

GRT Seminar Fall 2024 – Equivariant Mathematics

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Abstract

This semester, the GRT Seminar will focus on equivariant mathematics (especially homotopy theory). The goal is not necessarily to become homotopy theorists but rather to gain an appreciation for one of the most developed approaches to equivariance.

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1 (8/28) David Nadler – S^1 -Actions and Koszul Duality

A good reference for this material is GKM (Goresky–Kottwitz–Macpherson, “Equivariant cohomology, Koszul duality, and the localization theorem”). We’ll focus on the case of $G = S^1$ acting on a (reasonable) topological space X , e.g. a CW complex.

1.1 Linearization

We can “linearize” the space X by taking cochains $C^*(X)$, say with coefficients in $k = \mathbb{Q}$. Morally, this is like passing to “functions” on a space, except that in topology our notion of function is “locally constant.” This is not as uninteresting as it sounds, since we also include “derived” information.

Following this analogy, “distributions” on S^1 should act on “functions” on X by convolution. The right notion of “distributions” on S^1 is captured by the chain complex $C_{-*}(S^1)$. We may write

$$C_{-*}(S^1) = \begin{cases} k \cdot 1 & * = 0 \\ k \cdot \lambda & * = 1, \end{cases}$$

i.e. $C_{-*}(S^1) = k[\lambda]/\lambda^2$ with $|\lambda| = -1$. The multiplication $C_{-*}(S^1) \otimes_k C_{-*}(S^1) \rightarrow C_{-*}(S^1)$ is induced by the multiplication map $S^1 \times S^1 \rightarrow S^1$. We may view $C^*(X)$ as a $C_{-*}(S^1)$ -module. In the following examples we’ll often use Poincaré duality to identify $C^*(X) = C_{-*}(X)$.

Example 1.1. Let S^1 act on itself via translation. This linearizes to $C_{-*}(S^1) \curvearrowright C^*(S^1)$ by “sweeping out cochains along a chain.” Algebraically, we may write $C^*(S^1) = k[v]/v^2$ where $|v| = 1$, and the action satisfies $\lambda \cdot v = 1$. In topology, this is the “slant product.”

Example 1.2. Let S^1 act on $X = \text{pt}$. This linearizes to $C_{-*}(S^1) \curvearrowright k$ via the augmentation action.

Example 1.3. Let S^1 act on S^2 by rotation. The S^1 action here is trivial at the level of cohomology (any operation of degree -1 on $H_{-*}(S^2)$ is trivial). However, there is some interesting and subtle behavior: if we sweep out a (the cocycle corresponding to a) longitudinal line, we get all of S^1 . The next example exhibits a similar phenomenon which is more cohomologically meaningful.

Example 1.4. Let S^1 act on S^3 via the Hopf action. View $S^3 = \mathbb{R}^3 \cup \{\infty\}$. Let v be the cocycle corresponding to a point. Sweeping v out gives a circle (which we can take to be the compactified z -axis). This is still trivial – there is a disk D with $\partial D = \lambda \cdot v$. However, we have $\lambda \cdot D = S^3$.

We end up with a “secondary sweep” operation $\lambda_{(2)}$, defined somewhat like the snake lemma: if $\lambda \cdot v$ is trivial, we write $\lambda \cdot v = \partial D$, and set $\lambda_{(2)}v = \lambda D$. Note that $\lambda_{(2)}$ is defined in terms of the dg-algebras and dg-modules but cannot be constructed from the cohomology! We say that the action $C_{-*}(S^1) \curvearrowright C^*(S^3)$ is not *formal*.

1.2 Quotients and equivariant cohomology

From an action $S^1 \curvearrowright X$, we’d like to construct a quotient $[X/S^1]$. One way to do this that avoids issues with non-free actions is to choose a contractible space ES^1 on which S^1 acts freely and define

$$[X/S^1] = (X \times ES^1)/S^1.$$

This space comes with a natural map to $BS^1 = [\text{pt}/S^1] = (ES^1)/S^1$. In fact, we can take $ES^1 = S^\infty = \bigcup_n S^{2n+1}$. Then $BS^1 = \bigcup_n S^{2n} = \mathbb{CP}^\infty$.

By taking quotients, we’ve turned $S^1 \curvearrowright X$ into a space $[X/S^1]$ over BS^1 . Koszul duality is about going back and forth between these perspectives.

Definition 1.5. We define the *equivariant cochains* on X to be $C_{S^1}^*(X) = C^*([X/S^1])$. This is naturally a $C^*([\text{pt}/S^1])$ -module.

2 (9/4) David Nadler – Continued

Next week Peter Rowley will tell us about the G-Whitehead theorem and Elmendorf’s theorem.

