GRT Seminar Fall 2024 – Rozansky-Witten Theory

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Abstract

This semester, the GRT Seminar will focus on Rozansky-Witten theory.

Contents

1 9/5 (David Nadler) – Introduction

Our goal is to discuss Rozansky-Witten theory. Some related topics include:

- Quasicoherent sheaves of categories (as discussed last spring).
- Categories of matrix factorizations.[1]
- The cobordism hypothesis.
- Local structure theory of holomorphic symplectic varieties.

1.1 What is Rozansky-Witten theory?

Suppose we have a hyperkähler / holomorphic symplectic manifold X. This means that X has a holomorphic $(2,0)$-form ω satisfying the (complex analogues of) the usual symplectic form axioms. Given such an X, there is a conjectural 3-dimensional topological field theory $\mathcal{Z}_X$, called *Rozansky-Witten theory* with target X.

What we mean by 3d TFT is as follows:

- Given a closed 3-manifold[2] M^3, we obtain a number $\mathcal{Z}_X(M^3)$.
- Closed 2-manifolds M^2 give vector spaces $\mathcal{Z}_X(M^2)$.
- Closed 1-manifolds M^1 give categories[3] $\mathcal{Z}_X(M^1)$.

[1] In more detail: given a smooth variety X and a function $f : X \to \mathbb{A}^1$, we can construct a category MF_f which categorifies the vanishing cycles of f.

[2] Typically with some extra structure, e.g. an orientation

[3] As is standard for GRT, we use the implicit ∞ convention.

- Closed 0-manifolds M^0 give 2-categories $\mathcal{Z}_{\mathsf{X}}(\mathsf{M}^0)$.

In particular, $\mathcal{Z}_{\mathsf{X}}(\mathrm{pt})$ is a 2-category. The *cobordism hypothesis* tells us that we can recover the entire theory $\mathcal{Z}_{\mathsf{X}}$ from the "3-dualizable" 2-category $\mathcal{Z}_{\mathsf{X}}(\mathrm{pt})$. For purposes of geometric representation theory, we are most interested in the low-dimensional behavior, which captures more data about the theory.

Rozansky-Witten theory should satisfy something like:

- $\mathsf{Z}_{\mathsf{X}}(\mathsf{S}^2) = \mathcal{O}(\mathsf{X})$.[4]
- $\mathsf{Z}_{\mathsf{X}}(\mathsf{S}^1) = \mathsf{Coh}(\mathsf{X})$.

These end up inheriting interesting structure from the TFT.

1.2 Why do we care?

Recall that 2-dimensional mirror symmetry can be schematically understood as an equivalence between the following 2d TFTs:

- An A-model $\mathcal{A}$ arising from symplectic geometry
- A B-model $\mathcal{B}_{\mathsf{X}}$, coming from some Kähler manifold X, satisfying $\mathcal{B}_{\mathsf{X}}(\mathrm{pt}) \simeq \mathsf{Coh}(\mathsf{X})$.

In particular, $\mathcal{A}(\mathrm{pt})$ is often some category of geometric interest, and the equivalence $\mathcal{A}(\mathrm{pt}) \simeq \mathcal{B}_{\mathsf{X}}(\mathrm{pt})$ lets us resolve questions about $\mathcal{A}(\mathrm{pt})$.

There's an analogue in higher dimensions: we'd like to take a 3d TFT $\mathcal{Y}$ and give an equivalence $\mathcal{Y} \simeq \mathcal{Z}_{\mathsf{X}}$ for some holomorphic symplectic X. This would give an equivalence between some 2-category and $\mathcal{Z}_{\mathsf{X}}(\mathrm{pt})$.

Conjecture 1.1 (Teleman)**.** *Let G be a complex reductive group with maximal compact subgroup G_{c}. There is an equivalence between:*

- *A suitable 2-category of "categories with G_{c}-action."*
- *The Rozansky-Witten 2-category of $\mathsf{T}^*(\mathsf{G}^\vee/\mathsf{G}^\vee)$.*

Note that $\mathsf{T}^*(\mathsf{G}^\vee/\mathsf{G}^\vee)$ is stacky and non-proper, which makes it impossible for the corresponding 2-category to be 3-dualizable. Thus we typically won't obtain 3-manifold invariants from such a theory. That's terrible for 3-manifold topologists, but this isn't a 3-manifold seminar.

