# GRT Seminar Fa23-Sp24 Notes

October 5, 2023

**Abstract**

The seminar covers Ben-Zvi–Sakellaridis–Venkatesh, "Relative Langlands Duality."

## Contents



## 1 8/31 (David Nadler) – ???

I missed this day. If you have good notes from this day, send them to me and I will type them up.

## 2 9/7 (Elliot Kienzle) – Hamiltonian $\mathsf{G}$-Spaces and Quantization

Elliot's notes for his talks are available at `https://chessapig.github.io/files/notes/G-spaces.pdf`.

The original Langlands program studies a duality of Lie groups $\mathsf{G} \leftrightarrow \mathsf{G}^\vee$. Relative Langlands seeks to upgrade this to a duality of Hamiltonian $\mathsf{G}$-actions $(\mathsf{G} \curvearrowright \mathsf{M}) \leftrightarrow (\mathsf{G}^\vee \curvearrowright \mathsf{M}^\vee)$. This is proposed for hyperspherical varieties $\mathsf{M}$, of which a typical example is $\mathsf{M} = \mathsf{T}^*\mathsf{X}$ for $\mathsf{X}$ a spherical variety.

We can approach and motivate this using quantization. Start by considering the action of $\mathsf{G}$ on $\mathsf{L}^2(\mathsf{X})$ for $\mathsf{X}$ a spherical variety (discussed in an earlier paper of Sakellaridis-Venkatesh discussing "harmonic analysis on spherical varieties").

## 2.1 Symplectic geometry and quantization

The original motivation for symplectic geometry comes from classical mechanics. Suppose that we have a particle moving in $\mathbb{R}^n$. We can capture the data of the position and momentum using the cotangent bundle $T^*\mathbb{R}^n$. By Newton's second law, the time evolution of the particle is described by (the flow along) a vector field on $T^*\mathbb{R}^n$.

We can generalize this to a symplectic manifold $(M, \omega)$, which is a manifold $M$ with a closed, non-degenerate 2-form $\omega$. To make this easier to work with, we can fix a metric $\langle , \rangle$ on $M$ and write $\omega(x, y) = \langle x, Jy\rangle$ where $J^2 = -1$ (i.e. $J^2$ is an almost complex structure). We think of $J^2$ as "multiplication by $-i$." Given a Hamiltonian $H \in \mathcal{C}^\infty(M)$, we obtain a Hamiltonian vector field $X_H = J\nabla H$. More invariantly, we can define $X_H$ via the formula $\omega(X_H, -) = dH$.

Moving to quantum mechanics, we view a particle in $\mathbb{R}^n$ as a $\mathbb{C}$-valued function $\psi$ on $\mathbb{R}^n$ (not $T^*\mathbb{R}^n$). In this case, the Hilbert space is $L^2(\mathbb{R}^n)$. A free particle evolves according to Schrödinger's equation:

$$i\dot{\psi} = \Delta\psi.$$

We can summarize the classical and quantum pictures in the following table.

| | Classical | Quantum |
|---|---|---|
| State Space | Symplectic manifold $(M, \omega)$ | Hilbert space $\mathcal{H}$ |
| Observables | $f \in \mathcal{C}^\infty(M)$ | Bounded operators $A \in \mathrm{End}(\mathcal{H})$ |
| Evolution | Vector fields $X_H$ for $H \in \mathcal{C}^\infty(M)$ | Unitary operators $U(t) = e^{itA}$ for $A \in \mathrm{End}(\mathcal{H})$ |
| Lie Algebra of observables | Poisson bracket $\{f, g\} = X_f(g)$ | Commutator $[A, B]$ |

To obtain a quantum system from a classical system (heuristically), we pass from nonlinear evolution of points in $T^*M$ to linear evolution of functions on $M$. (This linearity is forced on us by our desire to have superposition of states.) The dream of quantization is, given a symplectic manifold $(M, \omega)$, to construct a Lie algebra homomorphism $(\mathcal{C}^\infty(M), \{,\}) \to (\mathrm{End}(\mathcal{H}), [,])$ for some Hilbert space $\mathcal{H}$. Unfortunately, this is impossible to do consistently / functorially in general. However, there are some cases in which we can get good answers.

We will focus on geometric quantization, which behaves (loosely) as follows:

- For $M = T^*X$, we obtain $\mathcal{H} = L^2(X)$.
- For $M$ a compact Kähler manifold, we obtain $\mathcal{H} = H^0(M, \mathcal{L})$ for some line bundle $\mathcal{L}$ on $M$.

