

DERIVED ALGEBRAIC GEOMETRY NOTES

1. LECTURE 1, 6/26 (D. ARINKIN) – DG-CATEGORIES

Arinkin’s lectures aim to treat DG-categories as they are used in practice (without completely ignoring the foundations). More focus will be given to the “key parts” of the theory than to technicalities. We will treat foundations and geometrically-flavored examples (e.g. QCoh from a DG-perspective).

1.1. Motivation. In classical algebra and algebraic geometry, we like to consider modules over a ring. We pass to resolutions to get things which behave better from a homological perspective. This leads to considering derived categories. What kind of objects are derived categories?

- (1) Derived categories are *triangulated categories*. This means that they are equipped with a notion of *distinguished triangles* (also called *exact triangles*), which capture the notion of mapping cones. However, this structure can be poorly behaved.
- (2) Derived categories are *DG-categories*, or (mostly equivalently) *stable ∞ -categories*. Arinkin considers the DG category perspective to be more concrete. Using this perspective changes something fundamental about the category (what we obtain is more complex than a typical 1-category). However, DG categories can be more robust, especially when considering the “category of DG-categories.”

1.2. DG-categories and examples. Fix a field k (often we can get away with working over a commutative ring, but this may require more caution). Let $C(k)$ be the monoidal category of complexes of k -vector spaces, equipped with the tensor product \otimes_k as monoidal structure.

Definition 1.1. A *DG-category* A consists of

- (1) A collection of objects $\text{Ob } A$
- (2) For all $x, y \in \text{Ob } A$, a chain complex of morphisms $\text{Hom}_A(x, y) \in \text{Ob } C(k)$
- (3) For all $x, y, z \in \text{Ob } A$, a composition law $\circ : \text{Hom}_A(x, y) \otimes \text{Hom}_A(y, z) \rightarrow \text{Hom}_A(x, z)$

such that

- (1) Composition is associative.
- (2) Composition is unital: for all $x \in A$, there exists a degree-zero cycle $\mathbb{1}_x \in \text{Hom}_A(x, x)$ (so $d(\mathbb{1}_x) = 0$) such that $\mathbb{1}_x \circ f = f$ and $g \circ \mathbb{1}_x = g$ for all f, g .

Example 1.2. We can consider the case where $\text{Hom}(x, y)$ is always concentrated in degree zero. This recovers the notion of a k -linear category.

Example 1.3. Let R be a k -algebra. We can turn $C(R)$ (the category of complexes of R -modules) into a DG-category by setting

$$\mathrm{Hom}(M^\bullet, N^\bullet)^p = \prod_q \mathrm{Hom}_R(M^p, N^{p+q})$$

with differential $d_{\mathrm{Hom}} = [d, -]$. It is crucial here that we use the product rather than the direct sum (otherwise we run into issues when the complexes are not bounded).

We can also construct some examples where the Hom-complexes are standard complexes computing Ext groups. This ties into the classical perspective on homological algebra, where derived functors are computed using explicit resolutions.

Example 1.4. Let R be a k -algebra. We can define a DG-category A with $\mathrm{Ob} A$ being R -modules and $\mathrm{Hom}_A(M, N)$ being the standard complex computing $\mathrm{Ext}^\bullet(M, N)$ via the bar resolution. We will not spell out the details here because this is not our emphasis.

Example 1.5. Let $k = \mathbb{C}$, and fix a complex manifold X . We can define a DG-category A with objects being holomorphic vector bundles on X and $\mathrm{Hom}_A(E, F)$ given by the Dolbeault complex

$$0 \longrightarrow \mathcal{C}^\infty(E^* \otimes F) \xrightarrow{\bar{\partial}} \Omega^{0,1}(E^* \otimes F) \xrightarrow{\bar{\partial}} \dots$$

As stated above, our main examples of interest are really *derived categories*. We still need to build up some more definitions to get a good understanding of these.

1.3. DG-Functors. One would like to have a good notion of functors between DG-categories. It turns out that the naïve definition of DG-functors works (and does not require homotopical corrections).

Definition 1.6. A DG-functor $F : A \rightarrow B$ consists of:

- (1) An assignment $F : \mathrm{Ob} A \rightarrow \mathrm{Ob} B$
- (2) For every $x, y \in \mathrm{Ob} A$, a chain map $\mathrm{Hom}_A(x, y) \rightarrow \mathrm{Hom}_B(F(x), F(y))$

which preserves identities and composition.

Remark 1.7. While the obvious notion of a DG-functor works fine, subtleties arise when one considers the notion of equivalence.

One can first define a notion of *strict equivalence*. A DG-functor $F : A \rightarrow B$ is a strict equivalence if

- (1) F is strictly fully faithful: for all $x, y \in \mathrm{Ob} A$, $\mathrm{Hom}_A(x, y) \rightarrow \mathrm{Hom}_B(F(x), F(y))$ is an isomorphism.
- (2) F is strictly essentially surjective: for all $y \in \mathrm{Ob} B$, there exists $x \in \mathrm{Ob} A$ such that $F(x)$ is isomorphic to y .

As one knows from the study of derived categories, the notion of “isomorphism” here is often too strong - one would rather consider quasi-isomorphisms or other weaker notions.

A better notion is that of *quasi-equivalence*. A DG-functor $F : A \rightarrow B$ is a quasi-equivalence if

- (1) F is quasi-fully faithful: for all $x, y \in \mathrm{Ob} A$, $\mathrm{Hom}_A(x, y) \rightarrow \mathrm{Hom}_B(F(x), F(y))$ is a quasi-isomorphism.

- (2) F is quasi-essentially surjective: for all $y \in \text{Ob } B$, there exists $x \in \text{Ob } A$ such that there exist maps $F(x) \rightarrow y$ and $y \rightarrow F(x)$ with the both compositions $F(x) \rightarrow F(x)$ and $y \rightarrow y$ homotopic to the relevant identities.

Exercise 1.8. For a DG-category A , let $\text{Ho}A$ be the homotopy category of A , i.e. the 1-category with the same objects as A and morphisms given by $\text{Hom}_{\text{Ho}A}(x, y) = H^0(\text{Hom}_A(x, y))$.

- (1) Check that $\text{Ho}A$ is a well-defined 1-category. (This uses the fact that the functor H^0 is lax monoidal.)
- (2) Show that a DG-functor $F : A \rightarrow B$ is quasi-essentially surjective if and only if $\text{Ho}F : \text{Ho}A \rightarrow \text{Ho}B$ is essentially surjective.

2. LECTURE 2, 6/26 (B. ANTIEAU) – MOTIVATION AND THE FUNCTOR OF POINTS

Antieau’s lectures aim to treat the intuition and geometry behind DAG, without getting lost in the details.

2.1. “Bird’s Eye View” and Examples. We begin with a classical proposal of Serre. Let X be a regular variety, with $W, Y \subset X$ and $W \subset Y$ being a finite set of points. We would like to count $W \cap Y$. In the case of transverse intersections, this is easy, but in general we have to count with multiplicity. If we move W and Y around, this count with multiplicity should not change. Serre proposed that the intersection multiplicity at a point $z \in W \cap Y$ is given by

$$\sum_{i \geq 0} (-1)^i \text{length}_{k(z)} \text{Tor}^i(\mathcal{O}(X)_z)(\mathcal{O}(X)_z/I_z, \mathcal{O}(Y)/I_z).$$

It is not immediately obvious why this is a reasonable definition, but it turns out to work well in good cases. One can show that (when working over a field) this intersection multiplicity is nonnegative.

Example 2.1. Consider the intersection in \mathbb{A}^2 of the curve $W = \{y = x^2\}$ with the x -axis Y . One can compute that the intersection multiplicity is 2 (coming exclusively from Tor^0). This makes sense: if we move the curves around, then we will get two actual intersection points.

For the Tor computation, we note $\mathcal{O}(W) = k[x, y]/(y - x^2)$ and $\mathcal{O}(Y) = k[x, y]/(y)$. Then $\mathcal{O}(Y) \otimes_{\mathcal{O}(X)} \mathcal{O}(W) = k[x]/(x^2)$, and there are no higher Tor’s. This can be interpreted geometrically using the common tangent vector to W and Y at the origin.

Example 2.2. For the first historical example with higher Tors, we work in $\mathbb{A}^4 = \text{Spec } k[w, x, y, z]$. Let Y be the union of planes $P_1 \cup P_2 = \{x = y = 0\} \cup \{z = w = 0\}$, and let W be the plane $\{x = z, y = w\}$. Then the Tor_0 term is 3-dimensional, but the “correct” intersection multiplicity should be 2 (if we perturb W , it should meet each plane P_i in a single point). Thus we must use the Tor^1 term to correct our intersection multiplicity.

The main idea of derived algebraic geometry is that we should understand intersection multiplicities geometrically by replacing $\mathcal{O}(Y) \otimes_{\mathcal{O}(X)} \mathcal{O}(W)$ by a *derived commutative ring* $\mathcal{O}(Y) \otimes_{\mathcal{O}(X)}^L \mathcal{O}(W)$. In the affine case, “Spec” of this should give the *derived intersection* $Y \cap^L W = Y \times_X^L W$. We will of course need to make sense of what this means.

Derived algebraic geometry also gives rise to interesting self-intersections, which we can use to understand what derived commutative rings and schemes should be.

Example 2.3. Consider $\{0\} \subset \mathbb{A}^1$, viewed as Spec of $k = k[x]/(x)$. The classical self-intersection is $\{0\} \cap \{0\} = \{0\}$. Note that this is not what we'd get if we moved the points (since two general points in \mathbb{A}^1 do not meet).

For the derived self-intersection, we have

$$\pi_i(k \otimes_{k[x]}^{\mathbf{L}} k) = \begin{cases} k & i = 0 \\ k & i = 1 \\ 0 & i > 0. \end{cases}$$

For derived commutative rings, we say $\pi_i = H_i = H^{-i}$. We can think of elements of π_i for $i > 0$ as higher nilpotents / “fuzz” (in addition to the nilpotents that appear in π_0 when working scheme-theoretically).

Recall that in classical geometry, for commutative k -algebras R and S , we have

$$\text{Hom}_{\text{Sch}_k}(\text{Spec } S, \text{Spec } R) = \text{Hom}_{\text{CAlg}_k}(R, S).$$

Note that the right hand side of this is a set. For our example, we expect to have

$$\text{Hom}_{\text{dSch}_k}(\{0\} \cap^{\mathbf{L}} \{0\}, \mathbb{A}^1) = \mathcal{O}(\{0\} \cap^{\mathbf{L}} \{0\}) = k \otimes_{k[x]}^{\mathbf{L}} k,$$

which should be a “space” with $\pi_0 = \pi_1 = k$ (not just a set).

Furthermore, instead of considering abelian categories of quasicoherent sheaves on $\{0\} \cap^{\mathbf{L}} \{0\}$, it makes much more sense to consider derived categories of quasicoherent sheaves. One can still make sense of the abelian categories, but they don't see the interesting higher homotopy groups.

Through considering this and other examples, we end up with some ideas about what “derived replacements” of concepts in classical algebraic geometry should be.

Classical AG	Derived AG
Commutative rings R	Derived commutative rings R
Affine schemes $\text{Spec } R$	Derived affine schemes $\text{Spec } R$
Sheaves of sets	Sheaves of homotopy types
Abelian categories $\text{QCoh}(X)$	DG-categories (or stable ∞ -categories) $\text{D}_{\text{qc}}(X)$

2.2. Presheaves and Yoneda. Let C be a category. Write $\text{Psh}(C)^{\heartsuit} = \text{Fun}(C^{\text{op}}, \text{Set})$ for the *category of presheaves of sets* on C . Objects of this category are functors $F : C^{\text{op}} \rightarrow \text{Set}$, and morphisms are natural transformations of functors.

There is a natural *Yoneda embedding* $h : C \rightarrow \text{Psh}(C)^{\heartsuit}$, defined by $h(X) = h_X$, where $h_X(Y) = \text{Hom}_C(Y, X)$.

Lemma 2.4 (Yoneda).

- (1) *The Yoneda embedding $h : C \rightarrow \text{Psh}(C)^{\heartsuit}$ is fully faithful, i.e. $\text{Hom}_C(X, Y) \cong \text{Hom}_{\text{Psh}(C)^{\heartsuit}}(h_X, h_Y)$ naturally in X, Y .*
- (2) *For any $F \in \text{Psh}(C)^{\heartsuit}$, there is a natural isomorphism $F(X) \cong \text{Hom}_{\text{Psh}(C)^{\heartsuit}}(h_X, F)$.*

Proof. The first statement follows from the second by taking $F = h_Y$ in the second. To prove the second statement, we can construct explicit natural isomorphisms as follows. To get $\text{Hom}_{\text{Psh}(C)^{\heartsuit}}(h_X, F) \rightarrow F(X)$, start with $f : h_X \rightarrow F$, and evaluate f on $\text{id}_X \in h_X(X)$ to get an element of $F(X)$. Conversely, given $g \in F(X)$, construct a natural transformation $h_X \rightarrow F$ by sending $a \in h_X(Y) = \text{Hom}_C(Y, X)$ to $F(a)(g) \in F(Y)$. One can check that these are both natural and mutually inverse. \square

Example 2.5. Let Δ^1 be the category with two objects $0, 1$ and one non-identity morphism $0 \rightarrow 1$. Then $\mathbf{Psh}(\Delta^1)^\heartsuit$ is the category of arrows in \mathbf{Set} . The functor h_0 corresponds to the arrow $\emptyset \rightarrow *$, and the functor h_1 corresponds to the arrow $* \rightarrow *$.

2.3. Topologies and sheaves.

Definition 2.6. A map $f : R \rightarrow S$ in \mathbf{CAlg}_k is *flat* if the functor $S \otimes_R (-) : \mathbf{Mod}_R \rightarrow \mathbf{Mod}_S$ is exact. We say that f is furthermore *faithfully flat* if $S \otimes_R (-)$ is conservative (i.e. for $g : M \rightarrow N$, if $S \otimes g : S \otimes_R M \rightarrow S \otimes_R N$ is an isomorphism, then g must have already been an isomorphism).

Example 2.7.

- (1) For any R , the map $R \rightarrow 0$ is flat but not faithfully flat.
- (2) The map $\mathbb{Z} \rightarrow \mathbb{Q}$ is flat but not faithfully flat.
- (3) Any field extension $K \rightarrow L$ is faithfully flat.
- (4) For $f, g \in R$ with $(f, g) = 1$, the map $R \rightarrow R[f^{-1}] \times R[g^{-1}]$ is faithfully flat. Note that this gives a Zariski cover of $\mathrm{Spec} R$.

Let $\mathbf{Aff}_k = \mathbf{CAlg}_k^{\mathrm{op}}$ be the *category of affine k -schemes*. We will define some topologies on \mathbf{Aff}_k .

Definition 2.8.

- (1) The *fpqc topology* on \mathbf{Aff}_k has coverings of $\mathrm{Spec} R$ generated by those of the form $\{\mathrm{Spec} S_i \rightarrow \mathrm{Spec} R\}_{i \in I}$ where I is finite, each $R \rightarrow S_i$ is flat, and $R \rightarrow \prod_i S_i$ is faithfully flat.¹
- (2) For the *fppf topology*, we require furthermore that each S_i is of finite presentation over R (i.e. $S_i = R[x_1, \dots, x_p]/(f_1, \dots, f_q)$ for some x_i and f_j).
- (3) For the *étale topology*, we require furthermore that each S_i is étale over R (i.e. S_i is smooth over R with $\Omega_{S_i/R}^1 = 0$).
- (4) For the *Zariski topology*, we require furthermore that each S_i is of the form $R[f^{-1}]$ for some $f \in R$.

3. LECTURE 3, 6/27 (B. ANTIEAU) – MORE ON FUNCTOR OF POINTS, INTRODUCTION TO ∞ -CATEGORIES

3.1. Topologies and sheaves, continued. Let τ be one of the topologies on \mathbf{Aff}_k mentioned last time. Define a full subcategory $\mathbf{Sh}_\tau(\mathbf{Aff}_k)^\heartsuit \subset \mathbf{Psh}(\mathbf{Aff}_k)^\heartsuit$, the *category of sheaves for the τ -topology*, by declaring $\mathcal{F} \in \mathbf{Sh}_\tau(\mathbf{Aff}_k)^\heartsuit$ if and only if for all τ -covers $\mathrm{Spec} S \rightarrow \mathrm{Spec} R$, the natural diagram

$$\mathcal{F}(R) \rightarrow \mathcal{F}(S) \rightrightarrows \mathcal{F}(S \otimes_R S)$$

is an equalizer diagram. In more pedestrian language, $\mathcal{F}(R)$ is the subset of elements $x \in \mathcal{F}(S)$ such that the two images of x in $\mathcal{F}(S \otimes_R S)$ agree. One should think of $\mathrm{Spec} S$ as a collection of opens of $\mathrm{Spec} R$. From this perspective, $\mathrm{Spec} S \otimes_R S$ is the collection of intersections of the opens appearing in $\mathrm{Spec} S$.

There was a question about whether we need to impose a condition on compatibility with products to account for the case of covers of the form $\{\mathrm{Spec} S_i \rightarrow \mathrm{Spec} R\}_{i \in I}$. Antieau claimed that this condition is superfluous and that it would be a good exercise to deduce this compatibility from the definition of $\mathbf{Sh}_\tau(\mathbf{Aff}_k)^\heartsuit$.

¹Note the “generated by” – we allow infinite covers.

Theorem 3.1 (Grothendieck). *The presheaf $\text{Spec } R \mapsto R$ is a sheaf for the τ -topologies.*

Proof. This follows from faithfully flat descent, i.e. the exactness of

$$R \rightarrow S \rightarrow S \otimes_R S$$

for a faithfully flat ring map $R \rightarrow S$. \square

Exercise 3.2. *We can view $\text{Spec } R$ as a presheaf on Aff_k via $(\text{Spec } R)(\text{Spec } S) = \text{Hom}_{\text{CAlg}_k}(R, S)$. Show that this is a τ -sheaf. Thus the topologies τ mentioned above are subcanonical (i.e. representable presheaves are sheaves).*

The inclusion $\text{Sh}_\tau(\text{Aff}_k)^\heartsuit \subset \text{Psh}(\text{Aff}_k)^\heartsuit$ admits a left adjoint, known as τ -sheafification and written a_τ . Here we are secretly ignoring some potential set-theoretic issues (which can be dealt with systematically using Grothendieck universes).

Remark 3.3. If $D \rightarrow \text{Psh}(C)^\heartsuit$ is a fully faithful map of categories that admits a left adjoint (which satisfies the mild set-theoretic hypothesis of being κ -accessible for some κ), we call D a *presentable category*. If furthermore the left adjoint $\text{Psh}(C)^\heartsuit \rightarrow D$ preserves finite limits, we call D a *topos*. Topoi generally behave like categories of sheaves.

3.2. Representable morphisms. For a scheme Z , we get a τ -sheaf $h_Z \in \text{Sh}_\tau(\text{Aff}_k)^\heartsuit$ via $h_Z(\text{Spec } R) = \text{Hom}_{\text{Sch}_k}(\text{Spec } R, Z)$.

Theorem 3.4. *The functor $h_{(-)} : \text{Sch}_k \rightarrow \text{Sh}_\tau(\text{Aff}_k)^\heartsuit$ is fully faithful.*

Thus we may identify h_Z with Z for any scheme Z . Note that this is not quite the Yoneda lemma (we are viewing a scheme as a τ -sheaf on affine schemes, not all schemes), but it is closely related. The key idea is that schemes are constructed from gluing affine schemes, and meditating on this enough yields a proof of the above theorem.

