GRT Seminar Fall 2024 – Categorical Representation Theory

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Abstract

This semester, the GRT Seminar will focus on "categorical representation theory." A good reference is "An informal introduction to categorical representation theory and the local geometric Langlands program" by Gurbir Dhillon.

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1 9/5 (David Nadler) – Introduction

Today will be fairly informal – we'll just talk about basic definitions and the example of Lusztig's character sheaves.

1.1 What is a representation of a group?

Consider the following well-studied classes of groups:

- Finite groups
- Groups over local fields (e.g. Lie groups, $\mathfrak{p}$-adic groups, . . .)

Many interesting examples (e.g. finite groups of Lie type) arise as the points of an algebraic group. These algebraic groups often have a *geometric* interpretation.

Representations of groups are interesting for two reasons:

- They allow us to understand abstract groups concretely.
- They allow us to study the symmetries of objects we already cared about.

We can describe a representation of a group as a homomorphism $G \to \mathrm{Aut}_{\mathcal{C}}(c)$ for some object c of a category $\mathcal{C}$.

Example 1.1. In the most classical setup, we take $\mathcal{C} = \mathsf{Vect}_{\mathbb{C}}$, the category of complex vector spaces. Here $\mathrm{Aut}_{\mathsf{Vect}_{\mathbb{C}}}(V) = \mathrm{GL}(V)$, the *general linear group.* Note that, at some point, we have to account for the presence of $\mathbb{C}$ here. We also have to ask: do we consider continuous representations? Algebraic representations?

Let's try to reformulate this in a way that makes the dependence on choices clearer. Given a group G and a coefficient field k, we can construct a *coalgebra of functions* $\mathrm{Fun}_k(G)$ and an *algebra of distributions* $\mathrm{Dist}_k(G)$. For a finite group G, these are the same. A representation of G can be interpreted as either a *comodule* over $\mathrm{Fun}_k(G)$ or a *module* over $\mathrm{Dist}_k(G)$.

The choices we make are:

- The coefficient field k
- The types of representation (i.e. types of "functions" or "distributions" considered)

Let's see this in an example.

Example 1.2. What are representations of S^1?

- One answer comes by viewing S^1 as a compact Lie group, so a representation V decomposes as $\oplus_{n\in\mathbb{Z}} V_n$, where $e^{i\theta}$ acts on V_n by multiplication by $e^{in\theta}$.
- If we use homotopy theory, all we can see about S^1 is the homotopy type – in particular, S^1 and $\mathbb{C}^\times$ should "have the same representation theory." The correct notion of a "distribution" here is $\mathrm{Dist}_{\mathbb{C}}(S^1) = C_{-\bullet}(S^1;\mathbb{C}) \simeq \mathbb{C}[\epsilon]$, where $\deg \epsilon = -1$ and $\epsilon^2 = 0$. Thus representations of S^1 are $\mathbb{C}[\epsilon]$-modules.

  Representations in the ordinary category $\mathsf{Vect}_{\mathbb{C}}$ are not interesting – ϵ must always act trivially. However, if we consider representations in the dg-category of $\mathbb{C}$-dg-vector spaces, we get a more interesting answer – the category of dg-representations of S^1 is $\mathbb{C}[\epsilon]$-dg-modules.

1.2 Why study categorical representation theory?

Categorical representation theory is based on an old "miracle."

Consider the representation theory of finite groups (especially those of Lie type). There are two classical ways to construct representations of such groups:

- Induction from abelian groups (e.g. tori)
- Prayer (e.g. "cuspidal representations")

Lusztig realized a good way to make sense of cuspidal representations. Recall that representations of a finite group G are equivalent to characters of class functions on G. So we can rephrase our problem: how do we construct characters of cuspidal representations? The trick is to instead construct character *sheaves* on the corresponding algebraic groups. Once we have the character sheaves, we can recover the character functions by decategorification.

The "miracle" is that character sheaves can be accessed via induction from tori! One can explain this by appealing to topological field theory: there's a tradeoff between asking seemingly "easy" questions about complicated manifolds and asking seemingly "hard" questions about simpler manifolds (e.g. points). In many cases, one can answer questions about the former by finding the right questions to ask about the latter.

