Notes from GCS 2024 Summer School

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Abstract

These are my notes from the "Categorical Symmetries in Quantum Field Theory" summer school, held at the University of Edinburgh from June 10-14, 2024. Lectures were given by Graeme Segal, Nitu Kitchloo, Ibou Bah, Claudia Scheimbauer, and Ryan Thorngren. All errors are my own.

Contents

Chapter 1

Graeme Segal – A Perspective on Quantum Field Theory

1.1 6/10 – Overview of QFT

1.1.1 From classical to quantum

In classical mechanics, one focuses one's attention on a fixed space M_0 and obtains a configuration space X from that. Dynamics are specified by giving a Lagrangian $L : TX \to \mathbb{R}$, which we assume is positive, inhomogeneous, and quadratic on tangent vectors. Under these hypotheses, the Lagrangian can be viewed as a generalized Riemannian metric, so that time evolution is given by a (generalized) "geodesic flow." This gives rise to a symplectic form ω and Poisson bracket $\{-,-\}$ on T^*X. Furthermore, from the Lagrangian L, we may extract a Hamiltonian $H : T^*X \to \mathbb{R}$ such that

$$\frac{d}{dt}f = \{H, f\} \quad \forall f \in \mathcal{C}^\infty(X).$$

Thus we may view a classical dynamical system as a triple (Y, ω, H) with (Y, ω) a symplectic manifold and $H : Y \to \mathbb{R}$ a function.

Quantum mechanics can be viewed as a complex noncommutative analogue of this. We replace (Y, ω, H) by a triple $(\mathcal{A}, \star, H)$ with $\mathcal{A}$ a (topological) $\mathbb{C}$-algebra, $\star$ an antilinear involution on $\mathcal{A}$, and $H \in \mathcal{A}$. For an observable $f \in \mathcal{A}$, we require

$$\frac{d}{dt}f = ih[H, f].$$

If the quantum system $(\mathcal{A}, \star, H)$ is a "quantization" of a classical system (Y, ω, H), then there is a deep relationship between the two. For example, we should view $\mathcal{A}$ as a completion / extension of $\mathcal{C}^\infty(Y)$, and we can relate the values of an observable to the eigenvalues of the corresponding operator.

The above picture privileges the time dimension, so it is fundamentally non-relativistic. Working relativistically requires us to move from finite-dimensional spaces to infinite-dimensional spaces. Heuristically, we take Y to be our space of fields on spacetime (typically $M = M_0 \times \mathbb{R}$), and we fix a Lagrangian $L : Y \to \mathbb{R}$. However, $\mathcal{A}$ should now be viewed as a space of functions on an "almost finite-dimensional manifold." This requires many corrections!

Example 1.1.1. Consider scalar field theory, with $M = M_0 \times \mathbb{R}$ and $Y = \mathcal{C}^\infty(M)$. Then $\mathcal{A}$ looks like functions on a stratified configuration space

$$\coprod_{n \geqslant 0} \mathrm{Conf}_n(M_0),$$

where $\mathrm{Conf}_n(M_0)$ is the space of sets of n distinct unordered points on M_0.

1.1.2 Approaching quantum field theory

Heuristically, we should "spread things out over spacetime" by attaching an algebra $\mathcal{O}_x$ to every point $x \in M$ and letting

$$\mathcal{A} \approx \bigotimes_{x \in M} \mathcal{O}_x.$$

This has been formalized in the approach of algebraic quantum field theory – see e.g. the definition of Haag, which assigns an algebra $\mathcal{A}_U$ to each open set $U \subset M$.

An alternative approach, which we will pursue, privileges the time dimension. In this approach, we define a d-dimensional quantum field theory as a symmetric monoidal functor

$$E : \mathsf{Cob}_{d-1} \to \mathsf{Vect}.$$

Here Cob_{d-1} is the symmetric monoidal category with:

- Objects: $(d-1)$-manifolds, often assumed compact or compact with boundary (and thought of as time slices)
- Morphisms $M_0 \to M_1$: d-dimensional cobordisms from M_0 to M_1 (thought of as controlling time evolution)
- Monoidal structure: disjoint union of manifolds / cobordisms

and Vect is the symmetric monoidal category of (possibly infinite-dimensional) topological vector spaces (with $\otimes$ as tensor product).

There are many other ways to view quantum field theory. For example, one could think of QFT as a continuum limit of lattice models.

QFT is traditionally seen as describing "the world except for gravity." That is, we think of spacetime as fixed, without allowing for the effects of gravity. However, QFT can tell us about what sorts of gravitational effects are possible.

The perspective on QFT that we will follow has the advantage that it allows for direct comparisons between QFTs. Thus, in principle, we may consider a "moduli space" of QFTs (of a given type). We can think of gravitational effects as acting upon this moduli space.

1.1.3 The perspective of Connes

We'd like to use geometry to understand the noncommutative world of QFT. This can be accessed by studying the spectrum of our Hamiltonian H. Operators which evolve slowly in time "nearly commute" with H and thus must be "nearly diagonal." We're mostly interested in studying such operators (as the time scale of humans is much slower than that of the universe). Thus, when studying QFT, we're interested in algebras which are "nearly commutative."

Connes was interested in finding a mildly noncommutative generalization of algebras of functions on a manifold M. Recall that we can define the Clifford algebra $\mathrm{Cliff}(T_m M)$ as the algebra with generators $\{\gamma_\xi\}_{\xi \in T_m M}$ and relations $\gamma_\xi^2 = -\|\xi\|^2$. Let $\mathcal{B}$ be a bundle of finite-dimensional algebras on M containing $\mathrm{Cliff}(T_m M)$, and take a connection on $\mathcal{B}$. Let $\not{D}$ be the Dirac operator

$$\not{D} = \sum_i \gamma_{\xi_i} \otimes \frac{\partial}{\partial x_i},$$

and let $H = \not{D}^2$. Connes gave a formula for $\operatorname{tr} \not{D}^2$ and extracted interesting physical objects, e.g. the Higgs field, from $\not{D}$. From Connes' perspective, the Dirac operator is "as good as" the metric.

1.1.4 Algebra from functorial QFT

Suppose we view spacetime M as a cobordism from M_0 to M_1. Fix a d-dimensional quantum field theory E, where $E = \dim M$. For $x \in M$, we can define a vector space of operators $\mathcal{O}_x$ by taking a small disk D_x about x and letting $\mathcal{O}_x = E_{\partial D_x}$. We can view a punctured copy of M as giving a cobordism

$$M \sqcup (\sqcup_i \partial D_{x_i}) \to M_1,$$

so that applying E gives $E_{M_0} \otimes (\otimes_i \mathcal{O}_{x_i}) \to E_{M_1}$. This gives a sort of "higher multiplication" on the spaces of operators $\mathcal{O}_{x_i}$.

We can make this much more precise in special cases. For example, if E is topological, then $\mathcal{O}_{x_i} = \mathcal{O}$ is independent of x_i. Taking $M_0 = \emptyset$ and $M_1 = S^{d-1}$, we obtain a family of multiplications $\mathcal{O}^{\otimes n} \to \mathcal{O}$.

Example 1.1.2. If $d = 2$, the cap / cup / pair-of-pants cobordisms equip $A = \mathcal{O}$ with the structure of a finite-dimensional commutative Frobenius algebra. This means that A is a unital algebra with a nondegenerate trace $\theta : A \to \mathbb{C}$.

It is interesting to extend this picture to non-topological 2-dimensional QFTs. Here, one can set up a moduli space of QFTs and obtain a gravitational flow. This was a major historical motivation for string theory.

The second lecture will discuss the importance of positive energy to this theory. The third lecture will discuss scaling, and the fourth lecture will focus on finding a suitable definition for Cob.

1.2 6/10 – Positive Energy

1.2.1 Warmup

Consider a quantum mechanical system with Hilbert space $\mathcal{H}$. The Hamiltonian is a self-adjoint operator $H : \mathcal{H} \to \mathcal{H}$, and we can integrate iH to get unitary operators $U_t = e^{iHt} : \mathcal{H} \to \mathcal{H}$ describing the time evolution of the system.

Assuming $H \geqslant 0$ has strong consequences for the theory. In this case, $U_- : \mathbb{R} \to \mathrm{End}(\mathcal{H})$ is the boundary value of the function on the complex upper half-plane also given by $t \mapsto e^{iHt}$.

To fit this into QFT, consider $\mathbb{C}$ as a bundle over $\mathbb{R}$, where the inner product on the base is Lorentzian and the inner product on the fiber is Riemannian. Think of $\mathbb{R}$ as our spacetime M and the fiber as $V = T_x M$.

We will introduce the notion of a *complex metric* on a real vector space V. This is a quadratic form $V \to \mathbb{C}$ satisfying a certain condition to be determined later.

1.2.2 Historical digression

The notion of a complex metric was introduced by Kontsevich and Segal. Segal was originally interested in loop groups $\mathcal{L}G = \mathrm{Map}(S^1, G)$. There is a nice class of "positive energy" representations of loop groups. These extend to representations of $\mathrm{Diff}^+(S^1) \ltimes \mathcal{L}G$ which are also "positive energy" in a certain sense. This notion effectively means that the actions respect the grading $\mathcal{H} = \oplus_{k \geqslant 0} \mathcal{H}_k$ coming from an S^1-action.

An even simpler illustrative case is that of "discrete series" representations of $\mathrm{PSL}_2(\mathbb{R})$. Embed $\mathrm{PSL}_2(\mathbb{R}) \hookrightarrow \mathrm{PSL}_2(\mathbb{C})$. Let $\mathrm{PSL}_2^{<}(\mathbb{C})$ be the subsemigroup of Möbius transformations sending the unit disk D to a proper subdisk of itself. Then $\mathrm{PSL}_2^{<}(\mathbb{C})$ acts by contraction on the discrete series representations.

There is a similar story for loop groups, if we consider the semigroup of holomorphic $f : D \to D$ with $f(D) \subset \mathring{D}$. The Kontsevich-Segal definition connects to this somehow (I didn't quite catch how).

1.2.3 Complex metrics

Definition 1.2.1. A *complex metric* on a real vector space V is a quadratic form $g : V \to \mathbb{C}$ such that, if λ_k are the eigenvalues of the matrix corresponding to g and $\lambda_k = e^{i\theta_k} |\lambda_k|$, then $\sum_i |\theta_i| < \pi$.

