstein, then $\mathsf{X} \cong \mathsf{Y}$.[2]

The Fano / anti-Fano assumptions here are rather strong. Another method of reconstructing X is as follows.

6.2. Balmer spectra.

Theorem 6.3 (Balmer)**.** *Let X be a variety (more generally, a qcqs scheme). Then we can reconstruct X from $(\mathsf{Perf}\,\mathsf{X}, \otimes^{\mathsf{L}}_{\mathcal{O}_{\mathsf{X}}})$.*

More precisely, for any tensor-triangulated (tt) category $(\mathcal{T}, \otimes)$, we can construct a ringed space $\operatorname{Spec}_{\otimes} \mathcal{T}$. Taking $(\mathcal{T}, \otimes) \cong (\mathsf{Perf}\,\mathsf{X}, \otimes_{\mathsf{X}})$, we obtain $\operatorname{Spec}_{\otimes} \mathcal{T} \cong \mathsf{X}$.

It is natural to ask if we can perform Balmer's construction without having the tensor structure when ω_{X} is anti-ample.

Definition 6.4 (Balmer)**.** Let $(\mathcal{T}, \otimes)$ be a tt-category. The *Balmer spectrum* $\operatorname{Spec}_{\otimes} \mathcal{T}$ is defined as follows. As a set, $\operatorname{Spec}_{\otimes} \mathcal{T}$ consists of *prime thick $\otimes$-ideals* $\mathcal{P} \subset \mathcal{T}$, meaning that:
(1) (Thick subcategory) $\mathcal{P}$ is a full triangulated subcategory of $\mathcal{T}$, such that if $\mathsf{X} \oplus \mathsf{Y} \in \mathcal{P}$, then $\mathsf{X} \in \mathcal{P}$ or $\mathsf{Y} \in \mathcal{P}$.
(2) ($\otimes$-ideal) If $\mathsf{X} \in \mathcal{P}$ and $\mathsf{Y} \in \mathcal{T}$, then $\mathsf{X} \otimes \mathsf{Y} \in \mathcal{P}$.
(3) (Prime) $\mathcal{P}$ is a proper $\otimes$-ideal such that, if $\mathsf{X} \otimes \mathsf{Y} \in \mathcal{P}$, then $\mathsf{X} \in \mathcal{P}$ or $\mathsf{Y} \in \mathcal{P}$.

We can equip this with a natural ringed space structure.

Example 6.5. If X is a qcqs scheme, then $\mathsf{X} \cong \operatorname{Spec}_{\otimes_{\mathcal{O}_{\mathsf{X}}}} \mathsf{Perf}\,\mathsf{X}$, where a (not necessarily closed) point $x \in \mathsf{X}$ corresponds to the prime

$$\mathsf{S}_{\mathsf{X}}(x) = \left\{ \mathcal{F} \in \mathsf{Perf}\,\mathsf{X} \,\middle|\, \mathcal{F}_x \simeq 0 \text{ in } \mathsf{Perf}(\mathcal{O}_{\mathsf{X},x}) \right\}$$

6.3. Matsui spectra.

Definition 6.6 (Matsui)**.** Let $\mathcal{T}$ be a triangulated category. The *Matsui spectrum* $\operatorname{Spec}_{\Delta} \mathcal{T}$ is defined as the set of prime thick subcategories $\mathcal{P}$, meaning that:
(1) $\mathcal{P}$ is a thick subcategory of $\mathcal{T}$.
(2) The collection $\{\mathcal{Q} \supsetneq \mathcal{P} \,|\, \mathcal{Q} \text{ thick}\}$ has a unique smallest element.

Example 6.7. If X is a curve and $x \in \mathsf{X}$ is a closed point, then $\mathsf{S}_{\mathsf{X}}(x) = \langle \kappa(y) \,|\, y \neq x \rangle$ is prime. The smallest thick subcategory containing $\mathsf{S}_{\mathsf{X}}(x)$ is obtained by adjoining $\kappa(x)$ to $\mathsf{S}_{\mathsf{X}}(x)$.

