GRT Seminar Fa23-Sp24 Notes

October 13, 2023

Abstract

The seminar covers Ben-Zvi–Sakellaridis–Venkatesh, "Relative Langlands Duality."

Contents

1 8/31 (David Nadler) – ??? 1

2 9/7 (Elliot Kienzle) – Hamiltonian G-Spaces and Quantization 2
- 2.1 Symplectic geometry and quantization . . . 2
- 2.2 G-Spaces . . . 2

3 9/14 (Elliot Kienzle) – Continued 3
- 3.1 Symplectic reduction . . . 3
- 3.2 (David) – Interlude . . . 4
- 3.3 (Elliot) – A few last words . . . 4

4 9/21 (Mark Macerato) – Hyperspherical Varieties 4
- 4.1 (David) – Multiplicity-freeness . . . 4
- 4.2 (Mark) – Towards hyperspherical varieties . . . 5

5 9/28 (Mark Macerato) – Continued 6
- 5.1 (David) – Groupoids and Hamiltonian G-spaces . . . 6
- 5.2 (Mark) – Coisotropic G-varieties . . . 6
- 5.3 Approaching hyperspherical varieties . . . 7

6 10/5 (Mark Macerato) – Continued 7
- 6.1 Pre-hyperspherical varieties . . . 7
- 6.2 (David) – Weinstein manifolds . . . 8
- 6.3 (Mark) – Hyperspherical varieties . . . 8

7 10/12 (Mark Macerato) – Continued 9
- 7.1 Refresher on the definition . . . 9
- 7.2 (David) – More on shearing . . . 10
- 7.3 (Mark) – A construction . . . 10

1 8/31 (David Nadler) – ???

I missed this day. If you have good notes from this day, send them to me and I will type them up.

2 9/7 (Elliot Kienzle) – Hamiltonian G-Spaces and Quantization

Elliot's notes for his talks are available at `https://chessapig.github.io/files/notes/G-spaces.pdf`.

The original Langlands program studies a duality of Lie groups $G \leftrightarrow G^\vee$. Relative Langlands seeks to upgrade this to a duality of Hamiltonian G-actions $(G \curvearrowright M) \leftrightarrow (G^\vee \curvearrowright M^\vee)$. This is proposed for hyperspherical varieties M, of which a typical example is $M = T^*X$ for X a spherical variety.

We can approach and motivate this using quantization. Start by considering the action of G on $L^2(X)$ for X a spherical variety (discussed in an earlier paper of Sakellaridis-Venkatesh discussing "harmonic analysis on spherical varieties").

2.1 Symplectic geometry and quantization

The original motivation for symplectic geometry comes from classical mechanics. Suppose that we have a particle moving in $\mathbb{R}^n$. We can capture the data of the position and momentum using the cotangent bundle $T^*\mathbb{R}^n$. By Newton's second law, the time evolution of the particle is described by (the flow along) a vector field on $T^*\mathbb{R}^n$.

We can generalize this to a symplectic manifold (M, ω), which is a manifold M with a closed, non-degenerate 2-form ω. To make this easier to work with, we can fix a metric $\langle , \rangle$ on M and write $\omega(x, y) = \langle x, Jy \rangle$ where $J^2 = -1$ (i.e. J^2 is an almost complex structure). We think of J^2 as "multiplication by $-i$." Given a Hamiltonian $H \in \mathcal{C}^\infty(M)$, we obtain a Hamiltonian vector field $X_H = J\nabla H$. More invariantly, we can define X_H via the formula $\omega(X_H, -) = dH$.

Moving to quantum mechanics, we view a particle in $\mathbb{R}^n$ as a $\mathbb{C}$-valued function ψ on $\mathbb{R}^n$ (not $T^*\mathbb{R}^n$). In this case, the Hilbert space is $L^2(\mathbb{R}^n)$. A free particle evolves according to Schrödinger's equation:

$$i\dot{\psi} = \Delta\psi.$$

We can summarize the classical and quantum pictures in the following table.

	Classical	Quantum
State Space	Symplectic manifold (M, ω)	Hilbert space $\mathcal{H}$
Observables	$f \in \mathcal{C}^\infty(M)$	Bounded operators $A \in \mathrm{End}(\mathcal{H})$
Evolution	Vector fields X_H for $H \in \mathcal{C}^\infty(M)$	Unitary operators $U(t) = e^{itA}$ for $A \in \mathrm{End}(\mathcal{H})$
Lie Algebra of observables	Poisson bracket $\{f, g\} = X_f(g)$	Commutator $[A, B]$

To obtain a quantum system from a classical system (heuristically), we pass from nonlinear evolution of points in T^*M to linear evolution of functions on M. (This linearity is forced on us by our desire to have superposition of states.) The dream of quantization is, given a symplectic manifold (M, ω), to construct a Lie algebra homomorphism $(\mathcal{C}^\infty(M), \{,\}) \to (\mathrm{End}(\mathcal{H}), [,])$ for some Hilbert space $\mathcal{H}$. Unfortunately, this is impossible to do consistently / functorially in general. However, there are some cases in which we can get good answers.

We will focus on geometric quantization, which behaves (loosely) as follows:

- For $M = T^*X$, we obtain $\mathcal{H} = L^2(X)$.
- For M a compact Kähler manifold, we obtain $\mathcal{H} = H^0(M, \mathcal{L})$ for some line bundle $\mathcal{L}$ on M.

2.2 G-Spaces

We want to incorporate symmetries into the previous picture. Suppose G is a compact Lie group / reductive algebraic group (depending on context). We say a symplectic G-space is a symplectic manifold (M, ω) with G-action preserving ω. We can hope to quantize this to a linear representation $G \curvearrowright \mathcal{H}$. (There are subtleties that arise here – for geometric quantization, we would like a G-equivariant polarization.)

