GRT Seminar Fa23-Sp24 Notes

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Abstract

The seminar covers Ben-Zvi–Sakellaridis–Venkatesh, "Relative Langlands Duality."

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1 8/31 (David Nadler) – ???

I missed this day. If you have good notes from this day, send them to me and I will type them up.

2 9/7 (Elliot Kienzle) – Hamiltonian G-Spaces and Quantization

Elliot's notes for his talks are available at `https://chessapig.github.io/files/notes/G-spaces.pdf`.

The original Langlands program studies a duality of Lie groups $G \leftrightarrow G^\vee$. Relative Langlands seeks to upgrade this to a duality of Hamiltonian G-actions $(G \curvearrowright M) \leftrightarrow (G^\vee \curvearrowright M^\vee)$. This is proposed for hyperspherical varieties M, of which a typical example is $M = T^*X$ for X a spherical variety.

We can approach and motivate this using quantization. Start by considering the action of G on $L^2(X)$ for X a spherical variety (discussed in an earlier paper of Sakellaridis-Venkatesh discussing "harmonic analysis on spherical varieties").

2.1 Symplectic geometry and quantization

The original motivation for symplectic geometry comes from classical mechanics. Suppose that we have a particle moving in $\mathbb{R}^n$. We can capture the data of the position and momentum using the cotangent bundle $T^*\mathbb{R}^n$. By Newton's second law, the time evolution of the particle is described by (the flow along) a vector field on $T^*\mathbb{R}^n$.

We can generalize this to a symplectic manifold (M, ω), which is a manifold M with a closed, non-degenerate 2-form ω. To make this easier to work with, we can fix a metric $\langle , \rangle$ on M and write $\omega(x, y) = \langle x, Jy \rangle$ where $J^2 = -1$ (i.e. J^2 is an almost complex structure). We think of J^2 as "multiplication by $-i$." Given a Hamiltonian $H \in \mathcal{C}^\infty(M)$, we obtain a Hamiltonian vector field $X_H = J\nabla H$. More invariantly, we can define X_H via the formula $\omega(X_H, -) = dH$.

Moving to quantum mechanics, we view a particle in $\mathbb{R}^n$ as a $\mathbb{C}$-valued function ψ on $\mathbb{R}^n$ (not $T^*\mathbb{R}^n$). In this case, the Hilbert space is $L^2(\mathbb{R}^n)$. A free particle evolves according to Schrödinger's equation:

$$i\dot{\psi} = \Delta\psi.$$

We can summarize the classical and quantum pictures in the following table.

	Classical	Quantum
State Space	Symplectic manifold (M, ω)	Hilbert space $\mathcal{H}$
Observables	$f \in \mathcal{C}^\infty(M)$	Bounded operators $A \in \mathrm{End}(\mathcal{H})$
Evolution	Vector fields X_H for $H \in \mathcal{C}^\infty(M)$	Unitary operators $U(t) = e^{itA}$ for $A \in \mathrm{End}(\mathcal{H})$
Lie Algebra of observables	Poisson bracket $\{f, g\} = X_f(g)$	Commutator $[A, B]$

To obtain a quantum system from a classical system (heuristically), we pass from nonlinear evolution of points in T^*M to linear evolution of functions on M. (This linearity is forced on us by our desire to have superposition of states.) The dream of quantization is, given a symplectic manifold (M, ω), to construct a Lie algebra homomorphism $(\mathcal{C}^\infty(M), \{,\}) \to (\mathrm{End}(\mathcal{H}), [,])$ for some Hilbert space $\mathcal{H}$. Unfortunately, this is impossible to do consistently / functorially in general. However, there are some cases in which we can get good answers.

We will focus on geometric quantization, which behaves (loosely) as follows:

- For $M = T^*X$, we obtain $\mathcal{H} = L^2(X)$.
- For M a compact Kähler manifold, we obtain $\mathcal{H} = H^0(M, \mathcal{L})$ for some line bundle $\mathcal{L}$ on M.

2.2 G-Spaces

We want to incorporate symmetries into the previous picture. Suppose G is a compact Lie group / reductive algebraic group (depending on context). We say a symplectic G-space is a symplectic manifold (M, ω) with

G-action preserving ω. We can hope to quantize this to a linear representation $\mathsf{G} \curvearrowright \mathcal{H}$. (There are subtleties that arise here – for geometric quantization, we would like a G-equivariant polarization.)

In general, it is better to consider Hamiltonian G-actions, where $\mathfrak{g}$ acts by Hamiltonian vector fields. This allows us to construct a moment map $\mu : \mathsf{M} \to \mathfrak{g}^*$ which is equivariant (with respect to the coadjoint action on $\mathfrak{g}^*$).

Let us start by understanding the coadjoint action $\mathsf{G} \curvearrowright \mathfrak{g}^*$ using Kirillov's "orbit method." For $\alpha \in \mathfrak{g}^*$, consider the coadjoint orbit $\mathcal{O}_\alpha$. This $\mathcal{O}_\alpha$ turns out to be a symplectic manifold (with "Kirillov-Kostant-Souriau" / "KKS" form) with Hamiltonian G-action, and the moment map $\mathcal{O}_\alpha \to \mathfrak{g}^*$ is just the inclusion.

Example 2.1. Consider $\mathsf{G} = \mathrm{SO}(3)$. The coadjoint action is just $\mathrm{SO}(3)$ acting on $\mathbb{R}^3$ by rotations. Thus the generic orbits are spheres S^2.

The orbits $\mathcal{O}_\alpha$ will look like generalized flag manifolds, and conversely every generalized flag manifold arises in this way. (This is the first place where our compactness hypothesis comes in).

Proposition 2.2. *A coadjoint orbit $\mathcal{O}_\alpha$ is quantizable if and only if α is in the orbit of an integer point of the root lattice $\mathfrak{t}^*_{\mathbb{Z}} \subset \mathfrak{t}^*$ (viewed as a subspace of $\mathfrak{g}^*$ via the Killing form).*

Example 2.3. Continuing on with our SO(3) example, we see that a symplectic sphere is quantizable if and only if it has integer area.

In these cases, the quantization of $\mathcal{O}_\alpha$ is $\mathsf{H}^0(\mathcal{O}_\alpha, \mathcal{L}_\alpha)$ where $\mathcal{L}_\alpha$ is the line bundle corresponding to the character α. By the Borel-Weil theorem, $\mathsf{H}^0(\mathcal{O}_\alpha, \mathcal{L}_\alpha)$ is the irrep V_α of G with highest weight $\mathcal{L}_\alpha$.

We can summarize this in the following table:

Classical	Quantum
Symplectic action $\mathsf{G} \curvearrowright \mathsf{M}$	Representation $\mathsf{G} \curvearrowright \mathcal{H}$
Coadjoint orbit $\mathcal{O}_\alpha$	Highest weight representation E_α

3 9/14 (Elliot Kienzle) – Continued

3.1 Symplectic reduction

Suppose we have a Hamiltonian action $\mathsf{G} \curvearrowright \mathsf{M}$. This yields a G-equivariant moment map $\mu : \mathsf{M} \to \mathfrak{g}^*$, and the image of μ will necessarily be a collection of coadjoint orbits $\mathcal{O}_\alpha$. We can use these orbits to decompose M.

First consider the orbit $\mathcal{O}_0 = \{0\}$. We note that $\mu^{-1}(0)$ is G-invariant, so we can consider the quotient $\mu^{-1}(0)/\mathsf{G}$. We define this to be the *symplectic quotient*: $\mathsf{M}/\!/\mathsf{G} := \mu^{-1}(0)/\mathsf{G}$.

