GRT Seminar Fall 2024 – Categorical Representation Theory

September 5, 2024

Abstract

This semester, the GRT Seminar will focus on "categorical representation theory." A good reference is the recent notes of Gurbir Dhillon.

Contents

1 9/5 (David Nadler) – Introduction

Today will be fairly informal – we'll just talk about basic definitions and the example of Lusztig's character sheaves.

1.1 What is a representation of a group?

Consider the following well-studied classes of groups:

- Finite groups
- Groups over local fields (e.g. Lie groups, p-adic groups, ...)

Many interesting examples (e.g. finite groups of Lie type) arise as the points of an algebraic group. These algebraic groups often have a *geometric* interpretation.

Representations of groups are interesting for two reasons:

- They allow us to understand abstract groups concretely.
- They allow us to study the symmetries of objects we already cared about.

We can describe a representation of a group as a homomorphism $\mathsf{G} \to \mathrm{Aut}_{\mathcal{C}}(\mathsf{c})$ for some object c of a category $\mathcal{C}$.

Example 1.1. In the most classical setup, we take $\mathcal{C} = \mathsf{Vect}_{\mathbb{C}}$, the category of complex vector spaces. Here $\mathrm{Aut}_{\mathsf{Vect}_{\mathbb{C}}}(\mathsf{V}) = \mathrm{GL}(\mathsf{V})$, the *general linear group.* Note that, at some point, we have to account for the presence of $\mathbb{C}$ here. We also have to ask: do we consider continuous representations? Algebraic representations?

Let's try to reformulate this in a way that makes the dependence on choices clearer. Given a group G and a coefficient field k, we can construct a *coalgebra of functions* $\mathrm{Fun}_{\mathsf{k}}(\mathsf{G})$ and an *algebra of distributions* $\mathrm{Dist}_{\mathsf{k}}(\mathsf{G})$. For a finite group G, these are the same. A representation of G can be interpreted as either a *comodule* over $\mathrm{Fun}_{\mathsf{k}}(\mathsf{G})$ or a *module* over $\mathrm{Dist}_{\mathsf{k}}(\mathsf{G})$.

The choices we make are:

- The coefficient field k

- The types of representation (i.e. types of "functions" or "distributions" considered)

Let's see this in an example.

Example 1.2. What are representations of S^1?

- One answer comes by viewing S^1 as a compact Lie group, so a representation V decomposes as $\oplus_{n\in\mathbb{Z}} V_n$, where $e^{i\theta}$ acts on V_n by multiplication by $e^{in\theta}$.

- If we use homotopy theory, all we can see about S^1 is the homotopy type – in particular, S^1 and $\mathbb{C}^\times$ should "have the same representation theory." The correct notion of a "distribution" here is $\mathrm{Dist}_{\mathbb{C}}(S^1) = C_{-\bullet}(S^1;\mathbb{C}) \simeq \mathbb{C}[\epsilon]$, where $\deg \epsilon = -1$ and $\epsilon^2 = 0$. Thus representations of S^1 are $\mathbb{C}[\epsilon]$-modules.

  Representations in the ordinary category $\mathsf{Vect}_{\mathbb{C}}$ are not interesting – ϵ must always act trivially. However, if we consider representations in the dg-category of $\mathbb{C}$-dg-vector spaces, we get a more interesting answer – the category of dg-representations of S^1 is $\mathbb{C}[\epsilon]$-dg-modules.

1.2 Why study categorical representation theory?

Categorical representation theory is based on an old "miracle."

Consider the representation theory of finite groups (especially those of Lie type). There are two classical ways to construct representations of such groups:

- Induction from abelian groups (e.g. tori)

- Prayer (e.g. "cuspidal representations")

Lusztig realized a good way to make sense of cuspidal representations. Recall that representations of a finite group G are equivalent to characters of class functions on G. So we can rephrase our problem: how do we construct characters of cuspidal representations? The trick is to instead construct character *sheaves* on the corresponding algebraic groups. Once we have the character sheaves, we can recover the character functions by decategorification.

The "miracle" is that character sheaves can be accessed via induction from tori! One can explain this by appealing to topological field theory: there's a tradeoff between asking seemingly "easy" questions about complicated manifolds and asking seemingly "hard" questions about simpler manifolds (e.g. points). In many cases, one can answer questions about the former by finding the right questions to ask about the latter.