Notes from GCS 2024 Summer School

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June 10-14, 2024

Abstract

These are my notes from the "Categorical Symmetries in Quantum Field Theory" summer school, held at the University of Edinburgh from June 10-14, 2024. Lectures were given by Graeme Segal, Nitu Kitchloo, Ibou Bah, Claudia Scheimbauer, and Ryan Thorngren. All errors are my own.

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Chapter 1

Graeme Segal – A Perspective on Quantum Field Theory

1.1 6/10 – Overview of QFT

1.1.1 From classical to quantum

In classical mechanics, one focuses one's attention on a fixed space M_0 and obtains a configuration space X from that. Dynamics are specified by giving a Lagrangian $L : TX \to \mathbb{R}$, which we assume is positive, inhomogeneous, and quadratic on tangent vectors. Under these hypotheses, the Lagrangian can be viewed as a generalized Riemannian metric, so that time evolution is given by a (generalized) "geodesic flow." This gives rise to a symplectic form ω and Poisson bracket $\{-,-\}$ on T^*X. Furthermore, from the Lagrangian L, we may extract a Hamiltonian $H : T^*X \to \mathbb{R}$ such that

$$\frac{d}{dt} f = \{H, f\} \quad \forall f \in \mathcal{C}^\infty(X).$$

Thus we may view a classical dynamical system as a triple (Y, ω, H) with (Y, ω) a symplectic manifold and $H : Y \to \mathbb{R}$ a function.

Quantum mechanics can be viewed as a complex noncommutative analogue of this. We replace (Y, ω, H) by a triple $(\mathcal{A}, \star, H)$ with $\mathcal{A}$ a (topological) $\mathbb{C}$-algebra, $\star$ an antilinear involution on $\mathcal{A}$, and $H \in \mathcal{A}$. For an observable $f \in \mathcal{A}$, we require

$$\frac{d}{dt} f = ih[H, f].$$

If the quantum system $(\mathcal{A}, \star, H)$ is a "quantization" of a classical system (Y, ω, H), then there is a deep relationship between the two. For example, we should view $\mathcal{A}$ as a completion / extension of $\mathcal{C}^\infty(Y)$, and we can relate the values of an observable to the eigenvalues of the corresponding operator.

The above picture privileges the time dimension, so it is fundamentally non-relativistic. Working relativistically requires us to move from finite-dimensional spaces to infinite-dimensional spaces. Heuristically, we take Y to be our space of fields on spacetime (typically $M = M_0 \times \mathbb{R}$), and we fix a Lagrangian $L : Y \to \mathbb{R}$. However, $\mathcal{A}$ should now be viewed as a space of functions on an "almost finite-dimensional manifold." This requires many corrections!

Example 1.1.1. Consider scalar field theory, with $M = M_0 \times \mathbb{R}$ and $Y = \mathcal{C}^\infty(M)$. Then $\mathcal{A}$ looks like functions on a stratified configuration space

$$\coprod_{n \geqslant 0} \mathrm{Conf}_n(M_0),$$

where $\mathrm{Conf}_n(M_0)$ is the space of sets of n distinct unordered points on M_0.

1.1.2 Approaching quantum field theory

Heuristically, we should "spread things out over spacetime" by attaching an algebra $\mathcal{O}_x$ to every point $x \in M$ and letting

$$\mathcal{A} \approx \bigotimes_{x \in M} \mathcal{O}_x.$$

This has been formalized in the approach of algebraic quantum field theory – see e.g. the definition of Haag, which assigns an algebra $\mathcal{A}_U$ to each open set $U \subset M$.