2.1 Review

Recall that we are focusing on actions $S^1 \curvearrowright X$ and linearizing by taking (co)chains valued in $k = \mathbb{Q}$. This induces an action $C_{-*}(S^1) \curvearrowright C^*(X)$. Write $C_{-*}(S^1) = k[\lambda]$, where λ is in degree -1 and acts by “sweeps.” On cohomology we may have “higher sweeps” $\lambda_{(n)}$ related to differentials in a spectral sequence.

Let

$$S^\infty = \bigcup_{n \geq 0} S^{2n+1} = ES^1 \subset \mathbb{C}^\infty.$$

We define $C_{S^1}^*(X) = C^*(X \times^{S^1} S^\infty)$. We have a natural map $X \times^{S^1} S^\infty \rightarrow \text{pt} \times^{S^1} S^\infty$, where

$$\text{pt} \times^{S^1} S^\infty = S^\infty / S^1 = \mathbb{C}P^\infty = BS^1.$$

This induces an action $C_{S^1}^*(\text{pt}) \curvearrowright C_{S^1}^*(X)$. We have $C_{S^1}^*(\text{pt}) = k[u]$ where u is in degree 2.

2.2 Koszul duality

Koszul duality lets us pass between the action $S^1 \curvearrowright X$ and the fibration $X \times^{S^1} ES^1 \rightarrow BS^1$. At the linear level, we pass between $k[\lambda] \curvearrowright C^*(X)$ and $k[u] \curvearrowright C_{S^1}^*(X)$.

Theorem 2.1 (Algebraic Koszul duality).

$$\text{Coh}(k[\lambda]) = \text{Perf}(k[u]).$$

Remark 2.2. Note that $\text{Coh}(k[\lambda]) \neq \text{Perf}(k[\lambda])$. For example, the augmentation module $k = k[\lambda]/(\lambda)$ is coherent but not perfect (any resolution must be infinite).

To understand Koszul duality, consider $C^*(S^\infty)$. We have an action $S^1 \curvearrowright S^\infty$ and a fibration $S^\infty \rightarrow BS^1$. Because the S^1 -action is fiberwise, the induced $k[\lambda]$ and $k[u]$ -actions on $C^*(S^\infty)$ commute. That is, $C^*(S^\infty)$ is a $(k[\lambda], k[u])$ -bimodule. We may write $C^*(S^\infty)$ as a free $(k[\lambda], k[u])$ -bimodule on one generator (with nontrivial differential so that cohomology is concentrated in degree 0). The Koszul duality equivalence is given by

$$M \mapsto M \otimes_{C_{-*}(S^1)} C^*(S^\infty) = \text{Hom}_{C_{-*}(S^1)}(C^*(S^\infty), M).$$

To check that this is an equivalence, it suffices to show that it sends the generator $k \in \text{Coh}(k[\lambda])$ to $k[u] \in \text{Perf}(k[u])$ (up to a shift) and that this is an equivalence on endomorphisms.

A related theorem gives an equivalence

$$\text{Perf}(k[\lambda]) = \text{Tors}(k[u])$$

where $\text{Tors}(k[u])$ consists of modules set-theoretically supported at zero.

Suppose X is a finite CW complex with S^1 -action. (We need some sort of hypothesis to ensure everything is finite.) Then Koszul duality exchanges $C^*(X)$ and $C^*(X \times^{S^1} S^\infty)$. The GKM paper explains how one may extract non-equivariant information from the equivariant cohomology.

2.3 Cohomology and spectral sequences

The Koszul duality theorem we mentioned is a statement about cochains. What happens if we take cohomology?

The fibration $X \hookrightarrow X \times^{S^1} S^\infty \rightarrow BS^1$ gives a Serre spectral sequence abutting to $H_{S^1}^*(X)$. The E_2 page of this spectral sequence has even columns given by $H^*(X)$ and odd columns all 0. Because this is an E_2 page, the differentials increase horizontal degree by 2 and decrease vertical degree by 1. These differentials are the sweeps $d_2 = \lambda_{(1)}$. The differentials on the E_3 page are trivial by degree reasons (they go from even columns to odd columns). On the E_4 page, the differentials are higher sweeps $d_4 = \lambda_{(2)}$. In general, differentials on odd pages are trivial, while differentials on even pages are higher sweeps (sweep, fill, then sweep again). This is secretly what’s powering the computation in the above proof of Koszul duality.

Example 2.3. Applying this method to the Hopf action $S^1 \curvearrowright S^3$, we get $H_{S^1}^*(S^3) = H^*(S^2)$ as expected.

We say that X is *equivariantly formal* if the spectral sequence degenerates at E_2 , i.e. if all sweeps are 0.

Example 2.4. The action $S^1 \curvearrowright S^2$ is equivariantly formal.