2 9/12 (Raymond Guo) – D-Modules

2.1 (David) – Brief orientation

Let G be an algebraic group, e.g. GL_n. When considering a "type of representations," we need to choose a notion of "group algebra" / "distributions." In categorical representation theory, we take the "group algebra" to be a monoidal category.

Example 2.1. The fundamental example to consider is the category $D(G)$ of D-modules on G.

We can find "representations" of $D(G)$ by considering G-varieties X. The action of G on X gives an action of $D(G)$ on $D(X)$.

Example 2.2. The fundamental example is $X = G/B$, the flag variety of G.

2.2 (Raymond) – Weyl algebras

Consider a homogeneous linear ordinary differential equation

$$\sum_{i=0}^{N} p_i(t)\partial^i f(t) = 0.$$

Here $\partial = d/dt$. We'd like to understand this algebraically.

To accomplish this, we work in the Weyl algebra $D_{\mathbb{A}^1} = \mathbb{C}\langle t, \partial\rangle/([\partial, t] - 1)$, where the relation comes from the Leibniz rule

$$\partial(f \cdot t) = 1 \cdot f + t\partial f.$$

In higher dimensions, we use the Weyl algebra

$$D_{\mathbb{A}^n} = \frac{\mathbb{C}\langle t_1, \dots, t_2, \partial_1, \dots, \partial_n\rangle}{([t_i, t_j], [\partial_i, \partial_j], [\partial_i, \partial_j], [\partial_i, t_i] - 1)}.$$

Here the relations are imposed for $i \neq j$. This allows us to work algebraically with linear PDEs.

Let's consider modules over $D_{\mathbb{A}^1}$.

Example 2.3. Letting ∂ act by d/dt, we see that $\mathbb{C}[t]$ is a $D_{\mathbb{A}^1}$-module.

Example 2.4. The module $D_{\mathbb{A}^1}/(D_{\mathbb{A}^1} \cdot t)$ can be understood as $\mathbb{C}[\delta]$ where t acts by $t\delta^n = -n\delta^{n-1}$.

David gave an exercise of finding all simple modules for $D_{\mathbb{A}^1}$.

2.3 D-modules on general varieties

Let X be a smooth affine variety over a field k of characteristic zero. The endomorphism algebra $\mathrm{Hom}_k(\mathcal{O}_X, \mathcal{O}_X)$ contains $\mathcal{O}_X(X)$ (embedded via $f \mapsto (a \mapsto f \cdot a)$) as well as

$$\mathrm{Vect}_X = \{F \in \mathrm{Hom}_k(\mathcal{O}_X, \mathcal{O}_X) \mid F(fg) = F(f)g + fF(g)\}.$$

Taken together, these generate a subalgebra $D_X = k\langle \mathcal{O}_X, \mathrm{Vect}_X\rangle \subset \mathrm{Hom}_k(\mathcal{O}_X, \mathcal{O}_X)$.

Example 2.5. One can check that $D_{\mathbb{A}^n}$ agrees with the presentation given in the previous section.

All of the above discussion globalizes to give a quasicoherent sheaf of k-linear algebras $\mathcal{D}_X$ on any (not necessarily affine) algebraic variety X. This is locally generated by $\mathcal{O}_X$ and the sheaf of vector fields / derivations $\mathcal{V}\mathrm{ect}_X$. A D-module on such an X is a sheaf of $\mathcal{D}_X$-modules which is quasicoherent as an $\mathcal{O}_X$-module.

Example 2.6. For any X, $\mathcal{O}_X$ may be viewed as a D-module by letting $\mathcal{D}_X$ act in the usual way. Because $\mathcal{V}\mathrm{ect}_X$ kills the constant function 1, which generates $\mathcal{O}_X$, we see $\mathcal{O}_X \simeq \mathcal{D}_X/\mathcal{D}_X \cdot \mathcal{V}\mathrm{ect}_X$. One may think of this as the trivial vector bundle equipped with the standard flat connection.

Maps $\mathcal{O}_X \to \mathcal{F}$ correspond to "flat" sections of $\mathcal{F}$, i.e. sections of $\mathcal{F}$ on which $\mathcal{V}\mathrm{ect}_X$ acts trivially. This can be interpreted as "solutions of a system of PDEs."