In general, it is better to consider Hamiltonian G-actions, where $\mathfrak{g}$ acts by Hamiltonian vector fields. This allows us to construct a moment map $\mu : M \to \mathfrak{g}^*$ which is equivariant (with respect to the coadjoint action on $\mathfrak{g}^*$).

Let us start by understanding the coadjoint action $\mathsf{G} \curvearrowright \mathfrak{g}^*$ using Kirillov's "orbit method." For $\alpha \in \mathfrak{g}^*$, consider the coadjoint orbit $\mathcal{O}_\alpha$. This $\mathcal{O}_\alpha$ turns out to be a symplectic manifold (with "Kirillov-Kostant-Souriau" / "KKS" form) with Hamiltonian G-action, and the moment map $\mathcal{O}_\alpha \to \mathfrak{g}^*$ is just the inclusion.

Example 2.1. Consider $\mathsf{G} = \mathrm{SO}(3)$. The coadjoint action is just $\mathrm{SO}(3)$ acting on $\mathbb{R}^3$ by rotations. Thus the generic orbits are spheres S^2.

The orbits $\mathcal{O}_\alpha$ will look like generalized flag manifolds, and conversely every generalized flag manifold arises in this way. (This is the first place where our compactness hypothesis comes in).

Proposition 2.2. *A coadjoint orbit $\mathcal{O}_\alpha$ is quantizable if and only if α is in the orbit of an integer point of the root lattice $\mathfrak{t}^*_{\mathbb{Z}} \subset \mathfrak{t}^*$ (viewed as a subspace of $\mathfrak{g}^*$ via the Killing form).*

Example 2.3. Continuing on with our SO(3) example, we see that a symplectic sphere is quantizable if and only if it has integer area.

In these cases, the quantization of $\mathcal{O}_\alpha$ is $\mathsf{H}^0(\mathcal{O}_\alpha, \mathcal{L}_\alpha)$ where $\mathcal{L}_\alpha$ is the line bundle corresponding to the character α. By the Borel-Weil theorem, $\mathsf{H}^0(\mathcal{O}_\alpha, \mathcal{L}_\alpha)$ is the irrep V_α of G with highest weight $\mathcal{L}_\alpha$.

We can summarize this in the following table:

Classical	Quantum
Symplectic action $\mathsf{G} \curvearrowright \mathsf{M}$	Representation $\mathsf{G} \curvearrowright \mathcal{H}$
Coadjoint orbit $\mathcal{O}_\alpha$	Highest weight representation E_α

3 9/14 (Elliot Kienzle) – Continued

3.1 Symplectic reduction

Suppose we have a Hamiltonian action $\mathsf{G} \curvearrowright \mathsf{M}$. This yields a G-equivariant moment map $\mu : \mathsf{M} \to \mathfrak{g}^*$, and the image of μ will necessarily be a collection of coadjoint orbits $\mathcal{O}_\alpha$. We can use these orbits to decompose M.

First consider the orbit $\mathcal{O}_0 = \{0\}$. We note that $\mu^{-1}(0)$ is G-invariant, so we can consider the quotient $\mu^{-1}(0)/\mathsf{G}$. We define this to be the *symplectic quotient*: $\mathsf{M}//\mathsf{G} := \mu^{-1}(0)/\mathsf{G}$.

We will assume that 0 is a regular value of the moment map and that G acts on $\mu^{-1}(0)$ freely. We can drop these assumptions if we consider things in a suitable derived / stacky sense.

Theorem 3.1 (Marsden-Weinstein)**.** *The symplectic quotient $\mathsf{M}//\mathsf{G}$ carries a natural symplectic structure.*

Example 3.2. If X is a (not necessarily symplectic) manifold with a G-action, then $\mathsf{T}^*\mathsf{X}//\mathsf{G} = \mathsf{T}^*(\mathsf{X}/\mathsf{G})$.

Example 3.3. Let $\mathsf{M} = \mathsf{T}^*\mathbb{R}^2 \cong \mathbb{C}^2$. This has a U(1)-action via

$$e^{i\theta}(z_1, z_2) = (e^{i\theta}z_1, e^{i\theta}z_2).$$

We can define a (shifted) moment map $\mu : \mathbb{C}^2 \to \mathbb{R}$ via

$$\mu(z_1, z_2) = |z_1|^2 + |z_2|^2 - 1.$$

Then $\mathbb{C}^2 // \mathrm{U}(1) = \mathsf{S}^3 / \mathrm{U}(1) = \mathsf{S}^2 = \mathbb{P}^1$ (consider the Hopf fibration).

Morally, we should think of every symplectic manifold as a symplectic reduction of a (possibly infinite-dimensional) affine space.

Note that

$$\dim \mathsf{M}//\mathsf{G} = \dim \mathsf{M} - 2 \dim \mathsf{G}.$$

The slogan is that "in symplectic geometry, groups act twice."

Theorem 3.4 (Guillemin-Sternberg, etc.)**.** *The geometric quantization of a symplectic quotient satisfies*

$$\mathcal{H}(\mathsf{M}//\mathsf{G}) = \mathcal{H}(\mathsf{M})^{\mathsf{G}},$$

where the right hand side is the subspace of G-invariant vectors in G.

We can also define the symplectic reduction along any coadjoint orbit $\mathcal{O}_\alpha$ as $M/\!/_\alpha G = \mu^{-1}(\mathcal{O}_\alpha)/G$. This gives a decomposition of M as

$$M = \cup_{\alpha\in\mu(M)}\mu^{-1}(\mathcal{O}_\alpha) = \cup_{\alpha\in\mu(M)}(G\text{-bundles over } M/\!/_\alpha G),$$

at least if we avoid critical points.

Elliot has some fancy art of this decomposition.

Let's focus on the simplest possible case:

Definition 3.5. A Hamiltonian G-space M is *multiplicity-free* if $\dim M/\!/_\alpha G = 0$ for all α.

