

# Notes from GCS 2024 Summer School

Notes by John S. Nolan

June 10-14, 2024

### **Abstract**

These are my notes from the “Categorical Symmetries in Quantum Field Theory” summer school, held at the University of Edinburgh from June 10-14, 2024. Lectures were given by Graeme Segal, Nitu Kitchloo, Ibou Bah, Claudia Scheimbauer, and Ryan Thorngren. All errors are my own.

# Contents

<b>1</b>	<b>Graeme Segal – A Perspective on Quantum Field Theory</b>	<b>2</b>
1.1	6/10 – Overview of QFT . . . . .	2
1.1.1	From classical to quantum . . . . .	2
1.1.2	Approaching quantum field theory . . . . .	3
1.1.3	The perspective of Connes . . . . .	3
1.1.4	Algebra from functorial QFT . . . . .	4
1.2	6/10 – Positive Energy . . . . .	4
1.2.1	Warmup . . . . .	4
1.2.2	Historical digression . . . . .	4
1.2.3	Complex metrics . . . . .	4
1.2.4	Application to QFT . . . . .	5
1.3	6/11 – Desired Properties of QFT and Scaling . . . . .	5
1.3.1	Renormalization . . . . .	5
1.3.2	Unitarity . . . . .	6
1.3.3	Locality . . . . .	6
1.3.4	Cluster decomposition . . . . .	6
1.3.5	Causality . . . . .	6
1.3.6	Scaling . . . . .	6
<b>2</b>	<b>Nitu Kitchloo – Symmetry Breaking, Surface Defects, and Link Homologies</b>	<b>8</b>
2.1	6/10 – Overview . . . . .	8
2.1.1	Artin braid groups . . . . .	8
2.1.2	Categorification . . . . .	9
2.1.3	Preliminaries: $U(\mathfrak{n})$ . . . . .	9
2.2	6/10 – Geometric interpretations for the $U(\mathfrak{n})$ case . . . . .	10
2.2.1	Categorification of $[Br_n]$ . . . . .	10
2.2.2	Topological interpretation of Bott-Samelson bimodules . . . . .	11
2.2.3	Interpretation of Rouquier’s chain complex . . . . .	11
2.2.4	Interpretation of $HH(bS_I)$ . . . . .	12
2.3	6/11 – Local systems and twisted cohomology . . . . .	12
2.3.1	Review of bundle theory . . . . .	12
2.3.2	Recap . . . . .	13
2.3.3	Recovering $\mathfrak{sl}_N$ -link homology . . . . .	13
2.3.4	Local systems . . . . .	14
<b>3</b>	<b>Ibou Bah – Lagrangian Methods for Categorical Symmetries</b>	<b>15</b>
3.1	6/11 – What is QFT? . . . . .	15
3.1.1	What should a QFT do? . . . . .	15
3.1.2	Unitarity, locality, and causality . . . . .	15
3.1.3	Extra features . . . . .	16
3.1.4	Where do QFTs come from? . . . . .	16
3.1.5	Lagrangians . . . . .	17

3.2	6/11 – Lagrangians in QFT . . . . .	17
3.2.1	Lagrangians . . . . .	17
3.2.2	Parameters . . . . .	18
3.2.3	Examples . . . . .	18
3.2.4	More remarks on Lagrangians . . . . .	18

# Chapter 1

## Graeme Segal – A Perspective on Quantum Field Theory

### 1.1 6/10 – Overview of QFT

#### 1.1.1 From classical to quantum

In classical mechanics, one focuses one's attention on a fixed space  $M_0$  and obtains a configuration space  $X$  from that. Dynamics are specified by giving a Lagrangian  $L : TX \rightarrow \mathbb{R}$ , which we assume is positive, inhomogeneous, and quadratic on tangent vectors. Under these hypotheses, the Lagrangian can be viewed as a generalized Riemannian metric, so that time evolution is given by a (generalized) “geodesic flow.” This gives rise to a symplectic form  $\omega$  and Poisson bracket  $\{-, -\}$  on  $T^*X$ . Furthermore, from the Lagrangian  $L$ , we may extract a Hamiltonian  $H : T^*X \rightarrow \mathbb{R}$  such that

$$\frac{d}{dt}f = \{H, f\} \quad \forall f \in \mathcal{C}^\infty(X).$$

Thus we may view a classical dynamical system as a triple  $(Y, \omega, H)$  with  $(Y, \omega)$  a symplectic manifold and  $H : Y \rightarrow \mathbb{R}$  a function.

Quantum mechanics can be viewed as a complex noncommutative analogue of this. We replace  $(Y, \omega, H)$  by a triple  $(\mathcal{A}, \star, H)$  with  $\mathcal{A}$  a (topological)  $\mathbb{C}$ -algebra,  $\star$  an antilinear involution on  $\mathcal{A}$ , and  $H \in \mathcal{A}$ . For an observable  $f \in \mathcal{A}$ , we require

$$\frac{d}{dt}f = i\hbar[H, f].$$

If the quantum system  $(\mathcal{A}, \star, H)$  is a “quantization” of a classical system  $(Y, \omega, H)$ , then there is a deep relationship between the two. For example, we should view  $\mathcal{A}$  as a completion / extension of  $\mathcal{C}^\infty(Y)$ , and we can relate the values of an observable to the eigenvalues of the corresponding operator.

The above picture privileges the time dimension, so it is fundamentally non-relativistic. Working relativistically requires us to move from finite-dimensional spaces to infinite-dimensional spaces. Heuristically, we take  $Y$  to be our space of fields on spacetime (typically  $M = M_0 \times \mathbb{R}$ ), and we fix a Lagrangian  $L : Y \rightarrow \mathbb{R}$ . However,  $\mathcal{A}$  should now be viewed as a space of functions on an “almost finite-dimensional manifold.” This requires many corrections!

**Example 1.1.1.** Consider scalar field theory, with  $M = M_0 \times \mathbb{R}$  and  $Y = \mathcal{C}^\infty(M)$ . Then  $\mathcal{A}$  looks like functions on a stratified configuration space

$$\coprod_{n \geq 0} \text{Conf}_n(M_0),$$

where  $\text{Conf}_n(M_0)$  is the space of sets of  $n$  distinct unordered points on  $M_0$ .

### 1.1.2 Approaching quantum field theory

Heuristically, we should “spread things out over spacetime” by attaching an algebra  $\mathcal{O}_x$  to every point  $x \in M$  and letting

$$\mathcal{A} \approx \bigotimes_{x \in M} \mathcal{O}_x.$$

This has been formalized in the approach of algebraic quantum field theory – see e.g. the definition of Haag, which assigns an algebra  $\mathcal{A}_U$  to each open set  $U \subset M$ .

An alternative approach, which we will pursue, privileges the time dimension. In this approach, we define a  $d$ -dimensional quantum field theory as a symmetric monoidal functor

$$E : \mathbf{Cob}_{d-1} \rightarrow \mathbf{Vect}.$$

Here  $\mathbf{Cob}_{d-1}$  is the symmetric monoidal category with:

- Objects:  $(d-1)$ -manifolds, often assumed compact or compact with boundary (and thought of as time slices)
- Morphisms  $M_0 \rightarrow M_1$ :  $d$ -dimensional cobordisms from  $M_0$  to  $M_1$  (thought of as controlling time evolution)
- Monoidal structure: disjoint union of manifolds / cobordisms

and  $\mathbf{Vect}$  is the symmetric monoidal category of (possibly infinite-dimensional) topological vector spaces (with  $\otimes$  as tensor product).

There are many other ways to view quantum field theory. For example, one could think of QFT as a continuum limit of lattice models.

QFT is traditionally seen as describing “the world except for gravity.” That is, we think of spacetime as fixed, without allowing for the effects of gravity. However, QFT can tell us about what sorts of gravitational effects are possible.

