GRT Seminar Fa23-Sp24 Notes

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Abstract

The seminar covers Ben-Zvi–Sakellaridis–Venkatesh, "Relative Langlands Duality."

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1 8/31 (David Nadler) – ???

I missed this day. If you have good notes from this day, send them to me and I will type them up.

2 9/7 (Elliot Kienzle) – Hamiltonian G-Spaces and Quantization

The original Langlands program studies a duality of Lie groups $G \leftrightarrow G^\vee$. Relative Langlands seeks to upgrade this to a duality of Hamiltonian G-actions $(G \curvearrowright M) \leftrightarrow (G^\vee \curvearrowright M^\vee)$. This is proposed for hyperspherical varieties M, of which a typical example is $M = T^*X$ for X a spherical variety.

We can approach and motivate this using quantization. Start by considering the action of G on $L^2(X)$ for X a spherical variety (discussed in an earlier paper of Sakellaridis-Venkatesh discussing "harmonic analysis on spherical varieties").

2.1 Symplectic geometry and quantization

The original motivation for symplectic geometry comes from classical mechanics. Suppose that we have a particle moving in $\mathbb{R}^n$. We can capture the data of the position and momentum using the cotangent bundle $T^*\mathbb{R}^n$. By Newton's second law, the time evolution of the particle is described by (the flow along) a vector field on $T^*\mathbb{R}^n$.

We can generalize this to a symplectic manifold (M, ω), which is a manifold M with a closed, non-degenerate 2-form ω. To make this easier to work with, we can fix a metric $\langle , \rangle$ on M and write $\omega(x, y) = \langle x, Jy \rangle$ where $J^2 = -1$ (i.e. J^2 is an almost complex structure). We think of J^2 as "multiplication by $-i$."

Given a Hamiltonian $H \in \mathcal{C}^\infty(M)$, we obtain a Hamiltonian vector field $X_H = J\nabla H$. More invariantly, we can define X_H via the formula $\omega(X_H, -) = dH$.

Moving to quantum mechanics, we view a particle in $\mathbb{R}^n$ as a $\mathbb{C}$-valued function ψ on $\mathbb{R}^n$ (not $T^*\mathbb{R}^n$). In this case, the Hilbert space is $L^2(\mathbb{R}^n)$. A free particle evolves according to Schrödinger's equation:

$$i\dot{\psi} = \Delta\psi.$$

We can summarize the classical and quantum pictures in the following table.

	Classical	Quantum
State Space	Symplectic manifold (M, ω)	Hilbert space $\mathcal{H}$
Observables	$f \in \mathcal{C}^\infty(M)$	Bounded operators $A \in \mathrm{End}(\mathcal{H})$
Evolution	Vector fields X_H for $H \in \mathcal{C}^\infty(M)$	Unitary operators $U(t) = e^{itA}$ for $A \in \mathrm{End}(\mathcal{H})$
Lie Algebra of observables	Poisson bracket $\{f, g\} = X_f(g)$	Commutator $[A, B]$

To obtain a quantum system from a classical system (heuristically), we pass from nonlinear evolution of points in T^*M to linear evolution of functions on M. (This linearity is forced on us by our desire to have superposition of states.) The dream of quantization is, given a symplectic manifold (M, ω), to construct a Lie algebra homomorphism $(\mathcal{C}^\infty(M), \{,\}) \to (\mathrm{End}(\mathcal{H}), [,])$ for some Hilbert space $\mathcal{H}$. Unfortunately, this is impossible to do consistently / functorially in general. However, there are some cases in which we can get good answers.

We will focus on geometric quantization, which behaves (loosely) as follows:

- For $M = T^*X$, we obtain $\mathcal{H} = L^2(X)$.
- For M a compact Kähler manifold, we obtain $\mathcal{H} = H^0(M, \mathcal{L})$ for some line bundle $\mathcal{L}$ on M.