Remark 3.6. If M is compact, then a Morse theory argument shows that $M/\!/_\alpha G = \mathrm{pt}$ for all α.

Here are some relevant examples.

Example 3.7. For a coadjoint orbit $\mathcal{O}_\alpha$, we have $\mathcal{O}_\alpha/\!/_\alpha G = \mathrm{pt}$, so coadjoint orbits are multiplicity-free. Here we are ignoring stacky / derived quotients even though the action is typically nonfree.

Example 3.8. Consider $\mathbb{P}^1$ with $\mathrm{U}(1)$ acting by rotation. Then μ is the height function on $\mathbb{P}^1 = S^2$. If the top height is 1 and the bottom height is -1, then $\mu^{-1}(1)$ and $\mu^{-1}(1)$ are both points. For any $x \in (-1,1)$, we have $\mu^{-1}(x) = S^1$ and therefore $\mathbb{P}^1/\!/_x \mathrm{U}(1) = \mathrm{pt}$. Thus this action is multiplicity-free.

Example 3.9. Let $\mathrm{U}(1)^2$ acts on $\mathbb{P}^2$ (extending the standard action on $\mathbb{A}^2 \subset \mathbb{P}^2$). The fibers of the moment map over points in the interior of $\mu(M)$ are 2-tori, which degenerate to circles on the boundary lines of $\mu(M)$ and points at the corners of $\mu(M)$.

A non-example is given by the $\mathrm{U}(1)$ action on $\mathbb{C}^2$ from earlier in the lecture. This is an obvious non-example because the dimension of the symplectic quotient is nonzero. The slogan is that "multiplicity-free manifolds have maximal symmetry."

3.2 (David) – Interlude

For a Lie group G, we have $T^*G = G \times \mathfrak{g}^*$. Consider $G \curvearrowright T^*G$ induced by the adjoint action of G on itself. We obtain a moment map $\mu : G \times \mathfrak{g}^* \to \mathfrak{g}^*$ given by the formula

$$\mu(g, x) = \mathrm{Ad}_g(x) - x.$$

Then $\mu^{-1}(0) = \{(g,x) \in G \times \mathfrak{g}^* \mid g \in G_x\}$, where G_x is the centralizer of $x \in G$.

The multiplicity-freeness property for a general Hamiltonian G-space M can be understood as the requirement that the centralizers G_x act transitively on the preimages $\mu^{-1}(x)$.

It is a good exercise to classify multiplicity-free Hamiltonian G-spaces for $G = \mathrm{U}(1)$ or $G = \mathrm{SU}(2)$.

3.3 (Elliot) – A few last words

Multiplicity-freeness has a useful consequence for quantization: if M is multiplicity-free, then each highest weight representation E_α appears in $\mathcal{H}(M)$ at most once. In fact, E_α will appear if and only if $\mathcal{O}_\alpha \in \mu(M)$.

We will be interested in hyperspherical varieties as a large family of multiplicity-free symplectic manifolds. More on that next time!

4 9/21 (Mark Macerato) – Hyperspherical Varieties

4.1 (David) – Multiplicity-freeness

There may have been minor errors in the discussion last time, but the basic ideas were right. Suppose for simplicity that T is an *abelian* Lie group, and consider the cotangent bundle $T^*T \cong T \times \mathfrak{t}^*$. The moment map for the translation action of T on itself is the projection $T \times \mathfrak{t}^* \to \mathfrak{t}^*$. This gives a (trivial) family of abelian groups over $\mathfrak{t}^*$.

If we have another Hamiltonian T-space X, we obtain a moment map $\mu_{\mathsf{X}} : \mathsf{X} \to \mathfrak{t}^*$. We can view our family of abelian groups over $\mathfrak{t}^*$ as acting fiberwise on X. The multiplicity-freeness condition is requiring that the orbits of this action are fiberwise discrete.

This story still works for non-abelian G (but you have to be careful about left versus right actions). In this case, the fiber over $\nu \in \mathfrak{g}^*$ will be given by the stabilizer G_ν.

Example 4.1. We can describe Hamiltonian U(1)-spaces as lying over $\mathfrak{u}(1) \cong \mathbb{R}$. The multiplicity-freeness condition implies that the fibers are (disjoint unions of) copies of S^1 and points. For example, we can consider the height function on the sphere, or the projection of a cylinder $\mathsf{S}^1 \times \mathbb{R}$, or many related examples – these all give multiplicity-free Hamiltonian U(1)-spaces.

Example 4.2. If we take $\mathsf{G} = \mathrm{SU}(2)$, we obtain a similar (but distinct) picture because $\mathfrak{su}(2)/\mathrm{SU}(2) \cong [0, \infty)$ (the SU(2)-orbits in $\mathfrak{su}(2)$ are spheres). The fibers of $\mathsf{T}^*\mathrm{SU}(2) \to \mathfrak{su}(2)$ are SU(2) (over 0) and S^1 (over points in $(0, \infty)$). We can analyze multiplicity-free Hamiltonian G-spaces as before.

In general, the left action $\mathsf{G} \curvearrowright \mathsf{T}^*\mathsf{G}$ (via $g \cdot (h, \nu) = (gh, \mathrm{Ad}_g \nu)$) is not multiplicity-free. Consider the moment map $\mathsf{T}^*\mathsf{G} \cong \mathsf{G} \times \mathfrak{g}^* \to \mathfrak{g}^*$ given by projection (this depends on how we trivialize $\mathsf{T}^*\mathsf{G}$). For a coadjoint orbit $\mathcal{O}$, the preimage $\mu^{-1}(\mathcal{O})$ is $\mathsf{G} \times \mathcal{O}$. The multiplicity-freeness here reduces to the question of whether the action $\mathsf{G}_\nu \curvearrowright \mathsf{G}$ has discrete orbits. This is not true in general (see e.g. the SU(2) example above), proving the claim.