We will assume that 0 is a regular value of the moment map and that G acts on $\mu^{-1}(0)$ freely. We can drop these assumptions if we consider things in a suitable derived / stacky sense.

Theorem 3.1 (Marsden-Weinstein)**.** *The symplectic quotient $\mathsf{M}/\!/\mathsf{G}$ carries a natural symplectic structure.*

Example 3.2. If X is a (not necessarily symplectic) manifold with a G-action, then $\mathsf{T}^*\mathsf{X}/\!/\mathsf{G} = \mathsf{T}^*(\mathsf{X}/\mathsf{G})$.

Example 3.3. Let $\mathsf{M} = \mathsf{T}^*\mathbb{R}^2 \cong \mathbb{C}^2$. This has a U(1)-action via

$$e^{i\theta}(z_1, z_2) = (e^{i\theta}z_1, e^{i\theta}z_2).$$

We can define a (shifted) moment map $\mu : \mathbb{C}^2 \to \mathbb{R}$ via

$$\mu(z_1, z_2) = |z_1|^2 + |z_2|^2 - 1.$$

Then $\mathbb{C}^2/\!/\,\mathrm{U}(1) = \mathsf{S}^3/\,\mathrm{U}(1) = \mathsf{S}^2 = \mathbb{P}^1$ (consider the Hopf fibration).

Morally, we should think of every symplectic manifold as a symplectic reduction of a (possibly infinite-dimensional) affine space.

Note that

$$\dim \mathsf{M}/\!/\mathsf{G} = \dim \mathsf{M} - 2\dim \mathsf{G}.$$

The slogan is that "in symplectic geometry, groups act twice."

Theorem 3.4 (Guillemin-Sternberg, etc.). *The geometric quantization of a symplectic quotient satisfies*

$$\mathcal{H}(M /\!/ G) = \mathcal{H}(M)^G,$$

where the right hand side is the subspace of G*-invariant vectors in* G.

We can also define the symplectic reduction along any coadjoint orbit $\mathcal{O}_\alpha$ as $M /\!/_\alpha G = \mu^{-1}(\mathcal{O}_\alpha)/G$. This gives a decomposition of M as

$$M = \cup_{\alpha \in \mu(M)} \mu^{-1}(\mathcal{O}_\alpha) = \cup_{\alpha \in \mu(M)} (G\text{-bundles over } M /\!/_\alpha G),$$

at least if we avoid critical points.

Elliot has some fancy art of this decomposition.

Let's focus on the simplest possible case:

Definition 3.5. A Hamiltonian G-space M is *multiplicity-free* if $\dim M /\!/_\alpha G = 0$ for all α.

Remark 3.6. If M is compact, then a Morse theory argument shows that $M /\!/_\alpha G = \mathrm{pt}$ for all α.

Here are some relevant examples.

Example 3.7. For a coadjoint orbit $\mathcal{O}_\alpha$, we have $\mathcal{O}_\alpha /\!/_\alpha G = \mathrm{pt}$, so coadjoint orbits are multiplicity-free. Here we are ignoring stacky / derived quotients even though the action is typically nonfree.

Example 3.8. Consider $\mathbb{P}^1$ with $\mathrm{U}(1)$ acting by rotation. Then μ is the height function on $\mathbb{P}^1 = S^2$. If the top height is 1 and the bottom height is -1, then $\mu^{-1}(1)$ and $\mu^{-1}(1)$ are both points. For any $x \in (-1, 1)$, we have $\mu^{-1}(x) = S^1$ and therefore $\mathbb{P}^1 /\!/_x \mathrm{U}(1) = \mathrm{pt}$. Thus this action is multiplicity-free.

Example 3.9. Let $\mathrm{U}(1)^2$ acts on $\mathbb{P}^2$ (extending the standard action on $\mathbb{A}^2 \subset \mathbb{P}^2$). The fibers of the moment map over points in the interior of $\mu(M)$ are 2-tori, which degenerate to circles on the boundary lines of $\mu(M)$ and points at the corners of $\mu(M)$.

A non-example is given by the $\mathrm{U}(1)$ action on $\mathbb{C}^2$ from earlier in the lecture. This is an obvious non-example because the dimension of the symplectic quotient is nonzero. The slogan is that "multiplicity-free manifolds have maximal symmetry."

3.2 (David) – Interlude

For a Lie group G, we have $T^*G = G \times \mathfrak{g}^*$. Consider $G \curvearrowright T^*G$ induced by the adjoint action of G on itself. We obtain a moment map $\mu : G \times \mathfrak{g}^* \to \mathfrak{g}^*$ given by the formula

$$\mu(g, x) = \mathrm{Ad}_g(x) - x.$$

Then $\mu^{-1}(0) = \{(g, x) \in G \times \mathfrak{g}^* \mid g \in G_x\}$, where G_x is the centralizer of $x \in G$.

The multiplicity-freeness property for a general Hamiltonian G-space M can be understood as the requirement that the centralizers G_x act transitively on the preimages $\mu^{-1}(x)$.

It is a good exercise to classify multiplicity-free Hamiltonian G-spaces for $G = \mathrm{U}(1)$ or $G = \mathrm{SU}(2)$.

3.3 (Elliot) – A few last words

Multiplicity-freeness has a useful consequence for quantization: if M is multiplicity-free, then each highest weight representation E_α appears in $\mathcal{H}(M)$ at most once. In fact, E_α will appear if and only if $\mathcal{O}_\alpha \in \mu(M)$.

We will be interested in hyperspherical varieties as a large family of multiplicity-free symplectic manifolds. More on that next time!

4 9/21 (Mark Macerato) – Hyperspherical Varieties

4.1 (David) – Multiplicity-freeness

There may have been minor errors in the discussion last time, but the basic ideas were right. Suppose for simplicity that T is an *abelian* Lie group, and consider the cotangent bundle $\mathsf{T}^*\mathsf{T} \cong \mathsf{T} \times \mathfrak{t}^*$. The moment map for the translation action of T on itself is the projection $\mathsf{T} \times \mathfrak{t}^* \to \mathfrak{t}^*$. This gives a (trivial) family of abelian groups over $\mathfrak{t}^*$.

If we have another Hamiltonian T-space X, we obtain a moment map $\mu_{\mathsf{X}} : \mathsf{X} \to \mathfrak{t}^*$. We can view our family of abelian groups over $\mathfrak{t}^*$ as acting fiberwise on X. The multiplicity-freeness condition is requiring that the orbits of this action are fiberwise discrete.

This story still works for non-abelian G (but you have to be careful about left versus right actions). In this case, the fiber over $\nu \in \mathfrak{g}^*$ will be given by the stabilizer G_ν.

Example 4.1. We can describe Hamiltonian U(1)-spaces as lying over $\mathfrak{u}(1) \cong \mathbb{R}$. The multiplicity-freeness condition implies that the fibers are (disjoint unions of) copies of S^1 and points. For example, we can consider the height function on the sphere, or the projection of a cylinder $\mathsf{S}^1 \times \mathbb{R}$, or many related examples – these all give multiplicity-free Hamiltonian U(1)-spaces.

Example 4.2. If we take $\mathsf{G} = \mathrm{SU}(2)$, we obtain a similar (but distinct) picture because $\mathfrak{su}(2)/\mathrm{SU}(2) \cong [0, \infty)$ (the SU(2)-orbits in $\mathfrak{su}(2)$ are spheres). The fibers of $\mathsf{T}^*\mathrm{SU}(2) \to \mathfrak{su}(2)$ are SU(2) (over 0) and S^1 (over points in $(0, \infty)$). We can analyze multiplicity-free Hamiltonian G-spaces as before.