Definition 3.5. A morphism $X \rightarrow Y$ in $\text{Sh}_\tau(\text{Aff}_k)^\heartsuit$ is *representable in schemes* if, for all $\text{Spec } R \rightarrow Y$, the fiber product $P = X \times_Y \text{Spec } R$ is representable (i.e. isomorphic to h_Z for some scheme Z).

We can generalize standard geometric properties of schemes to properties of representable morphisms by imposing said properties on base changes to (affine) schemes. For example:

Definition 3.6. A morphism of τ -sheaves $X \rightarrow Y$ is *open* if it is representable in schemes and for all $\text{Spec } R \rightarrow Y$, the base change $X \times_Y \text{Spec } R \rightarrow \text{Spec } R$ is a disjoint union of open embeddings of schemes.

The terminology here is a bit lazy and is designed primarily to make the next theorem straightforward. We would like to characterize the essential image of the embedding $h_{(-)} : \text{Sch}_k \rightarrow \text{Sh}_\tau(\text{Aff}_k)^\heartsuit$.

Theorem 3.7. *A τ -sheaf X is isomorphic to h_Z for some $Z \in \text{Sch}_k$ if and only if it admits an open surjective map $\coprod_i \text{Spec } S_i \rightarrow X$.*

3.3. Examples. Here we will define some schemes by their functors of points.

Example 3.8. Let $X \in \mathrm{Sh}_\tau(\mathrm{Aff}_k)^\vee$ be given by

$$X(\mathbf{R}) = \{(x, y) \in \mathbf{R}^2 \mid x^2 + y^2 = 1\}.$$

Then we can represent X by $\mathrm{Spec} k[x, y]/(x^2 + y^2 - 1)$.

Example 3.9. The additive group \mathbb{G}_a is given by $\mathbb{G}_a(\mathbf{R}) = \mathbf{R}$. In terms of coordinate rings, we have $\mathbb{G}_a = \mathrm{Spec} k[x]$.

Example 3.10. The multiplicative group \mathbb{G}_m is given by $\mathbb{G}_m(\mathbf{R}) = \mathbf{R}^\times$. In terms of coordinate rings, we have $\mathbb{G}_m = \mathrm{Spec} k[x, x^{-1}]$.

Example 3.11. We can naïvely try to define $X = \mathbb{P}_k^n$ via the functor of points $X(\mathbf{R}) = (\mathbf{R}^{n+1} \setminus \{0\})/\mathbf{R}^\times$. However, this fails – the X we define here is not even a functor!

Example 3.12. The correct definition of \mathbb{P}_k^n is via the functor of points

$$\mathbb{P}_k^n(\mathbf{R}) = \{(P, q) \mid P \text{ a rank 1 projective module over } \mathbf{R}, q : \mathbf{R}^{n+1} \twoheadrightarrow P\}.$$

Exercise 3.13. Use the theorem of the preceding section to show that \mathbb{P}_k^n is a scheme.

3.4. ∞ -categories. Let's move on to a new topic. As in Arinkin's lectures, we would like to replace 1-categories with categories “enriched in spaces up to weak homotopy equivalence”. Taking this literally (i.e. using actual topological spaces) turns out to be hard to work with, so we will instead use the formalism of quasi-categories.

Let Δ , the *simplex category*, be the category of nonempty finite totally ordered sets and order-preserving functors. Every object of Δ is isomorphic to some $[n] = \{0 \leq 1 \leq \dots \leq n\}$ for some $n \geq 0$. We are particularly interested in the *face maps* $\partial^i : [n] \rightarrow [n+1]$ for $i = 0, \dots, n+1$ (with ∂^i being the injective map missing the element $i \in [n+1]$ and otherwise bijective) and the *degeneracy maps* $s^i : [n] \rightarrow [n-1]$ for $i = 0, \dots, n+1$ (with s^i being the surjective map hitting the element i twice and otherwise bijective).

A *simplicial set* is a presheaf on Δ , and we write the category of simplicial sets as $\mathbf{sSet} = \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathbf{Set}) = \mathbf{Psh}(\Delta)^\vee$. More generally, the category of simplicial objects in a category \mathbf{C} is $\mathbf{sC} = \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathbf{C})$. Dually, the category of cosimplicial objects in \mathbf{C} is $\mathbf{cC} = \mathrm{Fun}(\Delta, \mathbf{C})$.

Example 3.14. Let X be a topological space. We can define the singular simplicial set of X as $\mathrm{Sing}_\bullet(X) = \mathrm{Hom}_{\mathbf{Top}}(\Delta_\bullet^{\mathrm{top}}, X)$. That is, $\mathrm{Sing}_n(X) = \mathrm{Hom}_{\mathbf{Top}}(\Delta_{\mathrm{top}}^n, X)$ where

$$\Delta_{\mathrm{top}}^n = \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid x_i \geq 0 \forall i, \text{ and } \sum_i x_i = 1\}.$$

Note that $\Delta_\bullet^{\mathrm{top}} \in \mathbf{cTop}$. Applying the free abelian group functor $\mathbb{Z}[-] : \mathbf{Set} \rightarrow \mathbf{Ab}$ to $\mathrm{Sing}_\bullet(X)$ gives a simplicial abelian group from X . Via the Dold-Kan correspondence, this corresponds to a connective chain complex $\mathbf{C}_\bullet(X, \mathbb{Z})$ which computes $\mathbf{H}_\bullet(X, \mathbb{Z})$.

Example 3.15. Let \mathbf{C} be a (small) category. We can define a simplicial set $\mathbf{N}_\bullet \mathbf{C}$, the *nerve of \mathbf{C}* , by setting $\mathbf{N}_n(\mathbf{C}) = \mathrm{Hom}_{\mathbf{Cat}}([n], \mathbf{C})$, where we view $[n]$ as the category $(0 \rightarrow 1 \rightarrow \dots \rightarrow n)$. Here $\mathbf{N}_0(\mathbf{C}) = \mathrm{Ob} \mathbf{C}$, $\mathbf{N}_1(\mathbf{C})$ is the set of all morphisms in \mathbf{C} , and $\mathbf{N}_2(\mathbf{C})$ is the set of all composable pairs of morphisms in \mathbf{C} .

Construction 3.16. Consider the left Kan extension $|-| : \mathbf{sSet} \rightarrow \mathbf{Top}$ in

$$\begin{array}{ccc} \Delta & \xrightarrow{\Delta_{\text{top}}^\bullet} & \mathbf{Top} \\ \downarrow [n] \dashrightarrow \Delta^n & \nearrow \Delta^n & \\ \mathbf{sSet} & & \end{array}$$

We call this functor *geometric realization*.

Example 3.17. Here $|\Delta^n| = \Delta_{\text{top}}^n$. Let $\partial\Delta^n$ be the subsimplicial set of Δ^n missing the unique nondegenerate n -simplex. Then $|\partial\Delta^n| \simeq S^{n-1}$.

We define the horn Λ_i^n as the subsimplicial set of Δ^n missing the i th face of Δ^n (i.e. the face opposite to the i th vertex).

Exercise 3.18. Consider the problem of extending a map of simplicial set $\Lambda_i^n \rightarrow X_\bullet$ along the inclusion $\Lambda_i^n \rightarrow \Delta^n$.

- (1) For a topological space Y and $X_\bullet = \text{Sing}_\bullet(Y)$, show that lifts exist for all horns Λ_i^n . Stated differently, Y is a Kan complex or ∞ -groupoid. This is easiest understood using the corresponding diagrams

$$\begin{array}{ccc} \Lambda_i^n & \longrightarrow & \text{Sing}_\bullet(Y) \\ \downarrow & & \downarrow \\ \Delta^n & \longrightarrow & \text{pt} \end{array} \quad \Leftrightarrow \quad \begin{array}{ccc} |\Lambda_i^n| & \longrightarrow & Y \\ \downarrow & & \downarrow \\ \Delta_{\text{top}}^n & \longrightarrow & \text{pt} \end{array}$$

- (2) For a category \mathcal{C} and $X_\bullet = \mathbf{N}_\bullet \mathcal{C}$, show that extensions exist for all inner horns (i.e. when $0 < i < n$).

Definition 3.19. An ∞ -category or quasi-category is a *weak Kan complex*, i.e. a simplicial set X_\bullet such that any map $\Lambda_i^n \rightarrow X_\bullet$ (with $0 < i < n$) extends to a map $\Delta^n \rightarrow X_\bullet$.

Thus singular simplicial sets and nerves are both examples of ∞ -categories.

Studying ∞ -categories will allow us to achieve our goal of unifying homotopy theory and category theory. Namely, there is a model structure on \mathbf{sSet} such that if we invert weak equivalences, we get a category (Quillen?)-equivalent to \mathbf{Top} with weak homotopy equivalences inverted. There was some discussion of “Joyal model structures” but I missed it.

4. LECTURE 4, 6/27 (D. ARINKIN) - DERIVED CATEGORIES

Recall what we did last time. If A is a DG-category, then $\text{Hom}_A(x, y)$ is a chain complex for all $x, y \in \text{Ob } A$. DG-functors are defined “naïvely, but (quasi-)equivalences are not. A quasi-equivalence is required to be quasi-fully faithful and quasi-essentially surjective. This comes from the natural weakening of the notion of “inverses,” i.e. requiring that quasi-equivalences are only invertible “up to homotopy.”

4.1. Construction of Derived Categories – Strategy. For a (DG-)ring R , we would like to move from the DG-category $C(R)$ of complexes of R -modules to the corresponding derived category $D(R)$.

Example 4.1. A DG-category with a single object x can be identified with a DG-ring R , namely $R = \text{End}(x)$.

There are two well-known approaches to constructing $D(R)$.

- (1) One approach is via localization: we invert quasi-isomorphisms in $C(R)$. To make this work in the DG-context, one can see Drinfeld's paper "DG-quotients of DG-categories." It may be surprising that we use quotients to construct localizations. In fact, when working with DG-categories, localizations and quotients are effectively equivalent: inverting a morphism f is the same as killing the object $\text{cone}(f)$.
- (2) Another approach is to resolve all objects of $C(R)$. This is analogous to choosing good representatives in each equivalence class in the quotient.

We will focus on the latter approach (using resolutions) for now.

4.2. The classical viewpoint.

Example 4.2. The bounded below derived category $D^-(R)$ is equivalent to the (DG-)category $C^-(R)^{\text{proj}}$ of bounded above complexes of projective modules. If one likes, one could take this as a definition of $D^-(R)$. One could also prove this using a different definition of $D^-(R)$ by showing that the notions of equivalence in $D^-(R)$ and $C^-(R)^{\text{proj}}$ match up.

An analogous approach (using injective modules) works for $D^+(R)$. It takes some more work to find an analogous description of $D(R)$, but this can still be treated classically.

The classical idea (due to Spaltenstein) says that we should resolve an unbounded complex

$$\dots \longrightarrow A^{-1} \longrightarrow A^0 \longrightarrow A^1 \longrightarrow \dots$$

by truncating A^\bullet and resolving the truncations. Consider first the clean truncation $\tau_{\leq 0} A^\bullet$, and resolve by a complex of projectives:

$$\begin{array}{ccccccc} \dots & \longrightarrow & A^{-1} & \longrightarrow & A^0 & \longrightarrow & d(A^0) \longrightarrow 0 \longrightarrow \dots \\ & & \uparrow & & \uparrow & & \uparrow \\ \dots & \longrightarrow & P_{(0)}^{-1} & \longrightarrow & P_{(0)}^0 & \longrightarrow & P_{(0)}^1 \longrightarrow 0 \longrightarrow \dots \end{array}$$

We can come up with a different resolution for the clean truncation $\tau_{\leq 1} A^\bullet$:

$$\begin{array}{ccccccc} \dots & \longrightarrow & A^{-1} & \longrightarrow & A^0 & \longrightarrow & A^1 \longrightarrow d(A^1) \longrightarrow 0 \longrightarrow \dots \\ & & \uparrow & & \uparrow & & \uparrow \\ \dots & \longrightarrow & P_{(1)}^{-1} & \longrightarrow & P_{(1)}^0 & \longrightarrow & P_{(1)}^1 \longrightarrow P_{(1)}^2 \longrightarrow 0 \longrightarrow \dots \end{array}$$

Since these are resolutions and we have a map $\tau_{\leq 0} A^\bullet \rightarrow \tau_{\leq 1} A^\bullet$, we obtain a map $P_{(0)}^\bullet \rightarrow P_{(1)}^\bullet$. Continuing inductively, we get a sequence of bounded above complexes of projectives $P_{(i)}^\bullet$ with maps $P_{(i)}^\bullet \rightarrow P_{(i+1)}^\bullet$. Using mapping cylinders, we can assume that each $P_{(i)}^\bullet \rightarrow P_{(i+1)}^\bullet$ is injective (at the chain level), so let $P^\bullet = \cup_i P_{(i)}^\bullet$. Thus we obtain an unbounded complex of projectives P^\bullet with a quasi-isomorphism $P^\bullet \rightarrow A^\bullet$.

The key property of P^\bullet that we will be using is *K-projectivity*, also called *homological projectivity*. This property states that for any acyclic complex C^\bullet , the Hom-complex $\text{Hom}(P^\bullet, C^\bullet)$ is acyclic. One can show that $D(R)$ is equivalent to the category of K-projective complexes in $C(R)$. In the DG-context, one either takes

this as a definition or proves the equivalence (starting from the definition of $D(R)$ as a quotient mentioned earlier).

Remark 4.3. Note that we could have replaced “projective” by “free” throughout, and nothing would have changed.

Note that homotopical projectivity does not follow from the fact that P^\bullet is a complex of projectives (in the unbounded case).

Example 4.4. Take $R = \mathbb{Z}/4\mathbb{Z}$, and let P^\bullet be the complex of projectives defined by $P^i = R$ for all i , with all differentials given by multiplication by 2. For $C^\bullet = P^\bullet$ (which is acyclic), we can compute that $\text{Hom}(P^\bullet, C^\bullet)$ is not acyclic (due to the presence of the identity map, which in this case is not homotopic to 0). If one wants to work over a field, one could instead use $k[x]/(x^2)$ or something similar. This example leads to much theory (related to categories of singularities, matrix factorizations, Eisenbud periodicity, etc.). The problem here is related to R not being regular.

Note however that bounded above complexes of projectives are homotopically projective. To prove that our original P^\bullet is homotopically projective, we would want to reduce to the case of bounded above complexes of projectives.

4.3. Semi-free modules. We will now study how to make this work in the DG-context. Let R be a DG-ring. The term “module” will now be used for a (potentially unbounded) complex of R -modules.

Definition 4.5. A *free module* is a chain complex of the form $\bigoplus_{n \in \mathbb{Z}} R^{I_n}[n]$. A *semi-free module* is a chain complex $M^\bullet = \bigcup_{i \geq 0} M_i^\bullet$ such that M_{i+1}/M_i is free for all i . In this case, the differential satisfies $d(M_{i+1}) \subset M_i$, i.e. is “strictly upper triangular.”

Example 4.6. If R is a classical ring, a bounded above complex of free modules

$$\dots \longrightarrow F^{-2} \longrightarrow F^{-1} \longrightarrow F^0 \longrightarrow 0 \longrightarrow \dots$$

is semi-free.²

Example 4.7. More generally (but still for R classical), consider an increasing union

$$\begin{array}{ccccccc} \dots & \longrightarrow & F_{(0)}^{-2} & \longrightarrow & F_{(0)}^{-1} & \longrightarrow & F_{(0)}^0 \longrightarrow 0 \longrightarrow \dots \\ & & \downarrow & & \downarrow & & \downarrow \\ \dots & \longrightarrow & F_{(1)}^{-2} & \longrightarrow & F_{(1)}^{-1} & \longrightarrow & F_{(1)}^0 \longrightarrow F_{(1)}^1 \longrightarrow \dots \\ & & \downarrow & & \downarrow & & \downarrow \\ \dots & \longrightarrow & \dots & \longrightarrow & \dots & \longrightarrow & \dots \longrightarrow \dots \end{array}$$

where each $F_{(i)}^k/F_{(i-1)}^k$ is free. Then the union $\bigcup_{i \geq 0} F_{(i)}^k$ is semi-free.

Proposition 4.8. *Semi-free modules are homotopically projective.*

Proposition 4.9. *Every module admits a semi-free resolution.*

²If R is not classical, this complex doesn’t really make sense.

5. LECTURE 5, 6/28 (D. ARINKIN) – MODULES

Last time, for a classical ring R , we defined the DG-category $C(R)$ of chain complexes of R -modules. We further defined a subclass of “semi-free” objects in $C(R)$. Our goal will be to examine these concepts for DG-rings and more generally DG-categories.

5.1. DG-Rings and their Modules. Let R be a DG-ring. We can view the data of R as:

- (1) An underlying complex $R \in C(k)$
- (2) A degree-zero multiplication map $\cdot : R \otimes_k R \rightarrow R$ such that:
 - (a) The (graded) product rule holds:

$$d(ab) = (da)b + (-1)^{|a|}adb,$$

(or equivalently \cdot is a chain map).

- (b) Multiplication is (strictly) associative.
- (c) Multiplication is (strictly) unital.

Given a DG-ring R , we can consider the category $C(R)$ of *DG-modules* (or *modules* for short), i.e. chain complexes $M \in C(k)$ equipped with a multiplication $R \otimes_k M \rightarrow M$ satisfying the usual module axioms together with a (graded) product rule. We can view $C(R)$ as a DG-category where $(\text{Hom}_R(M, N))^i$ is the vector space of R -linear maps $M \rightarrow N$ of degree i (with no regard for the differentials on M and N). The differential on $\text{Hom}_R(M, N)^\bullet$ is given by the (graded) commutator $[d, -]$. Note that $\text{Hom}_R(M, N)$ is naturally a subcomplex of the Hom-complex $\text{Hom}_k(M, N)$ defined in the first lecture.

We can still make sense of the notion of freeness and semi-freeness in this context:

Definition 5.1. Let R be a DG-ring and $M \in C(R)$. A *free R -module* is an R -module strictly isomorphic to one of the form $\bigoplus_{n \in \mathbb{Z}} R[n]^{\oplus I_n}$. A *semi-free R -module* is an R -module of the form $M = \bigcup_{i=0}^{\infty} M_i$ where $M_0 \subset M_1 \subset \dots$ with each M_i/M_{i-1} free.

If we ignore the differential d (i.e. pretend that R and M both have $d = 0$), then semi-free modules are free: $M \cong \bigoplus_{i=0}^{\infty} M_i/M_{i-1}$. However, in general, d has interesting upper triangular behavior.

Example 5.2. Suppose $M = M_1$ where $M_0 = Re_0$ and $M_1/M_0 = Re_1$. Then we can write $M \cong Re_1 \oplus Re_0$ (ignoring differentials). Here d satisfies $d(e_0) = 0$ and $d(e_1) = se_0$ for some $s \in R$. It is a good exercise to find the conditions s must satisfy for this to give a well-defined module.

As one sees from this example, semi-free modules can be constructed inductively. One can take these notions further and define “twisted complexes.”

