GRT Seminar Fall 2024 – Rozansky-Witten Theory

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Abstract

This semester, the GRT Seminar will focus on Rozansky-Witten theory.

Contents

1   1/23 (David Nadler) – Introduction

Our goal is to discuss Rozansky-Witten theory. Some related topics include:

- Quasicoherent sheaves of categories (as discussed last spring).

- Categories of matrix factorizations.[1]

- The cobordism hypothesis.

- Local structure theory of holomorphic symplectic varieties.

1.1   What is Rozansky-Witten theory?

Suppose we have a hyperkähler / holomorphic symplectic manifold X. This means that X has a holomorphic $(2,0)$-form ω satisfying the (complex analogues of) the usual symplectic form axioms. Given such an X, there is a conjectural 3-dimensional topological field theory $\mathcal{Z}_\mathsf{X}$, called *Rozansky-Witten theory* with target X.

What we mean by 3d TFT is as follows:

- Given a closed 3-manifold[2] M^3, we obtain a number $\mathcal{Z}_\mathsf{X}(\mathsf{M}^3)$.

- Closed 2-manifolds M^2 give vector spaces $\mathcal{Z}_\mathsf{X}(\mathsf{M}^2)$.

- Closed 1-manifolds M^1 give categories[3] $\mathcal{Z}_\mathsf{X}(\mathsf{M}^1)$.

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[1]In more detail: given a smooth variety X and a function $\mathsf{f}: \mathsf{X} \to \mathbb{A}^1$, we can construct a category MF_f which categorifies the vanishing cycles of f.

[2]Typically with some extra structure, e.g. an orientation

[3]As is standard for GRT, we use the implicit ∞ convention.

- Closed 0-manifolds M^0 give 2-categories $\mathcal{Z}_X(M^0)$.

In particular, $\mathcal{Z}_X(\mathrm{pt})$ is a 2-category. The *cobordism hypothesis* tells us that we can recover the entire theory $\mathcal{Z}_X$ from the "3-dualizable" 2-category $\mathcal{Z}_X(\mathrm{pt})$. For purposes of geometric representation theory, we are most interested in the low-dimensional behavior, which captures more data about the theory.

Rozansky-Witten theory should satisfy something like:

- $Z_X(S^2) = \mathcal{O}(X).$[4]

- $Z_X(S^1) = \mathsf{Coh}(X).$

These end up inheriting interesting structure from the TFT.

1.2   Why do we care?

Recall that 2-dimensional mirror symmetry can be schematically understood as an equivalence between the following 2d TFTs:

- An A-model $\mathcal{A}$ arising from symplectic geometry

- A B-model $\mathcal{B}_X$, coming from some Kähler manifold X, satisfying $\mathcal{B}_X(\mathrm{pt}) \simeq \mathsf{Coh}(X)$.

In particular, $\mathcal{A}(\mathrm{pt})$ is often some category of geometric interest, and the equivalence $\mathcal{A}(\mathrm{pt}) \simeq \mathcal{B}_X(\mathrm{pt})$ lets us resolve questions about $\mathcal{A}(\mathrm{pt})$.

There's an analogue in higher dimensions: we'd like to take a 3d TFT $\mathcal{Y}$ and give an equivalence $\mathcal{Y} \simeq \mathcal{Z}_X$ for some holomorphic symplectic X. This would give an equivalence between some 2-category and $\mathcal{Z}_X(\mathrm{pt})$.

Conjecture 1.1 (Teleman). *Let G be a complex reductive group with maximal compact subgroup G_c. There is an equivalence between:*

- *A suitable 2-category of "categories with G_c-action."*

- *The Rozansky-Witten 2-category of $T^*(G^\vee/G^\vee)$.*

Note that $T^*(G^\vee/G^\vee)$ is stacky and non-proper, which makes it impossible for the corresponding 2-category to be 3-dualizable. Thus we typically won't obtain 3-manifold invariants from such a theory. That's terrible for 3-manifold topologists, but this isn't a 3-manifold seminar.

Some other examples of interest for Rozansky-Witten theory include symplectic resolutions and cotangent bundles of smooth algebraic varieties.

1.3   What is the correct 2-category?

To rigorously construct Rozansky-Witten theory, we'd need to give a definition of the 2-category $RW_2 = \mathcal{Z}_X(\mathrm{pt})$. This was studied by Kapustin, Rozansky, and Saulina, but much is still unknown.

Roughly, we expect RW_2 to be a 2-category where:

- Objects are smooth Lagrangians $L \subset X$ (or some suitable generalization of these).

- 1-morphisms from L_1 to L_2 are given by some sort of category associated to $L_1 \cap L_2$. In the simplest possible case, where $X = T^*W$ is a cotangent bundle, L_1 is the zero-section, and L_2 is the graph of a differential df, then $L_1 \cap L_2$ is the critical locus of X and we assign $\mathrm{Hom}(L_1, L_2) = MF_f$, the matrix factorization category of f. Work of Joyce and many others has focused on understanding how much the local setting looks like this.

- 2-morphisms and higher are "natural compatibilities" between the 1-morphisms.

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[4]By our conventions, this is what is classically called $\mathbf{R}\Gamma(X, \mathcal{O})$, so there is interesting derived information.

One should think of the matrix factorization category MF_f as giving a categorical way to measure the critical locus of f. When the critical points of f are Morse, the category MF_f looks like a direct sum of copies of Vect (one for each critical point).

There is an important distinction between Rozansky-Witten theory and the 2d A-model. In the complex setting, there are no "instantons," so the theory is local and we don't run into the full difficulty of Floer theory. Thus Rozansky-Witten theory is a categorified version of Fukaya theory that avoids the need for instanton corrections.

1.4 An alternative viewpoint

If $X = T^*W$ is a cotangent bundle, then $\mathsf{ShvCat}(W)$, the 2-category of (quasicoherent) sheaves of categories on W, embeds into RW_2. The image of this embedding consists of "conic objects." Thus we can understand a key part of Rozansky-Witten theory, at least in this simple case.

The thesis (work in progress) of Enoch Yiu relates RW_2 to $\mathsf{ShvCat}(W \times \mathbb{A}^1)$.

2 1/30 (Daigo Ito) – Theory of Critical Points and Matrix Factorizations

Recall that we wanted to understand the Rozansky-Witten theory of a holomorphic symplectic variety M. By the cobordism hypothesis, it suffices to understand the 2-category $\mathrm{RW}_2(\mathsf{M})$. We expect $\mathrm{RW}_2(\mathsf{M})$ to have some vague properties as follows.

The objects of RW_2 should be holomorphic Lagrangians in M (possibly equipped with extra data). If $\mathsf{M} = T^*\mathsf{L}_1$, then we should have $\mathrm{Hom}_{\mathrm{RW}_2}(\mathsf{L}_1, \mathsf{L}_2) = \mathsf{MF}(\mathsf{L}_1, f)$, the category of *matrix factorizations* of f. This measures the local geometry of $p \in \mathsf{L}_1 \cap \mathsf{L}_2 = \mathrm{Crit}(f)$.

Recall the two key differences between this and Lagrangian Floer homology:

- There are no instantons, so the full subtleties of Floer theory don't appear.

- We are working at a higher category level.

Today we will recall the theory of critical points for a function $f : X \to \mathbb{A}^1$.

2.1 Milnor fibers

Let's start by considering a regular map $f : \mathbb{C}^n \to \mathbb{C}$. Assume that $0 \in \mathbb{C}$ is a critical value. Call $X_0 = f^{-1}(0)$ the special fiber – this is typically singular. For small $s \in \mathbb{C}$, let $X_s = f^{-1}(s)$ be the nearby fiber.

Theorem 2.1 (Milnor). *Let $x \in X_0$. For $\epsilon > 0$ sufficiently small, let $B(x, \epsilon)$ be the closed ball of radius ϵ centered at x, and let $S(x, \epsilon) = \partial B(x, \epsilon)$. Then:*

1. *$B(x, \epsilon) \cap X_0$ is homeomorphic to the cone over $K_x = S(x, \epsilon) \cap X_0$.*

2. *The map $\rho_f = \frac{f}{|f|} : S(x, \epsilon) \setminus K_x \to S^1$ is a locally trivial fibration. We call ρ_f the* Milnor fibration *and the fiber F_x the* Milnor fiber.

The Milnor fibers F_x degenerate to the cone over K_x.

Example 2.2. If x is nonsingular, then K_x is a sphere, so the cone over K_x is a ball. The Milnor fibers F_x are also balls.

The topology of the Milnor fibers reflects "how singular the point is" – a more singular point leads to a more complicated topology.

Example 2.3. Let $(X_0, x) = (z_1^2 - z_2^2 = 0, 0)$. Then F_x is homotopy equivalent to S^1. Looking at real points, the map f describes a family of hyperbolas degenerating to a union of lines. Here $\partial B = S^3$ and $K_x = S^1 \sqcup S^1$, so topologically K_x is a double cone. The Milnor fibers form a family of cylinders degenerating to this double cone.

Example 2.4. Let $(X_0, x) = (z_1^3 - z_2^2 = 0, 0)$. Then K_x is a trefoil knot

$$\{(r_1 e^{2\pi i t}, r_2 e^{2\pi i t}) \,|\, t \in \mathbb{R}\} \subset S_{r_1}^1 \times S_{r_2}^1.$$

The closures of the Milnor fibers are genus one "Seifert surfaces" for K_x. Thus the Milnor fibers are homotopy equivalent to $S^1 \wedge S^1$.

More generally, if (X, x) is an isolated hypersurface singularity, then we can write $F_x \simeq (S^n)^{\vee \mu_x}$, where μ_x is the *Milnor number*.[5] The S^n's here are the *vanishing cycles* of the singularity.

2.2 Monodromy

The singularity carries information beyond the Milnor fibers. We can capture some of this by looking at the monodromy.

Definition 2.5. The *monodromy* of f at x is the map $h_f : F_x \to F_x$ induced by circling around the base. This is a homeomorphism of F_x which restricts to the identity on ∂F_x. Note that h_f is only well-defined up to isotopy (fixing ∂F_x).

