# DAG NOTES

## 1. Lecture 1, 6/26 (D. Arinkin) – DG-Categories

Arinkin's lectures aim to treat DG-categories as they are used in practice (without completely ignoring the foundations). More focus will be given to the "key parts" of the theory than to technicalities. We will treat foundations and geometrically-flavored examples (e.g. $\mathsf{QCoh}$ from a DG-perspective).

### 1.1. Motivation.

In classical algebra and algebraic geometry, we like to consider modules over a ring. We pass to resolutions to get things which behave better from a homological perspective. This leads to considering derived categories. What kind of objects are derived categories?

(1) Derived categories are *triangulated categories*. This means that they are equipped with a notion of *distinguished triangles* (also called *exact triangles*), which capture the notion of mapping cones. However, this structure can be poorly behaved.
(2) Derived categories are *DG-categories*, or (mostly equivalently) *stable $\infty$-categories*. Arinkin considers the DG category perspective to be more concrete. Using this perspective changes something fundamental about the category (what we obtain is more complex than a typical 1-category). However, DG categories can be more robust, especially when considering the "category of DG-categories."

### 1.2. DG-categories and examples.

Fix a field $\mathsf{k}$ (often we can get away with working over a commutative ring, but this may require more caution). Let $\mathsf{C}(\mathsf{k})$ be the monoidal category of complexes of $\mathsf{k}$-vector spaces, equipped with the tensor product $\otimes_{\mathsf{k}}$ as monoidal structure.

**Definition 1.1.** A *DG-category* $\mathsf{A}$ consists of

(1) A collection of objects $\operatorname{Ob}\mathsf{A}$
(2) For all $x, y \in \operatorname{Ob}\mathsf{A}$, a chain complex of morphisms $\operatorname{Hom}_{\mathsf{A}}(x, y) \in \operatorname{Ob}\mathsf{C}(\mathsf{k})$
(3) For all $x, y, z \in \operatorname{Ob}\mathsf{A}$, a composition law $\circ : \operatorname{Hom}_{\mathsf{A}}(x, y) \otimes \operatorname{Hom}_{\mathsf{A}}(y, z) \to \operatorname{Hom}_{\mathsf{A}}(x, z)$

such that

(1) Composition is associative.
(2) Composition is unital: for all $x \in \mathsf{A}$, there exists a degree-zero cycle $\mathbb{1}_x \in \operatorname{Hom}_{\mathsf{A}}(x, x)$ (so $d(\mathbb{1}_x) = 0$) such that $\mathbb{1}_x \circ f = f$ and $g \circ \mathbb{1}_x = g$ for all $f, g$.

**Example 1.2.** We can consider the case where $\operatorname{Hom}(x, y)$ is always concentrated in degree zero. This recovers the notion of a $\mathsf{k}$-linear category.

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**Example 1.3.** Let $\mathsf{R}$ be a $\mathsf{k}$-algebra. We can turn $\mathsf{C}(\mathsf{R})$ (the category of complexes of $\mathsf{R}$-modules) into a DG-category by setting

$$\mathrm{Hom}(M^\bullet, N^\bullet)^p = \prod_q \mathrm{Hom}_R(M^p, N^{p+q})$$

with differential $d_{\mathrm{Hom}} = [d, -]$. It is crucial here that we use the product rather than the direct sum (otherwise we run into issues when the complexes are not bounded).

We can also construct some examples where the Hom-complexes are standard complexes computing Ext groups. This ties into the classical perspective on homological algebra, where derived functors are computed using explicit resolutions.

**Example 1.4.** Let $\mathsf{R}$ be a $\mathsf{k}$-algebra. We can define a DG-category $\mathsf{A}$ with $\mathrm{Ob}\,\mathsf{A}$ being $\mathsf{R}$-modules and $\mathrm{Hom}_A(M, N)$ being the standard complex computing $\mathrm{Ext}^\bullet(M, N)$ via the bar resolution. We will not spell out the details here because this is not our emphasis.

**Example 1.5.** Let $k = \mathbb{C}$, and fix a complex manifold $X$. We can define a DG-category $\mathsf{A}$ with objects being holomorphic vector bundles on $X$ and $\mathrm{Hom}_A(E, F)$ given by the Dolbeault complex

$$0 \longrightarrow \mathcal{C}^\infty(E^* \otimes F) \xrightarrow{\bar\partial} \Omega^{0,1}(E^* \otimes F) \xrightarrow{\bar\partial} \dots$$

As stated above, our main examples of interest are really *derived categories.* We still need to build up some more definitions to get a good understanding of these.