## 2.2 G-Spaces

We want to incorporate symmetries into the previous picture. Suppose $G$ is a compact Lie group / reductive algebraic group (depending on context). We say a symplectic $G$-space is a symplectic manifold $(M, \omega)$ with $G$-action preserving $\omega$. We can hope to quantize this to a linear representation $G \curvearrowright \mathcal{H}$. (There are subtleties that arise here – for geometric quantization, we would like a $G$-equivariant polarization.)

In general, it is better to consider Hamiltonian $G$-actions, where $\mathfrak{g}$ acts by Hamiltonian vector fields. This allows us to construct a moment map $\mu : M \to \mathfrak{g}^*$ which is equivariant (with respect to the coadjoint action on $\mathfrak{g}^*$).

Let us start by understanding the coadjoint action $G \curvearrowright \mathfrak{g}^*$ using Kirillov's "orbit method." For $\alpha \in \mathfrak{g}^*$, consider the coadjoint orbit $\mathcal{O}_\alpha$. This $\mathcal{O}_\alpha$ turns out to be a symplectic manifold (with "Kirillov-Kostant-Souriau" / "KKS" form) with Hamiltonian $G$-action, and the moment map $\mathcal{O}_\alpha \to \mathfrak{g}^*$ is just the inclusion.

**Example 2.1.** Consider $G = \mathrm{SO}(3)$. The coadjoint action is just $\mathrm{SO}(3)$ acting on $\mathbb{R}^3$ by rotations. Thus the generic orbits are spheres $S^2$.

The orbits $\mathcal{O}_\alpha$ will look like generalized flag manifolds, and conversely every generalized flag manifold arises in this way. (This is the first place where our compactness hypothesis comes in).

**Proposition 2.2.** *A coadjoint orbit $\mathcal{O}_\alpha$ is quantizable if and only if $\alpha$ is in the orbit of an integer point of the root lattice $\mathfrak{t}^*_{\mathbb{Z}} \subset \mathfrak{t}^*$ (viewed as a subspace of $\mathfrak{g}^*$ via the Killing form).*

**Example 2.3.** Continuing on with our SO(3) example, we see that a symplectic sphere is quantizable if and only if it has integer area.

In these cases, the quantization of $\mathcal{O}_\alpha$ is $H^0(\mathcal{O}_\alpha, \mathcal{L}_\alpha)$ where $\mathcal{L}_\alpha$ is the line bundle corresponding to the character $\alpha$. By the Borel-Weil theorem, $H^0(\mathcal{O}_\alpha, \mathcal{L}_\alpha)$ is the irrep $V_\alpha$ of $G$ with highest weight $\mathcal{L}_\alpha$.

We can summarize this in the following table:

| Classical | Quantum |
|---|---|
| Symplectic action $G \curvearrowright M$ | Representation $G \curvearrowright \mathcal{H}$ |
| Coadjoint orbit $\mathcal{O}_\alpha$ | Highest weight representation $E_\alpha$ |

# 3 9/14 (Elliot Kienzle) – Continued

## 3.1 Symplectic reduction

Suppose we have a Hamiltonian action $G \curvearrowright M$. This yields a $G$-equivariant moment map $\mu : M \to \mathfrak{g}^*$, and the image of $\mu$ will necessarily be a collection of coadjoint orbits $\mathcal{O}_\alpha$. We can use these orbits to decompose $M$.

First consider the orbit $\mathcal{O}_0 = \{0\}$. We note that $\mu^{-1}(0)$ is $G$-invariant, so we can consider the quotient $\mu^{-1}(0)/G$. We define this to be the *symplectic quotient*: $M/\!/G := \mu^{-1}(0)/G$.

We will assume that 0 is a regular value of the moment map and that $G$ acts on $\mu^{-1}(0)$ freely. We can drop these assumptions if we consider things in a suitable derived / stacky sense.

**Theorem 3.1** (Marsden-Weinstein)**.** *The symplectic quotient $M/\!/G$ carries a natural symplectic structure.*

**Example 3.2.** If $X$ is a (not necessarily symplectic) manifold with a $G$-action, then $T^*X/\!/G = T^*(X/G)$.

**Example 3.3.** Let $M = T^*\mathbb{R}^2 \cong \mathbb{C}^2$. This has a U(1)-action via

$$e^{i\theta}(z_1, z_2) = (e^{i\theta}z_1, e^{i\theta}z_2).$$

We can define a (shifted) moment map $\mu : \mathbb{C}^2 \to \mathbb{R}$ via

$$\mu(z_1, z_2) = |z_1|^2 + |z_2|^2 - 1.$$

Then $\mathbb{C}^2/\!/\,\mathrm{U}(1) = S^3/\,\mathrm{U}(1) = S^2 = \mathbb{P}^1$ (consider the Hopf fibration).