Example 2.6. For a Morse function $f = \sum_i x_i^2$, the Milnor fibers are homotopy equivalent to S^n. We understand the singularity by studying the monodromy of the Milnor fibers as we move around the singular point. This monodromy is a Dehn twist, "corkscrewing" the cylinder.

Theorem 2.7 (Thom-Sebastiani). *Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ and $g : (\mathbb{C}^{m+1}, 0) \to (\mathbb{C}, 0)$ be germs of hypersurface singularities. Define $f \boxplus g : (\mathbb{C}^{n+1} \times \mathbb{C}^{m+1}, 0) \to (\mathbb{C}, 0)$ by $(f \boxplus g)(x, y) = f(x) + g(y)$. Then there is a homotopy-commutative diagram*

$$
\begin{array}{ccc}
F_f * F_g & \xrightarrow{\;\sim\;} & F_{f \boxplus g} \\
\Big\downarrow{\scriptstyle h_f * h_g} & & \Big\downarrow{\scriptstyle h_{f \boxplus g}} \\
F_f * F_g & \xrightarrow{\;\sim\;} & F_{f \boxplus g},
\end{array}
$$

where $*$ is the join of spaces.

2.3 Preview

Next time we will introduce sheaves that describe the homology of these spaces. We get a fiber sequence

$$i^* \mathcal{F} \longrightarrow \psi_f \mathcal{F} \longrightarrow \phi_f \mathcal{F} \longrightarrow$$

of sheaves on X_0, where:

- $i : X_0 \to X$ is the inclusion,

- ψ_f is nearby cycles, and

- ϕ_f is vanishing cycles.

This will categorify to a sequence

$$\mathsf{Perf}(X_0) \longrightarrow D_{\mathrm{coh}}^b(X_0) \longrightarrow D_{\mathrm{sing}}(X_0).$$

where $D_{\mathrm{sing}}(X_0)$ agrees with $\mathsf{MF}(X, f)$ in nice cases.

3 2/6 (Daigo Ito) – Continued

Last time we discussed the construction of Milnor fibers, vanishing cycles, monodromy, and Thom-Sebastiani isomorphisms for $f : \mathbb{C}^{n+1} \to \mathbb{C}$. Today we would like to discuss and categorify these stories in a sheaf-theoretic framework.

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[5]There is an explicit formula for the Milnor number, but we won't write it here.

3.1 Vanishing and nearby cycles

Let $\mathbb{D} \subset \mathbb{C}$ be a small disk around 0. Let X be (an open subset of) a smooth algebraic variety over $\mathbb{C}$, and let $f : X \to \mathbb{D}$ be a map. Consider the diagram

$$
\begin{array}{ccccccc}
X_0 & \xrightarrow{\ i\ } & X & \xleftarrow{\ j\ } & X^* & \xleftarrow{\ \tilde{\pi}\ } & \tilde{X}^* \\
\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f^*} & & \downarrow \\
\{0\} & \longrightarrow & \mathbb{D} & \longleftarrow & \mathbb{D}^* & \xleftarrow{\ \pi\ } & \tilde{\mathbb{D}}^*
\end{array}
$$

where all squares are pullback squares and $\pi : \tilde{\mathbb{D}}^* \to \mathbb{D}^*$ is the universal cover $z \mapsto \exp(2\pi i z)$. Note that $X^* \simeq X_s$ for s small, $s \neq 0$.

Write $D_c^b(-)$ for the bounded constructible derived category of A-modules on a space (where A is some fixed coefficient ring, typically $\mathbb{Z}$ or $\mathbb{C}$). Recall that "constructible" means locally constant on the strata of a nice stratification and with finite-rank stalks.

Definition 3.1. The *nearby cycle functor* associated with f is $\psi_f : D_c^b(X) \to D_c^b(X_0)$, defined by[6]

$$
\mathcal{F} \mapsto i^*(j \circ \tilde{\pi})_*(j \circ \tilde{\pi})^*\mathcal{F}.
$$

Morally, we have a (very non-analytic) specialization map $\mathrm{sp} : X_s \to X_0$, and $\psi_f = \mathrm{sp}_*(\mathcal{F}|_{X_s})$. The previous definition is used to avoid referencing sp.

Example 3.2. Consider $f : \mathbb{D} \to \mathbb{D}$ by $f(z) = z^2$. For $\mathcal{F} = \underline{A}_{\mathbb{D}}$, we can compute $\psi_f(\mathcal{F}) = A_0 \oplus A_0$, reflecting the fact that the nearby fibers have two points. The monodromy map swaps the two factors: this can be seen directly using the specialization definition or by considering deck transformations using the formal definition.

Remark 3.3. David mentioned that one can actually rephrase this story so that the only f which we consider is projection to the first coordinate. The cost is that we are forced to work with arbitrarily complicated sheaves. The reverse (working with the constant sheaf but allowing arbitrarily complicated f) is not possible in general, though there is a related theory of "sheaves of geometric origin."

From the pushforward-pullback adjunction, there is a natural map $r : i^*\mathcal{F} \to i^*(j \circ \tilde{\pi})_*(j \circ \tilde{\pi})^*\mathcal{F} = \psi_f\mathcal{F}$.

Definition 3.4. We define the *vanishing cycle functor* $\phi_f : D_c^b(X) \to D_c^b(X_0)$ by $\phi_f(\mathcal{F}) = \mathrm{cone}(r)$, so there is a cofiber sequence

$$
i^*\mathcal{F} \longrightarrow \psi_f\mathcal{F} \longrightarrow \phi_f\mathcal{F} \longrightarrow .
$$

We call $\psi_f\underline{A}_X$ (resp. $\phi_f\underline{A}_X$) the *nearby (resp. vanishing) cycle complex* associated with f. From the cofiber sequence containing these, we obtain a long exact sequence (using $H^*(X) \cong H^*(X_0)$):

$$
\ldots \longrightarrow H^*(X_0) \longrightarrow H^*(X_s) \longrightarrow H^*(X, X_s) \longrightarrow \ldots
$$

This encompasses much of our discussion from last time.

Proposition 3.5. *For* $f : \mathbb{C}^{n+1} \to \mathbb{C}$*, if* X_0 *has only isolated singularities (so* $F_f \simeq \vee S^n$*), then*

$$
H^k(X_0, \phi_f\underline{A}_X) = \begin{cases} 0 & k \neq n \\ \oplus_{x \in \mathrm{Sing}(X_0)} H^n(F_{f,x}; A) & k = n. \end{cases}
$$

Remark 3.6. One can obtain the monodromy of nearby / vanishing cycles using the deck transformations of $\tilde{\mathbb{D}}^*$. There's also a Thom-Sebastiani theorem in the sheaf-theoretic setting.

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[6]All functors here are derived.

3.2 Singularity categories and matrix factorizations

We can categorify the preceding story using the exact sequence of categories:

$$\mathsf{Perf}(X_0) \longrightarrow D^b_{coh}(X_0) \longrightarrow D_{sing}(X_0),$$

where $D_{sing}(X_0)$ is defined as the quotient $D^b_{coh}(X_0)/\mathsf{Perf}(X_0)$. In nice cases, $D_{sing}(X_0)$ agrees with the *matrix factorization category* $\mathsf{MF}(X, f)$. The reason D_{sing} is called the "category of singularities" is the following:

Proposition 3.7. X_0 *is smooth if and only if* $\mathsf{Perf}(X_0) \simeq D^b_{coh}(X_0)$.

Example 3.8. If $x \in X_0$ is singular, then the skyscraper sheaf $k(x_0)$ is not in $\mathsf{Perf}(X)$.

To decategorify our exact sequence to the sheaf-theoretic statement above, we take "periodic cyclic homology."

4 2/13 (Will Fisher) – The Singular Category and Matrix Factorizations

Today's talk is based on Orlov's paper "Triangulated categories of singularities . . ." We work over a field k.
Last time, Daigo presented the exact sequence

$$\mathsf{Perf}(X_0) \longrightarrow D^b_{coh}(X_0) \longrightarrow D_{sing}(X_0).$$

One can think of this as a categorified version of the usual nearby / vanishing cycles exact sequence (which we recover by taking periodic cyclic homology): $D^b_{coh}(X_0)$ knows something about a "nearby smoothing" of X_0.

For nice $f : X \to \mathbb{A}^1$ with $X_0 = f^{-1}(0)$, we can identify $D_{sing}(X_0) \simeq \mathsf{MF}(X, f)$. Our goal is to discuss this result.

4.1 Definitions

First, we will need to define everything involved.

Definition 4.1. Let A be a commutative ring. A chain complex $M \in D(\mathsf{Mod}_A)$ is *perfect* if it is quasi-isomorphic to a bounded complex of finite projective modules. These form a subcategory $\mathsf{Perf}(A) \subset D(\mathsf{Mod}_A)$. Equivalently, $\mathsf{Perf}(A)$ is the smallest triangulated subcategory containing A and closed under and retracts.

Definition 4.2. If X is a scheme, then $\mathsf{Perf}(X)$ is the full subcategory of $D^b_{coh}(X)$ consisting of objects which are perfect affine locally, i.e. locally they can be written as a quotient of vector bundles.

We will assume X is a separated noetherian scheme of finite Krull dimension, and that $\mathsf{Coh}(X)$ "has enough vector bundles," i.e. for all $\mathcal{F} \in \mathsf{Coh}(X)$, there exists a vector bundle $\mathcal{P}$ and a surjection $\mathcal{P} \twoheadrightarrow \mathcal{F}$. These conditions will hold X is quasiprojective.

Definition 4.3. The category $D^b_{coh}(X) \subset D^b_{coh}(X)$ consists of objects with bounded and coherent cohomology.[7]

Under the above hypotheses, we have $D^b_{coh}(X) = D^b(\mathsf{Coh}(X))$.

Definition 4.4. The category $D_{sing}(X)$ is the Verdier quotient $D^b_{coh}(X)/\mathsf{Perf}(X)$, constructed by formally inverting morphisms in $D^b_{coh}(X)$ with cones in $\mathsf{Perf}(X)$.

