GRT Seminar Fall 2024 – Rozansky-Witten Theory

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Abstract

This semester, the GRT Seminar will focus on Rozansky-Witten theory.

Contents

1 9/5 (David Nadler) – Introduction

Our goal is to discuss Rozansky-Witten theory. Some related topics include:

- Quasicoherent sheaves of categories (as discussed last spring).
- Categories of matrix factorizations.[1]
- The cobordism hypothesis.
- Local structure theory of holomorphic symplectic varieties.

[1] In more detail: given a smooth variety X and a function $\mathsf{f} : \mathsf{X} \to \mathbb{A}^1$, we can construct a category MF_{f} which categorifies the vanishing cycles of f.

1.1 What is Rozansky-Witten theory?

Suppose we have a hyperkähler / holomorphic symplectic manifold X. This means that X has a holomorphic $(2,0)$-form ω satisfying the (complex analogues of) the usual symplectic form axioms. Given such an X, there is a conjectural 3-dimensional topological field theory $\mathcal{Z}_X$, called *Rozansky-Witten theory* with target X.

What we mean by 3d TFT is as follows:

- Given a closed 3-manifold[2] M^3, we obtain a number $\mathcal{Z}_X(M^3)$.
- Closed 2-manifolds M^2 give vector spaces $\mathcal{Z}_X(M^2)$.
- Closed 1-manifolds M^1 give categories[3] $\mathcal{Z}_X(M^1)$.
- Closed 0-manifolds M^0 give 2-categories $\mathcal{Z}_X(M^0)$.

In particular, $\mathcal{Z}_X(\mathrm{pt})$ is a 2-category. The *cobordism hypothesis* tells us that we can recover the entire theory $\mathcal{Z}_X$ from the "3-dualizable" 2-category $\mathcal{Z}_X(\mathrm{pt})$. For purposes of geometric representation theory, we are most interested in the low-dimensional behavior, which captures more data about the theory.

Rozansky-Witten theory should satisfy something like:

- $Z_X(S^2) = \mathcal{O}(X)$.[4]
- $Z_X(S^1) = \mathsf{Coh}(X)$.

These end up inheriting interesting structure from the TFT.

1.2 Why do we care?

Recall that 2-dimensional mirror symmetry can be schematically understood as an equivalence between the following 2d TFTs:

- An A-model $\mathcal{A}$ arising from symplectic geometry
- A B-model $\mathcal{B}_X$, coming from some Kähler manifold X, satisfying $\mathcal{B}_X(\mathrm{pt}) \simeq \mathsf{Coh}(X)$.

In particular, $\mathcal{A}(\mathrm{pt})$ is often some category of geometric interest, and the equivalence $\mathcal{A}(\mathrm{pt}) \simeq \mathcal{B}_X(\mathrm{pt})$ lets us resolve questions about $\mathcal{A}(\mathrm{pt})$.

There's an analogue in higher dimensions: we'd like to take a 3d TFT $\mathcal{Y}$ and give an equivalence $\mathcal{Y} \simeq \mathcal{Z}_X$ for some holomorphic symplectic X. This would give an equivalence between some 2-category and $\mathcal{Z}_X(\mathrm{pt})$.

Conjecture 1.1 (Teleman)**.** *Let G be a complex reductive group with maximal compact subgroup G_c. There is an equivalence between:*

- *A suitable 2-category of "categories with G_c-action."*
- *The Rozansky-Witten 2-category of $T^*(G^\vee/G^\vee)$.*

Note that $T^*(G^\vee/G^\vee)$ is stacky and non-proper, which makes it impossible for the corresponding 2-category to be 3-dualizable. Thus we typically won't obtain 3-manifold invariants from such a theory. That's terrible for 3-manifold topologists, but this isn't a 3-manifold seminar.

Some other examples of interest for Rozansky-Witten theory include symplectic resolutions and cotangent bundles of smooth algebraic varieties.

[2]Typically with some extra structure, e.g. an orientation
[3]As is standard for GRT, we use the implicit ∞ convention.
[4]By our conventions, this is what is classically called $\mathbf{R}\Gamma(X, \mathcal{O})$, so there is interesting derived information.

1.3 What is the correct 2-category?

To rigorously construct Rozansky-Witten theory, we'd need to give a definition of the 2-category $\mathrm{RW}_2 = \mathcal{Z}_X(\mathrm{pt})$. This was studied by Kapustin, Rozansky, and Saulina, but much is still unknown.

Roughly, we expect RW_2 to be a 2-category where:

- Objects are smooth Lagrangians $L \subset X$ (or some suitable generalization of these).
- 1-morphisms from L_1 to L_2 are given by some sort of category associated to $L_1 \cap L_2$. In the simplest possible case, where $X = T^*W$ is a cotangent bundle, L_1 is the zero-section, and L_2 is the graph of a differential df, then $L_1 \cap L_2$ is the critical locus of X and we assign $\mathrm{Hom}(L_1, L_2) = \mathsf{MF}_f$, the matrix factorization category of f. Work of Joyce and many others has focused on understanding how much the local setting looks like this.
- 2-morphisms and higher are "natural compatibilities" between the 1-morphisms.