We can equivalently state this as

$$-\pi \leqslant \sum_i \pm\theta_i \leqslant \pi.$$

One may view the θ_i as weights of the corresponding spin representation.

For $v \in V$, we have $g(v) \in \mathbb{C} \setminus (-\infty, 0)$, so $g(v)$ has a canonical square root. Thus we may assign a "complex length" in the positive half-plane to every $v \in V$.

Given an inner product $V \times V \to \mathbb{C}$, we get an inner product on $\wedge^k V$ with squared norm given by $\alpha \wedge (\star\alpha)$. Really, this is better thought of as a map $\wedge^k V \times \wedge^k V \to \wedge^d V$, for $d = \dim V$.

We'll change topics now.

1.2.4 Application to QFT

If we want to view a QFT as a functor E defined on a category of cobordisms, we should require:

- Cobordisms should be equipped with complex metrics.
- The induced maps $U_M : E_{M_0} \to E_{M_1}$ should be holomorphic in M (viewed as living in some complex analytic moduli space of complex metrics).
- The operators U_M should be *trace class*: there should be bases ξ_i for E_{M_0} and η_i for E_{M_1} with $\xi_i \mapsto \lambda_i \eta_i$ and $\sum_i |\lambda_i| < \infty$.

The case of Lorentzian metrics occurs "on the boundary" of the moduli space of complex metrics. Thus usual Lorentzian QFTs can be viewed as holomorphic degenerations of nice QFTs.

Some simpler examples are similar in principle:

- $\mathrm{Diff}^+(S^1)$ can be viewed as the boundary of the semigroup of contraction mappings.
- $\mathrm{U}(n) \subset \mathrm{GL}_n(\mathbb{C})$ can be viewed as the boundary of the semigroup of contraction operators $\mathbb{C}^n \to \mathbb{C}^n$.

However, $\mathrm{PSL}_2(\mathbb{R})$ is not compact, so we can't view it as a boundary.

Viewing our spacetime M as Lorentzian, we can ask for M to be "globally hyperbolic," so every point in M_0 can be connected to a point in M_1 via a timelike curve. This is similar to requiring that there are no black holes. The results of Kontsevich-Segal are best in the case of globally hyperbolic metrics.

Heuristically, we are requiring the existence of a Dirac operator and requiring that this Dirac operator is a contraction operator.

1.3 6/11 – Desired Properties of QFT; Scaling

In his first lecture, Ibou Bah proposed five properties that a (nice) QFT should satisfy: *unitarity*, *locality*, *causality*, *cluster decomposition*, and *renormalization*. We will first try to understand how these relate to Segal's perspective.

1.3.1 Renormalization

Historically, understanding renormalization was crucial to the development of QFT. This is somewhat arcane from a mathematical point of view. However, one should note that renormalizability of a Lagrangian does have a natural relationship to the calculus of variations.

Consider a σ-model where our fields are maps $M \to N$. If M and N have Riemannian metrics, we can define an energy functional. For $M = \mathbb{R}$ or S^1, this recovers the usual energy functional on paths. This is well-behaved and has a good Morse theory. When $\dim M = 2$, this is almost well-behaved, except there are problems with bubbling. For $\dim M > 2$, things are terrible.

The upshot is that (at least in this case?) renormalizability happens exactly when we expect things to be mathematically well-behaved.

1.3.2 Unitarity

If M is a cobordism from M_0 to M_1, we can reverse orientations on the boundary and view $\overline{M}$ as a cobordism from $\overline{M}_1$ to $\overline{M}_0$. When M is Riemannian, this doesn't significantly affect things. However, when M is Lorentzian, this changes the direction of time.

If $U_M : \mathcal{H}_{M_0} \to \mathcal{H}_{M_1}$ is the induced map, then we can think of $U_{\overline{M}} : \mathcal{H}_{\overline{M}_1} \to \mathcal{H}_{\overline{M}_0}$ as suitably "adjoint" to U_M. More precisely, we can write $\mathcal{H}_{M_i}$ and $\mathcal{H}_{\overline{M}_i}$ as duals, and this connects U_M to $U_{\overline{M}}$. We will assume that $U_{\overline{M}} = \overline{U_M}$. Reflection-positivity turns into the assumption that the pairing $\mathcal{H}_{M_0} \otimes \mathcal{H}_{\overline{M}_0} \to \mathbb{C}$ is positive (at least in nice cases?).

1.3.3 Locality

Given a point $x \in M$, we can view point operators as being defined on sufficiently small disks D about x. This gives algebras $\mathcal{O}_x = \lim_{x \in D} E_D$, where E is the QFT. In the first lecture here, we discussed how this relates to operadic multiplication. This is quite similar to the language of operator product expansion used in physics.

1.3.4 Cluster decomposition

Saying that two events are a long time apart is essentially saying that there is a long cylinder between them in spacetime. This can also be viewed as factorizing the cobordism defining spacetime into a product involving said long cylinder. Recall that the long cylinder in spacetime corresponds to time evolution e^{-iHt} in the QFT.

1.3.5 Causality

Consider a Lorentzian spacetime M with boundaries M_0 and M_1. Placing operations at points x in the interior of M gives operations $\mathcal{O}_x \otimes E_{M_0} \to E_{M_1}$.

We assume that our spacetime is globally hyperbolic, so every point in M_0 can be connected to a point in M_1 via a timelike path. This splits M into a family of time slices (all diffeomorphic to each other). Given distinct points x_i on the same time slice (so the x_i are spacelike separated), causality ensures that we can define

$$\mathcal{O}_{x_1,\dots,x_n} \otimes \check{E}_{M_0} \to \hat{E}_{M_1},$$

where $\check{-}$ denotes a dense subspace and $\hat{-}$ denotes a completion. This product is independent of the ordering. Using global hyperbolicity, we can extend this to any collection of spacelike separated points.

The notion of "small disk" makes complete sense in a Riemannian metric, but it seems a bit odd in a Lorentzian metric. To make sense of this, we recall our notion of complex metric $g : V \to \mathbb{C}$. If we pass to the complexification, we get a standard quadratic form $V_{\mathbb{C}} \to \mathbb{C}$. Thus we may consider V as a subspace of $\mathbb{C}^d \cong V_{\mathbb{C}}$. The space of allowable complex metrics can be viewed as a contractible subspace of the Grassmannian of real subspaces of V. One can make sense of the algebras $\mathcal{O}_x$ from this perspective.

1.3.6 Scaling

Let's return to the originally proposed topic of this lecture. Many things we'd like to prove about QFT can't be proved from the axioms we've given.

Viewing our spacetime M as a cobordism from M_0 to M_1, we should view a scalar field as a compactly-supported density on M. Such a scalar field should give an operator $E_{M_0} \to E_{M_1}$. This transformation $\mathrm{Dens}_c(M) \to \mathrm{Hom}(E_{M_0}, E_{M_1})$ can be called a *Wightman field*. It's not clear how to relate this to the bundle of algebras $\mathcal{O}$ we've been discussing earlier.

We also don't really know what to do about Lagrangians. From our definition, we can make sense of a moduli space of theories, and we get a renormalization group flow on this moduli space (by rescaling metrics). Reconciling our notion of fields (using this space) should help us make sense of Lagrangians. (It's not clear how global symmetries fit into this description.)

We should be able to fix these with a scaling axiom (defined in Riemannian signature). This should be something saying that our spaces of local operators $\mathsf{E}_{(r)}$ (defined on disks of radius r) tend to a limit $\mathsf{E}_{(0)}$. This would provide a filtration on the spaces $\mathcal{O}_x$ with associated graded $\mathsf{E}_{(0)}$. Using this should allow us to relate $\mathcal{O}_x$ with densities.

Extending our theories to codimension 2 should also help us prove things.

1.4 6/12 – More Desired Properties of QFT; Connection to Homotopy Theory

1.4.1 More on scaling

The notion of scaling seems quite common in physical treatments of QFT but is largely absent from mathematical treatments of the same.

Segal began by studying 2d CFT to understand T-duality. One version of this identifies a sigma-model with torus target T and another sigma-model with dual torus target T^*. This isomorphism isn't compatible with scaling: it interchanges short-distance and long-distance aspects of the theories.

1.4.2 Locality

Suppose that our theory sends a $(d-1)$-manifold M to a (topological) vector space E_M. To say that our theory is local, we'd like to say that E_M can be constructed "locally on M." More precisely, if $M = M_1 \cup_{\Sigma^{d-2}} M_2$, we'd like to construct E_M from E_{M_1}, E_{M_2}, and "$\mathsf{E}_{\Sigma^{d-2}}$." A naïve guess, ignoring the unknown E_Σ, fails: we typically have $\mathsf{E}_{M_1} \otimes \mathsf{E}_{M_2} \not\cong \mathsf{E}_M$. To understand this further, let's consider an example.

Example 1.4.1. Let G be a compact Lie group and $\mathcal{L}G$ the loop group of G. Assume E_M is an irrep of $\mathcal{L}G$. If we think of M as an interval with glued endpoints, then considering paths from any subinterval to G gives rise to a dense subspace of E_M. It's not clear how to combine such subspaces to get E_M.

In our case $M = M_1 \cup_{\Sigma^{d-2}} M_2$, we can construct E_M as "the space of operators on fields on all of M." The algebras of local operators $\mathcal{A}_{M_1}$ and $\mathcal{A}_{M_2}$ acts on E_M. If we enlarge M_1 and M_2 slightly past Σ (to get $M_1^+, M_2^+ \subset M$), we can get good results. Specifically, if $\mathcal{A}_{M_1}^{\mathrm{comm}}$ is the commutant of $\mathcal{A}_{M_1}$ acting on E_M, then $\mathcal{A}_{M_1}^{\mathrm{comm}}$ contains $\mathcal{A}_{M_2}$ and $\mathcal{A}_{M_1^+}$. The Tomita-Takesaki theorem tells us that $\mathcal{A}_{M_1}^{\mathrm{comm}}$ is the von Neumann algebra closure of $\mathcal{A}_{M_1}$ (?). Combining this with the action of $\mathcal{A}_{M_2}$ allows us to recover E_M from the behavior of the field theory M_1 and M_2. Here Σ gives us a bimodule relating $\mathcal{A}_{M_1}$ and $\mathcal{A}_{M_2}$.