1.3. **DG-Functors.** One would like to have a good notion of functors between DG-categories. It turns out that the naïve definition of DG-functors works (and does not require homotopical corrections).

**Definition 1.6.** A DG-functor $F : A \to B$ consists of:

(1) An assignment $F : \mathrm{Ob}\,A \to \mathrm{Ob}\,B$
(2) For every $x, y \in \mathrm{Ob}\,A$, a chain map $\mathrm{Hom}_A(x, y) \to \mathrm{Hom}_B(F(x), F(y))$

which preserves identities and composition.

**Remark 1.7.** While the obvious notion of a DG-functor works fine, subtleties arise when one considers the notion of equivalence.

One can first define a notion of *strict equivalence.* A DG-functor $F : A \to B$ is a strict equivalence if

(1) $F$ is strictly fully faithful: for all $x, y \in \mathrm{Ob}\,A$, $\mathrm{Hom}_A(x, y) \to \mathrm{Hom}_B(F(x), F(y))$ is an isomorphism.
(2) $F$ is strictly essentially surjective: for all $y \in \mathrm{Ob}\,B$, there exists $x \in \mathrm{Ob}\,A$ such that $F(x)$ is isomorphic to $y$.

As one knows from the study of derived categories, the notion of "isomorphism" here is often too strong - one would rather consider quasi-isomorphisms or other weaker notions.

A better notion is that of *quasi-equivalence.* A DG-functor $F : A \to B$ is a quasi-equivalence if

(1) $F$ is quasi-fully faithful: for all $x, y \in \mathrm{Ob}\,A$, $\mathrm{Hom}_A(x, y) \to \mathrm{Hom}_B(F(x), F(y))$ is a quasi-isomorphism.

(2) $\mathsf{F}$ is quasi-essentially surjective: for all $y \in \mathrm{Ob}\,\mathsf{B}$, there exists $x \in \mathrm{Ob}\,\mathsf{A}$ such that there exist maps $\mathsf{F}(x) \to y$ and $y \to \mathsf{F}(x)$ with the both compositions $\mathsf{F}(x) \to \mathsf{F}(x)$ and $y \to y$ homotopic to the relevant identities.

**Exercise 1.8.** *For a DG-category* $\mathsf{A}$*, let* $\mathsf{Ho}\,\mathsf{A}$ *be the* homotopy category *of* $\mathsf{A}$*, i.e. the* 1*-category with the same objects as* $\mathsf{A}$ *and morphisms given by* $\mathrm{Hom}_{\mathsf{Ho}\,\mathsf{A}}(x, y) = \mathrm{H}^0(\mathrm{Hom}_{\mathsf{A}}(x, y))$.

*(1) Check that* $\mathsf{Ho}\,\mathsf{A}$ *is a well-defined* 1*-category. (This uses the fact that the functor* $\mathrm{H}^0$ *is lax monoidal.)*
*(2) Show that a DG-functor* $\mathsf{F} : \mathsf{A} \to \mathsf{B}$ *is quasi-essentially surjective if and only if* $\mathsf{Ho}\,\mathsf{F} : \mathsf{Ho}\,\mathsf{A} \to \mathsf{Ho}\,\mathsf{B}$ *is essentially surjective.*

## 2. Lecture 2, 6/26 (B. Antieau) -

Antieau's lectures aim to treat the intuition and geometry behind DAG, without getting lost in the details.

### 2.1. **"Bird's Eye View" and Examples.**

We begin with a classical proposal of Serre. Let $X$ be a regular variety, with $W, Y \subset X$ and $W \subset Y$ being a finite set of points. We would like to count $W \cap Y$. In the case of transverse intersections, this is easy, but in general we have to count with multiplicity. If we move $W$ and $Y$ around, this count with multiplicity should not change. Serre proposed that the intersection multiplicity at a point $z \in W \cap Y$ is given by

$$\sum_{i \geqslant 0} (-1)^i \,\mathrm{length}_{k(z)}\, \mathrm{Tor}^i(\mathcal{O}(X)_z)(\mathcal{O}(X)_z/I_z, \mathcal{O}(Y)/I_z).$$

It is not immediately obvious why this is a reasonable definition, but it turns out to work well in good cases. One can show that (when working over a field) this intersection multiplicity is nonnegative.

**Example 2.1.** Consider the intersection in $\mathbb{A}^2$ of the curve $W = \{y = x^2\}$ with the $x$-axis $Y$. One can compute that the intersection multiplicity is 2 (coming exclusively from $\mathrm{Tor}^0$). This makes sense: if we move the curves around, then we will get two actual intersection points.