In general, the left action $\mathsf{G} \curvearrowright \mathsf{T}^*\mathsf{G}$ (via $g \cdot (h, \nu) = (gh, \mathrm{Ad}_g \nu)$) is not multiplicity-free. Consider the moment map $\mathsf{T}^*\mathsf{G} \cong \mathsf{G} \times \mathfrak{g}^* \to \mathfrak{g}^*$ given by projection (this depends on how we trivialize $\mathsf{T}^*\mathsf{G}$). For a coadjoint orbit $\mathcal{O}$, the preimage $\mu^{-1}(\mathcal{O})$ is $\mathsf{G} \times \mathcal{O}$. The multiplicity-freeness here reduces to the question of whether the action $\mathsf{G}_\nu \curvearrowright \mathsf{G}$ has discrete orbits. This is not true in general (see e.g. the SU(2) example above), proving the claim.

A later clarification: Really, we should think of $\mathsf{T}^*\mathsf{G} \rightrightarrows \mathfrak{g}^*$ as a groupoid, where the "source" and "target" maps are μ_L and μ_R (the moment maps for the left / right actions, respectively). Given a groupoid, we can obtain a group scheme (encapsulating the "automorphism groups of points") as a fiber product, e.g.

$$\begin{tikzcd} \{[X, g] = 0\} \arrow[r] \arrow[d] & \mathsf{T}^*\mathsf{G} \arrow[d] \\ \Delta \arrow[r] & \mathfrak{g}^* \times \mathfrak{g}^*. \end{tikzcd}$$

Understanding things from this perspective clears up the difficulties with left / right actions.

Hamiltonian G-spaces ($\mathsf{M} \to \mathfrak{g}^*$) will be module objects for this groupoid.

4.2 (Mark) – Towards hyperspherical varieties

We will change settings to algebraic geometry (following section 3 of Ben-Zvi–Sakellaridis-Venkatesh). Fix an algebraically closed field k of characteristic zero (e.g. $\mathbb{C}$ or $\overline{\mathbb{Q}_\ell}$). Let G be a connected reductive group over k.

Recall that a spherical variety is a normal G-variety X such that there exists a Borel subgroup $\mathsf{B} \subset \mathsf{G}$ with an open orbit in X. We can rephrase the last condition without picking a Borel: we require that G has an open orbit on $\mathsf{X} \times \mathrm{Fl}_{\mathsf{G}}$. If X is affine, this is equivalent to requiring that the coordinate ring $k[\mathsf{X}]$ is multiplicity-free as a G-module.

Example 4.3 ("Group case")**.** Let H be a connected reductive group and $\mathsf{G} = \mathsf{H} \times \mathsf{H}$. For $\mathsf{X} = \mathsf{H}$ and $\mathsf{G} \hookrightarrow \mathsf{X}$ via $(h_1, h_2) \cdot h = h_1 h h_2^{-1}$, H is a spherical variety.

If we fix a Borel $\mathsf{B} \subset \mathsf{H}$, we have a unipotent subgroup $\mathsf{U} \subset \mathsf{B}$ and a surjection $\mathsf{B} \twoheadrightarrow \mathsf{T} = \mathsf{B}/\mathsf{U}$. By Levi's theorem, this splits, giving $\mathsf{T} \hookrightarrow \mathsf{B} \subset \mathsf{G}$. We get a vector space decomposition $\mathfrak{g} = \mathfrak{u}^- \oplus \mathfrak{t} \oplus \mathfrak{u}$. Consider the open embedding $\mathsf{U}^- \times \mathsf{B} \to \mathsf{H}$ given by $(u, b) \mapsto ub$. The Borel subgroup $\mathsf{B}^- \times \mathsf{B} \subset \mathsf{G}$ has an open orbit in H. This leads to a Bruhat decomposition $\mathsf{H} = \sqcup_{w \in W} \mathsf{B}w\mathsf{B}$.

We can obtain Bruhat decompositions for more general spherical varieties. This is a rich theory that has been worked out by several authors (Knapp, Brion, etc.). But let's move on to hyperspherical varieties, which give a symplectic point of view.

Instead of a spherical variety X, let us consider $M = T^*X$ with the moment map $\mu : T^*X \to M$. For simplicity, we will assume our base spherical variety X is affine, smooth, and irreducible. In this case M is *coisotropic*, which means that the G-invariant function field $k(M)^G$ is Poisson-commutative.

Another way of saying this is as follows. Let $\mathfrak{c} = \mathfrak{g}^*//G \cong \mathfrak{g}//G$ be the "Chevalley space." Letting $\eta \in M$ be the generic point, we obtain a Stein factorization $M \to \mathfrak{c}_M \to \mathfrak{c}$. The map $\tilde{\mu} : M \to \mathfrak{c}_M$ has connected generic fiber, and $\mathfrak{c}_M \to \mathfrak{c}$ is finite. The second definition of "coisotropic" is that the group $G_{K(\mathfrak{c}_M)}$ acts on $M_{K(\mathfrak{c}_M)}$ with an open (hence dense) orbit.

Theorem 4.4 (Losev). *If M is a smooth Hamiltonian G-variety, then* all *of the fibers of $\tilde{\mu} : M \to \mathfrak{c}_M$ are connected.*[1]

A third definition of coisotropic is that the generic G-orbit on M is coisotropic in the usual sense.

"Coisotropic" is the algebraic geometry version of "multiplicity-free." Elliot gave a discussion of why this recovers the earlier condition in symplectic geometry, but it was a bit too fast to type up.

5 9/28 (Mark Macerato) – Continued

5.1 (David) – Groupoids and Hamiltonian G-spaces

Recall the homework problem of classifying multiplicity-free SU(2)-spaces.

The corrected general picture is as follows. Consider the cotangent bundle T^*G with natural Hamiltonian G-actions on the left and right. These yield moment maps $\mu_L, \mu_R : T^*G \to \mathfrak{g}^*$. If we trivialize $T^*G \cong G \times \mathfrak{g}^*$, these maps are given by $(g, X) \mapsto X$ and $(g, X) \mapsto \mathrm{Ad}_g X$.

We should think of $T^*G \rightrightarrows \mathfrak{g}^*$ as a groupoid. The "objects" are $X \in G$, and the "morphisms" are $g : X \to \mathrm{Ad}_g X$. Composition is given by group multiplication.

We may view any Hamiltonian G-space Y (with moment map $\mu : Y \to \mathfrak{g}^*$) as a module over this groupoid. Specifically, we have a natural map $T^*G \times_{\mathfrak{g}^*} Y \to Y$, the projection of the fiber product onto the second factor. On elements, this is given by $(g, X, y) \mapsto gy$, which lies in the fiber of Y over $\mathrm{Ad}_g X \in \mathfrak{g}^*$.

Consider the pullback

$$\begin{tikzcd} \mathcal{S} \arrow[r] \arrow[d] & T^*G \arrow[d] \\ \mathfrak{g}^* \arrow[r, "\Delta"] & \mathfrak{g}^* \times \mathfrak{g}^*. \end{tikzcd}$$

In equation, $\mathcal{S} = \{[g, X] = 0\}$. From the groupoid perspective, $\mathcal{S} \to \mathfrak{g}^*$ is obtained by only considering automorphisms of objects in our original groupoid (i.e. forgetting about isomorphisms between different objects). We can view $\mathcal{S} \to \mathfrak{g}^*$ as the relative group over $\mathfrak{g}^*$ with fibers given by stabilizers $\mathrm{Stab}_G(X)$.