Example 1.4.2. Returning to loop groups, suppose we decompose the interval $I = I_1 \cup_{\mathrm{pt}} I_2$. Consider the algebras of loops supported on each subinterval. To glue these together, we need to be able to extend these slightly across pt. This corresponds to a bimodule relating the algebras. In functional analysis, this corresponds to the "Tomita-Takesaki flow." This is related to the action of Poincaré boosts in special relativity.

1.4.3 Non-commutative geometry

We can reconstruct a manifold M from the behavior of modules over $\mathcal{C}^\infty(M)$. More generally, in non-commutative geometry, we study the category of modules over an algebra A. We say that Morita-equivalent algebras have the same geometry.

For example, we can study the K-theory of A. When $A = \mathcal{C}^\infty(M)$, the Chern character isomorphism lets us write

$$\mathsf{K}(A) \otimes \mathbb{Q} \cong \oplus_i H^{2i}(M; \mathbb{Q}).$$

The grading here relies on the existence of Adams operations (based on $\mathsf{E} \mapsto \wedge^n \mathsf{E}$, at least for line bundles).

In the non-commutative context, we don't have Adams operations. Instead, there's a formal group law, and we get lots of strangeness governed by chromatic homotopy theory. From a QFT perspective, the grading relies on some sort of scaling.

Rings of $\mathcal{C}^\infty$ functions also experience a sort of Bott periodicity. Consider the Schwartz space $\mathcal{C}^\infty_S(\mathbb{R}^2)$ of functions decaying sufficiently rapidly at infinity. We can construct a non-commutative deformation $\mathcal{C}^\infty_S(\mathbb{R}^2)_{(h)}$, which is isomorphic (via Fourier transformation) to the algebra of "smoothing operators" from $\mathcal{C}^\infty(\mathbb{R})$ to itself. By "smoothing operator," we mean a smooth function $K(x, y)$, viewed as an integral transform

$$(Kf)(x) = \int f(y)K(x, y)dy.$$

This algebra is Morita equivalent to $\mathbb{C}$ (?), and deforming again recovers $\mathcal{C}^\infty_S(\mathbb{R}^2)$ (?).

More generally, we can relate the geometry of X with a compactified geometry of $X \times \mathbb{R}^2$. This ends up producing Floer homotopy types.

1.4.4 The smooth homotopy category

Long-distance behavior in QFT looks somewhat like homotopy theory. Short-distance behavior in QFT is a bit more open to debate. Many people like to think about lattices. However, Segal prefers to think about the short-distance behavior in terms of (rings of functions of) manifolds.[1]

We can ask how to represent QFTs geometrically. Classically, Brown's representability theorem lets us view homotopy theory as captured by contravariant functors from the homotopy category of finite CW complexes to sets. The "smooth homotopy category" replaces sets here by manifolds (and interprets everything in terms of suitable model categories).

Example 1.4.3. Given a Lie algebra $\mathfrak{g}$, we can view the smooth homotopy type of $\mathfrak{g}$ as given by sending M to the moduli space of flat $\mathfrak{g}$-connections on the trivial bundle on M. Similarly, given a Lie group G, we can view the smooth homotopy type of G as given by sending M to the moduli space of flat G-bundles on M.

There's a paper of Kapranov which interprets the Lie algebra of a based loop group ΩX in this context. Curvature is captured by maps $\Omega X \to G$.

Classical homotopy theory is "perturbative:" it is built one step at a time from sets (by adding homotopies, higher homotopies, etc.). Analysis is also "perturbative:" we take higher derivatives one at a time. However, the theories are "perturbative" in different dirrections: homotopy theory gives the "long-distance perturbative theory" while analysis gives "short-distance perturbative theory."

[1] This may be more a matter of human taste in describing the universe than a fundamental fact of nature.

Chapter 2

Nitu Kitchloo – Symmetry Breaking, Surface Defects, and Link Homologies

2.1 6/10 – Overview

2.1.1 Artin braid groups

Let G be a compact connected Lie group with maximal torus T. Recall that this means $T \subset G$ is a maximal connected compact abelian subgroup.

Example 2.1.1. For $G = U(n)$, we can take T to be the subgroup Δ of diagonal matrices. We will return to this example throughout.

Given (G, T), we can define an *Artin braid group* $\mathrm{Br}(G, T)$.

Example 2.1.2. For $(G, T) = (U(n), \Delta)$, this is the usual braid group

$$\mathrm{Br}(G, T) = \mathrm{Br}_n := \langle \sigma_1, \ldots, \sigma_{n-1} \mid \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \quad \sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |i-j| > 1\rangle.$$

More generally, note that T acts on G (and its Lie algebra $\mathfrak{g}$) via the adjoint representation. Complexify $\mathfrak{g}$ to get a complex representation $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ of T. Let R be the set of roots of G, i.e. nonzero characters of T in $\mathfrak{g}_{\mathbb{C}}$. Let $\mathfrak{h} = \mathrm{Lie}(T) \otimes_{\mathbb{R}} \mathbb{C}$, and let

$$\mathcal{H} = \mathfrak{h}_{\mathbb{C}} - \cup_{\alpha \in \mathbb{R}} \mathfrak{h}_\alpha \text{ for } \mathfrak{h}_\alpha = \{h \in \mathfrak{h}_{\mathbb{C}} \mid \alpha(h) = 0\}.$$

In other words, $\mathcal{H}$ is the union of the interiors of the Weyl chambers. Note that $\mathcal{H}$ has a free action of the Weyl group $W(G, T) = N_T(G)/T$.

Definition 2.1.3. The *Artin braid group* $\mathrm{Br}(G, T)$ is $\pi_1(\mathcal{H}/W(G, T))$.

It is a fact that $\mathcal{H}/W(G, T)$ is a $K(\mathrm{Br}(G, T), 1)$. Let's see why this works for $G = U(n)$.

Example 2.1.4. For $(G, T) = (U(n), \Delta)$, we have

$$\cup_\alpha \mathfrak{h}_\alpha = \{(x_1, \ldots, x_n) \mid x_i = x_j \text{ for some } i \neq j\}.$$

Since $W(G, T) = \Sigma_n$ acting on $\mathfrak{h}_{\mathbb{C}} = \mathbb{C}^n$ by permuting coordinates, we see that $\mathcal{H}/W(G, T)$ is the moduli of n points in $\mathbb{C}$. This picture gives us the usual definition of braid groups.

Note that $\pi_0 \operatorname{Map}(S^1, \mathcal{H}/W(G, T))$ is the space of conjugacy classes of elements in $\mathrm{Br}(G, T)$, which we will write as $[\mathrm{Br}(G, T)]$.

Example 2.1.5. For $(G, T) = (U(n), \Delta)$, elements of $\mathrm{Br}(G, T)$ are represented by braid closures. We may view these as links in $\mathbb{R}^3$ using Markov's theorem.

Theorem 2.1.6 (Markov). *The map sending a braid to its closure expresses the set of equivalence classes of links in* $\mathbb{R}^3$ *as the set of equivalence classes of all braids,* $(\sqcup_{n\geqslant 1}\mathrm{Br}_n)/\sim$*, where* $\sim$ *is the equivalence relation generated by M1 and M2:*

1. *M1 (Conjugation):* $\sigma \sim \rho\sigma\rho^{-1}$ *for* $\sigma, \rho \in \mathrm{Br}_n$
2. *M2 (Stabilization):* $\sigma \sim \sigma\sigma_n^{\pm 1}$ *for* $\sigma \in \mathrm{Br}_n \subset \mathrm{Br}_{n+1}$

The individual maps from $[\mathrm{Br}_n]$ to the space of links are not faithful. That is, stabilization does cause us to lose some information.

2.1.2 Categorification

We'd like to categorify $\mathrm{Br}(G, T)$ and $[\mathrm{Br}(G, T)]$. In particular, we'd like a functor of ("discrete" / non-∞) 2-categories[1]

$$* \mathbin{/\!\!/} \mathrm{Br}(G, T) \to \mathrm{Gr}(\mathsf{Alg}).$$

Here $* \mathbin{/\!\!/} \mathrm{Br}(G, T)$ is the 2-category with:

- one object,
- $\mathrm{Br}(G, T)$ as 1-morphisms, and
- trivial 2-morphisms.

The right hand side, $\mathrm{Gr}(\mathsf{Alg})$, is the 2-category with:

- objects: graded commutative algebras,
- 1-morphisms: chain complexes of graded bimodules, and
- 2-morphisms: homotopy classes of chain maps

We also want a map $[\mathrm{Br}(G, T)] \to \mathsf{Ch}_{\bullet,\bullet}(\mathbb{Z})$.

This has been worked out by Soergel, Rouquier, Khovanov, and Khovanov-Rozansky. For $G = U(n)$, the last map above gives bigraded homological invariants of links. (This requires us to check that the stabilization move is satisfied.)

Being more ambitious, we can try to realize this using 3d and 4d topological gauge theories. This splits into subgoals:

1. Describe a configuration space of G-background fields on a 2-torus. (Here "background" means that we haven't "summed over these fields.")
2. Quantize by "summing over the background fields" to obtain the partition function of the 2-torus in 3d and 4d topological gauge theories with defects.
3. Identify the partition functions with the categorifications mentioned above.

2.1.3 Preliminaries: $U(n)$

We'll start out by looking at the case of $G = U(n)$. Let $R_n = H^\bullet_T(\mathrm{pt}) = \mathbb{Z}[x_1, \ldots, x_n]$ with $|x_i| = 2$. The Weyl group W is Σ_n, acting on R_n with σ_i swapping x_i and x_{i+1}.

Definition 2.1.7. The *Bott-Samelson* R_n*-bimodule* is the graded bimodule $\mathrm{bS}_i = R_n \otimes_{R_n^{\sigma_i}} R_n$, where $R_n^{\sigma_i}$ is the ring of σ_i-invariants in R_n. The category SBimod_n *Soergel bimodules* is the subcategory of R_n-bimodules generated by the bS_i's (allowing idempotent completion and grading shifts).

[1] Recent work extends this to a map of $(\infty, 2)$-categories.