One should think of the matrix factorization category MF_f as giving a categorical way to measure the critical locus of f. When the critical points of f are Morse, the category MF_f looks like a direct sum of copies of Vect (one for each critical point).

There is an important distinction between Rozansky-Witten theory and the 2d A-model. In the complex setting, there are no "instantons," so the theory is local and we don't run into the full difficulty of Floer theory. Thus Rozansky-Witten theory is a categorified version of Fukaya theory that avoids the need for instanton corrections.

1.4 An alternative viewpoint

If $X = T^*W$ is a cotangent bundle, then $\mathsf{ShvCat}(W)$, the 2-category of (quasicoherent) sheaves of categories on W, embeds into RW_2. The image of this embedding consists of "conic objects." Thus we can understand a key part of Rozansky-Witten theory, at least in this simple case.

The thesis (work in progress) of Enoch Yiu relates RW_2 to $\mathsf{ShvCat}(W \times \mathbb{A}^1)$.

2 1/30 (Daigo Ito) – Theory of Critical Points and Matrix Factorizations

Recall that we wanted to understand the Rozansky-Witten theory of a holomorphic symplectic variety M. By the cobordism hypothesis, it suffices to understand the 2-category $\mathrm{RW}_2(M)$. We expect $\mathrm{RW}_2(M)$ to have some vague properties as follows.

The objects of RW_2 should be holomorphic Lagrangians in M (possibly equipped with extra data). If $M = T^*L_1$, then we should have $\mathrm{Hom}_{\mathrm{RW}_2}(L_1, L_2) = \mathsf{MF}(L_1, f)$, the category of *matrix factorizations* of f. This measures the local geometry of $p \in L_1 \cap L_2 = \mathrm{Crit}(f)$.

Recall the two key differences between this and Lagrangian Floer homology:

- There are no instantons, so the full subtleties of Floer theory don't appear.
- We are working at a higher category level.

Today we will recall the theory of critical points for a function $f : X \to \mathbb{A}^1$.

2.1 Milnor fibers

Let's start by considering a regular map $f : \mathbb{C}^n \to \mathbb{C}$. Assume that $0 \in \mathbb{C}$ is a critical value. Call $X_0 = f^{-1}(0)$ the special fiber – this is typically singular. For small $s \in \mathbb{C}$, let $X_s = f^{-1}(s)$ be the nearby fiber.

Theorem 2.1 (Milnor). *Let $x \in X_0$. For $\epsilon > 0$ sufficiently small, let $B(x, \epsilon)$ be the closed ball of radius ϵ centered at x, and let $S(x, \epsilon) = \partial B(x, \epsilon)$. Then:*

1. *$B(x, \epsilon) \cap X_0$ is homeomorphic to the cone over $K_x = S(x, \epsilon) \cap X_0$.*

2. *The map* $\rho_f = \frac{f}{|f|} : S(x, \epsilon) \setminus K_x \to S^1$ *is a locally trivial fibration. We call* ρ_f *the* Milnor fibration *and the fiber* F_x *the* Milnor fiber*.*

The Milnor fibers F_x degenerate to the cone over K_x.

Example 2.2. If x is nonsingular, then K_x is a sphere, so the cone over K_x is a ball. The Milnor fibers F_x are also balls.

The topology of the Milnor fibers reflects "how singular the point is" – a more singular point leads to a more complicated topology.

Example 2.3. Let $(X_0, x) = (z_1^2 - z_2^2 = 0, 0)$. Then F_x is homotopy equivalent to S^1. Looking at real points, the map f describes a family of hyperbolas degenerating to a union of lines. Here $\partial B = S^3$ and $K_x = S^1 \sqcup S^1$, so topologically K_x is a double cone. The Milnor fibers form a family of cylinders degenerating to this double cone.

Example 2.4. Let $(X_0, x) = (z_1^3 - z_2^2 = 0, 0)$. Then K_x is a trefoil knot

$$\{(r_1 e^{2\pi i t}, r_2 e^{2\pi i t}) \,|\, t \in \mathbb{R}\} \subset S^1_{r_1} \times S^1_{r_2}.$$

The closures of the Milnor fibers are genus one "Seifert surfaces" for K_x. Thus the Milnor fibers are homotopy equivalent to $S^1 \wedge S^1$.

More generally, if (X, x) is an isolated hypersurface singularity, then we can write $F_x \simeq (S^n)^{\vee \mu_x}$, where μ_x is the *Milnor number*.[5] The S^n's here are the *vanishing cycles* of the singularity.