The perspective on QFT that we will follow has the advantage that it allows for direct comparisons between QFTs. Thus, in principle, we may consider a “moduli space” of QFTs (of a given type). We can think of gravitational effects as acting upon this moduli space.

### 1.1.3 The perspective of Connes

We’d like to use geometry to understand the noncommutative world of QFT. This can be accessed by studying the spectrum of our Hamiltonian  $H$ . Operators which evolve slowly in time “nearly commute” with  $H$  and thus must be “nearly diagonal.” We’re mostly interested in studying such operators (as the time scale of humans is much slower than that of the universe). Thus, when studying QFT, we’re interested in algebras which are “nearly commutative.”

Connes was interested in finding a mildly noncommutative generalization of algebras of functions on a manifold  $M$ . Recall that we can define the Clifford algebra  $\text{Cliff}(T_m M)$  as the algebra with generators  $\{\gamma_\xi\}_{\xi \in T_m M}$  and relations  $\gamma_\xi^2 = -\|\xi\|^2$ . Let  $\mathcal{B}$  be a bundle of finite-dimensional algebras on  $M$  containing  $\text{Cliff}(T_m M)$ , and take a connection on  $\mathcal{B}$ . Let  $\mathcal{D}$  be the Dirac operator

$$\mathcal{D} = \sum_i \gamma_{\xi_i} \otimes \frac{\partial}{\partial x_i},$$

and let  $H = \mathcal{D}^2$ . Connes gave a formula for  $\text{tr } \mathcal{D}^2$  and extracted interesting physical objects, e.g. the Higgs field, from  $\mathcal{D}$ . From Connes’ perspective, the Dirac operator is “as good as” the metric.

### 1.1.4 Algebra from functorial QFT

Suppose we view spacetime  $M$  as a cobordism from  $M_0$  to  $M_1$ . Fix a  $d$ -dimensional quantum field theory  $E$ , where  $E = \dim M$ . For  $x \in M$ , we can define a vector space of operators  $\mathcal{O}_x$  by taking a small disk  $D_x$  about  $x$  and letting  $\mathcal{O}_x = E_{\partial D_x}$ . We can view a punctured copy of  $M$  as giving a cobordism

$$M \sqcup (\sqcup_i \partial D_{x_i}) \rightarrow M_1,$$

so that applying  $E$  gives  $E_{M_0} \otimes (\otimes_i \mathcal{O}_{x_i}) \rightarrow E_{M_1}$ . This gives a sort of “higher multiplication” on the spaces of operators  $\mathcal{O}_{x_i}$ .

We can make this much more precise in special cases. For example, if  $E$  is topological, then  $\mathcal{O}_{x_i} = \mathcal{O}$  is independent of  $x_i$ . Taking  $M_0 = \emptyset$  and  $M_1 = S^{d-1}$ , we obtain a family of multiplications  $\mathcal{O}^{\otimes n} \rightarrow \mathcal{O}$ .

**Example 1.1.2.** If  $d = 2$ , the cap / cup / pair-of-pants cobordisms equip  $A = \mathcal{O}$  with the structure of a finite-dimensional commutative Frobenius algebra. This means that  $A$  is a unital algebra with a nondegenerate trace  $\theta : A \rightarrow \mathbb{C}$ .

It is interesting to extend this picture to non-topological 2-dimensional QFTs. Here, one can set up a moduli space of QFTs and obtain a gravitational flow. This was a major historical motivation for string theory.

The second lecture will discuss the importance of positive energy to this theory. The third lecture will discuss scaling, and the fourth lecture will focus on finding a suitable definition for **Cob**.

## 1.2 6/10 – Positive Energy

### 1.2.1 Warmup

Consider a quantum mechanical system with Hilbert space  $\mathcal{H}$ . The Hamiltonian is a self-adjoint operator  $H : \mathcal{H} \rightarrow \mathcal{H}$ , and we can integrate  $iH$  to get unitary operators  $U_t = e^{iHt} : \mathcal{H} \rightarrow \mathcal{H}$  describing the time evolution of the system.

Assuming  $H \geq 0$  has strong consequences for the theory. In this case,  $U_- : \mathbb{R} \rightarrow \text{End}(\mathcal{H})$  is the boundary value of the function on the complex upper half-plane also given by  $t \mapsto e^{iHt}$ .

To fit this into QFT, consider  $\mathbb{C}$  as a bundle over  $\mathbb{R}$ , where the inner product on the base is Lorentzian and the inner product on the fiber is Riemannian. Think of  $\mathbb{R}$  as our spacetime  $M$  and the fiber as  $V = T_x M$ .

We will introduce the notion of a *complex metric* on a real vector space  $V$ . This is a quadratic form  $V \rightarrow \mathbb{C}$  satisfying a certain condition to be determined later.

### 1.2.2 Historical digression

The notion of a complex metric was introduced by Kontsevich and Segal. Segal was originally interested in loop groups  $\mathcal{L}G = \text{Map}(S^1, G)$ . There is a nice class of “positive energy” representations of loop groups. These extend to representations of  $\text{Diff}^+(S^1) \ltimes \mathcal{L}G$  which are also “positive energy” in a certain sense. This notion effectively means that the actions respect the grading  $\mathcal{H} = \oplus_{k \geq 0} \mathcal{H}_k$  coming from an  $S^1$ -action.

An even simpler illustrative case is that of “discrete series” representations of  $\text{PSL}_2(\mathbb{R})$ . Embed  $\text{PSL}_2(\mathbb{R}) \hookrightarrow \text{PSL}_2(\mathbb{C})$ . Let  $\text{PSL}_2^<(\mathbb{C})$  be the subsemigroup of Möbius transformations sending the unit disk  $D$  to a proper subdisk of itself. Then  $\text{PSL}_2^<(\mathbb{C})$  acts by contraction on the discrete series representations.

There is a similar story for loop groups, if we consider the semigroup of holomorphic  $f : D \rightarrow D$  with  $f(D) \subset \overset{\circ}{D}$ . The Kontsevich-Segal definition connects to this somehow (I didn’t quite catch how).

### 1.2.3 Complex metrics

**Definition 1.2.1.** A *complex metric* on a real vector space  $V$  is a quadratic form  $g : V \rightarrow \mathbb{C}$  such that, if  $\lambda_k$  are the eigenvalues of the matrix corresponding to  $g$  and  $\lambda_k = e^{i\theta_k} |\lambda_k|$ , then  $\sum_i |\theta_i| < \pi$ .

We can equivalently state this as

$$-\pi \leq \sum_i \pm \theta_i \leq \pi.$$

One may view the  $\theta_i$  as weights of the corresponding spin representation.

For  $v \in V$ , we have  $g(v) \in \mathbb{C} \setminus (-\infty, 0)$ , so  $g(v)$  has a canonical square root. Thus we may assign a “complex length” in the positive half-plane to every  $v \in V$ .

Given an inner product  $V \times V \rightarrow \mathbb{C}$ , we get an inner product on  $\wedge^k V$  with squared norm given by  $\alpha \wedge (\star \alpha)$ . Really, this is better thought of as a map  $\wedge^k V \times \wedge^k V \rightarrow \wedge^d V$ , for  $d = \dim V$ .

We’ll change topics now.

### 1.2.4 Application to QFT

If we want to view a QFT as a functor  $E$  defined on a category of cobordisms, we should require:

- Cobordisms should be equipped with complex metrics.
- The induced maps  $U_M : E_{M_0} \rightarrow E_{M_1}$  should be holomorphic in  $M$  (viewed as living in some complex analytic moduli space of complex metrics).
- The operators  $U_M$  should be *trace class*: there should be bases  $\xi_i$  for  $E_{M_0}$  and  $\eta_i$  for  $E_{M_1}$  with  $\xi_i \mapsto \lambda_i \eta_i$  and  $\sum_i |\lambda_i| < \infty$ .