Theorem 2.1.8 (Soergel)**.** *The category* SBimod_n *categorfies the Hecke algebra* $H_q(A_n) = \mathbb{Z}[q, q^{-1}]\langle \mathrm{Br}_n \rangle / \mathrm{I}$*, where* I *is generated by the relations*

$$(\sigma_i - q^2)(\sigma_i + 1) = 0$$
$$\sigma_i^{-1} = q^{-2}\sigma_i + (q^2 - 1).$$

That is, there is an equivalence $K_0(\mathsf{SBimod}_n) \cong H_q(A_n)$*. Under this equivalence:*

- 1 *corresponds to* R_n*,*
- bS_i *corresponds to* $\sigma_i + 1$*, and*
- q *corresponds to grading shift.*

The chain complex $C_i = (\mathrm{bS}_i \to R_n)$, where the map is given by $\alpha \otimes \beta \mapsto \alpha\beta$, should be viewed as a categorification of σ_i.

Theorem 2.1.9 (Rouquier)**.** *The assignment* $\sigma_I \mapsto c_I := C_{i_1} \otimes_{R_n} C_{i_2} \otimes_{R_n} \cdots \otimes_{R_n} C_{i_k}$ *gives a well-defined map* $* /\!\!/ \mathrm{Br}_n \to \mathrm{Gr}(\mathsf{Alg})$*.*

Example 2.1.10. The braid $\sigma_1\sigma_2$ is sent to the chain complex $C_{12} = (\mathrm{bS}_1 \to R_3) \otimes (\mathrm{bS}_2 \to R_3)$, which can also be written

$$\mathrm{bS}_1 \otimes_{R_3} \mathrm{bS}_2 \longrightarrow \mathrm{bS}_1 \oplus \mathrm{bS}_2 \longrightarrow R_3.$$

Rouquier's theorem is nontrivial for two reasons:

- We need to show that the map, which *a priori* is defined only on braid words, is actually defined on the braid group.
- We need to make sure that this respects the triviality of the higher morphisms in $* /\!\!/ \mathrm{Br}_n$.

2.2 6/10 – Geometric Interpretations for the $\mathrm{U}(n)$ Case

Last time, we discussed a categorification of Br_n. We'd like to obtain a corresponding categorification of $[\mathrm{Br}_n]$.

2.2.1 Categorification of $[\mathrm{Br}_n]$

Recall the definition of Hochschild homology.

Definition 2.2.1. If M is a graded R_n-bimodule, then the *Hochschild homology* of M is

$$\mathrm{HH}(M) = \mathrm{Tor}^{\bullet,\bullet}_{R_n \otimes R_n^{\mathrm{op}}}(M, R_n)$$

We will consider the Hochschild homology $\mathrm{HH}(C_I)$ termwise.

Example 2.2.2. For $\sigma_1\sigma_2$, we discussed C_{12} in the last lecture. With the above convention, $\mathrm{HH}(C_{12})$ is

$$\mathrm{HH}(\mathrm{bS}_1 \otimes_{R_3} \mathrm{bS}_2) \longrightarrow \mathrm{HH}(\mathrm{bS}_1 \oplus \mathrm{bS}_2) \longrightarrow \mathrm{HH}(R_3).$$

Theorem 2.2.3 (Khovanov)**.** *The assignment* $\sigma_I \mapsto \mathrm{HH}(C_I)$ *extends to* $[\mathrm{Br}_n] \to \mathsf{Ch}_{\bullet,\bullet}(\mathbb{Z})$*. In fact, the output is naturally trigraded. Furthermore, this assignment satisfies M1 and M2, so it gives a homological invariant of links.*

This invariant can be identified with the trigraded Khovanov-Rozansky link homology. If $\mathcal{L}$ is the link $[\sigma_i]$, we write

$$\mathfrak{sl}_\infty(\mathcal{L}) = \mathrm{HHH}(\mathcal{L}) = \mathrm{HH}(C_I).$$

We may view $\mathfrak{sl}_\infty$ as a "limit" of bigraded homological invariants $\mathfrak{sl}_N(\mathcal{L})$ which can be obtained from $\mathrm{HH}(C_I)$ by adding a "matrix factorization" differential.

2.2.2 Topological interpretation of Bott-Samelson bimodules

Definition 2.2.4. For $i < n$, let $G_i \subset U(n)$ be the semisimple rank one compact parabolic consisting of matrices of the form

$$\begin{bmatrix} D_1 & 0 & 0 \\ 0 & U & 0 \\ 0 & 0 & D_2 \end{bmatrix}$$

with D_1 and D_2 diagonal and $U \in U(2)$ (placed in rows / columns i and $i+1$).

Note that $T \subset G_i \subset G = U(n)$ for all i.

Definition 2.2.5. Let BS_i be the balanced product $U(n) \times^T (G_i/T)$, considered with its natural $U(n)$-action.

We may obtain BS_i from a pullback square

$$\begin{tikzcd} BS_i \arrow[r, "\alpha"] \arrow[d, "\beta"] & U(n)/T \arrow[d] \\ U(n)/T \arrow[r] & U(n)/G_i \end{tikzcd}$$

where $\alpha : [g, g_i] \mapsto [gg_i]$ and $\beta : [g, g_i] \mapsto [g]$.

In cohomology, we get a map

$$H^\bullet_{U(n)}(U(n)/T) \otimes_{H^\bullet_{U(n)}(U(n)/G_i)} H^\bullet_{U(n)}(U(n)/T) \to H^\bullet_{U(n)}(BS_i).$$

A spectral sequence computation shows this is an isomorphism. Combining this with the standard isomorphisms $H^\bullet_G(G/K) \cong H^\bullet_K(\mathrm{pt})$ for any G and any $K \subset G$, we get

$$H^\bullet_{U(n)}(BS_i) \cong H^\bullet_T(\mathrm{pt}) \otimes_{H^\bullet_{G_i}(\mathrm{pt})} H^\bullet_T(\mathrm{pt}) \cong R_n \otimes_{R_n^{\sigma_i}} R_n = bS_i.$$

Thus, we can interpret the Bott-Samelson bimodule bS_i geometrically as the $U(n)$-equivariant cohomology of BS_i.

The pullback square defining BS_i is in fact a homotopy pullback square. Comparing with the standard definition of the homotopy pullback, we can view BS_i as the moduli space of $U(n)$-bundles on $[0,1]$ with a reduction of structure group to G_i on $[0,1]$ and to T on $\{0,1\}$.

2.2.3 Interpretation of Rouquier's chain complex

More generally, for a multi-index $I = (i_1, \ldots, i_k)$, define

$$BS_I = U(n) \times^T (G_{i_1} \times^T G_{i_2} \times^T \cdots \times^T G^{i_k}/T).$$

Then

$$H^\bullet_{U(n)}(BS_I) \cong bS_{i_1} \otimes_{R_n} bS_{i_2} \otimes_{R_n} \cdots \otimes_{R_n} bS_{i_k}.$$

We may view BS_I as the moduli space of $U(n)$-bundles on $[0,1]$ with $k+1$ marked points (including 0 and 1) with reduction of structure group to G_{i_j} on the jth interval and to T at each marked point.

If J is obtained from I by discarding indices, then we get $BS_J \hookrightarrow BS_I$ by replacing every factor G_{i_k} (for each removed index i_k) by T in the construction of BS_I. These replaced copies of T are "transparent defects." This inclusion respects composition of inclusions of multi-indices.

Rouquier's chain complex C_I is isomorphic to

$$H^\bullet_{U(n)}(BS_I) \longrightarrow \oplus_{|J|=k-1, J \subset I} H^\bullet_{U(n)}(L_J) \longrightarrow \ldots \longrightarrow H^\bullet_{U(n)}(U(n)/T).$$

All differentials here are obtained from (signed) inclusions $BS_J \to BS_{J'}$.

2.2.4 Interpretation of $\mathsf{HH}(\mathrm{bS}_\mathrm{I})$

To interpret $\mathsf{HH}(\mathsf{H}^\bullet_{\mathrm{U}(n)}(\mathrm{BS}_\mathrm{I})$, we introduce the following.

Definition 2.2.6. Let
$$\mathsf{L}_i = \mathrm{U}(n) \times^\mathsf{T} \mathsf{G}_i$$
where T acts on G_i by conjugation.

The map $\mathsf{L}_i \to \mathrm{BS}_i$ given by $[g, g_i] \mapsto [g, g_i]$ is a principal T-bundle. In fact, we have a pullback square

$$\begin{array}{ccc} \mathsf{EU}(n)\times^{\mathrm{U}(n)} & \longrightarrow & \mathsf{BT} \\ \downarrow & & \downarrow{\scriptstyle\Delta} \\ \mathsf{EU}(n)\times^{\mathrm{U}(n)}\mathrm{BS}_i & \longrightarrow & \mathsf{BT}\times\mathsf{BT} \end{array}$$

where both vertical arrows are trivial principal bundles.

Collapse of the Eilenberg-Moore spectral sequence at the E_2 page gives
$$\mathsf{H}^\bullet_{\mathrm{U}(n)}(\mathsf{L}_i) \cong \mathrm{Tor}_{\mathsf{H}^\bullet(\mathsf{BT}\times\mathsf{BT})}(\mathsf{H}^\bullet_{\mathrm{U}(n)}(\mathrm{BS}_i), \mathsf{H}^\bullet(\mathsf{BT})) = \mathsf{HH}(\mathsf{H}^\bullet_{\mathrm{U}(n)}(\mathrm{BS}_i)) = \mathsf{HH}(\mathrm{bS}_i).$$
Thus we obtain a geometric interpretation of $\mathsf{HH}(\mathrm{bS}_i)$. We may interpret L_i as the moduli space of $\mathrm{U}(n)$-bundles on S^1 with one marked point, where the structure group reduces to G_i away from the marked point and to T at the marked point.

More generally, for a multi-index I, we let
$$\mathsf{L}_\mathrm{I} = \mathrm{U}(n) \times^\mathsf{T} (\mathsf{G}_{i_1} \times^\mathsf{T} \mathsf{G}_{i_2} \times^\mathsf{T} \cdots \times^\mathsf{T} \mathsf{G}_{i_k})$$
where T acts by conjugation on G_{i_1} and G_{i_k}. An argument analogous to the above gives $\mathsf{H}^\bullet_{\mathrm{U}(n)}(\mathsf{L}_\mathrm{I}) \cong \mathsf{HH}(\mathrm{bS}_\mathrm{I})$.