2.4 Equivariant localization

We want to think of “ $\text{Spec } C_{S^1}^*(\text{pt})$ ” as something like an affine line. There is a torsion part (at “0”) Koszul dual to $\text{Perfk}[\lambda]$. The non-torsion part is governed by the augmentation module $k = C^*(\text{pt})$ for $k[\lambda]$. Note that this corresponds to S^1 -fixed points. The *Tate construction* gives $\text{Coh}(k[\lambda])/\text{Perf}(k[\lambda]) = \text{Perfk}[u, u^{-1}]$. These are two descriptions of “ $\text{Spec } C_{S^1}^*(\text{pt})$.”

Theorem 2.5 (Equivariant localization). *Let X be a finite CW complex and let $S^1 \curvearrowright X$. Then $C_{S^1}^*(X)[u^{-1}] = C_{S^1}^*(X^{S^1})[u^{-1}]$.*

The modern perspective is that equivariant localization allows us to compute $C_{S^1}^*(X)$ away from the origin. To compute $C_{S^1}^*(X)$ completely, we need to understand its behavior at the origin (i.e. the torsion part) and the gluing data.

3 (9/11) Peter Rowley – G-Spaces and the G-Whitehead Theorem

I wasn’t here for this day. I obtained notes from Swapnil Garg and will try to type them up soon.

4 (9/18) Peter Rowley – Elmendorf’s Theorem

4.1 Model categories

We will take a model-categorical perspective on Elmendorf’s theorem. Recall that a *model category* \mathcal{C} is a category with distinguished (non-full) subcategories of fibrations, cofibrations, and weak equivalences (all satisfying conditions we won’t get into). A trivial (co)fibration is a (co)fibration which is also a weak equivalence.

Definition 4.1. An object $A \in \mathcal{C}$ is *cofibrant* if $\emptyset \rightarrow A$ is a cofibration. Dually, an object $B \in \mathcal{C}$ is *fibrant* if $A \rightarrow \text{pt}$ is a cofibration.

These can be used to capture the homotopy theory of topological spaces:

Example 4.2. The category **Top** of (compactly generated weak Hausdorff) topological spaces has a model structure where:

- Fibrations are Serre fibrations.
- Weak equivalences are weak homotopy equivalences.
- Cofibrations are retracts of relative CW complexes. In particular, the cofibrant objects are precisely the CW complexes.

We may upgrade this to a G-equivariant version as follows.

Example 4.3. Let **GTop** be the category of (cgwh) G-spaces, where we say:

- $f : X \rightarrow Y$ is a fibration if $f^H : X^H \rightarrow Y^H$ is a fibration for all H .
- $f : X \rightarrow Y$ is a weak equivalence if $f^H : X^H \rightarrow Y^H$ is a weak equivalence for all H (equivalently, $\pi_n^H(f)$ is an isomorphism for all n and H).
- Cofibrations are retracts of relative G-CW complexes. In particular, cofibrant objects are G-CW complexes.

The right notion of adjunction between model categories is the following:

Definition 4.4. A *Quillen adjunction* between model categories is an adjunction

$$F : \mathcal{C} \rightleftarrows \mathcal{D} : G$$

such that F preserves cofibrations and trivial cofibrations, or equivalently G preserves fibrations and trivial fibrations.

Definition 4.5. The *homotopy category* $\mathrm{Ho}\mathcal{C}$ is $\mathcal{C}[W^{-1}]$ where W is the class of weak equivalences.

Under suitable hypotheses, we may define left and right derived functors between homotopy categories. These are essentially constructed by performing (co)fibrant replacement and then applying the functors as usual. We may use these to obtain a notion of *equivalence* of model categories.

Definition 4.6. A *Quillen equivalence* is a Quillen adjunction $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ such that $\mathbf{L}F$ and $\mathbf{R}G$ are mutually inverse equivalences $\mathrm{Ho}\mathcal{C} \simeq \mathrm{Ho}\mathcal{D}$.

4.2 Orbit categories

To state Elmendorf’s theorem, we need the following definition.

Definition 4.7. The *orbit category* \mathcal{O}_G is the full subcategory of \mathbf{GTop} on objects of the form G/H .

Note that $\mathrm{Map}_{\mathcal{O}_G}(G/H, G/K)$ corresponds to subconjugacy relations $g^{-1}Hg \subset K$ (since these correspond to maps $eH \mapsto gK$). In particular, we have

$$\mathrm{Map}_{\mathcal{O}_G}(G/H, G/H) = WH := N_G(H)/H.$$

Example 4.8. Let $G = \mathbb{Z}_2$. Then G has exactly two subgroups: e and \mathbb{Z}_2 . The category \mathcal{O}_G has:

- $\mathrm{End}(\mathbb{Z}_2/e) = \mathbb{Z}_2$
- $\mathrm{End}(\mathbb{Z}_2/\mathbb{Z}_2) = \mathrm{pt.}$
- $\mathrm{Map}(\mathbb{Z}_2/e, \mathbb{Z}_2/\mathbb{Z}_2) = \mathrm{pt.}$
- $\mathrm{Map}(\mathbb{Z}_2/\mathbb{Z}_2, \mathbb{Z}_2/e) = \emptyset$.