The "multiplicity-free" condition can now be restated: it means that the $\mathcal{S}$-action on Y relative to $\mathfrak{g}$ has only finitely many orbits.

For the exercise about SU(2), we have $\mathfrak{g}^* = \mathbb{R}^3$, and $\mathcal{S}$ has fiber SU(2) over the identity and U(1) over other fibers. We really only care about $\mathfrak{g}^*/\mathrm{SU}(2)$, which looks like a real ray $[0, \infty)$. This allows us to produce some examples of multiplicity-free Hamiltonian SU(2)-spaces - these spaces should have maps to $[0, \infty)$ with fibers over $X \in \mathfrak{g}^*/\mathrm{SU}(2) \cong [0, \infty)$ looking like (finite disjoint unions of) orbits of $\mathrm{Stab}_{\mathrm{SU}(2)}(X)$-actions.

Example 5.1. The 2-sphere S^2 has multiplicity-free SU(2)-action via the action coming from $\mathrm{SU}(2) \to \mathrm{SO}(3)$.

Example 5.2. The standard representation $\mathbb{C}^2$ has multiplicity-free SU(2)-action.

Example 5.3. The blowup of $\mathbb{C}^2$ at the origin (with a corrected symplectic form) has multiplicity-free SU(2)-action.

Are these all of the possible examples (up to finite covers)? It would be good to figure this out.

[1] This is the closest analogue in algebraic geometry of the connectedness theorem of Atiyah-Guillemin-Sternberg.

5.2 (Mark) – Coisotropic G-varieties

Recall our setup: G is connected and reductive, and M is a smooth affine Hamiltonian G-variety. We have a moment map $\mu : M \to \mathfrak{g}^*$, and we can compose this with a GIT quotient map to get $\overline{\mu} : M \to \mathfrak{c}_G$, where $\mathfrak{c}_G = \mathfrak{g}^*//G$ is called the Chevalley base. This admits a "Knop factorization"

$$\begin{tikzcd} M \arrow[rr, "\overline{\mu}"] \arrow[dr, "\tilde{\mu}_M"'] & & \mathfrak{c}_G \\ & \mathfrak{c}_M \arrow[ur, "\pi"'] & \end{tikzcd}$$

where π is finite and $\tilde{\mu}_M$ has generically connected fiber.

Definition 5.4. We say that M is *coisotropic* if any of the following equivalent conditions hold.

1. $k(M)^G$ is Poisson-commutative.[2]
2. The generic orbit of G on M is coisotropic.
3. The generic fiber of $\tilde{\mu}_M$ has a dense G-orbit.

Let's see why 1 and 2 are equivalent. Choose $f_1, \ldots, f_n \in K(M)$ which separate generic orbits (this is possible by a theorem of Rosenlicht). This yields $\underline{f} = (f_1, \ldots, f_n) : U \to \mathbb{A}^n$ (for $U \subset M$ open), and we can restrict this to a surjective smooth map $U' \to W$ such that U' is dense in U and $W \subset \mathbb{A}^n$ is a locally closed subvariety. Replace U by U'. The fibers of $\underline{f}$ are exactly the G-orbits in U. Therefore, for $x \in U$, we see that $df_1(x), \ldots, df_n(x)$ span the conormal space $T^*_U(G \cdot x)_x$. Thus $G \cdot x$ is coisotropic at x if and only if $T^*_U(G \cdot x)_x$ is isotropic, if and only if the $f_1, \ldots, f_n$ Poisson-commute at x.

5.3 Approaching hyperspherical varieties

Suppose that M is a smooth affine Hamiltonian G-variety as before. We will also require that M comes with a $\mathbb{G}_m$-action (equivalently, a grading on $k[M]$) such that

1. The $\mathbb{G}_m$-action on M commutes with the G-action.
2. The symplectic form ω on M has weight 2, i.e. $\lambda \cdot \omega = \lambda^2 \omega$.

David noted that this latter condition implies that ω is exact: if ν is the vector field generating the $\mathbb{G}_m$-action, then Cartan's magic formula (using that ω is closed) gives

$$2\omega = \mathcal{L}_\nu \omega = d(i_\nu \omega).$$

The 2 here is needed to ensure that we can construct a "$\mathbb{G}_m$-equivariant Kostant slice."

We want to define what it means for M to be hyperspherical. The first condition will be that M is coisotropic.

The second condition is that $\mu(M) \subset \mathfrak{g}^*$ meets the nilpotent cone $\mathcal{N}_G = \chi^{-1}(0)$ (for $\chi : \mathfrak{g}^* \twoheadrightarrow \mathfrak{g}^*//G$). Equivalently, $\overline{\mu})(M)$ contains $0 \in \mathfrak{c}_G$. This implies that $M//G \to \mathfrak{c}_M$ is surjective (it is always an open immersion, so we get $M//G = \mathfrak{c}_M$). There will be two more conditions (which we will discuss next time).

6 10/5 (Mark Macerato) – Continued

6.1 Pre-hyperspherical varieties

Let G be a connected reductive group and $G_{\text{gr}} = G \times \mathbb{G}_m$. We consider a smooth affine Hamiltonian G-variety with auxiliary $\mathbb{G}_m$-action governing the grading. This yields a map $M \to \mathfrak{g}^* \to \mathfrak{c}_G$, which has a Knop factorization $M \to \mathfrak{c}_M \to \mathfrak{c}_G$. Here $\mathbb{G}_m \curvearrowright \mathfrak{g}^*$ quadratically, and the map $M \to \mathfrak{g}^*$ is $\mathbb{G}_m$-equivariant.

[2] In this setup, we can replace this by the condition that $k[M]^G$ is Poisson commutative, since $\mathrm{Frac}\,k[M]^G = k(M)^G$.

Definition 6.1. We say that M is *pre-hyperspherical* if

1. M is coisotropic, i.e. $k(M)^G$ is Poisson commutative (equivalently, the generic fiber of $M \to \mathfrak{c}_M$ has a dense G-orbit),
2. $\mu(M) \cap \mathcal{N}_G \neq \emptyset$ (for $\mathcal{N}_G$ the nilpotent cone of G), and
3. The stabilizer of a generic point of M is connected.

Example 6.2. Let $G = \mathrm{Sp}_{2n}$ and $M = \mathbb{C}^{2n} \oplus \mathbb{C}^{2n}$. Here $\mu_M : \mathbb{C}^{2n} \oplus \mathbb{C}^{2n} \to (\mathbb{C}^{2n} \oplus \mathbb{C}^{2n})/\!/ \mathrm{Sp}_{2n} \cong \mathbb{A}^1$ via $(v, w) \mapsto \omega(v, w)$. Thus

$$\mu_M^{-1}(1) = \{(v, w) \in \mathbb{C}^{2n} \mid \omega(v, w) = 1\},$$

and Sp_{2n} acts transitively on this fiber. Meanwhile, $\mu^{-1}(0)$ can be decomposed as:

$$\mu^{-1}(0) = \{(v, w) \mid v, w \text{ lin. ind. } \ \omega(v, w) = 0\} \cup \{(v, w) \mid v, w \text{lin. dep., not both } 0\} \cup \{(0, 0)\}.$$

The first set here is the unique open orbit, and the last set is the unique closed orbit. The middle set contains a $\mathbb{P}^1$ worth of orbits. In particular, $\mu(M)$ meets $\mathcal{N}_G$. The stabilizer of a generic point of M can be identified with Sp_{2n-2}.

Proposition 6.3. *In general, if M is pre-hyperspherical, there exists a unique closed orbit $M_0 \subset M$ for $G_{\mathrm{gr}} = G \times \mathbb{G}_m$.*

We call M_0 the *core* of M.