Proposition 6.8 (Matsui)**.** *If X is a noetherian scheme, then $\operatorname{Spec}_{\otimes_{\mathcal{O}_{\mathsf{X}}}} \mathsf{Perf}\,\mathsf{X} \subset \operatorname{Spec}_{\Delta} \mathsf{Perf}\,\mathsf{X}$ as sets.*

Remark 6.9. It is not known yet whether this extends to arbitrary qcqs schemes.

Theorem 6.10 (Ito-Matsui)**.** *If X is a quasiprojective scheme, then $(\operatorname{Spec}_{\otimes_{\mathcal{O}_{\mathsf{X}}}} \mathsf{X})_{\mathrm{red}} \subset \operatorname{Spec}_{\Delta} \mathsf{Perf}\,\mathsf{X}$ is an open immersion of ringed spaces.*

Remark 6.11. It is not known yet whether we can attach a natural structure sheaf to $\operatorname{Spec}_{\Delta} \mathsf{Perf}\,\mathsf{X}$ that "sees the nilpotents of X."

[2] This is needed to ensure that our reconstruction gives the correct result! Note that we do not assume Y is Fano / anti-Fano.

6.4. **Proof of Bondal-Orlov.** Assume ω_X is (anti)ample. By the theorem on open immersions above, it suffices to specify the correct open subspace of the Matsui spectrum.

Let $\mathbb{S}(-) = - \otimes \omega_X[\dim X]$ be the Serre functor of $\mathsf{Perf}\, X$. This gives an automorphism of $\mathsf{Perf}\, X$.

Lemma 6.12. *The Serre functor can be uniquely determined by the triangulated category structure of* $\mathsf{Perf}\, X$.

With this lemma, we can define the *Serre-invariant locus* $\operatorname{Spec}^{\mathrm{Ser}} \mathsf{Perf}\, X \subset \operatorname{Spec}_\Delta \mathsf{Perf}\, X$ as

$$\operatorname{Spec}^{\mathrm{Ser}} \mathsf{Perf}\, X = \left\{ \mathcal{P} \in \operatorname{Spec}_\Delta \mathsf{Perf}\, X \,\middle|\, \mathbb{S}(\mathcal{P}) = \mathcal{P} \right\}$$

We claim $\operatorname{Spec}_{\otimes_{\mathcal{O}_X}} \mathsf{Perf}\, X = \operatorname{Spec}^{\mathrm{Ser}} \mathsf{Perf}\, X$.

Example 6.13. For $X = \mathbb{P}^1$, we have

$$\operatorname{Spec}_\Delta \mathsf{Perf}\, \mathbb{P}^1 = \operatorname{Spec}_{\otimes_{\mathcal{O}_{\mathbb{P}^1}}} \sqcup \coprod_{i \in \mathbb{Z}} \langle \mathcal{O}_{\mathbb{P}^1}(i) \rangle,$$

and $\mathbb{S}$ fixes $\operatorname{Spec}_{\otimes_{\mathcal{O}_{\mathbb{P}^1}}}$ while acting freely on $\coprod_{i \in \mathbb{Z}} \langle \mathcal{O}_{\mathbb{P}^1}(i) \rangle$. Thus the claim holds in this case.

In general, it is clear that $\operatorname{Spec}^{\mathrm{Ser}} \mathsf{Perf}\, X \subset \operatorname{Spec}_{\otimes_{\mathcal{O}_X}} \mathsf{Perf}\, X$. For the reverse inclusion, suppose $\mathbb{S}(\mathcal{P}) = \mathcal{P}$. Then, for all $n \in \mathbb{Z}$, we have $\mathcal{P} \otimes \omega_X^{\otimes n} = \mathcal{P}$.

Theorem 6.14 (Orlov). *If* ω_X *is (anti)ample, then* $\langle \omega_X^{\otimes n} \mid n \in \mathbb{Z} \rangle = \mathsf{Perf}\, X$.

It follows that, for all $\mathcal{F} \in \mathsf{Perf}\, X$, we have $\mathcal{F} \otimes \mathcal{P} \subset \mathcal{P}$, i.e. $\mathcal{P}$ is a $\otimes$-ideal.