More generally, G/e has G as its group of automorphisms.

4.3 Elmendorf’s theorem

Define a functor $\psi : \mathbf{GTop} \rightarrow \mathrm{Fun}(\mathcal{O}_G^{\mathrm{op}}, \mathbf{Top})$ by

$$\psi(X)(G/H) = \mathrm{Map}(G/H, X) = X^H.$$

We’d like to say that ψ induces a Quillen equivalence. To accomplish this, we need to define a model structure on $\mathrm{Fun}(\mathcal{O}_G^{\mathrm{op}}, \mathbf{Top})$. We may use the *projective model structure*, where fibrations and weak equivalences are defined pointwise.

Theorem 4.9 (Elmendorf). *The functor ψ is the right adjoint in a Quillen equivalence. The left adjoint is given by $\theta : \mathrm{Fun}(\mathcal{O}_G^{\mathrm{op}}, \mathbf{Top}) \rightarrow \mathbf{GTop}$ defined by $\theta(Y) = Y(G/e)$.*

At the level of ∞ -categories, we may say that ψ is an equivalence of $(\infty, 1)$ -categories. In practice, Elmendorf’s theorem lets us exchange the “algebra of G ” for the “geometry of \mathcal{O}_G .”

Let’s first understand why $\theta : \mathrm{Fun}(\mathcal{O}_G^{\mathrm{op}}, \mathbf{Top}) \rightleftarrows \mathbf{GTop} : \psi$ is a Quillen adjunction. It follows from the definitions that ψ preserves fibrations and trivial fibrations, so it suffices to show that we actually have an adjunction. That is, we need to show

$$\mathrm{Hom}_{\mathbf{GTop}}(Y(G/e), X) \cong \mathrm{Hom}(Y, X^{(-)}).$$

The backwards map is just evaluation at H . The forwards map is given by assembling the maps

$$Y(G/H) \rightarrow Y(G/e)^H \rightarrow X^H.$$

At the level of homotopy categories, $\mathbf{L}\theta$ is equivalent to the bar construction. More precisely: let $M : \mathcal{O}_G \rightarrow \mathbf{Top}$ be the forgetful map, and let $\Phi : \mathrm{Fun}(\mathcal{O}_G^{\mathrm{op}}, \mathbf{Top}) \rightarrow \mathbf{Top}$ be the geometric realization of the bar complex

$$\Phi(X) = |B_\bullet(X, \mathcal{O}_G, M)|.$$

Here $B_\bullet(X, \mathcal{O}_G, M)$ is the simplicial space

$$[n] \mapsto \sqcup_{G/H_{n-1} \rightarrow \dots \rightarrow G/H_0} X(G/H_0) \times M(G/H_{n-1}).$$

Then Φ gives an alternative construction of $\mathbf{L}\theta$ on homotopy categories.

One may understand this explicitly for $G = \mathbb{Z}_2$ but I couldn’t make the pictures for that, sorry!

5 (9/25) Gabriel Beiner – Bredon Cohomology

5.1 Review

Recall that, given a group G , we can construct an *orbit category* \mathcal{O}_G . The objects of \mathcal{O}_G are quotients G/H , and maps are given by

$$\mathrm{Mor}_{\mathcal{O}_G}(G/H, G/K) = \mathrm{Map}^G(G/H, G/K).$$

Elmendorf's theorem gives a (Quillen) equivalence

$$\begin{aligned} \mathbf{GTop} &\simeq \mathrm{Fun}(\mathcal{O}_G^{\mathrm{op}}, \mathbf{Top}) \\ X &\mapsto (G/H \mapsto X^H). \end{aligned}$$

Today we want to discuss a robust notion of cohomology for G -spaces which vastly generalizes the classical Borel cohomology.

5.2 Bredon cohomology

Suppose G is a finite group.

Definition 5.1. A *coefficient system* for G is $M \in \mathrm{Fun}(\mathcal{O}_G^{\mathrm{op}}, \mathbf{Ab})$.

Remark 5.2. If G is a topological group, we'd replace $\mathcal{O}_G^{\mathrm{op}}$ by the homotopy category $\mathbf{h}\mathcal{O}_G^{\mathrm{op}}$.

Example 5.3. For any G , there is a constant coefficient system $\underline{\mathbb{Z}}$ given by $G/H \mapsto \mathbb{Z}$.

Example 5.4. Given a G -space X , we may define coefficient systems:

- $\underline{\pi}_n(X)$ (for $n > 1$) by $G/H \mapsto \pi_n(X^H)$
- $\underline{H}_n(X)$ by $G/H \mapsto H_n(X^H)$.