5.2. Modules over DG-categories. Recall that we can view a DG-ring R as a DG-category with one object (say $*$ with $\text{End}(*) = R$). From this viewpoint, a right DG-module over R is a contravariant DG-functor from the corresponding category to $C(k)$.³ The free R -module $R \cdot 1$ is given by the representable functor h_* . The nice thing about this viewpoint is that the Yoneda embedding lets us start with a single free module and obtain all other modules as suitable colimits.

We can generalize this to talk about modules over a DG-category.

³The subtleties about covariance / contravariance and left / right cannot be avoided entirely.

Definition 5.3. Let \mathbf{A} be a DG-category. The *DG-category of left \mathbf{A} -modules* $\mathbf{C}(\mathbf{A})$ is given by $\mathrm{Fun}^{\mathrm{dg}}(\mathbf{A}, \mathbf{C}(\mathbf{k}))$. Here $\mathrm{Hom}_{\mathbf{A}}(M, N)$ is the complex of maps of functors $M \rightarrow N$ (not respecting the differential), with differential given by the (graded) commutator $[d, -]$. Right \mathbf{A} -modules are defined similarly, starting from \mathbf{A}^{op} .

Taking 0-cycles in Hom-complexes gives $Z^0(\mathbf{C}(\mathbf{A}))$, the “original” category of DG-functors from \mathbf{A} to \mathbf{C} . Morphisms in $Z^0(\mathbf{C}(\mathbf{A}))$ are genuine chain maps. We can also consider the homotopy category $\mathrm{HoC}(\mathbf{A}) = H^0(\mathbf{C}(\mathbf{A})) = |\mathbf{C}(\mathbf{A})|$.

One can think of a left \mathbf{A} -module M as a collection of complexes $M(x)$ for $x \in \mathbf{A}$ together with maps $M(x) \rightarrow M(y)$ for every map $x \rightarrow y$ in \mathbf{A} (respecting the unit and composition). If \mathbf{A} is small, one can think of M as a labeled diagram.

Definition 5.4. A basic free \mathbf{A} -module⁴ is given by a (co)representable functor $h_x = \mathrm{Hom}(x, -)$ for $x \in \mathbf{A}$. A general *free \mathbf{A} -module* is a direct sum of shifts of basic free modules h_{x_i} (for $x_i \in \mathbf{A}$). *Semi-free \mathbf{A} -modules* are defined in the same way as before (i.e. as countable increasing unions of \mathbf{A} -modules with all relevant subquotients free).

Note that if $M = \oplus_i h_{x_i}[d_i]$, then we can understand $M(a)$ concretely for any a :

$$M(a) = \oplus_i \mathrm{Hom}(x_i, a)[d_i].$$

6. LECTURE 6, 6/28 (B. ANTIEAU) – ???

6.1. More on ∞ -categories. Recall that an ∞ -category is a weak Kan complex, i.e. a simplicial set satisfying an inner horn filling condition. We can think of this as follows:

- (1) 0-simplices are objects.
- (2) 1-simplices are morphisms.
- (3) 2-simplices are homotopies / witnesses to composition.
- (4) 3-simplices homotopies between homotopies, etc.

Here’s another viewpoint. Let \mathbf{C} be an ∞ -category. For any $x, y \in \mathbf{C}$, we can define a Kan complex $\mathrm{Map}_{\mathbf{C}}^R(x, y)$ together with composition maps

$$\circ : \mathrm{Map}^R(x, y) \times \mathrm{Map}^R(y, z) \rightarrow \mathrm{Map}^R(x, z)$$

which are defined, associative, unital, etc. all up to homotopy. So ∞ -categories are “categories enriched in Kan complexes” in a suitable sense.

A functor of ∞ -categories is a just a map of the associated simplicial sets. However, this can be hard to specify directly (since we need to declare where simplices at all levels go). This happens even if we are considering maps $N(\mathbf{C}) \rightarrow D$, where $N(\mathbf{C})$ is the nerve of a 1-category and D is an ∞ -category.

6.2. The ∞ -category of spaces. Let S be the ∞ -category of *spaces* (or *anima*, if you’re hip). The idea is that S is to Cat_{∞} as Set is to Cat_1 . One can construct S as follows.

Let Cat_{Δ} be the category of *simplicial categories* (i.e. categories enriched in simplicial sets). In a simplicial category \mathbf{C} , morphisms form simplicial sets $\mathrm{Map}_{\mathbf{C}}(x, y)$, and composition, units, etc. are defined so that all identities hold on the nose.

There exists a *homotopy-coherent nerve functor* $N_{\bullet}^{\mathrm{hc}} : \mathrm{Cat}_{\Delta} \rightarrow \mathrm{sSet}$ given by $N_{\bullet}^{\mathrm{hc}}(\mathbf{C}) = \mathrm{Hom}_{\mathrm{Cat}_{\Delta}}([\mathbf{n}], \mathbf{C})$ (details including a definition of $[\mathbf{n}]$ can be found in

⁴This terminology is made up.

Kerodon or HTT). If $\mathrm{Map}_{\mathcal{C}}(x, y)$ is a Kan complex for all $x, y \in \mathcal{C}$, then $\mathbf{N}_{\bullet}^{\mathrm{hc}}(\mathcal{C})$ is an ∞ -category, and we have a natural equivalence

$$\mathrm{Map}_{\mathcal{C}}(x, y) \simeq \mathrm{Map}_{\mathbf{N}_{\bullet}^{\mathrm{hc}}(\mathcal{C})}^{\mathbf{R}}(x, y).$$

Recall that an equivalence of simplicial sets is just a (weak) homotopy equivalence of their geometric realizations.

Definition 6.1. The ∞ -category of spaces \mathbf{S} is $\mathbf{N}_{\bullet}^{\mathrm{hc}}(\mathbf{Kan})$, where \mathbf{Kan} is the simplicial category of Kan complexes.

Remark 6.2.

- (1) Like \mathbf{Set} , \mathbf{S} has all (homotopy) limits and colimits. Recall that homotopy limits and colimits are computed using fibrant / cofibrant replacements, much like how we use projective / injective resolutions in derived categories.
- (2) A functor of ∞ -categories $F : \mathcal{C} \rightarrow \mathcal{D}$ is fully faithful if, for all $x, y \in \mathcal{C}$, the induced map $\mathrm{Map}_{\mathcal{C}}(x, y) \rightarrow \mathrm{Map}_{\mathcal{D}}(F(x), F(y))$ is an equivalence in \mathbf{S} .⁵ Such an F is an equivalence if furthermore every $z \in \mathcal{D}$ is equivalent to $F(x)$ for some $x \in \mathcal{C}$.
- (3) We can view \mathbf{S} as obtained by inverting weak equivalences in \mathbf{Kan} as an ∞ -category (if we invert weak equivalences as a 1-category, we just get $\mathrm{Ho}(\mathbf{S})$).

6.3. **Limits and Colimits.** Omitted for time...

6.4. **Model Categories.**

Definition 6.3. A *model category* is a category \mathcal{C} with three distinguished classes of morphisms: *weak equivalences* W , *fibrations* F , and *cofibrations* C , all closed under composition. From these we also get two more classes: *trivial fibrations* $W \cap F$ and *trivial cofibrations* $W \cap C$. These are required to satisfy the following axioms:

- (1) \mathcal{C} has all limits and colimits.
- (2) W has a two-out-of-three property (if any two of f , g , and $g \circ f$ are in W , then the third is also in W).
- (3) W , F , and C are closed under retracts. (In particular they contain all isomorphisms.)
- (4) In the following diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ \uparrow i & & \downarrow p \\ B & \longrightarrow & Y, \end{array}$$

if i is a cofibration and p is a trivial fibration, or i is a trivial cofibration and p is a fibration, then the dashed lift exists.

- (5) Every $f : X \rightarrow Y$ in \mathcal{C} factorizes as a cofibration followed by a trivial fibration, and also as a trivial cofibration followed by a fibration.

In fact, it suffices to define W and C (resp. W and F), as the missing class of maps can be recovered as the class for which the lifting property of the fourth axiom holds.

⁵Here we write Map for the object of \mathbf{S} corresponding to the simplicial set $\mathrm{Map}^{\mathbf{R}}$.

We say an object $X \in \mathcal{C}$ is fibrant (resp. cofibrant) if the map from X to the final object is a fibration (resp. the map from the initial object to X is a cofibration).

Model categories give a nice theory of “inverting at weak equivalences.” Furthermore, model categories provide notions of (left and right) homotopy.

Definition 6.4. Consider the “fold map” $X \sqcup X \rightarrow X$, and factorize this as a cofibration $X \sqcup X \rightarrow X \times I$ followed by a trivial fibration $X \times I \rightarrow X$. Here $X \times I$ is notation – not a literal product of X with an interval (though this is the intuition). We say that maps $f, g : X \rightarrow Y$ are left homotopic, written $f \sim^\ell g$, if $f \sqcup g : X \sqcup X \rightarrow Y$ factors through $X \sqcup X \rightarrow X \times I$.

If X is cofibrant, then \sim^ℓ is an equivalence relation on $\text{Hom}_{\mathcal{C}}(X, Y)$ for all $Y \in \mathcal{C}$. We define $\pi_\ell(X, Y) = \text{Hom}_{\mathcal{C}}(X, Y) / \sim^\ell$. We can similarly define a notion of right homotopy \sim^r , which gives an equivalence relation on $\text{Hom}_{\mathcal{C}}(X, Y)$ (and hence a definition of $\pi_r(X, Y)$) when Y is fibrant. When X is cofibrant and Y is cofibrant, these notions coincide, and we let $\pi(X, Y) = \pi_r(X, Y) = \pi_\ell(X, Y)$.

Let \mathcal{C}^{cf} be the set of objects of \mathcal{C} which are both fibrant and cofibrant. We can make this into a category with $\text{Hom}_{\mathcal{C}^{\text{cf}}}(X, Y) = \pi(X, Y)$. This gives a construction of $\mathcal{C}[W^{-1}]$, i.e. $\mathcal{C}[W^{-1}] = \mathcal{C}^{\text{cf}}$.

Definition 6.5. An adjunction of model categories $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ is a Quillen adjunction if it satisfies the following equivalent conditions:

- (1) F preserves \mathcal{C} and $W \cap \mathcal{C}$, or
- (2) G preserves \mathcal{D} and $W \cap \mathcal{D}$, or
- (3) F preserves \mathcal{C} and G preserves F , or
- (4) F preserves $W \cap \mathcal{C}$ and G preserves $W \cap \mathcal{D}$.

For a Quillen equivalence, we can define a left derived functor $\mathbf{L}F$ by setting $\mathbf{L}F(Y)$ to be F evaluated on a cofibrant replacement of Y .

6.5. CDGAs. One category of “derived commutative rings” is given by $\text{CDga}_{\mathbb{Q}, \geq 0}$, the category of connective (i.e. homologically non-negatively graded) CDGAs. We say that a map $f : A_\bullet \rightarrow B_\bullet$ is an equivalence if it is a quasi-isomorphism of complexes.

One can show that $\text{CDga}_{\mathbb{Q}, \geq 0}$ has a model category structure with

- (1) W is equivalences (i.e. quasi-isomorphisms).
- (2) F is the class of $A_\bullet \rightarrow B_\bullet$ which are degreewise surjections in positive degrees (not necessarily 0).
- (3) \mathcal{C} is determined by W and F .

In this model category structure, all objects are fibrant (the map to the zero CDGA is always a levelwise surjection).

Why are we restricting to nonnegatively graded CDGAs? The positive homotopy / homology groups should capture “nilpotence” in derived algebraic geometry. By contrast, negative homotopy / homology groups capture stacky behavior. Another problem is that quasicohherent sheaf theory is poorly understood in the non-connective case.

Definition 6.6. Let $\text{DAlg}_{\mathbb{Q}}^{\text{cn}}$ be the ∞ -category associated to $\text{CDga}_{\mathbb{Q}, \geq 0}$.⁶

⁶To construct this, one would like a simplicial model structure on $\text{CDga}_{\mathbb{Q}, \geq 0}$, but this does not seem to appear directly in the literature. Indirect approaches are possible.

There exists an adjunction $\mathrm{DAlg}_{\mathbb{Q}}^{\mathrm{cn}} \rightleftarrows \mathrm{CAlg}_{\mathbb{Q}}$, where the top map is given by $A_{\bullet} \mapsto H_0(A) = \pi_0(A)$. This lets us reduce some questions about a connective CDGA to questions about an ordinary commutative ring. It is not clear what a useful analogue of this would be for non-connective CDGAs.

Example 6.7. There exists a Quillen adjunction

$$\mathrm{Sym}_{\mathbb{Q}} : \mathrm{Ch}_{\geq 0}(\mathbb{Q}) \rightleftarrows \mathrm{CDga}_{\mathbb{Q}, \geq 0} : \text{forget}.$$

(The model structure on $\mathrm{Ch}_{\geq 0}(\mathbb{Q})$ is defined with the same conditions used for $\mathrm{CDga}_{\mathbb{Q}, \geq 0}$, and every object in $\mathrm{Ch}_{\geq 0}(\mathbb{Q})$ is fibrant and cofibrant.) This adjunction yields an adjunction of ∞ -categories

$$\mathrm{Sym}_{\mathbb{Q}} : \mathrm{D}_{\geq 0}(\mathbb{Q}) \rightleftarrows \mathrm{DAlg}_{\mathbb{Q}}^{\mathrm{cn}} : \text{forget}.$$

One can compute the image of $\mathrm{Sym}_{\mathbb{Q}}$ explicitly:

$$\mathrm{Sym}_{\mathbb{Q}}(\mathbb{Q}[n]) \simeq \begin{cases} \mathbb{Q}[x_n] & n \text{ even} \\ \mathbb{Q}[x_n]/(x_n^2) & n \text{ odd,} \end{cases}$$

where the differentials are trivial in both cases.

Furthermore, we have

$$\mathrm{Map}_{\mathrm{DAlg}_{\mathbb{Q}}^{\mathrm{cn}}}(\mathrm{Sym}_{\mathbb{Q}}(\mathbb{Q}[n], A) \simeq \mathrm{Map}_{\mathrm{D}_{\geq 0}(\mathbb{Q})}(\mathbb{Q}[n], A) \simeq \mathrm{Map}_{\mathrm{D}_{\geq 0}(\mathbb{Q})}(\mathbb{Q}, \tau_{\geq 0}A[-n]).$$

Thus

$$\pi_i(\mathrm{Map}_{\mathrm{DAlg}_{\mathbb{Q}}^{\mathrm{cn}}}(\mathrm{Sym}_{\mathbb{Q}}(\mathbb{Q}[n], A)) \cong \pi_i(\mathrm{Map}_{\mathrm{D}_{\geq 0}(\mathbb{Q})}(\mathbb{Q}, \tau_{\geq 0}A[-n])) \cong \begin{cases} 0 & i < 0 \\ \pi_{i-n}(A) = H_{i-n}(A) & i \geq 0. \end{cases}$$

Example 6.8. We can also try to understand quotients of symmetric CDGAs. Say we want to construct something that behaves like $\mathbb{Q}[x_{2n}]/(x_{2n}^2)$. The naïve construction produces an object which is not cofibrant. It is better to instead consider the homotopy pushout $A = \mathbb{Q}[x_{2n}] \otimes_{\mathbb{Q}[x_{4n}]} (\mathbb{Q}[x_{4n}] \otimes \bigwedge_{\mathbb{Q}}(y_{4n+1}))$, where in the latter tensor factor we set $d(y_{4n+1}) = x_{4n}$. We end up with $A \cong \mathbb{Q}[x_{2n}] \otimes \bigwedge_{\mathbb{Q}}(y_{4n+1})$, with $d(y_{4n+1}) = x_{2n}$.

7. LECTURE 7, 6/29 (B. ANTIEAU) – CONSTRUCTION OF ∞ -CATEGORIES AND MORE DERIVED COMMUTATIVE RINGS

7.1. Bootstrapping. To clarify some points that may have been confusing in the last lecture, we will discuss some of the “bootstrapping” processes we use in the study of ∞ -categories.

Consider the 1-categories \mathbf{sSet} (of simplicial sets) and \mathbf{Cat}_{Δ} (of simplicial categories). Both of these are model categories:

- In the Joyal model structure on \mathbf{sSet} , W consists of “categorical equivalences” (arising from the following Quillen adjunction) and C consists of monomorphisms.⁷
- In the model structure on \mathbf{Cat}_{Δ} , W consists of \mathbf{sSet} -enriched equivalences, i.e. $F : C \rightarrow D$ such that $|\mathrm{Map}_C(x, y)| \rightarrow |\mathrm{Map}_D(F(x), F(y))|$ is a weak homotopy equivalence for all x, y , and F is essentially surjective up to \simeq .

⁷Last time, we considered the Quillen model structure, where W consists of f such that the geometric realization $|f|$ is a weak homotopy equivalence.

We have a Quillen adjunction

$$\text{Path}[-] : \mathbf{sSet} \leftrightarrow \mathbf{Cat}_\Delta : \mathbf{N}_\bullet^{\text{hc}}(-).$$

Note that maps of simplicial sets naturally form a simplicial set:

$$\text{Map}_{\mathbf{sSet}}(X_\bullet, Y_\bullet)_n = \text{Hom}_{\mathbf{sSet}}(X_\bullet \times \Delta^n, Y_\bullet).$$

If X_\bullet and Y_\bullet are ∞ -categories, then $\text{Map}_{\mathbf{sSet}}(X_\bullet, Y_\bullet)$ is an ∞ -category.

Let \mathbf{WKan} be the simplicial category with objects weak Kan complexes and morphisms given by the maximal sub-Kan complex of $\text{Map}_{\mathbf{sSet}}(X_\bullet, Y_\bullet)$. Then we can define \mathbf{Cat}_∞ , the ∞ -category of ∞ -categories as

$$\mathbf{Cat}_\infty = \mathbf{N}_\bullet^{\text{hc}}(\mathbf{WKan}).$$

Note that this construction loses information: we'd really rather have an $(\infty, 2)$ -category of $(\infty, 1)$ -categories, but this would require a good theory of $(\infty, 2)$ -categories.

Another approach to bootstrapping ∞ -categories is as follows. Suppose we have a “simplicial model category” \mathbf{M} – this implies in particular that the simplicial enrichment on the full subcategory \mathbf{M}^{cf} of fibrant-cofibrant objects is actually an enrichment in Kan complexes. Then $\mathbf{N}_\bullet^{\text{hc}}(\mathbf{M}^{\text{cf}})$ is an ∞ -category.

For a general model category \mathbf{M} , we can construct a “hammock localization” $L^H \mathbf{M}$, which is a simplicial category. We can furthermore apply a functor Ex^∞ which turns simplicial categories into categories enriched in Kan complexes. Applying $\mathbf{N}_\bullet^{\text{hc}}$ gives an ∞ -category $\mathbf{N}_\bullet^{\text{hc}}(\text{Ex}^\infty(L^H \mathbf{M}))$. More details can be found in a paper of Dwyer and Kan.

A third approach involves starting with a category \mathbf{M} with a notion of weak equivalences W and constructing an ∞ -category $\mathbf{N}_\bullet(\mathbf{M})[W^{-1}]$ satisfying the expected universal property of localization in ∞ -categories.