For $\mathsf{J} \subset \mathrm{I}$, we get an inclusion $\mathsf{L}_\mathsf{J} \to \mathsf{L}_\mathrm{I}$ that behaves well with respect to composition of inclusions. These assemble into a complex
$$\mathsf{H}^\bullet_{\mathrm{U}(n)}(\mathsf{L}_\mathrm{I}) \longrightarrow \oplus_{|\mathsf{J}|=k-1,\mathsf{J}\subset\mathrm{I}}\mathsf{H}^\bullet_{\mathrm{U}(n)}(\mathsf{L}_\mathsf{J}) \longrightarrow \dots$$
which is isomorphic to $\mathsf{HH}(\mathsf{C}_\mathrm{I})$.[2]

2.3 6/11 – Local Systems and Twisted Cohomology

2.3.1 Review of bundle theory

Let G be a topological group. Principal G-bundles are classified by pulling back a "universal bundle" $\mathsf{EG} \to \mathsf{BG}$. That is, for any principal G-bundle $\mathsf{E} \to \mathsf{B}$, there exists an $f : \mathsf{B} \to \mathsf{BG}$ (unique up to homotopy) such that $\mathsf{E} = f^*\mathsf{EG}$. To construct $\mathsf{EG} \to \mathsf{BG}$, just take any contractible space EG with free G-action, and let $\mathsf{BG} = \mathsf{EG}/\mathsf{G}$.

We will write a G-space X as $\mathsf{X} /\!\!/ \mathsf{G}$. From such a space, we may construct the *homotopy orbits* $\mathsf{X} /\!\!/^h \mathsf{G} = \mathsf{EG} \times^\mathsf{G} \mathsf{X} := (\mathsf{EG} \times \mathsf{X})/\mathsf{G}$, where G imposes the relation $(e, x) \sim (eg, g^{-1}x)$. We define the G-*equivariant cohomology* of X to be
$$\mathsf{H}^\bullet_\mathsf{G}(\mathsf{X}) := \mathsf{H}^\bullet(\mathsf{X} /\!\!/^h \mathsf{G}).$$

If $\mathsf{K} \subset \mathsf{G}$, then we may take EG as a model for EK. Because $\mathsf{EG} \times^\mathsf{G} (\mathsf{G} \times^\mathsf{K} \mathsf{Y}) = \mathsf{EG} \times^\mathsf{K} \mathsf{Y} = \mathsf{EK} \times^\mathsf{K} \mathsf{Y}$, we obtain
$$\mathsf{H}^\bullet_\mathsf{G}(\mathsf{G} \times^\mathsf{K} \mathsf{Y}) = \mathsf{H}^\bullet_\mathsf{K}(\mathsf{Y}).$$

Given a principal G-bundle $\mathsf{E} \to \mathsf{B}$, we may ask whether or not E is extended from a principal K-bundle, i.e. whether $\mathsf{E} \cong \mathsf{E}' \times^\mathsf{K} \mathsf{G}$ for some principal K-bundle $\mathsf{E}' \to \mathsf{B}$. This is equivalent to asking whether the classifying map $f_\mathsf{E} : \mathsf{B} \to \mathsf{BG}$ lifts to a map $f_{\mathsf{E}'} : \mathsf{B} \to \mathsf{BK}$. Note that $\mathsf{BK} \to \mathsf{BG}$ has fiber G/K, so in particular reductions of structure group of the trivial G-bundle to K correspond to maps $\mathsf{B} \to \mathsf{G}/\mathsf{K}$.

[2] This is a "cubical complex," related to crossing resolutions for nice presentations of knots.

2.3.2 Recap

Last time, we defined subgroups $G_i \subset U(n)$ for $1 \leqslant i < n$. For each multi-index $I = (i_1, \ldots, i_k$, we defined spaces $BS_I = U(n) \times^T (G_{i_1} \times^T G_{i_2} \times^T \cdots \times^T G_{i_k}/T)$. We showed

$$H^\bullet_{U(n)}(BS_I) = bS_{i_1} \otimes_{R_n} \cdots \otimes_{R_n} bS_{i_k}$$

for $bS_i = R_n \otimes_{R_n^{\sigma_i}} R_n$. The spaces BS_I are functorial with respect to inclusions of multi-indices.

Example 2.3.1. Let $I = \{i\}$ and $J = \emptyset$, so $BS_i = U(n) \times^T G_i/T$ and $BS_\emptyset = U(n) \times^T (T/T) = U(n)/T$. Note that $H^\bullet_{U(n)}(BS_i) = bS_i$ and $H^\bullet_{U(n)}(U(n)/T) = H^\bullet_T(pt) = R_n$. The inclusion $BS_\emptyset \to BS_i$ comes from $T \subset G_i$, and the induced map on cohomology is the multiplication map

$$bS_i = R_n \otimes_{R_n^{\sigma_i}} R_n \to R_n.$$

This is the chain complex C_i appearing in Rouquier's work.

We also defined spaces $L_I = U(n) \times^T (G_{i_1} \times^T G_{i_2} \times^T \ldots G_{i_k})$ where T acts on G_{i_1} and G_{i_k} by conjugation. These spaces may be viewed as moduli of $U(n)$-bundles on S^1 with $k+1$ marked points with reduction of structure group to G_{i_j} on the jth interval and to T at each marked point. The cohomologies can be used to construct a chain complex which recovers Khovanov's link homology. Specifically, we have $H^\bullet_{U(n)}(L_I) = HH(bS_I)$.

2.3.3 Recovering $\mathfrak{sl}_N$-link homology

There is a forgetful map $L_I /\!\!/ U(n) \to U(n) /\!\!/ U(n)$, where $U(n)$ acts on itself via the adjoint representation. This map is given by the formula

$$[g, g_1, \ldots, g_k] \mapsto g g_1 \ldots g_k g^{-1}.$$

If one thinks of the bundles appearing in the moduli description of L_I as having connections, this is essentially taking the product of the holonomies.

The functorial maps of $L_I /\!\!/ U(n)$ are compatible with the forgetful maps to $U(n) /\!\!/ U(n)$. In particular, Khovanov's chain complex

$$H^\bullet_{U(n)}(L_I) \longrightarrow \oplus_{|J|=k-1, J\subset I}(L_J) \longrightarrow \ldots$$

is a complex over the torsion-free $H^\bullet_{U(n)}(U(n))$. Let's think about what happens if we take termwise cohomology with respect to odd elements of $H^\bullet_{U(n)}(U(n))$.

There is a non-canonical ring isomorphism

$$H^\bullet_{U(n)}(U(n)) \cong H^\bullet_{U(n)}(pt) \otimes H^\bullet(U(n)).$$

The non-canonicalness of this is related to the fact that $U(n) /\!\!/^h U(n) \not\simeq BU(n) \times U(n)$. Essentially, it's not clear what to do on primitive Hopf algebra elements.

However, if we take $n \to \infty$ (and write $U = U(\infty)$), we do get a canonical equivalence

$$U /\!\!/^h U \simeq BU \times U.$$

Pulling back the elements of $H^\bullet(U)$ gives odd elements $d_1, d_3, \ldots$, and we can modify Khovanov's complex to get

$$H^\bullet(H_{U(n)}(L_I), d_{2N-1}) \longrightarrow \oplus_{|J|=k-1, J\subset I} H^\bullet(H_{U(n)}(L_J), d_{2n-1}) \longrightarrow \ldots$$

Here n and N are unrelated.

Theorem 2.3.2 (T. Gomez)**.** *This complex is isomorphic to the link homology $\mathfrak{sl}_N(\mathcal{L})$ for the link $\mathcal{L} = [\sigma_I]$.*

Example 2.3.3. Let $\mathcal{L}$ be the unknot, so $\mathcal{L} = [\sigma_1]$. Then $\mathfrak{sl}_\infty(\mathcal{L}) = H^\bullet_{U(1)}(U(1) \times^T T) = H^\bullet_T(T) = \mathbb{Z}[x_2, y_1]$. Here d_N corresponds to the odd class $y_1 x^N$. We get

$$H^\bullet(H_{U(1)}(L_I), d_N) \cong (\mathbb{Z}[x]/x^N)\langle y_1 \rangle.$$

2.3.4 Local systems

The above story was a bit ad hoc, so we'd like to fit it into a more topological context. We can think of $U(n)$ as $B\Omega U(n)$, so $\Omega U(n)$ is really acting everywhere. In fact, we'd like the second Markov move (stabilization) to play well with our theory, so we should pass to an ΩU-action. We can implement this by using a *twisted cohomology theory*

$$\mathcal{H} = \Omega U_+ \wedge H_{\mathbb{Z}} = (\mathbb{Z} \times BU)_+ \wedge H_{\mathbb{Z}}.$$

One can compute

$$\pi_\bullet \mathcal{H} = \pi_\bullet H[b_0^\pm, b_1, b_2, \dots]$$

with $|b_i| = 2i$.

Setting $b_0 = b_N = 1$ and $b_i = 0$ otherwise (and indicating this with the subscript N), one can calculate

$$\tilde{\mathcal{H}}^\bullet_{U(n)}(L_I)_N \simeq H^\bullet(H_{U(n)}(L_I), d_n).$$

2.4 6/12 – Spectral Sequence Invariants; Relationship to TQFT

2.4.1 Remarks on questions

Previously, we constructed (for $G = U(n)$) a G-space L_I such that $H_G(L_I) = HH(bS_I)$. There is a G-equivariant map $L_I \to G$ (where G acts on itself by convolution), and we can view this as a derived local system. For these purposes, we should view $G = B(\Omega G)$.

Example 2.4.1. Consider ΩG-twisted cohomology $\mathcal{H} = (\Omega U)_+ \wedge H_{\mathbb{Z}}$. Then $\pi_\bullet \mathcal{H} = (H_{\mathbb{Z}})_\bullet[b_0^{\pm 1}, b_1, \dots]$. Here the b_i control the possible twistings. Setting $b_0 = b_N = 1$ and $b_i = 0$ otherwise gives $\mathfrak{sl}_N$-link homology. We can consider the power series $\sum_i b_i x^i$, which can be viewed in a matrix factorization context as the derivative of our superpotential.