For a G -CW complex X , let $\mathrm{sk}_n X$ denote the n -skeleton (built inductively out of disks $G/H \times D^k$ for $k < n$). Let

$$\underline{\mathcal{C}}_n(X) = \underline{H}_n(\mathrm{sk}_n X, \mathrm{sk}_{n-1} X; \mathbb{Z})$$

This sends G/H to $H_n(\mathrm{sk}_n X^H, \mathrm{sk}_{n-1} X^H) = \mathcal{C}_n^{\mathrm{CW}}(X^H)$. Allowing n to vary, we get cellular chain complexes $\underline{\mathcal{C}}_\bullet(X)(G/H)$ for all H .

Definition 5.5. Let X be a G -CW complex and M a coefficient system for G . The *Bredon cohomology* of X with coefficients in M is

$$H_G^n(X; M) = H^n(\mathrm{Hom}_{\mathrm{Fun}(\mathcal{O}_G^{\mathrm{op}}, \mathbf{Ab})}(\underline{\mathcal{C}}_\bullet(X), M))$$

The *Bredon homology* of X with coefficients in M is

$$H_n^G(X; M) = \underline{\mathcal{C}}_\bullet(X) \otimes_{\mathcal{O}_G} M$$

where $\otimes_{\mathcal{O}_G}$ is a coend (sort of like the usual tensor product).

Remark 5.6. David suggested that we can instead take an ∞ -categorical perspective. Let's think of the orbit category as (the exit-path category of) a stratified space with strata $B(WH) = B(N_G(H)/H)$ for $H \subset G$. The coefficient systems are then constructible (co?)sheaves on this stratified space. The G -CW complexes X above give spaces \mathcal{X} over our stratified space (with fiber over $B(WH)$ given by X^H) Bredon cohomology computes the cohomology of the pullback of a constructible sheaf to \mathcal{X} .

5.3 Examples

Example 5.7. Let's consider \mathbb{Z}_2 acting on S^2 via rotation by π . We'll compute $H_{\mathbb{Z}_2}^*(S^2; \mathbb{Z})$.

To perform this computation, let's give S^2 the \mathbb{Z}_2 -CW structure with:

- Two 0-cells $\mathbb{Z}_2/\mathbb{Z}_2 \times D^0$ (the north and south poles)
- One 1-cell $\mathbb{Z}_2/e \times D^1$ (corresponding to the prime meridian and its opposite, which \mathbb{Z}_2 permutes)
- One 2-cell $\mathbb{Z}_2/e \times D^2$ (corresponding to the eastern and western hemispheres, which \mathbb{Z}_2 permutes).

The category $\mathcal{O}_{\mathbb{Z}_2}$ was described in the last talk. We may describe $\mathcal{C}_\bullet(X)$ as follows:

- For e -fixed points, the cellular chain complex $\mathcal{C}_\bullet(X^e)$ is

$$0 \longrightarrow \mathbb{Z}^2 \xrightarrow{d_2} \mathbb{Z}^2 \xrightarrow{d_1} \mathbb{Z}^2 \longrightarrow 0$$

The differentials are $d_2(x, y) = (x - y, y - x)$ and $d_1(u, v) = (u + v, -u - v)$. Here \mathbb{Z}_2 (acting on \mathbb{Z}_2/e by outer automorphisms) acts trivially on \mathcal{C}_0 and swaps the factors of \mathcal{C}_1 and \mathcal{C}_2 .

- For \mathbb{Z}_2 -fixed points, the cellular chain complex $\mathcal{C}_\bullet(X^{\mathbb{Z}_2})$ is

$$0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathbb{Z}^2 \longrightarrow 0.$$

The automorphism group of $\mathbb{Z}_2/\mathbb{Z}_2$ is trivial, so we don't need to care about the group actions. However, we do want to remember the homomorphism from this complex to the above complex (given by the identity on $\mathbb{Z}^2 = \mathcal{C}_0$).

To compute Bredon cohomology, we take $\text{Hom}(\mathcal{C}_\bullet(X), \underline{\mathbb{Z}})$, where

$$\underline{\mathbb{Z}}(\mathbb{Z}_2/H) = \begin{cases} \mathbb{Z} \text{ with trivial action} & H = e \\ \mathbb{Z} & H = \mathbb{Z}_2. \end{cases}$$

Some computation (which I didn't entirely catch) shows that $\text{Hom}(\mathcal{C}_\bullet(X), \underline{\mathbb{Z}})$ is

$$0 \longrightarrow \mathbb{Z}^2 \xrightarrow{\delta_1} \mathbb{Z} \xrightarrow{\delta_2} \mathbb{Z} \longrightarrow 0.$$

Here the middle \mathbb{Z} is generated by ϕ satisfying $\phi(u, v) = u + v$. The differential δ_1 satisfies $\delta_1(a, b) = a - b$, and $\delta_2 = 0$. Thus

$$H_{\mathbb{Z}_2}^\bullet(S^2; \underline{\mathbb{Z}}) = \begin{cases} \mathbb{Z} & \bullet = 0, 2 \\ 0 & \text{else.} \end{cases}$$

6 (10/2) Gabriel Beiner – Continued

Recall that, to define Bredon cohomology, we choose a coefficient system $M \in \text{Fun}(\mathcal{O}_G^{\text{op}}, \text{Ab})$. For a G -CW complex X , we define

$$H_G^\bullet(X, M) = H^n(\text{Hom}_{\text{Fun}(\mathcal{O}_G^{\text{op}}, \text{Ab})}(\underline{\mathcal{C}}_\bullet^{\text{CW}}(X), M)).$$

Our goal today is to compute some more examples.