2.2 Monodromy

The singularity carries information beyond the Milnor fibers. We can capture some of this by looking at the monodromy.

Definition 2.5. The *monodromy* of f at x is the map $h_f : F_x \to F_x$ induced by circling around the base. This is a homeomorphism of F_x which restricts to the identity on ∂F_x. Note that h_f is only well-defined up to isotopy (fixing ∂F_x).

Example 2.6. For a Morse function $f = \sum_i x_i^2$, the Milnor fibers are homotopy equivalent to S^n. We understand the singularity by studying the monodromy of the Milnor fibers as we move around the singular point. This monodromy is a Dehn twist, "corkscrewing" the cylinder.

Theorem 2.7 (Thom-Sebastiani)**.** *Let* $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ *and* $g : (\mathbb{C}^{m+1}, 0) \to (\mathbb{C}, 0)$ *be germs of hypersurface singularities. Define* $f \boxplus g : (\mathbb{C}^{n+1} \times \mathbb{C}^{m+1}, 0) \to (\mathbb{C}, 0)$ *by* $(f \boxplus g)(x, y) = f(x) + g(y)$*. Then there is a homotopy-commutative diagram*

$$\begin{tikzcd} F_f * F_g \arrow[r, "\sim"] \arrow[d, "h_f * h_g"] & F_{f \boxplus g} \arrow[d, "h_{f \boxplus g}"] \\ F_f * F_g \arrow[r, "\sim"] & F_{f \boxplus g}, \end{tikzcd}$$

where $*$ *is the join of spaces.*

2.3 Preview

Next time we will introduce sheaves that describe the homology of these spaces. We get a fiber sequence

$$i^* \mathcal{F} \longrightarrow \psi_f \mathcal{F} \longrightarrow \phi_f \mathcal{F} \longrightarrow$$

of sheaves on X_0, where:

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[5]There is an explicit formula for the Milnor number, but we won't write it here.

- $i : X_0 \to X$ is the inclusion,
- ψ_f is nearby cycles, and
- ϕ_f is vanishing cycles.

This will categorify to a sequence

$$\mathsf{Perf}(X_0) \longrightarrow D^b_{\mathrm{coh}}(X_0) \longrightarrow D_{\mathrm{sing}}(X_0).$$

where $D_{\mathrm{sing}}(X_0)$ agrees with $\mathsf{MF}(X, f)$ in nice cases.

3 2/6 (Daigo Ito) – Continued

Last time we discussed the construction of Milnor fibers, vanishing cycles, monodromy, and Thom-Sebastiani isomorphisms for $f : \mathbb{C}^{n+1} \to \mathbb{C}$. Today we would like to discuss and categorify these stories in a sheaf-theoretic framework.

3.1 Vanishing and nearby cycles

Let $\mathbb{D} \subset \mathbb{C}$ be a small disk around 0. Let X be (an open subset of) a smooth algebraic variety over $\mathbb{C}$, and let $f : X \to \mathbb{D}$ be a map. Consider the diagram

$$\begin{tikzcd} X_0 \arrow[r, "i"] \arrow[d] & X \arrow[d, "f"] & X^* \arrow[l, "j"'] \arrow[d, "f^*"] & \tilde{X^*} \arrow[l, "\tilde{\pi}"'] \arrow[d] \\ \{0\} \arrow[r] & \mathbb{D} & \mathbb{D}^* \arrow[l] & \tilde{\mathbb{D}^*} \arrow[l, "\pi"'] \end{tikzcd}$$

where all squares are pullback squares and $\pi : \tilde{\mathbb{D}^*} \to \mathbb{D}^*$ is the universal cover $z \mapsto \exp(2\pi i z)$. Note that $X^* \simeq X_s$ for s small, $s \neq 0$.

Write $D^b_c(-)$ for the bounded constructible derived category of A-modules on a space (where A is some fixed coefficient ring, typically $\mathbb{Z}$ or $\mathbb{C}$). Recall that "constructible" means locally constant on the strata of a nice stratification and with finite-rank stalks.

Definition 3.1. The *nearby cycle functor* associated with f is $\psi_f : D^b_c(X) \to D^b_c(X_0)$, defined by[6]

$$\mathcal{F} \mapsto i^*(j \circ \tilde{\pi})_*(j \circ \tilde{\pi})^*\mathcal{F}.$$

Morally, we have a (very non-analytic) specialization map $\mathrm{sp} : X_s \to X_0$, and $\psi_f = \mathrm{sp}_*(\mathcal{F}|_{X_s})$. The previous definition is used to avoid referencing sp.