2.2 G-Spaces

We want to incorporate symmetries into the previous picture. Suppose G is a compact Lie group / reductive algebraic group (depending on context). We say a symplectic G-space is a symplectic manifold (M, ω) with G-action preserving ω. We can hope to quantize this to a linear representation $G \curvearrowright \mathcal{H}$. (There are subtleties that arise here – for geometric quantization, we would like a G-equivariant polarization.)

In general, it is better to consider Hamiltonian G-actions, where $\mathfrak{g}$ acts by Hamiltonian vector fields. This allows us to construct a moment map $\mu : M \to \mathfrak{g}^*$ which is equivariant (with respect to the coadjoint action on $\mathfrak{g}^*$).

Let us start by understanding the coadjoint action $G \curvearrowright \mathfrak{g}^*$ using Kirillov's "orbit method." For $\alpha \in \mathfrak{g}^*$, consider the coadjoint orbit $\mathcal{O}_\alpha$. This $\mathcal{O}_\alpha$ turns out to be a symplectic manifold (with "Kirillov-Kostant-Souriau" / "KKS" form) with Hamiltonian G-action, and the moment map $\mathcal{O}_\alpha \to \mathfrak{g}^*$ is just the inclusion.

Example 2.1. Consider $G = \mathrm{SO}(3)$. The coadjoint action is just $\mathrm{SO}(3)$ acting on $\mathbb{R}^3$ by rotations. Thus the generic orbits are spheres S^2.

The orbits $\mathcal{O}_\alpha$ will look like generalized flag manifolds, and conversely every generalized flag manifold arises in this way. (This is the first place where our compactness hypothesis comes in).

Proposition 2.2. *A coadjoint orbit $\mathcal{O}_\alpha$ is quantizable if and only if α is in the orbit of an integer point of the root lattice $\mathfrak{t}^*_{\mathbb{Z}} \subset \mathfrak{t}^*$ (viewed as a subspace of $\mathfrak{g}^*$ via the Killing form).*

Example 2.3. Continuing on with our $\mathrm{SO}(3)$ example, we see that a symplectic sphere is quantizable if and only if it has integer area.

In these cases, the quantization of $\mathcal{O}_\alpha$ is $H^0(\mathcal{O}_\alpha, \mathcal{L}_\alpha)$ where $\mathcal{L}_\alpha$ is the line bundle corresponding to the character α. By the Borel-Weil theorem, $H^0(\mathcal{O}_\alpha, \mathcal{L}_\alpha)$ is the irrep V_α of G with highest weight $\mathcal{L}_\alpha$.

We can summarize this in the following table:

Classical	Quantum
Symplectic action $G \curvearrowright M$	Representation $G \curvearrowright \mathcal{H}$
Coadjoint orbit $\mathcal{O}_\alpha$	Highest weight representation E_α

3 9/14 (Elliot Kienzle) – Continued

3.1 Symplectic reduction

Suppose we have a Hamiltonian action $G \curvearrowright M$. This yields a G-equivariant moment map $\mu : M \to \mathfrak{g}^*$, and the image of μ will necessarily be a collection of coadjoint orbits $\mathcal{O}_\alpha$. We can use these orbits to decompose M.

First consider the orbit $\mathcal{O}_0 = \{0\}$. We note that $\mu^{-1}(0)$ is G-invariant, so we can consider the quotient $\mu^{-1}(0)/G$. We define this to be the *symplectic quotient*: $M//G := \mu^{-1}(0)/G$.

We will assume that 0 is a regular value of the moment map and that G acts on $\mu^{-1}(0)$ freely. We can drop these assumptions if we consider things in a suitable derived / stacky sense.

Theorem 3.1 (Marsden-Weinstein)**.** *The symplectic quotient* $M//G$ *carries a natural symplectic structure.*

Example 3.2. If X is a (not necessarily symplectic) manifold with a G-action, then $T^*X//G = T^*(X/G)$.