A final approach works for DG-categories. Starting from a DG-category \mathbf{C} , Lurie gives a direct construction of the DG-nerve $\mathbf{N}_\bullet^{\text{dg}}(\mathbf{C})$, which is an ∞ -category. Alternatively, using Dold-Kan (and systematically using good truncations), we can convert \mathbf{C} into a simplicially enriched category and instead construct $\mathbf{N}_\bullet^{\text{hc}}(\mathbf{C})$. There is a natural map $\mathbf{N}_\bullet^{\text{hc}}(\mathbf{C}) \rightarrow \mathbf{N}_\bullet^{\text{dg}}(\mathbf{C})$, which Lurie shows is a trivial fibration. The resulting ∞ -category will be stable if \mathbf{C} is pre-triangulated.

7.2. More on derived commutative rings. Recall some relevant model categories:

- $\mathbf{CDga}_{\mathbb{Q}, \geq 0}$ has W consisting of qisos and F consisting of degreewise surjections in positive degrees. The corresponding ∞ -category is $\mathbf{DAlg}_{\mathbb{Q}}^{\text{cn}}$.
- $\mathbf{Ch}_{\geq 0}(\mathbb{Q})$ has W consisting of qisos, F consisting of degreewise surjections in positive degrees, and C consisting of degreewise injections. The corresponding ∞ -category is $\mathbf{D}(\mathbb{Q})_{\geq 0}$.
- $\mathbf{Ch}(\mathbb{Q})$ has W consisting of qisos, F consisting of degreewise surjections, and C consisting of degreewise injections. The corresponding ∞ -category is $\mathbf{D}(\mathbb{Q})$.

Example 7.1. Why are we working over \mathbb{Q} ? The answer is that strange and undesirable phenomena can occur if we work naïvely over \mathbb{Z} . Considering the chain complex A_\bullet given by

$$\dots \longleftarrow 0 \longleftarrow \mathbb{Z} \xleftarrow{\text{id}} \mathbb{Z} \longleftarrow 0 \longleftarrow \dots$$

with \mathbb{Z} 's appearing in degree 1 and 2. Then $\mathrm{Sym}_{\mathbb{Z}}^{\mathrm{CDga}}(\mathbf{A}_{\bullet}) = \mathbb{Z}[x] \otimes \mathbb{Z}[y]/(2y^2)$ with $dx = y$ and $dy = 0$. In particular $d(x^n) = nx^{n-1}y$, and since n is not invertible, we obtain $H_{2n-1} \cong \mathbb{Z}/n$. But the chain complex we started with is acyclic and cofibrant, so we would hope to get a “symmetric algebra” which is also acyclic. This example shows that there is no model structure on $\mathrm{CDga}_{\mathbb{Z}, \geq 0}$ such that the usual monadic adjunction $\mathrm{Ch}_{\geq 0}(\mathbb{Z}) \leftrightarrow \mathrm{CDga}_{\mathbb{Z}, \geq 0}$ is a Quillen adjunction.

How can we make things work over \mathbb{Z} ?

7.3. Animated (simplicial) commutative rings. When working over \mathbb{Z} , we instead consider $\mathrm{sCAlg}_{\mathbb{Z}}$, the category of simplicial commutative rings, equipped with a model structure where W and F are as in the Quillen model structure on the underlying simplicial sets. (This definition makes sense over any commutative ring R .) We will write $\mathrm{DAlg}_{\mathbb{R}}^{\mathrm{cn}}$ for the ∞ -category obtained by localizing $\mathrm{sCAlg}_{\mathbb{R}}$ at weak equivalences. For $R = \mathbb{Q}$, this yields an ∞ -category equivalent to the one constructed earlier.

The usual “free algebra / abelian group” monadic adjunctions extend to Quillen adjunctions between sSet , sAb , and $\mathrm{sCAlg}_{\mathbb{R}}$. When working with the corresponding ∞ -categories, we consider the corresponding derived functors.

Example 7.2. (1) $\pi_{\bullet} \mathbf{L} \mathrm{Sym}(\mathbb{Z}[0]) = \mathbb{Z}[x]$ with $|x| = 0$.
 (2) $\pi_{\bullet} \mathbf{L} \mathrm{Sym}(\mathbb{Z}[1]) = \bigwedge_{\mathbb{Z}}(y) = \mathbb{Z}[y]/y^2$ with $|y| = 1$.
 (3) $\pi_{\bullet} \mathbf{L} \mathrm{Sym}(\mathbb{Z}[2]) = \Gamma_{\mathrm{pd}(z)}$, the divided power algebra in $\mathbb{Q}[z]$ spanned over \mathbb{Z} by the elements $z^n/n!$. Here $|z| = 2$.

The complexity of $\mathbf{L} \mathrm{Sym}$ can be understood from the following Lemma.

Lemma 7.3. *For $n \geq 2$, we have $\mathbf{L} \mathrm{Sym}(\mathbb{Z}[n]) = C_{\bullet}(K(\mathbb{Z}, n), \mathbb{Z})$.*

This can be proved using the bar resolution (for $K(\mathbb{Z}, n) = \mathrm{pt}/K(\mathbb{Z}, n-1)$) and the equivalence

$$C_{\bullet}(K(\mathbb{Z}, n), \mathbb{Z}) \simeq \mathbb{Z} \otimes_{C_{\bullet}(K(\mathbb{Z}, n), \mathbb{Z})}^{\mathbf{L}} \mathbb{Z}.$$

The adjunction $D(\mathbb{Z})_{\geq 0} \leftrightarrow \mathrm{DAlg}_{\mathbb{Z}}^{\mathrm{cn}}$ forms a commutative square of adjunctions with $\mathrm{Mod}_{\mathbb{Z}} \leftrightarrow \mathrm{CAlg}_{\mathbb{Z}}$ (where the maps from classical objects to derived objects are inclusions and the maps from derived objects to classical objects are π_0).

One way of understanding $\mathrm{DAlg}_{\mathbb{Z}}^{\mathrm{cn}}$ is through the following Theorem.

Theorem 7.4 (Quillen, Lurie). *The inclusion $\mathrm{CAlg}_{\mathbb{Z}}^{\mathrm{fp}, \mathrm{poly}} \hookrightarrow \mathrm{DAlg}_{\mathbb{Z}}^{\mathrm{cn}}$ (where the left hand side is the category of classical finitely generated polynomial rings) induces an equivalence*

$$\mathrm{Fun}_{\Sigma}(\mathrm{DAlg}_{\mathbb{Z}}^{\mathrm{cn}}, D) \simeq \mathrm{Fun}(\mathrm{CAlg}_{\mathbb{Z}}^{\mathrm{fp}, \mathrm{poly}}, D)$$

for any D with sifted colimits (where Fun_{Σ} denotes sifted colimit preserving functors).

Recall that $\mathrm{DAlg}_{\mathbb{Z}}^{\mathrm{cn}}$ is presentable, so it has all colimits. In this context, $F : \mathrm{DAlg}_{\mathbb{Z}}^{\mathrm{cn}} \rightarrow D$ preserves sifted colimits if it preserves filtered colimits and geometric realizations. “Filtered” here means ω -filtered, where a simplicial set I is κ -filtered (for some regular cardinal κ) if for every J with a κ -small set of nondegenerate simplices, if J^{\triangleright} denotes J with a cone point added on the right, then every map $J \rightarrow I$ factors through J^{\triangleright} . Directed colimits give one example of filtered colimits.

The above theorem lets us construct (sifted colimit-preserving) functors out of $\mathrm{DAlg}_{\mathbb{Z}}^{\mathrm{cn}}$ “by hand,” i.e. by defining them explicitly on $\mathrm{CAlg}_{\mathbb{Z}}^{\mathrm{fp}, \mathrm{poly}}$. For example, we can define de Rham cohomology in this way.

8. LECTURE 8, 6/29 (D. ARINKIN) – PERSPECTIVES ON MODULES

For a DG-category A , we defined the DG-category $C(A)$ of left A -modules (i.e. DG-functors $A \rightarrow C(k)$) and the subcategory of semi-free modules $C(A)^{\text{semifree}}$. Here are some useful perspectives on these categories.

First: the Yoneda embedding gives a functor $h : A \hookrightarrow C(A^{\text{op}})$, and the image of h generates the subcategory $C(A^{\text{op}})^{\text{semifree}}$ under shifts, direct sums, and countable extensions. Since $C(A^{\text{op}})^{\text{semifree}}$ is closed under shifts, direct sums, and finite extensions, it is also closed under cones and homotopy colimits. We write $C(A^{\text{op}})^{\text{semifree}} = \text{Ind}(A)$, the *ind-completion* of A .

Second: the DG-category $C(A)^{\text{semifree}}$ is “basically” the derived category $D(A)$. More precisely, note that we can define acyclicity and qisos in $C(A)$ pointwise. Then:

Proposition 8.1.

- (1) For all $M \in C(A)$, there is a qiso $F \rightarrow M$ for $F \in C(A)^{\text{semifree}}$.
- (2) For all $F \in C(A)^{\text{semifree}}$ and acyclic $N \in C(A)$, $\text{Hom}(F, N)$ is acyclic.

Corollary 8.2. The composite $C(A)^{\text{semifree}} \hookrightarrow C(A) \rightarrow C(A)/C(A)^{\text{acyclic}} = D(A)$ is a quasi-equivalence.

Third: Embedding into $C(A^{\text{op}})$ gives a “nicer version” of A . Suppose we have a DG-functor $F : A \rightarrow B$. We obtain a “restriction” functor $C(B^{\text{op}}) \rightarrow C(A^{\text{op}})$ via $(F : B^{\text{op}} \rightarrow C(k)) \mapsto (A^{\text{op}} \rightarrow B^{\text{op}} \rightarrow C(k))$. Note that the square

$$\begin{array}{ccc} A, & \xhookrightarrow{h} & C(A^{\text{op}}) \\ \downarrow F & & \downarrow \\ B & \xhookrightarrow{h} & C(B^{\text{op}}) \end{array}$$

does not commute. For example, the restriction of a free module may no longer be free.

We also have an “extension” functor $C(B^{\text{op}})C(A^{\text{op}})$ (defined using \otimes), which does form a commutative square

$$\begin{array}{ccc} A, & \xhookrightarrow{h} & C(A^{\text{op}}) \\ \downarrow F & & \uparrow \\ B & \xhookrightarrow{h} & C(B^{\text{op}}) \end{array}$$

In particular, suppose F is a quasi-equivalence. Recall from the problem sets that F need not have a “quasi-inverse” $B \rightarrow A$. However, let $h(A)$ and $h(B)$ be the respective quasi-essential images under the Yoneda embedding (both consisting of “free rank-1 unshifted complexes up to qiso”). Then extension gives a quasi-equivalence $h(A) \rightarrow h(B)$, and restriction (being the right adjoint of extension) gives an inverse quasi-equivalence. That is, passing to $h(-)$ creates quasi-inverses for quasi-equivalences. We can think of this as a form of fibrant replacement.

Definition 8.3. A *quasi-functor* $A \rightarrow B$ is a DG-functor $F : A \rightarrow C(B^{\text{op}})$ with image contained in $h(B)$.

We can think of a quasi-functor F as an “ (A, B) -bimodule” such that the underlying B -module lies in $h(B)$.

Remark 8.4. Given DG-categories A and B , we define a new DG-category $A \otimes B$ as follows:

- $\text{Ob}(A \otimes B) = \text{Ob}(A) \times \text{Ob}(B)$.
- $\text{Hom}_{A \otimes B}(a_1 \otimes b_1, a_2 \otimes b_2) = \text{Hom}_A(a_1, a_2) \otimes_k \text{Hom}_B(b_1, b_2)$.
- Composition involves a sign rule.

An (A, B) -bimodule is an $A \otimes B^{\text{op}}$ -module.

Recall that $C(A^{\text{op}})^{\text{semifree}}$ is generated by $h(A)$ under shifts, direct sums, and infinite extensions. We can also consider smaller subcategories. For example, the subcategory A^{pretr} generated by $h(A)$ under finitely iterated cones, finite direct sums (a special case of cones), and shifts gives the category of “finitely generated semifree modules.” We call A^{pretr} the *pre-triangulated completion* of A (because $\text{Ho}(A^{\text{pretr}})$ is triangulated).

Example 8.5. If R is a classical ring, the pre-triangulated completion of R (viewed as a one-object DG-category) gives the category of bounded complexes of free modules of finite rank.

Definition 8.6. If $h : A \rightarrow A^{\text{pretr}}$ is a quasi-equivalence, we say that A is a *pre-triangulated DG-category*.

Exercise 8.7. Describe explicit conditions for A to be pre-triangulated.

Remark 8.8. Perfect modules do not quite fit into the picture discussed so far – we also need to allow for (homotopy) direct summands.

DISCUSSION, 6/29 (K. SCHEFERS) - HOMOTOPICAL THINKING

The goal of this discussion is to teach some “folk ∞ -category theory” – i.e. intuition for ∞ -categories and homotopy theory.

A key slogan: “Never say ‘equals,’ only ‘equivalent.’” In practice, this means that we religiously avoid the word “is” when comparing two objects. This is essentially taking the notion of “equivalence” from 1-category theory to its logical extreme.

Example 8.9. In the setting of ∞ -categories, we have notions of objects, morphisms, functors, and composition. However, we do not have a notion of faithful functors. Let’s explain why.

In 1-category theory, a functor F is faithful if, whenever $f \neq g$, it follows that $F(f) \neq F(g)$. In ∞ -category theory, it is possible that $f \simeq g$ but $f \neq g$, so injectivity of $\text{Map}(x, y) \rightarrow \text{Map}(F(x), F(y))$ no longer makes sense.

If we pass to π_0 (i.e. to the homotopy category), we can regain some 1-categorical notions. As one expects, this loses a lot of information.

One might wonder how homotopy types arise out of this perspective. Suppose we have an ∞ -category (say a Kan complex) C and $x, y \in \text{Ob } C$. We cannot ask that $x = y$, only that we have an equivalence $f : x \rightarrow y$. This can be viewed as a 1-simplex from x to y . Similarly, given $f, g : x \rightarrow y$, we cannot ask that $f = g$, only that we have an equivalence $f \rightarrow g$. This would give a 2-simplex. Continuing on with higher morphisms would give a homotopy type corresponding to $\text{Map}(x, y)$.

9. LECTURE 9, 6/30 (D. ARINKIN) – DIFFERENT DG-CATEGORIES OF MODULES

9.1. Refresher on quasi-functors. Recall that we want to consider DG-categories up to quasi-equivalence. This can be made precise using a model structure on the category of DG-categories due to Tabuada.⁸

Theorem 9.1 (Toën). *After localization, the category of morphisms between $A, B \in \mathbf{DGCat}$ “is” the derived category of quasifunctors from A to B .*

Recall that a quasi-functor from A to B is an (A, B) -bimodule M (i.e. an $(A \otimes B^{\mathrm{op}})$ -module) such that, for any $a \in A$, the corresponding B^{op} -module is quasi-representable. Explicitly, if we view M as $M : A \otimes B^{\mathrm{op}} \rightarrow C(k)$ and fix $a \in A$, then the functor $M(a, -) : B^{\mathrm{op}} \rightarrow C(k)$ is qiso to h_b for some $b \in B$.

9.2. Various categories of modules. Let A be a DG-category. Recall that $C(A^{\mathrm{op}})$ contains a subcategory $\mathbf{Ind}(A) = C(A^{\mathrm{op}})^{\mathrm{semifree}}$ obtained from A using direct sums, cones, shifts, and countable extensions. We have a further subcategory $A^{\mathrm{pretr}} \subset \mathbf{Ind}(A)$ obtained from A using cones and shifts. A variant of this construction is the “Karoubi completion” $(A^{\mathrm{pretr}})^{\mathrm{Kar}}$, given by

$$\{x \in C(A^{\mathrm{op}}) \mid \exists y \text{ s.t. } x \oplus y \in A^{\mathrm{pretr}}\}.$$

Modules satisfying this condition are called *perfect*, and we also write this last category as $\mathbf{Perf}(A^{\mathrm{op}})$.

Thus, in increasing order, we have:

- (1) A , an arbitrary DG-category.
- (2) A^{pretr} , a pre-triangulated DG-category.
- (3) $\mathbf{Perf}(A^{\mathrm{op}})$, a Karoubian pre-triangulated DG-category.
- (4) $\mathbf{Ind}(A) = D(A^{\mathrm{op}})$, a cocomplete (pre-triangulated) DG-category.

If we consider a classical ring as a one-object DG-category, these four levels correspond to:

- (1) R as a one-object DG-category.
- (2) The DG-category of bounded complexes of finite-rank free R -modules.
- (3) The DG-category of bounded complexes of finite-rank projective R -modules.
- (4) The DG-derived category of R .

Remark 9.2. In a general DG-category, shifts and cones need not exist. However, those that exist are natural and “as unique as possible.” In particular, they are preserved by DG functors.

Example 9.3. Consider $C(A^{\mathrm{pretr}})$, the category of DG-functors $A^{\mathrm{pretr}} \rightarrow C(k)$. Such functors are determined by their values on $A \subset A^{\mathrm{pretr}}$, so $C(A^{\mathrm{pretr}}) \xrightarrow{\sim} C(A)$. Therefore, when considering categories of modules, we may as well assume that A is pretriangulated.

The above Example shows in particular that $(A^{\mathrm{pretr}})^{\mathrm{pretr}} \simeq A^{\mathrm{pretr}}$. Similarly, we can show that $((A^{\mathrm{pretr}})^{\mathrm{Kar}})^{\mathrm{pretr}} \simeq (A^{\mathrm{pretr}})^{\mathrm{Kar}}$. However, $\mathbf{Ind}(-)$ is not an idempotent operation, i.e. $\mathbf{Ind}(\mathbf{Ind}(A)) \not\simeq \mathbf{Ind}(A)$, as the construction of \mathbf{Ind} “freely

⁸One has to be cautious about the 2-category theory involved. Drinfeld asked the famous question, “What do DG-categories form?” We will not provide a complete answer to this question here.

adds colimits.” This can be explained by the fact that DG-functors do not automatically respect infinite direct sums. However, we do have $\mathrm{Ind}((A^{\mathrm{pretr}})^{\mathrm{Kar}}) \simeq \mathrm{Ind}(A)$.

9.3. Small and large DG-categories. We would like to separate DG-categories into “small” and “large” classes. The following gives a notion of what we will consider “large.”

Definition 9.4. A DG-category is (cocomplete) compactly generated if it is quasi-equivalent to $\mathrm{Ind}(A)$ for some A .

We can pass from “small” DG-categories to “large” DG-categories by taking Ind . To go from “large” DG-categories to “small” DG-categories, we pass to the subcategory of compact objects.

We would like these operations to be functorial in the DG-categories involved. For “small” categories, we consider DG (quasi-)functors, while for “large” categories we must consider *continuous* (i.e. colimit-preserving) DG (quasi-)functors.

Suppose that a DG-functor $F : \mathrm{Ind}(A_1) \rightarrow \mathrm{Ind}(A_2)$ is continuous. Then F is determined by its values on A_1 , but it need not be induced by a (quasi-)functor $A_1 \rightarrow A_2$. Thus Ind is not a full functor (i.e. more morphisms appear).

10. LECTURE 10, 6/30 (B. ANTIEAU) – MORE ∞ -CATEGORIES AND DERIVED COMMUTATIVE RINGS

Recall that for a commutative ring k , we define $\mathrm{DAlg}_k^{\mathrm{cn}} := \mathrm{sCalg}_k[W^{-1}]$.

Theorem 10.1 (Quillen, Lurie). *If D is an ∞ -category with sifted colimits, then*

$$\mathrm{Fun}_{\Sigma}(\mathrm{DAlg}_k^{\mathrm{cn}}, D) \xrightarrow{\sim} \mathrm{Fun}(\mathrm{Calg}_k^{\mathrm{fp}, \mathrm{poly}}, D),$$

where Fun_{Σ} denotes the ∞ -category of sifted colimit-preserving functors.