The case of Lorentzian metrics occurs “on the boundary” of the moduli space of complex metrics. Thus usual Lorentzian QFTs can be viewed as holomorphic degenerations of nice QFTs.

Some simpler examples are similar in principle:

- $\text{Diff}^+(S^1)$  can be viewed as the boundary of the semigroup of contraction mappings.
- $U(n) \subset GL_n(\mathbb{C})$  can be viewed as the boundary of the semigroup of contraction operators  $\mathbb{C}^n \rightarrow \mathbb{C}^n$ .

However,  $PSL_2(\mathbb{R})$  is not compact, so we can’t view it as a boundary.

Viewing our spacetime  $M$  as Lorentzian, we can ask for  $M$  to be “globally hyperbolic,” so every point in  $M_0$  can be connected to a point in  $M_1$  via a timelike curve. This is similar to requiring that there are no black holes. The results of Kontsevich-Segal are best in the case of globally hyperbolic metrics.

Heuristically, we are requiring the existence of a Dirac operator and requiring that this Dirac operator is a contraction operator.

## 1.3 6/11 – Desired Properties of QFT and Scaling

In his first lecture, Ibou Bah proposed five properties that a (nice) QFT should satisfy: *unitarity*, *locality*, *causality*, *cluster decomposition*, and *renormalization*. We will first try to understand how these relate to Segal’s perspective.

### 1.3.1 Renormalization

Historically, understanding renormalization was crucial to the development of QFT. This is somewhat arcane from a mathematical point of view. However, one should note that renormalizability of a Lagrangian does have a natural relationship to the calculus of variations.

Consider a  $\sigma$ -model where our fields are maps  $M \rightarrow N$ . If  $M$  and  $N$  have Riemannian metrics, we can define an energy functional. For  $M = \mathbb{R}$  or  $S^1$ , this recovers the usual energy functional on paths. This is well-behaved and has a good Morse theory. When  $\dim M = 2$ , this is almost well-behaved, except there are problems with bubbling. For  $\dim M > 2$ , things are terrible.

The upshot is that (at least in this case?) renormalizability happens exactly when we expect things to be mathematically well-behaved.



### 1.3.2 Unitarity

If  $M$  is a cobordism from  $M_0$  to  $M_1$ , we can reverse orientations on the boundary and view  $\overline{M}$  as a cobordism from  $\overline{M}_1$  to  $\overline{M}_0$ . When  $M$  is Riemannian, this doesn't significantly affect things. However, when  $M$  is Lorentzian, this changes the direction of time.

If  $U_M : \mathcal{H}_{M_0} \rightarrow \mathcal{H}_{M_1}$  is the induced map, then we can think of  $U_{\overline{M}} : \mathcal{H}_{\overline{M}_1} \rightarrow \mathcal{H}_{\overline{M}_0}$  as suitably “adjoint” to  $U_M$ . More precisely, we can write  $\mathcal{H}_{M_i}$  and  $\mathcal{H}_{\overline{M}_i}$  as duals, and this connects  $U_M$  to  $U_{\overline{M}}$ . We will assume that  $U_{\overline{M}} = \overline{U_M}$ . Reflection-positivity turns into the assumption that the pairing  $\mathcal{H}_{M_0} \otimes \mathcal{H}_{\overline{M}_0} \rightarrow \mathbb{C}$  is positive (at least in nice cases?).

### 1.3.3 Locality

Given a point  $x \in M$ , we can view point operators as being defined on sufficiently small disks  $D$  about  $x$ . This gives algebras  $\mathcal{O}_x = \lim_{x \in D} E_D$ , where  $E$  is the QFT. In the first lecture here, we discussed how this relates to operadic multiplication. This is quite similar to the language of operator product expansion used in physics.

### 1.3.4 Cluster decomposition

Saying that two events are a long time apart is essentially saying that there is a long cylinder between them in spacetime. This can also be viewed as factorizing the cobordism defining spacetime into a product involving said long cylinder. Recall that the long cylinder in spacetime corresponds to time evolution  $e^{-iHt}$  in the QFT.

### 1.3.5 Causality

Consider a Lorentzian spacetime  $M$  with boundaries  $M_0$  and  $M_1$ . Placing operations at points  $x$  in the interior of  $M$  gives operations  $\mathcal{O}_x \otimes E_{M_0} \rightarrow E_{M_1}$ .

We assume that our spacetime is globally hyperbolic, so every point in  $M_0$  can be connected to a point in  $M_1$  via a timelike path. This splits  $M$  into a family of time slices (all diffeomorphic to each other). Given distinct points  $x_i$  on the same time slice (so the  $x_i$  are spacelike separated), causality ensures that we can define

$$\mathcal{O}_{x_1, \dots, x_n} \otimes \check{E}_{M_0} \rightarrow \hat{E}_{M_1},$$

where  $\check{\phantom{x}}$  denotes a dense subspace and  $\hat{\phantom{x}}$  denotes a completion. This product is independent of the ordering. Using global hyperbolicity, we can extend this to any collection of spacelike separated points.

The notion of “small disk” makes complete sense in a Riemannian metric, but it seems a bit odd in a Lorentzian metric. To make sense of this, we recall our notion of complex metric  $g : V \rightarrow \mathbb{C}$ . If we pass to the complexification, we get a standard quadratic form  $V_{\mathbb{C}} \rightarrow \mathbb{C}$ . Thus we may consider  $V$  as a subspace of  $\mathbb{C}^d \cong V_{\mathbb{C}}$ . The space of allowable complex metrics can be viewed as a contractible subspace of the Grassmannian of real subspaces of  $V$ . One can make sense of the algebras  $\mathcal{O}_x$  from this perspective.

### 1.3.6 Scaling

Let's return to the originally proposed topic of this lecture. Many things we'd like to prove about QFT can't be proved from the axioms we've given.

Viewing our spacetime  $M$  as a cobordism from  $M_0$  to  $M_1$ , we should view a scalar field as a compactly-supported density on  $M$ . Such a scalar field should give an operator  $E_{M_0} \rightarrow E_{M_1}$ . This transformation  $\text{Dens}_c(M) \rightarrow \text{Hom}(E_{M_0}, E_{M_1})$  can be called a *Wightman field*. It's not clear how to relate this to the bundle of algebras  $\mathcal{O}$  we've been discussing earlier.

We also don't really know what to do about Lagrangians. From our definition, we can make sense of a moduli space of theories, and we get a renormalization group flow on this moduli space (by rescaling metrics). Reconciling our notion of fields (using this space) should help us make sense of Lagrangians. (It's not clear how global symmetries fit into this description.)

We should be able to fix these with a scaling axiom (defined in Riemannian signature). This should be something saying that our spaces of local operators  $E_{(r)}$  (defined on disks of radius  $r$ ) tend to a limit  $E_{(0)}$ . This would provide a filtration on the spaces  $\mathcal{O}_x$  with associated graded  $E_{(0)}$ . Using this should allow us to relate  $\mathcal{O}_x$  with densities.

Extending our theories to codimension 2 should also help us prove things.

## Chapter 2

# Nitu Kitchloo – Symmetry Breaking, Surface Defects, and Link Homologies

### 2.1 6/10 – Overview

#### 2.1.1 Artin braid groups

Let  $G$  be a compact connected Lie group with maximal torus  $T$ . Recall that this means  $T \subset G$  is a maximal connected compact abelian subgroup.

**Example 2.1.1.** For  $G = U(n)$ , we can take  $T$  to be the subgroup  $\Delta$  of diagonal matrices. We will return to this example throughout.

Given  $(G, T)$ , we can define an *Artin braid group*  $\text{Br}(G, T)$ .