An alternative approach, which we will pursue, privileges the time dimension. In this approach, we define a d-dimensional quantum field theory as a symmetric monoidal functor

$$E : \mathsf{Cob}_{d-1} \to \mathsf{Vect}.$$

Here Cob_{d-1} is the symmetric monoidal category with:

- Objects: $(d-1)$-manifolds, often assumed compact or compact with boundary (and thought of as time slices)
- Morphisms $M_0 \to M_1$: d-dimensional cobordisms from M_0 to M_1 (thought of as controlling time evolution)
- Monoidal structure: disjoint union of manifolds / cobordisms

and Vect is the symmetric monoidal category of (possibly infinite-dimensional) topological vector spaces (with $\otimes$ as tensor product).

There are many other ways to view quantum field theory. For example, one could think of QFT as a continuum limit of lattice models.

QFT is traditionally seen as describing "the world except for gravity." That is, we think of spacetime as fixed, without allowing for the effects of gravity. However, QFT can tell us about what sorts of gravitational effects are possible.

The perspective on QFT that we will follow has the advantage that it allows for direct comparisons between QFTs. Thus, in principle, we may consider a "moduli space" of QFTs (of a given type). We can think of gravitational effects as acting upon this moduli space.

1.1.3 The perspective of Connes

We'd like to use geometry to understand the noncommutative world of QFT. This can be accessed by studying the spectrum of our Hamiltonian H. Operators which evolve slowly in time "nearly commute" with H and thus must be "nearly diagonal." We're mostly interested in studying such operators (as the time scale of humans is much slower than that of the universe). Thus, when studying QFT, we're interested in algebras which are "nearly commutative."

Connes was interested in finding a mildly noncommutative generalization of algebras of functions on a manifold M. Recall that we can define the Clifford algebra $\mathrm{Cliff}(T_m M)$ as the algebra with generators $\{\gamma_\xi\}_{\xi \in T_m M}$ and relations $\gamma_\xi^2 = -\|\xi\|^2$. Let $\mathcal{B}$ be a bundle of finite-dimensional algebras on M containing $\mathrm{Cliff}(T_m M)$, and take a connection on $\mathcal{B}$. Let $\not{D}$ be the Dirac operator

$$\not{D} = \sum_i \gamma_{\xi_i} \otimes \frac{\partial}{\partial x_i},$$

and let $H = \not{D}^2$. Connes gave a formula for $\operatorname{tr} \not{D}^2$ and extracted interesting physical objects, e.g. the Higgs field, from $\not{D}$. From Connes' perspective, the Dirac operator is "as good as" the metric.

1.1.4 Algebra from functorial QFT

Suppose we view spacetime M as a cobordism from M_0 to M_1. Fix a d-dimensional quantum field theory E, where $E = \dim M$. For $x \in M$, we can define a vector space of operators $\mathcal{O}_x$ by taking a small disk D_x about x and letting $\mathcal{O}_x = E_{\partial D_x}$. We can view a punctured copy of M as giving a cobordism

$$M \sqcup (\sqcup_i \partial D_{x_i}) \to M_1,$$

so that applying E gives $E_{M_0} \otimes (\otimes_i \mathcal{O}_{x_i}) \to E_{M_1}$. This gives a sort of "higher multiplication" on the spaces of operators $\mathcal{O}_{x_i}$.

We can make this much more precise in special cases. For example, if E is topological, then $\mathcal{O}_{x_i} = \mathcal{O}$ is independent of x_i. Taking $M_0 = \emptyset$ and $M_1 = S^{d-1}$, we obtain a family of multiplications $\mathcal{O}^{\otimes n} \to \mathcal{O}$.

Example 1.1.2. If $d = 2$, the cap / cup / pair-of-pants cobordisms equip $A = \mathcal{O}$ with the structure of a finite-dimensional commutative Frobenius algebra. This means that A is a unital algebra with a nondegenerate trace $\theta : A \to \mathbb{C}$.

It is interesting to extend this picture to non-topological 2-dimensional QFTs. Here, one can set up a moduli space of QFTs and obtain a gravitational flow. This was a major historical motivation for string theory.

The second lecture will discuss the importance of positive energy to this theory. The third lecture will discuss scaling, and the fourth lecture will focus on finding a suitable definition for Cob.