Proof. Consider the GIT quotient $M \to M/\!/G_{\mathrm{gr}}$, and recall that closed orbits of G_{gr} correspond to points of $M/\!/G_{\mathrm{gr}}$. Thus it suffices to show that $M/\!/G_{\mathrm{gr}} = \mathrm{pt}$, or equivalently $k[M]^{G\times\mathbb{G}_m} = k$. Note

$$k[M]^{G\times\mathbb{G}_m} = k[\mathfrak{c}_M]^{\mathbb{G}_m} = k[\mathfrak{c}_m]_0,$$

the weight 0 component. By construction, $k[\mathfrak{c}_G] \to k[\mathfrak{c}_M]$ is finite (and therefore integral), so $k[\mathfrak{c}_M]_0 \to k[\mathfrak{c}_G]_0$ is integral (an exercise in counting degrees). Now we have

$$k[\mathfrak{c}_G]_0 = k[\mathfrak{c}_G]^{\mathbb{G}_m} = k[\mathfrak{g}^*]^{G\times\mathbb{G}_m} = k,$$

so $k[\mathfrak{c}_M]_0$ is integral over k. Since k is algebraically closed, we see $k[\mathfrak{c}_M]_0 = k$. $\square$

6.2 (David) – Weinstein manifolds

Suppose we have an exact symplectic manifold $(M, \omega = d\lambda)$. Take $Z = \omega^{-1}(\lambda)$ (this is a vector field on M). Then Z gives a flow on M, and the core M_0 is the subset of points of M which do not escape to infinity along this flow. Assuming M_0 is isotropic (this is implied by the Weinstein condition), the flow gives an action of $\mathbb{C}^\times$ on M_0.

Example 6.4. Consider the surface singularity $x^2 + y^2 + z^{n+1} = 0$. Let M be a symplectic resolution of this. The core M_0 is the chain of $\mathbb{P}^1$'s appearing as the zero fiber. In terms of geometric representation theory, we can call M_0 a "subregular Springer fiber."

Suppose M is G-Hamiltonian – then we are in a situation very similar to what Mark is talking about. That is, *pre-hyperspherical varieties are analogous to G-Hamiltonian Weinstein manifolds*. This precludes examples like the above, since the union of $\mathbb{P}^1$'s cannot be a single G-orbit. This fact is essentially kin to the last Proposition.

Example 6.5. Consider $G \times G$ acting on $M = T^*G$ by left and right translation. The core M_0 is the zero section.

6.3 (Mark) – Hyperspherical varieties

Let $\mu_M : M \to M//G$ be the GIT quotient map.

Proposition 6.6. *The core* M_0 *is the unique closed* G*-orbit in* $\mu_M^{-1}(0)$.

Proof. Note that G has a unique closed orbit $M_0' \subset \mu_M^{-1}(0)$ (by standard GIT). Since $\mathbb{G}_m$ commutes with G, the $\mathbb{G}_m$ action takes closed G-orbits to closed G-orbits. Therefore $\mathbb{G}_m$ preserves M_0', and we get $G \times \mathbb{G}_m \curvearrowright M_0'$. It follows that M_0' contains a closed G_{gr} orbit, hence contains M_0. But M_0' is itself a G-orbit, so $M_0 = M_0'$. □

Pick $x \in M_0$, and let $H = \mathrm{Stab}_G(x)$, so $M_0 = G/H$. Since M_0 is affine, H must be reductive. Since $\mu_M^{-1}(0)$ maps to $\mathcal{N}_G \subset \mathfrak{g}^*$, we get an element $f = \mu(x) \in \mathcal{N}_G$.

Because $\mathbb{G}_m \curvearrowright M_0 = G/H$, we get a homomorphism $\mathbb{G}_M \to \mathrm{Aut}_G(G/H) \cong N_G(H)/H$. In fact, this factors through $(Z_G(H)/Z(H))^0$ (where ^{0} denotes the connected component of the identity element). Let $\overline{\pi} : \mathbb{G}_M \to Z_G(H)/Z(H)$ be the induced map.

Definition 6.7. A pre-hyperspherical variety M is hyperspherical if

1. $\overline{\pi}$ lifts to a homomorphism $\pi : \mathbb{G}_m \to Z_G(H) \subset G$, which moreover lifts to a homomorphism $\rho : \mathrm{SL}_2 \to G$ such that
$$d\rho \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = f$$
under the standard identification $\mathfrak{g} \cong \mathfrak{g}^*$.

2. Consider the *sheared* $\mathbb{G}_m$*-action* $(\mathbb{G}_m)_{sh} \curvearrowright M$ induced by
$$\begin{aligned}(\mathbb{G}_m)_{sh} &\to G \times \mathbb{G}_m \\ g &\mapsto (\pi(g)^{-1}, g).\end{aligned}$$
By construction, $(\mathbb{G}_m)_{sh}$ fixes x, and thus we get $(\mathbb{G}_m)_{sh} \curvearrowright (T_xM_0)^\perp/(T_xM_0 \cap T_xM_0^\perp) := N_xM_0$, the *symplectic normal space*. The condition is that this $(\mathbb{G}_m)_{sh}$-action on N_xM_0 is given by linear scaling.

7 10/12 (Mark Macerato) – Continued

7.1 Refresher on the definition

Recall that M is a smooth affine graded Hamiltonian G-variety with moment map $\mu : M \to \mathfrak{g}^*$. The grading means that we have a $\mathbb{G}_m$ action (acting on ω_M with weight 2) such that the G and $\mathbb{G}_m$ actions commute. Therefore μ is $\mathbb{G}_m$-equivariant, where $\mathbb{G}_m$ acts on $\mathfrak{g}^*$ with weight 2.

Recall that we say M is pre-hyperspherical if:

1. M is coisotropic (i.e. $k(M)^G$ is commutative, equivalently generic G-orbits are cut out by Poisson-commuting G-invariant functions).

2. The image of μ meets the nilpotent cone $\mathcal{N}_G^*$. Thus there exists a unique closed $G \times \mathbb{G}_m$-orbit in M, the "core" M_0 of M. This is the unique closed G-orbit in $\tilde{\mu}_M^{-1}(0)$, where $\tilde{\mu}_M : M \to \mathfrak{c}_M = M//G$ and we write 0 for an element of $\mathfrak{c}_M$ mapping to $0 \in \mathfrak{c}_G = \mathfrak{g}^*//G$.)

3. The stabilizer of a generic point $m \in M$ is connected.

We now move on to the full hyperspherical condition. Fix $x \in M_0$, and let $f = \mu(x) \in \mathcal{N}_G^*$. Then $M_0 \cong G/H$ where $H = \mathrm{Stab}_G(x)$, and H is reductive. Let $\overline{\pi} : \mathbb{G}_m \to \mathrm{Aut}_G(G/H) \cong N_G(H)/H$ be the natural map.

Condition (4a) is that $\overline{\pi}$ lifts to a homomorphism $\pi : \mathbb{G}_m \to N_G(H) \cap [G, G]$.

Proposition 7.1. *The lift* π *factors through* $Z_G(H)$.

Proof. Let $\mathsf{t} = d\pi(1) \in \mathfrak{g}$, and let $\mathfrak{h} = \mathrm{Lie}(H)$. We need to show that t centralizes $\mathfrak{h}$. Identify $\mathfrak{g} \cong \mathfrak{g}^*$, so we can view f as an element of $\mathfrak{g}$. Then $[\mathsf{t}, f] = 2f$ (since $\mathbb{G}_m$ acts on $\mathfrak{g}^*$ with weight 2, and $\mu : M \to \mathfrak{g}^*$ is G-equivariant). By the Jacobson-Morozov theorem, there exists $e \in \mathfrak{g}$ such that $[\mathsf{t}, e] = -2e$ and $[f, e] = \mathsf{t}$, giving an embedding $\mathfrak{sl}_2 \hookrightarrow \mathfrak{g}$.