Example 3.2. Consider $f : \mathbb{D} \to \mathbb{D}$ by $f(z) = z^2$. For $\mathcal{F} = \underline{A}_{\mathbb{D}}$, we can compute $\psi_f(\mathcal{F}) = A_0 \oplus A_0$, reflecting the fact that the nearby fibers have two points. The monodromy map swaps the two factors: this can be seen directly using the specialization definition or by considering deck transformations using the formal definition.

Remark 3.3. David mentioned that one can actually rephrase this story so that the only f which we consider is projection to the first coordinate. The cost is that we are forced to work with arbitrarily complicated sheaves. The reverse (working with the constant sheaf but allowing arbitrarily complicated f) is not possible in general, though there is a related theory of "sheaves of geometric origin."

From the pushforward-pullback adjunction, there is a natural map $r : i^*\mathcal{F} \to i^*(j \circ \tilde{\pi})_*(j \circ \tilde{\pi})^*\mathcal{F} = \psi_f\mathcal{F}$.

Definition 3.4. We define the *vanishing cycle functor* $\phi_f : D^b_c(X) \to D^b_c(X_0)$ by $\phi_f(\mathcal{F}) = \mathrm{cone}(r)$, so there is a cofiber sequence

$$i^*\mathcal{F} \longrightarrow \psi_f\mathcal{F} \longrightarrow \phi_f\mathcal{F} \longrightarrow .$$

[6] All functors here are derived.

We call $\psi_f \underline{A}_X$ (resp. $\phi_f \underline{A}_X$) the *nearby (resp. vanishing) cycle complex* associated with f. From the cofiber sequence containing these, we obtain a long exact sequence (using $H^*(X) \cong H^*(X_0)$):

$$\ldots \longrightarrow H^*(X_0) \longrightarrow H^*(X_s) \longrightarrow H^*(X, X_s) \longrightarrow \ldots$$

This encompasses much of our discussion from last time.

Proposition 3.5. *For $f : \mathbb{C}^{n+1} \to \mathbb{C}$, if X_0 has only isolated singularities (so $F_f \simeq \vee S^n$), then*

$$H^k(X_0, \phi_f \underline{A}_X) = \begin{cases} 0 & k \neq n \\ \oplus_{x \in \mathrm{Sing}(X_0)} H^n(F_{f,x}; A) & k = n. \end{cases}$$

Remark 3.6. One can obtain the monodromy of nearby / vanishing cycles using the deck transformations of $\tilde{\mathbb{D}}^*$. There's also a Thom-Sebastiani theorem in the sheaf-theoretic setting.

3.2 Singularity categories and matrix factorizations

We can categorify the preceding story using the exact sequence of categories:

$$\mathsf{Perf}(X_0) \longrightarrow D^b_{\mathrm{coh}}(X_0) \longrightarrow D_{\mathrm{sing}}(X_0),$$

where $D_{\mathrm{sing}}(X_0)$ is defined as the quotient $D^b_{\mathrm{coh}}(X_0)/\mathsf{Perf}(X_0)$. In nice cases, $D_{\mathrm{sing}}(X_0)$ agrees with the *matrix factorization category* $\mathsf{MF}(X, f)$. The reason D_{sing} is called the "category of singularities" is the following:

Proposition 3.7. *X_0 is smooth if and only if $\mathsf{Perf}(X_0) \simeq D^b_{\mathrm{coh}}(X_0)$.*

Example 3.8. If $x \in X_0$ is singular, then the skyscraper sheaf $k(x_0)$ is not in $\mathsf{Perf}(X)$.

To decategorify our exact sequence to the sheaf-theoretic statement above, we take "periodic cyclic homology."

4 2/13 (Will Fisher) – The Singular Category and Matrix Factorizations

Today's talk is based on Orlov's paper "Triangulated categories of singularities ..." We work over a field k.

Last time, Daigo presented the exact sequence

$$\mathsf{Perf}(X_0) \longrightarrow D^b_{\mathrm{coh}}(X_0) \longrightarrow D_{\mathrm{sing}}(X_0).$$

One can think of this as a categorified version of the usual nearby / vanishing cycles exact sequence (which we recover by taking periodic cyclic homology): $D^b_{\mathrm{coh}}(X_0)$ knows something about a "nearby smoothing" of X_0.

For nice $f : X \to \mathbb{A}^1$ with $X_0 = f^{-1}(0)$, we can identify $D_{\mathrm{sing}}(X_0) \simeq \mathsf{MF}(X, f)$. Our goal is to discuss this result.

4.1 Definitions

First, we will need to define everything involved.

Definition 4.1. Let A be a commutative ring. A chain complex $M \in D(\mathsf{Mod}_A)$ is *perfect* if it is quasi-isomorphic to a bounded complex of finite projective modules. These form a subcategory $\mathsf{Perf}(A) \subset D(\mathsf{Mod}_A)$. Equivalently, $\mathsf{Perf}(A)$ is the smallest triangulated subcategory containing A and closed under and retracts.

