GRT Seminar Fall 2024 – Rozansky-Witten Theory

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Abstract

This semester, the GRT Seminar will focus on Rozansky-Witten theory.

Contents

1 9/5 (David Nadler) – Introduction

Our goal is to discuss Rozansky-Witten theory. Some related topics include:

- Quasicoherent sheaves of categories (as discussed last spring).
- Categories of matrix factorizations.[1]
- The cobordism hypothesis.
- Local structure theory of holomorphic symplectic varieties.

1.1 What is Rozansky-Witten theory?

Suppose we have a hyperkähler / holomorphic symplectic manifold X. This means that X has a holomorphic $(2,0)$-form ω satisfying the (complex analogues of) the usual symplectic form axioms. Given such an X, there is a conjectural 3-dimensional topological field theory $\mathcal{Z}_\mathsf{X}$, called *Rozansky-Witten theory* with target X.

What we mean by 3d TFT is as follows:

- Given a closed 3-manifold[2] M^3, we obtain a number $\mathcal{Z}_\mathsf{X}(\mathsf{M}^3)$.

[1] In more detail: given a smooth variety X and a function $\mathsf{f} : \mathsf{X} \to \mathbb{A}^1$, we can construct a category $\mathsf{MF_f}$ which categorifies the vanishing cycles of f.

[2] Typically with some extra structure, e.g. an orientation

- Closed 2-manifolds M^2 give vector spaces $\mathcal{Z}_X(M^2)$.
- Closed 1-manifolds M^1 give categories[3] $\mathcal{Z}_X(M^1)$.
- Closed 0-manifolds M^0 give 2-categories $\mathcal{Z}_X(M^0)$.

In particular, $\mathcal{Z}_X(\mathrm{pt})$ is a 2-category. The *cobordism hypothesis* tells us that we can recover the entire theory $\mathcal{Z}_X$ from the "3-dualizable" 2-category $\mathcal{Z}_X(\mathrm{pt})$. For purposes of geometric representation theory, we are most interested in the low-dimensional behavior, which captures more data about the theory.

Rozansky-Witten theory should satisfy something like:

- $Z_X(S^2) = \mathcal{O}(X)$.[4]
- $Z_X(S^1) = \mathsf{Coh}(X)$.

These end up inheriting interesting structure from the TFT.

1.2 Why do we care?

Recall that 2-dimensional mirror symmetry can be schematically understood as an equivalence between the following 2d TFTs:

- An A-model $\mathcal{A}$ arising from symplectic geometry
- A B-model $\mathcal{B}_X$, coming from some Kähler manifold X, satisfying $\mathcal{B}_X(\mathrm{pt}) \simeq \mathsf{Coh}(X)$.

In particular, $\mathcal{A}(\mathrm{pt})$ is often some category of geometric interest, and the equivalence $\mathcal{A}(\mathrm{pt}) \simeq \mathcal{B}_X(\mathrm{pt})$ lets us resolve questions about $\mathcal{A}(\mathrm{pt})$.

There's an analogue in higher dimensions: we'd like to take a 3d TFT $\mathcal{Y}$ and give an equivalence $\mathcal{Y} \simeq \mathcal{Z}_X$ for some holomorphic symplectic X. This would give an equivalence between some 2-category and $\mathcal{Z}_X(\mathrm{pt})$.

Conjecture 1.1 (Teleman)**.** *Let G be a complex reductive group with maximal compact subgroup G_c. There is an equivalence between:*

- *A suitable 2-category of "categories with G_c-action."*
- *The Rozansky-Witten 2-category of $T^*(G^\vee/G^\vee)$.*

Note that $T^*(G^\vee/G^\vee)$ is stacky and non-proper, which makes it impossible for the corresponding 2-category to be 3-dualizable. Thus we typically won't obtain 3-manifold invariants from such a theory. That's terrible for 3-manifold topologists, but this isn't a 3-manifold seminar.

Some other examples of interest for Rozansky-Witten theory include symplectic resolutions and cotangent bundles of smooth algebraic varieties.

1.3 What is the correct 2-category?

To rigorously construct Rozansky-Witten theory, we'd need to give a definition of the 2-category $\mathrm{RW}_2 = \mathcal{Z}_X(\mathrm{pt})$. This was studied by Kapustin, Rozansky, and Saulina, but much is still unknown.