Morally, we should think of every symplectic manifold as a symplectic reduction of a (possibly infinite-dimensional) affine space.

Note that

$$\dim M/\!/G = \dim M - 2\dim G.$$

The slogan is that "in symplectic geometry, groups act twice."

**Theorem 3.4** (Guillemin-Sternberg, etc.)**.** *The geometric quantization of a symplectic quotient satisfies*

$$\mathcal{H}(M/\!/G) = \mathcal{H}(M)^G,$$

*where the right hand side is the subspace of $G$-invariant vectors in $G$.*

We can also define the symplectic reduction along any coadjoint orbit $\mathcal{O}_\alpha$ as $M/\!/_\alpha G = \mu^{-1}(\mathcal{O}_\alpha)/G$. This gives a decomposition of $M$ as

$$M = \cup_{\alpha\in\mu(M)}\mu^{-1}(\mathcal{O}_\alpha) = \cup_{\alpha\in\mu(M)}(G\text{-bundles over } M/\!/_\alpha G),$$

at least if we avoid critical points.

Elliot has some fancy art of this decomposition.

Let's focus on the simplest possible case:

**Definition 3.5.** A Hamiltonian $G$-space $M$ is *multiplicity-free* if $\dim M/\!/_\alpha G = 0$ for all $\alpha$.

**Remark 3.6.** If $M$ is compact, then a Morse theory argument shows that $M/\!/_\alpha G = \mathrm{pt}$ for all $\alpha$.

Here are some relevant examples.

**Example 3.7.** For a coadjoint orbit $\mathcal{O}_\alpha$, we have $\mathcal{O}_\alpha/\!/_\alpha G = \mathrm{pt}$, so coadjoint orbits are multiplicity-free. Here we are ignoring stacky / derived quotients even though the action is typically nonfree.

**Example 3.8.** Consider $\mathbb{P}^1$ with $\mathrm{U}(1)$ acting by rotation. Then $\mu$ is the height function on $\mathbb{P}^1 = S^2$. If the top height is 1 and the bottom height is $-1$, then $\mu^{-1}(1)$ and $\mu^{-1}(1)$ are both points. For any $x \in (-1, 1)$, we have $\mu^{-1}(x) = S^1$ and therefore $\mathbb{P}^1/\!/_x \mathrm{U}(1) = \mathrm{pt}$. Thus this action is multiplicity-free.

**Example 3.9.** Let $\mathrm{U}(1)^2$ acts on $\mathbb{P}^2$ (extending the standard action on $\mathbb{A}^2 \subset \mathbb{P}^2$). The fibers of the moment map over points in the interior of $\mu(M)$ are 2-tori, which degenerate to circles on the boundary lines of $\mu(M)$ and points at the corners of $\mu(M)$.

A non-example is given by the $\mathrm{U}(1)$ action on $\mathbb{C}^2$ from earlier in the lecture. This is an obvious non-example because the dimension of the symplectic quotient is nonzero. The slogan is that "multiplicity-free manifolds have maximal symmetry."

### 3.2 (David) – Interlude

For a Lie group $G$, we have $T^*G = G \times \mathfrak{g}^*$. Consider $G \curvearrowright T^*G$ induced by the adjoint action of $G$ on itself. We obtain a moment map $\mu : G \times \mathfrak{g}^* \to \mathfrak{g}^*$ given by the formula

$$\mu(g, x) = \mathrm{Ad}_g(x) - x.$$

Then $\mu^{-1}(0) = \{(g, x) \in G \times \mathfrak{g}^* \mid g \in G_x\}$, where $G_x$ is the centralizer of $x \in G$.

The multiplicity-freeness property for a general Hamiltonian $G$-space $M$ can be understood as the requirement that the centralizers $G_x$ act transitively on the preimages $\mu^{-1}(x)$.

It is a good exercise to classify multiplicity-free Hamiltonian $G$-spaces for $G = \mathrm{U}(1)$ or $G = \mathrm{SU}(2)$.

### 3.3 (Elliot) – A few last words

Multiplicity-freeness has a useful consequence for quantization: if $M$ is multiplicity-free, then each highest weight representation $E_\alpha$ appears in $\mathcal{H}(M)$ at most once. In fact, $E_\alpha$ will appear if and only if $\mathcal{O}_\alpha \in \mu(M)$.

We will be interested in hyperspherical varieties as a large family of multiplicity-free symplectic manifolds. More on that next time!