Some other examples of interest for Rozansky-Witten theory include symplectic resolutions and cotangent bundles of smooth algebraic varieties.

1.3 What is the correct 2-category?

To rigorously construct Rozansky-Witten theory, we'd need to give a definition of the 2-category $\mathrm{RW}_2 = \mathcal{Z}_{\mathsf{X}}(\mathrm{pt})$. This was studied by Kapustin, Rozansky, and Saulina, but much is still unknown.

Roughly, we expect RW_2 to be a 2-category where:

- Objects are smooth Lagrangians $\mathsf{L} \subset \mathsf{X}$ (or some suitable generalization of these).
- 1-morphisms from L_1 to L_2 are given by some sort of category associated to $\mathsf{L}_1 \cap \mathsf{L}_2$. In the simplest possible case, where $\mathsf{X} = \mathsf{T}^*W$ is a cotangent bundle, L_1 is the zero-section, and L_2 is the graph of a differential df, then $\mathsf{L}_1 \cap \mathsf{L}_2$ is the critical locus of X and we assign $\mathrm{Hom}(\mathsf{L}_1, \mathsf{L}_2) = \mathsf{MF}_{\mathsf{f}}$, the matrix factorization category of f. Work of Joyce and many others has focused on understanding how much the local setting looks like this.
- 2-morphisms and higher are "natural compatibilities" between the 1-morphisms.

[4] By our conventions, this is what is classically called $\mathbf{R}\Gamma(\mathsf{X}, \mathcal{O})$, so there is interesting derived information.

One should think of the matrix factorization category MF_f as giving a categorical way to measure the critical locus of f. When the critical points of f are Morse, the category MF_f looks like a direct sum of copies of Vect (one for each critical point).

There is an important distinction between Rozansky-Witten theory and the 2d A-model. In the complex setting, there are no "instantons," so the theory is local and we don't run into the full difficulty of Floer theory. Thus Rozansky-Witten theory is a categorified version of Fukaya theory that avoids the need for instanton corrections.

1.4 An alternative viewpoint

If $X = T^*W$ is a cotangent bundle, then $\mathsf{ShvCat}(W)$, the 2-category of (quasicoherent) sheaves of categories on W, embeds into RW_2. The image of this embedding consists of "conic objects." Thus we can understand a key part of Rozansky-Witten theory, at least in this simple case.

The thesis (work in progress) of Enoch Yiu relates RW_2 to $\mathsf{ShvCat}(W \times \mathbb{A}^1)$.

2 1/30 (Daigo Ito) – Theory of Critical Points and Matrix Factorizations

Recall that we wanted to understand the Rozansky-Witten theory of a holomorphic symplectic variety M. By the cobordism hypothesis, it suffices to understand the 2-category $\mathrm{RW}_2(M)$. We expect $\mathrm{RW}_2(M)$ to have some vague properties as follows.

The objects of RW_2 should be holomorphic Lagrangians in M (possibly equipped with extra data). If $M = T^*L_1$, then we should have $\mathrm{Hom}_{\mathrm{RW}_2}(L_1, L_2) = \mathsf{MF}(L_1, f)$, the category of *matrix factorizations* of f. This measures the local geometry of $p \in L_1 \cap L_2 = \mathrm{Crit}(f)$.

Recall the two key differences between this and Lagrangian Floer homology:

- There are no instantons, so the full subtleties of Floer theory don't appear.
- We are working at a higher category level.

Today we will recall the theory of critical points for a function $f : X \to \mathbb{A}^1$.

2.1 Milnor fibers

Let's start by considering a regular map $f : \mathbb{C}^n \to \mathbb{C}$. Assume that $0 \in \mathbb{C}$ is a critical value. Call $X_0 = f^{-1}(0)$ the special fiber – this is typically singular. For small $s \in \mathbb{C}$, let $X_s = f^{-1}(s)$ be the nearby fiber.