**Example 2.1.2.** For  $(G, T) = (U(n), \Delta)$ , this is the usual braid group

$$\text{Br}(G, T) = \text{Br}_n := \langle \sigma_1, \dots, \sigma_{n-1} \mid \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \quad \sigma_i \sigma_j = \sigma_j \sigma_i \text{ for } |i - j| > 1 \rangle.$$

More generally, note that  $T$  acts on  $G$  (and its Lie algebra  $\mathfrak{g}$ ) via the adjoint representation. Complexify  $\mathfrak{g}$  to get a complex representation  $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$  of  $T$ . Let  $R$  be the set of roots of  $G$ , i.e. nonzero characters of  $T$  in  $\mathfrak{g}_{\mathbb{C}}$ . Let  $\mathfrak{h} = \text{Lie}(T) \otimes_{\mathbb{R}} \mathbb{C}$ , and let

$$\mathcal{H} = \mathfrak{h}_{\mathbb{C}} - \cup_{\alpha \in R} \mathfrak{h}_{\alpha} \text{ for } \mathfrak{h}_{\alpha} = \{\mathfrak{h} \in \mathfrak{h}_{\mathbb{C}} \mid \alpha(\mathfrak{h}) = 0\}.$$

In other words,  $\mathcal{H}$  is the union of the interiors of the Weyl chambers. Note that  $\mathcal{H}$  has a free action of the Weyl group  $W(G, T) = N_T(G)/T$ .

**Definition 2.1.3.** The *Artin braid group*  $\text{Br}(G, T)$  is  $\pi_1(\mathcal{H}/W(G, T))$ .

It is a fact that  $\mathcal{H}/W(G, T)$  is a  $K(\text{Br}(G, T), 1)$ . Let's see why this works for  $G = U(n)$ .

**Example 2.1.4.** For  $(G, T) = (U(n), \Delta)$ , we have

$$\cup_{\alpha} \mathfrak{h}_{\alpha} = \{(x_1, \dots, x_n) \mid x_i = x_j \text{ for some } i \neq j\}.$$

Since  $W(G, T) = \Sigma_n$  acting on  $\mathfrak{h}_{\mathbb{C}} = \mathbb{C}^n$  by permuting coordinates, we see that  $\mathcal{H}/W(G, T)$  is the moduli of  $n$  points in  $\mathbb{C}$ . This picture gives us the usual definition of braid groups.

Note that  $\pi_0 \text{Map}(S^1, \mathcal{H}/W(G, T))$  is the space of conjugacy classes of elements in  $\text{Br}(G, T)$ , which we will write as  $[\text{Br}(G, T)]$ .

**Example 2.1.5.** For  $(G, T) = (U(n), \Delta)$ , elements of  $\text{Br}(G, T)$  are represented by braid closures. We may view these as links in  $\mathbb{R}^3$  using Markov's theorem.

**Theorem 2.1.6** (Markov). *The map sending a braid to its closure expresses the set of equivalence classes of links in  $\mathbb{R}^3$  as the set of equivalence classes of all braids,  $(\sqcup_{n \geq 1} \text{Br}_n) / \sim$ , where  $\sim$  is the equivalence relation generated by M1 and M2:*

1. M1 (Conjugation):  $\sigma \sim \rho \sigma \rho^{-1}$  for  $\sigma, \rho \in \text{Br}_n$
2. M2 (Stabilization):  $\sigma \sim \sigma \sigma_n^{\pm 1}$  for  $\sigma \in \text{Br}_n \subset \text{Br}_{n+1}$

The individual maps from  $[\text{Br}_n]$  to the space of links are not faithful. That is, stabilization does cause us to lose some information.

### 2.1.2 Categorification

We'd like to categorify  $\text{Br}(\mathbf{G}, \mathbf{T})$  and  $[\text{Br}(\mathbf{G}, \mathbf{T})]$ . In particular, we'd like a functor of (“discrete” / non- $\infty$ ) 2-categories<sup>1</sup>

$$* // \text{Br}(\mathbf{G}, \mathbf{T}) \rightarrow \text{Gr}(\mathbf{Alg}).$$

Here  $* // \text{Br}(\mathbf{G}, \mathbf{T})$  is the 2-category with:

- one object,
- $\text{Br}(\mathbf{G}, \mathbf{T})$  as 1-morphisms, and
- trivial 2-morphisms.

The right hand side,  $\text{Gr}(\mathbf{Alg})$ , is the 2-category with:

- objects: graded commutative algebras,
- 1-morphisms: chain complexes of graded bimodules, and
- 2-morphisms: homotopy classes of chain maps

We also want a map  $[\text{Br}(\mathbf{G}, \mathbf{T})] \rightarrow \text{Ch}_{\bullet, \bullet}(\mathbb{Z})$ .

This has been worked out by Soergel, Rouquier, Khovanov, and Khovanov-Rozansky. For  $\mathbf{G} = \text{U}(\mathbf{n})$ , the last map above gives bigraded homological invariants of links. (This requires us to check that the stabilization move is satisfied.)

Being more ambitious, we can try to realize this using 3d and 4d topological gauge theories. This splits into subgoals:

1. Describe a configuration space of  $\mathbf{G}$ -background fields on a 2-torus. (Here “background” means that we haven’t “summed over these fields.”)
2. Quantize by “summing over the background fields” to obtain the partition function of the 2-torus in 3d and 4d topological gauge theories with defects.
3. Identify the partition functions with the categorifications mentioned above.

### 2.1.3 Preliminaries: $\text{U}(\mathbf{n})$

We’ll start out by looking at the case of  $\mathbf{G} = \text{U}(\mathbf{n})$ . Let  $\mathbf{R}_n = \text{H}_T^\bullet(\text{pt}) = \mathbb{Z}[x_1, \dots, x_n]$  with  $|x_i| = 2$ . The Weyl group  $W$  is  $\Sigma_n$ , acting on  $\mathbf{R}_n$  with  $\sigma_i$  swapping  $x_i$  and  $x_{i+1}$ .

**Definition 2.1.7.** The *Bott-Samelson  $\mathbf{R}_n$ -bimodule* is the graded bimodule  $\text{bS}_i = \mathbf{R}_n \otimes_{\mathbf{R}_n^{\sigma_i}} \mathbf{R}_n$ , where  $\mathbf{R}_n^{\sigma_i}$  is the ring of  $\sigma_i$ -invariants in  $\mathbf{R}_n$ . The category  $\text{SBimod}_n$  *Soergel bimodules* is the subcategory of  $\mathbf{R}_n$ -bimodules generated by the  $\text{bS}_i$ ’s (allowing idempotent completion and grading shifts).

<sup>1</sup>Recent work extends this to a map of  $(\infty, 2)$ -categories.

**Theorem 2.1.8** (Soergel). *The category  $\text{SBimod}_n$  categorifies the Hecke algebra  $H_q(A_n) = \mathbb{Z}[q, q^{-1}]\langle \text{Br}_n \rangle / I$ , where  $I$  is generated by the relations*

$$\begin{aligned} (\sigma_i - q^2)(\sigma_i + 1) &= 0 \\ \sigma_i^{-1} &= q^{-2}\sigma_i + (q^2 - 1). \end{aligned}$$

*That is, there is an equivalence  $K_0(\text{SBimod}_n) \cong H_q(A_n)$ . Under this equivalence:*

- 1 corresponds to  $R_n$ ,
- $bS_i$  corresponds to  $\sigma_i + 1$ , and
- $q$  corresponds to grading shift.

The chain complex  $C_i = (bS_i \rightarrow R_n)$ , where the map is given by  $\alpha \otimes \beta \mapsto \alpha\beta$ , should be viewed as a categorification of  $\sigma_i$ .