A later clarification: Really, we should think of $\mathsf{T}^*\mathsf{G} \rightrightarrows \mathfrak{g}^*$ as a groupoid, where the "source" and "target" maps are μ_{L} and μ_{R} (the moment maps for the left / right actions, respectively). Given a groupoid, we can obtain a group scheme (encapsulating the "automorphism groups of points") as a fiber product, e.g.

$$\begin{array}{ccc} \{[\mathsf{X}, g] = 0\} & \longrightarrow & \mathsf{T}^*\mathsf{G} \\ \big\downarrow & & \big\downarrow \\ \Delta & \longrightarrow & \mathfrak{g}^* \times \mathfrak{g}^*. \end{array}$$

Understanding things from this perspective clears up the difficulties with left / right actions.

Hamiltonian G-spaces ($\mathsf{M} \to \mathfrak{g}^*$) will be module objects for this groupoid.

4.2 (Mark) – Towards hyperspherical varieties

We will change settings to algebraic geometry (following section 3 of Ben-Zvi–Sakellaridis-Venkatesh). Fix an algebraically closed field k of characteristic zero (e.g. $\mathbb{C}$ or $\overline{\mathbb{Q}_\ell}$). Let G be a connected reductive group over k.

Recall that a spherical variety is a normal G-variety X such that there exists a Borel subgroup $\mathsf{B} \subset \mathsf{G}$ with an open orbit in X. We can rephrase the last condition without picking a Borel: we require that G has an open orbit on $\mathsf{X} \times \mathrm{Fl}_{\mathsf{G}}$. If X is affine, this is equivalent to requiring that the coordinate ring $\mathsf{k}[\mathsf{X}]$ is multiplicity-free as a G-module.

Example 4.3 ("Group case")**.** Let H be a connected reductive group and $\mathsf{G} = \mathsf{H} \times \mathsf{H}$. For $\mathsf{X} = \mathsf{H}$ and $\mathsf{G} \hookrightarrow \mathsf{X}$ via $(h_1, h_2) \cdot h = h_1 h h_2^{-1}$, H is a spherical variety.

If we fix a Borel $\mathsf{B} \subset \mathsf{H}$, we have a unipotent subgroup $\mathsf{U} \subset \mathsf{B}$ and a surjection $\mathsf{B} \twoheadrightarrow \mathsf{T} = \mathsf{B}/\mathsf{U}$. By Levi's theorem, this splits, giving $\mathsf{T} \hookrightarrow \mathsf{B} \subset \mathsf{G}$. We get a vector space decomposition $\mathfrak{g} = \mathfrak{u}^- \oplus \mathfrak{t} \oplus \mathfrak{u}$. Consider the open embedding $\mathsf{U}^- \times \mathsf{B} \to \mathsf{H}$ given by $(u, b) \mapsto ub$. The Borel subgroup $\mathsf{B}^- \times \mathsf{B} \subset \mathsf{G}$ has an open orbit in H. This leads to a Bruhat decomposition $\mathsf{H} = \sqcup_{w \in W} \mathsf{B} w \mathsf{B}$.

We can obtain Bruhat decompositions for more general spherical varieties. This is a rich theory that has been worked out by several authors (Knapp, Brion, etc.). But let's move on to hyperspherical varieties, which give a symplectic point of view.

Instead of a spherical variety X, let us consider $\mathsf{M} = \mathsf{T}^*\mathsf{X}$ with the moment map $\mu : \mathsf{T}^*\mathsf{X} \to \mathsf{M}$. For simplicity, we will assume our base spherical variety X is affine, smooth, and irreducible. In this case M is *coisotropic*, which means that the G-invariant function field $\mathsf{k}(\mathsf{M})^{\mathsf{G}}$ is Poisson-commutative.

Another way of saying this is as follows. Let $\mathfrak{c} = \mathfrak{g}^*/\!/\mathsf{G} \cong \mathfrak{g}/\!/\mathsf{G}$ be the "Chevalley space." Letting $\eta \in \mathsf{M}$ be the generic point, we obtain a Stein factorization $\mathsf{M} \to \mathfrak{c}_{\mathsf{M}} \to \mathfrak{c}$. The map $\tilde{\mu} : \mathsf{M} \to \mathfrak{c}_{\mathsf{M}}$ has connected generic fiber, and $\mathfrak{c}_M \to \mathfrak{c}$ is finite. The second definition of "coisotropic" is that the group $G_{K(\mathfrak{c}_M)}$ acts on $M_{K(\mathfrak{c}_M)}$ with an open (hence dense) orbit.

Theorem 4.4 (Losev)**.** *If* M *is a smooth Hamiltonian* G*-variety, then* all *of the fibers of* $\tilde{\mu} : M \to \mathfrak{c}_M$ *are connected.*[1]

A third definition of coisotropic is that the generic G-orbit on M is coisotropic in the usual sense.

"Coisotropic" is the algebraic geometry version of "multiplicity-free." Elliot gave a discussion of why this recovers the earlier condition in symplectic geometry, but it was a bit too fast to type up.

5 9/28 (Mark Macerato) – Continued

5.1 (David) – Groupoids and Hamiltonian G-spaces

Recall the homework problem of classifying multiplicity-free SU(2)-spaces.

The corrected general picture is as follows. Consider the cotangent bundle T^*G with natural Hamiltonian G-actions on the left and right. These yield moment maps $\mu_L, \mu_R : T^*G \to \mathfrak{g}^*$. If we trivialize $T^*G \cong G \times \mathfrak{g}^*$, these maps are given by $(g, X) \mapsto X$ and $(g, X) \mapsto \mathrm{Ad}_g X$.

