

# Notes from GCS 2024 Summer School

June 10-14, 2024

### **Abstract**

These are my notes from the “Categorical Symmetries in Quantum Field Theory” summer school, held at the University of Edinburgh from June 10-14, 2024. Lectures were given by Graeme Segal, Nitu Kitchloo, Ibou Bah, Claudia Scheimbauer, and Ryan Thorngren. All errors are my own.

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# Chapter 1

## Graeme Segal – A Perspective on Quantum Field Theory

### 1.1 6/10 – Overview of QFT

#### 1.1.1 From classical to quantum

In classical mechanics, one focuses one's attention on a fixed space  $M_0$  and obtains a configuration space  $X$  from that. Dynamics are specified by giving a Lagrangian  $L : TX \rightarrow \mathbb{R}$ , which we assume is positive, inhomogeneous, and quadratic on tangent vectors. Under these hypotheses, the Lagrangian can be viewed as a generalized Riemannian metric, so that time evolution is given by a (generalized) “geodesic flow.” This gives rise to a symplectic form  $\omega$  and Poisson bracket  $\{-, -\}$  on  $T^*X$ . Furthermore, from the Lagrangian  $L$ , we may extract a Hamiltonian  $H : T^*X \rightarrow \mathbb{R}$  such that

$$\frac{d}{dt}f = \{H, f\} \quad \forall f \in \mathcal{C}^\infty(X).$$

Thus we may view a classical dynamical system as a triple  $(Y, \omega, H)$  with  $(Y, \omega)$  a symplectic manifold and  $H : Y \rightarrow \mathbb{R}$  a function.

Quantum mechanics can be viewed as a complex noncommutative analogue of this. We replace  $(Y, \omega, H)$  by a triple  $(\mathcal{A}, \star, H)$  with  $\mathcal{A}$  a (topological)  $\mathbb{C}$ -algebra,  $\star$  an antilinear involution on  $\mathcal{A}$ , and  $H \in \mathcal{A}$ . For an observable  $f \in \mathcal{A}$ , we require

$$\frac{d}{dt}f = i\hbar[H, f].$$

If the quantum system  $(\mathcal{A}, \star, H)$  is a “quantization” of a classical system  $(Y, \omega, H)$ , then there is a deep relationship between the two. For example, we should view  $\mathcal{A}$  as a completion / extension of  $\mathcal{C}^\infty(Y)$ , and we can relate the values of an observable to the eigenvalues of the corresponding operator.

The above picture privileges the time dimension, so it is fundamentally non-relativistic. Working relativistically requires us to move from finite-dimensional spaces to infinite-dimensional spaces. Heuristically, we take  $Y$  to be our space of fields on spacetime (typically  $M = M_0 \times \mathbb{R}$ ), and we fix a Lagrangian  $L : Y \rightarrow \mathbb{R}$ . However,  $\mathcal{A}$  should now be viewed as a space of functions on an “almost finite-dimensional manifold.” This requires many corrections!

**Example 1.1.1.** Consider scalar field theory, with  $M = M_0 \times \mathbb{R}$  and  $Y = \mathcal{C}^\infty(M)$ . Then  $\mathcal{A}$  looks like functions on a stratified configuration space

$$\coprod_{n \geq 0} \text{Conf}_n(M_0),$$

where  $\text{Conf}_n(M_0)$  is the space of sets of  $n$  distinct unordered points on  $M_0$ .

### 1.1.2 Approaching quantum field theory

Heuristically, we should “spread things out over spacetime” by attaching an algebra  $\mathcal{O}_x$  to every point  $x \in M$  and letting

$$\mathcal{A} \approx \bigotimes_{x \in M} \mathcal{O}_x.$$

This has been formalized in the approach of algebraic quantum field theory – see e.g. the definition of Haag, which assigns an algebra  $\mathcal{A}_U$  to each open set  $U \subset M$ .

An alternative approach, which we will pursue, privileges the time dimension. In this approach, we define a  $d$ -dimensional quantum field theory as a symmetric monoidal functor

$$E : \mathbf{Cob}_{d-1} \rightarrow \mathbf{Vect}.$$

Here  $\mathbf{Cob}_{d-1}$  is the symmetric monoidal category with:

- Objects:  $(d-1)$ -manifolds, often assumed compact or compact with boundary (and thought of as time slices)
- Morphisms  $M_0 \rightarrow M_1$ :  $d$ -dimensional cobordisms from  $M_0$  to  $M_1$  (thought of as controlling time evolution)
- Monoidal structure: disjoint union of manifolds / cobordisms

and  $\mathbf{Vect}$  is the symmetric monoidal category of (possibly infinite-dimensional) topological vector spaces (with  $\otimes$  as tensor product).

There are many other ways to view quantum field theory. For example, one could think of QFT as a continuum limit of lattice models.

QFT is traditionally seen as describing “the world except for gravity.” That is, we think of spacetime as fixed, without allowing for the effects of gravity. However, QFT can tell us about what sorts of gravitational effects are possible.

The perspective on QFT that we will follow has the advantage that it allows for direct comparisons between QFTs. Thus, in principle, we may consider a “moduli space” of QFTs (of a given type). We can think of gravitational effects as acting upon this moduli space.

### 1.1.3 The perspective of Connes

We’d like to use geometry to understand the noncommutative world of QFT. This can be accessed by studying the spectrum of our Hamiltonian  $H$ . Operators which evolve slowly in time “nearly commute” with  $H$  and thus must be “nearly diagonal.” We’re mostly interested in studying such operators (as the time scale of humans is much slower than that of the universe). Thus, when studying QFT, we’re interested in algebras which are “nearly commutative.”

Connes was interested in finding a mildly noncommutative generalization of algebras of functions on a manifold  $M$ . Recall that we can define the Clifford algebra  $\text{Cliff}(T_m M)$  as the algebra with generators  $\{\gamma_\xi\}_{\xi \in T_m M}$  and relations  $\gamma_\xi^2 = -\|\xi\|^2$ . Let  $\mathcal{B}$  be a bundle of finite-dimensional algebras on  $M$  containing  $\text{Cliff}(T_m M)$ , and take a connection on  $\mathcal{B}$ . Let  $\mathcal{D}$  be the Dirac operator

$$\mathcal{D} = \sum_i \gamma_{\xi_i} \otimes \frac{\partial}{\partial x_i},$$

and let  $H = \mathcal{D}^2$ . Connes gave a formula for  $\text{tr } \mathcal{D}^2$  and extracted interesting physical objects, e.g. the Higgs field, from  $\mathcal{D}$ . From Connes’ perspective, the Dirac operator is “as good as” the metric.

### 1.1.4 Algebra from functorial QFT

Suppose we view spacetime  $M$  as a cobordism from  $M_0$  to  $M_1$ . Fix a  $d$ -dimensional quantum field theory  $E$ , where  $E = \dim M$ . For  $x \in M$ , we can define a vector space of operators  $\mathcal{O}_x$  by taking a small disk  $D_x$  about  $x$  and letting  $\mathcal{O}_x = E_{\partial D_x}$ . We can view a punctured copy of  $M$  as giving a cobordism

$$M \sqcup (\sqcup_i \partial D_{x_i}) \rightarrow M_1,$$

so that applying  $E$  gives  $E_{M_0} \otimes (\otimes_i \mathcal{O}_{x_i}) \rightarrow E_{M_1}$ . This gives a sort of “higher multiplication” on the spaces of operators  $\mathcal{O}_{x_i}$ .

We can make this much more precise in special cases. For example, if  $E$  is topological, then  $\mathcal{O}_{x_i} = \mathcal{O}$  is independent of  $x_i$ . Taking  $M_0 = \emptyset$  and  $M_1 = S^{d-1}$ , we obtain a family of multiplications  $\mathcal{O}^{\otimes n} \rightarrow \mathcal{O}$ .

**Example 1.1.2.** If  $d = 2$ , the cap / cup / pair-of-pants cobordisms equip  $A = \mathcal{O}$  with the structure of a finite-dimensional commutative Frobenius algebra. This means that  $A$  is a unital algebra with a nondegenerate trace  $\theta : A \rightarrow \mathbb{C}$ .

It is interesting to extend this picture to non-topological 2-dimensional QFTs. Here, one can set up a moduli space of QFTs and obtain a gravitational flow. This was a major historical motivation for string theory.

The second lecture will discuss the importance of positive energy to this theory. The third lecture will discuss scaling, and the fourth lecture will focus on finding a suitable definition for **Cob**.

## 1.2 6/10 – Positive Energy

### 1.2.1 Warmup

Consider a quantum mechanical system with Hilbert space  $\mathcal{H}$ . The Hamiltonian is a self-adjoint operator  $H : \mathcal{H} \rightarrow \mathcal{H}$ , and we can integrate  $iH$  to get unitary operators  $U_t = e^{iHt} : \mathcal{H} \rightarrow \mathcal{H}$  describing the time evolution of the system.

Assuming  $H \geq 0$  has strong consequences for the theory. In this case,  $U_- : \mathbb{R} \rightarrow \text{End}(\mathcal{H})$  is the boundary value of the function on the complex upper half-plane also given by  $t \mapsto e^{iHt}$ .

To fit this into QFT, consider  $\mathbb{C}$  as a bundle over  $\mathbb{R}$ , where the inner product on the base is Lorentzian and the inner product on the fiber is Riemannian. Think of  $\mathbb{R}$  as our spacetime  $M$  and the fiber as  $V = T_x M$ .

We will introduce the notion of a *complex metric* on a real vector space  $V$ . This is a quadratic form  $V \rightarrow \mathbb{C}$  satisfying a certain condition to be determined later.

### 1.2.2 Historical digression

The notion of a complex metric was introduced by Kontsevich and Segal. Segal was originally interested in loop groups  $\mathcal{L}G = \text{Map}(S^1, G)$ . There is a nice class of “positive energy” representations of loop groups. These extend to representations of  $\text{Diff}^+(S^1) \ltimes \mathcal{L}G$  which are also “positive energy” in a certain sense. This notion effectively means that the actions respect the grading  $\mathcal{H} = \oplus_{k \geq 0} \mathcal{H}_k$  coming from an  $S^1$ -action.

An even simpler illustrative case is that of “discrete series” representations of  $\text{PSL}_2(\mathbb{R})$ . Embed  $\text{PSL}_2(\mathbb{R}) \hookrightarrow \text{PSL}_2(\mathbb{C})$ . Let  $\text{PSL}_2^<(\mathbb{C})$  be the subsemigroup of Möbius transformations sending the unit disk  $D$  to a proper subdisk of itself. Then  $\text{PSL}_2^<(\mathbb{C})$  acts by contraction on the discrete series representations.

There is a similar story for loop groups, if we consider the semigroup of holomorphic  $f : D \rightarrow D$  with  $f(D) \subset \overset{\circ}{D}$ . The Kontsevich-Segal definition connects to this somehow (I didn’t quite catch how).

### 1.2.3 Complex metrics

**Definition 1.2.1.** A *complex metric* on a real vector space  $V$  is a quadratic form  $g : V \rightarrow \mathbb{C}$  such that, if  $\lambda_k$  are the eigenvalues of the matrix corresponding to  $g$  and  $\lambda_k = e^{i\theta_k} |\lambda_k|$ , then  $\sum_i |\theta_i| < \pi$ .

We can equivalently state this as

$$-\pi \leq \sum_i \pm \theta_i \leq \pi.$$

One may view the  $\theta_i$  as weights of the corresponding spin representation.

For  $v \in V$ , we have  $g(v) \in \mathbb{C} \setminus (-\infty, 0)$ , so  $g(v)$  has a canonical square root. Thus we may assign a “complex length” in the positive half-plane to every  $v \in V$ .

Given an inner product  $V \times V \rightarrow \mathbb{C}$ , we get an inner product on  $\wedge^k V$  with squared norm given by  $\alpha \wedge (\star \alpha)$ . Really, this is better thought of as a map  $\wedge^k V \times \wedge^k V \rightarrow \wedge^d V$ , for  $d = \dim V$ .

We’ll change topics now.

### 1.2.4 Application to QFT

If we want to view a QFT as a functor  $E$  defined on a category of cobordisms, we should require:

- Cobordisms should be equipped with complex metrics.
- The induced maps  $U_M : E_{M_0} \rightarrow E_{M_1}$  should be holomorphic in  $M$  (viewed as living in some complex analytic moduli space of complex metrics).
- The operators  $U_M$  should be *trace class*: there should be bases  $\xi_i$  for  $E_{M_0}$  and  $\eta_i$  for  $E_{M_1}$  with  $\xi_i \mapsto \lambda_i \eta_i$  and  $\sum_i |\lambda_i| < \infty$ .

The case of Lorentzian metrics occurs “on the boundary” of the moduli space of complex metrics. Thus usual Lorentzian QFTs can be viewed as holomorphic degenerations of nice QFTs.

Some simpler examples are similar in principle:

- $\text{Diff}^+(S^1)$  can be viewed as the boundary of the semigroup of contraction mappings.
- $U(n) \subset GL_n(\mathbb{C})$  can be viewed as the boundary of the semigroup of contraction operators  $\mathbb{C}^n \rightarrow \mathbb{C}^n$ .

However,  $PSL_2(\mathbb{R})$  is not compact, so we can’t view it as a boundary.

Viewing our spacetime  $M$  as Lorentzian, we can ask for  $M$  to be “globally hyperbolic,” so every point in  $M_0$  can be connected to a point in  $M_1$  via a timelike curve. This is similar to requiring that there are no black holes. The results of Kontsevich-Segal are best in the case of globally hyperbolic metrics.

Heuristically, we are requiring the existence of a Dirac operator and requiring that this Dirac operator is a contraction operator.

## 1.3 6/11 – Desired Properties of QFT; Scaling

In his first lecture, Ibou Bah proposed five properties that a (nice) QFT should satisfy: *unitarity*, *locality*, *causality*, *cluster decomposition*, and *renormalization*. We will first try to understand how these relate to Segal’s perspective.

### 1.3.1 Renormalization

Historically, understanding renormalization was crucial to the development of QFT. This is somewhat arcane from a mathematical point of view. However, one should note that renormalizability of a Lagrangian does have a natural relationship to the calculus of variations.

Consider a  $\sigma$ -model where our fields are maps  $M \rightarrow N$ . If  $M$  and  $N$  have Riemannian metrics, we can define an energy functional. For  $M = \mathbb{R}$  or  $S^1$ , this recovers the usual energy functional on paths. This is well-behaved and has a good Morse theory. When  $\dim M = 2$ , this is almost well-behaved, except there are problems with bubbling. For  $\dim M > 2$ , things are terrible.

The upshot is that (at least in this case?) renormalizability happens exactly when we expect things to be mathematically well-behaved.

### 1.3.2 Unitarity

If  $M$  is a cobordism from  $M_0$  to  $M_1$ , we can reverse orientations on the boundary and view  $\overline{M}$  as a cobordism from  $\overline{M}_1$  to  $\overline{M}_0$ . When  $M$  is Riemannian, this doesn't significantly affect things. However, when  $M$  is Lorentzian, this changes the direction of time.

If  $U_M : \mathcal{H}_{M_0} \rightarrow \mathcal{H}_{M_1}$  is the induced map, then we can think of  $U_{\overline{M}} : \mathcal{H}_{\overline{M}_1} \rightarrow \mathcal{H}_{\overline{M}_0}$  as suitably “adjoint” to  $U_M$ . More precisely, we can write  $\mathcal{H}_{M_i}$  and  $\mathcal{H}_{\overline{M}_i}$  as duals, and this connects  $U_M$  to  $U_{\overline{M}}$ . We will assume that  $U_{\overline{M}} = \overline{U_M}$ . Reflection-positivity turns into the assumption that the pairing  $\mathcal{H}_{M_0} \otimes \mathcal{H}_{\overline{M}_0} \rightarrow \mathbb{C}$  is positive (at least in nice cases?).

### 1.3.3 Locality

Given a point  $x \in M$ , we can view point operators as being defined on sufficiently small disks  $D$  about  $x$ . This gives algebras  $\mathcal{O}_x = \lim_{x \in D} E_D$ , where  $E$  is the QFT. In the first lecture here, we discussed how this relates to operadic multiplication. This is quite similar to the language of operator product expansion used in physics.

### 1.3.4 Cluster decomposition

Saying that two events are a long time apart is essentially saying that there is a long cylinder between them in spacetime. This can also be viewed as factorizing the cobordism defining spacetime into a product involving said long cylinder. Recall that the long cylinder in spacetime corresponds to time evolution  $e^{-iHt}$  in the QFT.

### 1.3.5 Causality

Consider a Lorentzian spacetime  $M$  with boundaries  $M_0$  and  $M_1$ . Placing operations at points  $x$  in the interior of  $M$  gives operations  $\mathcal{O}_x \otimes E_{M_0} \rightarrow E_{M_1}$ .

We assume that our spacetime is globally hyperbolic, so every point in  $M_0$  can be connected to a point in  $M_1$  via a timelike path. This splits  $M$  into a family of time slices (all diffeomorphic to each other). Given distinct points  $x_i$  on the same time slice (so the  $x_i$  are spacelike separated), causality ensures that we can define

$$\mathcal{O}_{x_1, \dots, x_n} \otimes \check{E}_{M_0} \rightarrow \hat{E}_{M_1},$$

where  $\check{\phantom{x}}$  denotes a dense subspace and  $\hat{\phantom{x}}$  denotes a completion. This product is independent of the ordering. Using global hyperbolicity, we can extend this to any collection of spacelike separated points.

The notion of “small disk” makes complete sense in a Riemannian metric, but it seems a bit odd in a Lorentzian metric. To make sense of this, we recall our notion of complex metric  $g : V \rightarrow \mathbb{C}$ . If we pass to the complexification, we get a standard quadratic form  $V_{\mathbb{C}} \rightarrow \mathbb{C}$ . Thus we may consider  $V$  as a subspace of  $\mathbb{C}^d \cong V_{\mathbb{C}}$ . The space of allowable complex metrics can be viewed as a contractible subspace of the Grassmannian of real subspaces of  $V$ . One can make sense of the algebras  $\mathcal{O}_x$  from this perspective.

### 1.3.6 Scaling

Let's return to the originally proposed topic of this lecture. Many things we'd like to prove about QFT can't be proved from the axioms we've given.

Viewing our spacetime  $M$  as a cobordism from  $M_0$  to  $M_1$ , we should view a scalar field as a compactly-supported density on  $M$ . Such a scalar field should give an operator  $E_{M_0} \rightarrow E_{M_1}$ . This transformation  $\text{Dens}_c(M) \rightarrow \text{Hom}(E_{M_0}, E_{M_1})$  can be called a *Wightman field*. It's not clear how to relate this to the bundle of algebras  $\mathcal{O}$  we've been discussing earlier.