Roughly, we expect RW_2 to be a 2-category where:

- Objects are smooth Lagrangians $L \subset X$ (or some suitable generalization of these).
- 1-morphisms from L_1 to L_2 are given by some sort of category associated to $L_1 \cap L_2$. In the simplest possible case, where $X = T^*W$ is a cotangent bundle, L_1 is the zero-section, and L_2 is the graph of a differential df, then $L_1 \cap L_2$ is the critical locus of X and we assign $\mathrm{Hom}(L_1, L_2) = \mathsf{MF}_f$, the matrix factorization category of f. Work of Joyce and many others has focused on understanding how much the local setting looks like this.

[3]As is standard for GRT, we use the implicit ∞ convention.

[4]By our conventions, this is what is classically called $\mathbf{R}\Gamma(X, \mathcal{O})$, so there is interesting derived information.

- 2-morphisms and higher are "natural compatibilities" between the 1-morphisms.

One should think of the matrix factorization category MF_f as giving a categorical way to measure the critical locus of f. When the critical points of f are Morse, the category MF_f looks like a direct sum of copies of Vect (one for each critical point).

There is an important distinction between Rozansky-Witten theory and the 2d A-model. In the complex setting, there are no "instantons," so the theory is local and we don't run into the full difficulty of Floer theory. Thus Rozansky-Witten theory is a categorified version of Fukaya theory that avoids the need for instanton corrections.

1.4 An alternative viewpoint

If $X = T^*W$ is a cotangent bundle, then $\mathsf{ShvCat}(W)$, the 2-category of (quasicoherent) sheaves of categories on W, embeds into RW_2. The image of this embedding consists of "conic objects." Thus we can understand a key part of Rozansky-Witten theory, at least in this simple case.

The thesis (work in progress) of Enoch Yiu relates RW_2 to $\mathsf{ShvCat}(W \times \mathbb{A}^1)$.

2 1/30 (Daigo Ito) – Theory of Critical Points and Matrix Factorizations

Recall that we wanted to understand the Rozansky-Witten theory of a holomorphic symplectic variety M. By the cobordism hypothesis, it suffices to understand the 2-category $\mathrm{RW}_2(M)$. We expect $\mathrm{RW}_2(M)$ to have some vague properties as follows.

The objects of RW_2 should be holomorphic Lagrangians in M (possibly equipped with extra data). If $M = T^*L_1$, then we should have $\mathrm{Hom}_{\mathrm{RW}_2}(L_1, L_2) = \mathsf{MF}(L_1, f)$, the category of *matrix factorizations* of f. This measures the local geometry of $p \in L_1 \cap L_2 = \mathrm{Crit}(f)$.

Recall the two key differences between this and Lagrangian Floer homology:

- There are no instantons, so the full subtleties of Floer theory don't appear.
- We are working at a higher category level.

Today we will recall the theory of critical points for a function $f : X \to \mathbb{A}^1$.

2.1 Milnor fibers

Let's start by considering a regular map $f : \mathbb{C}^n \to \mathbb{C}$. Assume that $0 \in \mathbb{C}$ is a critical value. Call $X_0 = f^{-1}(0)$ the special fiber – this is typically singular. For small $s \in \mathbb{C}$, let $X_s = f^{-1}(s)$ be the nearby fiber.

Theorem 2.1 (Milnor)**.** *Let $x \in X_0$. For $\epsilon > 0$ sufficiently small, let $B(x, \epsilon)$ be the closed ball of radius ϵ centered at x, and let $S(x, \epsilon) = \partial B(x, \epsilon)$. Then:*

1. *$B(x, \epsilon) \cap X_0$ is homeomorphic to the cone over $K_x = S(x, \epsilon) \cap X_0$.*
2. *The map $\rho_f = \frac{f}{|f|} : S(x, \epsilon) \setminus K_x \to S^1$ is a locally trivial fibration. We call ρ_f the* Milnor fibration *and the fiber F_x the* Milnor fiber*.*

The Milnor fibers F_x degenerate to the cone over K_x.

Example 2.2. If x is nonsingular, then K_x is a sphere, so the cone over K_x is a ball. The Milnor fibers F_x are also balls.

The topology of the Milnor fibers reflects "how singular the point is" – a more singular point leads to a more complicated topology.

Example 2.3. Let $(X_0, x) = (z_1^2 - z_2^2 = 0, 0)$. Then F_x is homotopy equivalent to S^1. Looking at real points, the map f describes a family of hyperbolas degenerating to a union of lines. Here $\partial B = S^3$ and $K_x = S^1 \sqcup S^1$, so topologically K_x is a double cone. The Milnor fibers form a family of cylinders degenerating to this double cone.