We should think of $T^*G \rightrightarrows \mathfrak{g}^*$ as a groupoid. The "objects" are $X \in G$, and the "morphisms" are $g : X \to \mathrm{Ad}_g X$. Composition is given by group multiplication.

We may view any Hamiltonian G-space Y (with moment map $\mu : Y \to \mathfrak{g}^*$) as a module over this groupoid. Specifically, we have a natural map $T^*G \times_{\mathfrak{g}^*} Y \to Y$, the projection of the fiber product onto the second factor. On elements, this is given by $(g, X, y) \mapsto gy$, which lies in the fiber of Y over $\mathrm{Ad}_g X \in \mathfrak{g}^*$.

Consider the pullback

$$\begin{array}{ccc} \mathcal{S} & \longrightarrow & T^*G \\ \downarrow & & \downarrow \\ \mathfrak{g}^* & \xrightarrow{\Delta} & \mathfrak{g}^* \times \mathfrak{g}^*. \end{array}$$

In equation, $\mathcal{S} = \{[g, X] = 0\}$. From the groupoid perspective, $\mathcal{S} \to \mathfrak{g}^*$ is obtained by only considering automorphisms of objects in our original groupoid (i.e. forgetting about isomorphisms between different objects). We can view $\mathcal{S} \to \mathfrak{g}^*$ as the relative group over $\mathfrak{g}^*$ with fibers given by stabilizers $\mathrm{Stab}_G(X)$.

The "multiplicity-free" condition can now be restated: it means that the $\mathcal{S}$-action on Y relative to $\mathfrak{g}$ has only finitely many orbits.

For the exercise about SU(2), we have $\mathfrak{g}^* = \mathbb{R}^3$, and $\mathcal{S}$ has fiber SU(2) over the identity and U(1) over other fibers. We really only care about $\mathfrak{g}^*/\mathrm{SU}(2)$, which looks like a real ray $[0, \infty)$. This allows us to produce some examples of multiplicity-free Hamiltonian SU(2)-spaces - these spaces should have maps to $[0, \infty)$ with fibers over $X \in \mathfrak{g}^*/\mathrm{SU}(2) \cong [0, \infty)$ looking like (finite disjoint unions of) orbits of $\mathrm{Stab}_{\mathrm{SU}(2)}(X)$-actions.

Example 5.1. The 2-sphere S^2 has multiplicity-free SU(2)-action via the action coming from SU(2) $\to$ SO(3).

Example 5.2. The standard representation $\mathbb{C}^2$ has multiplicity-free SU(2)-action.

Example 5.3. The blowup of $\mathbb{C}^2$ at the origin (with a corrected symplectic form) has multiplicity-free SU(2)-action.

Are these all of the possible examples (up to finite covers)? It would be good to figure this out.

5.2 (Mark) – Coisotropic G-varieties

Recall our setup: G is connected and reductive, and M is a smooth affine Hamiltonian G-variety. We have a moment map $\mu : M \to \mathfrak{g}^*$, and we can compose this with a GIT quotient map to get $\overline{\mu} : M \to \mathfrak{c}_G$, where

[1] This is the closest analogue in algebraic geometry of the connectedness theorem of Atiyah-Guillemin-Sternberg.

$\mathfrak{c}_G = \mathfrak{g}^*//G$ is called the Chevalley base. This admits a "Knop factorization"

$$\begin{tikzcd} M \arrow[rr, "\overline{\mu}"] \arrow[dr, "\tilde{\mu}_M"'] & & \mathfrak{c}_G \\ & \mathfrak{c}_M \arrow[ur, "\pi"'] & \end{tikzcd}$$

where π is finite and $\tilde{\mu}_M$ has generically connected fiber.

Definition 5.4. We say that M is *coisotropic* if any of the following equivalent conditions hold.

1. $k(M)^G$ is Poisson-commutative.[2]
2. The generic orbit of G on M is coisotropic.
3. The generic fiber of $\tilde{\mu}_M$ has a dense G-orbit.

Let's see why 1 and 2 are equivalent. Choose $f_1, \ldots, f_n \in K(M)$ which separate generic orbits (this is possible by a theorem of Rosenlicht). This yields $\underline{f} = (f_1, \ldots, f_n) : U \to \mathbb{A}^n$ (for $U \subset M$ open), and we can restrict this to a surjective smooth map $U' \to W$ such that U' is dense in U and $W \subset \mathbb{A}^n$ is a locally closed subvariety. Replace U by U'. The fibers of $\underline{f}$ are exactly the G-orbits in U. Therefore, for $x \in U$, we see that $df_1(x), \ldots, df_n(x)$ span the conormal space $T^*_U(G \cdot x)_x$. Thus $G \cdot x$ is coisotropic at x if and only if $T^*_U(G \cdot x)_x$ is isotropic, if and only if the $f_1, \ldots, f_n$ Poisson-commute at x.

5.3 Approaching hyperspherical varieties

Suppose that M is a smooth affine Hamiltonian G-variety as before. We will also require that M comes with a $\mathbb{G}_m$-action (equivalently, a grading on $k[M]$) such that

1. The $\mathbb{G}_m$-action on M commutes with the G-action.
2. The symplectic form ω on M has weight 2, i.e. $\lambda \cdot \omega = \lambda^2 \omega$.

David noted that this latter condition implies that ω is exact: if ν is the vector field generating the $\mathbb{G}_m$-action, then Cartan's magic formula (using that ω is closed) gives

$$2\omega = \mathcal{L}_\nu \omega = d(i_\nu \omega).$$

The 2 here is needed to ensure that we can construct a "$\mathbb{G}_m$-equivariant Kostant slice."

We want to define what it means for M to be hyperspherical. The first condition will be that M is coisotropic.