## 4 9/21 (Mark Macerato) – Hyperspherical Varieties

### 4.1 (David) – Multiplicity-freeness

There may have been minor errors in the discussion last time, but the basic ideas were right. Suppose for simplicity that $T$ is an *abelian* Lie group, and consider the cotangent bundle $T^*T \cong T \times \mathfrak{t}^*$. The moment map for the translation action of $T$ on itself is the projection $T \times \mathfrak{t}^* \to \mathfrak{t}^*$. This gives a (trivial) family of abelian groups over $\mathfrak{t}^*$.

If we have another Hamiltonian $T$-space $X$, we obtain a moment map $\mu_X : X \to \mathfrak{t}^*$. We can view our family of abelian groups over $\mathfrak{t}^*$ as acting fiberwise on $X$. The multiplicity-freeness condition is requiring that the orbits of this action are fiberwise discrete.

This story still works for non-abelian $G$ (but you have to be careful about left versus right actions). In this case, the fiber over $\nu \in \mathfrak{g}^*$ will be given by the stabilizer $G_\nu$.

**Example 4.1.** We can describe Hamiltonian $\mathrm{U}(1)$-spaces as lying over $\mathfrak{u}(1) \cong \mathbb{R}$. The multiplicity-freeness condition implies that the fibers are (disjoint unions of) copies of $S^1$ and points. For example, we can consider the height function on the sphere, or the projection of a cylinder $S^1 \times \mathbb{R}$, or many related examples – these all give multiplicity-free Hamiltonian $\mathrm{U}(1)$-spaces.

**Example 4.2.** If we take $G = SU(2)$, we obtain a similar (but distinct) picture because $\mathfrak{su}(2)/SU(2) \cong [0, \infty)$ (the SU(2)-orbits in $\mathfrak{su}(2)$ are spheres). The fibers of $T^* SU(2) \to \mathfrak{su}(2)$ are SU(2) (over 0) and $S^1$ (over points in $(0, \infty)$). We can analyze multiplicity-free Hamiltonian G-spaces as before.

In general, the left action $G \curvearrowright T^*G$ (via $g \cdot (h, \nu) = (gh, \mathrm{Ad}_g \nu)$) is not multiplicity-free. Consider the moment map $T^*G \cong G \times \mathfrak{g}^* \to \mathfrak{g}^*$ given by projection (this depends on how we trivialize $T^*G$). For a coadjoint orbit $\mathcal{O}$, the preimage $\mu^{-1}(\mathcal{O})$ is $G \times \mathcal{O}$. The multiplicity-freeness here reduces to the question of whether the action $G_\nu \curvearrowright G$ has discrete orbits. This is not true in general (see e.g. the SU(2) example above), proving the claim.

*A later clarification:* Really, we should think of $T^*G \rightrightarrows \mathfrak{g}^*$ as a groupoid, where the "source" and "target" maps are $\mu_L$ and $\mu_R$ (the moment maps for the left / right actions, respectively). Given a groupoid, we can obtain a group scheme (encapsulating the "automorphism groups of points") as a fiber product, e.g.

$$\begin{tikzcd} \{[X, g] = 0\} \arrow[r] \arrow[d] & T^*G \arrow[d] \\ \Delta \arrow[r] & \mathfrak{g}^* \times \mathfrak{g}^*. \end{tikzcd}$$

Understanding things from this perspective clears up the difficulties with left / right actions.

Hamiltonian G-spaces ($M \to \mathfrak{g}^*$) will be module objects for this groupoid.

## 4.2 (Mark) – Towards hyperspherical varieties

We will change settings to algebraic geometry (following section 3 of Ben-Zvi–Sakellaridis-Venkatesh). Fix an algebraically closed field $k$ of characteristic zero (e.g. $\mathbb{C}$ or $\overline{\mathbb{Q}_\ell}$). Let $G$ be a connected reductive group over $k$.

Recall that a spherical variety is a normal G-variety X such that there exists a Borel subgroup $B \subset G$ with an open orbit in X. We can rephrase the last condition without picking a Borel: we require that G has an open orbit on $X \times \mathrm{Fl}_G$. If X is affine, this is equivalent to requiring that the coordinate ring $k[X]$ is multiplicity-free as a G-module.

**Example 4.3** ("Group case")**.** Let H be a connected reductive group and $G = H \times H$. For $X = H$ and $G \hookrightarrow X$ via $(h_1, h_2) \cdot h = h_1 h h_2^{-1}$, H is a spherical variety.

If we fix a Borel $B \subset H$, we have a unipotent subgroup $U \subset B$ and a surjection $B \twoheadrightarrow T = B/U$. By Levi's theorem, this splits, giving $T \hookrightarrow B \subset G$. We get a vector space decomposition $\mathfrak{g} = \mathfrak{u}^- \oplus \mathfrak{t} \oplus \mathfrak{u}$. Consider the open embedding $U^- \times B \to H$ given by $(u, b) \mapsto ub$. The Borel subgroup $B^- \times B \subset G$ has an open orbit in H. This leads to a Bruhat decomposition $H = \sqcup_{w \in W} BwB$.