Example 2.4.2. We can also consider twisted K-theory. Note that K_G has twistings indexed by $\mathbb{N}$. Here (for $N \in \mathbb{N}$)

$$^N K_{U(1)}(L_\emptyset) = \mathbb{Z}[x^{\pm 1}]/(x^N - 1)$$

gives the corresponding invariant for the unknot. Freed-Hopkins-Teleman allows us to identify this with the ring controlling level N representations of the loop group $LU(1)$. There is a "level-rank duality" going on here relating this to $\mathbb{Z}[x]/(x^N)$.

Twisted K-theory is a functor defined on G-spaces X over G. This produces modules over $^0K_G(X)$.

2.4.2 Spectral sequence interpretation of Khovanov homology

Recall that an inclusion of multi-indices $J \subset I$ induces $L_J \subset L_I$. We used this to construct a complex producing Khovanov homology. We'd like to interpret this topologically. To this end, we construct a filtered G-equivariant space $F_\bullet(sL_I)$ with

$$\begin{aligned}
F_0(sL_I) &= \mathrm{cone}(\emptyset \to L_I) = L_I \\
F_1(sL_I) &= \mathrm{cone}\left(\cup_{|J|=k-1} L_J \to L_I\right) \\
F_2(sL_I) &= \mathrm{cone}\left(\cup_{|J|=k-1 \text{ or } k-2} L_J \to L_I\right) \\
&\cdots
\end{aligned}$$

We can compute $F_k(sL_I)/F_{k-1}(sL_I) = \vee_{|J|=n-r} \Sigma^r (L_J)_+$. This yields a spectral sequence with E_1 term $HH(C_I^\bullet)$ converging to $H^\bullet_{U(n)}(F_k(sL_I))$. This limit space is not particularly interesting – it's just a Thom space, and it counts the number of components of the link. However, the terms of the spectral sequence are more interesting.

Theorem 2.4.3. *The pages E_i for $i \geqslant 2$ are invariants of the link σ_I.*

We'd like to view the filtered G-space $\mathsf{F}_\bullet(s\mathsf{L}_I)$ as a link invariant itself. First, we need a notion of equivalence for such objects. We will consider filtered G-spaces as giving key examples of G-spectra.

Definition 2.4.4. Let $\mathsf{F}_\bullet$, $\mathsf{G}_\bullet$ be G-spectra. We say that a map $f_\bullet : \mathsf{F}_\bullet \to \mathsf{G}_\bullet$ is an *elementary quasi-equivalence* if its (co)fiber $\mathsf{Z}_\bullet$ is acyclic, i.e. the identity on $\mathsf{Z}_\bullet$ is homotopic to 0. More generally, we say $\mathsf{F}_\bullet$ and $\mathsf{G}_\bullet$ are *quasi-equivalent* if they are connected by a zigzag of elementary quasi-equivalences.

Theorem 2.4.5. *The filtered* G*-space* $\mathsf{F}_\bullet(s\mathsf{L}_I)$ *is an invariant of the link* σ_I *up to quasi-equivalence.*

2.4.3 Towards a TQFT interpretation

Suppose that M is a manifold with a G-bundle (thought of as a background field). This is classified by a map $f_\mathsf{M} : \mathsf{M} \to \mathsf{BG}$. For $\mathsf{M} = \mathsf{N} \times \mathsf{S}^1$, we may regard the classifying map as equivalent to a map $\hat{f}_\mathsf{N} : \mathsf{N} \to \mathsf{LBG} = \mathrm{Map}(\mathsf{S}^1, \mathsf{BG})$.

Proposition 2.4.6. *The free loop space* LBG *is equivalent to* $\mathsf{G} /\!\!/^h \mathsf{G} = \mathsf{EG} \times^\mathsf{G} \mathsf{G}$ *(where* G *acts on itself by conjugation).*

Proof. Decomposing $\mathsf{S}^1 = \mathsf{D}^+ \cup_{\mathsf{S}^0} \mathsf{D}^-$, we may view the data of a map $\mathsf{S}^1 \to \mathsf{BG}$ as consisting of a bundle on D^+, a bundle on D^-, and an equivalence of the two on S^0. Written another way, this is a homotopy fiber product

$$\begin{tikzcd} \mathsf{LBG} \arrow[r] \arrow[d] & \mathsf{BG} \arrow[d, "\Delta"] \\ \mathsf{BG} \arrow[r, "\Delta"] & \mathsf{BG} \times \mathsf{BG}. \end{tikzcd}$$

Writing

$$\begin{aligned} \mathsf{BG} &\simeq \mathsf{E}(\mathsf{G} \times \mathsf{G})/\Delta \\ &\simeq \mathsf{E}(\mathsf{G} \times \mathsf{G}) \times^{\mathsf{G}\times\mathsf{G}} (\mathsf{G} \times \mathsf{G})/\Delta \\ &\simeq \mathsf{E}(\mathsf{G} \times \mathsf{G}) \times^{\mathsf{G}\times\mathsf{G}} \mathsf{G}, \end{aligned}$$

we can write $\mathsf{LBG} = \mathsf{BG} \times^h_{\mathsf{BG}\times\mathsf{BG}} (\mathsf{E}(\mathsf{G} \times \mathsf{G}) \times^{\mathsf{G}\times\mathsf{G}} \mathsf{G})$, which gives the desired result. □

The upshot is that G-background fields on $\mathsf{N} \times \mathsf{S}^1$ are equivalent to $\mathsf{G} /\!\!/^h \mathsf{G}$-background fields on N. It follows that we should study $\mathsf{G} /\!\!/ \mathsf{G}$ and $\mathsf{G} /\!\!/^h \mathsf{G}$. If we let G/G be the space of bona-fide G-orbits, we get a map $\mathsf{G} /\!\!/ \mathsf{G} \to \mathsf{G}/\mathsf{G}$.

Recall that every element of G is conjugate to an element of T, so we can write $\mathsf{G}/\mathsf{G} = \mathsf{T}/W$, where $W = \mathsf{N}_\mathsf{G}(\mathsf{T})/\mathsf{T}$. For $\mathfrak{h} = \mathrm{Lie}(\mathsf{T})$, we have an exponential map $\exp : \mathfrak{h}/W \to \mathsf{T}/W$, where $\mathfrak{h}/W$ is equivalent to a Weyl chamber. We can interpret $\mathsf{G} /\!\!/ \mathsf{G}$ or $(\mathsf{G} /\!\!/^h \mathsf{G})$ as stacks / spaces over $\mathfrak{h}/W$. In the open parts of the Weyl chambers, the fibers look like $(\mathsf{G}/\mathsf{T}) /\!\!/ \mathsf{G}$. Along the walls of the Weyl chambers, we get jumps / defects where the fibers become $(\mathsf{G}/\mathsf{G}_i) /\!\!/ \mathsf{G}$.

We can also interpret the spaces L_I. Given I, inscribe S^1 in $\mathfrak{h}/W$ so that the circle hits the wall $\alpha_{i_j} = 0$ between the points x_j and x_{j+1}. Then L_I consists of sections

$$\begin{tikzcd} & \mathsf{G} /\!\!/ \mathsf{G} \arrow[d] \\ \mathsf{S}^1 \arrow[r] \arrow[ur, dashed] & \mathfrak{h}/W \end{tikzcd}$$

2.4.4 Homotopy TQFT interpretation

We may view L_I as $\mathsf{G} /\!\!/ \mathsf{G}$-background fields on S^1, or equivalently as G-background fields on $\mathsf{S}^1 \times \mathsf{S}^1$. Let $\mathsf{Cob}_n(\mathrm{pt} /\!\!/^h \mathsf{G})$ be the (∞, n)-category of manifolds with G-background. Let $\mathcal{H}$ be a commutative "global" cohomology theory – see the book of Stefan Schwede for details of what this means. Write $\mathcal{C}_n$ for the category of $\mathbb{E}_{n-1}$-algebras in $\mathcal{H}$-modules.

There is a functor $\mathsf{Cob}_n(\mathrm{pt} /\!\!/^h \mathsf{G}) \to \mathcal{C}_n$ given by $\mathrm{pt} /\!\!/ \mathsf{G} \mapsto \mathcal{H}$. Compactifying this along S^1 (i.e. considering values on manifolds of the form $\mathsf{M} = \mathsf{N} \times \mathsf{S}^1$) gives a functor $\mathsf{Cob}_{n-1}(\mathsf{G} /\!\!/^h \mathsf{G}) \to \mathcal{C}_{n-1}$, defined by $\mathsf{G} /\!\!/^h \mathsf{G} \mapsto (\mathcal{H} \times \mathsf{G}) /\!\!/ \mathsf{G}$. On S^1 with L_I background, we get $(\mathsf{L}_I \times \mathcal{H} \to \mathsf{L}_I) /\!\!/ \mathsf{G}$. This is an invertible theory, and we can gauge / take G-equivariant global sections to get

$$\mathcal{H}_{\mathsf{G}}(\mathsf{L}_I) \simeq \mathsf{HH}(\mathrm{bS}_I).$$

The inclusions $\mathsf{L}_J \hookrightarrow \mathsf{L}_I$ give Khovanov's complex.

Twisted theories give modules over $\mathcal{H}$, which doesn't quite make sense in the above story. Instead, we want to view our theory as being the boundary of a 4d theory. That is, Khovanov-Rozansky $\mathfrak{sl}_N$-link homology comes from a 3d theory on the boundary of a 4d theory.

Chapter 3

Ibou Bah – Lagrangian Methods for Categorical Symmetries

3.1 6/11 – What is QFT?

This will be a physics-y, intuitive series of lectures. We will start by giving some idea of QFT in general. In the second lecture, we will move on to finite symmetry groups and use these to construct topological operators. In the third lecture, we will discuss the action of finite symmetry groups on fields (with an emphasis on the example of $SU(N)$ Yang-Mills theory). Finally, we will discuss the general construction of topological operators using Lagrangians.