View $\mathfrak{g}$ as an $\mathfrak{sl}_2$-module via the above embedding. Then $\mathfrak{h}$ is spanned by highest weight vectors in $\mathfrak{g}$, so all of the weights of t on $\mathfrak{h}$ are nonnegative. Write $\mathfrak{h} = \mathfrak{h}_+ \oplus \mathfrak{h}_0$, where $\mathfrak{h}_+$ is the sume of the strictly positive t-weight spaces. We have $\mathfrak{h}_+ \subset \mathrm{im}(\mathrm{ad}(f)) \cap \mathfrak{g}^f$ (since H fixes x, we see that H fixes $f = \mu(x)$).

Note that $\mathrm{im}(\mathrm{ad}(f)) \cap \mathfrak{g}^f$ is an ideal in $\mathfrak{g}^f$ (by a direct Lie algebra computation), so $\mathrm{im}(\mathrm{ad}(f)) \cap \mathfrak{h}$ is an ideal of $\mathfrak{h}$. Furthermore, $\mathrm{im}(\mathrm{ad}(f)) \cap \mathfrak{g}^f$ is a nilpotent Lie algebra (since contained in the positive weight part of $\mathfrak{g}$). Thus $\mathrm{im}(\mathrm{ad}(f)) \cap \mathfrak{h}$ is a nilpotent ideal in $\mathfrak{h}$, but $\mathfrak{h}$ is reductive, so $\mathrm{im}(\mathrm{ad}(f)) \cap \mathfrak{h}$ must be central. Since $\mathrm{im}(\mathrm{ad}(f)) \cap \mathfrak{h} \subset [\mathfrak{h}, \mathfrak{h}]$, it follows that $\mathrm{im}(\mathrm{ad}(f)) \cap \mathfrak{h} = 0$, and thus $\mathfrak{h}_+ = 0$. We see that $\mathfrak{h} = \mathfrak{h}_0$, so in particular x centralizes $\mathfrak{h}$. ☐

Proposition 7.2. *The lift π is unique if it exists.*

To state condition (4b) for hyperspherical varieties, let $\tilde{\mathbb{G}}_m$ be $\mathbb{G}_m$ acting on M via the sheared action $\mathbb{G}_m \xrightarrow{(\pi(-)^{-1}, \mathrm{id})} G \times \mathbb{G}_m$. Thus $\tilde{\mathbb{G}}_m$ acts on the symplectic normal space $T_x M_0^{\perp}/T_x M_0$, where $T_x M_0^{\perp}$ is the symplectic orthogonal subspace. Condition (4b) is that $\tilde{\mathbb{G}}_m$ acts on the symplectic normal space by $t \cdot v = tv$ (i.e. the action has weight 1). This is a useful condition, though the geometric interpretation is not clear.

We are interested in considering sheared $\mathbb{G}_m$ actions more generally – this turns out to be the natural thing to do from the perspective of geometric Satake.

7.2 (David) – More on shearing

Consider the vector field $2p\partial_p - q\partial_q$ on $\mathbb{R}^2 = T^*\mathbb{R}$. This gives a hyperbolic $\mathbb{G}_m$-action and corresponds to the Liouville form $\lambda = 2p\,dq + q\,dp$. For $H = pq$, we get an action of $G = \mathbb{G}_m$ via the Hamiltonian vector field $X_H = p\partial_p - q\partial_q$. Subtracting X_H from our Liouville vector field, we get a $\tilde{\mathbb{G}}_m$-action via the vector field $p\partial_p$. This $\tilde{\mathbb{G}}_m$-action preserves the zero section! Thus, by considering shearing actions, we can preserve certain desirable isotropic / Lagrangian submanifolds.

7.3 (Mark) – A construction

Let $H \subset G$ be a reductive group and $H \curvearrowright S$ be a symplectic representation of H. Let $\pi : \mathbb{G}_m \to [G, G] \cap Z_G(H)$. Choose a nilpotent element $f \in \mathfrak{g}^*$.

We equip S with a commuting $\mathbb{G}_m$ action via scaling (of weight 1). Write $\mathfrak{g} = \mathfrak{j} \oplus \mathfrak{u}^- \oplus \mathfrak{g}_0 \oplus \mathfrak{u}$, where $\mathfrak{j}$ is the centralizer of π and f, $\mathfrak{u}^-$ is the subspace with negative π eigenvalues, $\mathfrak{g}_0$ is the subspace with zero π eigenvalues (but nonzero f eigenvalues), and $\mathfrak{u}$ is the subspace with positive π eigenvalues. Integrate $\mathfrak{u}$ to a unipotent subgroup $U \subset G$.

... And we're out of time – we will finish next week.

8 10/19 (Mark Macerato) – Concluded

The goal for today is to discuss the Ben-Zvi–Sakellaridis–Venkatesh construction of hyperspherical varieties via Whittaker induction. This will be a complicated story, but it covers several examples of interest.

8.1 Hamiltonian induction

Let G be an algebraic group and H a subgroup. We can view every G-variety as an H-variety by forgetting some of the action; this gives a restriction functor Res_H^G from G-varieties to H-varieties. More interestingly, this functor has a left adjoint, "induction" Ind_H^G, sending an H-variety to a G-variety. Specifically, for a G-variety Y, we have a diagonal H-action on $G \times Y$ via $h \cdot (g, y) = (gh^{-1}, hy)$. The induction is the balanced product $G \times^H Y := (G \times Y)/H$.

Every G-variety X gives a Hamiltonian G-variety T^*X (and similarly for H-varieties). The Hamiltonian condition here appears as follows: if $D(X)$ is differential operators on X, then we get a natural map $\mathcal{U}\mathfrak{g} \to$

$D(X)$. Taking associated gradeds, we get a map $\operatorname{Sym}\mathfrak{g} \to \mathcal{O}(T^*X)$. This corresponds (under Spec) to the moment map $T^*X \to \mathfrak{g}^*$.

Given a Hamiltonian G-variety, we can restrict to the H-action and get a Hamiltonian H-variety. This defines a "functor" $h\operatorname{Res}_H^G$ from Hamiltonian G-varieties to Hamiltonian H-varieties. The term "functor" here is in quotes because the correct notion of a morphism between symplectic varieties is not the standard notion of morphism of varieties. Instead, we need to consider (G-stable) Lagrangian correspondences as our morphisms.

To understand why, suppose we have a morphism $f : X \to Y$ of varieties. We do not obtain a natural morphism of varieties between T^*X and T^*Y. Instead, we get a Lagrangian correspondence

$$T^*X \xleftarrow{f^*} T^*Y \times_Y X \xrightarrow{\pi_1} T^*Y.$$

We can view $T^*Y \times_Y X$ as the conormal variety to $\Gamma_f \subset T^*X \times T^*Y$. Morally, we should think of Lagrangians as being the symplectic analogue of points (they give the maximum amount of information we can cut out by Poisson-commuting functions).