The second condition is that $\mu(M) \subset \mathfrak{g}^*$ meets the nilpotent cone $\mathcal{N}_G = \chi^{-1}(0)$ (for $\chi : \mathfrak{g}^* \twoheadrightarrow \mathfrak{g}^*//G$). Equivalently, $\overline{\mu})(M)$ contains $0 \in \mathfrak{c}_G$. This implies that $M//G \to \mathfrak{c}_M$ is surjective (it is always an open immersion, so we get $M//G = \mathfrak{c}_M$). There will be two more conditions (which we will discuss next time).

6 10/5 (Mark Macerato) – Continued

6.1 Pre-hyperspherical varieties

Let G be a connected reductive group and $G_{\mathrm{gr}} = G \times \mathbb{G}_m$. We consider a smooth affine Hamiltonian G-variety with auxiliary $\mathbb{G}_m$-action governing the grading. This yields a map $M \to \mathfrak{g}^* \to \mathfrak{c}_G$, which has a Knop factorization $M \to \mathfrak{c}_M \to \mathfrak{c}_G$. Here $\mathbb{G}_m \curvearrowright \mathfrak{g}^*$ quadratically, and the map $M \to \mathfrak{g}^*$ is $\mathbb{G}_m$-equivariant.

Definition 6.1. We say that M is *pre-hyperspherical* if

1. M is coisotropic, i.e. $k(M)^G$ is Poisson commutative (equivalently, the generic fiber of $M \to \mathfrak{c}_M$ has a dense G-orbit),

[2] In this setup, we can replace this by the condition that $k[M]^G$ is Poisson commutative, since $\mathrm{Frac}\,k[M]^G = k(M)^G$.

2. $\mu(M) \cap \mathcal{N}_G \neq \emptyset$ (for $\mathcal{N}_G$ the nilpotent cone of G), and

3. The stabilizer of a generic point of M is connected.

Example 6.2. Let $G = \mathrm{Sp}_{2n}$ and $M = \mathbb{C}^{2n} \oplus \mathbb{C}^{2n}$. Here $\mu_M : \mathbb{C}^{2n} \oplus \mathbb{C}^{2n} \to (\mathbb{C}^{2n} \oplus \mathbb{C}^{2n})/\!/ \mathrm{Sp}_{2n} \cong \mathbb{A}^1$ via $(v, w) \mapsto \omega(v, w)$. Thus

$$\mu_M^{-1}(1) = \{(v, w) \in \mathbb{C}^{2n} \mid \omega(v, w) = 1\},$$

and Sp_{2n} acts transitively on this fiber. Meanwhile, $\mu^{-1}(0)$ can be decomposed as:

$$\mu^{-1}(0) = \{(v, w) \mid v, w \text{ lin. ind. } \ \omega(v, w) = 0\} \cup \{(v, w) \mid v, w \text{lin. dep., not both } 0\} \cup \{(0, 0)\}.$$

The first set here is the unique open orbit, and the last set is the unique closed orbit. The middle set contains a $\mathbb{P}^1$ worth of orbits. In particular, $\mu(M)$ meets $\mathcal{N}_G$. The stabilizer of a generic point of M can be identified with Sp_{2n-2}.

Proposition 6.3. *In general, if M is pre-hyperspherical, there exists a unique closed orbit $M_0 \subset M$ for $G_{\mathrm{gr}} = G \times \mathbb{G}_m$.*

We call M_0 the *core* of M.

Proof. Consider the GIT quotient $M \to M/\!/G_{\mathrm{gr}}$, and recall that closed orbits of G_{gr} correspond to points of $M/\!/G_{\mathrm{gr}}$. Thus it suffices to show that $M/\!/G_{\mathrm{gr}} = \mathrm{pt}$, or equivalently $k[M]^{G \times \mathbb{G}_m} = k$. Note

$$k[M]^{G \times \mathbb{G}_m} = k[\mathfrak{c}_M]^{\mathbb{G}_m} = k[\mathfrak{c}_m]_0,$$

the weight 0 component. By construction, $k[\mathfrak{c}_G] \to k[\mathfrak{c}_M]$ is finite (and therefore integral), so $k[\mathfrak{c}_M]_0 \to k[\mathfrak{c}_G]_0$ is integral (an exercise in counting degrees). Now we have

$$k[\mathfrak{c}_G]_0 = k[\mathfrak{c}_G]^{\mathbb{G}_m} = k[\mathfrak{g}^*]^{G \times \mathbb{G}_m} = k,$$

so $k[\mathfrak{c}_M]_0$ is integral over k. Since k is algebraically closed, we see $k[\mathfrak{c}_M]_0 = k$. □

6.2 (David) – Weinstein manifolds

Suppose we have an exact symplectic manifold $(M, \omega = d\lambda)$. Take $Z = \omega^{-1}(\lambda)$ (this is a vector field on M). Then Z gives a flow on M, and the core M_0 is the subset of points of M which do not escape to infinity along this flow. Assuming M_0 is isotropic (this is implied by the Weinstein condition), the flow gives an action of $\mathbb{C}^\times$ on M_0.

Example 6.4. Consider the surface singularity $x^2 + y^2 + z^{n+1} = 0$. Let M be a symplectic resolution of this. The core M_0 is the chain of $\mathbb{P}^1$'s appearing as the zero fiber. In terms of geometric representation theory, we can call M_0 a "subregular Springer fiber."

Suppose M is G-Hamiltonian – then we are in a situation very similar to what Mark is talking about. That is, *pre-hyperspherical varieties are analogous to G-Hamiltonian Weinstein manifolds.* This precludes examples like the above, since the union of $\mathbb{P}^1$'s cannot be a single G-orbit. This fact is essentially kin to the last Proposition.

Example 6.5. Consider $G \times G$ acting on $M = T^*G$ by left and right translation. The core M_0 is the zero section.

6.3 (Mark) – Hyperspherical varieties

Let $\mu_M : M \to M/\!/G$ be the GIT quotient map.