We also don't really know what to do about Lagrangians. From our definition, we can make sense of a moduli space of theories, and we get a renormalization group flow on this moduli space (by rescaling metrics). Reconciling our notion of fields (using this space) should help us make sense of Lagrangians. (It's not clear how global symmetries fit into this description.)



We should be able to fix these with a scaling axiom (defined in Riemannian signature). This should be something saying that our spaces of local operators  $E_{(r)}$  (defined on disks of radius  $r$ ) tend to a limit  $E_{(0)}$ . This would provide a filtration on the spaces  $\mathcal{O}_x$  with associated graded  $E_{(0)}$ . Using this should allow us to relate  $\mathcal{O}_x$  with densities.

Extending our theories to codimension 2 should also help us prove things.

## 1.4 6/12 – More Desired Properties of QFT; Connection to Homotopy Theory

### 1.4.1 More on scaling

The notion of scaling seems quite common in physical treatments of QFT but is largely absent from mathematical treatments of the same.

Segal began by studying 2d CFT to understand T-duality. One version of this identifies a sigma-model with torus target  $T$  and another sigma-model with dual torus target  $T^*$ . This isomorphism isn't compatible with scaling: it interchanges short-distance and long-distance aspects of the theories.

### 1.4.2 Locality

Suppose that our theory sends a  $(d-1)$ -manifold  $M$  to a (topological) vector space  $E_M$ . To say that our theory is local, we'd like to say that  $E_M$  can be constructed "locally on  $M$ ." More precisely, if  $M = M_1 \cup_{\Sigma^{d-2}} M_2$ , we'd like to construct  $E_M$  from  $E_{M_1}$ ,  $E_{M_2}$ , and " $E_{\Sigma^{d-2}}$ ." A naïve guess, ignoring the unknown  $E_{\Sigma}$ , fails: we typically have  $E_{M_1} \otimes E_{M_2} \neq E_M$ . To understand this further, let's consider an example.

**Example 1.4.1.** Let  $G$  be a compact Lie group and  $\mathcal{L}G$  the loop group of  $G$ . Assume  $E_M$  is an irrep of  $\mathcal{L}G$ . If we think of  $M$  as an interval with glued endpoints, then considering paths from any subinterval to  $G$  gives rise to a dense subspace of  $E_M$ . It's not clear how to combine such subspaces to get  $E_M$ .

In our case  $M = M_1 \cup_{\Sigma^{d-2}} M_2$ , we can construct  $E_M$  as "the space of operators on fields on all of  $M$ ." The algebras of local operators  $\mathcal{A}_{M_1}$  and  $\mathcal{A}_{M_2}$  acts on  $E_M$ . If we enlarge  $M_1$  and  $M_2$  slightly past  $\Sigma$  (to get  $M_1^+, M_2^+ \subset M$ ), we can get good results. Specifically, if  $\mathcal{A}_{M_1}^{\text{comm}}$  is the commutant of  $\mathcal{A}_{M_1}$  acting on  $E_M$ , then  $\mathcal{A}_{M_1}^{\text{comm}}$  contains  $\mathcal{A}_{M_2}$  and  $\mathcal{A}_{M_1^+}$ . The Tomita-Takesaki theorem tells us that  $\mathcal{A}_{M_1}^{\text{comm}}$  is the von Neumann algebra closure of  $\mathcal{A}_{M_1}$  (?). Combining this with the action of  $\mathcal{A}_{M_2}$  allows us to recover  $E_M$  from the behavior of the field theory  $M_1$  and  $M_2$ . Here  $\Sigma$  gives us a bimodule relating  $\mathcal{A}_{M_1}$  and  $\mathcal{A}_{M_2}$ .

**Example 1.4.2.** Returning to loop groups, suppose we decompose the interval  $I = I_1 \cup_{\text{pt}} I_2$ . Consider the algebras of loops supported on each subinterval. To glue these together, we need to be able to extend these slightly across pt. This corresponds to a bimodule relating the algebras. In functional analysis, this corresponds to the "Tomita-Takesaki flow." This is related to the action of Poincaré boosts in special relativity.

### 1.4.3 Non-commutative geometry

We can reconstruct a manifold  $M$  from the behavior of modules over  $\mathcal{C}^\infty(M)$ . More generally, in non-commutative geometry, we study the category of modules over an algebra  $A$ . We say that Morita-equivalent algebras have the same geometry.

For example, we can study the K-theory of  $A$ . When  $A = \mathcal{C}^\infty(M)$ , the Chern character isomorphism lets us write

$$K(A) \otimes \mathbb{Q} \cong \bigoplus_i H^{2i}(M; \mathbb{Q}).$$

The grading here relies on the existence of Adams operations (based on  $E \mapsto \wedge^n E$ , at least for line bundles).

In the non-commutative context, we don't have Adams operations. Instead, there's a formal group law, and we get lots of strangeness governed by chromatic homotopy theory. From a QFT perspective, the grading relies on some sort of scaling.

Rings of  $\mathcal{C}^\infty$  functions also experience a sort of Bott periodicity. Consider the Schwartz space  $\mathcal{C}_S^\infty(\mathbb{R}^2)$  of functions decaying sufficiently rapidly at infinity. We can construct a non-commutative deformation  $\mathcal{C}_S^\infty(\mathbb{R}^2)_{(\hbar)}$ , which is isomorphic (via Fourier transformation) to the algebra of “smoothing operators” from  $\mathcal{C}^\infty(\mathbb{R})$  to itself. By “smoothing operator,” we mean a smooth function  $K(x, y)$ , viewed as an integral transform

$$(Kf)(x) = \int f(y)K(x, y)dy.$$

This algebra is Morita equivalent to  $\mathbb{C}$  (?), and deforming again recovers  $\mathcal{C}_S^\infty(\mathbb{R}^2)$  (?).

More generally, we can relate the geometry of  $X$  with a compactified geometry of  $X \times \mathbb{R}^2$ . This ends up producing Floer homotopy types.

#### 1.4.4 The smooth homotopy category

Long-distance behavior in QFT looks somewhat like homotopy theory. Short-distance behavior in QFT is a bit more open to debate. Many people like to think about lattices. However, Segal prefers to think about the short-distance behavior in terms of (rings of functions of) manifolds.<sup>1</sup>

We can ask how to represent QFTs geometrically. Classically, Brown’s representability theorem lets us view homotopy theory as captured by contravariant functors from the homotopy category of finite CW complexes to sets. The “smooth homotopy category” replaces sets here by manifolds (and interprets everything in terms of suitable model categories).

**Example 1.4.3.** Given a Lie algebra  $\mathfrak{g}$ , we can view the smooth homotopy type of  $\mathfrak{g}$  as given by sending  $M$  to the moduli space of flat  $\mathfrak{g}$ -connections on the trivial bundle on  $M$ . Similarly, given a Lie group  $G$ , we can view the smooth homotopy type of  $G$  as given by sending  $M$  to the moduli space of flat  $G$ -bundles on  $M$ .

There’s a paper of Kapranov which interprets the Lie algebra of a based loop group  $\Omega X$  in this context. Curvature is captured by maps  $\Omega X \rightarrow G$ .

Classical homotopy theory is “perturbative:” it is built one step at a time from sets (by adding homotopies, higher homotopies, etc.). Analysis is also “perturbative:” we take higher derivatives one at a time. However, the theories are “perturbative” in different directions: homotopy theory gives the “long-distance perturbative theory” while analysis gives “short-distance perturbative theory.”

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<sup>1</sup>This may be more a matter of human taste in describing the universe than a fundamental fact of nature.

## Chapter 2

# Nitu Kitchloo – Symmetry Breaking, Surface Defects, and Link Homologies

### 2.1 6/10 – Overview

#### 2.1.1 Artin braid groups

Let  $G$  be a compact connected Lie group with maximal torus  $T$ . Recall that this means  $T \subset G$  is a maximal connected compact abelian subgroup.

**Example 2.1.1.** For  $G = U(n)$ , we can take  $T$  to be the subgroup  $\Delta$  of diagonal matrices. We will return to this example throughout.

Given  $(G, T)$ , we can define an *Artin braid group*  $\text{Br}(G, T)$ .

**Example 2.1.2.** For  $(G, T) = (U(n), \Delta)$ , this is the usual braid group

$$\text{Br}(G, T) = \text{Br}_n := \langle \sigma_1, \dots, \sigma_{n-1} \mid \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \quad \sigma_i \sigma_j = \sigma_j \sigma_i \text{ for } |i - j| > 1 \rangle.$$

More generally, note that  $T$  acts on  $G$  (and its Lie algebra  $\mathfrak{g}$ ) via the adjoint representation. Complexify  $\mathfrak{g}$  to get a complex representation  $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$  of  $T$ . Let  $R$  be the set of roots of  $G$ , i.e. nonzero characters of  $T$  in  $\mathfrak{g}_{\mathbb{C}}$ . Let  $\mathfrak{h} = \text{Lie}(T) \otimes_{\mathbb{R}} \mathbb{C}$ , and let

$$\mathcal{H} = \mathfrak{h}_{\mathbb{C}} - \cup_{\alpha \in R} \mathfrak{h}_{\alpha} \text{ for } \mathfrak{h}_{\alpha} = \{\mathfrak{h} \in \mathfrak{h}_{\mathbb{C}} \mid \alpha(\mathfrak{h}) = 0\}.$$

In other words,  $\mathcal{H}$  is the union of the interiors of the Weyl chambers. Note that  $\mathcal{H}$  has a free action of the Weyl group  $W(G, T) = N_T(G)/T$ .

**Definition 2.1.3.** The *Artin braid group*  $\text{Br}(G, T)$  is  $\pi_1(\mathcal{H}/W(G, T))$ .

It is a fact that  $\mathcal{H}/W(G, T)$  is a  $K(\text{Br}(G, T), 1)$ . Let's see why this works for  $G = U(n)$ .

**Example 2.1.4.** For  $(G, T) = (U(n), \Delta)$ , we have

$$\cup_{\alpha} \mathfrak{h}_{\alpha} = \{(x_1, \dots, x_n) \mid x_i = x_j \text{ for some } i \neq j\}.$$

Since  $W(G, T) = \Sigma_n$  acting on  $\mathfrak{h}_{\mathbb{C}} = \mathbb{C}^n$  by permuting coordinates, we see that  $\mathcal{H}/W(G, T)$  is the moduli of  $n$  points in  $\mathbb{C}$ . This picture gives us the usual definition of braid groups.

Note that  $\pi_0 \text{Map}(S^1, \mathcal{H}/W(G, T))$  is the space of conjugacy classes of elements in  $\text{Br}(G, T)$ , which we will write as  $[\text{Br}(G, T)]$ .

**Example 2.1.5.** For  $(G, T) = (U(n), \Delta)$ , elements of  $\text{Br}(G, T)$  are represented by braid closures. We may view these as links in  $\mathbb{R}^3$  using Markov's theorem.

**Theorem 2.1.6** (Markov). *The map sending a braid to its closure expresses the set of equivalence classes of links in  $\mathbb{R}^3$  as the set of equivalence classes of all braids,  $(\sqcup_{n \geq 1} \text{Br}_n) / \sim$ , where  $\sim$  is the equivalence relation generated by M1 and M2:*

1. M1 (Conjugation):  $\sigma \sim \rho \sigma \rho^{-1}$  for  $\sigma, \rho \in \text{Br}_n$
2. M2 (Stabilization):  $\sigma \sim \sigma \sigma_n^{\pm 1}$  for  $\sigma \in \text{Br}_n \subset \text{Br}_{n+1}$

The individual maps from  $[\text{Br}_n]$  to the space of links are not faithful. That is, stabilization does cause us to lose some information.

### 2.1.2 Categorification

We'd like to categorify  $\text{Br}(\mathbf{G}, \mathbf{T})$  and  $[\text{Br}(\mathbf{G}, \mathbf{T})]$ . In particular, we'd like a functor of (“discrete” / non- $\infty$ ) 2-categories<sup>1</sup>

$$* // \text{Br}(\mathbf{G}, \mathbf{T}) \rightarrow \text{Gr}(\mathbf{Alg}).$$

Here  $* // \text{Br}(\mathbf{G}, \mathbf{T})$  is the 2-category with:

- one object,
- $\text{Br}(\mathbf{G}, \mathbf{T})$  as 1-morphisms, and
- trivial 2-morphisms.

The right hand side,  $\text{Gr}(\mathbf{Alg})$ , is the 2-category with:

- objects: graded commutative algebras,
- 1-morphisms: chain complexes of graded bimodules, and
- 2-morphisms: homotopy classes of chain maps

We also want a map  $[\text{Br}(\mathbf{G}, \mathbf{T})] \rightarrow \text{Ch}_{\bullet, \bullet}(\mathbb{Z})$ .

This has been worked out by Soergel, Rouquier, Khovanov, and Khovanov-Rozansky. For  $\mathbf{G} = \text{U}(\mathbf{n})$ , the last map above gives bigraded homological invariants of links. (This requires us to check that the stabilization move is satisfied.)

Being more ambitious, we can try to realize this using 3d and 4d topological gauge theories. This splits into subgoals:

1. Describe a configuration space of  $\mathbf{G}$ -background fields on a 2-torus. (Here “background” means that we haven’t “summed over these fields.”)
2. Quantize by “summing over the background fields” to obtain the partition function of the 2-torus in 3d and 4d topological gauge theories with defects.
3. Identify the partition functions with the categorifications mentioned above.

### 2.1.3 Preliminaries: $\text{U}(\mathbf{n})$

We’ll start out by looking at the case of  $\mathbf{G} = \text{U}(\mathbf{n})$ . Let  $\mathbf{R}_n = \text{H}_T^\bullet(\text{pt}) = \mathbb{Z}[x_1, \dots, x_n]$  with  $|x_i| = 2$ . The Weyl group  $W$  is  $\Sigma_n$ , acting on  $\mathbf{R}_n$  with  $\sigma_i$  swapping  $x_i$  and  $x_{i+1}$ .

**Definition 2.1.7.** The *Bott-Samelson  $\mathbf{R}_n$ -bimodule* is the graded bimodule  $\text{bS}_i = \mathbf{R}_n \otimes_{\mathbf{R}_n^{\sigma_i}} \mathbf{R}_n$ , where  $\mathbf{R}_n^{\sigma_i}$  is the ring of  $\sigma_i$ -invariants in  $\mathbf{R}_n$ . The category  $\mathbf{SBimod}_n$  *Soergel bimodules* is the subcategory of  $\mathbf{R}_n$ -bimodules generated by the  $\text{bS}_i$ ’s (allowing idempotent completion and grading shifts).

<sup>1</sup>Recent work extends this to a map of  $(\infty, 2)$ -categories.

**Theorem 2.1.8** (Soergel). *The category  $\text{SBimod}_n$  categorifies the Hecke algebra  $H_q(A_n) = \mathbb{Z}[q, q^{-1}]\langle \text{Br}_n \rangle / I$ , where  $I$  is generated by the relations*

$$\begin{aligned} (\sigma_i - q^2)(\sigma_i + 1) &= 0 \\ \sigma_i^{-1} &= q^{-2}\sigma_i + (q^2 - 1). \end{aligned}$$

*That is, there is an equivalence  $K_0(\text{SBimod}_n) \cong H_q(A_n)$ . Under this equivalence:*

- 1 corresponds to  $R_n$ ,
- $bS_i$  corresponds to  $\sigma_i + 1$ , and
- $q$  corresponds to grading shift.

The chain complex  $C_i = (bS_i \rightarrow R_n)$ , where the map is given by  $\alpha \otimes \beta \mapsto \alpha\beta$ , should be viewed as a categorification of  $\sigma_i$ .

**Theorem 2.1.9** (Rouquier). *The assignment  $\sigma_I \mapsto c_I := C_{i_1} \otimes_{R_n} C_{i_2} \otimes_{R_n} \cdots \otimes_{R_n} C_{i_k}$  gives a well-defined map  $* // \text{Br}_n \rightarrow \text{Gr}(\text{Alg})$ .*

**Example 2.1.10.** The braid  $\sigma_1\sigma_2$  is sent to the chain complex  $C_{12} = (bS_1 \rightarrow R_3) \otimes (bS_2 \rightarrow R_3)$ , which can also be written

$$bS_1 \otimes_{R_3} bS_2 \longrightarrow bS_1 \oplus bS_2 \longrightarrow R_3.$$

Rouquier's theorem is nontrivial for two reasons:

- We need to show that the map, which *a priori* is defined only on braid words, is actually defined on the braid group.
- We need to make sure that this respects the triviality of the higher morphisms in  $* // \text{Br}_n$ .

## 2.2 6/10 – Geometric Interpretations for the $U(n)$ Case

Last time, we discussed a categorification of  $\text{Br}_n$ . We'd like to obtain a corresponding categorification of  $[\text{Br}_n]$ .

### 2.2.1 Categorification of $[\text{Br}_n]$

Recall the definition of Hochschild homology.

**Definition 2.2.1.** If  $M$  is a graded  $R_n$ -bimodule, then the *Hochschild homology* of  $M$  is

$$\text{HH}(M) = \text{Tor}_{R_n \otimes R_n^{\text{op}}}^{\bullet, \bullet}(M, R_n)$$

We will consider the Hochschild homology  $\text{HH}(C_I)$  termwise.

**Example 2.2.2.** For  $\sigma_1\sigma_2$ , we discussed  $C_{12}$  in the last lecture. With the above convention,  $\text{HH}(C_{12})$  is

$$\text{HH}(bS_1 \otimes_{R_3} bS_2) \longrightarrow \text{HH}(bS_1 \oplus bS_2) \longrightarrow \text{HH}(R_3).$$

**Theorem 2.2.3** (Khovanov). *The assignment  $\sigma_I \mapsto \text{HH}(C_I)$  extends to  $[\text{Br}_n] \rightarrow \text{Ch}_{\bullet, \bullet}(\mathbb{Z})$ . In fact, the output is naturally trigraded. Furthermore, this assignment satisfies M1 and M2, so it gives a homological invariant of links.*

This invariant can be identified with the trigraded Khovanov-Rozansky link homology. If  $\mathcal{L}$  is the link  $[\sigma_i]$ , we write

$$\mathfrak{sl}_\infty(\mathcal{L}) = \text{HHH}(\mathcal{L}) = \text{HH}(C_I).$$

We may view  $\mathfrak{sl}_\infty$  as a “limit” of bigraded homological invariants  $\mathfrak{sl}_N(\mathcal{L})$  which can be obtained from  $\text{HH}(C_I)$  by adding a “matrix factorization” differential.