The Hamiltonian induction functor acts on cotangent bundles by

$$h\operatorname{Ind}_H^G T^*Y = T^*(G \times^H Y) = T^*(G \times Y)/\!/H = (T^*G \times T^*Y)/\!/H.$$

From this, we can understand what to do for a general Hamiltonian H-variety M. We define

$$h\operatorname{Ind}_H^G(M) = (T^*G \times M)/\!/H,$$

where we use the diagonal H-action (from $h \mapsto (h^{-1}, h) \in H \times H$). Then G-stable Lagrangian correspondences between $h\operatorname{Ind}_H^G Y$ and M correspond to H-stable Lagrangian correspondences between Y and $h\operatorname{Res}_H^G M$ (i.e. $h\operatorname{Ind}_H^G$ is left adjoint to $h\operatorname{Res}_H^G$).

8.2 (David) – The groupoid picture

Recall that we should think of Hamiltonian H-spaces $M \to \mathfrak{h}^*$ as coming with an action of the groupoid $T^*H \rightrightarrows \mathfrak{h}^*$. To construct the induced Hamiltonian G-space, we consider T^*G with its natural maps to $\mathfrak{h}^* \times \mathfrak{h}^*$, and we combine this with M as Mark was indicating above.

8.3 (Mark) – Sheared Hamiltonian G-spaces

Recall that we started by considering *graded* Hamiltonian G-variety i.e. those equipped with a commuting $\mathbb{G}_m$-action (of weight 2 on ω). To discuss Whittaker induction, it is better to generalize to *sheared* Hamiltonian G-variety.

Definition 8.1. Let $\pi : \mathbb{G}_m \to \operatorname{Aut}(G)$. A π*-sheared Hamiltonian* G*-variety* is a Hamiltonian G-variety M with a $\mathbb{G}_m$-action such that, for $\lambda \in \mathbb{G}_m$, $g \in G$, and $x \in M$, we have

$$\lambda \cdot (g \cdot x) = \pi(\lambda)(g)(\lambda \cdot x)$$

and such that $\mu : M \to \mathfrak{g}^*$ is $\mathbb{G}_m$-equivariant, where $\mathbb{G}_m$ acts on $\mathfrak{g}^*$ via

$$\lambda \cdot \xi = \lambda^2 \pi(\lambda)^t(\xi).$$

Example 8.2. Consider the action of $\mathbb{G}_a$ on pt, where we assume $\mu : \mathrm{pt} \to \mathbb{A}^1$ sends pt to $x \in \mathbb{A}^1$. We can consider the action $\pi : \mathbb{G}_m \curvearrowright \mathbb{G}_a$ by $\lambda \cdot x = \lambda^2 x$. Then pt is a π-sheared Hamiltonian $\mathbb{G}_a$-variety, but is not a graded Hamiltonian G-variety.

However, suppose that π arises from an inner automorphism, i.e. we have $\pi : \mathbb{G}_m \to G$ and the action is $\lambda \cdot g = \pi(\lambda) g \pi(\lambda)^{-1}$. Then we have an equivalence of categories between graded Hamiltonian G-varieties and π-sheared Hamiltonian G-varieties, given by sending a graded Hamiltonian G-variety $\mathbb{G}_m \curvearrowright M$ to the sheared variety with $\mathbb{G}_m$-action given by $(\pi(-)^{-1}, \mathrm{id}) : \mathbb{G}_m \hookrightarrow G \times \mathbb{G}_m$.

Example 8.3. Let $\mathbb{G}_m \curvearrowright \mathfrak{g}^*$ and consider $2\rho : \mathbb{G}_m G$. This gives the sheared action $\lambda \cdot \xi = 2\rho(-\lambda)\lambda^2\xi$.

8.4 Whittaker induction

Suppose G is connected and reductive, and say we have $\pi : \mathbb{G}_m \to [G, G]$. Let $f \in \mathfrak{g}^*$ be nilpotent and assume $\pi(\lambda) \cdot f = \lambda^{-2} f$. Consider a reductive subgroup $H \subset G$ such that H commutes with $\pi(\mathbb{G}_m)$ and H fixes f. We can write

$$\mathfrak{g} = \mathfrak{j} \oplus \mathfrak{g}_+ \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_-$$

where $\mathfrak{j}$ is the centralizer of $\mathrm{Lie}(\pi(\mathbb{G}_m))$ and f, and $\mathfrak{g}_{+/0/-}$ collects the other π-eigenspaces according to their eigenvalues (whether positive, zero, or negative). Another way to construct this is using the Jacobson-Morozov theorem as discussed last time.

Let $\mathfrak{u} = \mathfrak{g}_+$ and $\mathfrak{u}_+ = \oplus_{i \geqslant 1} \mathfrak{g}_i$ (the sum of eigenspaces of π-weight strictly greater than one).[3] Then $\mathfrak{u}$ and $\mathfrak{u}_+$ integrate to subgroups $U_+ \subset U \subset G$.

Whittaker induction sends graded Hamiltonian H-varieties to a graded Hamiltonian G-varieties, (but uses shearing and unshearing with respect to π in the process). Note that $H \cap U$ is trivial (since our hypotheses imply $\mathfrak{h} \subset \mathfrak{j}$), H normalizes U, and $HU \subset G$.[4]

Example 8.4. Let $G = \mathrm{GL}_n$, $\pi : \mathbb{G}_m \xrightarrow{2\rho} \mathrm{GL}_n$, and let H be trivial. Let

$$f = \begin{bmatrix} 0 & & & \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix}.$$

Then

$$U = \begin{bmatrix} 1 & * & \dots & * \\ & 1 & \dots & * \\ & & \ddots & * \\ & & & 1 \end{bmatrix}$$

and $HU = U$. In this case,

$$\begin{aligned} \mathrm{WhitInd}_{H,2\rho,f}^{\mathrm{GL}_n} &= h\,\mathrm{Ind}_{HU}^{G}(\mathrm{pt} \times (\mathfrak{u}/\mathfrak{u}_+)_f) \\ &= h\,\mathrm{Ind}_{U}^{\mathrm{GL}_n}(\mathrm{pt}_f) \\ &= T^* \mathrm{GL}_n /\!/_f U. \end{aligned}$$

Suppose M is a hyperspherical variety. Then $M = \mathrm{WhitInd}_{H,\pi,f}^{G} S$, where H is the stabilizer of the unique closed M_0 and S is the fiber of the symplectic normal bundle over any $x \in M_0$ (and π and f are as in the discussion from last time?).

Example 8.5. Let $M = T^*X$ where $X = G/H$. Taking $S = \mathrm{pt}$, π the trivial homomorphism, and $f = 0$, we obtain M via Whittaker induction.

Example 8.6. Let $G = \mathrm{GL}_{2n}$, $H = \mathrm{Sp}_{2n}$, and $V = \mathbb{C}^{2n} \oplus (\mathbb{C}^{2n})^*$. Then

$$\mathrm{WhitInd}_{H,1,0}^{G}(V) \simeq T^*((\mathrm{GL}_{2n} \times \mathbb{C}^{2n})/\mathrm{Sp}_{2n}),$$

and this contains as an open dense subset

$$\mathrm{WhitInd}_{1,2\rho,\psi}^{G}(\mathrm{pt}) \simeq T^*(\mathrm{GL}_{2n} /\!/_\psi U)$$

where $\psi : U \to \mathbb{G}_a$ takes a $2n \times 2n$ matrix in U and sums the entries directly above the diagonal 1s (except for the nth such entry, which is not contained in a diagonal block of the matrix).

[3] For ease of reading, one is recommended to focus on examples where $\mathfrak{u} = \mathfrak{u}_+$, but $\mathfrak{u}/\mathfrak{u}_+$ is a point with a nontrivial moment map.

[4] I had to leave at this point – many thanks to Yuji Okitani for graciously sharing his notes.

9 10/26 (Jeremy Taylor) – Relative Local Langlands and Examples

The goal of these lectures is to explain the next section of the paper and discuss the geometric Satake theorem.