We can obtain Bruhat decompositions for more general spherical varieties. This is a rich theory that has been worked out by several authors (Knapp, Brion, etc.). But let's move on to hyperspherical varieties, which give a symplectic point of view.

Instead of a spherical variety X, let us consider $M = T^*X$ with the moment map $\mu : T^*X \to M$. For simplicity, we will assume our base spherical variety X is affine, smooth, and irreducible. In this case M is *coisotropic*, which means that the G-invariant function field $k(M)^G$ is Poisson-commutative.

Another way of saying this is as follows. Let $\mathfrak{c} = \mathfrak{g}^*//G \cong \mathfrak{g}//G$ be the "Chevalley space." Letting $\eta \in M$ be the generic point, we obtain a Stein factorization $M \to \mathfrak{c}_M \to \mathfrak{c}$. The map $\tilde{\mu} : M \to \mathfrak{c}_M$ has connected generic fiber, and $\mathfrak{c}_M \to \mathfrak{c}$ is finite. The second definition of "coisotropic" is that the group $G_{K(\mathfrak{c}_M)}$ acts on $M_{K(\mathfrak{c}_M)}$ with an open (hence dense) orbit.

**Theorem 4.4** (Losev)**.** *If M is a smooth Hamiltonian G-variety, then* all *of the fibers of $\tilde{\mu} : M \to \mathfrak{c}_M$ are connected.*[1]

A third definition of coisotropic is that the generic G-orbit on M is coisotropic in the usual sense.

"Coisotropic" is the algebraic geometry version of "multiplicity-free." Elliot gave a discussion of why this recovers the earlier condition in symplectic geometry, but it was a bit too fast to type up.

[1] This is the closest analogue in algebraic geometry of the connectedness theorem of Atiyah-Guillemin-Sternberg.

# 5 9/28 (Mark Macerato) – Continued

## 5.1 (David) – Groupoids and Hamiltonian G-spaces

Recall the homework problem of classifying multiplicity-free SU(2)-spaces.

The corrected general picture is as follows. Consider the cotangent bundle $\mathsf{T}^*\mathsf{G}$ with natural Hamiltonian $\mathsf{G}$-actions on the left and right. These yield moment maps $\mu_\mathsf{L}, \mu_\mathsf{R} : \mathsf{T}^*\mathsf{G} \to \mathfrak{g}^*$. If we trivialize $\mathsf{T}^*\mathsf{G} \cong \mathsf{G} \times \mathfrak{g}^*$, these maps are given by $(g, X) \mapsto X$ and $(g, X) \mapsto \mathrm{Ad}_g X$.

We should think of $\mathsf{T}^*\mathsf{G} \rightrightarrows \mathfrak{g}^*$ as a groupoid. The "objects" are $X \in \mathsf{G}$, and the "morphisms" are $g : X \to \mathrm{Ad}_g X$. Composition is given by group multiplication.

We may view any Hamiltonian $\mathsf{G}$-space $\mathsf{Y}$ (with moment map $\mu : \mathsf{Y} \to \mathfrak{g}^*$) as a module over this groupoid. Specifically, we have a natural map $\mathsf{T}^*\mathsf{G} \times_{\mathfrak{g}^*} \mathsf{Y} \to \mathsf{Y}$, the projection of the fiber product onto the second factor. On elements, this is given by $(g, X, y) \mapsto gy$, which lies in the fiber of $\mathsf{Y}$ over $\mathrm{Ad}_g X \in \mathfrak{g}^*$.

Consider the pullback

$$\begin{tikzcd} \mathcal{S} \arrow[r] \arrow[d] & \mathsf{T}^*\mathsf{G} \arrow[d] \\ \mathfrak{g}^* \arrow[r, "\Delta"] & \mathfrak{g}^* \times \mathfrak{g}^*. \end{tikzcd}$$

In equation, $\mathcal{S} = \{[g, X] = 0\}$. From the groupoid perspective, $\mathcal{S} \to \mathfrak{g}^*$ is obtained by only considering automorphisms of objects in our original groupoid (i.e. forgetting about isomorphisms between different objects). We can view $\mathcal{S} \to \mathfrak{g}^*$ as the relative group over $\mathfrak{g}^*$ with fibers given by stabilizers $\mathrm{Stab}_\mathsf{G}(X)$.