1.2 6/10 – Positive Energy

1.2.1 Warmup

Consider a quantum mechanical system with Hilbert space $\mathcal{H}$. The Hamiltonian is a self-adjoint operator $H : \mathcal{H} \to \mathcal{H}$, and we can integrate iH to get unitary operators $U_t = e^{iHt} : \mathcal{H} \to \mathcal{H}$ describing the time evolution of the system.

Assuming $H \geqslant 0$ has strong consequences for the theory. In this case, $U_- : \mathbb{R} \to \mathrm{End}(\mathcal{H})$ is the boundary value of the function on the complex upper half-plane also given by $t \mapsto e^{iHt}$.

To fit this into QFT, consider $\mathbb{C}$ as a bundle over $\mathbb{R}$, where the inner product on the base is Lorentzian and the inner product on the fiber is Riemannian. Think of $\mathbb{R}$ as our spacetime M and the fiber as $V = T_x M$.

We will introduce the notion of a *complex metric* on a real vector space V. This is a quadratic form $V \to \mathbb{C}$ satisfying a certain condition to be determined later.

1.2.2 Historical digression

The notion of a complex metric was introduced by Kontsevich and Segal. Segal was originally interested in loop groups $\mathcal{L}G = \mathrm{Map}(S^1, G)$. There is a nice class of "positive energy" representations of loop groups. These extend to representations of $\mathrm{Diff}^+(S^1) \ltimes \mathcal{L}G$ which are also "positive energy" in a certain sense. This notion effectively means that the actions respect the grading $\mathcal{H} = \oplus_{k \geqslant 0} \mathcal{H}_k$ coming from an S^1-action.

An even simpler illustrative case is that of "discrete series" representations of $\mathrm{PSL}_2(\mathbb{R})$. Embed $\mathrm{PSL}_2(\mathbb{R}) \hookrightarrow \mathrm{PSL}_2(\mathbb{C})$. Let $\mathrm{PSL}_2^{<}(\mathbb{C})$ be the subsemigroup of Möbius transformations sending the unit disk D to a proper subdisk of itself. Then $\mathrm{PSL}_2^{<}(\mathbb{C})$ acts by contraction on the discrete series representations.

There is a similar story for loop groups, if we consider the semigroup of holomorphic $f : D \to D$ with $f(D) \subset \mathring{D}$. The Kontsevich-Segal definition connects to this somehow (I didn't quite catch how).

1.2.3 Complex metrics

Definition 1.2.1. A *complex metric* on a real vector space V is a quadratic form $g : V \to \mathbb{C}$ such that, if λ_k are the eigenvalues of the matrix corresponding to g and $\lambda_k = e^{i\theta_k} |\lambda_k|$, then $\sum_i |\theta_i| < \pi$.

We can equivalently state this as

$$-\pi \leqslant \sum_i \pm\theta_i \leqslant \pi.$$

One may view the θ_i as weights of the corresponding spin representation.

For $v \in V$, we have $g(v) \in \mathbb{C} \setminus (-\infty, 0)$, so $g(v)$ has a canonical square root. Thus we may assign a "complex length" in the positive half-plane to every $v \in V$.

Given an inner product $V \times V \to \mathbb{C}$, we get an inner product on $\wedge^k V$ with squared norm given by $\alpha \wedge (\star\alpha)$. Really, this is better thought of as a map $\wedge^k V \times \wedge^k V \to \wedge^d V$, for $d = \dim V$.

We'll change topics now.

1.2.4 Application to QFT

If we want to view a QFT as a functor E defined on a category of cobordisms, we should require:

- Cobordisms should be equipped with complex metrics.
- The induced maps $U_M : E_{M_0} \to E_{M_1}$ should be holomorphic in M (viewed as living in some complex analytic moduli space of complex metrics).
- The operators U_M should be *trace class*: there should be bases ξ_i for E_{M_0} and η_i for E_{M_1} with $\xi_i \mapsto \lambda_i \eta_i$ and $\sum_i |\lambda_i| < \infty$.