Proposition 6.6. *The core M_0 is the unique closed G-orbit in $\mu_M^{-1}(0)$.*

Proof. Note that G has a unique closed orbit $\mathsf{M}_0' \subset \mu_\mathsf{M}^{-1}(0)$ (by standard GIT). Since $\mathbb{G}_m$ commutes with G, the $\mathbb{G}_m$ action takes closed G-orbits to closed G-orbits. Therefore $\mathbb{G}_m$ preserves M_0', and we get $\mathsf{G} \times \mathbb{G}_m \curvearrowright \mathsf{M}_0'$. It follows that M_0' contains a closed G_{gr} orbit, hence contains M_0. But M_0' is itself a G-orbit, so $\mathsf{M}_0 = \mathsf{M}_0'$. $\square$

Pick $x \in \mathsf{M}_0$, and let $\mathsf{H} = \mathrm{Stab}_\mathsf{G}(x)$, so $\mathsf{M}_0 = \mathsf{G}/\mathsf{H}$. Since M_0 is affine, H must be reductive. Since $\mu_\mathsf{M}^{-1}(0)$ maps to $\mathcal{N}_\mathsf{G} \subset \mathfrak{g}^*$, we get an element $f = \mu(x) \in \mathcal{N}_\mathsf{G}$.

Because $\mathbb{G}_m \curvearrowright \mathsf{M}_0 = \mathsf{G}/\mathsf{H}$, we get a homomorphism $\mathbb{G}_\mathsf{M} \to \mathrm{Aut}_\mathsf{G}(\mathsf{G}/\mathsf{H}) \cong \mathsf{N}_\mathsf{G}(\mathsf{H})/\mathsf{H}$. In fact, this factors through $(\mathsf{Z}_\mathsf{G}(\mathsf{H})/\mathsf{Z}(\mathsf{H}))^0$ (where ^{0} denotes the connected component of the identity element). Let $\overline{\pi} : \mathbb{G}_\mathsf{M} \to \mathsf{Z}_\mathsf{G}(\mathsf{H})/\mathsf{Z}(\mathsf{H})$ be the induced map.

Definition 6.7. A pre-hyperspherical variety M is hyperspherical if

1. $\overline{\pi}$ lifts to a homomorphism $\pi : \mathbb{G}_m \to \mathsf{Z}_\mathsf{G}(\mathsf{H}) \subset \mathsf{G}$, which moreover lifts to a homomorphism $\rho : \mathrm{SL}_2 \to \mathsf{G}$ such that
$$d\rho \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = f$$
under the standard identification $\mathfrak{g} \cong \mathfrak{g}^*$.

2. Consider the *sheared* $\mathbb{G}_m$*-action* $(\mathbb{G}_m)_{sh} \curvearrowright \mathsf{M}$ induced by
$$\begin{aligned} (\mathbb{G}_m)_{sh} &\to \mathsf{G} \times \mathbb{G}_m \\ g &\mapsto (\pi(g)^{-1}, g). \end{aligned}$$
By construction, $(\mathbb{G}_m)_{sh}$ fixes x, and thus we get $(\mathbb{G}_m)_{sh} \curvearrowright (\mathsf{T}_x\mathsf{M}_0)^\perp/(\mathsf{T}_x\mathsf{M}_0 \cap \mathsf{T}_x\mathsf{M}_0^\perp) := \mathsf{N}_x\mathsf{M}_0$, the *symplectic normal space*. The condition is that this $(\mathbb{G}_m)_{sh}$-action on $\mathsf{N}_x\mathsf{M}_0$ is given by linear scaling.

7 10/12 (Mark Macerato) – Continued

7.1 Refresher on the definition

Recall that M is a smooth affine graded Hamiltonian G-variety with moment map $\mu : \mathsf{M} \to \mathfrak{g}^*$. The grading means that we have a $\mathbb{G}_m$ action (acting on ω_M with weight 2) such that the G and $\mathbb{G}_m$ actions commute. Therefore μ is $\mathbb{G}_m$-equivariant, where $\mathbb{G}_m$ acts on $\mathfrak{g}^*$ with weight 2.

Recall that we say M is pre-hyperspherical if:

1. M is coisotropic (i.e. $k(\mathsf{M})^\mathsf{G}$ is commutative, equivalently generic G-orbits are cut out by Poisson-commuting G-invariant functions).

2. The image of μ meets the nilpotent cone $\mathcal{N}_\mathsf{G}^*$. Thus there exists a unique closed $\mathsf{G} \times \mathbb{G}_m$-orbit in M, the "core" M_0 of M. This is the unique closed G-orbit in $\tilde{\mu}_\mathsf{M}^{-1}(0)$, where $\tilde{\mu}_\mathsf{M} : \mathsf{M} \to \mathfrak{c}_\mathsf{M} = \mathsf{M}//\mathsf{G}$ and we write 0 for an element of $\mathfrak{c}_\mathsf{M}$ mapping to $0 \in \mathfrak{c}_\mathsf{G} = \mathfrak{g}^*//\mathsf{G}$.)

3. The stabilizer of a generic point $m \in \mathsf{M}$ is connected.

We now move on to the full hyperspherical condition. Fix $x \in \mathsf{M}_0$, and let $f = \mu(x) \in \mathcal{N}_\mathsf{G}^*$. Then $\mathsf{M}_0 \cong \mathsf{G}/\mathsf{H}$ where $\mathsf{H} = \mathrm{Stab}_\mathsf{G}(x)$, and H is reductive. Let $\overline{\pi} : \mathbb{G}_m \to \mathrm{Aut}_\mathsf{G}(\mathsf{G}/\mathsf{H}) \cong \mathsf{N}_\mathsf{G}(\mathsf{H})/\mathsf{H}$ be the natural map.