### 2.2.2 Topological interpretation of Bott-Samelson bimodules

**Definition 2.2.4.** For  $i < n$ , let  $G_i \subset U(n)$  be the semisimple rank one compact parabolic consisting of matrices of the form

$$\begin{bmatrix} D_1 & 0 & 0 \\ 0 & U & 0 \\ 0 & 0 & D_2 \end{bmatrix}$$

with  $D_1$  and  $D_2$  diagonal and  $U \in U(2)$  (placed in rows / columns  $i$  and  $i+1$ ).

Note that  $T \subset G_i \subset G = U(n)$  for all  $i$ .

**Definition 2.2.5.** Let  $BS_i$  be the balanced product  $U(n) \times^T (G_i/T)$ , considered with its natural  $U(n)$ -action.

We may obtain  $BS_i$  from a pullback square

$$\begin{array}{ccc} BS_i & \xrightarrow{\alpha} & U(n)/T \\ \downarrow \beta & & \downarrow \\ U(n)/T & \longrightarrow & U(n)/G_i \end{array}$$

where  $\alpha : [g, g_i] \mapsto [gg_i]$  and  $\beta : [g, g_i] \mapsto [g]$ .

In cohomology, we get a map

$$H_{U(n)}^\bullet(U(n)/T) \otimes_{H_{U(n)}^\bullet(U(n)/G_i)} H_{U(n)}^\bullet(U(n)/T) \rightarrow H_{U(n)}^\bullet(BS_i).$$

A spectral sequence computation shows this is an isomorphism. Combining this with the standard isomorphisms  $H_G^\bullet(G/K) \cong H_K^\bullet(\text{pt})$  for any  $G$  and any  $K \subset G$ , we get

$$H_{U(n)}^\bullet(BS_i) \cong H_T^\bullet(\text{pt}) \otimes_{H_{G_i}^\bullet(\text{pt})} H_T^\bullet(\text{pt}) \cong R_n \otimes_{R_n^{\sigma_i}} R_n = bS_i.$$

Thus, we can interpret the Bott-Samelson bimodule  $bS_i$  geometrically as the  $U(n)$ -equivariant cohomology of  $BS_i$ .

The pullback square defining  $BS_i$  is in fact a homotopy pullback square. Comparing with the standard definition of the homotopy pullback, we can view  $BS_i$  as the moduli space of  $U(n)$ -bundles on  $[0, 1]$  with a reduction of structure group to  $G_i$  on  $[0, 1]$  and to  $T$  on  $\{0, 1\}$ .

### 2.2.3 Interpretation of Rouquier's chain complex

More generally, for a multi-index  $I = (i_1, \dots, i_k)$ , define

$$BS_I = U(n) \times^T (G_{i_1} \times^T G_{i_2} \times^T \dots \times^T G_{i_k}/T).$$

Then

$$H_{U(n)}^\bullet(BS_I) \cong bS_{i_1} \otimes_{R_n} bS_{i_2} \otimes_{R_n} \dots \otimes_{R_n} bS_{i_k}.$$

We may view  $BS_I$  as the moduli space of  $U(n)$ -bundles on  $[0, 1]$  with  $k+1$  marked points (including 0 and 1) with reduction of structure group to  $G_{i_j}$  on the  $j$ th interval and to  $T$  at each marked point.

If  $J$  is obtained from  $I$  by discarding indices, then we get  $BS_J \hookrightarrow BS_I$  by replacing every factor  $G_{i_k}$  (for each removed index  $i_k$ ) by  $T$  in the construction of  $BS_I$ . These replaced copies of  $T$  are “transparent defects.” This inclusion respects composition of inclusions of multi-indices.

Rouquier's chain complex  $C_I$  is isomorphic to

$$H_{U(n)}^\bullet(BS_I) \longrightarrow \oplus_{|J|=k-1, J \subset I} H_{U(n)}^\bullet(BS_J) \longrightarrow \dots \longrightarrow H_{U(n)}^\bullet(U(n)/T).$$

All differentials here are obtained from (signed) inclusions  $BS_J \rightarrow BS_{J'}$ .

### 2.2.4 Interpretation of $\mathrm{HH}(\mathrm{bS}_I)$

To interpret  $\mathrm{HH}(\mathrm{H}_{\mathrm{U}(\mathfrak{n})}^\bullet(\mathrm{BS}_I))$ , we introduce the following.

**Definition 2.2.6.** Let

$$\mathrm{L}_i = \mathrm{U}(\mathfrak{n}) \times^{\mathrm{T}} \mathrm{G}_i$$

where  $\mathrm{T}$  acts on  $\mathrm{G}_i$  by conjugation.

The map  $\mathrm{L}_i \rightarrow \mathrm{BS}_i$  given by  $[g, g_i] \mapsto [g, g_i]$  is a principal  $\mathrm{T}$ -bundle. In fact, we have a pullback square

$$\begin{array}{ccc} \mathrm{EU}(\mathfrak{n}) \times^{\mathrm{U}(\mathfrak{n})} & \longrightarrow & \mathrm{BT} \\ \downarrow & & \downarrow \Delta \\ \mathrm{EU}(\mathfrak{n}) \times^{\mathrm{U}(\mathfrak{n})} \mathrm{BS}_i & \longrightarrow & \mathrm{BT} \times \mathrm{BT} \end{array}$$

where both vertical arrows are trivial principal bundles.

Collapse of the Eilenberg-Moore spectral sequence at the  $E_2$  page gives

$$\mathrm{H}_{\mathrm{U}(\mathfrak{n})}^\bullet(\mathrm{L}_i) \cong \mathrm{Tor}_{\mathrm{H}^\bullet(\mathrm{BT} \times \mathrm{BT})}(\mathrm{H}_{\mathrm{U}(\mathfrak{n})}^\bullet(\mathrm{BS}_i), \mathrm{H}^\bullet(\mathrm{BT})) = \mathrm{HH}(\mathrm{H}_{\mathrm{U}(\mathfrak{n})}^\bullet(\mathrm{BS}_i)) = \mathrm{HH}(\mathrm{bS}_i).$$

Thus we obtain a geometric interpretation of  $\mathrm{HH}(\mathrm{bS}_I)$ . We may interpret  $\mathrm{L}_i$  as the moduli space of  $\mathrm{U}(\mathfrak{n})$ -bundles on  $S^1$  with one marked point, where the structure group reduces to  $\mathrm{G}_i$  away from the marked point and to  $\mathrm{T}$  at the marked point.

More generally, for a multi-index  $I$ , we let

$$\mathrm{L}_I = \mathrm{U}(\mathfrak{n}) \times^{\mathrm{T}} (\mathrm{G}_{i_1} \times^{\mathrm{T}} \mathrm{G}_{i_2} \times^{\mathrm{T}} \cdots \times^{\mathrm{T}} \mathrm{G}_{i_k})$$

where  $\mathrm{T}$  acts by conjugation on  $\mathrm{G}_{i_1}$  and  $\mathrm{G}_{i_k}$ . An argument analogous to the above gives  $\mathrm{H}_{\mathrm{U}(\mathfrak{n})}^\bullet(\mathrm{L}_I) \cong \mathrm{HH}(\mathrm{bS}_I)$ .

For  $J \subset I$ , we get an inclusion  $\mathrm{L}_J \rightarrow \mathrm{L}_I$  that behaves well with respect to composition of inclusions. These assemble into a complex

$$\mathrm{H}_{\mathrm{U}(\mathfrak{n})}^\bullet(\mathrm{L}_I) \longrightarrow \bigoplus_{|J|=k-1, J \subset I} \mathrm{H}_{\mathrm{U}(\mathfrak{n})}^\bullet(\mathrm{L}_J) \longrightarrow \cdots$$

which is isomorphic to  $\mathrm{HH}(\mathrm{C}_I)$ .<sup>2</sup>

## 2.3 6/11 – Local Systems and Twisted Cohomology

### 2.3.1 Review of bundle theory

Let  $G$  be a topological group. Principal  $G$ -bundles are classified by pulling back a “universal bundle”  $\mathrm{EG} \rightarrow \mathrm{BG}$ . That is, for any principal  $G$ -bundle  $E \rightarrow B$ , there exists an  $f : B \rightarrow \mathrm{BG}$  (unique up to homotopy) such that  $E = f^* \mathrm{EG}$ . To construct  $\mathrm{EG} \rightarrow \mathrm{BG}$ , just take any contractible space  $\mathrm{EG}$  with free  $G$ -action, and let  $\mathrm{BG} = \mathrm{EG}/G$ .

We will write a  $G$ -space  $X$  as  $X // G$ . From such a space, we may construct the *homotopy orbits*  $X //^h G = \mathrm{EG} \times^G X := (\mathrm{EG} \times X)/G$ , where  $G$  imposes the relation  $(e, x) \sim (eg, g^{-1}x)$ . We define the  *$G$ -equivariant cohomology* of  $X$  to be

$$\mathrm{H}_G^\bullet(X) := \mathrm{H}^\bullet(X //^h G).$$

If  $K \subset G$ , then we may take  $\mathrm{EG}$  as a model for  $\mathrm{EK}$ . Because  $\mathrm{EG} \times^G (G \times^K Y) = \mathrm{EG} \times^K Y = \mathrm{EK} \times^K Y$ , we obtain

$$\mathrm{H}_G^\bullet(G \times^K Y) = \mathrm{H}_K^\bullet(Y).$$

Given a principal  $G$ -bundle  $E \rightarrow B$ , we may ask whether or not  $E$  is extended from a principal  $K$ -bundle, i.e. whether  $E \cong E' \times^K G$  for some principal  $K$ -bundle  $E' \rightarrow B$ . This is equivalent to asking whether the classifying map  $f_E : B \rightarrow \mathrm{BG}$  lifts to a map  $f_{E'} : B \rightarrow \mathrm{BK}$ . Note that  $\mathrm{BK} \rightarrow \mathrm{BG}$  has fiber  $G/K$ , so in particular reductions of structure group of the trivial  $G$ -bundle to  $K$  correspond to maps  $B \rightarrow G/K$ .

<sup>2</sup>This is a “cubical complex,” related to crossing resolutions for nice presentations of knots.

### 2.3.2 Recap

Last time, we defined subgroups  $G_i \subset U(\mathfrak{n})$  for  $1 \leq i < \mathfrak{n}$ . For each multi-index  $I = (i_1, \dots, i_k)$ , we defined spaces  $BS_I = U(\mathfrak{n}) \times^T (G_{i_1} \times^T G_{i_2} \times^T \dots \times^T G_{i_k}/T)$ . We showed

$$H_{U(\mathfrak{n})}^\bullet(BS_I) = bS_{i_1} \otimes_{R_n} \dots \otimes_{R_n} bS_{i_k}$$

for  $bS_i = R_n \otimes_{R_n^{\sigma_i}} R_n$ . The spaces  $BS_I$  are functorial with respect to inclusions of multi-indices.

**Example 2.3.1.** Let  $I = \{i\}$  and  $J = \emptyset$ , so  $BS_i = U(\mathfrak{n}) \times^T G_i/T$  and  $BS_\emptyset = U(\mathfrak{n}) \times^T (T/T) = U(\mathfrak{n})/T$ . Note that  $H_{U(\mathfrak{n})}^\bullet(BS_i) = bS_i$  and  $H_{U(\mathfrak{n})}^\bullet(U(\mathfrak{n})/T) = H_T^\bullet(\text{pt}) = R_n$ . The inclusion  $BS_\emptyset \rightarrow BS_i$  comes from  $T \subset G_i$ , and the induced map on cohomology is the multiplication map

$$bS_i = R_n \otimes_{R_n^{\sigma_i}} R_n \rightarrow R_n.$$

This is the chain complex  $C_i$  appearing in Rouquier's work.

We also defined spaces  $L_I = U(\mathfrak{n}) \times^T (G_{i_1} \times^T G_{i_2} \times^T \dots \times^T G_{i_k})$  where  $T$  acts on  $G_{i_1}$  and  $G_{i_k}$  by conjugation. These spaces may be viewed as moduli of  $U(\mathfrak{n})$ -bundles on  $S^1$  with  $k+1$  marked points with reduction of structure group to  $G_{i_j}$  on the  $j$ th interval and to  $T$  at each marked point. The cohomologies can be used to construct a chain complex which recovers Khovanov's link homology. Specifically, we have  $H_{U(\mathfrak{n})}^\bullet(L_I) = HH(bS_I)$ .

### 2.3.3 Recovering $\mathfrak{sl}_N$ -link homology

There is a forgetful map  $L_I // U(\mathfrak{n}) \rightarrow U(\mathfrak{n}) // U(\mathfrak{n})$ , where  $U(\mathfrak{n})$  acts on itself via the adjoint representation. This map is given by the formula

$$[g, g_1, \dots, g_k] \mapsto gg_1 \dots g_k g^{-1}.$$

If one thinks of the bundles appearing in the moduli description of  $L_I$  as having connections, this is essentially taking the product of the holonomies.

The functorial maps of  $L_I // U(\mathfrak{n})$  are compatible with the forgetful maps to  $U(\mathfrak{n}) // U(\mathfrak{n})$ . In particular, Khovanov's chain complex

$$H_{U(\mathfrak{n})}^\bullet(L_I) \longrightarrow \oplus_{|J|=k-1, J \subset I} H_{U(\mathfrak{n})}^\bullet(L_J) \longrightarrow \dots$$

is a complex over the torsion-free  $H_{U(\mathfrak{n})}^\bullet(U(\mathfrak{n}))$ . Let's think about what happens if we take termwise cohomology with respect to odd elements of  $H_{U(\mathfrak{n})}^\bullet(U(\mathfrak{n}))$ .

There is a non-canonical ring isomorphism

$$H_{U(\mathfrak{n})}^\bullet(U(\mathfrak{n})) \cong H_{U(\mathfrak{n})}^\bullet(\text{pt}) \otimes H^\bullet(U(\mathfrak{n})).$$

The non-canonicalness of this is related to the fact that  $U(\mathfrak{n}) //^h U(\mathfrak{n}) \not\cong BU(\mathfrak{n}) \times U(\mathfrak{n})$ . Essentially, it's not clear what to do on primitive Hopf algebra elements.

However, if we take  $\mathfrak{n} \rightarrow \infty$  (and write  $U = U(\infty)$ ), we do get a canonical equivalence

$$U //^h U \simeq BU \times U.$$

Pulling back the elements of  $H^\bullet(U)$  gives odd elements  $d_1, d_3, \dots$ , and we can modify Khovanov's complex to get

$$H^\bullet(H_{U(\mathfrak{n})}(L_I), d_{2N-1}) \longrightarrow \oplus_{|J|=k-1, J \subset I} H^\bullet(H_{U(\mathfrak{n})}(L_J), d_{2N-1}) \longrightarrow \dots$$

Here  $\mathfrak{n}$  and  $N$  are unrelated.

**Theorem 2.3.2** (T. Gomez). *This complex is isomorphic to the link homology  $\mathfrak{sl}_N(\mathcal{L})$  for the link  $\mathcal{L} = [\sigma_I]$ .*

**Example 2.3.3.** Let  $\mathcal{L}$  be the unknot, so  $\mathcal{L} = [\sigma_1]$ . Then  $\mathfrak{sl}_\infty(\mathcal{L}) = H_{U(1)}^\bullet(U(1) \times^T T) = H_T^\bullet(T) = \mathbb{Z}[x_2, y_1]$ . Here  $d_N$  corresponds to the odd class  $y_1 x^N$ . We get

$$H^\bullet(H_{U(1)}(L_I), d_N) \cong (\mathbb{Z}[x]/x^N) \langle y_1 \rangle.$$



### 2.3.4 Local systems

The above story was a bit ad hoc, so we'd like to fit it into a more topological context. We can think of  $U(n)$  as  $B\Omega U(n)$ , so  $\Omega U(n)$  is really acting everywhere. In fact, we'd like the second Markov move (stabilization) to play well with our theory, so we should pass to an  $\Omega U$ -action. We can implement this by using a *twisted cohomology theory*

$$\mathcal{H} = \Omega U_+ \wedge H_{\mathbb{Z}} = (\mathbb{Z} \times BU)_+ \wedge H_{\mathbb{Z}}.$$

One can compute

$$\pi_{\bullet} \mathcal{H} = \pi_{\bullet} H[b_0^{\pm}, b_1, b_2, \dots]$$

with  $|b_i| = 2i$ .

Setting  $b_0 = b_N = 1$  and  $b_i = 0$  otherwise (and indicating this with the subscript  $N$ ), one can calculate

$$\tilde{\mathcal{H}}_{U(n)}^{\bullet}(L_I)_N \simeq H^{\bullet}(H_{U(n)}(L_I), d_N).$$

## 2.4 6/12 – Spectral Sequence Invariants; Relationship to TQFT

### 2.4.1 Remarks on questions

Previously, we constructed (for  $G = U(n)$ ) a  $G$ -space  $L_I$  such that  $H_G(L_I) = HH(bS_I)$ . There is a  $G$ -equivariant map  $L_I \rightarrow G$  (where  $G$  acts on itself by convolution), and we can view this as a derived local system. For these purposes, we should view  $G = B(\Omega G)$ .

**Example 2.4.1.** Consider  $\Omega G$ -twisted cohomology  $\mathcal{H} = (\Omega U)_+ \wedge H_{\mathbb{Z}}$ . Then  $\pi_{\bullet} \mathcal{H} = (H_{\mathbb{Z}})_{\bullet}[b_0^{\pm 1}, b_1, \dots]$ . Here the  $b_i$  control the possible twistings. Setting  $b_0 = b_N = 1$  and  $b_i = 0$  otherwise gives  $\mathfrak{sl}_N$ -link homology. We can consider the power series  $\sum_i b_i x^i$ , which can be viewed in a matrix factorization context as the derivative of our superpotential.

**Example 2.4.2.** We can also consider twisted  $K$ -theory. Note that  $K_G$  has twistings indexed by  $\mathbb{N}$ . Here (for  $N \in \mathbb{N}$ )

$${}^N K_{U(1)}(L_{\emptyset}) = \mathbb{Z}[x^{\pm 1}] / (x^N - 1)$$

gives the corresponding invariant for the unknot. Freed-Hopkins-Teleman allows us to identify this with the ring controlling level  $N$  representations of the loop group  $LU(1)$ . There is a “level-rank duality” going on here relating this to  $\mathbb{Z}[x]/(x^N)$ .