10.1. Presentable ∞ -categories.

Definition 10.2. An ∞ -category C is *presentable* if it is an accessible localization

$$\mathrm{Psh}(C_0) \hookrightarrow C$$

for some small ∞ -category C_0 , where $\mathrm{Psh}(C_0) = \mathrm{Fun}(C_0^{\mathrm{op}}, S)$. The “accessible localization” condition means that $C \rightarrow \mathrm{Psh}(C_0)$ is fully faithful and preserves κ -filtered colimits for some κ .

Here are some useful facts.

- (1) Presentable ∞ -categories have all colimits and limits.
- (2) Left adjoints (always) preserve colimits, and right adjoints (always) preserve colimits. The adjoint functor theorem below gives a converse to this.

Theorem 10.3 (Adjoint Functor Theorem).

- (1) *Let C be a presentable ∞ -category, D an ∞ -category with all colimits, and $F : C \rightarrow D$ a colimit-preserving functor. Then F admits a right adjoint.*
- (2) *Let C and D be presentable ∞ -categories and $G : D \rightarrow C$ a functor that preserves limits and κ -filtered colimits for some κ . Then G admits a left adjoint.*

The reason “ κ -filtered colimits” are so important here is that presentable ∞ -categories have generators which are “ κ -compact,” so behavior of presentable ∞ -categories can be understood by looking at these generators.

Presentability is often obvious in practice from the construction of an ∞ -category. Lurie showed in HTT that many relevant model categories have presentable homotopy-coherent nerves.

Here are some examples of adjunctions (I was a bit rushed and might have gotten the handedness wrong).

Example 10.4. The ∞ -category \mathcal{S} of spaces has a natural adjunction $\pi_0 : \mathcal{S} \leftrightarrow \mathbf{Set} : G$. Both categories here are presentable. However, G does not preserve colimits. For a simplicial set X_\bullet , the colimit in \mathbf{Set} is just the coequalizer of $X_1 \rightrightarrows X_0$, whereas the coequalizer in \mathcal{S} is the geometric realization $|X_\bullet|$. (In particular, one can consider the pushout $\mathrm{pt} \sqcup_{\mathcal{S}^0} \mathrm{pt}$.)

Example 10.5. Let I be a simplicial set, and write $S^I = \mathrm{Fun}(I, \mathcal{S})$. The constant functor $\mathcal{S} \rightarrow S^I$ preserves both limits and colimits. Therefore this constant functor has a left adjoint (viz. I -indexed colimits) and a right adjoint (viz. I -indexed limits).

Example 10.6. Similarly, the inclusion $\mathcal{S} \rightarrow \mathbf{Cat}_\infty$ preserves both limits and colimits. The right adjoint is the “groupoid core” functor, and the left adjoint is the “geometric realization” functor.

Example 10.7. The functor $\mathbf{Cat} \rightarrow \mathbf{Cat}_\infty$ preserves limits and has a left adjoint given by the “homotopy category” functor $C \mapsto \mathrm{Ho}C$.

Here are a few constructions we can make starting from an ∞ -category C . The constructions are listed in decreasing order of inclusion (i.e. $\mathrm{Ind}(C) \subset \mathrm{Psh}_\Sigma(C) \subset \mathrm{Psh}(C)$). Details can be found in Chapter 5 of HTT.

- (1) For any C , we can define $\mathrm{Psh}(C) = \mathrm{Fun}(C^{\mathrm{op}}, \mathcal{S})$. The corresponding Yoneda embedding preserves no colimits. For any presentable D , we have $\mathrm{Fun}(C, D) = \mathrm{Fun}^L(\mathrm{Psh}(C), D)$, the category of left adjoint functors. Every object of $\mathrm{Psh}(C)$ is a colimit of representables.
- (2) For any C with finite coproducts, we can define $\mathrm{Psh}_\Sigma(C) = \mathrm{Fun}^\Pi(C^{\mathrm{op}}, \mathcal{S})$, the category of finite-product-preserving presheaves. The corresponding Yoneda embedding preserves finite coproducts. For any presentable D , we have $\mathrm{Fun}(C, D) = \mathrm{Fun}_\Sigma(\mathrm{Psh}_\Sigma(C), D)$. We have $\mathrm{Fun}^\Pi(C, D) = \mathrm{Fun}^L(\mathrm{Psh}_\Sigma(C), D)$. Every object of $\mathrm{Psh}(C)$ is a geometric realization $|X_\bullet|$ where each X_n is a filtered colimit of representables.
- (3) For any C with finite coproducts, we can define $\mathrm{Ind}(C) = \mathrm{Fun}^{\mathrm{lex}}(C^{\mathrm{op}}, \mathcal{S})$, the category of finite-limit-preserving presheaves. The corresponding Yoneda embedding preserves finite colimits. For any presentable D , we have $\mathrm{Fun}(C, D) = \mathrm{Fun}_\omega(\mathrm{Ind}(C), D)$, the category of filtered-colimit-preserving functors. We have $\mathrm{Fun}^{\mathrm{rex}}(C, D) = \mathrm{Fun}^L(\mathrm{Ind}(C), D)$. Every object of $\mathrm{Psh}(C)$ is a filtered colimit of representables.

Remark 10.8. The filtered simplicial sets I are exactly those such that $\mathrm{colim} : S^I \rightarrow \mathcal{S}$ preserves finite limits. Similarly, I is sifted if and only if $\mathrm{colim} : S^I \rightarrow \mathcal{S}$ preserves finite coproducts.

10.2. Animation. Let's return to the Theorem from the start of the lecture. From the above perspective, we can reinterpret the theorem as coming from an equivalence

$$\mathbf{DAlg}_k^{\text{cn}} \simeq \mathbf{Psh}_\Sigma(\mathbf{CAlg}_k^{\text{fp,poly}}).$$

The \mathbf{Psh}_Σ construction is useful enough that people call it *animation* and write $\mathbf{An}(-) = \mathbf{Psh}_\Sigma(-)$. People like to think of this as capturing the “spirit” or “anima.”

Example 10.9. $\mathbf{S} \simeq \mathbf{An}(\mathbf{FinSet}) \simeq \mathbf{Psh}(\mathbf{pt}) \simeq \mathbf{Ind}(\text{compact spaces})$.

Example 10.10. An ∞ -categorical version of Dold-Kan gives $\mathbf{D}(\mathbb{Z})_{\geq 0} \simeq \mathbf{An}(\mathbf{Mod}_{\mathbb{Z}}^{\text{free,fp}})$.

10.3. \mathbb{E}_∞ -rings. Another useful flavor of “derived commutative rings” is given by \mathbb{E}_∞ -rings. These arise as algebras / modules (terminology is inconsistent in general) for a monad on $\mathbf{D}(k)$.

Recall that a monad T on an ∞ -category \mathbf{C} is just an algebra object in the monoidal category of endofunctors $\mathbf{End}(\mathbf{C})$ (what's the problem?). This means we have a composition $T^2 \rightarrow T$ and a unit $\text{id} \rightarrow T$ satisfying some (homotopy-)commutative diagrams. A T -algebra X is an object $X \in \mathbf{C}$ together with a map $TX \rightarrow X$ again satisfying some (homotopy-)commutative diagrams.

The “free \mathbb{E}_∞ -algebra” monad on $\mathbf{D}(k)$ is given by

$$T(M) = \bigoplus_{n \geq 0} (M^{\otimes n})_{h\Sigma_n},$$

where the subscript $h\Sigma_n$ refers to homotopy Σ_n -orbits. A T -algebra A therefore comes with a “product map”

$$\bigoplus_{n \geq 0} (A^{\otimes n})_{h\Sigma_n} \rightarrow A,$$

where factoring through homotopy Σ_n -orbits captures the commutativity of the product.

Example 10.11. The free \mathbb{E}_∞ -algebra on \mathbb{Z} is given by $\bigoplus_{n \geq 0} (\mathbb{Z}^{\otimes n})_{h\Sigma_n}$. Computing homology ends up giving the group homology groups $H_*(\Sigma_n, \mathbb{Z}^{\sigma(n)})$. These are “free” but complicated and very “non-flat,” explaining why many people prefer to consider simplicial commutative rings.

We write $\mathbf{CAlg}_k^{\mathbb{E}_\infty}$ for the ∞ -category of \mathbb{E}_∞ -rings. If we consider \mathbb{E}_∞ -algebras in $\mathbf{D}(k)_{\geq 0}$, we obtain a subcategory $\mathbf{CAlg}_k^{\mathbb{E}_\infty}$ of connective \mathbb{E}_∞ -rings. Note that we can now make sense of non-connective “derived commutative rings.”

10.4. Four flavors of derived commutative rings. Let us recall our previous categories of derived commutative rings and introduce a new example.

- (1) For $k \supset \mathbb{Q}$, we can define $\mathbf{DAlg}_k^{\text{cn}} = \mathbf{CDga}_{k, \geq 0}[W^{-1}]$. This is very concrete but does not work over \mathbb{Z} .
- (2) For any k , we can define $\mathbf{DAlg}_k^{\text{cn}} = \mathbf{sCAlg}_k[W^{-1}]$. When $k \supset \mathbb{Q}$, this is equivalent to the above construction. This works over \mathbb{Z} , and the free objects are not too bad, but it's hard to directly write down a simplicial commutative ring (doing so requires an infinite amount of information).
- (3) We can define $\mathbf{CAlg}_k^{\mathbb{E}_\infty}$ as above. This works over the sphere spectrum, but many complicated homology groups appear.
- (4) Bhatt and Mathew define a category \mathbf{DAlg}_k as follows. They show that \mathbf{LSym} extends to a monad on $\mathbf{D}(k)$. Then \mathbf{DAlg}_k is the ∞ -category of algebras over this monad. This accounts for examples such as $\mathbf{R}\Gamma(X, \mathcal{O})$ (for a scheme X) but is less concrete than the other approach.

DISCUSSION, 6/30 (D. ARINKIN) – WHAT IS WRONG WITH TRIANGULATED CATEGORIES?

There are three problems (maybe more) with triangulated categories:

- (1) Gluing does not work.
- (2) Cones are not functorial.
- (3) Categories are not well-behaved.

Let's discuss each of these in more detail.

For gluing, suppose X is a scheme. We can consider the category of quasicoherent sheaves $\mathrm{QCoh}(X)$ and the corresponding derived category $\mathrm{D}_{\mathrm{qcoh}}(X)$. We can construct every $\mathcal{F} \in \mathrm{QCoh}(X)$ from sheaves $\mathcal{F}_i \in \mathrm{QCoh}(\mathcal{U}_i)$ (for an open cover $\{\mathcal{U}_i\}$ of X) satisfying the cocycle condition. However, this does not work in the derived category.

Example 10.12. Suppose $X = \mathcal{U}_1 \cup \mathcal{U}_2$, with inclusions $j_i : \mathcal{U}_i \rightarrow X$ and $j_{12} : \mathcal{U}_{12} \rightarrow X$. Then, to glue \mathcal{F}_1 and \mathcal{F}_2 along \mathcal{F}_{12} , we would like to let $\mathcal{F} = \mathrm{cone}(j_{1*}\mathcal{F}_1 \oplus j_{2*}\mathcal{F}_2 \rightarrow j_{12*}\mathcal{F}_{12})[-1]$. However, in the triangulated context, this cone is defined up to non-natural isomorphism. This can be okay for gluing along two opens, but it ruins things when we try to iterate the process and glue along more opens.

The same example illustrates why non-functoriality of cones is troublesome. When we work in the DG-context, cones are functorial and everything works well.

To see the bad categorical behavior of triangulated categories, note that we would like for suitable functors between categories of quasicoherent sheaves to correspond functorially to Fourier-Mukai transforms.

Example 10.13. Let E be an elliptic curve. Then the Fourier-Mukai transform with kernel \mathcal{O}_Δ gives the identity functor. We can compute that $\mathrm{Ext}^2(\mathcal{O}_\Delta, \mathcal{O}_\Delta) \neq 0$. This should correspond to a natural transformation from id_E to $\mathrm{id}_E[2]$. But one can show that no such natural transformation exists when we work in the triangulated context!

11. LECTURE 11, 7/3 (B. ANTIEAU) – DERIVED STACKS

After our setup work, we can move on to something more geometric.

11.1. Prestacks. Recall that the functor of points perspective lets us work intelligently with presheaves which are not necessarily representable. Let's see how far we can take this approach.

Let k be a commutative ring, and write $\mathrm{DAlg}_k^{\mathrm{cn}}$ for $\mathrm{CDga}_k[W^{-1}]$ (when $\mathbb{Q} \subset k$) or $\mathrm{sAlg}_k[W^{-1}]$ (in general). We define the category of derived affine schemes over k to be $\mathrm{dAff}_k = (\mathrm{DAlg}_k^{\mathrm{cn}})^{\mathrm{op}}$. Define the ∞ -category of derived prestacks as

$$\mathrm{PreStk}_k = \mathrm{Psh}(\mathrm{dAff}_k) = \mathrm{Fun}(\mathrm{dAff}_k^{\mathrm{op}}, \mathcal{S}) = \mathrm{Fun}(\mathrm{DAlg}_k^{\mathrm{cn}}, \mathcal{S}).$$

The “stacky” behavior of objects here (e.g. the fact that points may have automorphisms) comes from the fact that spaces are ∞ -groupoids.

For any $R \in \mathrm{DAlg}_k^{\mathrm{cn}}$, we let $\mathrm{Spec} R$ be the corresponding representable functor in PreStk_k , i.e.

$$(\mathrm{Spec} R)(S) = \mathrm{Map}_{\mathrm{DAlg}_k^{\mathrm{cn}}}(R, S).$$

11.2. Topologies.

Definition 11.1. A map $R \rightarrow S$ in $\mathbf{DAlg}_k^{\text{cn}}$ is *flat* if⁹

- (1) The map of commutative rings $\pi_0 R \rightarrow \pi_0 S$ is flat, and
- (2) The action map $\pi_i R \otimes_{\pi_0 R} \pi_0 S \rightarrow \pi_i S$ is an isomorphism for all $i \in \mathbb{Z}$.

The upshot is that the underlying map of commutative rings is flat and that all $\pi_i S$ are induced from $\pi_i R$ freely.

Example 11.2. If $R \rightarrow S$ is flat and R is discrete / classical (i.e. $\pi_i R = 0$ for $i \neq 0$), then S is discrete and flat over R .

Example 11.3. The derived commutative ring $\mathbf{L}\text{Sym}_k(k[0])$ is flat over k : we have $\pi_* \mathbf{L}\text{Sym}_k(k[0]) = k[x]$ where $|x| = 0$. Note that the analogous statement is not true for the free \mathbb{E}_∞ -ring over \mathbb{Z} .

Example 11.4. The derived commutative ring $\mathbf{L}\text{Sym}_{\mathbb{Q}}(\mathbb{Q}[0])$ is not flat over \mathbb{Q} , since $\pi_* \mathbf{L}\text{Sym}_{\mathbb{Q}}(\mathbb{Q}[0]) = \mathbb{Q}[x_2]$ is not discrete. One possible intuition for this is that x_2 is an “infinitesimal” or “higher nilpotent,” though the analogy isn’t complete since degree zero nilpotents don’t present problems.

Example 11.5. The identity map $R \rightarrow R$, or more generally any equivalence $R \xrightarrow{\sim} S$, is flat.

Example 11.6. For $R \in \mathbf{DAlg}_k^{\text{cn}}$ and $f \in \pi_0 R$, we can construct a new derived commutative ring $R[1/f]$ as the (homotopy) pushout in the following square:

$$\begin{array}{ccc} R[x] & \xrightarrow{x \mapsto uf-1} & R[u] \\ \downarrow x \mapsto 0 & & \downarrow \\ R & \longrightarrow & R[1/f] \end{array}$$

This is a homotopical version of the classical construction. Here (and elsewhere), we write $R[x]$ (or $R[u]$) for $\mathbf{L}\text{Sym}_k R[0] = (\mathbf{L}\text{Sym}_k k[0]) \otimes_k R$. Note that $R[1/f]$ is flat over R , and $\pi_*(R[1/f]) = (\pi_* R)[1/f]$. (One can justify these last few statements using a base change argument which may be discussed in the exercises.)

We can lift our classical τ -topologies (i.e. fpqc, fppf, étale, or Zariski) to the derived context from π_0 . Namely, the τ -topology on \mathbf{dAff}_k is generated by finite covers $\{\text{Spec } S_i \rightarrow \text{Spec } R\}$ where each $R \rightarrow S_i$ is flat and $\{\text{Spec } \pi_0 S_i \rightarrow \text{Spec } \pi_0 R\}$ is a classical τ -cover.

11.3. Extensions of étale algebras. Let $R \in \mathbf{CAlg}_k$ and $I \subset R$ be a nilpotent ideal. Then the category of $\mathbf{Ét}_R$ of étale algebras over R is equivalent to the category $\mathbf{Ét}_{R/I}$ of étale algebras over R/I . For example, every finite separable field extension over k extends (uniquely, functorially, and essentially surjectively) to an étale algebra over $k[x]/(x^n)$. One can think of this as being an étale analogue of the equivalence of $\text{Spec } R$ and $\text{Spec } R/I$ as non-ringed topological spaces.

Theorem 11.7 (Lurie). *For $R \in \mathbf{DAlg}_{\mathbb{Z}}^{\text{cn}}$, we have an equivalence*

$$\mathbf{Ét}_R \simeq \mathbf{Ét}_{\pi_0 R}.$$

⁹Other notions of flatness do appear in the literature (e.g. one could ask that $\oplus_i \pi_i R \rightarrow \oplus_i \pi_i S$ is flat as a map of graded rings, but we won’t use this.)

Thus we can construct étale morphisms of commutative rings from étale morphisms of classical rings.

Example 11.8. Let $R = \mathbf{L}\mathrm{Sym}_{\mathbb{Q}}(\mathbb{Q}[2])$, so $\pi_* R = \mathbb{Q}[x_2]$ for $|x_2| = 2$. We can view R as a limit of its truncations, which are all nilpotent extensions of $\pi_0 R = \mathbb{Q}$ (i.e. R is “pro-nilpotent”). Namely, $R = \lim_n (\tau_{\leq 2n} R)$, where $\pi_* \tau_{\leq 2n} R = \mathbb{Q}[x_2]/(x_2^{n+1})$. Thus, extending étale extensions of $\pi_0 R$ along each truncation successively, we should expect the above theorem of Lurie to hold. The underlying homotopical idea here is “convergence of the Postnikov tower.” Note that the analogous statement does not hold for $\mathbf{L}\mathrm{Sym}_{\mathbb{Q}}(\mathbb{Q}[0])$, as all terms are concentrated in degree zero and we cannot “truncate higher powers away” homotopically.

11.4. Smoothness.

Definition 11.9. A morphism of derived commutative rings is *smooth* if it is flat and it is smooth on π_0 .

Note that smooth and faithful implies fppf.

Example 11.10. $\mathbf{L}\mathrm{Sym}_k(k[0])$ is smooth over k .