The case of Lorentzian metrics occurs "on the boundary" of the moduli space of complex metrics. Thus usual Lorentzian QFTs can be viewed as holomorphic degenerations of nice QFTs.

Some simpler examples are similar in principle:

- $\mathrm{Diff}^+(S^1)$ can be viewed as the boundary of the semigroup of contraction mappings.
- $\mathrm{U}(n) \subset \mathrm{GL}_n(\mathbb{C})$ can be viewed as the boundary of the semigroup of contraction operators $\mathbb{C}^n \to \mathbb{C}^n$.

However, $\mathrm{PSL}_2(\mathbb{R})$ is not compact, so we can't view it as a boundary.

Viewing our spacetime M as Lorentzian, we can ask for M to be "globally hyperbolic," so every point in M_0 can be connected to a point in M_1 via a timelike curve. This is similar to requiring that there are no black holes. The results of Kontsevich-Segal are best in the case of globally hyperbolic metrics.

Heuristically, we are requiring the existence of a Dirac operator and requiring that this Dirac operator is a contraction operator.

Chapter 2

Nitu Kitchloo – Symmetry operators, surface defects, and link homologies

2.1 6/10 – Overview

2.1.1 Artin braid groups

Let G be a compact connected Lie group with maximal torus T. Recall that this means $\mathsf{T} \subset \mathsf{G}$ is a maximal connected compact abelian subgroup.

Example 2.1.1. For $\mathsf{G} = \mathrm{U}(n)$, we can take T to be the subgroup Δ of diagonal matrices. We will return to this example throughout.

Given (G, T), we can define an *Artin braid group* $\mathrm{Br}(\mathsf{G}, \mathsf{T})$.

Example 2.1.2. For $(\mathsf{G}, \mathsf{T}) = (\mathrm{U}(n), \Delta)$, this is the usual braid group

$$\mathrm{Br}(\mathsf{G}, \mathsf{T}) = \mathrm{Br}_n := \langle \sigma_1, \dots, \sigma_{n-1} \,|\, \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \quad \sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |i-j| > 1\rangle.$$

More generally, note that T acts on G (and its Lie algebra $\mathfrak{g}$) via the adjoint representation. Complexify $\mathfrak{g}$ to get a complex representation $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ of T. Let R be the set of roots of G, i.e. nonzero characters of T in $\mathfrak{g}_{\mathbb{C}}$. Let $\mathfrak{h} = \mathrm{Lie}(\mathsf{T}) \otimes_{\mathbb{R}} \mathbb{C}$, and let

$$\mathcal{H} = \mathfrak{h}_{\mathbb{C}} - \cup_{\alpha \in \mathbb{R}} \mathfrak{h}_\alpha \text{ for } \mathfrak{h}_\alpha = \{h \in \mathfrak{h}_{\mathbb{C}} \,|\, \alpha(h) = 0\}.$$

In other words, $\mathcal{H}$ is the union of the interiors of the Weyl chambers. Note that $\mathcal{H}$ has a free action of the Weyl group $W(\mathsf{G}, \mathsf{T}) = \mathsf{N}_{\mathsf{T}}(\mathsf{G})/\mathsf{T}$.

Definition 2.1.3. The *Artin braid group* $\mathrm{Br}(\mathsf{G}, \mathsf{T})$ is $\pi_1(\mathcal{H}/W(\mathsf{G}, \mathsf{T}))$.

It is a fact that $\mathcal{H}/W(\mathsf{G}, \mathsf{T})$ is a $\mathsf{K}(\mathrm{Br}(\mathsf{G}, \mathsf{T}), 1)$. Let's see why this works for $\mathsf{G} = \mathrm{U}(n)$.