Example 3.3. Let $M = T^*\mathbb{R}^2 \cong \mathbb{C}^2$. This has a U(1)-action via

$$e^{i\theta}(z_1, z_2) = (e^{i\theta}z_1, e^{i\theta}z_2).$$

We can define a (shifted) moment map $\mu : \mathbb{C}^2 \to \mathbb{R}$ via

$$\mu(z_1, z_2) = |z_1|^2 + |z_2|^2 - 1.$$

Then $\mathbb{C}^2// \mathrm{U}(1) = S^3/\mathrm{U}(1) = S^2 = \mathbb{P}^1$ (consider the Hopf fibration).

Morally, we should think of every symplectic manifold as a symplectic reduction of a (possibly infinite-dimensional) affine space.

Note that

$$\dim M//G = \dim M - 2\dim G.$$

The slogan is that "in symplectic geometry, groups act twice."

Theorem 3.4 (Guillemin-Sternberg, etc.)**.** *The geometric quantization of a symplectic quotient satisfies*

$$\mathcal{H}(M//G) = \mathcal{H}(M)^G,$$

where the right hand side is the subspace of G*-invariant vectors in* G*.*

We can also define the symplectic reduction along any coadjoint orbit $\mathcal{O}_\alpha$ as $M//_\alpha G = \mu^{-1}(\mathcal{O}_\alpha)/G$. This gives a decomposition of M as

$$M = \cup_{\alpha\in\mu(M)}\mu^{-1}(\mathcal{O}_\alpha) = \cup_{\alpha\in\mu(M)}(G\text{-bundles over } M//_\alpha G),$$

at least if we avoid critical points.

Elliot has some fancy art of this decomposition.

Let's focus on the simplest possible case:

Definition 3.5. A Hamiltonian G-space M is *multiplicity-free* if $\dim M//_\alpha G = 0$ for all α.

Remark 3.6. If M is compact, then a Morse theory argument shows that $M//_\alpha G = \mathrm{pt}$ for all α.

Here are some relevant examples.

Example 3.7. For a coadjoint orbit $\mathcal{O}_\alpha$, we have $\mathcal{O}_\alpha//_\alpha G = \mathrm{pt}$, so coadjoint orbits are multiplicity-free. Here we are ignoring stacky / derived quotients even though the action is typically nonfree.

Example 3.8. Consider $\mathbb{P}^1$ with U(1) acting by rotation. Then μ is the height function on $\mathbb{P}^1 = S^2$. If the top height is 1 and the bottom height is -1, then $\mu^{-1}(1)$ and $\mu^{-1}(1)$ are both points. For any $x \in (-1, 1)$, we have $\mu^{-1}(x) = S^1$ and therefore $\mathbb{P}^1//_x \mathrm{U}(1) = \mathrm{pt}$. Thus this action is multiplicity-free.

Example 3.9. Let $\mathrm{U}(1)^2$ acts on $\mathbb{P}^2$ (extending the standard action on $\mathbb{A}^2 \subset \mathbb{P}^2$). The fibers of the moment map over points in the interior of $\mu(M)$ are 2-tori, which degenerate to circles on the boundary lines of $\mu(M)$ and points at the corners of $\mu(M)$.

A non-example is given by the U(1) action on $\mathbb{C}^2$ from earlier in the lecture. This is an obvious non-example because the dimension of the symplectic quotient is nonzero. The slogan is that "multiplicity-free manifolds have maximal symmetry."

3.2 (David) – Interlude

For a Lie group G, we have $\mathsf{T}^*\mathsf{G} = \mathsf{G} \times \mathfrak{g}^*$. Consider $\mathsf{G} \curvearrowright \mathsf{T}^*\mathsf{G}$ induced by the adjoint action of G on itself. We obtain a moment map $\mu : \mathsf{G} \times \mathfrak{g}^* \to \mathfrak{g}^*$ given by the formula

$$\mu(g, x) = \mathrm{Ad}_g(x) - x.$$

Then $\mu^{-1}(0) = \{(g, x) \in \mathsf{G} \times \mathfrak{g}^* \mid g \in \mathsf{G}_x\}$, where G_x is the centralizer of $x \in \mathsf{G}$.

The multiplicity-freeness property for a general Hamiltonian G-space M can be understood as the requirement that the centralizers G_x act transitively on the preimages $\mu^{-1}(x)$.