Example 4.5. Let $X = \operatorname{Spec} A$ for $A = k[x]/(x^2)$. The A-module $k = A/(x)$ is coherent but not perfect: it has the infinite resolution

$$\ldots A \xrightarrow{\cdot x} A \xrightarrow{\cdot x} 0.$$

We can use this to compute $\operatorname{Ext}^i(k, k) \cong k$ for $i \geqslant 0$, showing that k is not perfect. In particular, $D_{sing}(X)$ is nontrivial. This relates to Serre's criterion for regularity in algebraic geometry.

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[7]In general, it is better to treat this as a *property* than as *structure*.

4.2 Basic properties

We state without proof some relevant (but hard) facts about $\mathsf{D}_{\mathrm{sing}}(\mathsf{X})$:

1. (Auslander-Buchsbaum-Serre) If A is noetherian and finite-dimensional, then $\mathsf{D}_{\mathrm{sing}}(\operatorname{Spec} \mathsf{A}) = 0$ if and only if A is regular.

2. (Thomason-Trobaugh) If $\mathsf{U} \subset \mathsf{X}$ is an open subscheme containing the singular locus of X, then $\mathsf{D}_{\mathrm{sing}}(\mathsf{X}) \xrightarrow{\sim} \mathsf{D}_{\mathrm{sing}}(\mathsf{U})$.

It turns out that $\mathsf{D}_{\mathrm{sing}}(\mathsf{X})$ is smaller than we might initially expect:

Proposition 4.6. *Every object of* $\mathsf{D}_{\mathrm{sing}}(\mathsf{X})$ *is equivalent to* $\mathcal{F}[\mathsf{k}+1]$ *for some coherent sheaf* $\mathcal{F}$ *and some* $\mathsf{k} \in \mathbb{Z}$.

Proof. For $\mathsf{A}^{\bullet} \in \mathsf{D}_{\mathrm{coh}}^{\mathrm{b}}(\mathsf{X})$, choose a quasi-isomorphism $\mathsf{P}^{\bullet} \xrightarrow{\sim} \mathsf{A}^{\bullet}$ with $\mathsf{P}^{\bullet}$ a bounded above complex of vector bundles. The stupid truncations $\sigma^{\geqslant -\mathsf{k}}\mathsf{P}^{\bullet}$ (obtained by replacing $\mathsf{P}^{\bullet}$ by 0 for $\bullet < -\mathsf{k}$) are perfect. Let $\mathsf{f}_{-\mathsf{k}} : \sigma^{\geqslant -\mathsf{k}}\mathsf{P}^{\bullet} \to \mathsf{A}^{\bullet}$ be the natural map, and consider $\operatorname{cone}(\mathsf{f}_{-\mathsf{k}})$. The long exact sequence of cohomology sheaves for the cofiber sequence shows that, for $\mathsf{k} \gg 0$, we have $\mathcal{H}^{\mathsf{i}}(\operatorname{cone}(\mathsf{f}_{-\mathsf{k}})) = 0$ unless $\mathsf{i} = -\mathsf{k}-1$. Thus, taking $\mathcal{F} = \mathcal{H}^{-\mathsf{k}}(\sigma^{\geqslant -\mathsf{k}}\mathsf{P}^{\bullet})$, we obtain $\operatorname{cone}(\mathsf{f}_{-\mathsf{k}}) \simeq \mathcal{F}[\mathsf{k}+1]$. This gives $\mathsf{A}^{\bullet} \simeq \mathcal{F}[\mathsf{k}+1]$ in $\mathsf{D}_{\mathrm{sing}}(\mathsf{X})$. $\qquad\square$

We'd also like to be able to take right resolutions of coherent sheaves. If we were capable of "dualizing" in a way that preserves vector bundles, we could take a left resolution of $\mathcal{F}^{\vee}$ and dualize this to get a resolution of $\mathcal{F}^{\vee\vee} \simeq \mathcal{F}$.

If X is Gorenstein and satisfies our standing assumptions, then $\mathcal{O}_{\mathsf{X}}$ has a finite injective resolution and

$$\mathcal{F} \xrightarrow{\sim} \mathbf{R}\mathcal{H}\mathrm{om}(\mathbf{R}\mathcal{H}\mathrm{om}(\mathcal{F}, \mathcal{O}_{\mathsf{X}}), \mathcal{O}_{\mathsf{X}})$$

for $\mathcal{F} \in \mathsf{Coh}(\mathsf{X})$.

Lemma 4.7. *Let* X *be as above, and let* $\mathcal{F} \in \mathsf{Coh}(\mathsf{X})$. *TFAE:*

1. $\mathcal{E}\mathrm{xt}^{\mathsf{i}}(\mathcal{F}, \mathcal{O}_{\mathsf{X}}) = 0$ *for all* $\mathsf{i} > 0$.

2. *There exists a right resolution of* $\mathcal{F}$ *by vector bundles.*

Corollary 4.8. *Every* $\mathsf{A}^{\bullet} \in \mathsf{D}_{\mathrm{sing}}(\mathsf{X})$ *is equivalent to* $\mathcal{F}[0]$ *for some* $\mathcal{F} \in \mathsf{Coh}(\mathsf{X})$ *with* $\mathcal{E}\mathrm{xt}^{\mathsf{i}}(\mathcal{F}, \mathcal{O}_{\mathsf{X}}) = 0$ *for* $\mathsf{i} > 0$.

5   2/20 (Will Fisher) – Continued

5.1 Dualizing complexes

Let's review / clarify some points on dualizing complexes. This concerns something a bit different than the usual Serre duality context: we only care about the local setting and giving equivalences $\mathsf{D}_{\mathrm{coh}}^{\mathrm{b}}(\mathsf{X})^{\mathrm{op}} \xrightarrow{\sim} \mathsf{D}_{\mathrm{coh}}^{\mathrm{b}}(\mathsf{X})$.

Definition 5.1. If A is a noetherian ring, a *dualizing complex* on $\operatorname{Spec} \mathsf{A}$ is $\omega^{\bullet} \in \mathsf{D}_{\mathrm{coh}}^{\mathrm{b}}(\operatorname{Spec} \mathsf{A})$ such that:

1. $\omega^{\bullet}$ has finite injective dimension, and

2. $\mathsf{A} \to \mathbf{R}\operatorname{Hom}(\omega^{\bullet}, \omega^{\bullet})$ is a quasi-isomorphism.

If X is a locally noetherian scheme, a *dualizing complex* on X is $\omega^{\bullet} \in \mathsf{D}_{\mathrm{coh}}^{\mathrm{b}}(\mathsf{X})$ such that $\omega^{\bullet}$ is affine locally a dualizing complex.

If ω is a dualizing complex on X, then $\mathbf{R}\mathcal{H}\mathrm{om}(-, \omega^{\bullet}) : \mathsf{D}_{\mathrm{coh}}^{\mathrm{b}}(\mathsf{X})^{\mathrm{op}} \to \mathsf{D}_{\mathrm{coh}}^{\mathrm{b}}(\mathsf{X})$ is an equivalence (and in fact is its own inverse).

Remark 5.2. Peter Haine pointed out a few things:

- Dualizing complexes are unique up to tensoring with complete line bundles.

- A result of Kawasaki (proving a conjecture of Sharp) shows that the commutative rings which admit dualizing complexes are those of finite Krull dimension which arise as quotients of Gorenstein rings.

- Upper shriek functors preserve dualizing complexes in this sense.

5.2 Gorenstein schemes

In the Gorenstein case, dualizing complexes are easy to understand.

Theorem 5.3. *If X is locally noetherian and has a dualizing complex, TFAE:*

1. X *is Gorenstein.*

2. X *has an invertible dualizing complex.*

3. $\mathcal{O}_X[0]$ *is a dualizing complex.*

We have a few large classes of Gorenstein schemes:

Theorem 5.4. *Smooth schemes are dualizing.*

Theorem 5.5. *Local complete intersections in Gorenstein schemes are dualizing.*

5.3 Representing objects of $D_{\mathrm{sing}}(X)$

From now on we will assume X is Gorenstein. Our goal is to show that objects of $D_{\mathrm{sing}}(X)$ can be represented by particularly nice sheaves and complexes.

Proposition 5.6. *If $\mathcal{O}_X$ has a finite injective resolution, then there exists $n_0 > 0$ such that, for all $\mathcal{F} \in \mathsf{QCoh}(X)$ and all $i > n_0$, we have $\mathcal{E}xt^i(\mathcal{F}, \mathcal{O}_X) = 0$.*

Proposition 5.7. *For $\mathcal{F} \in \mathsf{Coh}(X)$, TFAE:*

1. $\mathcal{E}xt^i(\mathcal{F}, \mathcal{O}_X) = 0$ *for all $i > 0$.*

2. $\mathcal{F}$ *has a right resolution by vector bundles.*

Proof. We only prove $\Rightarrow$. To obtain the desired right resolution, take a left resolution $\mathcal{P}^\bullet \xrightarrow{\sim} \mathcal{H}om(\mathcal{F}, \mathcal{O}_X) = \mathbf{R}\mathcal{H}om(\mathcal{F}, \mathcal{O}_X)$. Then $\mathcal{F} \simeq \mathcal{H}om(\mathcal{H}om(\mathcal{F}, \mathcal{O}_X), \mathcal{O}_X) \simeq \mathcal{H}om(\mathcal{P}^\bullet, \mathcal{O}_X)$ gives the desired right resolution. $\square$

Theorem 5.8. *Every object in $D_{\mathrm{sing}}(X)$ is equivalent to $\mathcal{F}[0]$ for $\mathcal{F}$ coherent with $\mathcal{E}xt^i(\mathcal{F}, \mathcal{O}_X) = 0$ for all $i > 0$.*

Proof. Let $\mathcal{A}^\bullet \in D^b_{\mathrm{coh}}(X)$. Take a left resolution $\mathcal{P}^\bullet \xrightarrow{\sim} \mathcal{A}^\bullet$. For $k \gg 0$, we saw last time that if $\mathcal{G} = \mathcal{H}^{-k}(\sigma^{\geq -k}\mathcal{P}^\bullet)$, then $\mathcal{A}^\bullet \simeq \mathcal{G}[k+1]$ in $D_{\mathrm{sing}}(X)$. Since $\mathcal{O}_X$ has bounded injective resolution, $\mathbf{R}\mathcal{H}om(\mathcal{A}^\bullet, \mathcal{O}_X)$ is cohomologically bounded. Thus, for $k \gg 0$, we get

$$\mathcal{E}xt^i(\mathcal{G}, \mathcal{O}_X) \cong \mathcal{E}xt^{i+k+1}(\mathcal{A}^\bullet, \mathcal{O}_X) = 0$$

for $i > 0$. Now take a right resolution $\mathcal{G} \to \mathcal{Q}^\bullet$,[8] and let $\mathcal{F} = \mathcal{H}^k(\sigma^{\leq k}\mathcal{Q}^\bullet)$. Then it is clear that $\mathcal{F}$ has the desired properties. $\square$

5.4 Equivalence with matrix factorizations

Let $X = \operatorname{Spec} A$ be smooth, let $f : X \to \mathbb{A}^1$, and let $X_0 = f^{-1}(0) = \operatorname{Spec} A/(f)$. Write $j : X_0 \to X$ for the inclusion. We would like to show that $D_{\mathrm{sing}}(X_0) \simeq \mathsf{MF}(X, f)$, the matrix factorization category of f.