Recall that we can think of $\text{Fun}(\mathcal{O}_G^{\text{op}}, \text{Ab})$ as an exit path category. For $G = \mathbb{Z}_2$, the large “open” stratum corresponds to \mathbb{Z}_2/e , while the small “closed” stratum corresponds to \mathbb{Z}_2 .

One advantage of this approach is that we may choose M to recover different features of X : we can compute cohomology of X , X/G , or X^G through suitable choices of coefficient.

6.1 An example

Example 6.1. Let $G = \mathbb{Z}_4$ (with generator σ), and let X be the star-shaped graph with six edges (it looks like $*$). We may view X as (a union of segments on) the axes in xyz -space, and we let G rotate the xy -plane and swap the positive and negative z -axes. Then X has a natural G -CW complex structure, with:

- 0-cells:
 - A basepoint $\gamma = \mathbb{Z}_4/\mathbb{Z}_4$.
 - An orbit $\{\beta_1, \beta_2\} = \mathbb{Z}_4/\mathbb{Z}_2 \times D^0$ corresponding to endpoints on the z -axis.
 - An orbit $\{\alpha_i\} = \mathbb{Z}_4/\mathbf{e} \times D^0$ corresponding to endpoints on the x - and y -axes.
- 1-cells:
 - An orbit $\{b_1, b_2\} = \mathbb{Z}_4/\mathbb{Z}_2 \times D^1$ corresponding to segments on the z -axis.
 - An orbit $\{a_i\} = \mathbb{Z}_4/\mathbb{Z}_4 \times D^1$ corresponding to segments on the x - and y -axes.

The cellular chain complex here is given by:

- \mathbb{Z}_4/\mathbf{e} goes to $\mathbb{Z}^2 \oplus \mathbb{Z}^4 \rightarrow \mathbb{Z} \oplus \mathbb{Z}^2 \oplus \mathbb{Z}^4$. The differential is

$$\begin{bmatrix} -1 & -1 & -1 & -1 & -1 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

- $\mathbb{Z}_4/\mathbb{Z}_2$ goes to $\mathbb{Z}^2 \rightarrow \mathbb{Z} \oplus \mathbb{Z}^2$. The differential is

$$\begin{bmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- $\mathbb{Z}_4/\mathbb{Z}_4$ goes to $0 \rightarrow \mathbb{Z}$.

All of the transition maps (going from chain complexes below to chain complexes above) are given by the obvious inclusions. Here the stabilizers act by rotation on each copy of \mathbb{Z}^4 , swaps on each copy of \mathbb{Z}^2 , and the identity on each copy of \mathbb{Z} .

If we use the constant coefficient system \mathbb{Z} , we get the total cochain complex

$$\mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \oplus \mathrm{Hom}^{\mathbb{Z}_2}(\mathbb{Z}^2, \mathbb{Z}) \oplus \mathrm{Hom}^{\mathbb{Z}_4}(\mathbb{Z}^4, \mathbb{Z}) \rightarrow \mathrm{Hom}^{\mathbb{Z}_2}(\mathbb{Z}^2, \mathbb{Z}) \oplus \mathrm{Hom}^{\mathbb{Z}_4}(\mathbb{Z}^4, \mathbb{Z}).$$

This may be identified with

$$\mathbb{Z}^3 \xrightarrow{\delta} \mathbb{Z}^2$$

where

$$\delta = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Since δ has full rank, we get

$$H_{\mathbb{Z}_4}^\bullet(X) = H^\bullet(X/\mathbb{Z}_4) = \begin{cases} \mathbb{Z} & \bullet = 0 \\ 0 & \text{otherwise.} \end{cases}$$

6.2 Another example

We saw above that taking cohomology with respect to the constant coefficient system gives the cohomology of the usual quotient (not the Borel quotient). Let's see what happens for more interesting coefficient systems.

Example 6.2. Let \mathbb{Z}_2 act on S^2 by reflection across the equator. Here we have a G-CW structure with:

- 0-cell $\mathbb{Z}_2/\mathbb{Z}_2 \times D^0$ sitting on the equator
- 1-cell $\mathbb{Z}_2/\mathbb{Z}_2 \times D^0$ sitting on the equator (connecting the 0-cell to itself)
- 2-cell $\mathbb{Z}_2/e \times D^2$ giving the hemispheres.