9.1 The relative local Langlands conjecture

The relative local Langlands conjecture is that, given $\mathsf{G} \curvearrowright \mathsf{M}$ with M hyperspherical (in particular affine), there is a dual action $\check{\mathsf{G}} \curvearrowright \check{\mathsf{M}}$ (with $\check{\mathsf{G}}$ the Langlands dual of G and $\check{\mathsf{M}}$ some hyperspherical variety) satisfying an implicitly derived equivalence

$$\mathsf{Sh}(\mathsf{M}_{\mathcal{K}}/\mathsf{G}_{\mathcal{O}}) = \mathsf{QCoh}(\check{\mathsf{M}}/\check{\mathsf{G}}).$$

Here $\mathcal{K} = \mathsf{k}((\mathsf{t}))$ (so $\mathsf{M}_{\mathcal{K}}$ is the loop space of M) and $\mathcal{O} = \mathsf{k}[[\mathsf{t}]]$ (so $\mathsf{G}_{\mathcal{O}}$ is the arc space of G).

Example 9.1. For $\mathsf{M} = \mathrm{pt}$ and G arbitrary, we have

$$\check{\mathsf{M}} = \mathsf{T}^*(\check{\mathsf{G}}/(\check{\mathsf{N}}, \mathsf{f})) = \check{\mathsf{G}} \times (\mathsf{f} + \check{\mathfrak{g}}^e),$$

where $\check{\mathsf{N}}$ is strictly upper triangular matrices and f is a regular nilpotent lower triangular matrix (an element of $(\check{\mathfrak{n}})^- = \check{\mathfrak{n}}^*$). The relative local Langlands conjecture gives

$$\mathsf{Sh}(\mathrm{pt}/\mathsf{G}) = \mathsf{QCoh}(\mathsf{f} + \check{\mathfrak{g}}^e).$$

Note that $\mathsf{f} + \check{\mathfrak{g}}^e = \check{\mathfrak{g}}/\!/\check{\mathsf{G}}$.

Suppose that $\mathsf{M} = \mathsf{T}^*\mathsf{X}$. We can construct $\check{\mathsf{M}}$ from its Whittaker data as follows. For $\check{\mathsf{H}}_\mathsf{X} \subset \check{\mathsf{G}}$ a reductive subgroup, $\check{\mathsf{H}}_\mathsf{X} \curvearrowright \mathsf{S}_\mathsf{X}$ a symplectic representation, and $\mathsf{f} \in \check{\mathfrak{g}}$ nilpotent (all described later), we have

$$\check{\mathsf{M}} = \mathrm{WhitInd}^{\check{\mathsf{G}}}_{\check{\mathsf{H}}_\mathsf{X},\mathsf{f}}(\mathsf{S}_\mathsf{X}) = \big((\mathsf{S}_\mathsf{X} \times (\check{\mathfrak{u}}/\check{\mathfrak{u}}_+)_\mathsf{f}) \times_{\check{\mathfrak{h}}^*_\mathsf{X} \oplus \check{\mathfrak{u}}_\mathsf{f}} \mathsf{T}^*\mathsf{G}\big)/(\check{\mathsf{H}}_\mathsf{X}\check{\mathsf{U}}).$$

This is analogous to parabolic induction from the classical theory of linear algebraic groups – it is a process for constructing hyperspherical varieties for $\check{\mathsf{G}}$ from hyperspherical varieties for simpler subgroups. Note that if the $\mathfrak{sl}_2$-triple containing f has even weights for its action on $\check{\mathfrak{g}}$, then $\check{\mathfrak{u}}/\check{\mathfrak{u}}_+ = \mathrm{pt}$.

Coweights of $\check{\mathsf{H}}_\mathsf{X}$ should be spanned by T-weights in $\mathsf{k}[\mathsf{X}]^\mathsf{N}$, where $\mathsf{N} \subset \mathsf{G}$ is the standard nilpotent subgroup. The highest weights of the representation S_x can be determined by looking at divisors in X which are stable under G. Let P_X be the maximal subgroup of G stabilizing the open B-orbit in X, and take f to be a principal nilpotent element of the Lie algebra of the dual Levi $\check{\mathsf{L}}_\mathsf{X} \subset \check{\mathsf{P}}_\mathsf{X}$.

Example 9.2. If $\mathsf{S}_\mathsf{X} = 0$ and $\mathsf{f} = 0$, then $\check{\mathsf{M}} = \mathsf{T}^*(\check{\mathsf{G}}/\check{\mathsf{H}})$.

9.2 An example: $\mathsf{M} = \mathrm{pt}$

Example 9.3. Let $\mathsf{M} = \mathsf{T}^*\mathsf{X} = \mathrm{pt}$. In this case, we see that coweights of $\check{\mathsf{H}}_\mathsf{X}$ are zero, and thus $\check{\mathsf{H}}_\mathsf{x}$ must be 1. The highest weights of S_x must be empty, so $\mathsf{S}_\mathsf{X} = 0$. Here $\mathsf{P}_\mathsf{X} = \mathsf{G}$, so $\check{\mathsf{L}}_\mathsf{X} = \check{\mathsf{P}}_\mathsf{X} = \check{\mathsf{G}}$, and f is a principal nilpotent element of $\check{\mathsf{G}}$. The Whittaker induction formula yields

$$\check{\mathsf{M}} = \mathsf{T}^*(\check{\mathsf{G}}/(\check{\mathsf{N}}, \mathsf{f})) = \big((\mathsf{f} + \check{\mathfrak{n}}^\perp) \times \check{\mathsf{G}}\big)/\check{\mathsf{N}}.$$

There is an isomorphism $(\mathsf{f} + \check{\mathfrak{n}}^\perp) \Leftrightarrow \check{\mathsf{N}} \times (\mathsf{f} + \check{\mathfrak{g}}^e)$ given by $\mathrm{Ad}_n(x) \leftrightarrow (n, x)$. Thus we may identify $\check{\mathsf{M}} = (\mathsf{f} + \check{\mathfrak{g}}^e) \times \check{\mathsf{G}}$.

This has commuting $\mathbb{G}_m$-action given by

$$z \cdot (x, g) = \big(\mathrm{Ad}_{2\rho(z)}(z^2 x), g \cdot 2\rho(z)\big).$$

The core $\check{\mathsf{M}}_0$ (i.e. the unique closed $\check{\mathsf{G}} \times \mathbb{G}_m$-orbit) is $\mathsf{f} \times \check{\mathsf{G}}$ (which is isomorphic to $\check{\mathsf{G}}/\check{\mathsf{H}}_\mathsf{X} = \check{\mathsf{G}}$).

The automorphic side of relative local Langlands is

$$\mathsf{Sh}(\mathsf{M}_{\mathcal{K}}/\mathsf{G}_{\mathcal{O}}) = \mathsf{Sh}(\mathrm{pt}/\mathsf{G}) = \mathsf{Mod}_{\mathrm{Nilp}}(\mathsf{H}^*(\mathrm{pt}/\mathsf{G})) = \mathsf{Mod}_{\mathrm{Nilp}}\big((\mathrm{Sym}\,\mathfrak{g}[-2])^{\mathsf{G}}\big).$$

The spectral side of relative local Langlands is

$$\mathsf{QCoh}(\check{\mathsf{M}}/\check{\mathsf{G}}) = \mathsf{QCoh}(\mathsf{f} + \check{\mathfrak{g}}^e) = \mathsf{QCoh}(\check{\mathfrak{g}}/\!/\check{\mathsf{G}}).$$

We can identify the two, proving relative local Langlands in this case.