Proposition 5.9. *Let $\mathcal{F}$ be coherent on X_0 with $\mathcal{E}xt^i(\mathcal{F}, \mathcal{O}_{X_0}) = 0$ for all $i > 0$. Then there exist vector bundles $\mathcal{P}_0, \mathcal{P}_1$ on X with maps $p_0 : \mathcal{P}_0 \to \mathcal{P}_1$ and $p_1 : \mathcal{P}_1 \to \mathcal{P}_0$ such that:*

1. $p_0 p_1 = f \cdot \mathrm{id}_X$,

2. $p_1 p_0 = f \cdot \mathrm{id}_X$, *and*

3. $\operatorname{coker} p_1 = j_* \mathcal{F}$.

---

[8]This is where the Gorenstein hypothesis appears.

Sketch of proof. Choose a surjection $\mathcal{P}_0 \to j_* \mathcal{F}$ with $\mathcal{P}_0$ a vector bundle. Let $p_1 : \mathcal{P}_1 \to \mathcal{P}_0$ be the kernel of this surjection. Then the hypotheses on $\mathcal{F}$ imply that $\mathcal{P}_1$ is a vector bundle.[9] Because multiplication by f annihilates $j_* \mathcal{F}$, the composite

$$\mathcal{P}_0 \xrightarrow{\cdot f} \mathcal{P}_0 \to j_* \mathcal{F}$$

is zero, so it factors through p_1. That is, there exists $p_0 : \mathcal{P}_0 \to \mathcal{P}_1$ such that $p_1 p_0 = f \cdot \mathrm{id}$. The computation $p_1 p_0 p_1 = (- \cdot f) \circ p_1 = p_1 \circ (- \cdot f)$ implies $p_0 p_1 = \mathrm{id}$ as well. $\qquad\square$

Let $\mathcal{P}_0$ and $\mathcal{P}_1$ be as above. Then we get an exact sequence[10]

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{P}_1|_{X_0} \longrightarrow \mathcal{P}_0|_{X_0} \longrightarrow \mathcal{F} \longrightarrow 0.$$

In particular, we obtain $\mathcal{F} \simeq \mathcal{F}[2]$ in $D_{\mathrm{sing}}(X)$.

Definition 5.10. The *matrix factorization category* of (X, f) has objects given by pairs of vector bundles $\mathcal{P}_0, \mathcal{P}_1$ on X with maps $p_0 : \mathcal{P}_0 \to \mathcal{P}_1$ and $p_1 : \mathcal{P}_1 \to \mathcal{P}_0$ such that:

1. $p_0 p_1 = f \cdot \mathrm{id}_X$ and

2. $p_1 p_0 = f \cdot \mathrm{id}_X$.

Morphisms are defined to be homotopy classes of commutative squares.

There is a functor $\mathrm{coker} : \mathsf{MF}(X, f) \to D_{\mathrm{sing}}(X_0)$ sending $(\mathcal{P}_0, \mathcal{P}_1, p_0, p_1)$ to $\mathrm{coker}(p_1)$.

Theorem 5.11 (Orlov). *The functor* $\mathrm{coker} : \mathsf{MF}(X, f) \to D_{\mathrm{sing}}(X_0)$ *is an equivalence.*

Example 5.12. Let $A = \mathbb{C}[x]$, $f = x^n$, and $X_0 = \mathrm{Spec}\, \mathbb{C}[x]/(x^n)$. Then $D_{\mathrm{sing}}(X_0)$ has objects given by direct sums of $V_i = \mathbb{C}[x]/(x^i)$ for $1 \leqslant i \leqslant n-1$. Under the equivalence $D_{\mathrm{sing}}(X_0) \simeq \mathsf{MF}(X, f)$, V_i corresponds to $(\mathbb{C}[x], \mathbb{C}[x], x^{n-i}, x^i)$. One can decategorify this to obtain the usual nearby / vanishing cycle picture for f.

6 2/27 (Elliot Kienzle) – Matrix Factorizations

6.1 David – Opening remarks

Next week we will have a guest speaker. We will also eventually hear from Enoch Yiu, discussing his thesis work on Homs in the Rozansky-Witten 2-category involving conormals (in T^*X) to subvarieties of X.

6.2 Elliot – Quick review

Suppose we are given a function / superpotential $W : X \to \mathbb{C}$. We get a singular fiber X_0 and nearby fibers X_s. These fit into a long exact sequence

$$\ldots \longrightarrow H^\bullet(X_0) \longrightarrow H^\bullet(X_s) \longrightarrow H^\bullet(X, X_s) \longrightarrow \ldots$$

which categorifies to

$$\mathsf{Perf}(X_0) \longrightarrow D^b_{\mathrm{coh}}(X_0) \longrightarrow D_{\mathrm{sing}}(X_0) =\!=\!= \mathsf{MF}(X, W).$$

In the above, $\mathsf{MF}(X, W)$ is a category whose objects are "matrix factorizations" $P^\bullet = (P_0, P_1, p_0, p_1)$ where P_0 and P_1 are vector bundles on X, $p_0 : P_0 \to P_1$, $p_1 : P_1 \to P_0$, and $p_0 p_1 = W \cdot \mathrm{id}$ and $p_1 p_0 = W \cdot \mathrm{id}$. This allows us to factorize polynomials which we could not factorize in the original polynomial rings.

Example 6.1. Working on $\mathbb{R}^2$, we cannot factor $W = x^2 + y^2$ in $\mathbb{R}[x, y]$. However, we do have

$$\begin{bmatrix} x & y \\ -y & x \end{bmatrix} \begin{bmatrix} x & -y \\ y & x \end{bmatrix} = \begin{bmatrix} x^2 + y^2 & 0 \\ 0 & x^2 + y^2 \end{bmatrix}.$$

[9]More details can be found on Will Fisher's website.
[10]This is left as a non-obvious exercise.

One may view $P^\bullet$ as a $\mathbb{Z}/2$-graded vector space with an endomorphism of odd degree given by

$$\begin{bmatrix} 0 & p_0 \\ p_1 & 0 \end{bmatrix}.$$

In particular, we get a $\mathbb{Z}/2$-graded morphism space $\mathrm{Hom}^\bullet(P, Q)$ (defined without reference to the endomorphisms). There is a natural differential ∂ on $P^\bullet$ given by $\partial\phi = q\phi - (-1)^{|\phi|}\phi p$. We define $\mathrm{Mor}^\bullet(P, Q) = H^\bullet(\mathrm{Hom}^\bullet(P, Q), \partial)$. In particular, $\mathrm{Mor}^0(P, Q)$ consists of "chain maps modulo chain homotopy."

6.3 Boring matrix factorizations

Say that $\mathsf{MF}(X, W)$ is *boring* if every object is equivalent to $0 = (0 \rightleftarrows 0)$.

Lemma 6.2. *The following are equivalent for a matrix factorization* $P = (P_0 \rightleftarrows P_1)$:

1. P *is equivalent to* 0.

2. $\mathrm{id}_P \sim 0 \in \mathrm{Mor}(P, P)$.

3. *There exists odd* ϕ *such that* $\phi_1 p_0 + p_1 \phi_0 = \mathrm{id}_{P_0}$ *and* $p_1 \phi_1 + \phi_0 p_1 = \mathrm{id}_{P_1}$.

Example 6.3. The matrix factorization category $\mathsf{MF}(X, 1)$ is boring: given P, we can define a chain homotopy ϕ from id_P to 0 via $\phi = (0, p_0)$. More generally, if W is nonvanishing, then $\mathsf{MF}(X, W)$ is boring.

Example 6.4. The matrix factorization category $\mathsf{MF}(\mathbb{C}, z)$ is boring. For a special case of this claim, consider P with $P_0 = P_1 = \mathcal{O}$. At most one of p_0 and p_1 can vanish, so assume p_1 is nonvanishing. Then we can take $\phi_0 = p_0/z$ and $\phi_1 = 0$. The general case is encompassed by the following.

Example 6.5. The matrix factorization category $\mathsf{MF}(\mathbb{C}^n, z_i)$ is boring. To see this, note that $p_0 p_1 = z_i$ implies $(\partial_i p_0)p_1 + p_0(\partial_i p_1) = \mathrm{id}$. It's plausible that this generalizes to show that $\mathsf{MF}(\mathbb{C}^n, W)$ is boring whenever dW is nonvanishing.

The upshot is that, if W or dW is nonvanishing, then $\mathsf{MF}(X, W)$ is trivial.

6.4 The first interesting matrix factorization category

Let $X = V = \mathbb{C}^n$ and $W = Q(z, z) = z_1^2 + \cdots + z_n^2$. To do this, let us recall some of the history of quantum field theory.

