GRT Seminar Fall 2024 – Categorical Representation Theory

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Abstract

This semester, the GRT Seminar will focus on "categorical representation theory." A good reference is "An informal introduction to categorical representation theory and the local geometric Langlands program" by Gurbir Dhillon.

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1 9/5 (David Nadler) – Introduction

Today will be fairly informal – we'll just talk about basic definitions and the example of Lusztig's character sheaves.

1.1 What is a representation of a group?

Consider the following well-studied classes of groups:

- Finite groups
- Groups over local fields (e.g. Lie groups, p-adic groups, ...)

Many interesting examples (e.g. finite groups of Lie type) arise as the points of an algebraic group. These algebraic groups often have a *geometric* interpretation.

Representations of groups are interesting for two reasons:

- They allow us to understand abstract groups concretely.
- They allow us to study the symmetries of objects we already cared about.

We can describe a representation of a group as a homomorphism $G \to \mathrm{Aut}_{\mathcal{C}}(c)$ for some object c of a category $\mathcal{C}$.

Example 1.1. In the most classical setup, we take $\mathcal{C} = \mathsf{Vect}_{\mathbb{C}}$, the category of complex vector spaces. Here $\mathrm{Aut}_{\mathsf{Vect}_{\mathbb{C}}}(V) = \mathrm{GL}(V)$, the *general linear group*. Note that, at some point, we have to account for the presence of $\mathbb{C}$ here. We also have to ask: do we consider continuous representations? Algebraic representations?

Let's try to reformulate this in a way that makes the dependence on choices clearer. Given a group G and a coefficient field k, we can construct a *coalgebra of functions* $\mathrm{Fun}_k(G)$ and an *algebra of distributions* $\mathrm{Dist}_k(G)$. For a finite group G, these are the same. A representation of G can be interpreted as either a *comodule* over $\mathrm{Fun}_k(G)$ or a *module* over $\mathrm{Dist}_k(G)$.

The choices we make are:

- The coefficient field k
- The types of representation (i.e. types of "functions" or "distributions" considered)

Let's see this in an example.

Example 1.2. What are representations of S^1?

- One answer comes by viewing S^1 as a compact Lie group, so a representation V decomposes as $\oplus_{n\in\mathbb{Z}} V_n$, where $e^{i\theta}$ acts on V_n by multiplication by $e^{in\theta}$.
- If we use homotopy theory, all we can see about S^1 is the homotopy type – in particular, S^1 and $\mathbb{C}^\times$ should "have the same representation theory." The correct notion of a "distribution" here is $\mathrm{Dist}_{\mathbb{C}}(S^1) = C_{-\bullet}(S^1;\mathbb{C}) \simeq \mathbb{C}[\epsilon]$, where $\deg \epsilon = -1$ and $\epsilon^2 = 0$. Thus representations of S^1 are $\mathbb{C}[\epsilon]$-modules.

  Representations in the ordinary category $\mathsf{Vect}_{\mathbb{C}}$ are not interesting – ϵ must always act trivially. However, if we consider representations in the dg-category of $\mathbb{C}$-dg-vector spaces, we get a more interesting answer – the category of dg-representations of S^1 is $\mathbb{C}[\epsilon]$-dg-modules.

1.2 Why study categorical representation theory?

Categorical representation theory is based on an old "miracle."

Consider the representation theory of finite groups (especially those of Lie type). There are two classical ways to construct representations of such groups:

- Induction from abelian groups (e.g. tori)
- Prayer (e.g. "cuspidal representations")

Lusztig realized a good way to make sense of cuspidal representations. Recall that representations of a finite group G are equivalent to characters of class functions on G. So we can rephrase our problem: how do we construct characters of cuspidal representations? The trick is to instead construct character *sheaves* on the corresponding algebraic groups. Once we have the character sheaves, we can recover the character functions by decategorification.