Example 2.4. Let $(X_0, x) = (z_1^3 - z_2^2 = 0, 0)$. Then K_x is a trefoil knot

$$\{(r_1 e^{2\pi i t}, r_2 e^{2\pi i t}) \mid t \in \mathbb{R}\} \subset S^1_{r_1} \times S^1_{r_2}.$$

The closures of the Milnor fibers are genus one “Seifert surfaces” for K_x. Thus the Milnor fibers are homotopy equivalent to $S^1 \wedge S^1$.

More generally, if (X, x) is an isolated hypersurface singularity, then we can write $F_x \simeq (S^n)^{\vee \mu_x}$, where μ_x is the *Milnor number*.[5] The S^n's here are the *vanishing cycles* of the singularity.

2.2 Monodromy

The singularity carries information beyond the Milnor fibers. We can capture some of this by looking at the monodromy.

Definition 2.5. The *monodromy* of f at x is the map $h_f : F_x \to F_x$ induced by circling around the base. This is a homeomorphism of F_x which restricts to the identity on ∂F_x. Note that h_f is only well-defined up to isotopy (fixing ∂F_x).

Example 2.6. For a Morse function $f = \sum_i x_i^2$, the Milnor fibers are homotopy equivalent to S^n. We understand the singularity by studying the monodromy of the Milnor fibers as we move around the singular point. This monodromy is a Dehn twist, “corkscrewing” the cylinder.

Theorem 2.7 (Thom-Sebastiani). *Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ and $g : (\mathbb{C}^{m+1}, 0) \to (\mathbb{C}, 0)$ be germs of hypersurface singularities. Define $f \boxplus g : (\mathbb{C}^{n+1} \times \mathbb{C}^{m+1}, 0) \to (\mathbb{C}, 0)$ by $(f \boxplus g)(x, y) = f(x) + g(y)$. Then there is a homotopy-commutative diagram*

$$\begin{tikzcd} F_f * F_g \arrow[r, "\sim"] \arrow[d, "h_f * h_g"] & F_{f \boxplus g} \arrow[d, "h_{f \boxplus g}"] \\ F_f * F_g \arrow[r, "\sim"] & F_{f \boxplus g}, \end{tikzcd}$$

where $*$ is the join of spaces.

2.3 Preview

Next time we will introduce sheaves that describe the homology of these spaces. We get a fiber sequence

$$i^* \mathcal{F} \longrightarrow \psi_f \mathcal{F} \longrightarrow \phi_f \mathcal{F} \longrightarrow$$

of sheaves on X_0, where:

- $i : X_0 \to X$ is the inclusion,
- ψ_f is nearby cycles, and
- ϕ_f is vanishing cycles.

This will categorify to a sequence

$$\mathsf{Perf}(X_0) \longrightarrow D^b_{\mathrm{coh}}(X_0) \longrightarrow D_{\mathrm{sing}}(X_0).$$

where $D_{\mathrm{sing}}(X_0)$ agrees with $\mathsf{MF}(X, f)$ in nice cases.

3 2/6 (Daigo Ito) – Continued

Last time we discussed the construction of Milnor fibers, vanishing cycles, monodromy, and Thom-Sebastiani isomorphisms for $f : \mathbb{C}^{n+1} \to \mathbb{C}$. Today we would like to discuss and categorify these stories in a sheaf-theoretic framework.

[5] There is an explicit formula for the Milnor number, but we won't write it here.

3.1 Vanishing and nearby cycles

Let $\mathbb{D} \subset \mathbb{C}$ be a small disk around 0. Let X be (an open subset of) a smooth algebraic variety over $\mathbb{C}$, and let $f : X \to \mathbb{D}$ be a map. Consider the diagram

$$\begin{tikzcd}
X_0 \arrow[r, "i"] \arrow[d] & X \arrow[d, "f"] & X^* \arrow[l, "j"'] \arrow[d, "f^*"] & \tilde{X^*} \arrow[l, "\tilde{\pi}"'] \arrow[d] \\
\{0\} \arrow[r] & \mathbb{D} & \mathbb{D}^* \arrow[l] & \tilde{\mathbb{D}^*} \arrow[l, "\pi"']
\end{tikzcd}$$

where all squares are pullback squares and $\pi : \tilde{\mathbb{D}^*} \to \mathbb{D}^*$ is the universal cover $z \mapsto \exp(2\pi i z)$. Note that $X^* \simeq X_s$ for s small, $s \neq 0$.

Write $D^b_c(-)$ for the bounded constructible derived category of A-modules on a space (where A is some fixed coefficient ring, typically $\mathbb{Z}$ or $\mathbb{C}$). Recall that "constructible" means locally constant on the strata of a nice stratification and with finite-rank stalks.