Theorem 2.1 (Milnor)**.** *Let $x \in X_0$. For $\epsilon > 0$ sufficiently small, let $B(x, \epsilon)$ be the closed ball of radius ϵ centered at x, and let $S(x, \epsilon) = \partial B(x, \epsilon)$. Then:*

1. *$B(x, \epsilon) \cap X_0$ is homeomorphic to the cone over $K_x = S(x, \epsilon) \cap X_0$.*
2. *The map $\rho_f = \frac{f}{|f|} : S(x, \epsilon) \setminus K_x \to S^1$ is a locally trivial fibration. We call ρ_f the* Milnor fibration *and the fiber F_x the* Milnor fiber*.*

The Milnor fibers F_x degenerate to the cone over K_x.

Example 2.2. If x is nonsingular, then K_x is a sphere, so the cone over K_x is a ball. The Milnor fibers F_x are also balls.

The topology of the Milnor fibers reflects "how singular the point is" – a more singular point leads to a more complicated topology.

Example 2.3. Let $(X_0, x) = (z_1^2 - z_2^2 = 0, 0)$. Then F_x is homotopy equivalent to S^1. Looking at real points, the map f describes a family of hyperbolas degenerating to a union of lines. Here $\partial B = S^3$ and $K_x = S^1 \sqcup S^1$, so topologically K_x is a double cone. The Milnor fibers form a family of cylinders degenerating to this double cone.

Example 2.4. Let $(X_0, x) = (z_1^3 - z_2^2 = 0, 0)$. Then K_x is a trefoil knot

$$\{(r_1 e^{2\pi i t}, r_2 e^{2\pi i t}) \mid t \in \mathbb{R}\} \subset S^1_{r_1} \times S^1_{r_2}.$$

The closures of the Milnor fibers are genus one "Seifert surfaces" for K_x. Thus the Milnor fibers are homotopy equivalent to $S^1 \wedge S^1$.

More generally, if (X, x) is an isolated hypersurface singularity, then we can write $F_x \simeq (S^n)^{\vee \mu_x}$, where μ_x is the *Milnor number*.[5] The S^n's here are the *vanishing cycles* of the singularity.

2.2 Monodromy

The singularity carries information beyond the Milnor fibers. We can capture some of this by looking at the monodromy.

Definition 2.5. The *monodromy* of f at x is the map $h_f : F_x \to F_x$ induced by circling around the base. This is a homeomorphism of F_x which restricts to the identity on ∂F_x. Note that h_f is only well-defined up to isotopy (fixing ∂F_x).

Example 2.6. For a Morse function $f = \sum_i x_i^2$, the Milnor fibers are homotopy equivalent to S^n. We understand the singularity by studying the monodromy of the Milnor fibers as we move around the singular point. This monodromy is a Dehn twist, "corkscrewing" the cylinder.

Theorem 2.7 (Thom-Sebastiani)**.** *Let* $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ *and* $g : (\mathbb{C}^{m+1}, 0) \to (\mathbb{C}, 0)$ *be germs of hypersurface singularities. Define* $f \boxplus g : (\mathbb{C}^{n+1} \times \mathbb{C}^{m+1}, 0) \to (\mathbb{C}, 0)$ *by* $(f \boxplus g)(x, y) = f(x) + g(y)$. *Then there is a homotopy-commutative diagram*

$$\begin{tikzcd} F_f * F_g \arrow[r, "\sim"] \arrow[d, "h_f * h_g"] & F_{f\boxplus g} \arrow[d, "h_{f\boxplus g}"] \\ F_f * F_g \arrow[r, "\sim"] & F_{f\boxplus g}, \end{tikzcd}$$

where $*$ is the join of spaces.

2.3 Preview

Next time we will introduce sheaves that describe the homology of these spaces. We get a fiber sequence

$$i^*\mathcal{F} \longrightarrow \psi_f \mathcal{F} \longrightarrow \phi_f \mathcal{F} \longrightarrow$$

of sheaves on X_0, where:

- $i : X_0 \to X$ is the inclusion,
- ψ_f is nearby cycles, and
- ϕ_f is vanishing cycles.

This will categorify to a sequence

$$\mathsf{Perf}(X_0) \longrightarrow D^b\mathsf{Coh}(X_0) \longrightarrow D_{\mathrm{sing}}(X_0),$$

where $D_{\mathrm{sing}}(X_0)$ agrees with $\mathsf{MF}(X, f)$ in nice cases.

[5] There is an explicit formula for the Milnor number, but we won't write it here.