Twisted  $K$ -theory is a functor defined on  $G$ -spaces  $X$  over  $G$ . This produces modules over  ${}^0 K_G(X)$ .

### 2.4.2 Spectral sequence interpretation of Khovanov homology

Recall that an inclusion of multi-indices  $J \subset I$  induces  $L_J \subset L_I$ . We used this to construct a complex producing Khovanov homology. We'd like to interpret this topologically. To this end, we construct a filtered  $G$ -equivariant space  $F_{\bullet}(sL_I)$  with

$$\begin{aligned} F_0(sL_I) &= \text{cone}(\emptyset \rightarrow L_I) = L_I \\ F_1(sL_I) &= \text{cone}(\cup_{|J|=k-1} L_J \rightarrow L_I) \\ F_2(sL_I) &= \text{cone}(\cup_{|J|=k-1 \text{ or } k-2} L_J \rightarrow L_I) \\ &\dots \end{aligned}$$

We can compute  $F_k(sL_I)/F_{k-1}(sL_I) = \vee_{|J|=n-r} \Sigma^r(L_J)_+$ . This yields a spectral sequence with  $E_1$  term  $HH(C_I^{\bullet})$  converging to  $H_{U(n)}^{\bullet}(F_k(sL_I))$ . This limit space is not particularly interesting – it's just a Thom space, and it counts the number of components of the link. However, the terms of the spectral sequence are more interesting.

**Theorem 2.4.3.** *The pages  $E_i$  for  $i \geq 2$  are invariants of the link  $\sigma_I$ .*

We'd like to view the filtered  $G$ -space  $F_\bullet(sL_I)$  as a link invariant itself. First, we need a notion of equivalence for such objects. We will consider filtered  $G$ -spaces as giving key examples of  $G$ -spectra.

**Definition 2.4.4.** Let  $F_\bullet, G_\bullet$  be  $G$ -spectra. We say that a map  $f_\bullet : F_\bullet \rightarrow G_\bullet$  is an *elementary quasi-equivalence* if its (co)fiber  $Z_\bullet$  is acyclic, i.e. the identity on  $Z_\bullet$  is homotopic to 0. More generally, we say  $F_\bullet$  and  $G_\bullet$  are *quasi-equivalent* if they are connected by a zigzag of elementary quasi-equivalences.

**Theorem 2.4.5.** *The filtered  $G$ -space  $F_\bullet(sL_I)$  is an invariant of the link  $\sigma_1$  up to quasi-equivalence.*

### 2.4.3 Towards a TQFT interpretation

Suppose that  $M$  is a manifold with a  $G$ -bundle (thought of as a background field). This is classified by a map  $f_M : M \rightarrow BG$ . For  $M = N \times S^1$ , we may regard the classifying map as equivalent to a map  $\hat{f}_N : N \rightarrow LBG = \text{Map}(S^1, BG)$ .

**Proposition 2.4.6.** *The free loop space  $LBG$  is equivalent to  $G \parallel^h G = EG \times^G G$  (where  $G$  acts on itself by conjugation).*

*Proof.* Decomposing  $S^1 = D^+ \cup_{S^0} D^-$ , we may view the data of a map  $S^1 \rightarrow BG$  as consisting of a bundle on  $D^+$ , a bundle on  $D^-$ , and an equivalence of the two on  $S^0$ . Written another way, this is a homotopy fiber product

$$\begin{array}{ccc} LBG & \longrightarrow & BG \\ \downarrow & & \downarrow \Delta \\ BG & \xrightarrow{\Delta} & BG \times BG. \end{array}$$

Writing

$$\begin{aligned} BG &\simeq E(G \times G)/\Delta \\ &\simeq E(G \times G) \times^{G \times G} (G \times G)/\Delta \\ &\simeq E(G \times G) \times^{G \times G} G, \end{aligned}$$

we can write  $LBG = BG \times_{BG \times BG}^h (E(G \times G) \times^{G \times G} G)$ , which gives the desired result.  $\square$

The upshot is that  $G$ -background fields on  $N \times S^1$  are equivalent to  $G \parallel^h G$ -background fields on  $N$ . It follows that we should study  $G \parallel G$  and  $G \parallel^h G$ . If we let  $G/G$  be the space of bona-fide  $G$ -orbits, we get a map  $G \parallel G \rightarrow G/G$ .

Recall that every element of  $G$  is conjugate to an element of  $T$ , so we can write  $G/G = T/W$ , where  $W = N_G(T)/T$ . For  $\mathfrak{h} = \text{Lie}(T)$ , we have an exponential map  $\exp : \mathfrak{h}/W \rightarrow T/W$ , where  $\mathfrak{h}/W$  is equivalent to a Weyl chamber. We can interpret  $G \parallel G$  or  $(G \parallel^h G)$  as stacks / spaces over  $\mathfrak{h}/W$ . In the open parts of the Weyl chambers, the fibers look like  $(G/T) \parallel G$ . Along the walls of the Weyl chambers, we get jumps / defects where the fibers become  $(G/G_i) \parallel G$ .

We can also interpret the spaces  $L_I$ . Given  $I$ , inscribe  $S^1$  in  $\mathfrak{h}/W$  so that the circle hits the wall  $\alpha_{i_j} = 0$  between the points  $x_j$  and  $x_{j+1}$ . Then  $L_I$  consists of sections

$$\begin{array}{ccc} & G \parallel G & \\ & \downarrow & \\ S^1 & \xrightarrow{\quad} & \mathfrak{h}/W \end{array}$$

### 2.4.4 Homotopy TQFT interpretation

We may view  $L_I$  as  $G \parallel G$ -background fields on  $S^1$ , or equivalently as  $G$ -background fields on  $S^1 \times S^1$ . Let  $\text{Cob}_n(\text{pt} \parallel^h G)$  be the  $(\infty, n)$ -category of manifolds with  $G$ -background. Let  $\mathcal{H}$  be a commutative “global” cohomology theory – see the book of Stefan Schwede for details of what this means. Write  $\mathcal{C}_n$  for the category of  $\mathbb{E}_{n-1}$ -algebras in  $\mathcal{H}$ -modules.

There is a functor  $\text{Cob}_n(\text{pt} //^h G) \rightarrow \mathcal{C}_n$  given by  $\text{pt} // G \mapsto \mathcal{H}$ . Compactifying this along  $S^1$  (i.e. considering values on manifolds of the form  $M = N \times S^1$ ) gives a functor  $\text{Cob}_{n-1}(G //^h G) \rightarrow \mathcal{C}_{n-1}$ , defined by  $G //^h G \mapsto (\mathcal{H} \times G) // G$ . On  $S^1$  with  $L_I$  background, we get  $(L_I \times \mathcal{H} \rightarrow L_I) // G$ . This is an invertible theory, and we can gauge / take  $G$ -equivariant global sections to get

$$\mathcal{H}_G(L_I) \simeq \text{HH}(\text{bS}_I).$$

The inclusions  $L_J \hookrightarrow L_I$  give Khovanov's complex.

Twisted theories give modules over  $\mathcal{H}$ , which doesn't quite make sense in the above story. Instead, we want to view our theory as being the boundary of a 4d theory. That is, Khovanov-Rozansky  $\mathfrak{sl}_N$ -link homology comes from a 3d theory on the boundary of a 4d theory.

## Chapter 3

# Ibou Bah – Lagrangian Methods for Categorical Symmetries

### 3.1 6/11 – What is QFT?

This will be a physics-y, intuitive series of lectures. We will start by giving some idea of QFT in general. In the second lecture, we will move on to finite symmetry groups and use these to construct topological operators. In the third lecture, we will discuss the action of finite symmetry groups on fields (with an emphasis on the example of  $SU(N)$  Yang-Mills theory). Finally, we will discuss the general construction of topological operators using Lagrangians.

#### 3.1.1 What should a QFT do?

We don't actually know what a QFT *is*. However, we have an excellent physical idea of what a QFT should *do*. QFTs should have:

- A list of operators: point operators  $\{\mathcal{O}_i(x)\}$ , line operators  $\{U_\alpha(\gamma_p)\}$ , and other operators.
- A Hilbert space  $\mathcal{H}$  of states on spacetime  $M = \mathbb{R}_t \times Y$ .
- A partition function sending combinations / products of operators to complex numbers, e.g.  $\langle \mathcal{O}_i \mathcal{O}_j \rangle$ .

These should satisfy three key properties: *unitarity*, *locality*, and *causality*. Let's discuss each of these in more detail.

#### 3.1.2 Unitarity, locality, and causality

Unitarity can be interpreted as “preservation of probability.” For example, for many QFTs, we can construct an “S-matrix” governing scattering amplitudes of particles, and unitarity forces  $S^\dagger S = 1$ . This specific interpretation doesn't work for CFTs / TFTs. Instead, we use a notion of *reflection positivity*.

To explain this, recall that time evolution in quantum mechanics is governed by

$$|\psi\rangle \mapsto e^{-iHt} |\psi(t)\rangle$$

and

$$\mathcal{O}(x, 0) |\psi\rangle \mapsto \mathcal{O}(x, t) |\psi\rangle .$$

We should have

$$\langle \psi | \mathcal{O}^\dagger \mathcal{O} | \psi \rangle = \| \mathcal{O}(x, t) |\psi\rangle \| \geq 0 .$$

To make sense of  $\mathcal{O}^\dagger$ , we allow our time variable to take complex values. Say that  $\mathcal{O}(x, t+i\tau)^\dagger = \mathcal{O}(x, t-i\tau)$ . Then the above condition forces  $\langle \mathcal{O}(x, t-i\tau) \mathcal{O}(x, t+i\tau) \rangle \geq 0$ . Taking  $t = 0$ , we see that reflection positivity requires  $\langle \mathcal{O}(x, -i\tau) \mathcal{O}(x, i\tau) \rangle \geq 0$ . More generally, reflection positivity forces

$$\langle \mathcal{O}_i(x, -i\tau) \mathcal{O}_j(x, -i\tau) \dots \mathcal{O}_j(x, i\tau) \mathcal{O}_i(x, i\tau) \rangle \geq 0 .$$

Locality forces point operators to have an *operator product expansion*

$$\mathcal{O}_i(x_1)\mathcal{O}_j(x_2) = \sum_k c_{ij}^k(x_1, x_2)\mathcal{O}(x_1)$$

governing the behavior as  $x_1 \rightarrow x_2$ . (For conformal field theories, the  $c_{ij}^k$  should be meromorphic.) More generally, the interaction of operators should depend only on their behavior in a common tubular neighborhood. If two point operators are outside the same light cone, then we can find a reference frame where the two operators take place at the same time but at different points in space.

Finally, causality requires that any operators which are spacelike separated commute with each other. If two point operators are in the interior of the same light cone, then we can find a reference frame where the operators take place at the same point in space but at different points in time. Things are more complicated for extended operators.

### 3.1.3 Extra features

In many QFTs, we can expect other nice properties to hold:<sup>1</sup>

- Cluster decomposition:

$$\lim_{b \rightarrow \infty} \langle \mathcal{O}_i(x_1)\mathcal{O}_j(x_2) \dots \mathcal{O}_{i'}(x_1 + b)\mathcal{O}_{j'}(x_2 + b) \dots \rangle = \langle \mathcal{O}_i(x_1)\mathcal{O}_j(x_2) \dots \rangle \langle \mathcal{O}_{i'}(x_1 + b)\mathcal{O}_{j'}(x_2 + b) \dots \rangle.$$

- Renormalization: high energy / short distance / UV physics and low energy / long distance / IR physics decouple, at least up to finitely many parameters.

Axiomatic treatments can take these as additional axioms.

Gravity fails both of these conditions and locality. Black holes are UV phenomena which are still visible at low energy. Furthermore, black holes also break cluster decomposition. Whatever quantum gravity is, unitarity “must” be retained, since it’s fundamental to quantum physics.

The goal of the Simons collaboration is to understand the kinematics of operators in QFT. This is governed by symmetries! We can essentially think of this as understanding the *labels* of operators  $\mathcal{O}_i(x)$ ,  $U_\alpha(\gamma)$ , etc.

### 3.1.4 Where do QFTs come from?

There are many sources of QFTs.

**Example 3.1.1** (Lattices). Consider a lattice at which each point behaves according to quantum mechanics. Taking a continuum limit of the lattice gives a quantum field theory. This limit leads to the appearance of infinities and other strange phenomena. Nevertheless, it’s the most physically “rigorous” way of constructing QFTs.

**Example 3.1.2** (Geometric engineering). Start with a string theory, and “decouple gravity” by freezing some dimensions. This produces a QFT involving geometric objects. Class  $\mathcal{S}$  theories arise in this way.

**Example 3.1.3** (Lagrangians). We can construct many QFTs from Lagrangians. We’ll spend most of our time focusing on this perspective.

Each perspective has advantages over the others:

- Lattices make renormalization straightforward.
- Geometric engineering makes the geometric nature of objects more apparent.
- Lagrangians make locality explicit.

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<sup>1</sup>There’s also a last property of “analyticity” which is hard to formulate.

### 3.1.5 Lagrangians

A Lagrangian is a functional depending on the fields  $\phi_\alpha(x)$ :

$$L(x) = L(\phi_\alpha(x), \partial\phi_\alpha(x), \dots)$$

This is a classical object. Locality enforces that  $L$  only depends on the behavior of the fields at the given input point (unlike, say, Green's functions  $G(x_1, x_2)$ ). Locality also ensures that there are only finitely many derivative terms appearing in  $L$ .

Lorentz symmetry (which we will assume implicitly) tells us that our fields can be:

- Scalar fields  $\phi(x)$
- Vector fields / gauge fields  $A_\mu(x)$ , either:
  - Massless: living in the representation  $(1, -1)$ , or
  - Massive: living in  $(1, 0, 1)$
- Fermions  $\psi_\alpha$  (representations of a Clifford algebra)
- Metrics  $g_{\mu\nu}$

We won't consider gravitational theories, so we'll ignore  $g_{\mu\nu}$ .

## 3.2 6/11 – Lagrangians in QFT

Unitarity is true for closed quantum systems. For open quantum systems, where we “trace away” some information, we can drop this assumption. We must restore unitarity as we restore data to our system. This explains e.g. where the  $W$ -boson and Higgs boson come from: they're needed to restore unitarity in the Standard Model.

### 3.2.1 Lagrangians

Consider a Lagrangian

$$\mathcal{L}(x) = \mathcal{L}(\phi_\alpha(x), \partial\phi_\alpha(x), \dots).$$

From this (classical) data, we can define a (quantum) partition function

$$Z = \int D[\phi_\alpha] e^{i \int_M \sqrt{g} \mathcal{L}(\phi_\alpha)}$$

This can be thought of as a formal expression: heuristically, it is a sum over all possible field configurations.

We can obtain correlation functions, etc. by inserting local operators

$$\mathcal{O}_i(\phi_\alpha(x), \partial\phi_\alpha(x), \dots)$$

or extended operators

$$U_a(\gamma) = U\left(\int_\gamma F(\phi_\alpha(x)), \dots\right).$$

Namely, we declare

$$\langle \mathcal{O}_i \dots \rangle = \frac{1}{Z(0)} \int D[\phi_\alpha] e^{i \int \mathcal{L}} \mathcal{O}_i \dots$$

### 3.2.2 Parameters

In general, it is possible that our Lagrangian depends on some additional parameters, say  $\tau_a, \tau_b$ . The partition function then also depends on the  $\tau$ 's. We will assume that these parameters are *perturbative*: that small changes in our theory are governed by Taylor expansions in terms of the  $\tau$ 's. Often, there is a large parameter space for a given QFT, and there are many perturbative regions within this space corresponding to different weakly coupled versions of the theory. These *duality frames* can have different fields, different Lagrangians, and different gauge groups. (Gauge symmetry is a redundancy in parametrization – it is not a physical symmetry, but rather a “trick” enabling the use of local Lagrangians.)

However, the global symmetry, Hilbert space, and local operators are the same in every duality frame.<sup>2</sup> This may not always be visible (at least at the classical level) but it is true. Global symmetry permutes the operators, while gauge symmetry does nothing to the operators.

**Example 3.2.1.** We can view 4d  $\mathcal{N} = 4$  super-Yang-Mills as part of the same parameter space as type IIB supergravity on  $\text{AdS}_5 \times S^5$ . Thus these are versions of the same theory.

In addition to varying our parameters, we can also turn on background fields  $B_i(x)$ , i.e. add in new fields which do not fluctuate.

### 3.2.3 Examples

Consider a theory with one free massless scalar field  $\phi$ . The Lagrangian is

$$\mathcal{L} = \frac{1}{e^2} d\phi \wedge \star d\phi,$$

a single kinetic term. The partition function is

$$Z = \int D[\phi] \exp \left( \frac{i}{e^2} \int d\phi \wedge \star d\phi \right).$$

Taking the coupling parameter  $e \rightarrow 0$  yields  $d\phi = 0$ .

We can also add matter, incorporated via a term  $m^2 \phi^2 \text{Vol}(M)$  added to the Lagrangian. Taking  $m \rightarrow \infty$  gives  $\phi = 0$ .

The stress tensor of the theory is

$$T_{\mu\nu} = \frac{\delta \mathcal{L}}{\delta g^{\mu\nu}}.$$

In our above setup, there are kinetic terms, so  $T_{\mu\nu} \neq 0$  and the theory is not topological. However, the theory does become topological in certain limits.

Suppose we add a gauge field  $A_\mu$  transforming (under local gauge transformations) according to  $A_\mu \mapsto A_\mu + d\lambda$ . Taking  $\phi$  to be a complex scalar, setting  $D\phi = d\phi + iA\phi$ , and letting  $F = dA$ , we can define a Lagrangian

$$\mathcal{L} = D\phi \wedge \star D\phi + \frac{1}{e^2} F \wedge \star F.$$

This produces a non-topological gauge theory.

We can produce a topological theory by adding topological coupling terms:  $\theta F \wedge F$  or  $A \wedge d\phi \wedge F$ . These terms would have coefficients determined by the gauge theory. Alternatively, we could add boundary terms.

### 3.2.4 More remarks on Lagrangians

The constraints depend on the choice of gauge group. For example, having gauge group  $\mathbb{R}$  would allow for only local transformations (corresponding to exact 1-forms  $d\lambda$ ). Gauge group  $U(1)$  would allow more interesting global transformations (e.g.  $\omega \in H^1(-; \mathbb{Z})$  allowing  $A_\mu \mapsto A_\mu + \omega$ ).