Definition 4.2. If X is a scheme, then $\mathsf{Perf}(X)$ is the full subcategory of $D^b_{\mathrm{coh}}(X)$ consisting of objects which are perfect affine locally, i.e. locally they can be written as a quotient of vector bundles.

We will assume X is a separated noetherian scheme of finite Krull dimension, and that $\mathsf{Coh}(\mathsf{X})$ "has enough vector bundles," i.e. for all $\mathcal{F} \in \mathsf{Coh}(\mathsf{X})$, there exists a vector bundle $\mathcal{P}$ and a surjection $\mathcal{P} \twoheadrightarrow \mathcal{F}$. These conditions will hold X is quasiprojective.

Definition 4.3. The category $\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathsf{X}) \subset \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathsf{X})$ consists of objects with bounded and coherent cohomology.[7]

Under the above hypotheses, we have $\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathsf{X}) = \mathsf{D}^{\mathrm{b}}(\mathsf{Coh}(\mathsf{X}))$.

Definition 4.4. The category $\mathsf{D}_{\mathrm{sing}}(\mathsf{X})$ is the Verdier quotient $\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathsf{X})/\mathsf{Perf}(\mathsf{X})$, constructed by formally inverting morphisms in $\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathsf{X})$ with cones in $\mathsf{Perf}(\mathsf{X})$.

Example 4.5. Let $\mathsf{X} = \operatorname{Spec} \mathsf{A}$ for $\mathsf{A} = \mathsf{k}[\mathsf{x}]/(\mathsf{x}^2)$. The A-module $\mathsf{k} = \mathsf{A}/(\mathsf{x})$ is coherent but not perfect: it has the infinite resolution

$$\dots \mathsf{A} \xrightarrow{\cdot \mathsf{x}} \mathsf{A} \xrightarrow{\cdot \mathsf{x}} 0.$$

We can use this to compute $\operatorname{Ext}^{\mathsf{i}}(\mathsf{k}, \mathsf{k}) \cong \mathsf{k}$ for $\mathsf{i} \geqslant 0$, showing that k is not perfect. In particular, $\mathsf{D}_{\mathrm{sing}}(\mathsf{X})$ is nontrivial. This relates to Serre's criterion for regularity in algebraic geometry.

4.2 Basic properties

We state without proof some relevant (but hard) facts about $\mathsf{D}_{\mathrm{sing}}(\mathsf{X})$:

1. (Auslander-Buchsbaum-Serre) If A is noetherian and finite-dimensional, then $\mathsf{D}_{\mathrm{sing}}(\operatorname{Spec} \mathsf{A}) = 0$ if and only if A is regular.

2. (Thomason-Trobaugh) If $\mathsf{U} \subset \mathsf{X}$ is an open subscheme containing the singular locus of X, then $\mathsf{D}_{\mathrm{sing}}(\mathsf{X}) \xrightarrow{\sim} \mathsf{D}_{\mathrm{sing}}(\mathsf{U})$.

It turns out that $\mathsf{D}_{\mathrm{sing}}(\mathsf{X})$ is smaller than we might initially expect:

Proposition 4.6. *Every object of $\mathsf{D}_{\mathrm{sing}}(\mathsf{X})$ is equivalent to $\mathcal{F}[\mathsf{k}+1]$ for some coherent sheaf $\mathcal{F}$ and some $\mathsf{k} \in \mathbb{Z}$.*

Proof. For $\mathsf{A}^\bullet \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathsf{X})$, choose a quasi-isomorphism $\mathsf{P}^\bullet \xrightarrow{\sim} \mathsf{A}^\bullet$ with $\mathsf{P}^\bullet$ a bounded above complex of vector bundles. The stupid truncations $\sigma^{\geqslant -\mathsf{k}}\mathsf{P}^\bullet$ (obtained by replacing $\mathsf{P}^\bullet$ by 0 for $\bullet < -\mathsf{k}$) are perfect. Let $\mathsf{f}_{-\mathsf{k}} : \sigma^{\geqslant -\mathsf{k}}\mathsf{P}^\bullet \to \mathsf{A}^\bullet$ be the natural map, and consider $\operatorname{cone}(\mathsf{f}_{-\mathsf{k}})$. The long exact sequence of cohomology sheaves for the cofiber sequence shows that, for $\mathsf{k} \gg 0$, we have $\mathcal{H}^{\mathsf{i}}(\operatorname{cone}(\mathsf{f}_{-\mathsf{k}})) = 0$ unless $\mathsf{i} = -\mathsf{k}-1$. Thus, taking $\mathcal{F} = \mathcal{H}^{-\mathsf{k}}(\sigma^{\geqslant -\mathsf{k}}\mathsf{P}^\bullet)$, we obtain $\operatorname{cone}(\mathsf{f}_{-\mathsf{k}}) \simeq \mathcal{F}[\mathsf{k}+1]$. This gives $\mathsf{A}^\bullet \simeq \mathcal{F}[\mathsf{k}+1]$ in $\mathsf{D}_{\mathrm{sing}}(\mathsf{X})$. □

We'd also like to be able to take right resolutions of coherent sheaves. If we were capable of "dualizing" in a way that preserves vector bundles, we could take a left resolution of $\mathcal{F}^\vee$ and dualize this to get a resolution of $\mathcal{F}^{\vee\vee} \simeq \mathcal{F}$.