Theorem 6.15 (Matsui). *A $\otimes$-ideal is a prime $\otimes$-ideal if and only if it is a prime thick subcategory.*

Thus $\mathcal{P}$ is a prime $\otimes_{\mathcal{O}_X}$-ideal of $\mathsf{Perf}\, X$, i.e. a point of $\operatorname{Spec}_{\otimes_X} \mathsf{Perf}\, X$. This concludes the reconstruction of X from $\mathsf{Perf}\, X$.

To see that $\mathsf{Perf}\, X \simeq \mathsf{Perf}\, Y$ implies $X \simeq Y$, note that, because the Serre functor commutes with any equivalence, we must have

$$X \cong \operatorname{Spec}^{\mathrm{Ser}} \mathsf{Perf}\, X \cong \operatorname{Spec}^{\mathrm{Ser}} \mathsf{Perf}\, Y.$$

We know that Y embeds as an open subspace of $\operatorname{Spec}^{\mathrm{Ser}} \mathsf{Perf}\, Y$. Thus Y is a closed (because proper) and open subspace of X. But X is connected, so $X \cong Y$.

6.5. **More applications.** Using these methods, we can also recover a theorem of Favero.

Theorem 6.16 (Favero). *Let X and Y be projective varieties. Suppose there is an equivalence $\Phi : \mathsf{Perf}\, X \xrightarrow{\sim} \mathsf{Perf}\, Y$, an (anti)ample line bundle $\mathcal{A}$ on X, and a line bundle $\mathcal{L}$ on Y, such that $\Phi(- \otimes \mathcal{A}) \simeq \Phi(-) \otimes \mathcal{L}$. Then $X \cong Y$.*

Proof. By the above, we have $X \subset (\operatorname{Spec}_\Delta \mathsf{Perf}\, X)^{-\otimes \mathcal{A}} = (\operatorname{Spec}_\Delta \mathsf{Perf}\, Y)^{-\otimes \mathcal{L}}$. The same argument as above gives $X \simeq Y$. □

The above proof is much shorter than Favero's.

Remark 6.17. We do not yet know how to reconstruct the tensor product $\otimes_{\mathcal{O}_X}$ directly from the Serre functor.

7. 10/1b (Noah Olander) – Fully Faithful Functors and Dimension

We'll write X and Y for smooth projective varieties over some field k. We write $\mathsf{D}^b(X) := \mathsf{D}^b(\mathsf{Coh}(X)) \simeq \mathsf{Perf}\, X$, viewed as a k-linear triangulated category. All functors considered will be exact and k-linear.

7.1. **The main theorem.** Our goal is to prove the following:

Theorem 7.1 (Theorem 1). *If there exists a fully faithful functor $\mathsf{F} : \mathsf{D}^b(X) \to \mathsf{D}^b(Y)$, then $\dim X \leqslant \dim Y$.*

Let's give some examples of fully faithful functors to explain what we mean:

1. If $f : Y \to X$ satisfies $\mathsf{R}f_*\mathcal{O}_Y = \mathcal{O}_X$, then $\mathsf{L}f^* : \mathsf{D}^b(Y) \to \mathsf{D}^b(X)$ is fully faithful. This applies for Y a projective bundle over X, a blowup of X with smooth center, and many other cases. In this case, f must be surjective, proving the theorem in this case.
2. (Kuznetsov) There exist fully faithful functors from derived categories of K3 surfaces to derived categories of cubic fourfolds.

In these and other examples, the theorem is obvious. But the result in general wasn't known before!

We will prove the theorem by proving a weaker version of Orlov's conjecture, stated as follows. Let Rdim denote the Rouquier dimension of a triangulated category, to be defined later.

Conjecture 7.2 (Orlov)**.** *For a smooth projective variety* X*, we have* $\operatorname{Rdim} D^b(X) = \dim X$.

In this generality, the conjecture implies Theorem 1.

7.2. **Rouquier dimension.** Let $\mathcal{T}$ be a triangulated category and $S \subset \mathcal{T}$. Let $\langle S \rangle_{d+1} \subset \mathcal{T}$ consist of objects in $\mathcal{T}$ built by taking direct sums, shifts, passing to direct summands, and *taking at most* d *cones.*

Definition 7.3. The *Rouquier dimension* $\operatorname{Rdim} \mathcal{T}$ is the smallest integer d such that there exists a single object $G \in \mathcal{T}$ with $\mathcal{T} = \langle G \rangle_{d+1}$.