Condition (4a) is that $\overline{\pi}$ lifts to a homomorphism $\pi : \mathbb{G}_m \to \mathsf{N}_\mathsf{G}(\mathsf{H}) \cap [\mathsf{G}, \mathsf{G}]$.

Proposition 7.1. *The lift* π *factors through* $\mathsf{Z}_\mathsf{G}(\mathsf{H})$.

Proof. Let $t = d\pi(1) \in \mathfrak{g}$, and let $\mathfrak{h} = \mathrm{Lie}(\mathsf{H})$. We need to show that t centralizes $\mathfrak{h}$. Identify $\mathfrak{g} \cong \mathfrak{g}^*$, so we can view f as an element of $\mathfrak{g}$. Then $[t, f] = 2f$ (since $\mathbb{G}_m$ acts on $\mathfrak{g}^*$ with weight 2, and $\mu : \mathsf{M} \to \mathfrak{g}^*$ is G-equivariant). By the Jacobson-Morozov theorem, there exists $e \in \mathfrak{g}$ such that $[t, e] = -2e$ and $[f, e] = t$, giving an embedding $\mathfrak{sl}_2 \hookrightarrow \mathfrak{g}$.

View $\mathfrak{g}$ as an $\mathfrak{sl}_2$-module via the above embedding. Then $\mathfrak{h}$ is spanned by highest weight vectors in $\mathfrak{g}$, so all of the weights of t on $\mathfrak{h}$ are nonnegative. Write $\mathfrak{h} = \mathfrak{h}_+ \oplus \mathfrak{h}_0$, where $\mathfrak{h}_+$ is the sume of the strictly positive t-weight spaces. We have $\mathfrak{h}_+ \subset \mathrm{im}(\mathrm{ad}(f)) \cap \mathfrak{g}^f$ (since H fixes x, we see that H fixes $f = \mu(x)$).

Note that $\mathrm{im}(\mathrm{ad}(f)) \cap \mathfrak{g}^f$ is an ideal in $\mathfrak{g}^f$ (by a direct Lie algebra computation), so $\mathrm{im}(\mathrm{ad}(f)) \cap \mathfrak{h}$ is an ideal of $\mathfrak{h}$. Furthermore, $\mathrm{im}(\mathrm{ad}(f)) \cap \mathfrak{g}^f$ is a nilpotent Lie algebra (since contained in the positive weight part of $\mathfrak{g}$). Thus $\mathrm{im}(\mathrm{ad}(f)) \cap \mathfrak{h}$ is a nilpotent ideal in $\mathfrak{h}$, but $\mathfrak{h}$ is reductive, so $\mathrm{im}(\mathrm{ad}(f)) \cap \mathfrak{h}$ must be central. Since $\mathrm{im}(\mathrm{ad}(f)) \cap \mathfrak{h} \subset [\mathfrak{h}, \mathfrak{h}]$, it follows that $\mathrm{im}(\mathrm{ad}(f)) \cap \mathfrak{h} = 0$, and thus $\mathfrak{h}_+ = 0$. We see that $\mathfrak{h} = \mathfrak{h}_0$, so in particular x centralizes $\mathfrak{h}$. □

Proposition 7.2. *The lift π is unique if it exists.*

To state condition (4b) for hyperspherical varieties, let $\tilde{\mathbb{G}}_m$ be $\mathbb{G}_m$ acting on M via the sheared action $\mathbb{G}_m \xrightarrow{(\pi(-)^{-1}, \mathrm{id})} G \times \mathbb{G}_m$. Thus $\tilde{\mathbb{G}}_m$ acts on the symplectic normal space $T_x M_0^{\perp} / T_x M_0$, where $T_x M_0^{\perp}$ is the symplectic orthogonal subspace. Condition (4b) is that $\tilde{\mathbb{G}}_m$ acts on the symplectic normal space by $t \cdot v = tv$ (i.e. the action has weight 1). This is a useful condition, though the geometric interpretation is not clear.

We are interested in considering sheared $\mathbb{G}_m$ actions more generally – this turns out to be the natural thing to do from the perspective of geometric Satake.

7.2 (David) – More on shearing

Consider the vector field $2p\partial_p - q\partial_q$ on $\mathbb{R}^2 = T^*\mathbb{R}$. This gives a hyperbolic $\mathbb{G}_m$-action and corresponds to the Liouville form $\lambda = 2p\,dq + q\,dp$. For $H = pq$, we get an action of $G = \mathbb{G}_m$ via the Hamiltonian vector field $X_H = p\partial_p - q\partial_q$. Subtracting X_H from our Liouville vector field, we get a $\tilde{\mathbb{G}}_m$-action via the vector field $p\partial_p$. This $\tilde{\mathbb{G}}_m$-action preserves the zero section! Thus, by considering shearing actions, we can preserve certain desirable isotropic / Lagrangian submanifolds.

7.3 (Mark) – A construction

Let $H \subset G$ be a reductive group and $H \curvearrowright S$ be a symplectic representation of H. Let $\pi : \mathbb{G}_m \to [G, G] \cap Z_G(H)$. Choose a nilpotent element $f \in \mathfrak{g}^*$.

We equip S with a commuting $\mathbb{G}_m$ action via scaling (of weight 1). Write $\mathfrak{g} = \mathfrak{j} \oplus \mathfrak{u}^- \oplus \mathfrak{g}_0 \oplus \mathfrak{u}$, where $\mathfrak{j}$ is the centralizer of π and f, $\mathfrak{u}^-$ is the subspace with negative π eigenvalues, $\mathfrak{g}_0$ is the subspace with zero π eigenvalues (but nonzero f eigenvalues), and $\mathfrak{u}$ is the subspace with positive π eigenvalues. Integrate $\mathfrak{u}$ to a unipotent subgroup $U \subset G$.

... And we're out of time – we will finish next week.