11.5. Faithfully flat descent.

Construction 11.11. Let $R \rightarrow S$ be a morphism in $\mathrm{DAlg}_k^{\mathrm{cn}}$. There exists an “Amitsur” / “descent” / “Čech” complex, i.e. an augmented cosimplicial diagram

$$R \rightarrow S \rightrightarrows S \otimes_R S \rightarrow \dots$$

Really, $S^\bullet = (S \rightrightarrows S \otimes_R S \rightarrow \dots)$, as an object of $\mathrm{cDAlg}_k^{\mathrm{cn}} = \mathrm{Fun}(\Delta, \mathrm{DAlg}_k^{\mathrm{cn}})$, is a cosimplicial object in an ∞ -category, which takes some work to set up honestly.

Remark 11.12. If $R \rightarrow S$ is faithfully flat (i.e. faithful on π_0 and flat), then

$$R \xrightarrow{\sim} \mathrm{Tot} S^\bullet := \lim_{\Delta} S^\bullet.$$

The term “totalization” (and the notation Tot) is historically used for limits of cosimplicial objects. This claim follows from classical results: on π_0 , this is faithfully flat descent, while for higher π_i , this follows from our definition of flatness.

11.6. (Derived) stacks.

Definition 11.13. We define the *category of derived stacks* $\mathrm{Stk}_k \subset \mathrm{PreStk}_k$ as the full subcategory of τ -sheaves $\mathrm{Sh}_\tau(\mathrm{dAff}_k) \subset \mathrm{Psh}(\mathrm{dAff}_k)$. Here \mathcal{F} is a τ -sheaf if, for every τ -cover $\mathrm{Spec} S \rightarrow \mathrm{Spec} R$, the induced map $\mathcal{F}(R) \rightarrow \mathrm{Tot} \mathcal{F}(S^\bullet)$ is an equivalence of spaces.

Example 11.14. Any affine derived scheme $\mathrm{Spec} R$ is a derived stack (for any of our τ -topologies). To see this, note that for any $S \rightarrow T$, we get a descent complex $S \rightarrow T^\bullet$, and

$$(\mathrm{Spec} R)(S) \rightarrow \mathrm{Tot}((\mathrm{Spec} R)(T^\bullet))$$

can be recast as

$$\mathrm{Hom}(R, S) \rightarrow \mathrm{Tot} \mathrm{Hom}(R, T^\bullet) \simeq \mathrm{Hom}(R, \mathrm{Tot}(T^\bullet)) \simeq \mathrm{Hom}(R, S),$$

which is an equivalence by faithfully flat descent.

Example 11.15. Let $\Omega^\infty \mathcal{O} := \operatorname{Spec} k[t]$ where $|t| = 0$. Then

$$(\Omega^\infty \mathcal{O})(S) = \operatorname{Map}_{\operatorname{DAlg}_k^{\text{en}}}(k[t], S) \simeq \Omega^\infty S,$$

the “underlying space of S ” (this is an object of S). Here $\pi_i \Omega^\infty(S) \cong \pi_i(S)$ for all $i \geq 0$.

Example 11.16. Let $\operatorname{GL}_1 := \operatorname{Spec} k[t^{\pm 1}]$. We can view $\operatorname{GL}_1(S)$ as the pullback

$$\begin{array}{ccc} \operatorname{GL}_1(S) & \longrightarrow & \Omega^\infty \\ \downarrow & & \downarrow \\ \pi_0(S)^\times & \longrightarrow & \pi_0 S. \end{array}$$

Here

$$\pi_i(\operatorname{GL}_1(S)) \cong \begin{cases} (\pi_0 S)^\times & i = 0 \\ \pi_i S & i \geq 1. \end{cases}$$

Example 11.17. Let $\mathbb{G}_m(S) := \pi_0(S)^\times$. Then \mathbb{G}_m is a derived stack but is not representable or geometric in any sense.

Example 11.18. Let $X = \operatorname{Spec} \mathbb{Z}[x]/(x^2)$ for $|x| = 0$. The pushout square

$$\begin{array}{ccc} \mathbb{Z}[x] & \xrightarrow{x \mapsto x^2} & \mathbb{Z}[x] \\ \downarrow x \mapsto 0 & & \downarrow \\ \mathbb{Z} & \longrightarrow & \mathbb{Z}[x]/(x^2) \end{array}$$

gives rise to a pullback square of derived stacks

$$\begin{array}{ccc} X(S) & \longrightarrow & \Omega^\infty S \\ \downarrow & & \downarrow \\ \operatorname{pt} & \longrightarrow & \Omega^\infty S, \end{array}$$

where the rightmost vertical map acts by $r \mapsto r^2$ on π_0 and $s \mapsto 2s$ on π_i . This yields short exact sequences

$$0 \longrightarrow \frac{\pi_1 S}{2} \longrightarrow \pi_0 X(0) \longrightarrow \{r \in \pi_0 S \mid r^2 = 0\} \longrightarrow 0$$

and

$$0 \longrightarrow \frac{\pi_{i+1} S}{2} \pi_i X(S) \longrightarrow (\pi_i)(S)[2] \longrightarrow 0,$$

allowing us to compute $\pi_i X(S)$.

12. LECTURE 12, 7/3 (D. ARINKIN) – COMPACTNESS AND DG-CATEGORIES

There are two main flavors of DG-categories we will consider:

- Pre-triangulated Karoubian DG-categories, exemplified by $\operatorname{Perf}(\mathbf{A})$. These are relatively “small,” and allow us to take cones, shifts, and direct summands. We like to consider arbitrary DG (quasi-)functors between such DG-categories.

- Cocomplete DG-categories, exemplified by $\text{Ind}(\mathcal{A}) = \mathcal{D}(\mathcal{A}^{\text{op}})$. These are relatively “large,” and allow us to take infinite direct sums in addition to the above. We like to consider continuous DG (quasi-)functors between such DG-categories (i.e. those DG-functors that respect arbitrary direct sums and hence arbitrary direct colimits).

Remark 12.1. If we have cones and infinite direct sums, we can obtain arbitrary colimits. This is discussed in the exercises. The idea is that the colimit over a diagram $M : I \rightarrow \mathcal{A}$ (with I a 1-category) is given by

$$\text{cone}(\oplus_{\alpha \rightarrow \beta} M_{\alpha} \rightarrow \oplus_{\beta \in I} M_{\beta}).$$

We can extend to this notion to computing geometric realizations of simplicial sets.

12.1. A key fact. For any DG-category \mathcal{A} and any cocomplete DG-category \mathcal{B} , we have

$$\text{Fun}(\text{Ind}(\mathcal{A}), \mathcal{B})^{\text{cont}} = \text{Fun}(\mathcal{A}, \mathcal{B}).$$

Example 12.2. Let R_1 and R_2 be DG-rings. Note that $\mathcal{D}(R_1) = \text{Ind}(\text{Perf}(R_1)) = \text{Ind}(R_1^{\text{op}})$, so

$$\text{Fun}(\mathcal{D}(R_1), \mathcal{D}(R_2))^{\text{cont}} = \text{Fun}(R_1^{\text{op}}, \mathcal{D}(R_2)) \simeq \mathcal{D}(R_1^{\text{op}} \otimes R_2).$$

We are cheating a bit here, involving the fact that we are working with \mathcal{D} and quasi-functors rather than \mathcal{C} and functors. Luckily, the two ways in which we cheat cancel each other out.

We can furthermore consider $\mathcal{D}(R_2) = \text{Ind}(\text{Perf}(R_2)) = \text{Ind}(R_2^{\text{op}})$ and use this perspective to construct smaller subcategories of $\mathcal{D}(R_1^{\text{op}} \otimes R_2)$ as categories of functors.

Example 12.3. Recall that $\text{Ind}(\mathcal{A}) = \text{Fun}(\mathcal{A}^{\text{op}}, \mathcal{D}(k))$ (since k is a field, there is not much difference between $\mathcal{C}(k)$ and $\mathcal{D}(k)$). Then $\text{Ind}(\mathcal{A}) = \text{Fun}(\text{Ind}(\mathcal{A}^{\text{op}}), \mathcal{D}(k))^{\text{cont}}$.

12.2. Compactly generated DG-categories. Last time, we declared that a DG-category \mathcal{B} is compactly generated if $\mathcal{B} \simeq \text{Ind}(\mathcal{A})$ for some \mathcal{A} .

We would like to disentangle the terms “compact” and “generation” here. Let \mathcal{B} be a cocomplete DG-category.

Definition 12.4. An object $x \in \mathcal{B}$ is compact if $\text{Hom}_{\mathcal{B}}(x, -)$ respects (infinite) direct sums. That is, the natural map

$$\oplus_{\alpha} \text{Hom}(x, x_{\alpha}) \rightarrow \text{Hom}(x, \oplus_{\alpha} x_{\alpha})$$

is a qiso. Note that $\text{Hom}_{\mathcal{B}}(x, -)$ always respects infinite direct products (when they exist). One can think of “compactness” as meaning that every map from x to an infinite direct sum always factors (up to qiso) as a map to a finite subsum.

Remark 12.5. In the abelian category RMod^{\heartsuit} :

- $\text{Hom}(M, -)$ respects infinite direct sums if M is finitely generated.
- $\text{Hom}(M, -)$ respects filtered colimits if M is finitely presented.

In a cocomplete DG-category, these notions coincide.

Proposition 12.6. Let \mathcal{A} be a DG-category and $F \in \mathcal{D}(\mathcal{A})$. Then F is compact if and only if F is perfect.

The proof is left as an exercise.

If \mathcal{B} is compactly generated, then $\mathcal{B} \simeq \text{Ind}(\mathcal{B}^c)$, where \mathcal{B}^c is the full subcategory of compact objects of \mathcal{B} . This explains the name!

Remark 12.7. Suppose $B \simeq \text{Ind}(A)$. We have

$$\text{Hom}_B(\oplus_\alpha x_\alpha, \oplus_\beta x'_\beta) = \prod_\alpha \text{Hom}(x_\alpha, \oplus_\beta x'_\beta).$$

If $x_\alpha, x'_\beta \in A$ for all α, β , then this becomes

$$\prod_\alpha \oplus_\beta \text{Hom}_A(x_\alpha, x'_\beta),$$

so we can think of a homomorphism as an infinite matrix with only finitely many nonzero entries in each column.

Corollary 12.8. *Let A be a DG-category, B a cocomplete DG-category, and $F : A \rightarrow B$ a DG-functor. Write $\tilde{F} : \text{Ind}(A) \rightarrow B$ for the corresponding continuous DG-functor. Suppose that F is (quasi-)fully faithful and that $F(A) \subset B^c$. Then \tilde{F} is also (quasi-)fully faithful. Moreover, \tilde{F} is an equivalence if and only if $A^\perp = 0$ (i.e. if $x \in B$ and $\text{Hom}(F(y), x)$ is acyclic for all $y \in A$, then $x \simeq 0$).*

We can reformulate the former statement as follows. Let B be cocomplete and $A \subset B^c$ a full subcategory. Then $\text{Ind}(A)$ is a full subcategory of B (the full cocomplete subcategory generated by A).

Example 12.9. Let B be a cocomplete DG-category and $x \in B$. Then the DG-ring $R = \text{End}_B(x)$ gives a full subcategory A of B (the full subcategory with one object, namely x). This extends to a continuous functor $\text{Ind}(A) = D(R^{\text{op}}) \rightarrow B$, given by $-\otimes_R x$. If x is compact, this functor is fully faithful.

Example 12.10. The claim about equivalences in the Corollary above gives $\text{Ind}(A) \simeq \text{Ind}((A^{\text{pretr}})^{\text{Kar}})$.

Example 12.11. Let $\text{Ob } A = \{x_1, \dots, x_n\}$. Then $\text{Ind}(A)$ is generated by the compact object $x_1 \oplus \dots \oplus x_n$. Letting $R = \text{End}(x_1 \oplus \dots \oplus x_n)$, we obtain $\text{Ind}(A) \simeq D(R^{\text{op}})$.

13. DISCUSSION, 7/3 (B. ANTIEAU) –

How can we explicitly compute the ring structure on $H_\bullet(\mathbb{CP}^\infty, \mathbb{Q})$?

Note that $\mathbb{CP}^\infty \simeq BS^1 \simeq K(\mathbb{Z}, 2)$, and \mathbb{CP}^∞ has a product map $\mathbb{CP}^\infty \times \mathbb{CP}^\infty \rightarrow \mathbb{CP}^\infty$ (corresponding to the tensor product of complex line bundles).¹⁰ As an algebra, we have

$$C_\bullet(\mathbb{CP}^\infty, \mathbb{Q}) \simeq \mathbb{Q} \otimes_{C_\bullet(S^1, \mathbb{Q})} \mathbb{Q}.$$

(using the derived tensor product of algebras here). Let's explain why.

How do we compute $R \otimes_T S$, the pushout in $D\text{Alg}_k^{\text{en}}$? We can understand this as the geometric realization of

$$\cdots \rightarrow R \otimes_k T \otimes_k S \rightrightarrows R \otimes_k S,$$

i.e. the geometric realization of the simplicial derived commutative ring $(R \otimes_k T^\bullet \otimes_k S)$ (induced by the bar construction). In general, this argument shows that any ∞ -category C with finite coproducts and geometric realizations has pushouts. The bar construction can also be used to construct BG as the geometric realization of the simplicial object $B_\bullet G = \text{pt} \times G^\bullet \times \text{pt}$.

¹⁰If one wants to work strictly, one can view $B^2\mathbb{Z}$ as a simplicial object and consider the codiagonal map.

We can obtain the algebra structure via the fold maps $T \sqcup T \rightarrow T$. In particular, we have

$$C_{\bullet}(BS^1, \mathbb{Q}) \simeq C_{\bullet}(|B_{\bullet}S^1|, \mathbb{Q}) \simeq |C_{\bullet}(B_{\bullet}S^1, \mathbb{Q})| \simeq |C_{\bullet}(BS^1, \mathbb{Q})^{\otimes \bullet}| \simeq \mathbb{Q} \otimes_{C_{\bullet}(S^1, \mathbb{Q})} \mathbb{Q}.$$

This gives one way of computing $C_{\bullet}(BS^1, \mathbb{Q})$.

An alternative approach is to view $\mathbf{L}\mathrm{Sym}_{\mathbb{Z}}(\mathbb{Z}[2])$ as a divided power algebra on a degree two generator. Yet another way is to try to use the cell structure on \mathbb{CP}^{∞} .

14. LECTURE 13, 7/5 (D. ARINKIN) – $\mathrm{QCoh}(X)$ AS A DG-CATEGORY

Last time, we discussed compactly generated DG-categories. Recall that, for a cocomplete DG-category B , an object $x \in B$ is compact if and only if $\mathrm{Hom}_B(x, -)$ is continuous (i.e. commutes with arbitrary colimits, or equivalently commutes with direct sums). Given any DG-category A , we can construct a cocomplete, compactly generated DG-category $\mathrm{Ind}(A)$. Conversely, given a cocomplete DG-category B , we obtain a pre-triangulated, Karoubian DG-category B^c consisting of compactly generated objects of B . Thus we obtain an equivalence between pre-triangulated, Karoubian DG-categories and cocomplete, compactly generated DG-categories.

If B is cocomplete, then we say $A \subset B^c$ generates B if $A^{\perp} = 0$. In this case, the natural map $\mathrm{Ind}(A) \rightarrow B$ is an equivalence. Furthermore, we obtain $B^c = (A^{\mathrm{pretr}})^{\mathrm{Kar}}$, i.e. B^c can be obtained from A by taking cones and direct summands. Conversely, if B is compactly generated, then any $A \subset B^c$ with $B^c = (A^{\mathrm{pretr}})^{\mathrm{Kar}}$ generates B .

14.1. $\mathrm{QCoh}(X)$. We are interested in studying the (derived) category of quasicoherent sheaves $\mathrm{QCoh}(X)$ on a scheme X . One classical construction of this is as the derived category of an abelian category $\mathrm{QCoh}(X)^{\heartsuit}$ (objects of which are constructed by gluing along affines). This works fine for nice schemes but doesn't behave well in general.

A better construction is to take the derived category of arbitrary \mathcal{O}_X -modules and then restrict to the full subcategory of objects with quasicoherent cohomology. This solves some problems with the other approach:

- $\mathrm{QCoh}(X)^{\heartsuit}$ typically doesn't have enough projectives (more generally, "nice" resolutions by quasicoherent sheaves may not exist).
- Injectives in $\mathrm{QCoh}(X)^{\heartsuit}$ are not a priori injective as sheaves (leading to some question of how to define sheaf cohomology – whether to use the derived pushforward of $\mathrm{QCoh}(X)$ or $\mathrm{Sh}(X)$).

If we work with DG-categories, we can actually define $\mathrm{QCoh}(X)$ directly by gluing.

Definition 14.1. For a (derived) prestack X , we define

$$\mathrm{QCoh}(X) = \lim_{\mathrm{Spec} R \rightarrow X} D(R).$$

If X is a scheme, it suffices to take the limit over affine opens $\mathrm{Spec} R \subset X$ (so in particular, when X is classical, we only need to consider classical rings R). The limit appearing here is a limit of (cocomplete) DG-categories which is indexed by an ∞ -category.

Example 14.2. If X is a quasicompact separated scheme, then we can write $X = \bigcup_{i=1}^n U_i$ with each $U_i = \mathrm{Spec} R_i$ affine and all intersections also affine. An object $\mathcal{F} \in \mathrm{QCoh}(X)$ is specified by:

- (1) A collection $\mathcal{F}_i \in \mathrm{QCoh}(\mathcal{U}_i) = D(\mathcal{R}_i)$.
- (2) A collection of quasi-isomorphisms $\mathcal{F}_i|_{\mathcal{U}_i \cap \mathcal{U}_j} \xrightarrow{\sim} \mathcal{F}_j|_{\mathcal{U}_i \cap \mathcal{U}_j}$ in $D(\mathcal{R}_{ij})$.
- (3) Further compatibilities of these isomorphisms (chain homotopies and higher homotopies) on triple intersections, quadruple intersections, etc.

For X quasicompact, this process terminates after taking finitely many intersections (but can still be annoying).

Example 14.3. Suppose $X = \mathcal{U} \cup \mathcal{V}$. Then the limit definition lets us write

$$\mathrm{QCoh}(X) = \mathrm{QCoh}(\mathcal{U}) \times_{\mathrm{QCoh}(\mathcal{U} \cap \mathcal{V})} \mathrm{QCoh}(\mathcal{V}).$$

Thus an object of $\mathrm{QCoh}(X)$ is given by a pair of objects of $\mathrm{QCoh}(\mathcal{U})$ and $\mathrm{QCoh}(\mathcal{V})$ together with a quasi-isomorphism of the restrictions to $\mathrm{QCoh}(\mathcal{U} \cap \mathcal{V})$. One can compute Hom in this category using the Čech complex.

14.2. Compact generation of $\mathrm{QCoh}(X)$. We can apply the above arguments to understanding compact objects in $\mathrm{QCoh}(X)$.

Definition 14.4. An object $\mathcal{F} \in \mathrm{QCoh}(X)$ is perfect if $\mathcal{F}|_{\mathrm{Spec} \mathcal{R}}$ is perfect for all $\mathrm{Spec} \mathcal{R} \rightarrow X$. (It suffices to check this property on an open cover of X .)

Corollary 14.5. Let X be a quasicompact separated scheme. Then

$$(\mathrm{QCoh}(X))^c = \mathrm{Perf}(X).$$

The key idea here is that we can compute Hom using the Čech complex, and that in this case infinite direct sums and finite products commute.

We would like to generally understand when $\mathrm{QCoh}(X)$ is compactly generated. This is important and not immediate from the definitions (limits of compactly generated DG-categories need not be compactly generated).