The "multiplicity-free" condition can now be restated: it means that the $\mathcal{S}$-action on $\mathsf{Y}$ relative to $\mathfrak{g}$ has only finitely many orbits.

For the exercise about SU(2), we have $\mathfrak{g}^* = \mathbb{R}^3$, and $\mathcal{S}$ has fiber SU(2) over the identity and U(1) over other fibers. We really only care about $\mathfrak{g}^*/\,\mathrm{SU}(2)$, which looks like a real ray $[0, \infty)$. This allows us to produce some examples of multiplicity-free Hamiltonian SU(2)-spaces - these spaces should have maps to $[0, \infty)$ with fibers over $X \in \mathfrak{g}^*/\,\mathrm{SU}(2) \cong [0, \infty)$ looking like (finite disjoint unions of) orbits of $\mathrm{Stab}_{\mathrm{SU}(2)}(X)$-actions.

**Example 5.1.** The 2-sphere $\mathsf{S}^2$ has multiplicity-free SU(2)-action via the action coming from SU(2) $\to$ SO(3).

**Example 5.2.** The standard representation $\mathbb{C}^2$ has multiplicity-free SU(2)-action.

**Example 5.3.** The blowup of $\mathbb{C}^2$ at the origin (with a corrected symplectic form) has multiplicity-free SU(2)-action.

Are these all of the possible examples (up to finite covers)? It would be good to figure this out.

## 5.2 (Mark) – Coisotropic G-varieties

Recall our setup: $\mathsf{G}$ is connected and reductive, and $\mathsf{M}$ is a smooth affine Hamiltonian $\mathsf{G}$-variety. We have a moment map $\mu : \mathsf{M} \to \mathfrak{g}^*$, and we can compose this with a GIT quotient map to get $\overline{\mu} : \mathsf{M} \to \mathfrak{c}_\mathsf{G}$, where $\mathfrak{c}_\mathsf{G} = \mathfrak{g}^*/\!/\mathsf{G}$ is called the Chevalley base. This admits a "Knop factorization"

$$\begin{tikzcd} \mathsf{M} \arrow[rr, "\overline{\mu}"] \arrow[dr, "\tilde{\mu}_\mathsf{M}"'] & & \mathfrak{c}_\mathsf{G} \\ & \mathfrak{c}_\mathsf{M} \arrow[ur, "\pi"'] & \end{tikzcd}$$

where $\pi$ is finite and $\tilde{\mu}_\mathsf{M}$ has generically connected fiber.

**Definition 5.4.** We say that $\mathsf{M}$ is *coisotropic* if any of the following equivalent conditions hold.

1. $\mathsf{k}(\mathsf{M})^\mathsf{G}$ is Poisson-commutative.[2]
2. The generic orbit of $\mathsf{G}$ on $\mathsf{M}$ is coisotropic.

[2]In this setup, we can replace this by the condition that $\mathsf{k}[\mathsf{M}]^\mathsf{G}$ is Poisson commutative, since $\mathrm{Frac}\,\mathsf{k}[\mathsf{M}]^\mathsf{G} = \mathsf{k}(\mathsf{M})^\mathsf{G}$.

3. The generic fiber of $\tilde{\mu}_M$ has a dense $G$-orbit.

Let's see why 1 and 2 are equivalent. Choose $f_1, \dots, f_n \in K(M)$ which separate generic orbits (this is possible by a theorem of Rosenlicht). This yields $\underline{f} = (f_1, \dots, f_n) : U \to \mathbb{A}^n$ (for $U \subset M$ open), and we can restrict this to a surjective smooth map $U' \to W$ such that $U'$ is dense in $U$ and $W \subset \mathbb{A}^n$ is a locally closed subvariety. Replace $U$ by $U'$. The fibers of $\underline{f}$ are exactly the $G$-orbits in $U$. Therefore, for $x \in U$, we see that $df_1(x), \dots, df_n(x)$ span the conormal space $T^*_U(G \cdot x)_x$. Thus $G \cdot x$ is coisotropic at $x$ if and only if $T^*_U(G \cdot x)_x$ is isotropic, if and only if the $f_1, \dots, f_n$ Poisson-commute at $x$.

## 5.3 Approaching hyperspherical varieties

Suppose that $M$ is a smooth affine Hamiltonian $G$-variety as before. We will also require that $M$ comes with a $\mathbb{G}_m$-action (equivalently, a grading on $k[M]$) such that

1. The $\mathbb{G}_m$-action on $M$ commutes with the $G$-action.

2. The symplectic form $\omega$ on $M$ has weight 2, i.e. $\lambda \cdot \omega = \lambda^2 \omega$.

David noted that this latter condition implies that $\omega$ is exact: if $v$ is the vector field generating the $\mathbb{G}_m$-action, then Cartan's magic formula (using that $\omega$ is closed) gives

$$2\omega = \mathcal{L}_v \omega = d(i_v \omega).$$

The 2 here is needed to ensure that we can construct a "$\mathbb{G}_m$-equivariant Kostant slice."