Example 2.7. Let $x \in X$ be a closed point with maximal ideal $\mathfrak{m}_x$. Then $\delta_x = \mathcal{D}_X/\mathcal{D}_X \cdot \mathfrak{m}_x$ is a D-module on X. (Some authors twist this by a line.) Note that δ_x is supported (set-theoretically, as a quasicoherent sheaf) only at x.

If we take a more sophisticated perspective, δ_x is supported on every infinitesimal neighborhood of x. Maps $\delta_x \to \mathcal{F}$ correspond to sections f of $\mathcal{F}$ which are annihilated by $\mathfrak{m}_x$. Taking the image of ∂_i under such a map recovers (up to a sign) the derivatives of f.

Example 2.8. David suggested that, given a smooth variety X and a smooth subvariety $Y \subset X$, we can construct a D-module of "delta functions supported on Y."

2.4 Functoriality

Suppose $f : X \to Y$ is a morphism of algebraic varieties. We would like to be able to push forward and pull back D-modules along $\mathcal{F}$.

Definition 2.9. For a morphism $f : X \to Y$, let $\mathcal{D}_{X\to Y} = \mathcal{O}_X \otimes_{\mathcal{O}_Y} \mathcal{D}_Y$. Given a D-module $\mathcal{F}$ on Y, we define $f^*\mathcal{F} = \mathcal{D}_{X\to Y} \otimes_{\mathcal{D}_Y} \mathcal{F}$.

As a quasicoherent sheaf, $f^*\mathcal{F}$ is just the usual $f^*\mathcal{F} := \mathcal{O}_X \otimes_{\mathcal{O}_Y} \mathcal{F}$, but now we have a compatible $\mathcal{D}_X$-action.

Remark 2.10. One must be very careful about handedness: do we consider *left* or *right* D-modules? Because $\mathcal{D}_X$ is noncommutative, these two are different. We have been working with left D-modules so far. A standard left D-module is $\mathcal{O}_X = \mathcal{D}/\mathcal{D} \cdot \mathcal{V}\mathrm{ect}_X$, with sections looking like f. Stated differently, functions are a left D-module.

A standard right D-module is Ω_X^{top}, with sections looking like $f \cdot dx_1 \dots dx_n$ on charts. Here $\mathcal{D}_X$ acts (in a chart) by Lie derivatives. We can write $\Omega_x^{\mathrm{top}} = \det(T_X)^{\pm 1} \otimes (\mathcal{V}\mathrm{ect}_X \cdot \mathcal{D}_X)\backslash\mathcal{D}$. Stated differently, distributions are a right D-module. Note that we need to work with distributions to be able to pushforward.

We can convert between left and right D-modules by $\mathcal{F} \leftrightarrow \Omega_X^{\mathrm{top}} \otimes_{\mathcal{D}_X} \mathcal{F}$.

3 9/19 (Raymond Guo) – Continued

3.1 Refresher

Suppose $X = \operatorname{Spec} A$ is a smooth affine variety. Recall that we have a module of vector fields / derivations

$$\mathrm{Vect}_X = \{v : \mathcal{O}_X \to \mathcal{O}_X \mid v(fg) = v(f)g + fv(g)\}.$$

the module of Kähler differentials $\Omega_{A/k}$ is the universal receiver of k-linear derivations from A. That is,

$$\mathrm{Der}_k(A, M) \cong \mathrm{Hom}_A(\Omega_{A/k}, M).$$

In particular, $\mathrm{Vect}_X \cong \mathrm{Der}_k(A, A) \cong \mathrm{Hom}_A(\Omega_{A/k}, A)$.

We can globalize / sheafify all of these definitions to obtain sheaves of vector fields and differentials. Thus, in the following, we let X be any smooth variety.