For the Tor computation, we note $\mathcal{O}(W) = k[x, y]/(y - x^2)$ and $\mathcal{O}(Y) = k[x, y]/(y)$. Then $\mathcal{O}(Y) \otimes_{\mathcal{O}(X)} \mathcal{O}(W) = k[x]/(x^2)$, and there are no higher Tor's. This can be interpreted geometrically using the common tangent vector to $W$ and $Y$ at the origin.

**Example 2.2.** For the first historical example with higher Tors, we work in $\mathbb{A}^4 = \mathrm{Spec}\, k[w, x, y, z]$. Let $Y$ be the union of planes $P_1 \cup P_2 = \{x = y = 0\} \cup \{z = w = 0\}$, and let $W$ be the plane $\{x = z, y = w\}$. Then the $\mathrm{Tor}_0$ term is 3-dimensional, but the "correct" intersection multiplicity should be 2 (if we perturb $W$, it should meet each plane $P_i$ in a single point). Thus we must use the $\mathrm{Tor}^1$ term to correct our intersection multiplicity.

The main idea of derived algebraic geometry is that we should understand intersection multiplicities geometrically by replacing $\mathcal{O}(Y) \otimes_{\mathcal{O}(X)} \mathcal{O}(W)$ by a *derived commutative ring* $\mathcal{O}(Y) \otimes^{\mathbf{L}}_{\mathcal{O}(X)} \mathcal{O}(W)$. In the affine case, "Spec" of this should give the *derived intersection* $Y \cap^{\mathbf{L}} W = Y \times^{\mathbf{L}}_X W$. We will of course need to make sense of what this means.

Derived algebraic geometry also gives rise to interesting self-intersections, which we can use to understand what derived commutative rings and schemes should be.

**Example 2.3.** Consider $\{0\} \subset \mathbb{A}^1$, viewed as Spec of $\mathsf{k} = \mathsf{k}[\mathsf{x}]/(\mathsf{x})$. The classical self-intersection is $\{0\} \cap \{0\} = \{0\}$. Note that this is not what we'd get if we moved the points (since two general points in $\mathbb{A}^1$ do not meet).

For the derived self-intersection, we have

$$\pi_i(k \otimes^{\mathbf{L}}_{k[x]} k) = \begin{cases} k & i = 0 \\ k & i = 1 \\ 0 & i > 0. \end{cases}$$

For derived commutative rings, we say $\pi_i = H_i = H^{-i}$. We can think of elements of $\pi_i$ for $i > 0$ as higher nilpotents / "fuzz" (in addition to the nilpotents that appear in $\pi_0$ when working scheme-theoretically).

Recall that in classical geometry, for commutative $k$-algebras $R$ and $S$, we have

$$\mathrm{Hom}_{\mathsf{Sch}_k}(\mathrm{Spec}\, S, \mathrm{Spec}\, R) = \mathrm{Hom}_{\mathsf{CAlg}_k}(R, S).$$

Note that the right hand side of this is a set. For our example, we expect to have

$$\mathrm{Hom}_{\mathsf{dSch}_k}(\{0\} \cap^{\mathbf{L}} \{0\}, \mathbb{A}^1) = \mathcal{O}(\{0\} \cap^{\mathbf{L}} \{0\}) = k \otimes^{\mathbf{L}}_{k[x]} k,$$

which should be a "space" with $\pi_0 = \pi_1 = k$ (not just a set).

Furthermore, instead of considering abelian categories of quasicoherent sheaves on $\{0\} \cap^{\mathbf{L}} \{0\}$, it makes much more sense to consider derived categories of quasicoherent sheaves. One can still make sense of the abelian categories, but they don't see the interesting higher homotopy groups.

Through considering this and other examples, we end up with some ideas about what "derived replacements" of concepts in classical algebraic geometry should be.

| Classical AG | Derived AG |
|---|---|
| Commutative rings $R$ | Derived commutative rings $R$ |
| Affine schemes $\mathrm{Spec}\, R$ | Derived affine schemes $\mathrm{Spec}\, R$ |
| Sheaves of sets | Sheaves of homotopy types |
| Abelian categories $\mathsf{QCoh}(X)$ | DG-categories (or stable $\infty$-categories) $D_qc(X)$ |

2.2. **Presheaves and Yoneda.** Let $\mathsf{C}$ be a category. Write $\mathsf{Psh}(\mathsf{C})^{\heartsuit} = \mathsf{Fun}(\mathsf{C}^{\mathrm{op}}, \mathsf{Set})$ for the *category of presheaves of sets* on $\mathsf{C}$. Objects of this category are functors $F : \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$, and morphisms are natural transformations of functors.