We want to define what it means for $M$ to be hyperspherical. The first condition will be that $M$ is coisotropic.

The second condition is that $\mu(M) \subset \mathfrak{g}^*$ meets the nilpotent cone $\mathcal{N}_G = \chi^{-1}(0)$ (for $\chi : \mathfrak{g}^* \twoheadrightarrow \mathfrak{g}^*//G$). Equivalently, $\overline{\mu})(M)$ contains $0 \in \mathfrak{c}_G$. This implies that $M//G \to \mathfrak{c}_M$ is surjective (it is always an open immersion, so we get $M//G = \mathfrak{c}_M$). There will be two more conditions (which we will discuss next time).

# 6 10/5 (Mark Macerato) – Continued

## 6.1 Pre-hyperspherical varieties

Let $G$ be a connected reductive group and $G_{\text{gr}} = G \times \mathbb{G}_m$. We consider a smooth affine Hamiltonian $G$-variety with auxiliary $\mathbb{G}_m$-action governing the grading. This yields a map $M \to \mathfrak{g}^* \to \mathfrak{c}_G$, which has a Knop factorization $M \to \mathfrak{c}_M \to \mathfrak{c}_G$. Here $\mathbb{G}_m \curvearrowright \mathfrak{g}^*$ quadratically, and the map $M \to \mathfrak{g}^*$ is $\mathbb{G}_m$-equivariant.

**Definition 6.1.** We say that $M$ is *pre-hyperspherical* if

1. $M$ is coisotropic, i.e. $k(M)^G$ is Poisson commutative (equivalently, the generic fiber of $M \to \mathfrak{c}_M$ has a dense $G$-orbit),

2. $\mu(M) \cap \mathcal{N}_G \neq \emptyset$ (for $\mathcal{N}_G$ the nilpotent cone of $G$), and

3. The stabilizer of a generic point of $M$ is connected.

**Example 6.2.** Let $G = \mathrm{Sp}_{2n}$ and $M = \mathbb{C}^{2n} \oplus \mathbb{C}^{2n}$. Here $\mu_M : \mathbb{C}^{2n} \oplus \mathbb{C}^{2n} \to (\mathbb{C}^{2n} \oplus \mathbb{C}^{2n}) // \mathrm{Sp}_{2n} \cong \mathbb{A}^1$ via $(v, w) \mapsto \omega(v, w)$. Thus

$$\mu_M^{-1}(1) = \{(v, w) \in \mathbb{C}^{2n} \mid \omega(v, w) = 1\},$$

and $\mathrm{Sp}_{2n}$ acts transitively on this fiber. Meanwhile, $\mu^{-1}(0)$ can be decomposed as:

$$\mu^{-1}(0) = \{(v, w) \mid v, w \text{ lin. ind. } \omega(v, w) = 0\} \cup \{(v, w) \mid v, w \text{lin. dep., not both } 0\} \cup \{(0, 0)\}.$$

The first set here is the unique open orbit, and the last set is the unique closed orbit. The middle set contains a $\mathbb{P}^1$ worth of orbits. In particular, $\mu(M)$ meets $\mathcal{N}_G$. The stabilizer of a generic point of $M$ can be identified with $\mathrm{Sp}_{2n-2}$.

**Proposition 6.3.** *In general, if* $M$ *is pre-hyperspherical, there exists a unique closed orbit* $M_0 \subset M$ *for* $G_{\mathrm{gr}} = G \times \mathbb{G}_m$.

We call $M_0$ the *core* of $M$.

*Proof.* Consider the GIT quotient $M \to M//G_{\mathrm{gr}}$, and recall that closed orbits of $G_{\mathrm{gr}}$ correspond to points of $M//G_{\mathrm{gr}}$. Thus it suffices to show that $M//G_{\mathrm{gr}} = \mathrm{pt}$, or equivalently $k[M]^{G\times\mathbb{G}_m} = k$. Note

$$k[M]^{G\times\mathbb{G}_m} = k[\mathfrak{c}_M]^{\mathbb{G}_m} = k[\mathfrak{c}_m]_0,$$

the weight 0 component. By construction, $k[\mathfrak{c}_G] \to k[\mathfrak{c}_M]$ is finite (and therefore integral), so $k[\mathfrak{c}_M]_0 \to k[\mathfrak{c}_G]_0$ is integral (an exercise in counting degrees). Now we have

$$k[\mathfrak{c}_G]_0 = k[\mathfrak{c}_G]^{\mathbb{G}_m} = k[\mathfrak{g}^*]^{G\times\mathbb{G}_m} = k,$$

so $k[\mathfrak{c}_M]_0$ is integral over $k$. Since $k$ is algebraically closed, we see $k[\mathfrak{c}_M]_0 = k$. $\square$