3.2 Distributions

Let $n = \dim X$, and let $\Omega_X^{\mathrm{top}} = \wedge^n \Omega_X$. Given a vector field ξ and a top form ω, we define the *Lie derivative* $\mathrm{Lie}(\xi)(\omega)$ by

$$\mathrm{Lie}(\xi)(\omega)(\xi_1, \dots, \xi_n) = \xi\big(\omega(\xi_1, \dots, \xi_n)\big) - \sum_{i=1}^{n} \omega(\xi_1, \dots, [\xi, \xi_i], \dots).$$

This satisfies

$$\begin{aligned}
(\mathrm{Lie}(f\xi))(\omega) &= (\mathrm{Lie}(\xi))(f\omega)\\
(\mathrm{Lie}(\xi))(f\omega) &= (\xi(f))\omega + f\,\mathrm{Lie}(\xi)(\omega)\\
(\mathrm{Lie}([\xi_1, \xi_2]))(\omega) &= -[\mathrm{Lie}(\xi_1), \mathrm{Lie}(\xi_2)]\omega.
\end{aligned}$$

Thus we obtain a *right* action of $\mathcal{D}_X$ on Ω_X^{top}.

Remark 3.1. David explains this as follows. In calculus, functions $f(t)$ have a left action by differential operators. Distributions $g(t)dt$ can be paired with functions via integration. Because functions have a left action by differential operators, the dual of functions (i.e. distributions) must admit a right action. Ansuman pointed out that the presence of minus signs in the above equations can be understood from this perspective: it's just the usual integration by parts!

3.3 Passing between left and right

There is a pair of equivalences

$$\Omega_X^{\mathrm{top}} \otimes_{\mathcal{O}_X} (-) : \mathcal{D}_X\text{-Mod} \leftrightarrow \mathcal{D}_X^{\mathrm{op}}\text{-Mod} : (-) \otimes_{\mathcal{O}_X} (\Omega_X^{\mathrm{top}})^{-1}$$

Here $\mathcal{D}_X$ acts on $M \otimes_{\mathcal{O}_X} \Omega_X^{\mathrm{top}}$ (on the right) via something like $\xi \cdot (m\omega) = -\xi(m)\omega + m\xi(\omega)$.

Remark 3.2. David explained that we can consider $\mathcal{D}\mathrm{iff}(\mathcal{O}_X, \Omega_X^{\mathrm{top}})$, the sheaf of "differential operators from $\mathcal{O}_X$ to Ω_X^{top}." The left action of $\mathcal{D}_X$ on $\mathcal{O}_X$ and the right action of $\mathcal{D}_X$ on Ω_X^{top} gives us a right $\mathcal{D}_X \otimes \mathcal{D}_X$-action on $\mathcal{D}\mathrm{iff}(\mathcal{O}_X, \Omega_X^{\mathrm{top}})$ (so we may view this as a $(\mathcal{D}_X^{\mathrm{op}}, \mathcal{D}_X)$-bimodule). The functor $\mathcal{D}_X\text{-Mod} \to \mathcal{D}_X^{\mathrm{op}}\text{-Mod}$ can also be described as $\mathcal{D}\mathrm{iff}(\mathcal{O}_X, \Omega_X^{\mathrm{top}}) \otimes_{\mathcal{D}_X} (-)$. The inverse functor admits an analogous description.

3.4 Functoriality and convolution

Let's focus on the affine case, so everything is a module. Given $f : X \to Y$, let $\mathcal{D}_{X\to Y} = \mathcal{O}_X \otimes_{\mathcal{O}_Y} \mathcal{D}_Y$. We may define a pullback of $\mathcal{D}$-modules via

$$f^*M = \mathcal{D}_{X\to Y} \otimes_{\mathcal{D}_Y} M.$$

This can also be interpreted as tensoring with the bimodule $\mathcal{D}\mathrm{iff}(f^*\mathcal{O}_Y, \mathcal{O}_X)$.

To define a pushforward of $\mathcal{D}$-modules, we need to pass from left to right $\mathcal{D}$-modules. We define

$$f_*(M) = (\Omega_X^{\mathrm{top}} \otimes_{\mathcal{O}_X} M) \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y}) \otimes_{\mathcal{O}_Y} (\Omega_X^{\mathrm{top}})^{-1}$$

As above, this can be interpreted as tensoring with a bimodule (in this case $\mathcal{D}\mathrm{iff}(f^*\Omega_Y^{\mathrm{top}}, \Omega_X^{\mathrm{top}})$).

We may use these to define convolution products.