We will make use of a variant notion.

Definition 7.4. The *countable Rouquier dimension* $\operatorname{CRdim} \mathcal{T}$ is the smallest integer d such that there exists a countable subset $S \subset \operatorname{ob} \mathcal{T}$ with $\langle S \rangle_{d+1}$.

Example 7.5. Let R be a Dedekind domain and $\mathcal{T}$ the bounded derived category of finitely generated R-modules. Then $\operatorname{Rdim} \mathcal{T} = 1$, with $\mathcal{T} = \langle R \rangle_2$. To see this, let $K \in \mathcal{T}$. Then we may write $K = \oplus_{i \in \mathbb{Z}} H^i(K)[-i]$, so WLOG K is a finitely generated R-module concentrated in degree zero. Then K admits a resolution

$$0 \longrightarrow P \longrightarrow R^{\oplus d} \longrightarrow K \longrightarrow 0$$

where P is projective. As P and $R^{\oplus d}$ both lie in $\langle K \rangle_1$ (since P can be obtained as a direct summand of a free module), we have $K \in \langle R \rangle_2$. It follows that $\operatorname{Rdim} \mathcal{T} \leqslant 1$. The converse is left as an exercise.

Note that if R is countable, then $\operatorname{CRdim} \mathcal{T} = 0$, so CRdim and Rdim don't agree.

Example 7.6. Let X be a smooth projective curve of genus $g \geqslant 1$. Let $\mathcal{L}$ be a line bundle on X of degree $\geqslant 8g$. Orlov showed $D^b(X) = \langle \mathcal{L}^{-1} \oplus \mathcal{O} \oplus \mathcal{L} \oplus \mathcal{L}^{\otimes 2} \rangle_2$. Thus $\operatorname{Rdim} X \leqslant 1$, and equality holds by the following.

Lemma 7.7 (Rouquier)**.** *We have* $\operatorname{Rdim} D^b(X) \geqslant \dim X$.

The same argument also shows $\operatorname{CRdim} D^b(X) \geqslant \dim X$ if the ground field k is uncountable.

Lemma 7.8. *Let* $F : \mathcal{T} \to \mathcal{T}'$ *be a fully faithful exact functor with an exact right adjoint. Then* $\operatorname{Rdim} \mathcal{T} \leqslant \operatorname{Rdim} \mathcal{T}'$.

Proof. If R is the right adjoint, then $R \circ F = \mathrm{id}$, so R is essentially surjective. If G is an optimal generator for $\mathcal{T}$, then $R(G)$ generates $\operatorname{Rdim} \mathcal{T}$ in (at most) the same number of steps. □

This lemma (combined with a result of Bondal and van den Bergh on existence of adjoints) implies that Theorem 1 holds if Orlov's conjecture holds. The lemma also holds with Rdim replaced by CRdim.

7.3. **A weaker version of Orlov's conjecture.** Theorem 1 also follows from:[3]

Theorem 7.9 (Theorem 2)**.** *Let the ground field* k *be uncountable. Then* $\operatorname{CRdim} D^b(X) \leqslant \dim X$. *More precisely,* $D^b(X) = \langle \{\mathcal{O}_X(i)\}_{i \in \mathbb{Z}} \rangle_{\dim X + 1}$.

Theorem 7.10 (Theorem 3)**.** *Given maps* $K_0 \to K_1 \to \cdots \to K_{\dim X + 1}$ *in* $D^b(X)$ *such that* $H^n(K_i) \to H^n(K_{i+1})$ *is zero for all* i *and* n*, the composite map* $K_0 \to K_{\dim X + 1}$ *is zero.*

Maps $K_i \to K_{i+1}$ which are nonzero despite all maps $H^n(K_i) \to H^n(K_{i+1})$ being zero are called "ghosts" in topology. They are relatively easy to find.