The "miracle" is that character sheaves can be accessed via induction from tori! One can explain this by appealing to topological field theory: there's a tradeoff between asking seemingly "easy" questions about complicated manifolds and asking seemingly "hard" questions about simpler manifolds (e.g. points). In many cases, one can answer questions about the former by finding the right questions to ask about the latter.

2 9/12 (Raymond Guo) – D-Modules

2.1 (David) – Brief orientation

Let G be an algebraic group, e.g. GL_n. When considering a "type of representations," we need to choose a notion of "group algebra" / "distributions." In categorical representation theory, we take the "group algebra" to be a monoidal category.

Example 2.1. The fundamental example to consider is the category $D(G)$ of D-modules on G.

We can find "representations" of $D(G)$ by considering G-varieties X. The action of G on X gives an action of $D(G)$ on $D(X)$.

Example 2.2. The fundamental example is $X = G/B$, the flag variety of G.

2.2 (Raymond) – Weyl algebras

Consider a homogeneous linear ordinary differential equation

$$\sum_{i=0}^{N} p_i(t)\partial^i f(t) = 0.$$

Here $\partial = d/dt$. We'd like to understand this algebraically.

To accomplish this, we work in the Weyl algebra $D_{\mathbb{A}^1} = \mathbb{C}\langle t, \partial\rangle/([\partial, t] - 1)$, where the relation comes from the Leibniz rule

$$\partial(f \cdot t) = 1 \cdot f + t\partial f.$$

In higher dimensions, we use the Weyl algebra

$$D_{\mathbb{A}^n} = \frac{\mathbb{C}\langle t_1, \dots, t_2, \partial_1, \dots, \partial_n\rangle}{([t_i, t_j], [\partial_i, \partial_j], [\partial_i, \partial_j], [\partial_i, t_i] - 1)}.$$

Here the relations are imposed for $i \neq j$. This allows us to work algebraically with linear PDEs.

Let's consider modules over $D_{\mathbb{A}^1}$.

Example 2.3. Letting ∂ act by d/dt, we see that $\mathbb{C}[t]$ is a $D_{\mathbb{A}^1}$-module.

Example 2.4. The module $D_{\mathbb{A}^1}/(D_{\mathbb{A}^1} \cdot t)$ can be understood as $\mathbb{C}[\delta]$ where t acts by $t\delta^n = -n\delta^{n-1}$.

David gave an exercise of finding all simple modules for $D_{\mathbb{A}^1}$.

2.3 D-modules on general varieties

Let X be a smooth affine variety over a field k of characteristic zero. The k-linear endomorphism algebra $\mathrm{Hom}_k(\mathcal{O}_X, \mathcal{O}_X)$ contains $\mathcal{O}_X(X)$ (embedded via $f \mapsto (a \mapsto f \cdot a)$) as well as

$$\mathrm{Vect}_X = \{F \in \mathrm{Hom}_k(\mathcal{O}_X, \mathcal{O}_X) \,|\, F(fg) = F(f)g + fF(g)\}.$$

Taken together, these generate a subalgebra $D_X = k\langle \mathcal{O}_X, \mathrm{Vect}_X \rangle \subset \mathrm{Hom}_k(\mathcal{O}_X, \mathcal{O}_X)$.

Example 2.5. One can check that $D_{\mathbb{A}^n}$ agrees with the presentation given in the previous section.

All of the above discussion globalizes to give a quasicoherent sheaf of k-linear algebras $\mathcal{D}_X$ on any (not necessarily affine) algebraic variety X. This is locally generated by $\mathcal{O}_X$ and the sheaf of vector fields / derivations $\mathcal{V}\mathrm{ect}_X$. A D-module on such an X is a sheaf of $\mathcal{D}_X$-modules which is quasicoherent as an $\mathcal{O}_X$-module.

Example 2.6. For any X, $\mathcal{O}_X$ may be viewed as a D-module by letting $\mathcal{D}_X$ act in the usual way. Because $\mathcal{V}\mathrm{ect}_X$ kills the constant function 1, which generates $\mathcal{O}_X$, we see $\mathcal{O}_X \simeq \mathcal{D}_X/\mathcal{D}_X \cdot \mathcal{V}\mathrm{ect}_X$. One may think of this as the trivial vector bundle equipped with the standard flat connection.