Example 2.1.4. For $(\mathsf{G}, \mathsf{T}) = (\mathrm{U}(n), \Delta)$, we have

$$\cup_\alpha \mathfrak{h}_\alpha = \{(x_1, \dots, x_n) \,|\, x_i = x_j \text{ for some } i \neq j\}.$$

Since $W(\mathsf{G}, \mathsf{T}) = \Sigma_n$ acting on $\mathfrak{h}_{\mathbb{C}} = \mathbb{C}^n$ by permuting coordinates, we see that $\mathcal{H}/W(\mathsf{G}, \mathsf{T})$ is the moduli of n points in $\mathbb{C}$. This picture gives us the usual definition of braid groups.

Note that $\pi_0 \,\mathrm{Map}(\mathsf{S}^1, \mathcal{H}/W(\mathsf{G}, \mathsf{T}))$ is the space of conjugacy classes of elements in $\mathrm{Br}(\mathsf{G}, \mathsf{T})$, which we will write as $[\mathrm{Br}(\mathsf{G}, \mathsf{T})]$.

Example 2.1.5. For $(\mathsf{G}, \mathsf{T}) = (\mathrm{U}(n), \Delta)$, elements of $\mathrm{Br}(\mathsf{G}, \mathsf{T})$ are represented by braid closures. We may view these as links in $\mathbb{R}^3$ using Markov's theorem.

Theorem 2.1.6 (Markov)**.** *The map sending a braid to its closure expresses the set of equivalence classes of links in $\mathbb{R}^3$ as the set of equivalence classes of all braids, $(\sqcup_{n\geqslant 1}\mathrm{Br}_n)/\sim$, where $\sim$ is the equivalence relation generated by M1 and M2:*

1. *M1 (Conjugation):* $\sigma \sim \rho\sigma\rho^{-1}$ *for* $\sigma, \rho \in \mathrm{Br}_n$
2. *M2 (Stabilization):* $\sigma \sim \sigma\sigma_n^{\pm 1}$ *for* $\sigma \in \mathrm{Br}_n \subset \mathrm{Br}_{n+1}$

The individual maps from $[\mathrm{Br}_n]$ to the space of links are not faithful. That is, stabilization does cause us to lose some information.

2.1.2 Categorification

We'd like to categorify $\mathrm{Br}(G, T)$ and $[\mathrm{Br}(G, T)]$. In particular, we'd like a functor of ("discrete" / non-∞) 2-categories[1]

$$* /\!\!/ \mathrm{Br}(G, T) \to \mathrm{Gr}(\mathsf{Alg}).$$

Here $* /\!\!/ \mathrm{Br}(G, T)$ is the 2-category with:

- one object,
- $\mathrm{Br}(G, T)$ as 1-morphisms, and
- trivial 2-morphisms.

The right hand side, $\mathrm{Gr}(\mathsf{Alg})$, is the 2-category with:

- objects: graded commutative algebras,
- 1-morphisms: chain complexes of graded bimodules, and
- 2-morphisms: homotopy classes of chain maps

We also want a map $[\mathrm{Br}(G, T)] \to \mathsf{Ch}_{\bullet,\bullet}(\mathbb{Z})$.

This has been worked out by Soergel, Rouquier, Khovanov, and Khovanov-Rozansky. For $G = U(n)$, the last map above gives bigraded homological invariants of links. (This requires us to check that the stabilization move is satisfied.)

Being more ambitious, we can try to realize this using 3d and 4d topological gauge theories. This splits into subgoals:

1. Describe a configuration space of G-background fields on a 2-torus. (Here "background" means that we haven't "summed over these fields.")
2. Quantize by "summing over the background fields" to obtain the partition function of the 2-torus in 3d and 4d topological gauge theories with defects.
3. Identify the partition functions with the categorifications mentioned above.