Dirac wanted to find a square root of the Laplacian $\Delta = \partial_1^2 + \cdots + \partial_n^2$. That is, he wanted $\slashed{D}$ such that $\slashed{D}^2 = \Delta$. This $\slashed{D}$ does not exist as a scalar function: it only exists as a *matrix* of linear differential operators If we replace ∂_i^2 by x_i, we end up with a corresponding matrix factorization of our W.

Our goal is to prove the following:

Theorem 6.6. *Let* $\mathrm{C}\ell(V, Q)$ *be the Clifford algebra of* (V, Q). *Then* $\mathsf{MF}(\mathbb{C}^n, Q) \simeq \mathrm{C}\ell(V, Q)$-$\mathsf{Mod}$.

David pointed out that *Knörrer periodicity* tells us that

$$\mathrm{C}\ell(\mathbb{C}^n, Q)\text{-}\mathsf{Mod} \simeq \begin{cases} \mathsf{MF}(\mathbb{C}, x^2) & n \text{ odd} \\ \mathsf{MF}(\mathbb{C}^2, x^2 + y^2) & n \text{ even.} \end{cases}$$

The category $\mathsf{MF}(\mathbb{C}, x^2)$ can be identified with $\mathbb{Z}/2$-graded $\mathbb{C}[\epsilon]/(\epsilon^2 - 1)$-modules, where $|\epsilon| = 1$. The category $\mathsf{MF}(\mathbb{C}^2, x^2 + y^2)$ can be identified with $\mathbb{Z}/2$-graded vector spaces. One can interpret this in terms of vanishing cycles: for $(\mathbb{C}, x^2)$ we have nontrivial monodromy (-1), whereas for $(\mathbb{C}^2, x^2 + y^2)$ we have trivial monodromy.

7 3/6 (Lucas Mason-Brown) – Some Progress on the Problem of the Unitary Dual

This is based on joint wirk with Losev, Matvieievskyi, and Davis.

7.1 Review of unitary representation theory

Let $G_\mathbb{R}$ be a real reductive group (i.e. the real points of a reductive algebraic group over $\mathbb{R}$, or a Lie group which is isogenous to one). A *unitary representation* of $G_\mathbb{R}$ is a complex Hilbert space V (typically infinite-dimensional) with a continuous representation $G_\mathbb{R} \times V \to V$ by unitary operators.[11] Such a representation is *irreducible* if it has no nontrivial *closed* invariant subspaces.

Example 7.1. The trivial representation of any $G_\mathbb{R}$ is unitary.

Example 7.2. If $G_\mathbb{R}$ is compact, then every continuous finite-dimensional representation V of $G_\mathbb{R}$ is unitary (with respect to an inner product defined via averaging).

Example 7.3. Let X be a smooth manifold with an action of $G_\mathbb{R}$ by diffeomorphisms. On X, there is a real line bundle of *half-densities* $\mathcal{L}^{1/2} = \mathrm{Fr}(TX) \times^{\mathrm{GL}_n(\mathbb{R})} \mathbb{R}_{|\det|^{1/2}}$. Local sections of $\mathcal{L}^{1/2}$ look like $f(x_1, \ldots, x_n) |dx_1 \wedge \cdots \wedge dx_n|^{1/2}$. There is a real inner product on $\Gamma_c(X, \mathcal{L}^{1/2})$ given by $\langle s_1, s_2 \rangle = \int_X s_1 \otimes s_2$. Taking the completion of $\Gamma_c(X, \mathcal{L}^{1/2}) \otimes_\mathbb{R} \mathbb{C}$, we obtain a unitary $G_\mathbb{R}$-representation $L^2(X)$.

If we let $P_\mathbb{R}$ be the real points of a parabolic, and $X = G_\mathbb{R}/P_\mathbb{R}$, then $L^2(X)$ is the (unnormalized) "unitary induction" of the trivial $P_\mathbb{R}$-representation. We can twist this to get $L^2(X, V) = \mathrm{Ind}_{P_\mathbb{R}}^{G_\mathbb{R}} V$.

Example 7.4. Let $G_\mathbb{R}$ be the metaplectic group $\mathrm{Mp}(2n, \mathbb{R})$, a non-linear double cover of the real symplectic group. This has an "oscillator representation" on $L^2(\mathbb{R}^n)$ which does not come from an action of $G_\mathbb{R}$ on $\mathbb{R}^n$. The oscillator representation is constructed using the Stone-von Neumann theorem.

7.2 The problem of the unitary dual

An important open problem in unitary representation theorem is to classify the space $\Pi_u(G_\mathbb{R})$ of irreducible unitary $G_\mathbb{R}$-representations (up to unitary equivalence). Harish-Chandra proposed the following approach. Fix a maximal compact subgroup $K_\mathbb{R} \subset G_\mathbb{R}$, and let K be the complexification of $K_\mathbb{R}$.

Example 7.5. If $G_\mathbb{R} = \mathrm{SL}_n(\mathbb{R})$, we can take $K_\mathbb{R} = \mathrm{SO}_n(\mathbb{R})$ and $K = \mathrm{SO}_n(\mathbb{C})$.

For a unitary $G_\mathbb{R}$-representation, we can define the *Harish-Chandra module*

$$\mathrm{HC}(V) = \{v \in V \,|\, \dim \mathrm{span}\, K_\mathbb{R} v < \infty\}.$$

Harish-Chandra showed that $\mathrm{HC}(V)$ is a $(\mathfrak{g}, K)$-module with a $(\mathfrak{g}_\mathbb{R}, K_\mathbb{R})$-invariant positive definite inner product.

Theorem 7.6 (Harish-Chandra). *The assignment* $V \mapsto \mathrm{HC}(V)$ *gives a bijection between* $\Pi_u(G_\mathbb{R})$ *and* $\Pi_u(\mathfrak{g}, K)$ *(the set of irreducible unitarizable* $(\mathfrak{g}, K)$*-modules).*

Note that $\Pi_u(\mathfrak{g}, K)$ is quite algebraic, while $\Pi_u(G_\mathbb{R})$ is quite analytic! We can embed $\Pi_u(\mathfrak{g}, K)$ into $\Pi(\mathfrak{g}, K)$, the space of all irreducible $(\mathfrak{g}, K)$-modules. This latter space is well-understood via Langlands parameters / Beilinson-Bernstein localization / etc. Thus the question turns into understanding the embedding $\Pi_u(\mathfrak{g}, K) \hookrightarrow \Pi(\mathfrak{g}, K)$.

This splits into two subquestions. The first subquestion, understanding which irreducible $(\mathfrak{g}, K)$-modules admit a $(\mathfrak{g}, K)$-invariant bilinear form, was solved in the 1970s. The remaining subquestion, understanding when this bilinear form is Hermitian, is open.

7.3 Conjectural solution

Conjecture 7.7. *For every twisted Levi subgroup* $L_\mathbb{R} \subset G_\mathbb{R}$, *there are:*

1. *Standard inductive procedures* I *producing finitely many elements of* $\Pi_u(\mathfrak{g}, K)$ *from a given element of* $\Pi_u(L_\mathbb{R})$. *(These include parabolic induction, cohomological induction, and complementary series.)*

2. *A finite set of explicit "rigid" cuspidal representations* $\Pi_{\mathrm{rig}}(L_\mathbb{R})$.

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[11]In general, asking for $G_\mathbb{R} \to U(V)$ to be continuous is not the right continuity condition.

We may write $\Pi_{\mathfrak{u}}(G_{\mathbb{R}}) = \cup_{L_{\mathbb{R}}} I(\Pi_{\mathfrak{u}}(L_{\mathbb{R}}))$.

To make this conjecture worthwhile, we should explain what we mean by rigid representations. Let $\mathcal{O} \subset \mathfrak{g}^*$ be a nilpotent coadjoint orbit in the dual of the complexified Lie algebra. Then $\mathbb{C}[\mathcal{O}]$ is a finitely generated graded Poisson algebra. A quantization of $\mathbb{C}[\mathcal{O}]$ is a nonnegatively filtered algebra $\mathcal{A} = \cup_m \mathcal{A}_{\leqslant m}$ such that $[\mathcal{A}_{\leqslant m}, \mathcal{A}_{\leqslant n}] \subset \mathcal{A}_{\leqslant m+n-1}$ together with an isomorphism of graded Poisson algebras $\tau : \operatorname{gr}(\mathcal{A}) \xrightarrow{\sim} \mathbb{C}[\mathcal{O}]$. This gives a unique quantum comoment map, i.e. a filtered algebra homomorphism $\mathcal{U}(\mathfrak{g}) \to \mathcal{A}$ reducing to $\operatorname{Sym}\mathfrak{g} \to \mathbb{C}[\mathcal{O}]$ after passing to associated graded algebras.

Theorem 7.8 (Namikawa, Losev). *The set of quantizations of* $\mathbb{C}[\mathcal{O}]$ *is in natural bijection with points of the Namikawa space* $\mathfrak{h}_{\mathcal{O}}$, *a finite-dimensional complex vector space constructed from the geometry of* $\mathcal{O}$.[12]

Definition 7.9. The *canonical quantization* $\mathcal{A}_0$ of $\mathbb{C}[\mathcal{O}]$ is the quantization of $\mathbb{C}[\mathcal{O}]$ corresponding to $0 \in \mathfrak{h}_{\mathcal{O}}$. The *unipotent ideal* for $\mathcal{O}$ is $I_0(\mathcal{O}) = \ker(\mathcal{U}(\mathfrak{g}) \to \mathcal{A}_0)$.

Proposition 7.10. *The following are equivalent:*

1. $\operatorname{Spec} \mathbb{C}[\mathcal{O}]$ *is smooth in codimension 2 and* $H^2(\mathcal{O}, \mathbb{C}) = 0$.

2. $\mathfrak{h}_{\mathcal{O}} = 0$.

3. $\mathcal{O}$ *is not induced from a proper Levi subgroup.*

4. $I_0(\mathcal{O})$ *is not induced.*

In this case, we say $\mathcal{O}$ is *rigid*.