## 6.2 (David) – Weinstein manifolds

Suppose we have an exact symplectic manifold $(M, \omega = d\lambda)$. Take $Z = \omega^{-1}(\lambda)$ (this is a vector field on $M$). Then $Z$ gives a flow on $M$, and the core $M_0$ is the subset of points of $M$ which do not escape to infinity along this flow. Assuming $M_0$ is isotropic (this is implied by the Weinstein condition), the flow gives an action of $\mathbb{C}^\times$ on $M_0$.

**Example 6.4.** Consider the surface singularity $x^2 + y^2 + z^{n+1} = 0$. Let $M$ be a symplectic resolution of this. The core $M_0$ is the chain of $\mathbb{P}^1$'s appearing as the zero fiber. In terms of geometric representation theory, we can call $M_0$ a "subregular Springer fiber."

Suppose $M$ is $G$-Hamiltonian – then we are in a situation very similar to what Mark is talking about. That is, *pre-hyperspherical varieties are analogous to* $G$*-Hamiltonian Weinstein manifolds.* This precludes examples like the above, since the union of $\mathbb{P}^1$'s cannot be a single $G$-orbit. This fact is essentially kin to the last Proposition.

**Example 6.5.** Consider $G \times G$ acting on $M = T^*G$ by left and right translation. The core $M_0$ is the zero section.

## 6.3 (Mark) – Hyperspherical varieties

Let $\mu_M : M \to M//G$ be the GIT quotient map.

**Proposition 6.6.** *The core* $M_0$ *is the unique closed* $G$*-orbit in* $\mu_M^{-1}(0)$.

*Proof.* Note that $G$ has a unique closed orbit $M_0' \subset \mu_M^{-1}(0)$ (by standard GIT). Since $\mathbb{G}_m$ commutes with $G$, the $\mathbb{G}_m$ action takes closed $G$-orbits to closed $G$-orbits. Therefore $\mathbb{G}_m$ preserves $M_0'$, and we get $G \times \mathbb{G}_m \curvearrowright M_0'$. It follows that $M_0'$ contains a closed $G_{\mathrm{gr}}$ orbit, hence contains $M_0$. But $M_0'$ is itself a $G$-orbit, so $M_0 = M_0'$. $\square$

Pick $x \in M_0$, and let $H = \mathrm{Stab}_G(x)$, so $M_0 = G/H$. Since $M_0$ is affine, $H$ must be reductive. Since $\mu_M^{-1}(0)$ maps to $\mathcal{N}_G \subset \mathfrak{g}^*$, we get an element $f = \mu(x) \in \mathcal{N}_G$.

Because $\mathbb{G}_m \curvearrowright M_0 = G/H$, we get a homomorphism $\mathbb{G}_M \to \mathrm{Aut}_G(G/H) \cong N_G(H)/H$. In fact, this factors through $(Z_G(H)/Z(H))^0$ (where $^0$ denotes the connected component of the identity element). Let $\overline{\pi} : \mathbb{G}_M \to Z_G(H)/Z(H)$ be the induced map.

**Definition 6.7.** A pre-hyperspherical variety $M$ is hyperspherical if

1. $\overline{\pi}$ lifts to a homomorphism $\pi : \mathbb{G}_m \to Z_G(H) \subset G$, which moreover lifts to a homomorphism $\rho : \mathrm{SL}_2 \to G$ such that
$$d\rho \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = f$$
under the standard identification $\mathfrak{g} \cong \mathfrak{g}^*$.

2. Consider the *sheared* $\mathbb{G}_m$*-action* $(\mathbb{G}_m)_{sh} \curvearrowright M$ induced by

$$\begin{aligned}(\mathbb{G}_m)_{sh} &\to G \times \mathbb{G}_m \\ g &\mapsto (\pi(g)^{-1}, g).\end{aligned}$$

By construction, $(\mathbb{G}_m)_{sh}$ fixes $x$, and thus we get $(\mathbb{G}_m)_{sh} \curvearrowright (T_x M_0)^\perp/(T_x M_0 \cap T_x M_0^\perp) := N_x M_0$, the *symplectic normal space*. The condition is that this $(\mathbb{G}_m)_{sh}$-action on $N_x M_0$ is given by linear scaling.