**Theorem 2.1.9** (Rouquier). *The assignment  $\sigma_I \mapsto c_I := C_{i_1} \otimes_{R_n} C_{i_2} \otimes_{R_n} \cdots \otimes_{R_n} C_{i_k}$  gives a well-defined map  $* // \text{Br}_n \rightarrow \text{Gr}(\text{Alg})$ .*

**Example 2.1.10.** The braid  $\sigma_1\sigma_2$  is sent to the chain complex  $C_{12} = (bS_1 \rightarrow R_3) \otimes (bS_2 \rightarrow R_3)$ , which can also be written

$$bS_1 \otimes_{R_3} bS_2 \longrightarrow bS_1 \oplus bS_2 \longrightarrow R_3.$$

Rouquier's theorem is nontrivial for two reasons:

- We need to show that the map, which *a priori* is defined only on braid words, is actually defined on the braid group.
- We need to make sure that this respects the triviality of the higher morphisms in  $* // \text{Br}_n$ .

## 2.2 6/10 – Geometric interpretations for the $U(n)$ case

Last time, we discussed a categorification of  $\text{Br}_n$ . We'd like to obtain a corresponding categorification of  $[\text{Br}_n]$ .

### 2.2.1 Categorification of $[\text{Br}_n]$

Recall the definition of Hochschild homology.

**Definition 2.2.1.** If  $M$  is a graded  $R_n$ -bimodule, then the *Hochschild homology* of  $M$  is

$$\text{HH}(M) = \text{Tor}_{R_n \otimes R_n^{\text{op}}}^{\bullet, \bullet}(M, R_n)$$

We will consider the Hochschild homology  $\text{HH}(C_I)$  termwise.

**Example 2.2.2.** For  $\sigma_1\sigma_2$ , we discussed  $C_{12}$  in the last lecture. With the above convention,  $\text{HH}(C_{12})$  is

$$\text{HH}(bS_1 \otimes_{R_3} bS_2) \longrightarrow \text{HH}(bS_1 \oplus bS_2) \longrightarrow \text{HH}(R_3).$$

**Theorem 2.2.3** (Khovanov). *The assignment  $\sigma_I \mapsto \text{HH}(C_I)$  extends to  $[\text{Br}_n] \rightarrow \text{Ch}_{\bullet, \bullet}(\mathbb{Z})$ . In fact, the output is naturally trigraded. Furthermore, this assignment satisfies M1 and M2, so it gives a homological invariant of links.*

This invariant can be identified with the trigraded Khovanov-Rozansky link homology. If  $\mathcal{L}$  is the link  $[\sigma_i]$ , we write

$$\mathfrak{sl}_{\infty}(\mathcal{L}) = \text{HHH}(\mathcal{L}) = \text{HH}(C_I).$$

We may view  $\mathfrak{sl}_{\infty}$  as a “limit” of bigraded homological invariants  $\mathfrak{sl}_N(\mathcal{L})$  which can be obtained from  $\text{HH}(C_I)$  by adding a “matrix factorization” differential.

### 2.2.2 Topological interpretation of Bott-Samelson bimodules

**Definition 2.2.4.** For  $i < n$ , let  $G_i \subset U(n)$  be the semisimple rank one compact parabolic consisting of matrices of the form

$$\begin{bmatrix} D_1 & 0 & 0 \\ 0 & U & 0 \\ 0 & 0 & D_2 \end{bmatrix}$$

with  $D_1$  and  $D_2$  diagonal and  $U \in U(2)$  (placed in rows / columns  $i$  and  $i+1$ ).

Note that  $T \subset G_i \subset G = U(n)$  for all  $i$ .

**Definition 2.2.5.** Let  $BS_i$  be the balanced product  $U(n) \times^T (G_i/T)$ , considered with its natural  $U(n)$ -action.

We may obtain  $BS_i$  from a pullback square

$$\begin{array}{ccc} BS_i & \xrightarrow{\alpha} & U(n)/T \\ \downarrow \beta & & \downarrow \\ U(n)/T & \longrightarrow & U(n)/G_i \end{array}$$

where  $\alpha : [g, g_i] \mapsto [gg_i]$  and  $\beta : [g, g_i] \mapsto [g]$ .

In cohomology, we get a map

$$H_{U(n)}^\bullet(U(n)/T) \otimes_{H_{U(n)}^\bullet(U(n)/G_i)} H_{U(n)}^\bullet(U(n)/T) \rightarrow H_{U(n)}^\bullet(BS_i).$$

A spectral sequence computation shows this is an isomorphism. Combining this with the standard isomorphisms  $H_G^\bullet(G/K) \cong H_K^\bullet(\text{pt})$  for any  $G$  and any  $K \subset G$ , we get

$$H_{U(n)}^\bullet(BS_i) \cong H_T^\bullet(\text{pt}) \otimes_{H_{G_i}^\bullet(\text{pt})} H_T^\bullet(\text{pt}) \cong R_n \otimes_{R_n^{\sigma_i}} R_n = bS_i.$$

Thus, we can interpret the Bott-Samelson bimodule  $bS_i$  geometrically as the  $U(n)$ -equivariant cohomology of  $BS_i$ .

The pullback square defining  $BS_i$  is in fact a homotopy pullback square. Comparing with the standard definition of the homotopy pullback, we can view  $BS_i$  as the moduli space of  $U(n)$ -bundles on  $[0, 1]$  with a reduction of structure group to  $G_i$  on  $[0, 1]$  and to  $T$  on  $\{0, 1\}$ .

### 2.2.3 Interpretation of Rouquier's chain complex

More generally, for a multi-index  $I = (i_1, \dots, i_k)$ , define

$$BS_I = U(n) \times^T (G_{i_1} \times^T G_{i_2} \times^T \dots \times^T G_{i_k}/T).$$

Then

$$H_{U(n)}^\bullet(BS_I) \cong bS_{i_1} \otimes_{R_n} bS_{i_2} \otimes_{R_n} \dots \otimes_{R_n} bS_{i_k}.$$

We may view  $BS_I$  as the moduli space of  $U(n)$ -bundles on  $[0, 1]$  with  $k+1$  marked points (including 0 and 1) with reduction of structure group to  $G_{i_j}$  on the  $j$ th interval and to  $T$  at each marked point.

If  $J$  is obtained from  $I$  by discarding indices, then we get  $BS_J \hookrightarrow BS_I$  by replacing every factor  $G_{i_k}$  (for each removed index  $i_k$ ) by  $T$  in the construction of  $BS_I$ . These replaced copies of  $T$  are “transparent defects.” This inclusion respects composition of inclusions of multi-indices.

Rouquier's chain complex  $C_I$  is isomorphic to

$$H_{U(n)}^\bullet(BS_I) \longrightarrow \oplus_{|J|=k-1, J \subset I} \longrightarrow \dots \longrightarrow H_{U(n)}^\bullet(U(n)/T).$$

All differentials here are obtained from (signed) inclusions  $BS_J \rightarrow BS_{J'}$ .

### 2.2.4 Interpretation of $\mathrm{HH}(\mathrm{bS}_I)$

To interpret  $\mathrm{HH}(\mathrm{H}_{\mathrm{U}(\mathfrak{n})}^\bullet(\mathrm{BS}_I))$ , we introduce the following.

**Definition 2.2.6.** Let

$$\mathrm{L}_i = \mathrm{U}(\mathfrak{n}) \times^{\mathrm{T}} \mathrm{G}_i$$

where  $\mathrm{T}$  acts on  $\mathrm{G}_i$  by conjugation.

The map  $\mathrm{L}_i \rightarrow \mathrm{BS}_i$  given by  $[g, g_i] \mapsto [g, g_i]$  is a principal  $\mathrm{T}$ -bundle. In fact, we have a pullback square

$$\begin{array}{ccc} \mathrm{EU}(\mathfrak{n}) \times^{\mathrm{U}(\mathfrak{n})} & \longrightarrow & \mathrm{BT} \\ \downarrow & & \downarrow \Delta \\ \mathrm{EU}(\mathfrak{n}) \times^{\mathrm{U}(\mathfrak{n})} \mathrm{BS}_i & \longrightarrow & \mathrm{BT} \times \mathrm{BT} \end{array}$$

where both vertical arrows are trivial principal bundles.