Definition 7.11. A *rigid* $(\mathfrak{g}, K)$-*module* is an irreducible $(\mathfrak{g}, K)$-module M such that $\operatorname{Ann}_{\mathcal{U}(\mathfrak{g})}(M) = I(\mathcal{O})$ for $\mathcal{O}$ rigid.

Example 7.12. If $\mathcal{O} = 0$, the corresponding rigid representations are exactly the one-dimensional representations of $G_{\mathbb{R}}$ that factor through the component group of $G_{\mathbb{R}}$.

Example 7.13. The oscillator representation of $\operatorname{Mp}_{2n}(\mathbb{R})$ is rigid.

Example 7.14. There is an interesting rigid representation related to rigid orbits of for the real split E_8-group.

The speaker and collaborators showed that rigid representations are unitarizable using Hodge theory.

8   3/13 (Elliot Kienzle) – Continued

Write $\mathsf{MF}(V, Q) = \mathsf{MF}(\mathbb{C}^n, z_1^2 + \cdots + z_n^2)$. Our goal is to discuss the connection between this category and Clifford algebras.

8.1   Clifford algebras

Let V be an n-dimensional complex vector space with nondegenerate quadratic form Q. Write $T(V) = k \oplus V \oplus V^{\otimes 2} \oplus \ldots$ for the tensor algebra of V.

Definition 8.1. The nth *Clifford algebra* is

$$\operatorname{C\ell}_n = \operatorname{C\ell}(V, Q) = \frac{T(V)}{v \otimes v - Q(v, v) \otimes 1}$$

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[12]Specifically, $\mathfrak{h}_{\mathcal{O}}$ is the second cohomology of a $\mathbb{Q}$-factorial terminal resolution of $\mathcal{O}$ (e.g. a symplectic resolution of $\mathcal{O}$, if one exists).

As a vector space, $\mathrm{C}\ell_n \simeq \Lambda^\bullet V$, the exterior algebra of V. However, the product is "deformed" by

$$(v, w) \mapsto v \wedge w - Q(v, w).$$

We still have a $\mathbb{Z}/2$-grading of $\mathrm{C}\ell_n$ corresponding to the "degree of monomials" (which is well-defined modulo 2).

Note that $\mathrm{C}\ell_0 \cong \mathbb{C}$ and $\mathrm{C}\ell_1 \cong \mathbb{C} \times \mathbb{C}$.

Theorem 8.2 (Clifford algebra periodicity)**.** *We have $\mathrm{C}\ell_{n+2} \cong \mathrm{Mat}_2(\mathrm{C}\ell_n)$, the ring of 2×2 matrices with entries in $\mathrm{C}\ell_n$.*

This implies $\mathrm{C}\ell_n$ is semisimple, and $\mathrm{C}\ell_{n+2}$-Mod $\simeq \mathrm{C}\ell_n$-Mod (where these refer to *graded* modules). Taking M_n to be the representation ring of $\mathrm{C}\ell_n$, we get

$$M_n = \begin{cases} \mathbb{Z} & n \text{ even} \\ \mathbb{Z}^2 & \text{odd} \end{cases}$$

One can view this as an algebraic version of complex Bott periodicity.

8.2 Connection with matrix factorizations

Every Clifford module M gives rise to a matrix factorization in $\mathsf{MF}(V, Q)$. More precisely, write $M \cong M^0 \oplus M^1$, and view each M^i as (the fiber of) a trivial vector bundle on V. We can define maps via these vector bundles via $(v, m) \mapsto v \cdot m$ for $v \in V$, $m \in M_i$. This gives a matrix factorization $M^0 \rightleftarrows M^1$ of Q.

Theorem 8.3. *This induces an equivalence $\mathrm{C}\ell_n$-Mod $\simeq \mathsf{MF}(V, Q)$.*

8.3 Matrix factorizations and topology

We'd like to extract topological information about vanishing cycles from matrix factorizations. For a nice topological space X, let $K^0(X)$ be the Grothendieck group of vector bundles on X, i.e. the group generated by vector bundles on X with relations given by $[U] + [V] = [W]$ whenever W is an extension of U by V. For a nice subspace $Y \subset X$, we define *relative K-theory* via

$$K^0(X, Y) = \tilde{K}^0(X/Y) = \ker(K^0(X/Y) \to K^0(\mathrm{pt})).$$

We may extend K-theory to a cohomology theory via $\tilde{K}^{-n}(X) = \tilde{K}^0(\Sigma^n X)$. The *Chern character* gives homomorphisms $^{\mathrm{ch}} : K^0(X) \to H^0(X)$ and $K^0(X, Y) \to H^0(X, Y)$.

Given $\sigma : V_1 \to V_2$ over X such that σ is an isomorphism over $Y \subset X$, then we can construct a class $\chi(\sigma) \in K^0(X, Y)$. To accomplish this, we construct a vector bundle V on $M = (Y \times [0, 1]) \sqcup_{Y \times \{0,1\}} (X \times \{0, 1\})$ [13] by gluing V_1 and V_2 together using σ. We can also construct a vector bundle V_2' which is given by V_2 everywhere. The difference $[V] - [V_2']$ gives a class in $K^0(M)$, and this class in fact lives in $K^0(M, X) = K^0(X, Y)$ (where we mod out by the factor on which V and V_2' agree). This is our $\chi(\sigma)$.

Applying this to $\mathsf{MF}(V, Q)$, we see that, for every Clifford module M, we obtain an *Atiyah-Bott-Shapiro* class $\mathrm{ABS}(M) \in K^0(B, Q^{-1}(1) \cap B)$ where B is a ball. Note that $K^0(B, Q^{-1}(1) \cap B) \cong \tilde{K}^0(S^n)$. Thus we end up with a map $\mathrm{ABS} : \mathcal{M}_n \to \tilde{K}^0(S^n)$.

Theorem 8.4 (Atiyah-Bott-Shapiro)**.** *Let $i : \mathrm{C}\ell_n \hookrightarrow \mathrm{C}\ell_{n+1}$ be the inclusion. Then $\ker(\mathrm{ABS}) = i^* \mathcal{M}_{n+1}$. It follows that $\tilde{K}^0(S^n) \cong \mathcal{M}_n / i^* \mathcal{M}_{n+1}$.*

Note that the Atiyah-Bott-Shapiro construction, as presented here, secretly uses matrix factorizations!

We can also interpret Knörrer periodicity of matrix factorization categories as a categorification of Bott periodicity:

Theorem 8.5 (Knörrer periodicity)**.** *There is an equivalence*

$$\mathsf{MF}(X, W) \simeq \mathsf{MF}(X \times \mathbb{C}^2, W + x^2 + y^2).$$

Taking K-theory and using the Thom-Sebastiani isomorphism gives Bott periodicity.

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[13]Elliot drew a picture which I will not attempt to replicate here.

8.4 From matrix factorizations to Clifford modules

Let's see how to obtain a Clifford module from a matrix factorization. Recall that we have $\mathsf{MF}(V, Q) \simeq \mathsf{D}_{\mathrm{sing}}(Q^{-1}(0))$. The skyscraper sheaf at the origin of $\mathbb{C}$ corresponds to $\mathbb{C} \rightleftarrows \mathbb{C}$, where both maps are multiplication by z. Taking the nth external tensor power of this gives an object $\delta \in \mathsf{MF}(V^n, Q)$, with $\mathrm{End}^*(\delta) = (P \rightleftarrows Q)$. Our map $\mathsf{MF}(V, Q) \to \mathrm{C}\ell_n\text{-Mod}$ is just $\mathrm{Hom}(\delta, -)$.

9 3/20 (Yuji Okitani) – The Rozansky-Witten 2-Category

Following the paper by Kapustin and Rozansky, we will attempt to explain the idea behind the Rozansky-Witten 2-category.

9.1 The construction

Consider Rozansky-Witten theory with target T^*X for X a smooth algebraic variety. We construct a 2-category $\mathrm{RW}_2(\mathsf{T}^*X)$[14] with:

1. Objects: potentials $W : X \to \mathbb{A}^1$, or rather their graphs $\Gamma_{dW} \subset \mathsf{T}^*X$.

2. Morphism categories: $[W_1, W_2] = \mathsf{MF}(X, W_2 - W_1)$ (corresponding to Lagrangian intersections). Note that:

    (a) $\mathsf{MF}(X, W)$ localizes to $\mathrm{Crit}(W) = \{dW = 0\}$.
    (b) $\mathsf{MF}(X, W) = \prod_{s \in \mathbb{A}^1} \mathsf{D}^b_{\mathrm{coh}}(X_s)/\mathsf{Perf}(X_s)$.

3. Composition: $\mathsf{MF}(X, W_2 - W_1) \otimes \mathsf{MF}(X, W_3 - W_2) \to \mathsf{MF}(X, W_3 - W_1)$ by tensor product of periodic complexes:

$$(P^0, P^1, d_P) \otimes (Q^0, Q^1, d_Q) = \big((P^0 \otimes Q^0) \oplus (P^1 \otimes Q^1), (P^1 \otimes Q^0) \oplus (P^0 \otimes Q^1), d_{P \otimes Q}\big)$$

where the differential $d_{P \otimes Q}$ is constructed using the usual sign rule. It is a good exercise to work this all out.

9.2 Some examples

Example 9.1. Let's consider a triangle shape in $\mathsf{T}^*\mathbb{A}^1$ given by graphs Γ_{dW_i} where $W_1 = 0$, $W_2 = x^2$, and $W_3 = -(x-2)^2 + 2$. Then:

- $[W_1, W_2] = \mathsf{MF}(\mathbb{A}^1, x^2)$, generated by $(\mathbb{C}[x], \mathbb{C}[x])$ with morphisms given by multiplication by x.

- $[W_2, W_3] = \mathsf{MF}(\mathbb{A}^2, -2(x-1)^2)$, generated by $(\mathbb{C}[x], \mathbb{C}[x])$ with morphisms given by multiplication by $x - 2$ and $-2(x-2)$.

- $[W_1, W_3] = 0$.

We don't end up with a problem here due to "locality in X." Basically: for the W_i to have interesting composition, we need all differences $W_i - W_j$ to share a common critical value. Note that this requires choosing consistent representatives W_i of the graphs Γ_{dW_i}.