If X is Gorenstein and satisfies our standing assumptions, then $\mathcal{O}_{\mathsf{X}}$ has a finite injective resolution and

$$\mathcal{F} \xrightarrow{\sim} \mathbf{R}\mathcal{H}\mathrm{om}(\mathbf{R}\mathcal{H}\mathrm{om}(\mathcal{F}, \mathcal{O}_{\mathsf{X}}), \mathcal{O}_{\mathsf{X}})$$

for $\mathcal{F} \in \mathsf{Coh}(\mathsf{X})$.

Lemma 4.7. *Let X be as above, and let $\mathcal{F} \in \mathsf{Coh}(\mathsf{X})$. TFAE:*

1. $\mathcal{E}\mathrm{xt}^{\mathsf{i}}(\mathcal{F}, \mathcal{O}_{\mathsf{X}}) = 0$ *for all* $\mathsf{i} > 0$.

2. *There exists a right resolution of $\mathcal{F}$ by vector bundles.*

Corollary 4.8. *Every $\mathsf{A}^\bullet \in \mathsf{D}_{\mathrm{sing}}(\mathsf{X})$ is equivalent to $\mathcal{F}[0]$ for some $\mathcal{F} \in \mathsf{Coh}(\mathsf{X})$ with $\mathcal{E}\mathrm{xt}^{\mathsf{i}}(\mathcal{F}, \mathcal{O}_{\mathsf{X}}) = 0$ for $\mathsf{i} > 0$.*

[7] In general, it is better to treat this as a *property* than as *structure*.

5 2/20 (Will Fisher) – Continued

5.1 Dualizing complexes

Let's review / clarify some points on dualizing complexes. This concerns something a bit different than the usual Serre duality context: we only care about the local setting and giving equivalences $\mathsf{D}^b_{\mathrm{coh}}(X)^{\mathrm{op}} \xrightarrow{\sim} \mathsf{D}^b_{\mathrm{coh}}(X)$.

Definition 5.1. If A is a noetherian ring, a *dualizing complex* on $\operatorname{Spec} A$ is $\omega^\bullet \in \mathsf{D}^b_{\mathrm{coh}}(\operatorname{Spec} A)$ such that:

1. $\omega^\bullet$ has finite injective dimension, and
2. $A \to \mathbf{R}\operatorname{Hom}(\omega^\bullet, \omega^\bullet)$ is a quasi-isomorphism.

If X is a locally noetherian scheme, a *dualizing complex* on X is $\omega^\bullet \in \mathsf{D}^b_{\mathrm{coh}}(X)$ such that $\omega^\bullet$ is affine locally a dualizing complex.

If ω is a dualizing complex on X, then $\mathbf{R}\mathcal{H}\mathrm{om}(-, \omega^\bullet) : \mathsf{D}^b_{\mathrm{coh}}(X)^{\mathrm{op}} \to \mathsf{D}^b_{\mathrm{coh}}(X)$ is an equivalence (and in fact is its own inverse).

Remark 5.2. Peter Haine pointed out a few things:

- Dualizing complexes are unique up to tensoring with complete line bundles.
- A result of Kawasaki (proving a conjecture of Sharp) shows that the commutative rings which admit dualizing complexes are those of finite Krull dimension which arise as quotients of Gorenstein rings.
- Upper shriek functors preserve dualizing complexes in this sense.

5.2 Gorenstein schemes

In the Gorenstein case, dualizing complexes are easy to understand.

Theorem 5.3. *If X is locally noetherian and has a dualizing complex, TFAE:*

1. *X is Gorenstein.*
2. *X has an invertible dualizing complex.*
3. *$\mathcal{O}_X[0]$ is a dualizing complex.*

We have a few large classes of Gorenstein schemes:

Theorem 5.4. *Smooth schemes are dualizing.*

Theorem 5.5. *Local complete intersections in Gorenstein schemes are dualizing.*

5.3 Representing objects of $\mathsf{D}_{\mathrm{sing}}(X)$

From now on we will assume X is Gorenstein. Our goal is to show that objects of $\mathsf{D}_{\mathrm{sing}}(X)$ can be represented by particularly nice sheaves and complexes.