We can split a typical Lagrangian into a *kinetic term*, an *interaction term*, and a *topological term*. The kinetic term reflects the behavior of free fields. The interaction term consists of a polynomial in the  $\phi_\alpha$ , and the signs of the coefficients are constrained by locality and causality. The topological term is (broadly) independent of the metric.

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<sup>2</sup>The term “global” is not related to this, however.

### 3.3 6/12 – Introduction to BF Theory

We start by clarifying something from last time.

#### 3.3.1 Jumping global symmetry

The global symmetry can jump as we move throughout the moduli space of theories.

**Example 3.3.1.** Suppose we have a theory with two scalar fields  $\phi_1$  and  $\phi_2$  and Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_i)^2 + m_1^2 \phi_1^2 + m_2^2 \phi_2^2.$$

When  $m_1 = m_2$ , the symmetry of this theory increases to  $SO(2)$ . This leads to new and interesting physics.

#### 3.3.2 Definition of BF Theory

BF theory is the TFT with fields  $\mathbf{b}$  (a  $p$ -form) and  $\mathbf{c}$  (a codimension  $p+1$  form) and action  $S = N \int_M \mathbf{b} \wedge d\mathbf{c}$  (for some fixed  $N$ ). The partition function is then

$$Z = \int D[\mathbf{b}]D[\mathbf{c}] \exp \left( -2\pi i N \int_M \mathbf{b} \wedge d\mathbf{c} \right)$$

For specificity, we will take  $d = 5$ , letting  $\mathbf{b}$  and  $\mathbf{c}$  be 2-forms. However, our discussion will make sense in any dimension.

We can allow different classes of gauge symmetries:

- R-gauge:  $\mathbf{b} \mapsto \mathbf{b} + d\lambda_1$
- $U(1)$  gauge:  $\mathbf{b} \mapsto \mathbf{b} + \omega$  for  $\omega \in H^2(M; \mathbb{Z})$
- Trivial gauge:  $\mathbf{b} \mapsto \mathbf{b} + \omega$  for  $\omega \in H^2(M; \mathbb{R})$ .

We may consider independent transformations for  $\mathbf{c}$ .

BF theory has observables (where  $\gamma'$  has codimension  $p+1$  and  $\gamma$  has dimension  $p$ ):

$$W_{\mathbf{c}}(\gamma) = \exp \left( 2\pi i \oint_{\gamma} \mathbf{c} \right)$$

$$W_{\mathbf{b}}(\gamma') = \exp \left( 2\pi i \oint_{\gamma'} \mathbf{b} \right).$$

These operators have different behaviors with respect to different gauge symmetries:

- R-gauge: operators are gauge invariant for all  $\mathbf{b}$  (or  $\mathbf{c}$ ).
- $U(1)$  gauge:  $\mathbf{b}$  (or  $\mathbf{c}$ ) must have holonomy in  $U(1)$ .
- Trivial gauge: none of these observables are gauge invariant.

We will focus on the case of  $U(1)$  gauge symmetry.

Note that BF theory doesn't have any "iterated gauge symmetries" or "Stückelberg fields." Thus we do not need to worry about some of the more complicated aspects of gauge theories.



### 3.3.3 Linking numbers from correlation functions

We'd like to compute

$$\begin{aligned}
\langle W_c(\gamma) W_b(\gamma') \rangle &= \int D[b] D[c] \exp \left( -S + 2\pi i \int_{\gamma} c + 2\pi i \int_{\gamma'} b \right) \\
&= \int D[b] D[c] \exp \left( -iN2\pi(b \wedge dc + b \wedge dc_0) + 2\pi i \int_{\gamma} (c + c_0) + 2\pi i \int_{\gamma'} b \right) \\
&= \int D[b] D[c] \exp \left( -iN2\pi b \wedge dc + 2\pi i \int_{\gamma} c \right) \exp \left( \frac{2\pi i}{N} \int_{\gamma} c_0 \right) \\
&= Z \exp \left( \frac{2\pi i}{N} L(\gamma, \gamma') \right)
\end{aligned}$$

where  $L(\gamma, \gamma')$  is the linking number of  $\gamma$  and  $\gamma'$ .

Under  $\gamma \mapsto \gamma + N\beta$ , we have  $W_c(\gamma) \sim W_c(\gamma + N\beta)$ , and thus  $W_c^N(\gamma) = 1$ . The analogous statement holds for  $W_b(\gamma')$ . The upshot is that the theory actually has a  $\mathbb{Z}_N \times \mathbb{Z}_N$  gauge symmetry, coming from the maps

$$H^1(-; \mathbb{Z}_N) \longrightarrow H^1(-; U(1)) \longrightarrow H^1(-; U(1))$$

where the last map is induced by  $a \mapsto a^N$ .

We can write

$$W_c(\gamma) W_b(\gamma') = \exp \left( \frac{2\pi i}{N} L(\gamma, \gamma') \right) W_b(\gamma') W_c(\gamma).$$

This can be read as saying that the operators  $W_c(\gamma)$  are topological operators generating a  $\mathbb{Z}_N$  global symmetry, where the charged objects are the  $W_b(\gamma')$ . The corresponding statement with the roles of  $b$  and  $c$  reversed is also true.

### 3.3.4 Quantizing BF theory

Classically, BF theory seems trivial: the equations of motion for  $b$  give  $Ndc = 0$ , and the equations of motion for  $c$  give  $Ndb = 0$ . Thus classical solutions are just flat connections. Interesting behavior only appears at the quantum level.

Canonical quantization for  $b$  and  $c$  gives

$$[b(x), c(y)] = \frac{2\pi i}{N} \delta(x - y) \text{Vol}(M).$$

We can write

$$e^{2\pi i \int_{\gamma} b} e^{2\pi i \int_{\gamma'} c} = e^{[2\pi i \int_{\gamma} b, 2\pi i \int_{\gamma'} c]} e^{2\pi i \int_{\gamma'} c} e^{2\pi i \int_{\gamma} b}$$

Combining this with the above construction of commutators recovers our previous results.

### 3.3.5 Cheeger-Simons maps for more general gauge theories

In general, consider a  $p$ -form  $A$  with gauge symmetry  $A \mapsto A + \omega$  for  $\omega \in H^p(M; \mathbb{Z})$ . We have a natural operator given by the Cheeger-Simons map

$$\chi(\Sigma_p) = \exp \left( 2\pi i \int_{\Sigma_p} A \right).$$

Field configurations for which  $\chi$  is gauge-invariant are classified by the differential cohomology  $\check{H}^{p+2}(M)$ , which fits into the hexagon

$$\begin{array}{ccccc}
& & \Omega^p / \Omega_{\mathbb{Z}}^p & \xrightarrow{d} & \Omega^{p+1} & & 0 \\
& \nearrow & \searrow & & \nearrow & \searrow & \\
H^p(M; \mathbb{R}) & & \check{H} & & & & H^{p+1}(M; \mathbb{R}) \\
& \searrow & \nearrow & & \searrow & \nearrow & \\
& & H^p(M; \mathbb{R}/\mathbb{Z}) & \xrightarrow{\beta} & H^{p+1}(M; \mathbb{Z}) & & 0 \\
0 & \nearrow & & & & \searrow & 
\end{array}$$

This is a lot larger than what one would naïvely expect. Things are much simpler in BF theory where we only consider flat connections. One can see this written up in a paper of Freed, Moore, and Segal.

### 3.4 6/12 – Global symmetries in variants of BF theory

#### 3.4.1 Simultaneous BF theories

Let's see what happens when we combine two versions of BF theory at once. Consider the 5d theory with fields  $b_2$ ,  $c_2$ ,  $A_1$ , and  $c_3$  and action

$$S = 2\pi i \int N b_2 \wedge d c_1 + 2N A_1 \wedge d c_3.$$

The first term gives a  $\mathbb{Z}_N^{(1)} \times \mathbb{Z}_N^{(1)}$  gauge symmetry, and the second term gives a  $\mathbb{Z}_{2N}^{(0)} \times \mathbb{Z}_{2N}^{(2)}$  gauge symmetry. From the operators  $W_c$  and  $W_b$  corresponding to the first term, we get  $\mathbb{Z}_N^{(2)}$  and  $\mathbb{Z}_N^{(2)}$  global symmetries. From the (similarly defined) operators  $V_A$  and  $V_{c_3}$ , we get  $\mathbb{Z}_N^{(3)}$  and  $\mathbb{Z}_N^{(1)}$  global symmetries.

#### 3.4.2 Interacting terms

Can we get non-invertible global symmetries from something like this Yes: add in non-dynamical fields  $\phi$  and  $\beta$ , and add an interaction term to the action:

$$S = 2\pi i \int N b_2 \wedge d c_1 + 2N A_1 \wedge d c_3 + \frac{1}{2N} (d\phi - 2N A) \wedge (d\beta - N b)^2.$$

This forces the gauge symmetry groups to interact with each other.

The new equations of motion include

$$2N d c_3 - (d\beta - N b)^2 = 0.$$

This equation implies that our original  $V_{c_3}$  is no longer a topological operator. We seek to correct this somehow. If  $\Sigma_3$  is a boundary, say  $\Sigma_3 = \partial Y$ , then we can require

$$V_{c_3}(\Sigma_3) = \exp \left( \frac{1}{2} N \int_Y b^2 \right).$$

In general, we can define a topological operator

$$V_{c_3}(\Sigma_3) = \int d[a] \exp \left( 2\pi i \int_{\Sigma_3} c_3 + 2\pi i \int_{\Sigma_3} N a \wedge da + a \wedge b \right)$$

Essentially, we have an anomaly on the bulk (breaking the  $\mathbb{Z}_N^{(2)}$  symmetry), and we define a new TQFT on the defect to cancel this out. This is “stacking.” There are other theories we could use to cancel the anomaly, but the above method is “minimal.”

### 3.4.3 Fusion rules

We can compute some fusion rules of this new  $V$  operator:

$$\begin{aligned} V(\Sigma_3) \otimes V^\dagger(\Sigma_3) &= \int d[a]d[a'] \exp \left( 2\pi i \int_{\Sigma_3} N(a da - a' da') + (a - a') \wedge b \right) \\ &= \int d[\alpha]d[\alpha'] \exp \left( 2\pi i N \int_{\Sigma_3} \alpha' \wedge d\alpha + \alpha d\alpha + \alpha \wedge b \right) \end{aligned}$$

or something similar. Ultimately, this localizes to

$$V \otimes V^\dagger = A(\Sigma_3) \sum_{b \in H^2(\Sigma_3; \mathbb{Z}_N)} e^{2\pi i \int_{\Sigma_3} b}.$$

for some TQFT coefficient  $A(\Sigma_3)$  (depending only on  $\Sigma_3$ , not on the bulk). The sum here is a “condensation defect.” Working this out precisely is left to the reader as an exercise (though if anyone reading this has all the details, feel free to add them).

The upshot is that the Lagrangian approach allows us to understand much of what’s going on with categorical symmetries. We can even get the TQFT coefficients out of this. Adding more interaction terms can give even more complicated fusion rules (which are still accessible via the Lagrangian presentation).

### 3.4.4 Super Yang-Mills

In general, we can consider the action of a 5d symmetry TFT on 4d  $SU(N)$  or  $PSU(N)$  super-Yang-Mills. This 4d theory has interesting topological operators: Wilson lines and ’t Hooft lines. Fully understanding the bulk TFT allows us to work with the symmetries of the boundary theory.

## Chapter 4

# Claudia Scheimbauer – Higher Categories for TQFT

### 4.1 6/13 – Motivation, Examples, and First Definition

There are many examples that shape how we might want to think about higher categories.

#### 4.1.1 Example 1: Categories

The collection of all categories naturally forms a 2-category. Instead of demanding isomorphism of categories, we typically only want to ask for “equivalence.” Formalizing this requires a notion of “natural isomorphism.”

To encapsulate this information, we need a structure with three layers:

- 0-morphisms / objects: categories,
- 1-morphisms: functors, and
- 2-morphisms: natural transformations.

We also need notions of composition of 1-morphisms and 2-morphisms.

#### 4.1.2 Example 2: Bordisms

In the Segal-Atiyah approach to QFT, we assemble the possible “spacetimes” into a “category”  $\mathbf{Cob}_{n,n-1}$  controlling cutting and pasting along codimension 1 slices. The category  $\mathbf{Cob}_{n,n-1}$  has:

- 0-morphisms / objects: closed  $(n-1)$ -manifolds with boundary, and
- 1-morphisms  $M_0 \rightarrow M_1$ :  $n$ -dimensional smooth bordisms (i.e. manifolds  $\Sigma$  with boundary together with a diffeomorphism  $\partial\Sigma \xrightarrow{\sim} M_0 \sqcup M_1$ ).

Here composition is via gluing along the common boundary (but see below!). We are often interested in adding additional structure to the objects of  $\mathbf{Cob}_{n,n-1}$ : Riemannian metrics, flat  $G$ -bundles, etc.

A (topological) quantum field theory is a symmetric monoidal functor  $Z : \mathbf{Cob}_{n,n-1} \rightarrow \mathbf{Vect}$ . In the topological case, we require that all structures appearing in  $\mathbf{Cob}$  are topological. In more general cases, we may consider modifications of the target, e.g. replacing  $\mathbf{Vect}$  by the category of topological vector spaces. We’ll focus on the case of TQFT.

As defined above, composition of bordisms does not necessarily produce a smooth bordism. There are a few ways to fix this. For example, we can consider bordisms only up to diffeomorphism, in which case we can adjust things so that composites inherit smooth structures. Alternatively, we can enhance bordisms by adding the data of collars around the boundary.

The former approach reflects the fact that bordisms really live in a higher category, with:

- 0-morphisms: closed  $(n - 1)$ -manifolds,
- 1-morphisms:  $n$ -dimensional bordisms, and
- 2-morphisms: diffeomorphisms between bordisms up to isotopy.

This allows us access to mapping class groups, which are interesting! From a physics perspective, we often want the values of our field theory to vary smoothly in the moduli space of bordisms. Really, we want to consider the space of diffeomorphisms as well, so we should allow 3-morphisms corresponding to isotopies, 4-morphisms corresponding to isotopies of isotopies, etc.

We often take our bordisms to be pointing “forward in time” – however, there’s no mathematical reason to prefer a specific direction. In particular, we should be able to compute  $Z$  by slicing up our manifolds along different directions. Note that, if we compare slices taken in two different directions, we typically get codimension 2 submanifolds along the intersections of the slices. Iterating this process produces closed submanifolds of higher codimension, and we should explain what  $Z$  assigns to such manifolds. To this end, we can expand our bordism category to a  $k$ -category, allowing

- 0-morphisms: closed  $(n - k)$ -manifolds,
- 1-morphisms:  $(n - k + 1)$ -dimensional bordisms,
- ...
- $(n - 1)$ -morphisms:  $(n - 1)$ -dimensional bordisms of bordisms of ... ,
- $n$ -morphisms:  $n$ -dimensional bordisms of bordisms of ...

Including the diffeomorphism groups discussed earlier, we should really consider an  $(\infty, k)$ -category  $\mathbf{Bord}_{n, n-k}$ .<sup>1</sup>

### 4.1.3 Example 3: Targets

If we want to define a TQFT as a functor out of  $\mathbf{Bord}_{n, n-k}$ , the codomain should also be an  $(\infty, k)$ -category. What  $(\infty, k)$ -categories should we consider?

**Example 4.1.1.** For a first example, consider functors from  $\mathbf{Cob}_{3,2,1}$ . Following Segal’s lectures, we may send  $S^1$  to an algebra, 1-dimensional bordisms to bimodules, and 2-dimensional bordisms of bordisms to bimodule homomorphisms. That is, we take functors valued in the Morita bicategory  $\mathbf{Alg}$ , with:

- 0-morphisms: algebras (over our favorite field),
- 1-morphisms  $A \rightarrow B$ :  $(A, B)$ -bimodules, and
- 2-morphisms: bimodule homomorphisms.

In particular, the TQFT sends 2-manifolds or 2-dimensional bordisms to scalars or more generally linear maps.

We can define other targets similarly.

### 4.1.4 What is a (weak) 2-category?

Now that we’ve completed our motivation, we should find an actual mathematical definition of higher categories.

Let’s first recall how we define monoidal categories:

**Definition 4.1.2.** A *monoidal category* is:

- A category  $\mathcal{M}$ , with

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<sup>1</sup>For non- $\infty$  categories, we will write  $\mathbf{Cob}$ . For  $(\infty, -)$ -categories, we will write  $\mathbf{Bord}$ .

- A tensor product functor  $\otimes : \mathcal{M} \times \mathcal{M} \rightarrow \mathcal{M}$ ,
- A unit object  $\mathbb{1}$ ,
- An associator natural transformation  $(A \otimes B) \otimes C \xrightarrow{\sim} A \otimes (B \otimes C)$ , and
- Left and right unitor natural transformations  $\mathbb{1} \otimes A \xrightarrow{\sim} A \xleftarrow{\sim} A \otimes \mathbb{1}$ .

These are required to satisfy:

- A “pentagon axiom” ensuring nice behavior of the associator, and
- A “triangle axiom” ensuring nice behavior of the unitors.

We say that the monoidal category  $\mathcal{M}$  is *strict* if the associators and unitors are the identity maps.

The definition of weak 2-categories is quite similar!

**Definition 4.1.3.** A *weak 2-category* or *bicategory*  $\mathcal{B}$  consists of:

- A collection of objects  $\text{ob } \mathcal{B}$ ,
- For all  $X, Y \in \text{ob } \mathcal{B}$ , a category of morphisms  $\text{Hom}_{\mathcal{B}}(X, Y)$ ,
- For all  $X, Y, Z \in \text{ob } \mathcal{B}$ , a composition functor  $\circ : \text{Hom}_{\mathcal{B}}(Y, Z) \times \text{Hom}_{\mathcal{B}}(X, Y) \rightarrow \text{Hom}_{\mathcal{B}}(X, Z)$ ,
- For all  $X \in \text{ob } \mathcal{B}$ , a functor  $*$   $\rightarrow \text{Hom}_{\mathcal{B}}(X, X)$  (corresponding to  $\text{id}_X$  and  $\text{id}_{\text{id}_X}$ ),
- An associator (natural isomorphism between threefold composites), and
- Unitors (natural isomorphisms between morphisms and their composites with the identity)

satisfying the pentagon and triangle axioms. We say that a 2-category  $\mathcal{B}$  is *strict* if the associators and unitors are the identities.