Collapse of the Eilenberg-Moore spectral sequence at the  $\mathrm{E}_2$  page gives

$$\mathrm{H}_{\mathrm{U}(\mathfrak{n})}^\bullet(\mathrm{L}_i) \cong \mathrm{Tor}_{\mathrm{H}^\bullet(\mathrm{BT} \times \mathrm{BT})}(\mathrm{H}_{\mathrm{U}(\mathfrak{n})}^\bullet(\mathrm{BS}_i), \mathrm{H}^\bullet(\mathrm{BT})) = \mathrm{HH}(\mathrm{H}_{\mathrm{U}(\mathfrak{n})}^\bullet(\mathrm{BS}_i)) = \mathrm{HH}(\mathrm{bS}_i).$$

Thus we obtain a geometric interpretation of  $\mathrm{HH}(\mathrm{bS}_I)$ . We may interpret  $\mathrm{L}_i$  as the moduli space of  $\mathrm{U}(\mathfrak{n})$ -bundles on  $S^1$  with one marked point, where the structure group reduces to  $\mathrm{G}_i$  away from the marked point and to  $\mathrm{T}$  at the marked point.

More generally, for a multi-index  $I$ , we let

$$\mathrm{L}_I = \mathrm{U}(\mathfrak{n}) \times^{\mathrm{T}} (\mathrm{G}_{i_1} \times^{\mathrm{T}} \mathrm{G}_{i_2} \times^{\mathrm{T}} \cdots \times^{\mathrm{T}} \mathrm{G}_{i_k})$$

where  $\mathrm{T}$  acts by conjugation on  $\mathrm{G}_{i_1}$  and  $\mathrm{G}_{i_k}$ . An argument analogous to the above gives  $\mathrm{H}_{\mathrm{U}(\mathfrak{n})}^\bullet(\mathrm{L}_I) \cong \mathrm{HH}(\mathrm{bS}_I)$ .

For  $J \subset I$ , we get an inclusion  $\mathrm{L}_J \rightarrow \mathrm{L}_I$  that behaves well with respect to composition of inclusions. These assemble into a complex

$$\mathrm{H}_{\mathrm{U}(\mathfrak{n})}^\bullet(\mathrm{L}_I) \longrightarrow \oplus_{|J|=k-1, J \subset I} \mathrm{H}_{\mathrm{U}(\mathfrak{n})}^\bullet(\mathrm{L}_J) \longrightarrow \cdots$$

which is isomorphic to  $\mathrm{HH}(\mathrm{C}_I)$ .<sup>2</sup>

## 2.3 6/11 – Local systems and twisted cohomology

### 2.3.1 Review of bundle theory

Let  $G$  be a topological group. Principal  $G$ -bundles are classified by pulling back a “universal bundle”  $\mathrm{EG} \rightarrow \mathrm{BG}$ . That is, for any principal  $G$ -bundle  $E \rightarrow B$ , there exists an  $f : B \rightarrow \mathrm{BG}$  (unique up to homotopy) such that  $E = f^* \mathrm{EG}$ . To construct  $\mathrm{EG} \rightarrow \mathrm{BG}$ , just take any contractible space  $\mathrm{EG}$  with free  $G$ -action, and let  $\mathrm{BG} = \mathrm{EG}/G$ .

We will write a  $G$ -space  $X$  as  $X // G$ . From such a space, we may construct the *homotopy orbits*  $X //^h G = \mathrm{EG} \times^G X := (\mathrm{EG} \times X)/G$ , where  $G$  imposes the relation  $(e, x) \sim (eg, g^{-1}x)$ . We define the  *$G$ -equivariant cohomology* of  $X$  to be

$$\mathrm{H}_G^\bullet(X) := \mathrm{H}^\bullet(X //^h G).$$

If  $K \subset G$ , then we may take  $\mathrm{EG}$  as a model for  $\mathrm{EK}$ . Because  $\mathrm{EG} \times^G (G \times^K Y) = \mathrm{EG} \times^K Y = \mathrm{EK} \times^K Y$ , we obtain

$$\mathrm{H}_G^\bullet(G \times^K Y) = \mathrm{H}_K^\bullet(Y).$$

Given a principal  $G$ -bundle  $E \rightarrow B$ , we may ask whether or not  $E$  is extended from a principal  $K$ -bundle, i.e. whether  $E \cong E' \times^K G$  for some principal  $K$ -bundle  $E' \rightarrow B$ . This is equivalent to asking whether the classifying map  $f_E : B \rightarrow \mathrm{BG}$  lifts to a map  $f_{E'} : B \rightarrow \mathrm{BK}$ . Note that  $\mathrm{BK} \rightarrow \mathrm{BG}$  has fiber  $G/K$ , so in particular reductions of structure group of the trivial  $G$ -bundle to  $K$  correspond to maps  $B \rightarrow G/K$ .

<sup>2</sup>This is a “cubical complex,” related to crossing resolutions for nice presentations of knots.

### 2.3.2 Recap

Last time, we defined subgroups  $G_i \subset U(\mathfrak{n})$  for  $1 \leq i < \mathfrak{n}$ . For each multi-index  $I = (i_1, \dots, i_k)$ , we defined spaces  $BS_I = U(\mathfrak{n}) \times^T (G_{i_1} \times^T G_{i_2} \times^T \dots \times^T G_{i_k}/T)$ . We showed

$$H_{U(\mathfrak{n})}^\bullet(BS_I) = bS_{i_1} \otimes_{R_n} \dots \otimes_{R_n} bS_{i_k}$$

for  $bS_i = R_n \otimes_{R_n^{\sigma_i}} R_n$ . The spaces  $BS_I$  are functorial with respect to inclusions of multi-indices.

**Example 2.3.1.** Let  $I = \{i\}$  and  $J = \emptyset$ , so  $BS_i = U(\mathfrak{n}) \times^T G_i/T$  and  $BS_\emptyset = U(\mathfrak{n}) \times^T (T/T) = U(\mathfrak{n})/T$ . Note that  $H_{U(\mathfrak{n})}^\bullet(BS_i) = bS_i$  and  $H_{U(\mathfrak{n})}^\bullet(U(\mathfrak{n})/T) = H_T^\bullet(\text{pt}) = R_n$ . The inclusion  $BS_\emptyset \rightarrow BS_i$  comes from  $T \subset G_i$ , and the induced map on cohomology is the multiplication map

$$bS_i = R_n \otimes_{R_n^{\sigma_i}} R_n \rightarrow R_n.$$

This is the chain complex  $C_i$  appearing in Rouquier's work.

We also defined spaces  $L_I = U(\mathfrak{n}) \times^T (G_{i_1} \times^T G_{i_2} \times^T \dots \times^T G_{i_k})$  where  $T$  acts on  $G_{i_1}$  and  $G_{i_k}$  by conjugation. These spaces may be viewed as moduli of  $U(\mathfrak{n})$ -bundles on  $S^1$  with  $k+1$  marked points with reduction of structure group to  $G_{i_j}$  on the  $j$ th interval and to  $T$  at each marked point. The cohomologies can be used to construct a chain complex which recovers Khovanov's link homology. Specifically, we have  $H_{U(\mathfrak{n})}^\bullet(L_I) = HH(bS_I)$ .

### 2.3.3 Recovering $\mathfrak{sl}_N$ -link homology

There is a forgetful map  $L_I // U(\mathfrak{n}) \rightarrow U(\mathfrak{n}) // U(\mathfrak{n})$ , where  $U(\mathfrak{n})$  acts on itself via the adjoint representation. This map is given by the formula

$$[g, g_1, \dots, g_k] \mapsto gg_1 \dots g_k g^{-1}.$$

If one thinks of the bundles appearing in the moduli description of  $L_I$  as having connections, this is essentially taking the product of the holonomies.