Proposition 5.6. *If $\mathcal{O}_X$ has a finite injective resolution, then there exists $n_0 > 0$ such that, for all $\mathcal{F} \in \mathsf{QCoh}(X)$ and all $i > n_0$, we have $\mathcal{E}\mathrm{xt}^i(\mathcal{F}, \mathcal{O}_X) = 0$.*

Proposition 5.7. *For $\mathcal{F} \in \mathsf{Coh}(X)$, TFAE:*

1. *$\mathcal{E}\mathrm{xt}^i(\mathcal{F}, \mathcal{O}_X) = 0$ for all $i > 0$.*
2. *$\mathcal{F}$ has a right resolution by vector bundles.*

Proof. We only prove $\Rightarrow$. To obtain the desired right resolution, take a left resolution $\mathcal{P}^\bullet \xrightarrow{\sim} \mathcal{H}\mathrm{om}(\mathcal{F}, \mathcal{O}_X) = \mathbf{R}\mathcal{H}\mathrm{om}(\mathcal{F}, \mathcal{O}_X)$. Then $\mathcal{F} \simeq \mathcal{H}\mathrm{om}(\mathcal{H}\mathrm{om}(\mathcal{F}, \mathcal{O}_X), \mathcal{O}_X) \simeq \mathcal{H}\mathrm{om}(\mathcal{P}^\bullet, \mathcal{O}_X)$ gives the desired right resolution. $\square$

Theorem 5.8. *Every object in $\mathsf{D}_{\mathrm{sing}}(\mathsf{X})$ is equivalent to $\mathcal{F}[0]$ for $\mathcal{F}$ coherent with $\mathcal{E}\mathrm{xt}^{\mathrm{i}}(\mathcal{F}, \mathcal{O}_{\mathsf{X}}) = 0$ for all $\mathrm{i} > 0$.*

Proof. Let $\mathcal{A}^\bullet \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathsf{X})$. Take a left resolution $\mathcal{P}^\bullet \xrightarrow{\sim} \mathcal{A}^\bullet$. For $\mathrm{k} \gg 0$, we saw last time that if $\mathcal{G} = \mathcal{H}^{-\mathrm{k}}(\sigma^{\geqslant -\mathrm{k}}\mathcal{P}^\bullet)$, then $\mathcal{A}^\bullet \simeq \mathcal{G}[\mathrm{k}+1]$ in $\mathsf{D}_{\mathrm{sing}}(\mathsf{X})$. Since $\mathcal{O}_{\mathsf{X}}$ has bounded injective resolution, $\mathbf{R}\mathcal{H}\mathrm{om}(\mathcal{A}^\bullet, \mathcal{O}_{\mathsf{X}})$ is cohomologically bounded. Thus, for $\mathrm{k} \gg 0$, we get

$$\mathcal{E}\mathrm{xt}^{\mathrm{i}}(\mathcal{G}, \mathcal{O}_{\mathsf{X}}) \cong \mathcal{E}\mathrm{xt}^{\mathrm{i}+\mathrm{k}+1}(\mathcal{A}^\bullet, \mathcal{O}_{\mathsf{X}}) = 0$$

for $\mathrm{i} > 0$. Now take a right resolution $\mathcal{G} \to \mathcal{Q}^\bullet$,[8] and let $\mathcal{F} = \mathcal{H}^{\mathrm{k}}(\sigma^{\leqslant \mathrm{k}}\mathcal{Q}^\bullet)$. Then it is clear that $\mathcal{F}$ has the desired properties. $\square$

5.4 Equivalence with matrix factorizations

Let $\mathsf{X} = \operatorname{Spec} \mathsf{A}$ be smooth, let $\mathsf{f} : \mathsf{X} \to \mathbb{A}^1$, and let $\mathsf{X}_0 = \mathsf{f}^{-1}(0) = \operatorname{Spec} \mathsf{A}/(\mathsf{f})$. Write $\mathsf{j} : \mathsf{X}_0 \to \mathsf{X}$ for the inclusion. We would like to show that $\mathsf{D}_{\mathrm{sing}}(\mathsf{X}_0) \simeq \mathsf{MF}(\mathsf{X}, \mathsf{f})$, the matrix factorization category of f.

Proposition 5.9. *Let $\mathcal{F}$ be coherent on X_0 with $\mathcal{E}\mathrm{xt}^{\mathrm{i}}(\mathcal{F}, \mathcal{O}_{\mathsf{X}_0}) = 0$ for all $\mathrm{i} > 0$. Then there exist vector bundles $\mathcal{P}_0, \mathcal{P}_1$ on X with maps $\mathsf{p}_0 : \mathcal{P}_0 \to \mathcal{P}_1$ and $\mathsf{p}_1 : \mathcal{P}_1 \to \mathcal{P}_0$ such that:*

1. $\mathsf{p}_0\mathsf{p}_1 = \mathsf{f} \cdot \mathrm{id}_{\mathsf{X}}$,
2. $\mathsf{p}_1\mathsf{p}_0 = \mathsf{f} \cdot \mathrm{id}_{\mathsf{X}}$, *and*
3. $\operatorname{coker} \mathsf{p}_1 = \mathsf{j}_*\mathcal{F}$.