There is a natural *Yoneda embedding* $h : \mathsf{C} \to \mathsf{Psh}(\mathsf{C})^{\heartsuit}$, defined by $h(X) = h_X$, where $h_X(Y) = \mathrm{Hom}_{\mathsf{C}}(Y, X)$.

**Lemma 2.4** (Yoneda)**.**

1. *The Yoneda embedding* $h : \mathsf{C} \to \mathsf{Psh}(\mathsf{C})^{\heartsuit}$ *is fully faithful, i.e.* $\mathrm{Hom}_{\mathsf{C}}(X, Y) \cong \mathrm{Hom}_{\mathsf{Psh}(\mathsf{C})^{\heartsuit}}(h_X, h_Y)$ *naturally in* $X, Y$.
2. *For any* $F \in \mathsf{Psh}(\mathsf{C})^{\heartsuit}$*, there is a natural isomorphism* $F(X) \cong \mathrm{Hom}_{\mathsf{Psh}(\mathsf{C})^{\heartsuit}}(h_X, F)$.

*Proof.* The first statement follows from the second by taking $F = h_Y$ in the second. To prove the second statement, we can construct explicit natural isomorphisms as follows. To get $\mathrm{Hom}_{\mathsf{Psh}(\mathsf{C})^{\heartsuit}}(h_X, F) \to F(X)$, start with $f : h_X \to F$, and evaluate $f$ on $\mathrm{id}_X \in h_X(X)$ to get an element of $F(X)$. Conversely, given $g \in F(X)$, construct a natural transformation $h_X \to F$ by sending $a \in h_X(Y) = \mathrm{Hom}_{\mathsf{C}}(Y, X)$ to $F(a)(g) \in F(Y)$. One can check that these are both natural and mutually inverse. $\square$

**Example 2.5.** Let $\Delta^1$ be the category with two objects $0, 1$ and one non-identity morphism $0 \to 1$. Then $\mathsf{Psh}(\Delta^1)^\heartsuit$ is the category of arrows in $\mathsf{Set}$. The functor $h_0$ corresponds to the arrow $\emptyset \to *$, and the functor $h_1$ corresponds to the arrow $* \to *$.

### 2.3. **Topologies and Sheaves.**

**Definition 2.6.** A map $f : R \to S$ in $\mathsf{CAlg}_k$ is *flat* if the functor $S \otimes_R (-) : \mathsf{Mod}_R \to \mathsf{Mod}_S$ is exact. We say that $f$ is furthermore *faithfully flat* if $S \otimes_R (-)$ is conservative (i.e. for $g : M \to N$, if $S \otimes g : S \otimes_R M \to S \otimes_R N$ is an isomorphism, then $g$ must have already been an isomorphism).

**Example 2.7.**

(1) For any $R$, the map $R \to 0$ is flat but not faithfully flat.
(2) The map $\mathbb{Z} \to \mathbb{Q}$ is flat but not faithfully flat.
(3) Any field extension $K \to L$ is faithfully flat.
(4) For $f, g \in R$ with $(f, g) = 1$, the map $R \to R[f^{-1}] \times R[g^{-1}]$ is faithfully flat. Note that this gives a Zariski cover of $\operatorname{Spec} R$.

Let $\mathsf{Aff}_k = \mathsf{CAlg}^{\mathrm{op}}$ be the *category of affine* $k$*-schemes*. We will define some topologies on $\mathsf{Aff}_k$.

**Definition 2.8.**

(1) The *fpqc topology* on $\mathsf{Aff}_k$ has coverings of $\operatorname{Spec} R$ generated by those of the form $\{\operatorname{Spec} S_i \to \operatorname{Spec} R\}_{i \in I}$ where I is finite, each $R \to S_i$ is flat, and $R \to \prod_i S_i$ is faithfully flat.[1]
(2) For the *fppf topology*, we require furthermore that each $S_i$ is of finite presentation over $R$ (i.e. $S_i = R[x_1, \ldots, x_p]/(f_1, \ldots, f_q)$ for some $x_i$ and $f_j$.)
(3) For the *étale topology*, we require furthermore that each $S_i$ is étale over $R$ (i.e. $S_i$ is smooth over $R$ with $\Omega^1_{S_i/R} = 0$).
(4) For the *Zariski topology*, we require furthermore that each $S_i$ is of the form $R[f^{-1}]$ for some $f \in R$.

[1]Note the "generated by" – we allow infinite covers.