The functorial maps of  $L_I // U(\mathfrak{n})$  are compatible with the forgetful maps to  $U(\mathfrak{n}) // U(\mathfrak{n})$ . In particular, Khovanov's chain complex

$$H_{U(\mathfrak{n})}^\bullet(L_I) \longrightarrow \oplus_{|J|=k-1, J \subset I} H_{U(\mathfrak{n})}^\bullet(L_J) \longrightarrow \dots$$

is a complex over the torsion-free  $H_{U(\mathfrak{n})}^\bullet(U(\mathfrak{n}))$ . Let's think about what happens if we take termwise cohomology with respect to odd elements of  $H_{U(\mathfrak{n})}^\bullet(U(\mathfrak{n}))$ .

There is a non-canonical ring isomorphism

$$H_{U(\mathfrak{n})}^\bullet(U(\mathfrak{n})) \cong H_{U(\mathfrak{n})}^\bullet(\text{pt}) \otimes H^\bullet(U(\mathfrak{n})).$$

The non-canonicalness of this is related to the fact that  $U(\mathfrak{n}) //^h U(\mathfrak{n}) \not\cong BU(\mathfrak{n}) \times U(\mathfrak{n})$ . Essentially, it's not clear what to do on primitive Hopf algebra elements.

However, if we take  $\mathfrak{n} \rightarrow \infty$  (and write  $U = U(\infty)$ ), we do get a canonical equivalence

$$U //^h U \simeq BU \times U.$$

Pulling back the elements of  $H^\bullet(U)$  gives odd elements  $d_1, d_3, \dots$ , and we can modify Khovanov's complex to get

$$H^\bullet(H_{U(\mathfrak{n})}(L_I), d_{2N-1}) \longrightarrow \oplus_{|J|=k-1, J \subset I} H^\bullet(H_{U(\mathfrak{n})}(L_J), d_{2N-1}) \longrightarrow \dots$$

Here  $\mathfrak{n}$  and  $N$  are unrelated.

**Theorem 2.3.2** (T. Gomez). *This complex is isomorphic to the link homology  $\mathfrak{sl}_N(\mathcal{L})$  for the link  $\mathcal{L} = [\sigma_I]$ .*

**Example 2.3.3.** Let  $\mathcal{L}$  be the unknot, so  $\mathcal{L} = [\sigma_1]$ . Then  $\mathfrak{sl}_\infty(\mathcal{L}) = H_{U(1)}^\bullet(U(1) \times^T T) = H_T^\bullet(T) = \mathbb{Z}[x_2, y_1]$ . Here  $d_N$  corresponds to the odd class  $y_1 x^N$ . We get

$$H^\bullet(H_{U(1)}(L_I), d_N) \cong (\mathbb{Z}[x]/x^N) \langle y_1 \rangle.$$



### 2.3.4 Local systems

The above story was a bit ad hoc, so we'd like to fit it into a more topological context. We can think of  $U(\mathfrak{n})$  as  $B\Omega U(\mathfrak{n})$ , so  $\Omega U(\mathfrak{n})$  is really acting everywhere. In fact, we'd like the second Markov move (stabilization) to play well with our theory, so we should pass to an  $\Omega U$ -action. We can implement this by using a *twisted cohomology theory*

$$\mathcal{H} = \Omega U_+ \wedge H_{\mathbb{Z}} = (\mathbb{Z} \times BU)_+ \wedge H_{\mathbb{Z}}.$$

One can compute

$$\pi_{\bullet} \mathcal{H} = \pi_{\bullet} H[b_0^{\pm}, b_1, b_2, \dots]$$

with  $|b_i| = 2i$ .

Setting  $b_0 = b_N = 1$  and  $b_i = 0$  otherwise (and indicating this with the subscript  $N$ ), one can calculate

$$\tilde{\mathcal{H}}_{U(\mathfrak{n})}^{\bullet}(L_I)_N \simeq H^{\bullet}(H_{U(\mathfrak{n})}(L_I), d_n).$$

## Chapter 3

# Ibou Bah – Lagrangian Methods for Categorical Symmetries

### 3.1 6/11 – What is QFT?

This will be a physics-y, intuitive series of lectures. We will start by giving some idea of QFT in general. In the second lecture, we will move on to finite symmetry groups and use these to construct topological operators. In the third lecture, we will discuss the action of finite symmetry groups on fields (with an emphasis on the example of  $SU(N)$  Yang-Mills theory). Finally, we will discuss the general construction of topological operators using Lagrangians.

#### 3.1.1 What should a QFT do?

We don't actually know what a QFT *is*. However, we have an excellent physical idea of what a QFT should *do*. QFTs should have:

- A list of operators: point operators  $\{\mathcal{O}_i(x)\}$ , line operators  $\{U_\alpha(\gamma_p)\}$ , and other operators.
- A Hilbert space  $\mathcal{H}$  of states on spacetime  $M = \mathbb{R}_t \times Y$ .
- A partition function sending combinations / products of operators to complex numbers, e.g.  $\langle \mathcal{O}_i \mathcal{O}_j \rangle$ .

These should satisfy three key properties: *unitarity*, *locality*, and *causality*. Let's discuss each of these in more detail.

#### 3.1.2 Unitarity, locality, and causality

Unitarity can be interpreted as “preservation of probability.” For example, for many QFTs, we can construct an “S-matrix” governing scattering amplitudes of particles, and unitarity forces  $S^\dagger S = 1$ . This specific interpretation doesn't work for CFTs / TFTs. Instead, we use a notion of *reflection positivity*.

To explain this, recall that time evolution in quantum mechanics is governed by

$$|\psi\rangle \mapsto e^{-iHt} |\psi(t)\rangle$$

and

$$\mathcal{O}(x, 0) |\psi\rangle \mapsto \mathcal{O}(x, t) |\psi\rangle .$$

We should have

$$\langle \psi | \mathcal{O}^\dagger \mathcal{O} | \psi \rangle = \|\mathcal{O}(x, t) |\psi\rangle\| \geq 0.$$

To make sense of  $\mathcal{O}^\dagger$ , we allow our time variable to take complex values. Say that  $\mathcal{O}(x, t+i\tau)^\dagger = \mathcal{O}(x, t-i\tau)$ . Then the above condition forces  $\langle \mathcal{O}(x, t-i\tau) \mathcal{O}(x, t+i\tau) \rangle \geq 0$ . Taking  $t = 0$ , we see that reflection positivity requires  $\langle \mathcal{O}(x, -i\tau) \mathcal{O}(x, i\tau) \rangle \geq 0$ . More generally, reflection positivity forces

$$\langle \mathcal{O}_i(x, -i\tau) \mathcal{O}_j(x, -i\tau) \dots \mathcal{O}_j(x, i\tau) \mathcal{O}_i(x, i\tau) \rangle \geq 0.$$

Locality forces point operators to have an *operator product expansion*

$$\mathcal{O}_i(x_1)\mathcal{O}_j(x_2) = \sum_k c_{ij}^k(x_1, x_2)\mathcal{O}(x_1)$$

governing the behavior as  $x_1 \rightarrow x_2$ . (For conformal field theories, the  $c_{ij}^k$  should be meromorphic.) More generally, the interaction of operators should depend only on their behavior in a common tubular neighborhood. If two point operators are outside the same light cone, then we can find a reference frame where the two operators take place at the same time but at different points in space.

Finally, causality requires that any operators which are spacelike separated commute with each other. If two point operators are in the interior of the same light cone, then we can find a reference frame where the operators take place at the same point in space but at different points in time. Things are more complicated for extended operators.