Example 7.11. Let $\xi \in H^1(\mathbb{P}, \mathcal{O}(-2))$ be nonzero. Then ξ gives a nonzero element of $\operatorname{Hom}(\mathcal{O}, \mathcal{O}(-2)[1])$, despite the fact that $\mathcal{O}$ and $\mathcal{O}(-2)[1]$ are concentrated in different cohomological degrees.

Proof of Theorem 3. Consider the spectral sequence $E_1^{p,q} = \prod_n \operatorname{Ext}^{2p+q}(H^n(K), H^{n-p}(L))$ which converges to $\operatorname{Ext}^{p+q}(K, L)$. This gives a filtration $F^\bullet$ on $\operatorname{Hom}(K, L)$. We can see that $F^{\dim X + 1} = 0$, F^1 consists of ghost maps, and $F^r \circ F^s \subset F^{r+s}$. The result follows from these properties. □

[3] Some dexterity is needed to reduce the general case to the case where k is uncountable, but this is not too difficult.

Proof of Theorem 2. Let $K_0 = K \in D^b(X)$. Choose a finite direct sum $\oplus \mathcal{O}_X(n_i)^{\oplus d_i}[e_i]$ together with a map $\phi_0 : \oplus \mathcal{O}_X(n_i)^{\oplus d_i}[e_i] \to K_0$ which is surjective on cohomology. Let $K_1 = \operatorname{cone} \phi_0$, so $K_0 \to K_1$ is a ghost. Repeat this inductively to get $K_0 \to \cdots \to K_{\dim X+1}$.

We claim that $\operatorname{cone}(K_0 \to K_j) \in \langle \{\mathcal{O}_X(i)\}_{i\in\mathbb{Z}} \rangle_j$. For $j = 1$, this is true by construction. For $j > 1$, we use the octahedral axiom to construct an exact triangle

$$\operatorname{cone}(K_0 \to K_{j-1}) \to \operatorname{cone}(K_0 \to K_j) \to \operatorname{cone}(K_{j-1} \to K_j) \to$$

and use this to deduce the claim.

The theorem follows by considering $K_{\dim X+1} \oplus K_0[1] \simeq \operatorname{cone}(K_0 \xrightarrow{0} K_{\dim X+1}) \in \langle \{\mathcal{O}_X(i)\}_{i\in\mathbb{Z}} \rangle_{\dim X+1}$. □

8. 10/8 (Hannah Larson) – The Chow Rings of Moduli Spaces of Pointed Hyperelliptic Curves

This is related to the speaker's prior talk in this seminar, but knowledge of that talk is not necessary for this.

8.1. Refresher on Chow rings.

Let X be a smooth variety. The *Chow ring* $A^*(X)$ is the ring of $\mathbb{Z}$-linear combinations of irreducible subvarieties $Y \subset X$, modulo rational equivalence. Recall that Y_1 and Y_2 are rationally equivalent (written $Y_1 \sim Y_2$) if there is a subvariety $Z \subset X \times \mathbb{P}^1$ such that the forgetful map $p : Z \to \mathbb{P}^1$ is flat and for some $t_1, t_2 \in \mathbb{P}^1$ we have $p^{-1}(t_1) = Y_1$ and $p^{-1}(t_2) = Y_2$. The Chow ring is graded by codimension. Addition in the Chow ring corresponds to taking unions, and multiplication corresponds to taking (suitable) intersections.

We will need to know a few computational facts about Chow rings:

(1) $A^*(\mathbb{A}^n) = \mathbb{Z}$, concentrated in codimension 0. For example, the subvariety $V(x^2 + y^2 - 1) \subset \mathbb{A}^2$ is rationally equivalent to $\varnothing$ via the family $V((tx^2) + (ty)^2 - 1) \subset \mathbb{A}^2 \times \mathbb{P}^1_t$.

(2) There is an *excision property*: if $Z \subset X$ is closed of codimension c, then there is an exact sequence

$$A^{*-c}(Z) \longrightarrow A^*(X) \longrightarrow A^*(U) \longrightarrow 0$$

where $U = X \setminus Z$.