Let G be an algebraic group (with multiplication $m : G^2 \to G$) over $\mathbb{C}$. At closed points $a \in G$, we may construct δ-modules $\delta_a = \mathcal{D}_G/\mathcal{D}_G \cdot \mathfrak{m}_{G,a}$. The convolution product of δ_a and δ_b is

$$m_*\big(\pi_1^*\delta_a \otimes_{\mathcal{O}_{G\times G}} \pi_2^*\delta_b\big) = \delta_{m(a,b)}$$

In general, the convolution product defines a monoidal structure $\star$ on $\mathcal{D}_G$-Mod. The monoidal category $(\mathcal{D}_G\text{-Mod}, \star)$ is the *smooth categorical group algebra.* Actions of G on a space (e.g. G/B) give rise to module categories for the smooth categorical group algebra.

4 9/26 (Brian Yang) – Beilinson-Bernstein Localization

Let G be a semisimple Lie group with Lie algebra $\mathfrak{g}$. Let $B \subset G$ be a Borel subgroup, so G/B is the flag variety.

Beilinson-Bernstein localization gives an adjunction between:

- Representations of $\mathfrak{g}$ with central character 0 (this will be defined later).
- D-modules on G/B.

We can go from D-modules to representations by taking global sections. The left adjoint of this is Beilinson-Bernstein localization.

Really, we should think of our representations as living in category $\mathcal{O}$, which includes the finite dimensional representations and the Verma modules. This essentially consists of "highest weight" representations which can possibly be infinite-dimensional. We'd like to connect this with D-modules on the flag varieties.

4.1 Review of $\mathfrak{sl}_2$

Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$, with basis

$$e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \qquad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Let V be a representation of $\mathfrak{g}$. We say that $v \in V$ is a *heighest weight* vector of weight $n \geqslant 0$ if, letting $v_k = \frac{1}{k!} f^k v$ for $k \geqslant 0$, we have

$$hv_k = (n-2k)v_k, \qquad fv_k = (k+1)v_{k+1}, \qquad ev_k = (n-k+1)v_{k-1} \text{ with } ev_0 = 0.$$

In particular, the weight n is the eigenvalue of h corresponding to the eigenvector v.

For fixed n, we define the *Verma module* $M(n) = \mathrm{Span}_{\mathbb{C}}\{v_0, v_1, \dots\}$ as the free (infinite-dimensional) $\mathfrak{g}$-representation with a highest weight vector of weight n. Letting $\mathfrak{b} = \mathrm{Span}_{\mathbb{C}}\{e, h\}$, we can also define $M(n)$ as $\mathcal{U}\mathfrak{g} \otimes_{\mathcal{U}\mathfrak{b}} \mathbb{C}_n \cdot v_0$.

We may view $M(-n-2)$ as $\mathrm{Span}_{\mathbb{C}}\{v_{n+1}, v_{n+2}, \dots\} \subset M(n)$. The quotient $V(n) = M(n)/M(-n-2)$ is the unique irrep of dimension $n+1$. We call the resolution

$$0 \longrightarrow M(-n-2) \longrightarrow M(n) \longrightarrow V(n) \longrightarrow 0$$

the *Bernstein-Gelfand-Gelfand resolution.*

4.2 D-modules on $\mathbb{P}^1$ and representations

Let $G = \mathrm{SL}_2(\mathbb{C})$. The flag variety of G is $X = \mathbb{P}^1$. We may write $X = U_\infty \cup U_0$ where $U_\infty = \{[w:1]\}$ and $U_0 = \{[1:z]\}$. The sheaf $\mathcal{D}_X$ satisfies $\mathcal{D}_X(U_0) = \mathbb{C}\langle z, \partial_z\rangle$ and $\mathcal{D}_X(U_\infty) = \mathbb{C}\langle w, \partial_w\rangle$.

We have an action of G on X by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} [x:y] = [ax+by : cx+dy].$$

This induces a Lie algebra homomorphism $\mathfrak{g} \to \mathrm{Vect}(X)$.

In the U_0 chart, we can compute

$$\begin{aligned} e \cdot f(z) &= \frac{d}{dt}\bigg|_{t=0} \exp(te) f(z) \\ &= \frac{d}{dt}\bigg|_{t=0} f(\exp(-te)z) \\ &= \frac{d}{dt}\bigg|_{t=0} f\left(\begin{bmatrix} 1 & -t \\ 0 & 1 \end{bmatrix} [1:z]\right) \\ &= z^2 f'(z). \end{aligned}$$

That is, e acts by $z^2\partial_z$.