In general, we need the flexibility of “weak” / “non-strict” constructions. Most examples that arise in practice are not strict!

## 4.2 6/13 – Formal Definition

### 4.2.1 Examples of weak 2-categories

Some examples we’ve considered are (possibly weak) 2-categories.

**Example 4.2.1.** Categories, functors, and natural transformations form a strict 2-category.

**Example 4.2.2.** The collection of cobordisms down to codimension 2,  $\text{Cob}_{n,n-1,n-2}$ , forms a weak 2-category.

**Example 4.2.3.** Algebras, bimodules, and bimodule homomorphisms form a weak 2-category.

Checking the examples can be difficult, especially when we move into higher-dimensional cases. Furthermore, while weak 2-categories and strict 2-categories are (in some suitable sense) equivalent, this breaks down in higher dimensions. We’d like to find a suitable framework for dealing with higher category theory.

### 4.2.2 Simplicial sets and redefining categories

There are many definitions of  $(\infty, 1)$ -categories in the literature, but all are equivalent. The most common model is that of *quasicategory*, called  $\infty$ -category by Lurie. We'll use a somewhat different definition that is more suited to our needs. Let's first consider the case of ordinary categories.

Let  $\Delta$  be the skeletal category of nonempty linearly ordered finite sets, i.e. the category with:

- Objects:  $[0]$ ,  $[1]$ ,  $[2]$ , etc. where  $[n] = \{0 < 1 < \dots < n\}$
- Morphisms: (non-strictly) order-preserving maps.

**Example 4.2.4.** From  $[1]$  to  $[2]$ , there are six maps:

- $0 \mapsto 0, 1 \mapsto 1$ ,
- $0 \mapsto 0, 1 \mapsto 2$ ,
- $0 \mapsto 1, 1 \mapsto 2$ , and
- $0 \mapsto k, 1 \mapsto k$  for  $k = 0, 1$ , or  $2$ .

**Example 4.2.5.** From  $[2]$  to  $[1]$ , there are four maps:

- $0 \mapsto 0, 1 \mapsto 1, 2 \mapsto 1$ ,
- $0 \mapsto 0, 1 \mapsto 0, 2 \mapsto 1$ , and
- $0 \mapsto k, 1 \mapsto k, 2 \mapsto k$  for  $k = 0$  or  $1$ .

**Definition 4.2.6.** A *simplicial set* is a functor  $X_\bullet : \Delta^{\text{op}} \rightarrow \mathbf{Set}$ . More generally, a *simplicial object* in a category  $\mathcal{C}$  is a functor  $X_\bullet : \Delta^{\text{op}} \rightarrow \mathcal{C}$ . We write  $X_n$  for the image of  $[n]$  under  $X_\bullet$ .

**Example 4.2.7.** Let  $\mathcal{C}$  be a small category. The *nerve* of  $\mathcal{C}$  is the simplicial set  $N\mathcal{C} : \Delta^{\text{op}} \rightarrow \mathbf{Set}$  with  $N\mathcal{C}_0 = \text{ob } \mathcal{C}$  and  $(N\mathcal{C})_i$  the set of composable sequences of  $i$  morphisms. For example,

$$(N\mathcal{C})_2 = \sqcup_{X,Y,Z \in \text{ob } \mathcal{C}} \text{Hom}_{\mathcal{C}}(Y, Z) \times \text{Hom}_{\mathcal{C}}(X, Y) = \text{mor } \mathcal{C} \times_{s, \text{ob } \mathcal{C}, t} \text{mor } \mathcal{C}$$

The structure maps are given by source maps, target maps, composition, and diagonals.

For every  $i \in \{0, \dots, n-1\}$ , there is a morphism  $[1] \rightarrow [n]$  in  $\Delta$  given by  $0 \mapsto i$  and  $1 \mapsto i+1$ . In a simplicial set  $X$  these induce maps  $\gamma_i : X_n \rightarrow X_1$ .

**Example 4.2.8.** In  $N\mathcal{C}$ , these  $\gamma_i$  are given by  $(f_1, \dots, f_n) \mapsto f_i$ .

We may use the above discussion to arrive at a new definition of a category.

**Definition 4.2.9.** A *category* is a simplicial set  $X_\bullet$  such that the natural maps  $X_n \rightarrow X_1 \times_{X_0} X_1 \times_{X_0} \dots \times_{X_0} X_1$  (induced by the  $\gamma_i$ ) are bijections. A *monoid* is a category with  $X_0 = *$ .

### 4.2.3 Category objects

The advantage of the above definition is that it allows us to generalize the notion of category.

**Definition 4.2.10.** Let  $\mathcal{S}$  be a category with finite limits. A *category object* in  $\mathcal{S}$  is a simplicial object  $X_\bullet : \Delta^{\text{op}} \rightarrow \mathcal{S}$  satisfying the *Segal condition*: the natural maps  $X_n \rightarrow X_1 \times_{X_0} \dots \times_{X_0} X_1$  are isomorphisms. If  $\mathcal{S}$  has a notion of weak equivalence, then we take homotopy fiber products and require that the natural maps are weak equivalences.

**Example 4.2.11.** In  $\mathbf{Cat}$  with weak equivalences given by isomorphisms, the category objects are strict 2-categories. We can use this to produce an inductive definition of strict  $n$ -category. Namely, strict  $n$ -categories are category objects in  $\mathbf{Cat}_{n-1}$  (the category of strict  $(n-1)$ -categories) with weak equivalences given by isomorphisms.

**Example 4.2.12.** In  $\mathbf{Cat}$  with weak equivalences given by the usual equivalences, the category objects are weak 2-categories.

We can also consider category objects in the  $(\infty, 1)$ -category  $\mathbf{Spaces}$  of spaces (i.e.  $\infty$ -groupoids). If we are careful enough, we can avoid circular reasoning.

**Definition 4.2.13.** A *Segal space* (or *pre- $(\infty, 1)$ -category* or *flagged  $(\infty, 1)$ -category*) is a category object in  $\mathbf{Spaces}$ , i.e. a simplicial space  $X_\bullet : \Delta^{\text{op}} \rightarrow \mathbf{Spaces}$  satisfying the Segal condition  $X_n \xrightarrow{\sim} X_1 \times_{X_0} \cdots \times_{X_0} X_1$ . An  $(\infty, 1)$ -category is a flagged  $(\infty, 1)$ -category satisfying a “univalence equals completeness axiom”  $X_0 \xrightarrow{\text{sim}} X_1^{\text{inv}}$ .<sup>2</sup>

#### 4.2.4 Bordisms

We would like to understand bordisms from this perspective. Specifically, we seek to construct a simplicial object  $X_\bullet = (\mathbf{Bord}_{d,d+1})_\bullet$  where the  $X_n$  are composable sequences of  $n$  bordisms.

From another perspective, we can see  $X_n$  as the space of large bordisms with certain “allowed” cuts.

**Definition 4.2.14.** Let  $X_n$  be the “space” with each point labeled by

- An interval  $(a, b) \subset \mathbb{R}$ ,
- An  $n$ -dimensional submanifold  $M \subset (a, b) \times \mathbb{R}^\infty$ , and
- Closed subintervals  $I_0, \dots, I_n \subset (a, b)$  with endpoints  $a = a_0, b_0, \dots, a_n, b_n = b$  (giving collars of the cut values)

such that

- $a_0 \leq \dots \leq a_n$  and  $b_0 \leq \dots \leq b_n$ ,
- The projection  $\pi : M \rightarrow (a, b)$  is submersive over each  $I_i$ , and
- $\pi$  is proper.

The upshot here is that we avoid thinking about singularities by focusing entirely on  $n$ -dimensional objects (with  $(n-1)$ -dimensional objects viewed as cylinders). The restriction that  $\pi$  is submersive over each  $I_i$  avoids cutting at a Morse critical point (where the slice may be singular, e.g. a figure-eight). The properness of  $\pi$  ensures that all fibers are compact, so we don’t have to worry about anything going off to infinity.

**Example 4.2.15.** For  $X_0$ , we have  $I_0 = (a, b)$ , so  $\pi : M \rightarrow (a, b)$  is a proper submersion. Ehresmann’s fibration theorem implies that  $\pi$  is a fiber bundle, so contractibility of  $(a, b)$  tells us that  $M \cong M_0 \times (a, b)$  for some  $M_0$ . Thus  $X_0$ , the space of “objects,” can be identified with the space of  $(n-1)$ -manifolds.

### 4.3 6/14 – More Bordisms; Factorization Algebras

#### 4.3.1 Bordisms, continued

Last time, we sketched the construction of a Segal space  $X_\bullet = (\mathbf{Bord}_{d,d+1})_\bullet$ .<sup>3</sup> More precisely, we described the points of  $X_n$  as bordisms with  $n$  allowed cut loci. The topology is hard to describe precisely but comes from allowing “wiggling” of both the bordism and the intervals.

We should also describe the structure maps of the simplicial space. The face maps  $X_i \rightarrow X_{i-1}$  are given by forgetting cuts (for inner faces) or discarding the outer parts of the manifold (for outer faces). Degeneracy maps  $X_i \rightarrow X_{i+1}$  correspond to repeating intervals. These maps (and the intervals themselves) are essentially bookkeeping.

<sup>2</sup>This is essentially saying that we can reconstruct the objects from the equivalences.

<sup>3</sup>This is not complete in high dimensions due to the existence of  $H$ -cobordisms. Details can be found in Lurie’s paper on the cobordism hypothesis.



### 4.3.2 Extended bordisms and the cobordism hypothesis

Recall that  $(\infty, 1)$ -categories are defined as simplicial spaces  $\Delta^{\text{op}} \rightarrow \mathbf{Spaces}$  satisfying the Segal condition and a completeness condition. Similarly, we define  $(\infty, k)$ -categories to be  $k$ -simplicial spaces  $(\Delta^{\text{op}})^{\times k} \rightarrow \mathbf{Spaces}$  satisfying a Segal condition, a completeness condition, and “globularity.” From this perspective, defining  $(\infty, k)$ -categories is not too much harder than defining  $(\infty, 1)$ -categories.

To define an  $(\infty, k)$ -category of bordisms  $\mathbf{Bord}_{d, d-k}$ , we:

- Embed  $M$  in  $\mathbb{R}^k \times \mathbb{R}^\infty$ ,
- Take the “intervals” labeling our cuts to be rectangles in  $\mathbb{R}^k$ , and
- Modify the submersivity condition on  $\pi$ .

For  $k = d$ , we get an  $(\infty, d)$ -category  $\mathbf{Bord}_d := \mathbf{Bord}_{d, 0}$ . We say that a topological field theory is *fully extended* if it is defined on  $\mathbf{Bord}_d$ .

The *cobordism hypothesis* (due to Baez-Dolan, Lurie, and others) is a way of formalizing the notion that (fully extended) TFTs are fully local. This can be stated as follows. Let  $\mathcal{C}$  be a symmetric monoidal  $(\infty, d)$ -category. Write  $\mathbf{Bord}_d^{\text{fr}}$  for the *framed* bordism category. Then evaluation at pt gives an equivalence

$$\mathbf{Fun}^\otimes(\mathbf{Bord}_d^{\text{fr}}, \mathcal{C}) \xrightarrow{\sim} (\mathcal{C}^{d\text{-dualizable}})^\simeq$$

between the  $(\infty, d)$ -category of symmetric monoidal functors from  $\mathbf{Bord}_d^{\text{fr}}$  to  $\mathcal{C}$  and the maximal  $\infty$ -groupoid of  $d$ -dualizable objects in  $\mathcal{C}$ . For more details, see the lectures of David Jordan at the GCS summer school last year.

There are also variants allowing for different tangential structures on the TFTs. The above discussion doesn’t account for anomalous theories; corrections are needed to deal with these.

### 4.3.3 TFT targets

What should we take as the codomain  $\mathcal{C}$  of a TFT?

**Example 4.3.1.** One weak 2-category that shows up frequently in practice is the category  $\mathbf{Alg}_1(\mathbf{Vect})$  (denoted  $\mathbf{Alg}$  earlier) with:

- Objects: algebras,
- 1-morphisms: bimodules, and
- 2-morphisms: bimodule homomorphisms.

Here composition is given by the relative tensor product:  ${}_B N_C \circ {}_A M_B = {}_A M \otimes_B N_C$ . Note that this is only weakly associative.

**Example 4.3.2.** Trying to define a 3-categorical example, we can let  $\mathbf{Alg}_1(\mathbf{Cat}^{\text{lin}})$  be the weak 3-category with:

- Objects: tensor categories,
- 1-morphisms: bimodule categories,
- 2-morphisms: functors,
- 3-morphisms: natural transformations.

This is tricky – we first have to figure out what we mean by “weak 3-category.” However, this example is relevant to practical examples such as Witten-Reshetikhin-Turaev theories and Turaev-Viro theories.

**Example 4.3.3.** Recent work has made progress in defining  $\mathbf{Alg}_2(\mathbf{Cat}^{\text{lin}})$  and  $\mathbf{Alg}_3(\mathbf{Cat}^{\text{lin}})$ . This gets progressively harder as the subscript increases!

It seems difficult (but worthwhile) to understand what’s going on here. We’ll instead think about a different approach.

### 4.3.4 (Pointless) factorization algebras

Factorization algebras show up frequently in studies of TFT:

- They give a variation of algebraic QFT (which assigns an algebra  $\mathcal{A}(\mathcal{U})$  to every open subset  $\mathcal{U}$  of spacetime).
- They axiomatize observables in (perturbative) QFT – the work of Costello-Gwilliam has explained BV quantization from this perspective.
- They encapsulate  $A_\infty$ -algebras, bimodules,  $\mathbb{E}_n$ -algebras, etc.

We'll try to use these to understand TFT.

**Definition 4.3.4.** Let  $X$  be a topological space. The operad  $\text{Open}(X)^\otimes$  of opens in  $X$  has open sets of  $X$  as colors and one operation  $(\mathcal{U}_1, \dots, \mathcal{U}_n; V)$  if the  $\mathcal{U}_i$  are disjoint subsets of  $V$ . A *prefactorization algebra* on  $X$  with values in  $(\mathcal{S}, \otimes)$  (often  $\mathcal{S} = \text{Ch}$ ) is an algebra  $\mathcal{A}$  for this operad. In more detail, for every  $\mathcal{U}_1 \sqcup \dots \sqcup \mathcal{U}_n \subset V$ , we have  $\mathcal{A}(\mathcal{U}_1) \otimes \dots \otimes \mathcal{A}(\mathcal{U}_n) \rightarrow \mathcal{A}(V)$  “coherently.” That is, for  $\mathcal{O}_1 \sqcup \dots \sqcup \mathcal{O}_n \subset \mathcal{U}_1$ ,  $\mathcal{O}'_1 \sqcup \dots \subset \mathcal{U}_2$ ,  $\dots$ , and  $\mathcal{U}_1 \sqcup \mathcal{U}_2 \sqcup \dots \subset V$ , the triangle

$$\begin{array}{ccc} \mathcal{A}(\mathcal{O}_1) \otimes \dots \otimes \mathcal{A}(\mathcal{O}_n) \otimes \mathcal{A}(\mathcal{O}'_1) \otimes \dots & \xrightarrow{\quad\quad\quad} & \mathcal{A}(\mathcal{U}_1) \otimes \mathcal{A}(\mathcal{U}_2) \otimes \dots \\ & \searrow \quad \quad \quad \swarrow & \\ & \mathcal{A}(V) & \end{array}$$

commutes (up to coherent homotopy). This last condition is encapsulated nicely by the language of operads and  $(\infty, 1)$ -categories.

**Remark 4.3.5.** This makes sense with  $\text{Open}(X)$  replaced by any poset of opens in  $X$ .

Let's see how this captures the examples we've seen so far.

**Example 4.3.6.** Let  $X = \mathbb{R}$  and let  $A$  be an associative algebra over a field  $k$ . We can define  $\mathcal{A}$  on the poset  $\text{Int}$  of intervals in  $X$  via the assignment  $\mathcal{A}(I) = A$ , with  $\mathcal{A}(\emptyset) = k$ . For nonempty  $I$ , the map  $\mathcal{A}(\emptyset) \rightarrow \mathcal{A}(I)$ , equivalently  $k \rightarrow A$ , gives the unit of  $A$ . For nonempty  $I_1, I_2 \subset I_3$ , the map  $\mathcal{A}(I_1) \otimes \mathcal{A}(I_2) \rightarrow \mathcal{A}(I_3)$ , equivalently  $A \otimes A \rightarrow A$ , gives the multiplication on  $A$ . Coherence of  $\mathcal{A}$  corresponds to associativity of  $A$ .

**Example 4.3.7.** Let  $p \in X = \mathbb{R}$ , let  $M$  be an  $(A, B)$ -bimodule, and let  $m \in M$ . Define a prefactorization algebra on  $\text{Int}$  by  $\mathcal{A}(\emptyset) = k$

$$\mathcal{A}((x, y)) = \begin{cases} A & y \leq p \\ M & x < p < y \\ B & p < x. \end{cases}$$

This obtains a natural structure of prefactorization algebra. (The maps  $k \rightarrow M$  are just specified by the choice of  $m$ .)

**Example 4.3.8.** Let  $X = \mathbb{R}^2$  and consider the poset of disks within the unit disk. The corresponding operad is the  $\mathbb{E}_2$ -operad. The corresponding prefactorization algebras in  $\text{Vect}$  are commutative algebras, while the corresponding prefactorization algebras in  $\text{Cat}$  are braided monoidal categories.

## 4.4 6/14 – Factorization algebras and the Morita category

### 4.4.1 Factorization algebras

We've talked about prefactorization algebras – what does it mean to drop the “pre?”

**Definition 4.4.1.** A *factorization algebra* on a topological space  $X$  with values in  $(\mathcal{S}, \otimes)$  is a prefactorization algebra  $\mathcal{A}$  such that:

1.  $\mathcal{A}$  is multiplicative, i.e.  $\mathcal{A}(\mathbf{U}_1) \otimes \cdots \otimes \mathcal{A}(\mathbf{U}_n) \xrightarrow{\sim} \mathcal{A}(\mathbf{U}_1 \sqcup \cdots \sqcup \mathbf{U}_n)$  and  $\mathcal{A}(\emptyset) \simeq \mathbb{1}$ .
2.  $\mathcal{A}$  satisfies descent for Weiss covers.