2.1.3 Preliminaries: $U(n)$

We'll start out by looking at the case of $G = U(n)$. Let $R_n = H^{\bullet}_T(\mathrm{pt}) = \mathbb{Z}[x_1, \ldots, x_n]$ with $|x_i| = 2$. The Weyl group W is Σ_n, acting on R_n with σ_i swapping x_i and x_{i+1}.

Definition 2.1.7. The *Bott-Samelson* R_n*-bimodule* is the graded bimodule $\mathrm{bS}_i = R_n \otimes_{R_n^{\sigma_i}} R_n$, where $R_n^{\sigma_i}$ is the ring of σ_i-invariants in R_n. The category SBimod_n *Soergel bimodules* is the subcategory of R_n-bimodules generated by the bS_i's (allowing idempotent completion and grading shifts).

[1] Recent work extends this to a map of $(\infty, 2)$-categories.

Theorem 2.1.8 (Soergel)**.** *The category* SBimod_n *categorfies the Hecke algebra* $\mathsf{H}_q(A_n) = \mathbb{Z}[q, q^{-1}]\langle \mathrm{Br}_n \rangle / \mathrm{I}$*, where* I *is generated by the relations*

$$(\sigma_i - q^2)(\sigma_i + 1) = 0$$
$$\sigma_i^{-1} = q^{-2}\sigma_i + (q^2 - 1).$$

That is, there is an equivalence $K_0(\mathsf{SBimod}_n) \cong \mathsf{H}_q(A_n)$*. Under this equivalence:*

- 1 *corresponds to* R_n*,*
- bS_i *corresponds to* $\sigma_i + 1$*, and*
- q *corresponds to grading shift.*

The chain complex $C_i = (\mathrm{bS}_i \to R_n)$, where the map is given by $\alpha \otimes \beta \mapsto \alpha\beta$, should be viewed as a categorification of σ_i.

Theorem 2.1.9 (Rouquier)**.** *The assignment* $\sigma_I \mapsto c_I := C_{i_1} \otimes_{R_n} C_{i_2} \otimes_{R_n} \cdots \otimes_{R_n} C_{i_k}$ *gives a well-defined map* $* /\!\!/ \mathrm{Br}_n \to \mathrm{Gr}(\mathsf{Alg})$.

Example 2.1.10. The braid $\sigma_1\sigma_2$ is sent to the chain complex $C_{12} = (\mathrm{bS}_1 \to R_3) \otimes (\mathrm{bS}_2 \to R_3)$, which can also be written

$$\mathrm{bS}_1 \otimes_{R_3} \mathrm{bS}_2 \longrightarrow \mathrm{bS}_1 \oplus \mathrm{bS}_2 \longrightarrow R_3.$$

Rouquier's theorem is nontrivial for two reasons:

- We need to show that the map, which *a priori* is defined only on braid words, is actually defined on the braid group.
- We need to make sure that this respects the triviality of the higher morphisms in $* /\!\!/ \mathrm{Br}_n$.

2.2 6/10 – Geometric interpretations for the $\mathrm{U}(n)$ case

Last time, we discussed a categorification of Br_n. We'd like to obtain a corresponding categorification of $[\mathrm{Br}_n]$.

2.2.1 Categorification of $[\mathrm{Br}_n]$

Recall the definition of Hochschild homology.

Definition 2.2.1. If M is a graded R_n-bimodule, then the *Hochschild homology* of M is

$$\mathsf{HH}(M) = \mathrm{Tor}^{\bullet;\bullet}_{R_n \otimes R_n^{\mathrm{op}}}(M, R_n)$$

We will consider the Hochschild homology $\mathsf{HH}(C_I)$ termwise.