It is a good exercise to classify multiplicity-free Hamiltonian G-spaces for $\mathsf{G} = \mathrm{U}(1)$ or $\mathsf{G} = \mathrm{SU}(2)$.

3.3 (Elliot) – A few last words

Multiplicity-freeness has a useful consequence for quantization: if M is multiplicity-free, then each highest weight representation E_α appears in $\mathcal{H}(\mathsf{M})$ at most once. In fact, E_α will appear if and only if $\mathcal{O}_\alpha \in \mu(\mathsf{M})$.

We will be interested in hyperspherical varieties as a large family of multiplicity-free symplectic manifolds. More on that next time!

4 9/21 (Mark Macerato) – Hyperspherical varieties

4.1 (David) – Multiplicity-freeness

There may have been minor errors in the discussion last time, but the basic ideas were right. Suppose for simplicity that T is an *abelian* Lie group, and consider the cotangent bundle $\mathsf{T}^*\mathsf{T} \cong \mathsf{T} \times \mathfrak{t}^*$. The moment map for the translation action of T on itself is the projection $\mathsf{T} \times \mathfrak{t}^* \to \mathfrak{t}^*$. This gives a (trivial) family of abelian groups over $\mathfrak{t}^*$.

If we have another Hamiltonian T-space X, we obtain a moment map $\mu_\mathsf{X} : \mathsf{X} \to \mathfrak{t}^*$. We can view our family of abelian groups over $\mathfrak{t}^*$ as acting fiberwise on X. The multiplicity-freeness condition is requiring that the orbits of this action are fiberwise discrete.

This story still works for non-abelian G (but you have to be careful about left versus right actions). In this case, the fiber over $\nu \in \mathfrak{g}^*$ will be given by the stabilizer G_ν.

Example 4.1. We can describe Hamiltonian $\mathrm{U}(1)$-spaces as lying over $\mathfrak{u}(1) \cong \mathbb{R}$. The multiplicity-freeness condition implies that the fibers are (disjoint unions of) copies of S^1 and points. For example, we can consider the height function on the sphere, or the projection of a cylinder $\mathsf{S}^1 \times \mathbb{R}$, or many related examples – these all give multiplicity-free Hamiltonian $\mathrm{U}(1)$-spaces.

Example 4.2. If we take $\mathsf{G} = \mathrm{SU}(2)$, we obtain a similar (but distinct) picture because $\mathfrak{su}(2)/\mathrm{SU}(2) \cong [0, \infty)$ (the $\mathrm{SU}(2)$-orbits in $\mathfrak{su}(2)$ are spheres). The fibers of $\mathsf{T}^*\mathrm{SU}(2) \to \mathfrak{su}(2)$ are $\mathrm{SU}(2)$ (over 0) and S^1 (over points in $(0, \infty)$). We can analyze multiplicity-free Hamiltonian G-spaces as before.

In general, the left action $\mathsf{G} \curvearrowright \mathsf{T}^*\mathsf{G}$ (via $g \cdot (h, \nu) = (gh, \mathrm{Ad}_g\,\nu)$) is not multiplicity-free. Consider the moment map $\mathsf{T}^*\mathsf{G} \cong \mathsf{G} \times \mathfrak{g}^* \to \mathfrak{g}^*$ given by projection (this depends on how we trivialize $\mathsf{T}^*\mathsf{G}$). For a coadjoint orbit $\mathcal{O}$, the preimage $\mu^{-1}(\mathcal{O})$ is $\mathsf{G} \times \mathcal{O}$. The multiplicity-freeness here reduces to the question of whether the action $\mathsf{G}_\nu \curvearrowright \mathsf{G}$ has discrete orbits. This is not true in general (see e.g. the $\mathrm{SU}(2)$ example above), proving the claim.