3.1.1 What should a QFT do?

We don't actually know what a QFT *is*. However, we have an excellent physical idea of what a QFT should *do*. QFTs should have:

- A list of operators: point operators $\{\mathcal{O}_i(x)\}$, line operators $\{U_\alpha(\gamma_p)\}$, and other operators.
- A Hilbert space $\mathcal{H}$ of states on spacetime $M = \mathbb{R}_t \times Y$.
- A partition function sending combinations / products of operators to complex numbers, e.g. $\langle \mathcal{O}_i \mathcal{O}_j \rangle$.

These should satisfy three key properties: *unitarity*, *locality*, and *causality*. Let's discuss each of these in more detail.

3.1.2 Unitarity, locality, and causality

Unitarity can be interpreted as "preservation of probability." For example, for many QFTs, we can construct an "S-matrix" governing scattering amplitudes of particles, and unitarity forces $S^\dagger S = 1$. This specific interpretation doesn't work for CFTs / TFTs. Instead, we use a notion of *reflection positivity.*

To explain this, recall that time evolution in quantum mechanics is governed by

$$|\psi\rangle \mapsto e^{-iHt} |\psi(t)\rangle$$

and

$$\mathcal{O}(x, 0) |\psi\rangle \mapsto \mathcal{O}(x, t) |\psi\rangle .$$

We should have

$$\langle\psi| \mathcal{O}^\dagger \mathcal{O} |\psi\rangle = ||\mathcal{O}(x, t) |\psi\rangle|| \geqslant 0.$$

To make sense of $\mathcal{O}^\dagger$, we allow our time variable to take complex values. Say that $\mathcal{O}(x, t+i\tau)^\dagger = \mathcal{O}(x, t-i\tau)$. Then the above condition forces $\langle \mathcal{O}(x, t-i\tau)\mathcal{O}(x, t+i\tau)\rangle \geqslant 0$. Taking $t = 0$, we see that reflection positivity requires $\langle \mathcal{O}(x, -i\tau)\mathcal{O}(x, i\tau)\rangle \geqslant 0$. More generally, reflection positivity forces

$$\langle \mathcal{O}_i(x, -i\tau)\mathcal{O}_j(x, -i\tau) \dots \mathcal{O}_j(x, i\tau)\mathcal{O}_i(x, i\tau)\rangle \geqslant 0.$$

Locality forces point operators to have an *operator product expansion*

$$\mathcal{O}_i(x_1)\mathcal{O}_j(x_2) = \sum_k c_{ij}^k(x_1, x_2)\mathcal{O}(x_1)$$

governing the behavior as $x_1 \to x_2$. (For conformal field theories, the c_{ij}^k should be meromorphic.) More generally, the interaction of operators should depend only on their behavior in a common tubular neighborhood. If two point operators are outside the same light cone, then we can find a reference frame where the two operators take place at the same time but at different points in space.

Finally, causality requires that any operators which are spacelike separated commute with each other. If two point operators are in the interior of the same light cone, then we can find a reference frame where the operators take place at the same point in space but at different points in time. Things are more complicated for extended operators.

3.1.3 Extra features

In many QFTs, we can expect other nice properties to hold:[1]

- Cluster decomposition:

$$\lim_{b\to\infty}\langle\mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\ldots\mathcal{O}_{i'}(x_1+b)\mathcal{O}_{j'}(x_2+b)\ldots\rangle = \langle\mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\ldots\rangle\langle\mathcal{O}_{i'}(x_1+b)\mathcal{O}_{j'}(x_2+b)\ldots\rangle.$$

- Renormalization: high energy / short distance / UV physics and low energy / long distance / IR physics decouple, at least up to finitely many parameters.

Axiomatic treatments can take these as additional axioms.

Gravity fails both of these conditions and locality. Black holes are UV phenomena which are still visible at low energy. Furthermore, black holes also break cluster decomposition. Whatever quantum gravity is, unitarity "must" be retained, since it's fundamental to quantum physics.

The goal of the Simons collaboration is to understand the kinematics of operators in QFT. This is governed by symmetries! We can essentially think of this as understanding the *labels* of operators $\mathcal{O}_i(x)$, $U_\alpha(\gamma)$, etc.

3.1.4 Where do QFTs come from?

There are many sources of QFTs.

Example 3.1.1 (Lattices)**.** Consider a lattice at which each point behaves according to quantum mechanics. Taking a continuum limit of the lattice gives a quantum field theory. This limit leads to the appearance of infinities and other strange phenomena. Nevertheless, it's the most physically "rigorous" way of constructing QFTs.

Example 3.1.2 (Geometric engineering)**.** Start with a string theory, and "decouple gravity" by freezing some dimensions. This produces a QFT involving geometric objects. Class $\mathcal{S}$ theories arise in this way.

Example 3.1.3 (Lagrangians)**.** We can construct many QFTs from Lagrangians. We'll spend most of our time focusing on this perspective.

Each perspective has advantages over the others:

- Lattices make renormalization straightforward.
- Geometric engineering makes the geometric nature of objects more apparent.
- Lagrangians make locality explicit.

[1] There's also a last property of "analyticity" which is hard to formulate.

3.1.5 Lagrangians

A Lagrangian is a functional depending on the fields $\phi_\alpha(x)$:

$$L(x) = L(\phi_\alpha(x), \partial\phi_\alpha(x), \dots)$$

This is a classical object. Locality enforces that L only depends on the behavior of the fields at the given input point (unlike, say, Green's functions $G(x_1, x_2)$). Locality also ensures that there are only finitely many derivative terms appearing in L.

Lorentz symmetry (which we will assume implicitly) tells us that our fields can be:

- Scalar fields $\phi(x)$
- Vector fields / gauge fields $A_\mu(x)$, either:
  - Massless: living in the representation $(1, -1)$, or
  - Massive: living in $(1, 0, 1)$
- Fermions ψ_a (representations of a Clifford algebra)
- Metrics $g_{\mu\nu}$

We won't consider gravitational theories, so we'll ignore $g_{\mu\nu}$.

3.2 6/11 – Lagrangians in QFT

Unitarity is true for closed quantum systems. For open quantum systems, where we "trace away" some information, we can drop this assumption. We must restore unitarity as we restore data to our system. This explains e.g. where the W-boson and Higgs boson come from: they're needed to restore unitarity in the Standard Model.

3.2.1 Lagrangians

Consider a Lagrangian

$$\mathcal{L}(x) = \mathcal{L}(\phi_\alpha(x), \partial\phi_\alpha(x), \dots).$$

From this (classical) data, we can define a (quantum) partition function

$$Z = \int D[\phi_\alpha] e^{i\int_M \sqrt{g}\mathcal{L}(\phi_\alpha)}$$

This can be thought of as a formal expression: heuristically, it is a sum over all possible field configurations.

We can obtain correlation functions, etc. by inserting local operators

$$\mathcal{O}_i(\phi_\alpha(x), \partial\phi_\alpha(x), \dots)$$

or extended operators

$$U_a(\gamma) = U\left(\int_\gamma F(\phi_\alpha(x)), \dots\right).$$

Namely, we declare

$$\langle \mathcal{O}_i \dots \rangle = \frac{1}{Z(0)} \int D[\phi_\alpha] e^{i\int \mathcal{L}} \mathcal{O}_i \dots.$$

3.2.2 Parameters

In general, it is possible that our Lagrangian depends on some additional parameters, say τ_a, τ_b. The partition function then also depends on the τ's. We will assume that these parameters are *perturbative*: that small changes in our theory are governed by Taylor expansions in terms of the τ's. Often, there is a large parameter space for a given QFT, and there are many perturbative regions within this space corresponding to different weakly coupled versions of the theory. These *duality frames* can have different fields, different Lagrangians, and different gauge groups. (Gauge symmetry is a redundancy in parametrization – it is not a physical symmetry, but rather a "trick" enabling the use of local Lagrangians.)

However, the global symmetry, Hilbert space, and local operators are the same in every duality frame.[2] This may not always be visible (at least at the classical level) but it is true. Global symmetry permutes the operators, while gauge symmetry does nothing to the operators.

Example 3.2.1. We can view 4d $\mathcal{N} = 4$ super-Yang-Mills as part of the same parameter space as type IIB supergravity on $\mathrm{AdS}_5 \times S^5$. Thus these are versions of the same theory.

In addition to varying our parameters, we can also turn on background fields $B_i(x)$, i.e. add in new fields which do not fluctuate.

3.2.3 Examples

Consider a theory with one free massless scalar field ϕ. The Lagrangian is

$$\mathcal{L} = \frac{1}{e^2} d\phi \wedge \star d\phi,$$

a single kinetic term. The partition function is

$$Z = \int D[\phi] \exp\left(\frac{i}{e^2}\int d\phi \wedge \star d\phi\right).$$

Taking the coupling parameter $e \to 0$ yields $d\phi = 0$.

We can also add matter, incorporated via a term $m^2\phi^2\,\mathrm{Vol}(M)$ added to the Lagrangian. Taking $m \to \infty$ gives $\phi = 0$.

The stress tensor of the theory is

$$T_{\mu\nu} = \frac{\delta \mathcal{L}}{\delta g^{\mu\nu}}.$$

In our above setup, there are kinetic terms, so $T_{\mu\nu} \neq 0$ and the theory is not topological. However, the theory does become topological in certain limits.

Suppose we add a gauge field A_μ transforming (under local gauge transformations) according to $A_\mu \mapsto A_\mu + d\lambda$. Taking ϕ to be a complex scalar, setting $D\phi = d\phi + iA\phi$, and letting $F = dA$, we can define a Lagrangian

$$\mathcal{L} = D\phi \wedge \star D\phi + \frac{1}{e^2} F \wedge \star F.$$

This produces a non-topological gauge theory.

We can produce a topological theory by adding topological coupling terms: $\theta F \wedge F$ or $A \wedge d\phi \wedge F$. These terms would have coefficients determined by the gauge theory. Alternatively, we could add boundary terms.

3.2.4 More remarks on Lagrangians

The constraints depend on the choice of gauge group. For example, having gauge group $\mathbb{R}$ would allow for only local transformations (corresponding to exact 1-forms $d\lambda$). Gauge group U(1) would allow more interesting global transformations (e.g. $\omega \in H^1(-;\mathbb{Z})$ allowing $A_\mu \mapsto A_\mu + \omega$).