We can make similar computations for other elements / charts:

- In U_0, we have $e \mapsto z^2\partial_z$, $f \mapsto -\partial_z$, and $h \mapsto 2z\partial_z$.
- In U_∞, we have $e \mapsto -\partial_w$, $f \mapsto w^2\partial_w$, and $h \mapsto -2w\partial_w$.

These can all be interpreted geometrically: e is a vector field with a zero of order two at 0, f is a vector field with a zero of order two at ∞, and h generates the scaling action of $\mathbb{C}^\times$.

Using the homomorphism $\mathfrak{g} \to \mathrm{Vect}(X)$, we see that the global sections of any D-module on $\mathbb{P}^1$ give a representation of $\mathfrak{g}$!

Example 4.1. Letting $\mathcal{O}_X$ be the structure sheaf with the usual action of $\mathcal{D}_X$, we see that $\Gamma(X, \mathcal{O}_X) = \mathbb{C}$ consists of constant functions. Because $\mathrm{Vect}(X)$ acts trivially on constant functions, we get $\Gamma(X, \mathcal{O}_X) \cong V(0)$ as $\mathfrak{g}$-representations.

Example 4.2. Let

$$\mathsf{N} = \left\{ \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix} \right\}$$

act on X by restricting the G-action to N. This has two orbits: $\{0\}$ and U_∞. We can pushforward the structure sheaves of the orbits to X and obtain D-modules on X.

Let $\mathcal{M}_{\mathrm{pt}} = i_{\mathrm{pt}\,*}\mathcal{O}_{\mathrm{pt}}$. To compute this, note that

$$\mathcal{D}_{\mathrm{pt}\to U_0}(U_0) = \mathbb{C}[z]/(z) \otimes_{\mathbb{C}[z]} \mathbb{C}\langle z, \partial_z \rangle = \oplus_{i\geqslant 0}\mathbb{C}\cdot \delta\partial_z^i$$

where δ denotes a (formal) δ-function at 0. Since $U_0 \cong \mathbb{A}^1$ has trivial canonical bundle, we also get $\mathcal{D}_{U_0\leftarrow \mathrm{pt}}(U_0) = \oplus_{i\geqslant 0}\mathbb{C}\cdot \partial_z^i\delta$. It follows that

$$\Gamma(X, \mathcal{M}_{\mathrm{pt}}) = \mathcal{M}_{\mathrm{pt}}(U_0) = \oplus_{i\geqslant 0}\mathbb{C}\cdot\partial_z^i\delta = M(-2).$$

Here the Lie algebra action is computed by taking the basis $m_k = \frac{(-1)^k}{k!}\partial_z^k\delta$ and using the fact that $z\delta = 0$.

David made some comments about why we can think of δ as a δ-function: because $z\delta = 0$, any map $\mathcal{M}_{\mathrm{pt}} \to \mathcal{F}$ must send δ to some f satisfying $zf = 0$.

We'd also like to understand the pushforward of the structure sheaf of the open orbit. For this, it's useful to start with some general remarks. If $i : U \hookrightarrow X$ is an open immersion and $\mathcal{M}$ is a $\mathcal{D}_U$-module, then $i_*\mathcal{M}(V) = \mathcal{M}(U \cap V)$. The $\mathcal{D}_X$-module structure on $i_*\mathcal{M}$ agrees with the $\mathcal{D}_U$-module structure on $\mathcal{M}$. That is, pushforwards of D-modules along open immersions are just the usual pushforwards with the "obvious" D-module structure!

Example 4.3. Let $\mathcal{M}_{\mathrm{open}} = i_{U_\infty *}\mathcal{O}_{U_\infty}$. Then $\Gamma(X, \mathcal{M}_{\mathrm{open}}) = \Gamma(U_\infty, \mathcal{O}_{U_\infty}) = \mathbb{C}[w]$ with its usual $\mathcal{D}_{U_\infty}$-module structure. Choosing the basis $n_k = (-1)^k w^k$, we can compute the Lie algebra action and see that $\Gamma(X, \mathcal{M}_{\mathrm{open}}) = M(0)^\vee$, the dual Verma module.