A later clarification: Really, we should think of $\mathsf{T}^*\mathsf{G} \rightrightarrows \mathfrak{g}^*$ as a groupoid, where the "source" and "target" maps are μ_L and μ_R (the moment maps for the left / right actions, respectively). Given a groupoid, we can obtain a group scheme (encapsulating the "automorphism groups of points") as a fiber product, e.g.

$$\begin{tikzcd} \{[X, g] = 0\} \arrow[r] \arrow[d] & \mathsf{T}^*\mathsf{G} \arrow[d] \\ \Delta \arrow[r] & \mathfrak{g}^* \times \mathfrak{g}^*. \end{tikzcd}$$

Understanding things from this perspective clears up the difficulties with left / right actions.

Hamiltonian G-spaces $(\mathsf{M} \to \mathfrak{g}^*)$ will be module objects for this groupoid.

4.2 (Mark) – Hyperspherical varieties

We will change settings to algebraic geometry (following section 3 of Ben-Zvi–Sakellaridis-Venkatesh). Fix an algebraically closed field k of characteristic zero (e.g. $\mathbb{C}$ or $\overline{\mathbb{Q}_\ell}$). Let G be a connected reductive group over k.

Recall that a spherical variety is a normal G-variety X such that there exists a Borel subgroup $B \subset G$ with an open orbit in X. We can rephrase the last condition without picking a Borel: we require that G has an open orbit on $X \times \mathrm{Fl}_G$. If X is affine, this is equivalent to requiring that the coordinate ring $k[X]$ is multiplicity-free as a G-module.

Example 4.3 ("Group case"). Let H be a connected reductive group and $G = H \times H$. For $X = H$ and $G \hookrightarrow X$ via $(h_1, h_2) \cdot h = h_1 h h_2^{-1}$, H is a spherical variety.

If we fix a Borel $B \subset H$, we have a unipotent subgroup $U \subset B$ and a surjection $B \twoheadrightarrow T = B/U$. By Levi's theorem, this splits, giving $T \hookrightarrow B \subset G$. We get a vector space decomposition $\mathfrak{g} = \mathfrak{u}^- \oplus \mathfrak{t} \oplus \mathfrak{u}$. Consider the open embedding $U^- \times B \to H$ given by $(u, b) \mapsto ub$. The Borel subgroup $B^- \times B \subset G$ has an open orbit in H. This leads to a Bruhat decomposition $H = \sqcup_{w \in W} BwB$.

We can obtain Bruhat decompositions for more general spherical varieties. This is a rich theory that has been worked out by several authors (Knapp, Brion, etc.). But let's move on to hyperspherical varieties, which give a symplectic point of view.

Instead of a spherical variety X, let us consider $M = T^*X$ with the moment map $\mu : T^*X \to M$. For simplicity, we will assume our base spherical variety X is affine, smooth, and irreducible. In this case M is *coisotropic*, which means that the G-invariant function field $k(M)^G$ is Poisson-commutative.

Another way of saying this is as follows. Let $\mathfrak{c} = \mathfrak{g}^*//G \cong \mathfrak{g}//G$ be the "Chevalley space." Letting $\eta \in M$ be the generic point, we obtain a Stein factorization $M \to \mathfrak{c}_M \to \mathfrak{c}$. The map $\tilde{\mu} : M \to \mathfrak{c}_M$ has connected generic fiber, and $\mathfrak{c}_M \to \mathfrak{c}$ is finite. The second definition of "coisotropic" is that the group $G_{K(\mathfrak{c}_M)}$ acts on $M_{K(\mathfrak{c}_M)}$ with an open (hence dense) orbit.

Theorem 4.4 (Losev). *If M is a smooth Hamiltonian G-variety, then* all *of the fibers of $\tilde{\mu} : M \to \mathfrak{c}_M$ are connected.*[1]

A third definition of coisotropic is that the generic G-orbit on M is coisotropic in the usual sense.

"Coisotropic" is the algebraic geometry version of "multiplicity-free." Elliot gave a discussion of why this recovers the earlier condition in symplectic geometry, but it was a bit too fast to type up.

[1] This is the closest analogue in algebraic geometry of the connectedness theorem of Atiyah-Guillemin-Sternberg.