Here a *Weiss cover* is a collection  $\mathcal{U}$  of open sets of  $X$  such that for all finite subsets  $S \subset X$ , there exists  $\mathbf{U}_S \in \mathcal{U}$  such that  $S \subset \mathbf{U}_S$ . Descent means that, if  $\mathcal{U}$  is a Weiss cover of  $V$ , then

$$\operatorname{colim}_{\{\mathbf{U}_1, \dots, \mathbf{U}_n\} \in \mathcal{U}} \mathcal{A}(\mathbf{U}_1 \cap \dots \cap \mathbf{U}_n) \xrightarrow{\sim} \mathcal{A}(V).$$

The Weiss condition is a bit strange and hard to think about. However, it gives us a way to extend from a prefactorization algebra on a basis to a factorization algebra on the whole space. This can be implemented using Kan extensions – see work of Karlsson-Scheimbauer-Walde.

**Example 4.4.2.** Given an associative algebra  $A$ , we get a prefactorization algebra  $\mathcal{A}$  on  $\operatorname{Int}$  as before. Multiplicativity tells us how to extend this to  $\operatorname{Int}^{\sqcup}$  (finite disjoint unions of intervals). The Weiss condition tells us how to extend to a factorization algebra on all of  $\operatorname{Open}(\mathbb{R})$ .

## 4.4.2 Constructibility

We are particularly interested in prefactorization algebras that interact well with stratified manifold structures.

**Definition 4.4.3.** Let  $X$  be a conically smooth manifold.<sup>4</sup> A prefactorization algebra  $\mathcal{A}$  on  $X$  is *constructible* if, for every inclusion  $\mathbf{U} \subset \mathbf{V}$  of abstractly isomorphic disks (accounting for the stratification), we get an equivalence  $\mathcal{A}(\mathbf{U}) \xrightarrow{\sim} \mathcal{A}(\mathbf{V})$ .

**Theorem 4.4.4** (Lurie). *Equip  $\mathbb{R}^n$  with the trivial stratification  $\emptyset \subset \mathbb{R}^n$ . There is an equivalence*

$$\operatorname{Fact}_{\mathbb{R}^n}^{\operatorname{cstr}} \simeq \mathbb{E}_n\text{-Alg}.$$

**Theorem 4.4.5** (Ayala-Francis-Tanaka). *Equip  $\mathbb{R}$  with the stratification  $\emptyset \subset \{\mathbf{p}\} \subset \mathbb{R}$ . There is an equivalence between  $\operatorname{Fact}_{\mathbb{R}}^{\operatorname{cstr}}$  and the  $(\infty, 1)$ -category of pointed bimodules.*

Constructible factorization algebras glue, so it suffices to define them locally. That is:

**Theorem 4.4.6** (Karlsson-Scheimbauer-Walde). *Given an open cover  $\mathcal{U}$  of  $X$ , there is an equivalence*

$$\operatorname{colim}_{\{\mathbf{U}_1, \dots, \mathbf{U}_n\} \subset \mathcal{U}} \operatorname{Fact}_{\mathbf{U}_1 \cap \dots \cap \mathbf{U}_n}^{\operatorname{cstr}} \xrightarrow{\sim} \operatorname{Fact}_X^{\operatorname{cstr}}.$$

**Example 4.4.7.** We can glue our “pointed bimodule” example to get a picture of smashing defects on the real line.

The preceding theorem is surprisingly difficult to prove – it had been claimed for years, but a proof is only appearing now. It is not known whether the result extends to all factorization algebras. The ultimate reason is that Weiss covers are hard to work with!

One can also prove other useful results in the constructible context.

**Theorem 4.4.8** (KSW). *The category  $\operatorname{Fact}_X^{\operatorname{cstr}}$  has a symmetric monoidal structure given (at a naïve level) by  $(\mathcal{F} \otimes \mathcal{G})(\mathbf{U}) = \mathcal{F}(\mathbf{U}) \otimes \mathcal{G}(\mathbf{U})$ .*

It’s also possible to understand constructible factorization algebras on a cone.

**Theorem 4.4.9** (Brav-Rozenblyum, KSW). *Let  $C(Z) = Z \times [0, 1)/(Z \times \{0\})$ . There is a pullback square*

$$\begin{array}{ccc} \operatorname{Fact}_{C(Z)}^{\operatorname{cstr}} & \longrightarrow & \operatorname{Fact}_{Z \times (0, 1)}^{\operatorname{cstr}} \\ \downarrow & & \downarrow \\ \operatorname{Fact}_{[0, 1]}^{\operatorname{cstr}} & \longrightarrow & \operatorname{Fact}_{(0, 1)}^{\operatorname{cstr}}. \end{array}$$

<sup>4</sup>This is a “nice” stratified space, e.g. a stratification of an  $n$ -dimensional manifold by smooth submanifolds.

One can understand the terms in this pullback individually:

- $\mathbf{Fact}_{[0,1]}^{\text{ctr}}$  consists of algebras together with a pointed left module.
- $\mathbf{Fact}_{(0,1)}^{\text{ctr}} \simeq \mathbb{E}_1\text{-Alg}$  by Lurie’s result above.
- $\mathbf{Fact}_{\mathbb{Z} \times (0,1)}^{\text{ctr}} \simeq \mathbf{Fact}_{\mathbb{Z}}^{\text{ctr}}(\mathbb{E}_1\text{-Alg})$  by work in progress of Švraka.

A few more comments on this last equivalence: Ayala-Berry proved that

$$\mathbf{Fact}_{X \times Y} \simeq \mathbf{Fact}_X(\mathbf{Fact}_Y)$$

in the non-constructible case. The work of Švraka extends this to the constructible case.

#### 4.4.3 Pointless factorization algebras

We’d like to use the above results to understand the Morita bicategory and other important target categories in TFT. However, what we’ve done so far requires *pointed* bimodules, which aren’t quite what we want! Work in progress of Karlsson fixes this.

Let  $X$  be a stratified space, and let  $X_0$  be a collection of chosen points (forming a 0-dimensional stratum). Replace  $\mathbf{Open}(X)$  by the poset  $\mathbf{Open}^m(X)$  consisting of all open sets of  $X$  but with only inclusions  $U \subset V$  such that  $U \cup X_0 \xrightarrow{\sim} V \cap X_0$  abstractly. Karlsson’s work redoes the above theorems in this context.

#### 4.4.4 The Morita category

From the above perspective, we can construct the Morita category as the Segal space  $X_\bullet : \Delta^{\text{op}} \rightarrow \mathbf{Spaces}$  with:

- $X_0$  consisting of constructible factorization algebras on  $\mathbb{R}$  (equivalently,  $\mathbb{E}_1$ -algebras),
- $X_1$  consisting of constructible factorization algebras on  $\mathbb{R}$  with one marked point (almost...), i.e. bimodules,
- $X_2$  consisting of constructible factorization algebras on  $\mathbb{R}$  with two marked points (almost...), i.e. pairs of compatible bimodules,
- ...

The face maps come from either forgetting the leftmost or rightmost algebra and bimodule or tensoring adjacent pairs of bimodules. From the factorization algebra viewpoint, this corresponds to a pushforward. Degeneracy maps require some cleverness – for example, we can use an approach like that used for bordisms above. One can also modify the definitions slightly to get an easier definition. There’s some hard bookkeeping so that we can allow the points to vary.

We can also replace  $\mathbf{Spaces}$  by  $\mathbf{Cat}_{(\infty,1)}$  or other interesting targets. This recovers other notions of Morita categories.

This construction is due to Calaque-Scheimbauer and is in Scheimbauer’s thesis. Work of Karlsson will extend this to the pointless case. Haugseng accomplishes a similar goal without factorization algebras.

#### 4.4.5 Outlook

From this perspective, we can construct a pointed version of  $\mathbf{Alg}_n$  based on a stratification of  $\mathbb{R}^n$ . To understand dualizability, work of Karlsson uses the geometry of factorization algebras (as opposed to the operadic approach of Haugseng). Future work will compare Haugseng’s approach to the pointless version.

There is also much work on understanding unitarity in this context. Many people involved in this task will be at the conference.

## Chapter 5

# Ryan Thorngren – Symmetry Breaking, Higher Berry Phase, and the Long Exact Sequence

These talks are based on joint work with A. Debray, S. Devalapurkar, C. Krulewski, Y. L. Liu, N. Pacheco-Tallaj, available at <https://arxiv.org/abs/2309.16749> and <https://arxiv.org/abs/2405.04649>. A sketch of what we'd like to discuss:

1. Symmetry breaking defects including  $\rho$ -defects, domain walls, and vortices
2. Gapless modes and defect / bulk anomaly matching
3. Family anomalies, higher Berry phase, phase diagrams
4. Incorporating symmetries, index theory
5. Long exact sequence connecting everything

### 5.1 6/13 – Symmetry Breaking and Defects

#### 5.1.1 Symmetry breaking

There are multiple types of “symmetry breaking” in physics. We'll consider spontaneous symmetry breaking and explicit symmetry breaking.

In *spontaneous symmetry breaking*, there is a global symmetry (e.g. a Lie group  $G$ ), but the ground states are not symmetric (or perhaps their symmetries lie in a subgroup  $H \subset G$ ).

**Example 5.1.1.** Consider four vertices which lie in a square. We'd like to connect the vertices by a collection of strings of shortest total length. There are two solutions, both somewhat between an H shape and an X shape. Each solution has  $D_4$  symmetry rather than the  $D_8$  symmetry of the point configuration.

**Example 5.1.2.** We'd like to minimize the potential  $V = -x^2 + x^4$  on  $\mathbb{R}$ . The potential is preserved by the  $\mathbb{Z}_2$  reflection symmetry of  $\mathbb{R}$ . However, this swaps the minima, so the symmetry is broken to the trivial group.

In *explicit symmetry breaking*, a system which has a symmetry  $G$  is subjected to small perturbations breaking the  $G$ -symmetry.

**Example 5.1.3.** In the last example, we can change our potential to  $V = \epsilon x - x^2 + x^4$ . This breaks the symmetry between the minima.

### 5.1.2 Spatial locality

**Example 5.1.4.** Consider the problem of minimizing

$$E = \int [(\nabla\phi)^2 - \phi^2 + \phi^4] dx.$$

This is just given by  $\phi$  taking a constant value at a minimum of  $V(x) = -x^2 + x^4$ . Let's make things more interesting and assume we're doing thermodynamics or quantum physics, so that we want to understand “nearly minimal” configurations as well. This allows  $\phi$  to flip between different minima along *domain walls*.

Define the *correlation function*

$$\langle \phi(0)\phi(x) \rangle_c = \langle \phi(0)\phi(x) \rangle - \langle \phi(0) \rangle \langle \phi(x) \rangle.$$

The value of this is determined by the presence or absence of domain walls.

**Example 5.1.5.** Let  $G = U(1)$  and  $H = 1$ . Consider a complex scalar field  $\phi : X \rightarrow \mathbb{C}$  and the potential  $V = -|\phi|^2 + |\phi|^4$ . For  $\phi = re^{ikx}$ , we get energy scaling with  $k$ . These configurations are known as *Goldstone modes*.

For  $\phi = z$ , we get logarithmic energy, with  $\partial_\theta \phi$  and  $(\partial_\theta \phi)^2$ . This spoils the order in 2d, and we get “quasi-long-range order” where the correlation function decays according to a power law.

### 5.1.3 Symmetry defects

Suppose that our symmetry group  $G$  acts transitively on the space of ground states (which can then be identified with  $G/H$ ). For spontaneous symmetry breaking, codimension  $k$  defects correspond to  $\pi_{k-1}(G/H)$ .

Explicit symmetry breaking is given by a linear perturbation living in a representation  $\rho$  of  $G$ .

**Example 5.1.6.** For  $V = \epsilon x - x^2 + x^4$ , the perturbation transforms according to the sign representation of  $\mathbb{Z}_2$ .

More generally, we can write  $\Delta V = \int d^d x \epsilon_a \mathcal{O}^a(x)$ .

**Example 5.1.7.** For

$$E = \int d^d x [(\nabla \vec{\phi})^2 - \|\vec{\phi}\|^2 + \|\vec{\phi}\|^4 + \vec{\epsilon} \cdot \vec{\phi}],$$

the perturbation transforms with the vector representation of  $O(N)$ .

We can construct a “ $\rho$ -defect” (for  $\dim \rho = k$ ) as

$$\epsilon_a(y_1, \dots, y_k, x_{k+1}, \dots, x_d) = f\left(\sum_{i=1}^k y_i^2\right) \cdot \frac{y_a}{|y_a|}$$

for some scalar-valued  $f$ , typically growing linearly near zero and with or without a cutoff at infinity. This  $\epsilon$  “winds around a  $(k-1)$ -sphere  $S(\rho) \subset \rho$ .” The defect has codimension  $k$ .

The explicit symmetry breaking defects we've seen so far are examples of  $\rho$ -defects.

### 5.1.4 Jackiw-Rebba zero modes

Consider 1+1d QFT with (real or complex) two-component fermions. Say  $\gamma^0 = iY$  and  $\gamma^1 = X$ . We take the Lagrangian

$$\mathcal{L} = i\psi^\dagger \gamma^0 \gamma^\mu \psi + im\psi^\dagger \gamma^0 \psi.$$

The equations of motion are

$$(\gamma^\mu \partial_\mu + m)\psi = 0,$$

so we get solutions

$$\psi^i(x, t) = u^i(E, k) e^{iEt + ikx}$$

where  $E = \sqrt{k^2 + m^2}$ .

The lowest energy state ( $k = 0$ ) has energy  $m$  – we call this a *mass gap*. Letting our mass vary with space according to an odd function, we get gapped systems both left and right of the origin. We get a 0-dimensional  $\rho$ -defect (with  $\rho$  the sign representation) for time-reversal symmetry  $\psi \mapsto \gamma^0 \psi$  and  $m \mapsto -m$ .

If  $m(x) = x$ , the equations of motion

$$(\gamma^\mu \partial_\mu + x) \psi = 0$$

have a time-independent solution  $\psi_\pm = u_\pm e^{-\mp x^2/2}$  where  $\gamma^1 u_\pm = \pm u_\pm$ . With the  $+$  sign, this is normalizable; with the  $-$  sign, this is not. Somehow this ties in to anomalies.

**Exercise 5.1.8.** For a  $2+1d$  theory with real 2-component fermions and  $\gamma^2 = Z$ , show that for

$$\psi^i(t, x_1, y_2) = f(x_1) \chi^i(t, y_2),$$

we can choose  $f(x_1)$  such that  $\chi^i$  is chiral and satisfies the massless Majorana equation.

**Exercise 5.1.9.** For a  $3+1d$  theory with complex 2-component Weyl fermions, considering

$$x_1 \bar{\psi} \psi + x_2 i \bar{\psi} \gamma^3 \psi,$$

there is a real chiral massless fermion along the vortex line.

We'd like to understand how these behave as we add deformations.

## 5.2 6/13 – ('t Hooft) Anomalies

### 5.2.1 General theory

Suppose we have a quantum system with  $D$ -dimensional spacetime  $X$  and background  $G$ -gauge field / symmetry  $A$ . In general,  $Z(X, A)$  transforms via

$$Z(X, A^g) = e^{i\Omega(X, A, g)} Z(X, A).$$

One can think of  $Z$  as a *section of a line bundle* on the space of connections on  $X$  (rather than as a *function*) on said space). Stated better, one may view  $|Z(X, A)\rangle$  as an element of a Hilbert space of an invertible  $(D+1)$ -dimensional TQFT on  $X \times [0, 1]$ . Here  $X \times [0, 1]$  (with  $e^{i\Omega(\dots)}$ ) connects  $X \times 0$  (with  $Z(X, A)$ ) to  $X \times 1$  (with  $Z(X, A^g)$ ).

“Cobordism theory” tells us that these form a group  $\Omega_G^{D+1}$ . The anomaly TQFTs are determined by their partition functions  $\omega(\Omega^{D+1}, A)$ . Furthermore,  $e^{i\omega(\dots)}$  behaves like the holonomy of a connection. The actual topological quantity here is the Chern number of this connection. We get a universal coefficient sequence

$$1 \longrightarrow \Omega_{D+1}^{G, \text{tors}} \longrightarrow \Omega_G^{D+1} \longrightarrow \text{Hom}(\Omega_{D+2}^G, \mathbb{Z}) \longrightarrow 1$$

Here  $\Omega_r^G$  is the bordism group of  $r$ -manifolds with  $G$ -connection. As with the usual universal coefficient sequence, the above sequence splits, but not naturally.

### 5.2.2 Adding structure

We'd like to consider manifolds with more structure. For bosonic theories, we want an orientation. For fermionic theories, we want a spin structure. We'll refer to these and similar topological structures as *s-structures* on the tangent bundle  $TX$ . In general, we'll “twist” these as follows.

Take a *twisting vector bundle*  $\eta$  on  $BG$ , and view our gauge field as giving  $A : X \rightarrow BG$ . We say that an  *$\eta$ -twisted s-structure* is an *s-structure* on  $TX \oplus A^* \eta$ .

The class  $w_1(\eta) : G \rightarrow \mathbb{Z}_2$  controls which elements reverse the orientation of spacetime.

**Example 5.2.1.** Let  $G = U(1)$ , let  $s = SO$  be orientation structure, and let  $\eta$  be trivial. Then

$$\Omega_{U(1), SO}^3 = \text{Hom}(\Omega + 4^{U(1), SO}, \mathbb{Z}) = \mathbb{Z}^{c_1^2} \oplus \mathbb{Z}^{p_1}.$$

In general, we have a useful map  $H^{D+2}(\text{BG}; \mathbb{Z}) \rightarrow \Omega_{G,s}^{D+1}$ .

**Example 5.2.2.** Majorana zero modes are associated with the Arf invariant in  $\Omega_{\text{Spin}}^2 = \mathbb{Z}_2$ .

**Example 5.2.3.** Consider time reversal symmetry with  $T^2 = 1$ . Let  $w_2(\eta) = 0$  and  $w_1(\eta) \neq 0$  – say  $\eta$  is the sign representation of  $\mathbb{Z}_2$ . Note

$$0 = w_1(\text{TX} \oplus A^*\eta) = w_1(\text{TX}) + w_1(A^*\eta) = w_1(\text{TX}) + A$$

and

$$0 = w_2(\text{TX} \oplus A^*\eta) = w_2(\text{TX}) + w_1^2(\text{TX}).$$

We end up with  $\text{Pin}^-$  appearing somehow. Note  $\Omega_{\text{Pin}^-}^2 = \Omega_{\mathbb{Z}_2, \text{Spin}, \text{sgn}}^2 = \mathbb{Z}_8$ .