### 3.1.3 Extra features

In many QFTs, we can expect other nice properties to hold:<sup>1</sup>

- Cluster decomposition:

$$\lim_{b \rightarrow \infty} \langle \mathcal{O}_i(x_1)\mathcal{O}_j(x_2) \dots \mathcal{O}_{i'}(x_1 + b)\mathcal{O}_{j'}(x_2 + b) \dots \rangle = \langle \mathcal{O}_i(x_1)\mathcal{O}_j(x_2) \dots \rangle \langle \mathcal{O}_{i'}(x_1 + b)\mathcal{O}_{j'}(x_2 + b) \dots \rangle.$$

- Renormalization: high energy / short distance / UV physics and low energy / long distance / IR physics decouple, at least up to finitely many parameters.

Axiomatic treatments can take these as additional axioms.

Gravity fails both of these conditions and locality. Black holes are UV phenomena which are still visible at low energy. Furthermore, black holes also break cluster decomposition. Whatever quantum gravity is, unitarity “must” be retained, since it’s fundamental to quantum physics.

The goal of the Simons collaboration is to understand the kinematics of operators in QFT. This is governed by symmetries! We can essentially think of this as understanding the *labels* of operators  $\mathcal{O}_i(x)$ ,  $U_\alpha(\gamma)$ , etc.

### 3.1.4 Where do QFTs come from?

There are many sources of QFTs.

**Example 3.1.1** (Lattices). Consider a lattice at which each point behaves according to quantum mechanics. Taking a continuum limit of the lattice gives a quantum field theory. This limit leads to the appearance of infinities and other strange phenomena. Nevertheless, it’s the most physically “rigorous” way of constructing QFTs.

**Example 3.1.2** (Geometric engineering). Start with a string theory, and “decouple gravity” by freezing some dimensions. This produces a QFT involving geometric objects. Class  $\mathcal{S}$  theories arise in this way.

**Example 3.1.3** (Lagrangians). We can construct many QFTs from Lagrangians. We’ll spend most of our time focusing on this perspective.

Each perspective has advantages over the others:

- Lattices make renormalization straightforward.
- Geometric engineering makes the geometric nature of objects more apparent.
- Lagrangians make locality explicit.

---

<sup>1</sup>There’s also a last property of “analyticity” which is hard to formulate.

### 3.1.5 Lagrangians

A Lagrangian is a functional depending on the fields  $\phi_\alpha(x)$ :

$$L(x) = L(\phi_\alpha(x), \partial\phi_\alpha(x), \dots)$$

This is a classical object. Locality enforces that  $L$  only depends on the behavior of the fields at the given input point (unlike, say, Green's functions  $G(x_1, x_2)$ ). Locality also ensures that there are only finitely many derivative terms appearing in  $L$ .

Lorentz symmetry (which we will assume implicitly) tells us that our fields can be:

- Scalar fields  $\phi(x)$
- Vector fields / gauge fields  $A_\mu(x)$ , either:
  - Massless: living in the representation  $(1, -1)$ , or
  - Massive: living in  $(1, 0, 1)$
- Fermions  $\psi_\alpha$  (representations of a Clifford algebra)
- Metrics  $g_{\mu\nu}$

We won't consider gravitational theories, so we'll ignore  $g_{\mu\nu}$ .

## 3.2 6/11 – Lagrangians in QFT

Unitarity is true for closed quantum systems. For open quantum systems, where we “trace away” some information, we can drop this assumption. We must restore unitarity as we restore data to our system. This explains e.g. where the  $W$ -boson and Higgs boson come from: they're needed to restore unitarity in the Standard Model.

### 3.2.1 Lagrangians

Consider a Lagrangian

$$\mathcal{L}(x) = \mathcal{L}(\phi_\alpha(x), \partial\phi_\alpha(x), \dots).$$

From this (classical) data, we can define a (quantum) partition function

$$Z = \int D[\phi_\alpha] e^{i \int_M \sqrt{g} \mathcal{L}(\phi_\alpha)}$$

This can be thought of as a formal expression: heuristically, it is a sum over all possible field configurations.

We can obtain correlation functions, etc. by inserting local operators

$$\mathcal{O}_i(\phi_\alpha(x), \partial\phi_\alpha(x), \dots)$$

or extended operators

$$U_a(\gamma) = U\left(\int_\gamma F(\phi_\alpha(x)), \dots\right).$$

Namely, we declare

$$\langle \mathcal{O}_i \dots \rangle = \frac{1}{Z(0)} \int D[\phi_\alpha] e^{i \int \mathcal{L}} \mathcal{O}_i \dots$$

### 3.2.2 Parameters

In general, it is possible that our Lagrangian depends on some additional parameters, say  $\tau_a, \tau_b$ . The partition function then also depends on the  $\tau$ 's. We will assume that these parameters are *perturbative*: that small changes in our theory are governed by Taylor expansions in terms of the  $\tau$ 's. Often, there is a large parameter space for a given QFT, and there are many perturbative regions within this space corresponding to different weakly coupled versions of the theory. These *duality frames* can have different fields, different Lagrangians, and different gauge groups. (Gauge symmetry is a redundancy in parametrization – it is not a physical symmetry, but rather a “trick” enabling the use of local Lagrangians.)

However, the global symmetry, Hilbert space, and local operators are the same in every duality frame.<sup>2</sup> This may not always be visible (at least at the classical level) but it is true. Global symmetry permutes the operators, while gauge symmetry does nothing to the operators.

**Example 3.2.1.** We can view 4d  $\mathcal{N} = 4$  super-Yang-Mills as part of the same parameter space as type IIB supergravity on  $\text{AdS}_5 \times S^5$ . Thus these are versions of the same theory.

In addition to varying our parameters, we can also turn on background fields  $B_i(x)$ , i.e. add in new fields which do not fluctuate.

### 3.2.3 Examples

Consider a theory with one free massless scalar field  $\phi$ . The Lagrangian is

$$\mathcal{L} = \frac{1}{e^2} d\phi \wedge \star d\phi,$$

a single kinetic term. The partition function is

$$Z = \int D[\phi] \exp \left( \frac{i}{e^2} \int d\phi \wedge \star d\phi \right).$$

Taking the coupling parameter  $e \rightarrow 0$  yields  $d\phi = 0$ .

We can also add matter, incorporated via a term  $m^2 \phi^2 \text{Vol}(M)$  added to the Lagrangian. Taking  $m \rightarrow \infty$  gives  $\phi = 0$ .

The stress tensor of the theory is

$$T_{\mu\nu} = \frac{\delta \mathcal{L}}{\delta g^{\mu\nu}}.$$

In our above setup, there are kinetic terms, so  $T_{\mu\nu} \neq 0$  and the theory is not topological. However, the theory does become topological in certain limits.

Suppose we add a gauge field  $A_\mu$  transforming (under local gauge transformations) according to  $A_\mu \mapsto A_\mu + d\lambda$ . Taking  $\phi$  to be a complex scalar, setting  $D\phi = d\phi + iA\phi$ , and letting  $F = dA$ , we can define a Lagrangian

$$\mathcal{L} = D\phi \wedge \star D\phi + \frac{1}{e^2} F \wedge \star F.$$

This produces a non-topological gauge theory.

We can produce a topological theory by adding topological coupling terms:  $\theta F \wedge F$  or  $A \wedge d\phi \wedge F$ . These terms would have coefficients determined by the gauge theory. Alternatively, we could add boundary terms.

### 3.2.4 More remarks on Lagrangians

The constraints depend on the choice of gauge group. For example, having gauge group  $\mathbb{R}$  would allow for only local transformations (corresponding to exact 1-forms  $d\lambda$ ). Gauge group  $U(1)$  would allow more interesting global transformations (e.g.  $\omega \in H^1(-; \mathbb{Z})$  allowing  $A_\mu \mapsto A_\mu + \omega$ ).

We can split a typical Lagrangian into a *kinetic term*, an *interaction term*, and a *topological term*. The kinetic term reflects the behavior of free fields. The interaction term consists of a polynomial in the  $\phi_\alpha$ , and the signs of the coefficients are constrained by locality and causality. The topological term is (broadly) independent of the metric.

---

<sup>2</sup>The term “global” is not related to this, however.