These can be used to compute $A^*(\mathbb{P}^n) = \mathbb{Z}[\zeta]/(\zeta^{n+1})$, using induction and the fact that

$$\mathbb{P}^n = \mathbb{A}^n \sqcup \mathbb{P}^{n-1} = \mathbb{A}^n \sqcup \mathbb{A}^{n-1} \sqcup \cdots \sqcup \mathrm{pt}.$$

To make this calculation work, one does need to know that the class of a point in $A^*(\mathbb{P}^n)$ is nonzero, which follows from proper pushforward being well-defined. We will use similar stratification-based methods today.

8.2. Moduli of curves.

Let $\mathcal{M}_g$ be the (coarse) moduli space of curves. One way to stratify this by gonality.

Definition 8.1. The *gonality* of a curve C is the minimal k such that there exists a degree k map $C \to \mathbb{P}^1$.

For curves of genus $g \neq 0$, the minimum gonality is 2. Curves of gonality 2 are called *hyperelliptic* and determine a closed subvariety $\mathcal{H}_g \subset \mathcal{M}_g$. Using the Riemann-Hurwitz formula, we see that if C is a hyperelliptic curve of genus g, the ramification divisor R of any degree 2 map $C \to \mathbb{P}^1$ satisfies $\deg R = 2g+2$. In particular, the coarse moduli space $\mathcal{H}_g$ corresponds to the space of collections of $2g + 2$ distinct points on $\mathbb{P}^1$, all modulo automorphisms of $\mathbb{P}^1$.[4] This can be written as

$$(\mathbb{P}^{2g+2} \setminus \Delta)/\mathrm{PGL}_2$$

where Δ is the locus where the points collide. In particular, we note that the coarse moduli space of $\mathcal{H}_g$ is unirational.

We can compute the Chow ring of this space (with rational coefficients) using equivariant intersection theory. The answer is

$$A^*(\mathcal{H}_g) \otimes \mathbb{Q} = \mathbb{Q}$$

Since we are using rational coefficients, the coarse moduli space gives the same answer as the stacky computation.

[4] This is not quite true for the stacks, but we will not concern ourselves with this.

8.3. **Introducing marked points.** Let $\mathcal{H}_{g,n}$ be the moduli space of smooth hyperelliptic curves with n distinct marked points. We have good reason to care about these: the boundary of the compactification $\overline{\mathcal{M}}_{g'}$ can be stratified by products of moduli spaces $\mathcal{M}_{g,n}$ of curves with marked points, and the $\mathcal{H}_{g,n}$ appear in the stratifications of these moduli spaces by gonality.

Let's consider $\mathcal{H}_{g,1}$ first. For a hyperelliptic curve C with $f : C \to \mathbb{P}^1$, if p is a marked point on C, there is an automorphism of C taking p to the other point $\overline{p} \in f^{-1}(f(p))$. Thus all we really care about is the image $f(p)$, i.e. $\mathcal{H}_{g,1}$ has coarse moduli space given by the space of collections of $2g+2$ distinct points on $\mathbb{P}^1$ together with one distinguished point on $\mathbb{P}^1$. As a variety, this is

$$((\mathbb{P}^2 \setminus \Delta) \times \mathbb{P}^1)/\mathrm{PGL}_2.$$

The forgetful map $\mathcal{H}_{g,1} \to \mathcal{H}_g$ is a $\mathbb{P}^1$-bundle, and we get

$$A^*(\mathcal{H}_{g,1}) \otimes \mathbb{Q} = \mathbb{Q}[\phi]/(\phi^2).$$

As n grows, the spaces $\mathcal{H}_{g,n}$ become more complicated. This can be made precise:

Theorem 8.2 (Barros-Mullane, Schwarz). *For $n \geqslant 4g+7$, the space $\mathcal{H}_{g,n}$ is of general type.*

In particular, it is not possible to find a dominant map from a rational variety to $\mathcal{H}_{g,n}$ (i.e. a "parametrization") for n large. Nevertheless, $\mathcal{H}_{g,n}$ is easier to understand in low degrees:

Theorem 8.3 (Casnati). *For $n \leqslant 2g+8$, the space $\mathcal{H}_{g,n}$ is rational.*

Theorem 8.4 (Canning-Larson). *For $n \leqslant 2g+6$, we have*

$$A^*(\mathcal{H}_{g,n}) \otimes \mathbb{Q} = \frac{\mathbb{Q}[\phi_1, \dots, \phi_n]}{(\phi_1, \dots, \phi_n)^2}$$

where the classes ϕ_i all lie in codimension 1. Furthermore, $\mathcal{H}_{g,n}$ is rational for $n \leqslant 3g+6$.