### 5.2.3 Defect anomaly matching

Consider a gauge group  $G$  with a representation  $\rho$ . Assume our theory has operators  $\mathcal{O}^a$  transforming with  $\rho$  such that, for large  $\epsilon_a \in \rho$ , the theory perturbed by  $\epsilon_a \mathcal{O}^a$  is non-degenerately gapped. We call such a theory  $\rho$ -gappable. This produces a  $\rho$ -defect local  $(D - k)$  dimensional system. How can we relate the anomalies with the bulk?

On  $X^{D+1}$  with background field  $A$ , we want to choose a section  $\phi$  of  $A^*\rho$ . Let  $Y = \phi^*(0)$ ; for generic  $\phi$ , this is a codimension  $k$  submanifold. The theory is trivialized away from  $Y$ , so we can write

$$\omega(X^{D+1}, A, \xi) = \alpha(Y^{D-k+1}, A|_Y, \xi|_Y).$$

This lets us reconstruct the bulk anomaly  $\omega$  from the defect  $\alpha$ . Here  $\xi$  is an  $\eta$ -twisted  $s$ -structure on  $\text{TX}$ .

Note that  $\text{TX}|_Y = \text{TY} + \text{NY}$ , where  $\text{NY}$  is the vector bundle associated to  $A^*\rho|_Y = A|_Y^*\rho$ . Thus  $\xi|_Y$  is actually an  $(\eta + \rho)$ -twisted  $s$ -structure.

We end up with a linear map

$$\text{Def}_\rho : \Omega_{G,s,\eta+\rho}^{D-k+1} \rightarrow \Omega_{G,s,\eta}^{D+1}.$$

This satisfies  $\text{Def}_\rho \alpha = \omega$ . Note that, if this map is not injective, then  $\alpha$  is not uniquely defined by  $\omega$ . Similarly, if the map is not surjective, there may not be an  $\alpha$  corresponding to  $\omega$ . (This latter case is covered by the possibility that the theory is not  $\rho$ -gappable.)

The extra twisting here arises from the fact that, in the presence of a  $\rho$ -defect, we don't really have  $G$ -symmetry. One can understand this in examples involving CRT symmetry.

**Example 5.2.4.** Consider a 3+1d complex 4-component Dirac fermion  $\psi_L, \psi_R$  with  $U(1)_L$  background field. Take  $\eta$  to be trivial. We get an anomaly

$$\frac{1}{6}c_{1,L}^3 - \frac{1}{24}c_{1,L}p_1(\text{TX}^6).$$

Let  $\rho$  be the charge 1 representation. Take a section  $\phi$  of the bundle associated with  $A^*\rho$ ; then  $Y = \phi^{-1}(0)$  is Poincaré dual to  $c_{1,L}$ . We can write

$$\int_{X^6} \frac{1}{6}c_{1,L}^3 - \frac{1}{24}c_{1,L}p_1(\text{TX}^6) = \int_Y \frac{1}{6}c_{1,L}^2 - \frac{1}{24}p_1(\text{TX}).$$

We can write  $p_1(\text{TX})|_Y = p_1(\text{TY} \oplus A^*\rho) = p_1(\text{TY}) + p_1(A^*\rho)$ . As  $\mathbb{C} \otimes A^*\rho = A^*\rho \oplus \overline{A^*\rho}$ , we get  $p_1(\text{TX})|_Y = p_1(\text{TY}) + c_{1,L}^2$ . It follows that the defect anomaly is

$$\frac{1}{8}c_1^2 - \frac{1}{24}p_1(\text{TY}) \in \Omega_{\text{Spin}^c}^3.$$

This matches the anomaly of the 1+1d Weyl fermion with charge 1 under  $U(1)$ .



## 5.3 6/14 – Defect Anomalies and Families

### 5.3.1 Review

Last time, we constructed a defect anomaly map

$$\text{Def}_\rho : \Omega_{G,s,\eta+\rho}^{D-k+1} \rightarrow \Omega_{G,s,\eta}^{D+1}.$$

This sends the anomaly of the  $\rho$ -defect to the bulk anomaly.

**Example 5.3.1.** For a 3+1d  $U(1)$ -symmetric Dirac fermion, where the map is

$$\text{Def}_\rho : \Omega_{\text{Spin}^c}^3 \rightarrow \Omega_{U(1),\text{Spin}}^5.$$

We can show that this is an isomorphism

$$\mathbb{Z}^{c_1^2} \oplus \mathbb{Z}^{c_1^2/8 - p_1/24} \rightarrow \mathbb{Z}^{c_1^3/6 - c_1 p_1/24} \oplus \mathbb{Z}^{c_1^3}.$$

More generally, we want to understand the kernel and image of  $\text{Def}_\rho$ .

### 5.3.2 Families of QFTs

**Example 5.3.2.** Consider a 1+1d Dirac fermion with

$$a\bar{\psi}\psi + ib\bar{\psi}\gamma^c\psi.$$

Let's understand the behavior of the theory as a function of  $(a, b)$ . At  $a = b = 0$ , this is gapless. Away from  $(0, 0)$ , this theory has trivial phase.

We can ask whether the gapless point is protected, or whether it will disappear under small perturbations. With  $U(1)$  symmetry, this point is protected: as we move (“adiabatically slowly”) along a circle around the gapless point, we get a current  $j = \partial_t \theta / 2\pi$ . Moving around a full circle, one unit of charge gets “pumped” via the “Thouless pump.” This charge is quantized, so we can't get rid of the gapless point.

This is analogous to Berry phase.

**Example 5.3.3.** Consider a two-state system with Hamiltonian

$$H = a \cdot \sigma^x + b \cdot \sigma^y + c \cdot \sigma^z.$$

At  $a = b = c = 0$ , we have a twofold degenerate state. This is topologically protected: moving around it corresponds to a line bundle on  $S^2$  with Chern number 1. In optics, such a point is called a “diabolical point.”

In QFT, we can also find nontrivial families without symmetry.

**Example 5.3.4.** Consider the 1+1d CFT  $SU(2)_1$  (where 1 is the “level”). Add an  $SO(4)$  vector to break the symmetries and get a trivial theory. We get an interesting  $S^3 \subset \mathbb{R}^4$ . We may take an effective field theory  $\phi : X \rightarrow S^3$  which contains a Wess-Zumino–Witten theory at level 1. This theory has a quantized term protecting the behavior of our original theory at the origin. We may call this a “higher Berry phase.”

We want to generalize our cobordism theory to include such  $\phi$ . Let  $\Omega_{G,s,\eta}^D(M)$  classify cobordism invariants of  $D$ -manifolds  $X$  with  $G$ -background field,  $\eta$ -twisted  $s$ -structure, and  $\phi : X \rightarrow M$ . This can also be viewed as parametrizing  $G$ -symmetric theories of invertible theories.

**Example 5.3.5.** We can write  $\Omega_{U(1),SO}^2(S^1) = \Omega_{U(1),SO}^2 \oplus \Omega_{U(1),SO}^1$ . Here  $\Omega_{U(1),SO}^1 = \mathbb{Z}$  is what gets pumped. We can write

$$e^{iS_{\text{eff}}} = e^{i \int_{X^2} \wedge \frac{d\phi}{2\pi}}$$

where  $j_x = \partial_t \phi / 2\pi$  is our current. Note that this is bosonic: we get similar phenomena in both bosonic and fermionic cases.

Family anomalies happen at boundaries of nontrivial invertible families.

**Example 5.3.6.** Consider a Thouless pump with boundary (looking something like an interval). The current flows in one direction, accumulating charge at the endpoint.

Suppose in particular that we have a level crossing along the ray  $\theta = \pi$  in parameter space. At the origin, we have a bulk diabolical point. Away from the origin and  $\theta = \pi$ , we have a unique ground state. There are also two rays on which we have doubly degenerate ground states.

The above is largely academic; the following is more exciting.

**Example 5.3.7.** Consider 1+1d QED with  $N$  charge 1 bosons (giving  $\mathbb{CP}^{n-1}$ ), a  $\theta$ -angle, and  $\text{PSU}(N)$  flavor symmetry. For  $N$  even, there is an anomaly at  $\theta = \pi$ . This corresponds to  $u_3 \in H^3(\text{BPSU}(N), U(1)) = \mathbb{Z}_N$ .

For  $N$  odd, we may choose  $k$  such that  $2k \equiv 1 \pmod{N}$ . We end up with  $u_3 = du_2/N$ , giving a counterterm  $ku_2/N$  canceling the anomaly. There's still a family anomaly

$$\frac{1}{N} \int \frac{d\phi}{2\pi} u_2(\text{PSU}(N))$$

coming from the fact that there's no choice of counterterm over the whole family.

Similar stories can be told in other contexts.

### 5.3.3 Explicit symmetry breaking in families

In general, we want to study the explicit symmetry breaking family we get on a big sphere  $S(\rho)$  in the representation  $\rho$ . However,  $G$  acts on  $S(\rho)$  because we broke the symmetry. Thus we don't have a family of symmetric theories on  $S(\rho)$  but rather a “ $G$ -equivariant family on  $S(\rho)$ .” These can also be studied using bordism.

Say we have an anomaly-free  $G$ -equivariant  $M$ -family of invertible theories where  $G \curvearrowright M$ . Via the cobordism hypothesis, this can be viewed as an invertible TQFT for manifolds equipped with a  $G$ -gauge field  $A$ , a (possibly  $\eta$ -twisted)  $s$ -structure  $\xi$ , and a section  $\phi$  of  $P_A \times^G M$ . This latter map is equivalent to a map  $X \rightarrow M/_h G$ , as can be seen from the pullback square

$$\begin{array}{ccc} P_A \times_G M & \longrightarrow & M/_h G \\ \downarrow & & \downarrow \\ X & \longrightarrow & BG. \end{array}$$

Such TQFTs are classified by  $\Omega_{G,s,\eta}^D(M)$ .

## 5.4 6/14 – The long exact sequence and applications

### 5.4.1 The long exact sequence

Recall that we wanted to compute the kernel and image of the defect map  $\text{Def}_\rho : \Omega_{G,s,\eta+\rho}^{D-k+1} \rightarrow \Omega_{G,s,\eta}^{D+1}$ . This can be done using the *symmetry breaking long exact sequence*

$$\dots \longrightarrow \Omega_{G,s,\eta}^D(S(\rho)) \xrightarrow{\text{Ind}_\rho} \Omega_{G,s,\eta+\rho}^{D-k+1} \xrightarrow{\text{Def}_\rho} \Omega_{G,s,\eta}^{D+1} \xrightarrow{\text{Res}_\rho} \Omega_{G,s,\eta}^{D+1}(S(\rho)) \longrightarrow \dots$$

Here  $\text{Ind}_\rho$  is the *index map* and  $\text{Res}_\rho$  is the *residual family map*.

### 5.4.2 Examples: Majorana fermions

**Example 5.4.1.** Consider a 2+1d Majorana fermion  $\psi$ . We have a time reversal symmetry  $T : \psi \mapsto \gamma^0 \psi$  with  $T^2 = (-1)^F$ . This has an anomaly in  $\Omega_{\text{Pin}^+}^4 = \mathbb{Z}_{16}$ . Taking  $\rho$  to be the sign representation and considering  $\text{im} \bar{\psi} \psi$  with  $T$  odd, we have two solutions  $\psi_+$  (right-moving) and  $\psi_-$  (left-moving) corresponding

to eigenvectors of  $\gamma^1$ . Only one of these is normalizable, but the choice depends on the signs involved. We end up with a gravitational anomaly  $\pm 1 \in \mathbb{Z} = \Omega_{\text{Spin}}^3$ , where the sign agrees with  $\psi_{\pm}$ .

Since we are dealing with Majorana fermions, charge conjugation acts trivially, so CRT sends

$$\psi(x_1, y_2, t) \mapsto \gamma^1 \gamma^0 \psi(-x_1, y_2, -t).$$

The symmetry on the defect is  $U := RT^2 : \psi \mapsto -\gamma^1 \psi$ . We can write  $\Omega_{\text{Spin}, \mathbb{Z}_2}^3 = \mathbb{Z} \oplus \mathbb{Z}_8$  where  $\mathbb{Z}_8$  controls the number of right movers minus the number of left movers. The right-moving chiral mode has anomaly  $(1, 1)$ , and the left-moving mode has anomaly  $(-1, 0)$ .

The defect map here is  $\text{Def}_{\rho} : \Omega_{\text{Spin}, \mathbb{Z}_2}^3 \rightarrow \Omega_{\text{Pin}^+}^4$ , and we can write this numerically as

$$\begin{aligned} \mathbb{Z} \oplus \mathbb{Z}_8 &\rightarrow \mathbb{Z}_{16} \\ (a, b) &\mapsto -a + 2b. \end{aligned}$$

The kernel is  $\mathbb{Z} \cdot (2, 1)$ , looking like two right movers, one of which is charged.

**Example 5.4.2.** Let's modify the preceding example to have  $\psi_1$  and  $\psi_2$  with  $T : \psi_k \mapsto (-1)^k \gamma^0 \psi_k$ . This theory is anomaly free. Here  $i\bar{\psi}_1 \psi_1 - i\bar{\psi}_2 \psi_2$  lives in the sign representation.

Solving this theory, we get two right movers, one of which is charged. For  $m_1 > 0$  and  $m_2 < 0$ , we get  $1 \in \mathbb{Z} = \Omega_{\text{Spin}}^3$ , corresponding to  $p + ip$ . For  $m_1 < 0$  and  $m_2 > 0$ , we get  $-1 \in \mathbb{Z} = \Omega_{\text{Spin}}^3$ , corresponding to  $p - ip$ . The symmetry exchanges the movers, so the eigenvalues must be  $\pm 1$ .

### 5.4.3 Index maps

Let's consider  $\Omega_{G, s, \eta}^D(S(\rho))$ . This classifies anomaly-free  $G$ -equivariant invertible families on  $S(\rho)$ . The index map  $\text{Ind}_{\rho} : \Omega_{G, s, \eta}^D(S(\rho)) \rightarrow \Omega_{G, s, \eta + \rho}^{D-k+1}$  corresponds to compactification on  $S(\rho)$ . We can compute

$$\text{Ind}_{\rho}(\beta)(Y^{D-k+1}, A, \xi) = \beta(Y^{D-k+1} \times_A S(\rho), \pi^* A, \pi^* \xi, \phi)$$

where  $\phi : Y^{D-k+1} \times_A S(\rho) \rightarrow S(\rho)$ .

**Example 5.4.3.** Returning to our example of two Majorana fermions, the domain of  $\text{Ind}_{\rho}$  is  $\Omega_{\text{Pin}^+}^3(S^0)$ . Here  $S^0$  corresponds to the two points / theories  $p - ip$  and  $p + ip$ . The  $T$ -symmetry exchanges these two, so we have  $\Omega_{\text{Pin}^+}^3(S^0) = \mathbb{Z}$ . The index map corresponds to stacking the theories.

### 5.4.4 Residual family maps

Recall that, for the  $\rho$ -defect to be local, the theory must be non-degenerately gapped on  $S(\rho)$ . The map  $\text{Res}_{\rho} : \Omega_{G, s, \eta}^{D+1} \rightarrow \Omega_{G, s, \rho}^{D+1}(S(\rho))$  breaks the symmetry of the  $(D+1)$ -dimensional anomaly theory and tracks the  $S(\rho)$ -family. Mathematically,

$$\text{Res}_{\rho}(\omega)(X^{D+1}, A, \xi, \phi) = \omega(X^{D+1}, A, \xi).$$

By ignoring  $\phi$ , we are allowing more flexibility in constructing counterterms.

This map is often zero – for example, it is zero in the Majorana fermion example we considered.

**Example 5.4.4.** Consider a 2+1d Majorana fermion with  $T^2 = (-1)^F$ , taking  $\rho$  to be two times the sign defect. Can we find  $T$ -odd operators  $\mathcal{O}_1, \mathcal{O}_2$  such that, for  $r \gg 0$  and all  $\theta$ , the theory plus  $r \cos \theta \cdot \mathcal{O}_1 + r \sin \theta \mathcal{O}_2$  is trivially gapped?

No! To compute the residual family anomaly, we can use any choice of  $\mathcal{O}_1$  and  $\mathcal{O}_2$ , e.g.

$$\mathcal{O}_1 = \mathcal{O}_2 = i\bar{\psi}\psi.$$

The theory is not  $\rho$ -gapped: any circle around the origin contains two gapless points (on the line  $m = 0$ ). One of these points adds a  $p + ip$ , while the other point adds a  $p - ip$ . This corresponds to a 3+1d family on  $S(\rho)$  where one point pumps  $p + ip$  to 2 and the other pumps  $p - ip$ . Time reversal symmetry acts as a  $\pi$  rotation. There's a  $\mathbb{Z}_2$  invariant, and the residual family map  $\Omega_{\text{Pin}^+}^4 \rightarrow \Omega_{\text{Pin}^+}^4(S(\rho))$ , equivalently  $\mathbb{Z}_{16} \rightarrow \mathbb{Z}_2$ , is reduction mod 2.

### 5.4.5 LES computations for nice groups

It is easy to compute the symmetry breaking LES when the groups involved are uncomplicated.

**Example 5.4.5.** Consider  $\text{Pin}^+$  with  $\rho$  being twice the sign representation. We can compute the cobordism groups as follows (with homomorphisms being computed using standard homological algebra):

D	$\Omega_{\text{Pin}^-}^{D-2}$	$\Omega_{\text{Pin}^+}^D$	$\Omega_{\text{Pin}^+}^D(S(\rho)) = \tilde{\Omega}_{\text{Spin}}^{D+1}(\mathbb{RP}^2)$
0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
1	0	0	$\mathbb{Z}_2$
2	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_4$
3	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
4	$\mathbb{Z}_8$	$\mathbb{Z}_{16}$	$\mathbb{Z}_2$

**Example 5.4.6.** For  $\text{Pin}^+$  with  $\rho$  being the sign representation, we can also compute the cobordism groups. Standard homological algebra again gives a (partially complete) description of the maps.

D	$\Omega_{\mathbb{Z}_2, \text{Spin}}^{D-1}$	$\Omega_{\text{Pin}^+}^D$	$\Omega_{\mathbb{Z}_2, \text{Spin}}^{D-1} = \Omega_{\text{Spin}}^D(S^0)$
-1	0	0	$\mathbb{Z}$
0	$\mathbb{Z}$	$\mathbb{Z}_2$	0
1	0	0	$\mathbb{Z}_2$
2	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
3	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}$
4	$\mathbb{Z}_8$	$\mathbb{Z}_{16}$	0