Definition 3.1. The *nearby cycle functor* associated with f is $\psi_f : D^b_c(X) \to D^b_c(X_0)$, defined by[6]

$$\mathcal{F} \mapsto i^*(j \circ \tilde{\pi})_*(j \circ \tilde{\pi})^*\mathcal{F}.$$

Morally, we have a (very non-analytic) specialization map $\mathrm{sp} : X_s \to X_0$, and $\psi_f = \mathrm{sp}_*(\mathcal{F}|_{X_s})$. The previous definition is used to avoid referencing sp.

Example 3.2. Consider $f : \mathbb{D} \to \mathbb{D}$ by $f(z) = z^2$. For $\mathcal{F} = \underline{A}_{\mathbb{D}}$, we can compute $\psi_f(\mathcal{F}) = A_0 \oplus A_0$, reflecting the fact that the nearby fibers have two points. The monodromy map swaps the two factors: this can be seen directly using the specialization definition or by considering deck transformations using the formal definition.

Remark 3.3. David mentioned that one can actually rephrase this story so that the only f which we consider is projection to the first coordinate. The cost is that we are forced to work with arbitrarily complicated sheaves. The reverse (working with the constant sheaf but allowing arbitrarily complicated f) is not possible in general, though there is a related theory of "sheaves of geometric origin."

From the pushforward-pullback adjunction, there is a natural map $r : i^*\mathcal{F} \to i^*(j \circ \tilde{\pi})_*(j \circ \tilde{\pi})^*\mathcal{F} = \psi_f\mathcal{F}$.

Definition 3.4. We define the *vanishing cycle functor* $\phi_f : D^b_c(X) \to D^b_c(X_0)$ by $\phi_f(\mathcal{F}) = \mathrm{cone}(r)$, so there is a cofiber sequence

$$i^*\mathcal{F} \longrightarrow \psi_f\mathcal{F} \longrightarrow \phi_f\mathcal{F} \longrightarrow .$$

We call $\psi_f\underline{A}_X$ (resp. $\phi_f\underline{A}_X$) the *nearby (resp. vanishing) cycle complex* associated with f. From the cofiber sequence containing these, we obtain a long exact sequence (using $H^*(X) \cong H^*(X_0)$):

$$\ldots \longrightarrow H^*(X_0) \longrightarrow H^*(X_s) \longrightarrow H^*(X, X_s) \longrightarrow \ldots$$

This encompasses much of our discussion from last time.

Proposition 3.5. *For $f : \mathbb{C}^{n+1} \to \mathbb{C}$, if X_0 has only isolated singularities (so $F_f \simeq \vee S^n$), then*

$$H^k(X_0, \phi_f\underline{A}_X) = \begin{cases} 0 & k \neq n \\ \oplus_{x \in \mathrm{Sing}(X_0)} H^n(F_{f,x}; A) & k = n. \end{cases}$$

Remark 3.6. One can obtain the monodromy of nearby / vanishing cycles using the deck transformations of $\tilde{\mathbb{D}^*}$. There's also a Thom-Sebastiani theorem in the sheaf-theoretic setting.

[6] All functors here are derived.

3.2 Singularity categories and matrix factorizations

We can categorify the preceding story using the exact sequence of categories:

$$\mathsf{Perf}(\mathsf{X}_0) \longrightarrow \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathsf{X}_0) \longrightarrow \mathsf{D}_{\mathrm{sing}}(\mathsf{X}_0),$$

where $\mathsf{D}_{\mathrm{sing}}(\mathsf{X}_0)$ is defined as the quotient $\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathsf{X}_0)/\mathsf{Perf}(\mathsf{X}_0)$. In nice cases, $\mathsf{D}_{\mathrm{sing}}(\mathsf{X}_0)$ agrees with the *matrix factorization category* $\mathsf{MF}(\mathsf{X}, \mathsf{f})$. The reason $\mathsf{D}_{\mathrm{sing}}$ is called the "category of singularities" is the following:

Proposition 3.7. X_0 *is smooth if and only if* $\mathsf{Perf}(\mathsf{X}_0) \simeq \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathsf{X}_0)$.

Example 3.8. If $\mathsf{x} \in \mathsf{X}_0$ is singular, then the skyscraper sheaf $\mathsf{k}(\mathsf{x}_0)$ is not in $\mathsf{Perf}(\mathsf{X})$.

To decategorify our exact sequence to the sheaf-theoretic statement above, we take "periodic cyclic homology."