The classes ψ_i here are tautological classes. Over $\mathcal{H}_{g,n}$, there is a universal hyperelliptic curve $f : \mathcal{C} \to \mathcal{H}_{g,n}$ with disjoint sections $\sigma_1, \dots, \sigma_n : \mathcal{H}_{g,n} \to \mathcal{C}$. One defines $\psi_i = c_1(\sigma_i^* \omega_f)$. This is useful for understanding $A^*(\mathcal{M}_{g,n}) \otimes \mathbb{Q}$: the only contributions to this coming from $\mathcal{H}_{g,n}$ will be tautological!

Canning and Larson also have some similar results for higher gonality.

9. 10/15a (Smita Rajan) – Kinematic Varieties for Massless Particles

This is joint work with Svala Sverrisdottir and Bernd Sturmfels. The goal is to understand scattering amplitudes in theoretical physics.

9.1. **Background.** One can specify a QFT by specifying a Lagrangian $\mathcal{L}$. From this Lagrangian, one can extract Feynman diagrams, which describe the interaction of particles. "Tree-level" Feynman diagrams (those without loops) are the easiest to work with and produce the largest physical effects. To compute scattering amplitudes and other physical quantities from Feynman diagrams, we need to compute Feynman integrals. These are hard in general, but if we just care about tree-level scattering amplitudes, we can use the "spinor-helicity formalism" (which we'll discuss later).

The spinor-helicity formalism is well-understood in 4 ($= 3+1$) dimensions. However, extending it to higher dimensions turns out to be an interesting problem because of the relations between variables.

9.2. **Particles in d-dimensional spacetime.**

Definition 9.1. By d-dimensional spacetime, we mean the real vector space $\mathbb{R}^d$ with the Lorentzian inner product

$$x \cdot y = -x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

We consider a configuration of n particles in d dimensions. Each particle has a momentum $p_i = (p_{i,1}, \dots, p_{i,d})$. For massless particles, we obtain n quadric relations ($p_i^2 = 0$ for $1 \leqslant i \leqslant n$). We also impose momentum conservation, giving d linear relations ($\sum_{i=1}^n p_{ij} = 0$ for $1 \leqslant j \leqslant d$).

Algebraically, we focus on the polynomial ring $\mathbb{C}[p_{ij}]$ in nd variables.[5] Let $I_{d,n}$ be the ideal generated by the aforementioned n quadrics and d linear forms. Then $V(I_{d,n})$ is irreducible and of the expected dimension $(nd - n - d)$.

[5] The complexification occurs here for physical reasons – certain quantities aren't well-defined if we only work with real numbers.

Theorem 9.2. *The ideal* $I_{d,n}$ *is prime and is a complete intersection if* $\max(d,n) \geqslant 4$.

Proof. First consider the case $d \geqslant 5$. By eliminating the variable p_n, we can write $I_{d,n} = J + \langle f \rangle$, where J is generated by $n-1$ quadrics and

$$f = \left(\sum_{i=1}^{n-1} p_i\right) \cdot \left(\sum_{i=1}^{n-1} p_i\right).$$

We can write

$$\frac{\mathbb{C}[p_1, \ldots, p_{n-1}]}{J} \cong \frac{\mathbb{C}[p_1]}{p_1 \cdot p_1} \otimes \cdots \otimes \frac{\mathbb{C}[p_{n-1}]}{p_{n-1} \cdot p_{n-1}}.$$

A Hartshorne exercise shows that each of the factors has trivial divisor class group, so the ring $\mathbb{C}[p_1, \ldots, p_{n-1}]/J$ is a UFD. One can use this to show that the ideal in question is prime.

For $d = 3$ and $n \geqslant 4$, we can use Serre's criterion to show that $\mathbb{C}[p_i]/\langle p_i \cdot p_i \rangle$ is normal. This can be used to show the result.

