GRT Seminar Fall 2024 – Rozansky-Witten Theory

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Abstract

This semester, the GRT Seminar will focus on Rozansky-Witten theory.

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1 9/5 (David Nadler) – Introduction

Our goal is to discuss Rozansky-Witten theory. Some related topics include:

- Quasicoherent sheaves of categories (as discussed last spring).
- Categories of matrix factorizations.[1]
- The cobordism hypothesis.
- Local structure theory of holomorphic symplectic varieties.

1.1 What is Rozansky-Witten theory?

Suppose we have a hyperkähler / holomorphic symplectic manifold X. This means that X has a holomorphic $(2,0)$-form ω satisfying the (complex analogues of) the usual symplectic form axioms. Given such an X, there is a conjectural 3-dimensional topological field theory $\mathcal{Z}_{\mathsf{X}}$, called *Rozansky-Witten theory* with target X.

What we mean by 3d TFT is as follows:

- Given a closed 3-manifold[2] M^3, we obtain a number $\mathcal{Z}_{\mathsf{X}}(\mathsf{M}^3)$.
- Closed 2-manifolds M^2 give vector spaces $\mathcal{Z}_{\mathsf{X}}(\mathsf{M}^2)$.
- Closed 1-manifolds M^1 give categories[3] $\mathcal{Z}_{\mathsf{X}}(\mathsf{M}^1)$.
- Closed 0-manifolds M^0 give 2-categories $\mathcal{Z}_{\mathsf{X}}(\mathsf{M}^0)$.

[1] In more detail: given a smooth variety X and a function $\mathsf{f} : \mathsf{X} \to \mathbb{A}^1$, we can construct a category MF_f which categorifies the vanishing cycles of f.

[2] Typically with some extra structure, e.g. an orientation

[3] As is standard for GRT, we use the implicit ∞ convention.

In particular, $\mathcal{Z}_X(\mathrm{pt})$ is a 2-category. The *cobordism hypothesis* tells us that we can recover the entire theory $\mathcal{Z}_X$ from the "3-dualizable" 2-category $\mathcal{Z}_X(\mathrm{pt})$. For purposes of geometric representation theory, we are most interested in the low-dimensional behavior, which captures more data about the theory.

Rozansky-Witten theory should satisfy something like:

- $Z_X(S^2) = \mathcal{O}(X)$.[4]
- $Z_X(S^1) = \mathsf{Coh}(X)$.

These end up inheriting interesting structure from the TFT.

1.2 Why do we care?

Recall that 2-dimensional mirror symmetry can be schematically understood as an equivalence between the following 2d TFTs:

- An A-model $\mathcal{A}$ arising from symplectic geometry
- A B-model $\mathcal{B}_X$, coming from some Kähler manifold X, satisfying $\mathcal{B}_X(\mathrm{pt}) \simeq \mathsf{Coh}(X)$.

In particular, $\mathcal{A}(\mathrm{pt})$ is often some category of geometric interest, and the equivalence $\mathcal{A}(\mathrm{pt}) \simeq \mathcal{B}_X(\mathrm{pt})$ lets us resolve questions about $\mathcal{A}(\mathrm{pt})$.

There's an analogue in higher dimensions: we'd like to take a 3d TFT $\mathcal{Y}$ and give an equivalence $\mathcal{Y} \simeq \mathcal{Z}_X$ for some holomorphic symplectic X. This would give an equivalence between some 2-category and $\mathcal{Z}_X(\mathrm{pt})$.

Conjecture 1.1 (Teleman)**.** *Let G be a complex reductive group with maximal compact subgroup G_c. There is an equivalence between:*

- *A suitable 2-category of "categories with G_c-action."*
- *The Rozansky-Witten 2-category of $T^*(G^\vee/G^\vee)$.*

Note that $T^*(G^\vee/G^\vee)$ is stacky and non-proper, which makes it impossible for the corresponding 2-category to be 3-dualizable. Thus we typically won't obtain 3-manifold invariants from such a theory. That's terrible for 3-manifold topologists, but this isn't a 3-manifold seminar.

Some other examples of interest for Rozansky-Witten theory include symplectic resolutions and cotangent bundles of smooth algebraic varieties.

1.3 What is the correct 2-category?

To rigorously construct Rozansky-Witten theory, we'd need to give a definition of the 2-category $\mathrm{RW}_2 = \mathcal{Z}_X(\mathrm{pt})$. This was studied by Kapustin, Rozansky, and Saulina, but much is still unknown.

Roughly, we expect RW_2 to be a 2-category where:

- Objects are smooth Lagrangians $L \subset X$ (or some suitable generalization of these).
- 1-morphisms from L_1 to L_2 are given by some sort of category associated to $L_1 \cap L_2$. In the simplest possible case, where $X = T^*W$ is a cotangent bundle, L_1 is the zero-section, and L_2 is the graph of a differential df, then $L_1 \cap L_2$ is the critical locus of X and we assign $\mathrm{Hom}(L_1, L_2) = \mathsf{MF}_f$, the matrix factorization category of f. Work of Joyce and many others has focused on understanding how much the local setting looks like this.
- 2-morphisms and higher are "natural compatibilities" between the 1-morphisms.

One should think of the matrix factorization category MF_f as giving a categorical way to measure the critical locus of f. When the critical points of f are Morse, the category MF_f looks like a direct sum of copies of Vect (one for each critical point).

There is an important distinction between Rozansky-Witten theory and the 2d A-model. In the complex setting, there are no "instantons," so the theory is local and we don't run into the full difficulty of Floer theory. Thus Rozansky-Witten theory is a categorified version of Fukaya theory that avoids the need for instanton corrections.

[4] By our conventions, this is what is classically called $\mathbf{R}\Gamma(X, \mathcal{O})$, so there is interesting derived information.

1.4 An alternative viewpoint

If $\mathsf{X} = \mathsf{T}^*W$ is a cotangent bundle, then $\mathsf{ShvCat}(W)$, the 2-category of (quasicoherent) sheaves of categories on W, embeds into RW_2. The image of this embedding consists of "conic objects." Thus we can understand a key part of Rozansky-Witten theory, at least in this simple case.

The thesis (work in progress) of Enoch Yiu relates RW_2 to $\mathsf{ShvCat}(W \times \mathbb{A}^1)$.