Example 9.2. Let X be a vector space V equipped with a quadratic form $Q : V \to \mathbb{C}$. Elliot gave an equivalence $\mathsf{MF}(V, Q) \simeq \mathrm{C}\ell_n\text{-Mod}$. One can view this as a deformed version of Koszul duality (for $Q = 0$, this relates $\mathrm{Sym}\,V^*$ to ΛV).

We expect that $\otimes : \mathsf{MF}(V, Q_1) \otimes \mathsf{MF}(V, Q_2) \to \mathsf{MF}(V, Q_1 + Q_2)$ should correspond to a convolution product $\star : \mathrm{C}\ell(V, Q_1)\text{-Mod} \otimes \mathrm{C}\ell(V, Q_2)\text{-Mod} \to \mathrm{C}\ell(V, Q_1 + Q_2)\text{-Mod}$. The convolution product here is induced by

$$\mathrm{C}\ell(V, Q_1 + Q_2) \to \mathrm{C}\ell(V, Q_1) \otimes \mathrm{C}\ell(V, Q_2)$$
$$v \mapsto 1 \otimes v + v \otimes 1.$$

We can work this out explicitly in examples.

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[14]This is only part of the "true" Rozansky-Witten 2-category

Note that $[W, W] = \mathsf{MF}(X, 0)$ is a 2-periodic version of $\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$. For an arbitrary Lagrangian L, we can guess that $[\mathsf{L}, \mathsf{L}]$ should look like a 2-periodic version of $\mathsf{Coh}(\mathsf{L})$, with composition given by tensor product. In general, we expect that the category of "2-quasicoherent sheaves" on X embeds as the "conic part" of $\mathrm{RW}_2(\mathsf{T}^*X)$. Next time, Enoch will discuss how to allow conic and non-conic parts of $\mathrm{RW}_2(\mathsf{T}^*X)$ to interact.

10   4/3 (Enoch Yiu) – Sheaves of Categories with Non-Conic Singular Support

Let X be a smooth algebraic variety. We want to understand Rozansky-Witten theory on T^*X in terms of module categories over $\mathsf{QCoh}(X)$, $\mathsf{Coh}(X)$, or $\mathsf{Perf}(X)$.

10.1   Singular support of sheaves of categories

To define a notion of singular support, we should first recall some relevant notions.

1. The *loop space* of X is the derived fiber product $\mathcal{L}X = \mathrm{Hom}(S^1, X) = X \times_{X \times X} X$. If X is a smooth algebraic variety, then the Hochschild-Kostant-Rosenberg isomorphism gives an equivalence

$$\mathcal{L}X \simeq \mathsf{T}X[-1] = \mathrm{Spec}_{\mathcal{O}_X} \Omega_X^{-\bullet}.$$

   Note that $\mathcal{L}X$ is an algebra over X.

2. Note that
$$\mathrm{Hom}_{\mathsf{Perf}(S)}(\mathsf{Perf}(X), \mathsf{Coh}(Y)) \simeq \mathsf{Coh}(X \times_S Y).$$

   Taking $X = Y$ and $S = X \times X$, we see that

$$\mathrm{End}_{\mathsf{Perf}(X) \otimes \mathsf{Perf}(X)^{\mathrm{op}}}(\mathsf{Perf}(X)) \simeq \mathsf{Coh}(\mathcal{L}X).$$

3. We can compute
$$\mathrm{Ext}^*_{\mathrm{Sym}_{\mathcal{O}_X} \Omega_X[1]}(\mathcal{O}_X, \mathcal{O}_X) = \mathrm{Sym}_{\mathcal{O}_X} \Omega_X^\vee[-2] = \mathcal{O}_{\mathsf{T}^*X[2]}.$$

   This gives a Koszul duality equivalence between $\mathsf{Coh}(\mathcal{L}X)$ and a deformation of $\mathsf{Perf}(\mathsf{T}^*X[2])$.

4. If R is an algebra and M is a module, then we get a homomorphism $Z(R) \to \mathrm{End}(M)$. This can be understood in terms of a "whistle" cobordism in TQFT.

Definition 10.1. Let $M \in \mathsf{Perf}(X)\text{-}\mathsf{Mod}$. Then $\mathrm{End}(M)$ is a $\mathsf{Coh}(\mathcal{L}X)$-module. This induces a map

$$\mathrm{End}_{\mathsf{Coh}(\mathcal{L}X)}(\mathcal{O}_X) \to \mathrm{End}_{\mathrm{End}(M)}(\mathbb{1}_{\mathrm{End}(M)}).$$

We may think of $\mathrm{Ext}^*(\mathbb{1}_{\mathrm{End}(M)}, \mathbb{1}_{\mathrm{End}(M)})$ as a topological (i.e. non-quasicoherent) sheaf of categories on $\mathsf{T}^*X[2]$. The singular support $\mathrm{SS}(M)$ is the support of $\mathrm{Ext}^*(\mathbb{1}_{\mathrm{End}(M)}, \mathbb{1}_{\mathrm{End}(M)})$ over $\mathsf{T}^*X[2]$. This is a "conic Lagrangian" in $\mathsf{T}^*X[2]$.

Here $\mathcal{O}_{\mathsf{T}^*X[2]}$ is really an $\mathbb{E}_3$-algebra. We are passing to Ext^*'s so that we can work with commutative algebras rather than $\mathbb{E}_3$-algebras.

Example 10.2. Let $i : Z \hookrightarrow X$ be a closed embedding of smooth varieties. Then $\mathsf{Perf}(Z)$ is a $\mathsf{Perf}(X)$-module, and
$$\mathrm{End}_{\mathsf{Perf}(X)}(\mathsf{Perf}(Z)) = \mathsf{Coh}(Z \times_X Z) = \mathsf{Coh}(\mathsf{N}_Z X[-1]).$$

We may view the shifted normal bundle $\mathsf{N}_Z X[-1]$ as $\mathcal{L}_Z X / \mathcal{L}Z$, where $\mathcal{L}_Z X = Z \times_{X \times X} Z$ (?). We can compute $\mathrm{Ext}^*(\mathbb{1}_{\mathrm{End}(\mathsf{Perf}(Z))}) = \mathcal{O}_{\mathsf{T}^*_Z X[2]}$.

10.2   The conic part of RW_2

The proposal is essentially that the conic part of $\mathrm{RW}_2(\mathsf{T}^*X)$ is $\mathsf{Perf}(X)$-Mod.

Let $M, N \in \mathsf{Perf}(X)$-Mod. Then $\mathrm{Hom}(M, N)$ is a $\mathsf{Coh}(\mathcal{L}X)$-module, and it defines a sheaf on $\mathsf{T}^*X[2]$. The support (over $\mathsf{T}^*X[2]$) of $\mathrm{Hom}(M, N)$ is contained in $\mathrm{SS}(M) \cap \mathrm{SS}(N)$. This agrees with expectations about RW_2!

Example 10.3. Let $M = \mathsf{Perf}(X)$ and $N = \mathsf{Perf}(x)$ for some $x \in X$. Then $\mathrm{Hom}_{\mathsf{Perf}(X)}(\mathsf{Perf}(X), \mathsf{Perf}(x)) = \mathsf{Coh}(X) \simeq \mathsf{Vect}$. This confirms our intuition: the intersection of the singular supports here is just a point, so the Hom is just Vect.

Given $f : X \to Y$, we obtain $f^* : \mathsf{Perf}(Y) \to \mathsf{Perf}(X)$ and hence a restriction map $\mathsf{Perf}(X)$-Mod $\to$ $\mathsf{Perf}(Y)$-Mod. There is also an induction map $\mathsf{Perf}(Y)$-Mod $\to \mathsf{Perf}(X)$-Mod. We can use these to define an external tensor product

$$M \boxtimes N = \mathrm{Ind}_{p_1} M \otimes \mathrm{Ind}_{p_2} N.$$

11   4/10 (Enoch Yiu) – Continued

Last time, we discussed viewing $\mathsf{Perf}(X)$ as the conic part of $\mathrm{RW}_2(\mathsf{T}^*X)$. Now, we'd like to understand the non-conic part of the Rozansky-Witten theory.

11.1   Towards the non-conic part of Rozansky-Witten theory

Consider $X \times \mathbb{A}^1$, and equip $\mathsf{T}^*(X \times \mathbb{A}^1)$ with coordinates $(x \in X, y \in \mathbb{A}^1, \xi \in \mathsf{T}^*_x X, \tau \in \mathsf{T}^*_y Y)$. Write $\mu : \mathsf{T}^*(X \times \mathbb{A}^1) \to (\mathbb{A}^1)^\vee$ for the projection. Then we can write $\mathsf{T}^*X = \mu^{-1}(1^\vee)/\mathbb{A}^1$. In the shifted case, this doesn't quite work: there is not a point "$1^\vee$" of $(\mathbb{A}^1)^\vee[\pm 2]$. Regardless, this gives a rough / moral picture of what we want to do.

Let

$$\mathcal{C}^{\mathrm{pre}}(X) = \mathsf{Perf}(X \times \mathbb{A}^1)\text{-Mod}/\{M \,|\, \mathrm{SS}(M) \subset \{\tau = 0\}\}.$$

This is our analogue of $\mu^{-1}(1^\vee)$.

Define $\rho : \mathsf{T}^*X \times \mathsf{T}^*_{\tau \neq 0} \to \mathsf{T}^*X$ by

$$\rho(x, \xi, y, \tau) = \left(x, \frac{\xi}{\tau}\right),$$

and let $\mu\,\mathrm{supp}(M) := \rho(\mathrm{SS}(M) \cap \{\tau \neq 0\})$.

Suppose $L \subset \mathsf{T}^*X$ is an exact Lagrangian, i.e. if λ is the canonical 1-form on T^*X, then there exists $f : L \to \mathbb{A}^1$ such that $\lambda|_L = df$. Then we obtain a conic Lagrangian

$$\Lambda = \{(x, c\xi, f(x, \xi), c \cdot 1) \,|\, (x, \xi) \in L\} \subset \mathsf{T}^*(X \times \mathbb{A}^1).$$

We may recover L as $\rho(\Lambda)$. David pointed out that this is analogous to the relationship between projective varieties and their affine cones.