Note that, for $X = \mathcal{U} \cup \mathcal{V}$, we always have

$$\mathrm{QCoh}(X) = \mathrm{QCoh}(\mathcal{U}) \times_{\mathrm{QCoh}(\mathcal{U} \cap \mathcal{V})} \mathrm{QCoh}(\mathcal{V}) \text{ and } \mathrm{Perf}(X) = \mathrm{Perf}(\mathcal{U}) \times_{\mathrm{Perf}(\mathcal{U} \cap \mathcal{V})} \mathrm{Perf}(\mathcal{V}).$$

However, this tells us nothing about whether the latter category generates the former.

One of the first affirmative answers to the question of compact generation is the following.

Theorem 14.6 (Thomason-Trobaugh). *If X is a quasicompact and separated scheme, then $\mathrm{Perf}(X)$ generates $\mathrm{QCoh}(X)$.*

Proof. By induction, we can reduce to the case $X = \mathcal{U} \cup \mathcal{V}$ where \mathcal{U} and $\mathcal{U} \cap \mathcal{V}$ are “perfect” (i.e. satisfy the conclusions of the theorem) and \mathcal{V} is affine. The outline is as follows.

- (1) Take generators $\mathcal{F}_\alpha \in \mathrm{Perf}(\mathcal{U})$. (If \mathcal{U} is affine, we can just take $\mathcal{F} = \mathcal{O}_{\mathcal{U}}$.) Extend these generators to $\tilde{\mathcal{F}}_\alpha \in \mathrm{Perf}(X)$. This is a technical process and can be done in a weak sense that suffices for the purposes of the proof.
- (2) The $\tilde{\mathcal{F}}_\alpha$ ’s should generate $\mathrm{Perf}(X)$ “modulo” sheaves supported on $Y := X \setminus \mathcal{U}$. Note that Y is closed in \mathcal{V} . So take enough perfect objects \mathcal{G}_β of

$$\mathrm{QCoh}(\mathcal{V})_Y = \{\mathcal{F} \in \mathrm{QCoh}(\mathcal{V}) \mid \mathcal{F}|_{\mathcal{U} \cap \mathcal{V}} = 0\}$$

to generate $\mathrm{QCoh}(\mathcal{V})_Y$. If $Y = \{f_1 = \cdots = f_k = 0\}$, then we can take \mathcal{G} to be the complex $\bigotimes_{i=1}^k (f_i : \mathcal{O}_{\mathcal{V}} \rightarrow \mathcal{O}_{\mathcal{V}})$. Extend these to $\tilde{\mathcal{G}}_\alpha \in \mathrm{Perf}(X)$ such that $\tilde{\mathcal{G}}_\alpha|_{\mathcal{U}} = 0$.

- (3) The collection $\{\tilde{\mathcal{F}}_\alpha\} \cup \{\tilde{\mathcal{G}}_\beta\}$ generates $\mathbf{QCoh}(X)$. This is comparatively easy to show. \square

15. LECTURE 14, 7/5 (B. ANTIEAU) - MORE ON DERIVED STACKS

Recall that the category of derived stacks (for the τ -topology) is defined as $\mathbf{Stk}_k = \mathbf{Sh}_\tau(\mathbf{dAff}_k)$.

15.1. Homotopy sheaves. Given a derived stack X , we can consider the assignment $R \mapsto \pi_0(X(R))$, which gives a presheaf of sets on \mathbf{dAff}_k . Write $\pi_0^\tau(X)$ for the sheafification of this presheaf. To capture higher homotopy groups, fix a point $x : \mathrm{Spec} k \rightarrow X$ and consider the assignments $R \mapsto \pi_i(X(R), x)$ (where we precompose the map x with the structure map $\mathrm{Spec} R \rightarrow \mathrm{Spec} k$). This yields a presheaf which we sheafify to gain sheaves of groups $\pi_i^\tau(X, x)$ (in fact, sheaves of abelian groups for $i \geq 2$).

Example 15.1. Recall that we defined $\Omega^\infty \mathcal{O}(R) = \Omega^\infty R$, the “underlying space / homotopy type” of R . Here $\pi_i(\Omega^\infty R, 0) \simeq H_i(R)$, so

$$\pi_i^\tau(\Omega^\infty \mathcal{O})(R) = H_i(R) = \pi_i(R).$$

Example 15.2. For $\mathbb{G}_m(R) = (\pi_0 R)^\times$, we get $\pi_0^\tau \mathbb{G}_m = \mathbb{G}_m$ and $\pi_i^\tau \mathbb{G}_m = 0$ for $i > 0$.

Example 15.3. For $\mathrm{GL}_1 = \mathrm{Spec} \mathbb{Z}[t^{\pm 1}]$, the pullback diagram

$$\begin{array}{ccc} \mathrm{GL}_1(R) & \longrightarrow & \Omega^\infty R \\ \downarrow & & \downarrow \\ (\pi_0 R)^\times & \longrightarrow & \pi_0 R \end{array}$$

gives $\pi_0^\tau \mathrm{GL}_1(R) = \mathbb{G}_m(R)$ and $\pi_i^\tau \mathrm{GL}_1(R) = \pi_i \Omega^\infty \mathcal{O}$ for $i \geq 0$.

15.2. Derived stacks and classical stacks. Recall that we have an adjunction

$$\mathbf{DAlg}_k^{\mathrm{cn}} \rightleftarrows \mathbf{CAlg}_k.$$

Passing to opposites, we get an adjunction

$$\iota : \mathbf{Aff}_k \rightleftarrows \mathbf{dAff}_k : \mathrm{Spec} \pi_0.$$

We can obtain a further adjunction between “higher stacks” (i.e. $\mathbf{HStk} = \mathbf{Sh}_\tau(\mathbf{Aff}_k)$) and derived stacks (i.e. \mathbf{Stk}_k). Namely, ι induces a “classical locus” functor

$$\iota^* = (-)^{\mathrm{cl}} : \mathbf{HStk}_k \rightarrow \mathbf{Stk}_k,$$

given by $X^{\mathrm{cl}}(R) = X(R)$, where X^{cl} is only defined on discrete commutative rings. The functor $(-)^{\mathrm{cl}}$ admits both a left adjoint ι_* and a right adjoint $\iota_!$. Here ι_* is defined on a derived commutative ring R by

$$(\iota_* X)(R) = X(\pi_0 R)$$

The left Kan extension $\iota_!$ is defined on a discrete commutative ring by

$$\iota_!(R) = X(R)$$

and is extended to derived commutative rings by taking colimits.

Example 15.4.

$$\mathbb{G}_m^{\text{cl}} \simeq \text{GL}_1^{\text{cl}} \simeq \mathbb{G}_m^{\heartsuit},$$

where the latter is \mathbb{G}_m viewed as a higher stack.

Example 15.5.

$$(\text{Spec } R)^{\text{cl}} \simeq \text{Spec } \pi_0 R.$$

Example 15.6. One can compute using adjunctions that

$$(\iota_* \text{Spec } R)(\text{Spec } S) = \text{Hom}(R, \pi_0 S).$$

Definition 15.7. A derived stack X is *classical* if $\iota_! X^{\text{cl}} \xrightarrow{\sim} X$.

Example 15.8. The derived stack $\text{GL}_1 = \text{Spec } \mathbb{Z}[t^{\pm 1}]$ is classical, even though it has higher homotopy sheaves when viewed as an object of Stk_k . The idea is that derived commutative rings can have interesting homotopies between units. Due to this and similar phenomena, one should be careful when taking homotopy sheaves (doing so is not really a “geometric” operation).

15.3. Geometric (derived) stacks: desired properties. We would like to find a collection of “nice” objects in Stk_k . By “nice,” we mean:

- (1) Affines $\text{Spec } R$ are nice.
- (2) τ -locally affine stacks should be nice.
- (3) Quotients by group schemes of nice stacks should be nice.

15.4. Algebraic spaces. Let τ be the étale topology. The slogan to remember for algebraic spaces is that a scheme is to the Zariski topology as an algebraic space is to the étale topology. Thus, one should view an algebraic space as being “étale locally affine.”

Definition 15.9. A derived stack X is a *derived algebraic space with affine diagonal* (DASAD) if there exists a surjection $\coprod_i \text{Spec } S_i \rightarrow X$ such that, for any $\text{Spec } R \rightarrow X$, the pullback $P = (\coprod_i \text{Spec } S_i) \times_X \text{Spec } R$ is $\text{Spec } T$ for an étale R -algebra T . Each $\text{Spec } S_i \rightarrow X$ should be thought of as “étale and affine.” For surjectivity, recall first that a map of sheaves of sets $Y \rightarrow X$ is surjective if for all $x \in X(\mathcal{U})$, there exists a cover $V \rightarrow \mathcal{U}$ such that $x|_V \in X(V)$ lifts to $Y(V)$. The same definition works for sheaves of spaces (i.e. derived stacks): a map $Y \rightarrow X$ of derived stacks is surjective if $\pi_0^\tau Y \rightarrow \pi_0^\tau X$ is surjective.

Remark 15.10. This definition of surjectivity is related to a remarkable property of S : given $Y \rightarrow X$ in S , if we take the Čech cover $Y_\bullet \rightarrow X$ where Y_\bullet is the simplicial object given degreewise by $Y \times_X Y \times_X \cdots \times_X Y$ (starting from $Y_0 = Y$), then $|Y_\bullet| \rightarrow X$ is an equivalence if and only if $\pi_0 Y \rightarrow \pi_0 X$ is an equivalence. The result here is known as “effectivity of groupoids.”

Example 15.11. A separated classical scheme X is a DASAD.

Definition 15.12. A morphism $X \rightarrow Y$ in Stk_k is a *relative DASAD* (i.e. RDASAD) if, for all $\text{Spec } R \rightarrow Y$, the pullback $X \times_Y \text{Spec } R$ is a DASAD.

Definition 15.13. A morphism $X \rightarrow \text{Spec } S$ in Stk_k is an *étale derived algebraic space* over $\text{Spec } S$ if there is a surjection $\coprod_i \text{Spec } S_i \rightarrow X$ such that

- (1) Each S_i is étale over S .
- (2) Each $\text{Spec } S_i \rightarrow X$ is a RDASAD.

Definition 15.14. A morphism $X \rightarrow Y$ in \mathbf{Stk}_k is *representable étale* if, for all $\text{Spec } R \rightarrow Y$, the pullback $X \times_Y \text{Spec } R$ is an étale algebraic space over $\text{Spec } R$.

Definition 15.15. A *derived algebraic space* (DAS) is an object $X \in \mathbf{Stk}_k$ such that there exists a surjection $\coprod_i \text{Spec } S_i \rightarrow X$ where each $\text{Spec } S_i \rightarrow X$ is representable étale.

Definition 15.16. A morphism $X \rightarrow Y$ in \mathbf{Stk}_k is a relative derived algebraic space if, for all $\text{Spec } R \rightarrow Y$, the pullback $X \times_Y \text{Spec } R$ is a DAS.

Definition 15.17. A DAS X is smooth over $\text{Spec } k$ if there exists a surjection $\coprod_i \text{Spec } S_i \rightarrow X$ such that each $\text{Spec } S_i \rightarrow X$ is representable étale and each S_i is a smooth k -algebra.

The reason we need so many definitions is due to the fact that not everything is separable.

Remark 15.18. Let $X \in \mathbf{Stk}_k$ be a DAS and $R \in \mathbf{CAlg}_k$. Then $X(R)$ is a set. (If we take R to be a derived commutative ring, we may get higher homotopy groups, but those would really be coming from R rather than X itself.)

The classical locus of a derived algebraic space is a classical algebraic space.

15.5. $B_\tau G$. Suppose G is an algebraic group scheme over k . We would like to say that $B_\tau G$, the stack of τ -locally trivial principal G -torsors, is geometric. For G smooth, this will be the Platonic example of an Artin stack.

A G -torsor is a map $Y \rightarrow X$ which is G -equivariant (for some G -action on Y and the trivial G -action on X) such that the natural map $G \times Y \rightarrow Y \times_X Y$ (given by $(g, y) \mapsto (gy, y)$) is an equivalence. If fix $x \in X$ and consider a point $y \in Y_x$ of the fiber, this map gives an equivalence $G \simeq Y_x$. The “ τ -locally trivial” condition here means that there exists a τ -cover $U \rightarrow X$ such that $G \times U \rightarrow Y \times_X U$ gives a G -equivariant equivalence. Note that the topology τ matters here – different τ will give different answers.

We have $H_\tau^1(X, G) = \pi_0(B_\tau G(X))$ and $H_\tau^0(X, G) = \pi_1(B_\tau G(X)) \simeq G(X)$.

One can also interpret $B_\tau G = \text{pt}/G$ (similarly to the construction of homotopy quotients of topological spaces). In the setting of derived stacks, one can consider the presheaf of spaces $R \mapsto B(G(R))$. We define $B_\tau G$ as the τ -sheafification of this presheaf.

15.6. Geometric derived stacks.

Definition 15.19. (1) A *relative 0-stack* is a relative DAS $X \rightarrow Y$. This is furthermore *smooth* if the fibers are smooth.

(2) A *relative n -stack* is a morphism $X \rightarrow Y$ in \mathbf{Stk}_k such that, for all $\text{Spec } R \rightarrow Y$, the pullback $P = X \times_Y \text{Spec } R$ admits a surjection $\coprod_i \text{Spec } S_i \rightarrow P$ which is a smooth relative $(n-1)$ -stack. This is furthermore *smooth* if we can choose the S_i to be smooth over R .

(3) An *n -stack* is a relative n -stack over $\text{Spec } k$.

Example 15.20. (1) BGL_1 is a smooth 1-stack.

(2) If G is a smooth affine group scheme and $G \curvearrowright X = \text{Spec } R$, then X/G is a 1-stack which is smooth if X is.

(3) $B^n GL_1$ is a smooth n -stack.

Remark 15.21. We have

$$\pi_1^{\acute{e}t} B^n GL_1 = \begin{cases} 0 & i < n \\ \mathbb{G}_m & i = n \\ \pi_{i-n} \Omega^\infty \mathcal{O} & i > n. \end{cases}$$

Also $(B^n GL_1)^{cl} \simeq K(\mathbb{G}_m^\vee, n)$.

16. DISCUSSION, 7/5 (MULTIPLE PRESENTERS)

16.1. J. Mundinger – ring structure on $H_\bullet(\mathbb{CP}^\infty)$. Here's a way to compute the ring structure on $H_\bullet(\mathbb{CP}^\infty, \mathbb{Z})$. We will assume as known the following:

- $H^\bullet(\mathbb{CP}^\infty, \mathbb{Z}) \cong \mathbb{Z}[c]$ where $|c| = 2$.
- $\mathbb{CP}^\infty = BS^1$.

Recall that we have a group structure on \mathbb{CP}^∞ given by

$$\nabla : BS^1 \times BS^1 \xrightarrow{\sim} B(S^1 \times S^1) \rightarrow BS^1.$$

Viewing BS^1 as classifying complex line bundles, ∇ is given (up to homotopy) by the tensor product of line bundles.

To compute the action of ∇ on homology, note that the dual map on homology is given by

$$\nabla^* : H^2(\mathbb{CP}^\infty, \mathbb{Z}) \rightarrow H^2(\mathbb{CP}^\infty \times \mathbb{CP}^\infty, \mathbb{Z}) \quad \mapsto c \otimes 1 + 1 \otimes c.$$

This comes from the formula for Chern classes of tensor products:

$$c_1(L \otimes L') = c_1(L) + c_1(L').$$

Note that $H^{2n}(\mathbb{CP}^\infty, \mathbb{Z})$ is generated by c^{2n} . Let $(c^n)^\vee$ be the dual generator of H_{2n} . We note

$$\begin{aligned} \langle \nabla_*^{n-1}(c^\vee \otimes \cdots \otimes c^\vee), c^n \rangle &= \langle c^\vee \otimes \cdots \otimes c^\vee, (\nabla^*)^{n-1} c^n \rangle \\ &= \langle c^\vee \otimes \cdots \otimes c^\vee, (c \otimes 1 \otimes \cdots \otimes 1) + \cdots + (1 \otimes \cdots \otimes 1 \otimes c^\vee) \rangle = n!. \end{aligned}$$

Therefore $(c^\vee)^n = n!(c^n)^\vee$, so our homology ring is a divided power algebra.

16.2. D. Arinkin – More on QCoh. Let X be a scheme. How can we connect our definition of $\text{QCoh}(X)$ (as a homotopy limit) to $D_{qc}(X)$, the derived category of sheaves on X with quasicoherent cohomology?

An object \mathcal{F} in $\text{QCoh}(X)$ is given by:

- (1) Objects $\mathcal{F}_R \in D(R)$ for every $\text{Spec } R \rightarrow X$
- (2) Quasi-isomorphisms $R' \otimes_{R''} \mathcal{F}_{R''} \xrightarrow{\sim} \mathcal{F}_{R'}$ for every $\text{Spec } R'' \rightarrow \text{Spec } R'$
- (3) Higher homotopies, et cetera...

Similarly, to define a (non-derived) \mathcal{O}_X -module on X , we specify the sheaf on open affines by:

- (1) Objects $M_R \in (\text{Mod}_R)^\heartsuit$
- (2) Morphisms $R' \otimes_{R''} M_{R''} \rightarrow M_{R'}$, etc.

We end up with a lax limit rather than a true limit (since the morphisms are not required to be isomorphisms).

16.3. D. Arinkin and B. Antieau – A derived category which does not come from an abelian category. Consider the DG-category of \mathbb{C} -local systems on S^2 . One can define this as

$$\mathrm{Loc}(S^2) = \lim_{\mathcal{U} \subset S^2} \mathrm{Loc}(\mathcal{U}) = \mathrm{Fun}(S^2, D(\mathbb{C})).$$

The corresponding abelian category is Vect^\heartsuit (because $\pi_1(S^2)$ is trivial). However, $\mathrm{Loc}(S^2)$ is not trivial – one can compute $\mathrm{Hom}_{\mathrm{Loc}(S^2)}(\underline{\mathbb{C}}, \underline{\mathbb{C}}) = C^*(S^2; \mathbb{C})$, so we have a nontrivial $\mathrm{Ext}^2(\underline{\mathbb{C}}, \underline{\mathbb{C}})$. Thus looking only at the abelian category loses a lot of information.

One can interpret Vect^\heartsuit as the heart of a t-structure on Loc (induced by the standard t-structure on $D(\mathbb{C})$). We then obtain

$$\mathrm{Fun}(S^2, D(\mathbb{C})^\heartsuit) = \mathrm{Fun}(\mathrm{Ho}(S^2), D(\mathbb{C})) = \mathrm{Fun}(\tau_{\leq 1} S^2, D(\mathbb{C})^\heartsuit) = D(\mathbb{C})^\heartsuit.$$

This t-structure approach can also be used to understand the relationship between $\mathrm{QCoh}(X)$ and $D_{\mathrm{qc}}(X)$.

17. LECTURE 15, 7/6 (B. ANTIEAU) – THE COTANGENT COMPLEX

We would like to prove that some stacks of interest are “geometric.”