The cellular chain complex sends:

- \mathbb{Z}_2/e to $\mathbb{Z}^2 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}$.
- $\mathbb{Z}_2/\mathbb{Z}_2$ to $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}$.

The transition maps between orbits are inclusions. Here \mathbb{Z}_2 acts by swapping on the copy of \mathbb{Z}^2 and acts trivially on the copies of \mathbb{Z} .

Consider the coefficient system $\mathbb{Z} \rightarrow \mathbb{Z}[i]$, where \mathbb{Z}_2 acts on $\mathbb{Z}[i]$ by conjugation. This is the sum of the constant coefficient system and a copy of $i\mathbb{Z}$ supported at $\mathbb{Z}_2/e \in \mathcal{O}_G$. We can compute that the corresponding cochain complex is

$$\mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow \mathbb{Z}^2$$

and by understanding the maps we see

$$H_{\mathbb{Z}_2}^\bullet(S^2; M) = \begin{cases} \mathbb{Z} & \bullet = 0, 2 \\ 0 & \text{otherwise.} \end{cases}$$

6.3 Some last remarks

Bredon cohomology may be characterized by Eilenberg–Steenrod axioms similarly to ordinary cohomology. However, the dimension axiom is replaced by the axiom that

$$H_G^\bullet(G/H, M) = \begin{cases} M(G/H) & \bullet = 0 \\ 0 & \text{otherwise} \end{cases}$$

We can define more general cohomology theories by relaxing the dimension axiom.

However, in general we want to do better: cohomology should be graded by the representation ring $RO(G)$ rather than just \mathbb{Z} . This can be done using *Mackey functors*.

The analogue of the universal coefficient theorem for Bredon cohomology gives a spectral sequence

$$\text{Ext}^{p,q}(\underline{\mathcal{C}}_\bullet^{CW}(X), M) \Rightarrow H^{p+q}(X; M).$$

7 (10/9) Gabriel Beiner – Smith Theory

Our goal today is to discuss the following result in the context of equivariant homotopy theory.

Theorem 7.1. *Let G be a finite p -group. Suppose X is a finite G -CW complex which is an \mathbb{F}_p -cohomology n -sphere (i.e. $H^\bullet(X; \mathbb{F}_p) = H^\bullet(S^n; \mathbb{F}_p)$). Then X^G is either empty or a \mathbb{F}_p -homology for $m \leq n$.*

One can also show the following claims:

- If p is odd, then $n - m$ is even.
- If p is odd and n is even, then X^G is nonempty.

We'll follow an approach similar to Smith's original proof but with some simplifications using Bredon cohomology.

7.1 Proof of the Smith inequality

If $H \subset G$ is a normal subgroup, then we have $X^G = (X^H)^{G/H}$. Arguing by induction, we reduce to proving the result in the case $G = \mathbb{Z}/p$. We don't actually need to assume that X is an \mathbb{F}_p -homology sphere until the end of the proof, so let's drop this assumption for the moment.

The idea of the proof is to find coefficient systems L , M , and N such that:

- $H^\bullet(X; L) = \tilde{H}^\bullet((X/X^G)/G; \mathbb{F}_p)$,
- $H^\bullet(X; M) = H^\bullet(X; \mathbb{F}_p)$, and
- $H^\bullet(X; N) = H^\bullet(X^G; \mathbb{F}_p)$.

By the Eilenberg–Steenrod axioms, it will suffice to check that these results hold when X is a 0-cell.

The orbit category for \mathbb{Z}/p is $\mathbb{Z}/p \rightarrow \mathbf{e}$, where \mathbb{Z}/p has automorphism group \mathbb{Z}/p . We'll take:

- L is $\mathbb{F}_p \leftarrow 0$, where \mathbb{Z}/p acts trivially on the first factor.
- M is $\mathbb{F}_p[G] \leftarrow \mathbb{F}_p$.
- N is $0 \leftarrow \mathbb{F}_p$.

To check the above homology equivalences, we plug in $X = G \times D^0$ and $X = G/G \times D^0$.

Consider the augmentation $\mathbb{F}_p[G] \rightarrow \mathbb{F}_p$ (given by $g \mapsto 1$), and let $I = \ker(\mathbb{F}_p[G] \rightarrow \mathbb{F}_p)$ be the augmentation ideal. This gives rise to coefficient systems $I^n = (I^n \leftarrow 0)$. We have two natural short exact sequences of coefficient systems:

$$0 \longrightarrow I \longrightarrow M \longrightarrow L \oplus N \longrightarrow 0$$

and

$$0 \longrightarrow L \longrightarrow M \longrightarrow I \oplus N \longrightarrow 0.$$

To understand the latter short exact sequence, note that:

- At \mathbf{e} , the SES becomes $0 \rightarrow \mathbb{F}_p \rightarrow \mathbb{F}_p$.
- At \mathbb{Z}/p , the SES becomes $((t-1)^{p-1}) \rightarrow \mathbb{F}_p[t]/(t^p-1) \xrightarrow{\cdot(t-1)} (t-1)$.