The $d = 4$ case was handled in the paper "Spinor Helicity Varieties." □

9.3. The Clifford algebra and spinors.

Definition 9.3. Let V be a vector space equipped with a bilinear form B. The Clifford algebra is

$$C\ell(V) = T(V)/\langle v \otimes w + w \otimes v - 2B(v, w) \rangle.$$

We will write $C\ell(1, d-1)$ for the Clifford algebra corresponding to the Lorentzian inner product η on $\mathbb{C}^d$. This can be written explicitly as

$$C\ell(1, d-1) = \frac{\mathbb{C}\langle \gamma_1, \ldots, \gamma_d \rangle}{\langle \gamma_i\gamma_j + \gamma_j\gamma_i - 2\eta_{ij} \rangle}$$

Remark 9.4. The associated graded of the Clifford algebra is just the usual exterior algebra.

We'd like to construct a matrix representation of $C\ell(1, n-1)$ of dimension 2^k, where $k = \lfloor d/2 \rfloor$. For $d = 2$, we send γ_1 and γ_2 to the matrices

$$\Gamma_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad \Gamma_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

We can define the representations in dimensions $d > 2$ inductively.

For $d = 2k$, we send γ_i for $1 \leqslant i \leqslant d-2$ to

$$\Gamma_{2k,i} = \Gamma_{k-1,i} \otimes \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

We can also define matrices for γ_{d-1} and γ_d. Similar arguments work for odd d.

These representations give rise to the *spinor representations* of the Lie algebra $\mathfrak{so}(1, d-1)$. Specifically, send the ijth generator of $\mathfrak{so}(1, d-1)$ to $\Sigma_{ij} = \frac{1}{4}[\Gamma_i, \Gamma_j]$.

9.4. Back to particles.

For each particle, we construct a *momentum space Dirac matrix*

$$P_i = -p_{i1}\Gamma_1 + p_{i2}\Gamma_2 + \cdots + p_{id}\Gamma_d.$$

Definition 9.5. The *charge conjugation matrix* C is an equivariant map from the spinor representation of $\mathfrak{so}(1, d-1)$ to its dual satisfying

$$C\Gamma_i = -\Gamma_i^T C.$$

Example 9.6. For $d = 4$, we have

$$C = \begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{bmatrix}$$

We'd like to parametrize the column space of the matrix P_i by the variables

$$z_i = (z_{i,1}, \ldots, z_{i,2^{k-2}}, 0, \ldots, 0, z_{i,2^{k-1}+1}, \ldots, z_{2^k})^T.$$

We define the *spinor helicity variable* as $|i\rangle = P_i z_i$. We also let $\langle i| = (P_i z_i)^T$.

We can use these to define *spinor brackets*:

- Order 2: $\langle ij \rangle = \langle i| \, C \, |j\rangle = z_i^T P_i^T C P_j z_j$
- Order 3: $\langle ijk \rangle = \langle i| \, C P_j \, |k\rangle$
- Higher order: defined similarly.

Theorem 9.7. *These spinor brackets are invariant under the* $\mathfrak{so}(1, d-1)$ *action in the spinor representation (via conjugation by the* P*-matrices and left multiplication on the* z*-variables).*

Consider matrices S and T_j, where $S_{ij} = \langle ij \rangle$ and $(T_j)_{ik} = \langle ijk \rangle$.

Definition 9.8. The *kinematic variety* $K^{(2)}_{d,n}$ is the variety of possible matrices S as above.

Theorem 9.9. *For* $d = 3$*, the ideal of* $K^{(2)}_{3,n}$ *is generated by the* 4×4 *Pfaffians of a skew-symmetric matrix* $n \times n$ *matrix, so* $K^{(2)}_{3,n} \cong \mathrm{Gr}(2, n)$*. For* $d = 4$ *and* $d = 5$*, we have* $K^{(2)}_{4,n} \cong K^{(2)}_{5,n}$*, and these varieties are cut out by* 6×6 *Pfaffians of a skew-symmetric* $n \times n$ *matrix. In particular,* $K^{(2)}_{4,n} \cong K^{(2)}_{5,n}$ *is the first secant variety of* $\mathrm{Gr}(2, n)$.