Theorem 17.1. *Let X be a geometrically connected smooth proper scheme over k . Then $\mathrm{Pic}_{X/k}$, the stack of line bundles on X , is a 1-stack. Furthermore, the cotangent complex at a point \mathcal{L} is $\mathbf{R}\Gamma(X, \mathcal{O}_X)^\vee[-1]$.*

Here $\mathrm{Pic}_{X/k}$ is defined on a derived commutative ring R by

$$\mathrm{Pic}_{X/k}(R) \simeq \mathrm{Pic}(D(X_R)),$$

where the latter indicates the collection of invertible objects in $D(X_R)$.

The above theorem implies that the cotangent complex is concentrated in cohomological degrees $[-1, d-1]$ for $d = \dim(X/k)$. The -1 degree here captures the stackiness of $\mathrm{Pic}_{X/k}$, while the positive degree terms capture derived structure.

Corollary 17.2. *If $H^i(X, \mathcal{O}_X) \neq 0$ for some $i \geq 3$, then $\mathrm{Pic}_{X/k}$ is not classical.*

This corollary might also hold for $i = 2$, but the proof would be harder. We can recover the classical Picard stack as $(\mathrm{Pic}_{X/k})^{\mathrm{cl}}$.

17.1. Kähler differentials. To understand the above theorem, we must first understand the cotangent complex. We begin by recalling the theory of Kähler differentials.

Definition 17.3. Let $k \rightarrow R$ be a map of classical commutative rings. A k -derivation of R into an R -module M is a function $d : R \rightarrow M$ such that

- (1) $d(r+s) = d(r) + d(s)$
- (2) $d(rs) = sd(r) + rd(s)$
- (3) $d(\lambda) = 0$ for all $\lambda \in k$.

These form an R -module $\mathrm{Der}_k(R, M)$. There exists a universal k -derivation $R \rightarrow \Omega_{R/k}^1$ such that

$$\mathrm{Hom}(\Omega_{R/k}^1, M) \cong \mathrm{Der}_k(R, M)$$

for all R -modules M .

Example 17.4. If $k \rightarrow R$ is surjective, then $\Omega_{R/k}^1 = 0$.

Example 17.5.

$$\Omega_{k[x]/k}^1 = k[x]dx.$$

Example 17.6. Letting $I = \ker(R \otimes_k R \rightarrow R)$, we have $\Omega_{R/k}^1 \cong I/I^2$ via $adb \mapsto a \otimes b - ab \otimes 1$.

17.2. Infinitesimal liftings. Let $\mathbf{CAlg}_{k//R} = (\mathbf{CAlg}_k)_R$, the category of classical commutative k -algebras equipped with a map to R .

For an R -module M , one can define an algebra structure on $R \oplus M$ via the action of R on M (setting $M \cdot M = 0$). We call this the *trivial square-zero extension* of R by M . This satisfies

$$\mathrm{Hom}_{\mathbf{CAlg}_{k//R}}(R, R \oplus M) \cong \mathrm{Der}_k(R, M) \cong \mathrm{Hom}_R(\Omega_{R/k}^1, M).$$

Example 17.7. The dual numbers $R[x]/(x^2)$ give a trivial square-zero extension of R .

Example 17.8. The ring $\mathbb{Z}/p^2\mathbb{Z}$ is not a trivial square-zero extension of $\mathbb{Z}/p\mathbb{Z}$.

A general *square-zero extension* $\tilde{S} \rightarrow S$ is defined by the condition that, if $I = \ker(\tilde{S} \rightarrow S)$, then $I^2 = 0$. The set of k -algebra lifts of $R \rightarrow S$ to $R \rightarrow \tilde{S}$ is a $\mathrm{Der}_k(R, I)$ -torsor, i.e. is either empty or has a free transitive $\mathrm{Der}_k(R, I)$ -action. The action here is just by $\phi \mapsto \phi + d$ for ϕ a lift and d a derivation.

17.3. The cotangent complex.

Definition 17.9 (Quillen-Illusie). Let $R \in \mathbf{DAlg}_k^{\mathrm{cn}}$, and write $R = |R_\bullet|$ for R_\bullet a simplicial commutative ring with each R_i polynomial (in possibly infinitely many variables). Define the *cotangent complex* of R over k as $\mathbb{L}_{R/k} = \Omega_{R_\bullet/k}^1 \otimes_{R_\bullet} R$. This is a simplicial R -module, and we can take the geometric realization to view it as an object of $D(R)_{\geq 0}$.

This gives an explicit construction, but we can also give more abstract definitions that “get to the heart of things.”

Construction 17.10. Recall $\mathbf{DAlg}_k \simeq \mathbf{Psh}_\Sigma(\mathrm{Poly}_k^{\mathrm{fp}})$.

(1) Consider the diagram

$$\begin{array}{ccc} \mathrm{Poly}_k^{\mathrm{fp}} & \xrightarrow{R \mapsto \Omega_{R/k}^1} & D(k)_{\geq 0} \\ \downarrow & \nearrow \text{dashed} & \\ \mathbf{Psh}_\Sigma(\mathrm{Poly}_k^{\mathrm{fp}}) & & \end{array}$$

Here we obtain $\mathbb{L}_{-/k}$ as the left Kan extension of Ω^1 . Note that this only produces the k -module structure on $\mathbb{L}_{-/k}$! We want an R -module structure.

(2) A smarter variation of this comes from looking at the diagram

$$\begin{array}{ccccc} & & D(-)_{\geq 0} & \longleftarrow & D(R)_{\geq 0} \\ & \nearrow & \downarrow & & \downarrow \\ \mathrm{Poly}_k^{\mathrm{fp}} & \longrightarrow & \mathbf{DAlg}_k^{\mathrm{cn}} & \xleftarrow{R} & \mathrm{pt} \end{array}$$

where the right square is a pullback square, and $D(-)_{\geq 0}$ is constructed using “straightening-unstraightening” (similar to the Grothendieck construction

for turning a stack into a fibered category). Objects of $D_{\geq 0}$ are given by pairs $(R \in \mathrm{DAlg}_k^{\mathrm{cn}}, M \in D(R))$ with morphisms $(R, S) \rightarrow (M, N)$ given by pairs $(R \rightarrow S, S \otimes_R M \rightarrow N)$. The diagonal arrow is $R \mapsto (R, \Omega_{R/k}^1)$. In this context, $\mathbb{L}_{-/k}$ is defined as the left Kan extension of $\mathrm{CAlg}_k^{\mathrm{fp}, \mathrm{poly}} \rightarrow D(-)_{\geq 0}$ along the map to $\mathrm{DAlg}_k^{\mathrm{cn}}$.

It is immediate from this construction that $\mathbb{L}_{-/k}$ preserves sifted colimits. It preserves all colimits if and only if $\Omega_{-/k}^1$ preserves all colimits in $\mathrm{Poly}_k^{\mathrm{fp}}$. This latter hypothesis is true, as one can show in the exercises. (Note that colimits in $D(-)_{\geq 0}$ are given by $(R, M) \coprod (S, N) = (R \otimes_k S, (M \otimes_k S) \oplus (R \otimes_k N))$.) Therefore:

Proposition 17.11. *The functor $\mathbb{L}_{-/k}$ preserves all colimits.*

Example 17.12. Let $R = k[x]/x^2$. We can present R as the pushout

$$\begin{array}{ccc} k[y] & \xrightarrow{y \mapsto x^2} & k[x] \\ \downarrow y \mapsto 0 & & \downarrow \\ k & \longrightarrow & R. \end{array}$$

Here $\mathbb{L}_{k[y]/k} = k[y]dy$, $\mathbb{L}_{k[x]/k} = k[x]dx$, and $\mathbb{L}_{k/k} = 0$. Thus we have a pushout square of cotangent complexes

$$\begin{array}{ccc} Rdy & \xrightarrow{dy \mapsto 2xdx} & Rdx \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathbb{L}_{R/k}. \end{array}$$

If 2 is invertible in k , we obtain

$$H_i(\mathbb{L}_{R/k}) = \begin{cases} k & i = 0, 1 \\ 0 & \text{otherwise.} \end{cases}$$

17.4. Square-zero extensions. We can also understand cotangent complexes using square-zero extensions.

Let $\mathrm{DAlg}_{k/R}^{\mathrm{cn}} = (\mathrm{DAlg}_k^{\mathrm{cn}})_R$. For $M \in D(R)_{\geq 0}$, we define the space of k -linear derivations $R \rightarrow M$ as

$$\mathrm{Der}_k(R, M) := \mathrm{Map}_{\mathrm{DAlg}_{k/R}^{\mathrm{cn}}}(R, R \oplus M).$$

Proposition 17.13. *The functor $\mathrm{Der}_k(R, -)$ is representable, i.e. there exists $\mathbb{L}_{R/k} \in D(R)_{\geq 0}$ such that*

$$\mathrm{Der}_k(R, -) \simeq \mathrm{Map}_R(\mathbb{L}_{R/k}, -).$$

Furthermore, this $\mathbb{L}_{R/k}$ agrees with the previous definition.

Proof. This is mostly an exercise. The key idea is that, for $S \rightarrow T$ over R , we have a fiber sequence

$$\mathrm{Map}_{\mathrm{DAlg}_{k/R}^{\mathrm{cn}}}(S, T) \rightarrow \mathrm{Map}_{\mathrm{DAlg}_k^{\mathrm{cn}}}(S, T) \rightarrow \mathrm{Map}_{\mathrm{DAlg}_k^{\mathrm{cn}}}(S, R).$$

□

The following is very useful for computations.

Exercise 17.14 (Conormal sequence). *Let $k \rightarrow R \rightarrow S$ in $\mathrm{DAlg}_k^{\mathrm{cn}}$. Then there is a cofiber sequence*

$$S \otimes_R \mathbb{L}_{R/k} \rightarrow \mathbb{L}_{S/k} \rightarrow \mathbb{L}_{S/R}.$$

Example 17.15. To compute $\mathbb{L}_{k[x]/k}$, we consider $k \rightarrow k[x] \rightarrow k$ and note that

$$k \otimes_{k[x]} \mathbb{L}_{k[x]/k} \rightarrow \mathbb{L}_{k/k} \rightarrow \mathbb{L}_{k/k[x]}.$$

The first term is k and the second term is 0 , so we get $\mathbb{L}_{k/k[x]} \simeq k[1]$.

Here are some useful facts for classical commutative rings:

- For $R \in \mathrm{CAlg}_k$, we have $H^0(\mathbb{L}_{R/k}) \cong \Omega_{R/k}^1$.
- For a surjection $R \rightarrow S$ in CAlg_k (with kernel I), we have $H_1(\mathbb{L}_{S/R}) \cong I/I^2$.

We can recover some exact sequences in Hartshorne from these together with the above conormal sequence.

Definition 17.16. A *square-zero extension* $\tilde{S} \rightarrow S$ (of S by M) in $\mathrm{DAlg}_k^{\mathrm{cn}}$ is a pullback square

$$\begin{array}{ccc} \tilde{S} & \longrightarrow & S \\ \downarrow & & \downarrow \text{0 derivation} \\ S & \xrightarrow{\text{any derivation}} & S \oplus M[1] \end{array}$$

Example 17.17. We can construct \mathbb{Z}/p^2 as a square-zero extension

$$\begin{array}{ccc} \mathbb{Z}/p^2 & \longrightarrow & \mathbb{F}_p \\ \downarrow & & \downarrow \\ \mathbb{F}_p & \longrightarrow & \mathbb{F}_p \oplus \mathbb{F}_p[1] \end{array}$$

in $\mathrm{DAlg}_{\mathbb{Z}}^{\mathrm{cn}}$. The derivation here is $d : \mathbb{F}_p \rightarrow (\mathbb{F}_p)[1]$ constructed from a map $\mathbb{L}_{\mathbb{F}_p/\mathbb{Z}} \simeq \mathbb{F}_p[1] \rightarrow \mathbb{F}_p[1]$ induced by multiplication by an element of \mathbb{F}_p^\times .

Example 17.18. If $R \in \mathrm{DAlg}_k^{\mathrm{cn}}$, then

$$\begin{array}{ccc} \tau_{\leq n} R & \longrightarrow & \tau_{\leq n-1} R \\ \downarrow & & \downarrow \\ \tau_{\leq n-1} R & \longrightarrow & \tau_{\leq n-1} R \oplus H_n(R)[n+1] \end{array}$$

is a square-zero extension.

Proposition 17.19. *If \tilde{S} is a square-zero extension of S by $M \in D(S)_{\geq 0}$, then $\mathrm{Map}_{\mathrm{DAlg}_{k/S}^{\mathrm{cn}}}(R, \tilde{S})$ is a $\mathrm{Der}_k(R, M)$ -torsor. The obstruction to a lift existing is in $\mathrm{Map}_R(\mathbb{L}_{R/k}, M[1])$.*

17.5. Lecture 16, 7/6 (D. Arinkin) – Working with (Compactly Generated) DG-Categories. Last time, we defined

$$\mathrm{QCoh}(X) = \lim_{\mathrm{Spec} R \rightarrow X} D(R).$$

If X quasicompact and separated, then: $\mathrm{Perf}(X) = \mathrm{QCoh}(X)^c$, where $\mathrm{Perf}(X)$ is defined locally, i.e.

$$\mathrm{Perf}(X) = \lim_{\mathrm{Spec} R \rightarrow X} \mathrm{Perf}(R).$$

By the theorem of Thomason-Trobaugh, $\text{Perf}(X)$ generates $\text{QCoh}(X)$. The key step in the proof of this theorem is that, if $X = U \cup V$ with V affine, then we can (partially) extend from $\text{Perf}(U)$ to $\text{Perf}(X)$.¹¹ Really, we can't extend in general due to K-theoretic obstructions, but we can extend up to taking direct summands. More precisely, every $\mathcal{F} \in \text{Perf}(U)$ is a direct summand of some $\tilde{F}|_U$ where $\tilde{F} \in \text{Perf}(X)$.

17.6. Representability and adjoint functor theorems. Let $B = \text{Ind}(A) = D(A^{\text{op}})$ be compactly generated. Then B has products (and all limits) computed pointwise on A . We can view $B = \text{Fun}(A^{\text{op}}, D(k))$.

Proposition 17.20 (Brown representability theorem). *A functor $\phi : B^{\text{op}} \rightarrow D(k)$ is representable if and only if ϕ sends direct sums in B to direct products in k .*

Proof. The forward direction is obvious. For the backward direction, use the fact that ϕ is determined by $\phi|_{A^{\text{op}}}$. (The result is then clear from the construction of B .) \square

Corollary 17.21. *If B_1 is compactly generated and B_2 is cocomplete, then any continuous functor $F : B_1 \rightarrow B_2$ has a right adjoint F^R .*

Proof. For $x \in B_2$, the functor $\text{Hom}(F(-), x) : B_1^{\text{op}} \rightarrow D(k)$ is representable (by $F^R(x)$). \square

Remark 17.22. Note that F^R need not be continuous. In fact, F^R is continuous if and only if $F(B_1^c) \subset B_2^c$ (since we can probe direct sums using compact generators of B_1). In this case, $F(B_1^c)$ generates B_2 if and only if F^R is conservative (i.e. $F^R(x) \simeq 0$ implies $x \simeq 0$).

17.7. Quotients. The following proposition is valid for DG-categories or triangulated categories (though we prefer the former language).

Proposition 17.23. *Let B be a pretriangulated DG-category and $i : A \hookrightarrow B$ the inclusion of a full subcategory. The following are equivalent:*

- (1) *i admits a right adjoint i^R .*
- (2) *There exists a full subcategory $C \subset B$ such that:*
 - (a) *$A \perp C$ (i.e. $\text{Hom}(A, C) = 0$), and*
 - (b) *Every $b \in B$ fits into an exact triangle*

$$a \rightarrow b \rightarrow c \rightarrow a[1]$$

with $a \in A$ and $c \in C$.

The proof is a nice exercise, and the following properties may give some hints for the construction of i^R .

- (1) The triangle appearing in the Proposition is unique.
- (2) $A = {}^\perp C$ and $C = A^\perp$.
- (3) For i^R , the unit $\text{id}_A \rightarrow i^R \circ i$ is an isomorphism (because i is fully faithful).
- (4) $C = \ker(i^R)$ (i.e. the subcategory mapped by i^R to 0).
- (5) We can identify $A \simeq B/C$ via i^R (and also $B/A \simeq C$).

¹¹This is the actual TT theorem, and the claim about compact generation is really a corollary of this.

The above provides a “short exact sequence of DG-categories”

$$A \rightleftarrows B \rightleftarrows C.$$

We can call C a *localization* of B and A a *colocalization* of C .

Example 17.24. For a DG-category A , we have a decomposition of $C(A)$ into $C(A)^{\text{semifree}}$ and $C(A)^{\text{acyclic}}$. This gives a short exact sequence

$$C(A)^{\text{semifree}} \rightleftarrows C(A) \rightleftarrows C(A)^{\text{acyclic}},$$

explaining why $C(A)^{\text{semifree}} \simeq D(A)$.

Now suppose B is cocomplete and A is generated by some collection of compact objects $A^c \subset B^c$. Then $A \hookrightarrow B$ has a continuous right adjoint (by our above adjoint functor theorem). Setting $C = A^\perp$ gives a short exact sequence

$$A \rightleftarrows B \rightleftarrows C.$$

As C is closed under infinite direct sums (because A is compactly generated), all four functors in this short exact sequence preserve infinite direct sums.

As an application, we see that the following notions of “compact generation” are equivalent:

- (1) $C = 0$.
- (2) $(A^c)^\perp = 0$.
- (3) $\text{Ind}(A^c) = B$.

Now suppose furthermore that B is compactly generated. Then:

- (1) The natural map $p : B \rightarrow C$ preserves compact objects, so its right adjoint is continuous.
- (2) The image $p(B^c)$ generates C (because the right adjoint $C \rightarrow B$ is conservative).
- (3) In particular, C is compactly generated, and $C^c = (p(B^c)^{\text{pretr}})^{\text{Kar}} = \text{Perf}(p(B^c)^{\text{op}})$.

Example 17.25. Let $X = \text{Spec } R$, and let $Y \subset X$ be closed such that $U = X \setminus Y$ is quasicompact (equivalently, $Y = V(f_1, \dots, f_n)$ as a set). Note that $\text{QCoh}(X) = \text{Ind}(\mathcal{O}_X) = D(R)$. Consider the full subcategory $\text{QCoh}(X)_Y$ of quasicoherent sheaves supported on Y , which we can also write as

$$\text{QCoh}(X)_Y = \text{Ind}(\otimes_{i=1}^n (\mathcal{O}_X \xrightarrow{f_i} \mathcal{O}_X))$$

If $j : U \rightarrow X$ is the inclusion, then the right orthogonal to $\text{QCoh}(X)_Y$ is $j_* \text{QCoh}(U)$. It follows that $j_* \text{QCoh}(U) = \text{Ind}(\mathcal{O}_U)$ is compactly generated. (Note that U is only quasi-affine in general, and \mathcal{O}_U will in general have nontrivial derived endomorphisms.) The Thomason-Trobaugh theorem is essentially telling us that, instead of viewing $j_* \text{QCoh}(U)$ as $(j_*)^*(\text{Perf}(X))^{\text{pretr}}^{\text{Kar}}$, we can instead drop the “pre-triangulated completion” part and only consider the Karoubi completion.

Exercise 17.26. Suppose we have a short exact sequence of DG-categories

$$A \rightleftarrows B \rightleftarrows C.$$

If $\{G_\beta\} \subset A^c$ generate A and $\{F_\alpha\} \subset B^c$ are such that $\{p(F_\alpha)\}$ generate C , then $\{F_\alpha\} \cup \{G_\beta\}$ generate B .