Both of these are exact, so the sequence is an SES.

Remark 7.2. David pointed out that these relate to (and partially generalize) the short exact sequence building $\mathbb{F}_2[\Sigma_2]$ as a self-extension of the trivial representation. Note that this short exact sequence does not split (because we are working with \mathbb{F}_2 coefficients).

The SESes of coefficient systems give long exact sequences (where coefficients not mentioned are implicitly in \mathbb{F}_p)

$$\dots \longrightarrow H_G^q(X; I) \longrightarrow H^q(X) \longrightarrow \tilde{H}^q((X/X^G)/G) \oplus H^q(X^G) \longrightarrow \dots$$

and

$$\dots \longrightarrow \tilde{H}^q((X/X^G)/G) \longrightarrow H^q(X) \longrightarrow H_G^q(X; I) \oplus H^q(X^G) \longrightarrow \dots$$

Let's focus on Betti numbers for simplicity. Write:

- $a_q = \dim \tilde{H}^q((X/X^G)/G)$,
- $b_q = \dim H^q(X)$,
- $\bar{a}_q = H_G^q(X; I)$, and
- $c_q = \dim H^q(X^G)$.

Our long exact sequences give inequalities $a_q + c_q \leq b_q + \bar{a}_{q+1}$ and $\bar{a}_q + c_q \leq b_q + a_{q+1}$. Combining these gives the key inequality (for $q, r \geq 0$ and r odd):

$$a_q + c_q + c_{q+1} + \cdots + c_{q+r} \leq b_q + b_{q+1} + \cdots + b_{q+r} + a_{q+r+1}.$$

Starting at $q = 0$ and continuing to $r \gg 0$, this gives $\sum_i c_i \leq \sum_i b_i$. In other words,

$$\sum_i \text{rk } H^i(X^G; \mathbb{F}_p) \leq \sum_i \text{rk } H^i(X; \mathbb{F}_p).$$

This is the ‘‘Smith inequality’’ and does not use anything other than the fact that X is a finite G -CW complex.

7.2 Finishing the proof of Smith’s theorem

Now taking X to be an \mathbb{F}_p -homology sphere, we see that

$$\sum_i \text{rk } H^i(X^G; \mathbb{F}_p) \leq 2.$$

To see that X^G is an \mathbb{F}_p -homology sphere, it suffices to show that $\sum_i \text{rk } H^i(X^G; \mathbb{F}_p) \neq 1$.

To show this, first note that $I^n/I^{n+1} \cong \mathbb{F}_p$. From the short exact sequence

$$0 \longrightarrow I^{n+1} \longrightarrow I^n \longrightarrow L \longrightarrow 0,$$

we get $\chi(X; I^n) = \chi(X; I^{n+1}) + \chi(X; L)$. Summing these up gives

$$\sum_{i=1}^{p-1} \chi(X; I^i) = \sum_{i=1}^{p-1} (\chi(X; I^{i+1}) + \chi(X; L))$$

and hence (using $I^p = 0$)

$$\chi(X; I) = \chi(X; I^p) + (p-1)\chi(X; L) = (p-1)\chi(X; L).$$

Using $\chi(X; M) = \chi(X; I) + \chi(X; L) + \chi(X; N)$, we obtain

$$\chi(X; M) = \chi(X; N) + p\chi(X; L).$$

We may rewrite this as $\chi(X) = \chi(X^G) + p\tilde{\chi}((X/X^G)/G)$. This forces $\chi(X) \equiv \chi(X^G) \pmod{p}$. (Again, this claim doesn’t need X to be an \mathbb{F}_p -homology sphere.) We never have $1 \equiv 2 \pmod{p}$, so we must have $\sum_i \text{rk } H^i(X^G; \mathbb{F}_p) \neq 1$.

7.3 Harnack’s curve theorem

Here’s an application of Smith theory to algebraic geometry.

Theorem 7.3 (Harnack). *Let $X_{\mathbb{R}}$ be the real locus of a smooth projective plane curve of genus g . Then the number of connected components of $X_{\mathbb{R}}$ is bounded above by $g + 1$.*

Proof. We have $X_{\mathbb{R}} = (X_{\mathbb{C}})^{\mathbb{Z}/2}$, so

$$\sum_i \text{rk } H^i(X_{\mathbb{R}}; \mathbb{F}_2) \leq \sum_i \text{rk } H^i(X_{\mathbb{C}}; \mathbb{F}_2).$$

The former is twice the number of connected components of $X_{\mathbb{R}}$ (which are copies of S^1), and the latter is $2 + 2g$. \square