Sketch of proof. Choose a surjection $\mathcal{P}_0 \to \mathsf{j}_*\mathcal{F}$ with $\mathcal{P}_0$ a vector bundle. Let $\mathsf{p}_1 : \mathcal{P}_1 \to \mathcal{P}_0$ be the kernel of this surjection. Then the hypotheses on $\mathcal{F}$ imply that $\mathcal{P}_1$ is a vector bundle.[9] Because multiplication by f annihilates $\mathsf{j}_*\mathcal{F}$, the composite

$$\mathcal{P}_0 \xrightarrow{\cdot \mathsf{f}} \mathcal{P}_0 \to \mathsf{j}_*\mathcal{F}$$

is zero, so it factors through p_1. That is, there exists $\mathsf{p}_0 : \mathcal{P}_0 \to \mathcal{P}_1$ such that $\mathsf{p}_1\mathsf{p}_0 = \mathsf{f} \cdot \mathrm{id}$. The computation $\mathsf{p}_1\mathsf{p}_0\mathsf{p}_1 = (- \cdot \mathsf{f}) \circ \mathsf{p}_1 = \mathsf{p}_1 \circ (- \cdot \mathsf{f})$ implies $\mathsf{p}_0\mathsf{p}_1 = \mathrm{id}$ as well. $\square$

Let $\mathcal{P}_0$ and $\mathcal{P}_1$ be as above. Then we get an exact sequence[10]

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{P}_1|_{\mathsf{X}_0} \longrightarrow \mathcal{P}_0|_{\mathsf{X}_0} \longrightarrow \mathcal{F} \longrightarrow 0.$$

In particular, we obtain $\mathcal{F} \simeq \mathcal{F}[2]$ in $\mathsf{D}_{\mathrm{sing}}(\mathsf{X})$.

Definition 5.10. The *matrix factorization category* of (X, f) has objects given by pairs of vector bundles $\mathcal{P}_0, \mathcal{P}_1$ on X with maps $\mathsf{p}_0 : \mathcal{P}_0 \to \mathcal{P}_1$ and $\mathsf{p}_1 : \mathcal{P}_1 \to \mathcal{P}_0$ such that:

1. $\mathsf{p}_0\mathsf{p}_1 = \mathsf{f} \cdot \mathrm{id}_{\mathsf{X}}$ and
2. $\mathsf{p}_1\mathsf{p}_0 = \mathsf{f} \cdot \mathrm{id}_{\mathsf{X}}$.

Morphisms are defined to be homotopy classes of commutative squares.

There is a functor $\operatorname{coker} : \mathsf{MF}(\mathsf{X}, \mathsf{f}) \to \mathsf{D}_{\mathrm{sing}}(\mathsf{X}_0)$ sending $(\mathcal{P}_0, \mathcal{P}_1, \mathsf{p}_0, \mathsf{p}_1)$ to $\operatorname{coker}(\mathsf{p}_1)$.

Theorem 5.11 (Orlov)**.** *The functor* $\operatorname{coker} : \mathsf{MF}(\mathsf{X}, \mathsf{f}) \to \mathsf{D}_{\mathrm{sing}}(\mathsf{X}_0)$ *is an equivalence.*

Example 5.12. Let $\mathsf{A} = \mathbb{C}[\mathsf{x}]$, $\mathsf{f} = \mathsf{x}^{\mathsf{n}}$, and $\mathsf{X}_0 = \operatorname{Spec} \mathbb{C}[\mathsf{x}]/(\mathsf{x}^{\mathsf{n}})$. Then $\mathsf{D}_{\mathrm{sing}}(\mathsf{X}_0)$ has objects given by direct sums of $\mathsf{V}_{\mathrm{i}} = \mathbb{C}[\mathsf{x}]/(\mathsf{x}^{\mathrm{i}})$ for $1 \leqslant \mathrm{i} \leqslant \mathsf{n}-1$. Under the equivalence $\mathsf{D}_{\mathrm{sing}}(\mathsf{X}_0) \simeq \mathsf{MF}(\mathsf{X}, \mathsf{f})$, V_{i} corresponds to $(\mathbb{C}[\mathsf{x}], \mathbb{C}[\mathsf{x}], \mathsf{x}^{\mathsf{n}-\mathrm{i}}, \mathsf{x}^{\mathrm{i}})$. One can decategorify this to obtain the usual nearby / vanishing cycle picture for f.

[8] This is where the Gorenstein hypothesis appears.
[9] More details can be found on Will Fisher's website.
[10] This is left as a non-obvious exercise.