11.2   Examples

Example 11.1. Suppose $i : Z \to X$ is a closed embedding, so $\mathrm{SS}(\mathsf{Perf}(Z)) = \mathsf{T}^*_Z X$. We also have $\mathsf{Perf}(Z) \boxtimes \mathsf{Perf}(0) = \mathsf{Perf}(Z \times 0)$, so $\mathrm{SS}(\mathsf{Perf}(Z \times 0)) = \mathsf{T}^*_{Z \times 0}(X \times \mathbb{A}^1)$ and $\mu\,\mathrm{supp}(\mathsf{Perf}(Z \times 0)) = \mathsf{T}^*_Z X$ as desired.

Example 11.2. Let $X = \mathbb{A}^n$ or $\mathbb{G}^n_m$. Take $Z = \{y + f(x) = 0\} \subset X \times \mathbb{A}^1$. Then $\mathrm{SS}(\mathsf{Perf}(Z)) = \{(x, -f(x), c \cdot df, c)\}$ and $\mu\,\mathrm{supp}(\mathsf{Perf}(Z)) = \Gamma_{df}$.

Remark 11.3. David mentioned the following conjecture of Teleman: the 2-category of "stable categories with S^1-action" is equivalent to $\mathrm{RW}_2(\mathsf{T}^*\mathbb{G}_m)$. Taking the "underlying category" in the former should correspond to taking Hom from the zero section in the latter. Taking the "fixed point category" should correspond to taking Hom from the conormal $\mathsf{T}^*_1\mathbb{G}_m$. Since there are objects that have no intersection with the zero section, there should be interesting "S^1-categories with trivial underlying category." This is not so strange (it's analogous to the fact that pt and ES^1 are both contractible but are not equivalent as S^1-equivariant spaces).

Example 11.4. Let $N = \{f_1 = \cdots = f_m = 0\} \subset \mathbb{A}^n$, and let $\alpha \in \Omega^1(N)$. If α is closed, then $\alpha + T_N^* \mathbb{A}^n$ is a Lagrangian; if α is exact, then $\alpha + T_N^* \mathbb{A}^n$ is an exact Lagrangian. Suppose $\alpha = dg$. Taking $Z = \{y + g(x) = f_1(x) = \cdots = f_m(x) = 0\}$, we get $\mu \operatorname{supp}(\mathsf{Perf}(Z)) = dg + T_N^* \mathbb{A}^n$.

12   4/17 (Wenyuan Li) – Regular Functions on the Moduli of Microlocal Sheaves

This is a guest lecture based on joint work with Roger Casals (and partly with Chris Kuo). Fix $G = \mathrm{GL}(n, \mathbb{C})$.

12.1   Moduli of microlocal sheaves

A simple example of the sort of space we want to consider is the following:

Example 12.1 (Moduli of local systems)**.** The *character variety* of a real surface Σ is $\mathrm{Hom}(\pi_1(\Sigma), G)/G^{\mathrm{ad}}$. This has a symplectic form, the *Goldman bracket*. (For $n = 1$ this is induced by the intersection form on $H_1(\Sigma)$.) Regular functions on this space are given by traces of monodromy around some loop.

A more complicated and interesting example is:

Example 12.2 (Moduli of Stokes data)**.** Consider a punctured real surface Σ. We want to consider the moduli $M(\Sigma, \beta)$ of local systems on Σ together with complete flags around each puncture such that the monodromy of the local system preserves each flag. We can identify this space with the moduli of locally constant sheaves on Σ with generic rank n and with fibers over concentric circles around the punctures of decreasing rank. The concentric circles arise from solutions to $\mathcal{D}$-modules with irregular singularities. There are some pictures but I wasn't able to copy them down here.

12.2   The non-abelianization map

Studying rank n local systems is difficult. Instead we try to reduce to studying rank 1 local systems as follows.

Given a punctured surface Σ, we consider a "nice" n-fold branched cover $L \to \Sigma$. There exists a (highly nontrivial) map $\mathrm{Loc}_1(L) \to \mathrm{Loc}_n(\Sigma)$, which gives an open subset of $\mathrm{Loc}_n(\Sigma)$. Essentially, we view L as giving a Lagrangian in $T^*\Sigma$. Crossings near punctures extend to the interior of Σ and give a constructible sheaf in $\Sigma \times \mathbb{R}$. This can also be constructed combinatorially using "ideal triangulations."

Anyway, the important thing is that we can understand an open subset of $\mathrm{Loc}_n(\Sigma)$ in terms of $\mathrm{Loc}_1(L)$. The monodromies along loops in L give regular functions on open subsets of $M(\Sigma, \beta)$. Intersection forms for L give closed 2-forms on open subsets of $M(\Sigma, \beta)$. We want to ask: when do such objects extend to all of $M(\Sigma, \beta)$? .

12.3   Symplectic geometry

In terms of microlocal sheaves, we are interested in the category $\mathcal{C} = \mathsf{Sh}_\Lambda(\Sigma) = \mathrm{Fuk}_\Lambda(T^*\Sigma)$. Given a nice $L \subset T^*\Sigma$, we get a category $\mathcal{C}_L \subset \mathcal{C}$ consisting of rank 1 local systems $\mathcal{L}$ on L. The monodromies are given by $m_\gamma : \mathcal{C}_L \to \mathbb{C}$ for γ lying along L.

We should view $\mathsf{Sh}_\Lambda(\Sigma)$ as the global (co)sections of a constructible cosheaf of categories on the "Lagrangian skeleton." The Lagrangian skeleton is a stratified Lagrangian subset that is a deformation retract of $(T^*\Sigma, \Lambda)$.

The non-abelianization map suggests that instead we want to work with $L \subset T^*\Sigma$. Note that L is not a Lagrangian skeleton – it has different topology from Σ! We can fix this by filling in loops (with Lagrangian disks) to kill parts of the first homology of L. The main theorem is that m_γ extends to all of $M(\Sigma, \beta)$ if and only if γ has nonnegative intersection with the boundaries of all such Lagrangian disks.

13   4/24 (Enoch Yiu) – Continued

13.1   Refresher

We let
$$\mathcal{C}^{\mathrm{pre}}(X) = \mathsf{Perf}(X \times \mathbb{A}^1)\text{-}\mathsf{Mod}/\{M \,|\, \mathrm{SS}(M) \subset \{\tau = 0\}\}.$$

Recall that this is an analogue of $\mu^{-1}(1^{\vee})$, and we'd like to think of $\mathcal{C}(X)$ as an analogue of $\mu^{-1}(1^{\vee})/\mathbb{A}^1$. Consider $\rho : T^*X \times T^*_{\tau \neq 0}\mathbb{A}^1 \to T^*X$ via $\rho(x, \xi, y, t) = (x, \xi/\tau)$ – this gives a correspondence between exact Lagrangians in T^*X and conic Lagrangians in $T^*X \times T^*\mathbb{A}^1$. We'd like to think of $\mathcal{C}(X)$ as "$\mathsf{Coh}(\mathbb{A}^1)$-invariant $\mathcal{C}^{\mathrm{pre}}(X)$-modules."

13.2   Definitions

Let $N = \{f_1(x) = \cdots = f_m(x) = 0\} \subset X$, and consider $Z = \{y + g(x) = 0\} \cap (\mathbb{A}^1 \times N)$. Note that $\mu \operatorname{supp}(Z) = dg + T^*_N X$. These give "hybrid Lagrangians" in $T^*(X \times \mathbb{A}^1)$. The category $\mathcal{C}(X)$ has $\operatorname{Ob} \mathcal{C}(X) = \{\mathsf{Perf}(Z) \,|\, Z \text{ as above}\}$, where each $\mathsf{Perf}(Z)$ is viewed as a $\mathsf{Perf}(X \times \mathbb{A}^1)$-module category.

Letting $T_c : X \times \mathbb{A}^1 \to X \times \mathbb{A}^1$ be the translation, we get $\operatorname{Res}_{T_c} : \mathsf{Perf}(X \times \mathbb{A}^1)\text{-}\mathsf{Mod} \to \mathsf{Perf}(X \times \mathbb{A}^1)\text{-}\mathsf{Mod}$. We define the morphisms in $\mathcal{C}(X)$ by

$$\operatorname{Hom}_{\mathcal{C}(X)}(M, N) = \oplus_{c \in \mathbb{A}^1} \operatorname{Hom}_{\mathcal{C}^{\mathrm{pre}}(X)}(M, \operatorname{Res}_{T_c}(N)).$$

For $M, N \in \mathcal{C}(X)$, we define

$$\mu \operatorname{Hom}(M, N) = \mathsf{Perf}(\{\tau \neq 0\}) \otimes_{\mathsf{Perf}(T^*(X \times \mathbb{A}^1)[2])} \operatorname{Hom}_{\mathcal{C}(X)}(M, N).$$

Example 13.1. Let $C_f = \mathsf{Perf}(\{y + f(x) = 0\})$, so $\mu \operatorname{supp} C_f = \Gamma_{df}$. We compute

$$\mu \operatorname{Hom}(C_f, C_g) = \oplus_{c \in \mathbb{A}^1} k[\ldots, \tau, \tau^{-1}]\text{-}\mathsf{Mod} \otimes_{k[\ldots, \tau]\text{-}\mathsf{Mod}} \operatorname{Hom}_{\mathcal{C}^{\mathrm{pre}}(X)}(C_f, \operatorname{Res}_{T_c} C_g).$$

We have $\operatorname{Hom}_{\mathsf{Perf}(X \times \mathbb{A}^1)}(C_f, \operatorname{Res}_{T_c} C_g) = \mathsf{Coh}(\{y + f(x) = 0\} \times_{X \times \mathbb{A}^1} \{y + g(x) = c\})$. Eventually this gives $\mu \operatorname{Hom}(C_f, C_g) = \mathsf{MF}(X, f - g)$.