Example 2.2.2. For $\sigma_1\sigma_2$, we discussed C_{12} in the last lecture. With the above convention, $\mathsf{HH}(C_{12})$ is

$$\mathsf{HH}(\mathrm{bS}_1 \otimes_{R_3} \mathrm{bS}_2) \longrightarrow \mathsf{HH}(\mathrm{bS}_1 \oplus \mathrm{bS}_2) \longrightarrow \mathsf{HH}(R_3).$$

Theorem 2.2.3 (Khovanov)**.** *The assignment* $\sigma_I \mapsto \mathsf{HH}(C_I)$ *extends to* $[\mathrm{Br}_n] \to \mathsf{Ch}_{\bullet,\bullet}(\mathbb{Z})$*. In fact, the output is naturally trigraded. Furthermore, this assignment satisfies M1 and M2, so it gives a homological invariant of links.*

This invariant can be identified with the trigraded Khovanov-Rozansky link homology. If $\mathcal{L}$ is the link $[\sigma_i]$, we write

$$\mathfrak{sl}_\infty(\mathcal{L}) = \mathsf{HHH}(\mathcal{L}) = \mathsf{HH}(C_I).$$

We may view $\mathfrak{sl}_\infty$ as a "limit" of bigraded homological invariants $\mathfrak{sl}_N(\mathcal{L})$ which can be obtained from $\mathsf{HH}(C_I)$ by adding a "matrix factorization" differential.

2.2.2 Topological interpretation of Bott-Samelson bimodules

Definition 2.2.4. For $i < n$, let $G_i \subset U(n)$ be the semisimple rank one compact parabolic consisting of matrices of the form

$$\begin{bmatrix} D_1 & 0 & 0 \\ 0 & U & 0 \\ 0 & 0 & D_2 \end{bmatrix}$$

with D_1 and D_2 diagonal and $U \in U(2)$ (placed in rows / columns i and $i+1$).

Note that $T \subset G_i \subset G = U(n)$ for all i.

Definition 2.2.5. Let BS_i be the balanced product $U(n) \times^T (G_i/T)$, considered with its natural $U(n)$-action.

We may obtain BS_i from a pullback square

$$\begin{tikzcd} BS_i \arrow[r, "\alpha"] \arrow[d, "\beta"] & U(n)/T \arrow[d] \\ U(n)/T \arrow[r] & U(n)/G_i \end{tikzcd}$$

where $\alpha : [g, g_i] \mapsto [gg_i]$ and $\beta : [g, g_i] \mapsto [g]$.

In cohomology, we get a map

$$H^\bullet_{U(n)}(U(n)/T) \otimes_{H^\bullet_{U(n)}(U(n)/G_i)} H^\bullet_{U(n)}(U(n)/T) \to H^\bullet_{U(n)}(BS_i).$$

A spectral sequence computation shows this is an isomorphism. Combining this with the standard isomorphisms $H^\bullet_G(G/K) \cong H^\bullet_K(pt)$ for any G and any $K \subset G$, we get

$$H^\bullet_{U(n)}(BS_i) \cong H^\bullet_T(pt) \otimes_{H^\bullet_{G_i}(pt)} H^\bullet_T(pt) \cong R_n \otimes_{R_n^{\sigma_i}} R_n = bS_i.$$

Thus, we can interpret the Bott-Samelson bimodule bS_i geometrically as the $U(n)$-equivariant cohomology of BS_i.

The pullback square defining BS_i is in fact a homotopy pullback square. Comparing with the standard definition of the homotopy pullback, we can view BS_i as the moduli space of $U(n)$-bundles on $[0,1]$ with a reduction of structure group to G_i on $[0,1]$ and to T on $\{0,1\}$.

2.2.3 Interpretation of Rouquier's chain complex

More generally, for a multi-index $I = (i_1, \dots, i_k)$, define

$$BS_I = U(n) \times^T (G_{i_1} \times^T G_{i_2} \times^T \cdots \times^T G^{i_k}/T).$$

Then

$$H^\bullet_{U(n)}(BS_I) \cong bS_{i_1} \otimes_{R_n} bS_{i_2} \otimes_{R_n} \cdots \otimes_{R_n} bS_{i_k}.$$

We may view BS_I as the moduli space of $U(n)$-bundles on $[0,1]$ with $k+1$ marked points (including 0 and 1) with reduction of structure group to G_{i_j} on the jth interval and to T at each marked point.