We can split a typical Lagrangian into a *kinetic term*, an *interaction term*, and a *topological term*. The kinetic term reflects the behavior of free fields. The interaction term consists of a polynomial in the ϕ_α, and the signs of the coefficients are constrained by locality and causality. The topological term is (broadly) independent of the metric.

[2] The term "global" is not related to this, however.

3.3 6/12 – Introduction to BF Theory

We start by clarifying something from last time.

3.3.1 Jumping global symmetry

The global symmetry can jump as we move throughout the moduli space of theories.

Example 3.3.1. Suppose we have a theory with two scalar fields ϕ_1 and ϕ_2 and Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_i)^2 + m_1^2\phi_1^2 + m_2^2\phi_2^2.$$

When $m_1 = m_2$, the symmetry of this theory increases to SO(2). This leads to new and interesting physics.

3.3.2 Definition of BF Theory

BF theory is the TFT with fields b (a p-form) and c (a codimension $p+1$ form) and action $S = N\int_M b \wedge dc$ (for some fixed N). The partition function is then

$$Z = \int D[b]D[c]\exp\left(-2\pi i N \int_M b \wedge dc\right)$$

For specificity, we will take $d = 5$, letting b and c be 2-forms. However, our discussion will make sense in any dimension.

We can allow different classes of gauge symmetries:

- $\mathbb{R}$-gauge: $b \mapsto b + d\lambda_1$
- U(1) gauge: $b \mapsto b + \omega$ for $\omega \in H^2(M;\mathbb{Z})$
- Trivial gauge: $b \mapsto b + \omega$ for $\omega \in H^2(M;\mathbb{R})$.

We may consider independent transformations for c.

BF theory has observables (where γ' has codimension $p+1$ and γ has dimension p):

$$W_c(\gamma) = \exp\left(2\pi i \oint_\gamma c\right)$$
$$W_b(\gamma') = \exp\left(2\pi i \oint_{\gamma'} b\right).$$

These operators have different behaviors with respect to different gauge symmetries:

- $\mathbb{R}$-gauge: operators are gauge invariant for all b (or c).
- U(1) gauge: b (or c) must have holonomy in U(1).
- Trivial gauge: none of these observables are gauge invariant.

We will focus on the case of U(1) gauge symmetry.

Note that BF theory doesn't have any "iterated gauge symmetries" or "Stückelberg fields." Thus we do not need to worry about some of the more complicated aspects of gauge theories.

3.3.3 Linking numbers from correlation functions

We'd like to compute

$$\begin{aligned}\langle W_c(\gamma)W_b(\gamma)\rangle &= \int D[b]D[c]\exp\left(-S + 2\pi i\int_\gamma c + 2\pi i\int_\gamma b\right)\\ &= \int D[b]D[c]\exp\left(-iN2\pi(b\wedge dc + b\wedge dc_0) + 2\pi i\int_\gamma (c+c_0) + 2\pi i\int_{\gamma'} b\right)\\ &= \int D[b]D[c]\exp\left(-iN2\pi b\wedge dc + 2\pi i\int_\gamma c\right)\exp\left(\frac{2\pi i}{N}\int_\gamma c_0\right)\\ &= Z\exp\left(\frac{2\pi i}{N}L(\gamma,\gamma')\right)\end{aligned}$$

where $L(\gamma,\gamma')$ is the linking number of γ and γ'.

Under $\gamma\mapsto\gamma+N\beta$, we have $W_c(\gamma)\sim W_c(\gamma+N\beta)$, and thus $W_c^N(\gamma)=1$. The analogous statement holds for $W_b(\gamma')$. The upshot is that the theory actually has a $\mathbb{Z}_N\times\mathbb{Z}_N$ gauge symmetry, coming from the maps

$$H^1(-;\mathbb{Z}_N)\longrightarrow H^1(-;U(1))\longrightarrow H^1(-;U(1))$$

where the last map is induced by $a\mapsto a^N$.

We can write

$$W_c(\gamma)W_b(\gamma') = \exp\left(\frac{2\pi i}{N}L(\gamma,\gamma')\right)W_b(\gamma')W_c(\gamma).$$

This can be read as saying that the operators $W_c(\gamma)$ are topological operators generating a $\mathbb{Z}_N$ global symmetry, where the charged objects are the $W_b(\gamma')$. The corresponding statement with the roles of b and c reversed is also true.

3.3.4 Quantizing BF theory

Classically, BF theory seems trivial: the equations of motion for b give $Ndc=0$, and the equations of motion for c give $Ndb=0$. Thus classical solutions are just flat connections. Interesting behavior only appears at the quantum level.

Canonical quantization for b and c gives

$$[b(x),c(y)] = \frac{2\pi i}{N}\delta(x-y)\,\mathrm{Vol}(M).$$

We can write

$$e^{2\pi i\int_\gamma b}e^{2\pi i\int_{\gamma'} c} = e^{\left[2\pi i\int_\gamma b,2\pi i\int_{\gamma'} c\right]}e^{2\pi i\int_{\gamma'} c}e^{2\pi i\int_\gamma b}$$

Combining this with the above construction of commutators recovers our previous results.

3.3.5 Cheeger-Simons maps for more general gauge theories

In general, consider a p-form A with gauge symmetry $A\mapsto A+\omega$ for $\omega\in H^p(M;\mathbb{Z})$. We have a natural operator given by the Cheeger-Simons map

$$\chi(\Sigma_p) = \exp\left(2\pi i\int_{\Sigma_p} A\right).$$

Field configurations for which χ is gauge-invariant are classified by the differential cohomology $\check{H}^{p+2}(M)$, which fits into the hexagon

$$\begin{tikzcd}
& & & & 0 \\
& \Omega^p/\Omega^p_{\mathbb{Z}} \arrow[rr, "d"] \arrow[dr] & & \Omega^{p+1} \arrow[ur] \arrow[dr] & \\
H^p(M;\mathbb{R}) \arrow[ur] \arrow[dr] & & \check{H} \arrow[ur] \arrow[dr] & & H^{p+1}(M;\mathbb{R}) \\
& H^p(M;\mathbb{R}/\mathbb{Z}) \arrow[rr, "\beta"] \arrow[ur] & & H^{p+1}(M;\mathbb{Z}) \arrow[ur] \arrow[dr] & \\
0 \arrow[ur] & & & & 0
\end{tikzcd}$$

This is a lot larger than what one would naïvely expect. Things are much simpler in BF theory where we only consider flat connections. One can see this written up in a paper of Freed, Moore, and Segal.

3.4 6/12 – Global symmetries in variants of BF theory

3.4.1 Simultaneous BF theories

Let's see what happens when we combine two versions of BF theory at once. Consider the 5d theory with fields b_2, c_2, A_1, and c_3 and action

$$S = 2\pi i \int N b_2 \wedge dc_1 + 2NA_1 \wedge dc_3.$$

The first term gives a $\mathbb{Z}_N^{(1)} \times \mathbb{Z}_N^{(1)}$ gauge symmetry, and the second term gives a $\mathbb{Z}_{2N}^{(0)} \times \mathbb{Z}_{2N}^{(2)}$ gauge symmetry. From the operators W_c and W_b corresponding to the first term, we get $\mathbb{Z}_N^{(2)}$ and $\mathbb{Z}_N^{(2)}$ global symmetries. From the (similarly defined) operators V_A and V_{c_3}, we get $\mathbb{Z}_N^{(3)}$ and $\mathbb{Z}_N^{(1)}$ global symmetries.

3.4.2 Interacting terms

Can we get non-invertible global symmetries from something like this Yes: add in non-dynamical fields ϕ and β, and add an interaction term to the action:

$$S = 2\pi i \int N b_2 \wedge dc_1 + 2NA_1 \wedge dc_3 + \frac{1}{2N}(d\phi - 2NA) \wedge (d\beta - Nb)^2.$$

This forces the gauge symmetry groups to interact with each other.

The new equations of motion include

$$2N dc_3 - (d\beta - Nb)^2 = 0.$$

This equation implies that our original V_{c_3} is no longer a topological operator. We seek to correct this somehow. If Σ_3 is a boundary, say $\Sigma_3 = \partial Y$, then we can require

$$V_{c_3}(\Sigma_3) = \exp\left(\frac{1}{2} N \int_Y b^2\right).$$

In general, we can define a topological operator

$$V_{c_3}(\Sigma_3) = \int d[a] \exp\left(2\pi i \int_{\Sigma_3} c_3 + 2\pi i \int_{\Sigma_3} Na \wedge da + a \wedge b\right)$$

Essentially, we have an anomaly on the bulk (breaking the $\mathbb{Z}_N^{(2)}$ symmetry), and we define a new TQFT on the defect to cancel this out. This is "stacking." There are other theories we could use to cancel the anomaly, but the above method is "minimal."

3.4.3 Fusion rules

We can compute some fusion rules of this new V operator:

$$\begin{aligned} V(\Sigma_3) \otimes V^\dagger(\Sigma_3) &= \int d[a]d[a'] \exp\left(2\pi i \int_{\Sigma_3} N(a da - a' da') + (a - a') \wedge b\right) \\ &= \int d[\alpha]d[\alpha'] \exp\left(2\pi i N \int_{\Sigma_3} \alpha' \wedge d\alpha + \alpha d\alpha + \alpha \wedge b\right) \end{aligned}$$

or something similar. Ultimately, this localizes to

$$V \otimes V^\dagger = A(\Sigma_3) \sum_{b \in H^2(\Sigma_3; \mathbb{Z}_N)} e^{2\pi i \int_{\Sigma_3} b}.$$

for some TQFT coefficient $A(\Sigma_3)$ (depending only on Σ_3, not on the bulk). The sum here is a "condensation defect." Working this out precisely is left to the reader as an exercise (though if anyone reading this has all the details, feel free to add them).

The upshot is that the Lagrangian approach allows us to understand much of what's going on with categorical symmetries. We can even get the TQFT coefficients out of this. Adding more interaction terms can give even more complicated fusion rules (which are still accessible via the Lagrangian presentation).

3.4.4 Super Yang-Mills

In general, we can consider the action of a 5d symmetry TFT on 4d SU(N) or PSU(N) super-Yang-Mills. This 4d theory has interesting topological operators: Wilson lines and 't Hooft lines. Fully understanding the bulk TFT allows us to work with the symmetries of the boundary theory.