If J is obtained from I by discarding indices, then we get $BS_J \hookrightarrow BS_I$ by replacing every factor G_{i_k} (for each removed index i_k) by T in the construction of BS_I. These replaced copies of T are "transparent defects." This inclusion respects composition of inclusions of multi-indices.

Rouquier's chain complex C_I is isomorphic to

$$H^\bullet_{U(n)}(BS_I) \longrightarrow \oplus_{|J|=k-1, J\subset I} \longrightarrow \dots \longrightarrow H^\bullet_{U(n)}(U(n)/T).$$

All differentials here are obtained from (signed) inclusions $BS_J \to BS_{J'}$.

2.2.4 Interpretation of $\mathsf{HH}(\mathrm{bS}_I)$

To interpret $\mathsf{HH}(\mathsf{H}^\bullet_{U(n)}(\mathrm{BS}_I)$, we introduce the following.

Definition 2.2.6. Let

$$\mathsf{L}_i = U(n) \times^{\mathsf{T}} \mathsf{G}_i$$

where T acts on G_i by conjugation.

The map $\mathsf{L}_i \to \mathrm{BS}_i$ given by $[g, g_i] \mapsto [g, g_i]$ is a principal T-bundle. In fact, we have a pullback square

$$\begin{tikzcd}
\mathsf{EU}(n) \times^{U(n)} \arrow[r] \arrow[d] & \mathsf{BT} \arrow[d, "\Delta"] \\
\mathsf{EU}(n) \times^{U(n)} \mathrm{BS}_i \arrow[r] & \mathsf{BT} \times \mathsf{BT}
\end{tikzcd}$$

where both vertical arrows are trivial principal bundles.

Collapse of the Eilenberg-Moore spectral sequence at the E_2 page gives

$$\mathsf{H}^\bullet_{U(n)}(\mathsf{L}_i) \cong \mathrm{Tor}_{\mathsf{H}^\bullet(\mathsf{BT}\times\mathsf{BT})}(\mathsf{H}^\bullet_{U(n)}(\mathrm{BS}_i), \mathsf{H}^\bullet(\mathsf{BT})) = \mathsf{HH}(\mathsf{H}^\bullet_{U(n)}(\mathrm{BS}_i)) = \mathsf{HH}(\mathrm{bS}_i).$$

Thus we obtain a geometric interpretation of $\mathsf{HH}(\mathrm{bS}_i)$. We may interpret L_i as the moduli space of $U(n)$-bundles on S^1 with one marked point, where the structure group reduces to G_i away from the marked point and to T at the marked point.

More generally, for a multi-index I, we let

$$\mathsf{L}_I = U(n) \times^{\mathsf{T}} (\mathsf{G}_{i_1} \times^{\mathsf{T}} \mathsf{G}_{i_2} \times^{\mathsf{T}} \cdots \times^{\mathsf{T}} \mathsf{G}_{i_k})$$

where T acts by conjugation on G_{i_1} and G_{i_k}. An analogous argument to the above gives $\mathsf{H}^\bullet_{U(n)}(\mathsf{L}_I) \cong \mathsf{HH}(\mathrm{bS}_I)$.

For $\mathsf{J} \subset I$, we get an inclusion $\mathsf{L}_\mathsf{J} \to \mathsf{L}_I$ that behaves well with respect to composition of inclusions. These assemble into a complex

$$\mathsf{H}^\bullet_{U(n)}(\mathsf{L}_I) \longrightarrow \oplus_{|\mathsf{J}|=k-1, \mathsf{J}\subset I}(\mathsf{L}_\mathsf{J}) \longrightarrow \dots$$

which is isomorphic to $\mathsf{HH}(\mathsf{C}_I)$.