Maps $\mathcal{O}_X \to \mathcal{F}$ correspond to "flat" sections of $\mathcal{F}$, i.e. sections of $\mathcal{F}$ on which $\mathcal{V}\mathrm{ect}_X$ acts trivially. This can be interpreted as "solutions of a system of PDEs."

Example 2.7. Let $x \in X$ be a closed point with maximal ideal $\mathfrak{m}_x$. Then $\delta_x = \mathcal{D}_X/\mathcal{D}_X \cdot \mathfrak{m}_x$ is a D-module on X. (Some authors twist this by a line.) Note that δ_x is supported (set-theoretically, as a quasicoherent sheaf) only at x.

If we take a more sophisticated perspective, δ_x is supported on every infinitesimal neighborhood of x. Maps $\delta_x \to \mathcal{F}$ correspond to sections f of $\mathcal{F}$ which are annihilated by $\mathfrak{m}_x$. Taking the image of ∂_i under such a map recovers (up to a sign) the derivatives of f.

Example 2.8. David suggested that, given a smooth variety X and a smooth subvariety $Y \subset X$, we can construct a D-module of "delta functions supported on Y."

2.4 Functoriality

Suppose $f : X \to Y$ is a morphism of algebraic varieties. We would like to be able to push forward and pull back D-modules along $\mathcal{F}$.

Definition 2.9. For a morphism $f : X \to Y$, let $\mathcal{D}_{X\to Y} = \mathcal{O}_X \otimes_{\mathcal{O}_Y} \mathcal{D}_Y$. Given a D-module $\mathcal{F}$ on Y, we define $f^*\mathcal{F} = \mathcal{D}_{X\to Y} \otimes_{\mathcal{D}_Y} \mathcal{F}$.

As a quasicoherent sheaf, $f^*\mathcal{F}$ is just the usual $f^*\mathcal{F} := \mathcal{O}_X \otimes_{\mathcal{O}_Y} \mathcal{F}$, but now we have a compatible $\mathcal{D}_X$-action.

Remark 2.10. One must be very careful about handedness: do we consider *left* or *right* D-modules? Because $\mathcal{D}_X$ is noncommutative, these two are different. We have been working with left D-modules so far. A standard left D-module is $\mathcal{O}_X = \mathcal{D}/\mathcal{D} \cdot \mathcal{V}\mathrm{ect}_X$, with sections looking like f. Stated differently, functions are a left D-module.

A standard right D-module is Ω_X^{top}, with sections looking like $f \cdot dx_1 \dots dx_n$ on charts. Here $\mathcal{D}_X$ acts (in a chart) by Lie derivatives. We can write $\Omega_x^{\mathrm{top}} = \det(T_X)^{\pm 1} \otimes (\mathcal{V}\mathrm{ect}_X \cdot \mathcal{D}_X)\backslash\mathcal{D}$. Stated differently, distributions are a right D-module. Note that we need to work with distributions to be able to pushforward.

We can convert between left and right D-modules by $\mathcal{F} \leftrightarrow \Omega_X^{\mathrm{top}} \otimes_{\mathcal{D}_X} \mathcal{F}$.

3 9/19 (Raymond Guo) – Continued

3.1 Refresher

Suppose $X = \mathrm{Spec}\, A$ is a smooth affine variety. Recall that we have a module of vector fields / derivations

$$\mathrm{Vect}_X = \{v : \mathcal{O}_X \to \mathcal{O}_X \,|\, v(fg) = v(f)g + fv(g)\}.$$

the module of Kähler differentials $\Omega_{A/k}$ is the universal receiver of k-linear derivations from A. That is,

$$\mathrm{Der}_k(A, M) \cong \mathrm{Hom}_A(\Omega_{A/k}, M).$$

In particular, $\mathrm{Vect}_X \cong \mathrm{Der}_k(A, A) \cong \mathrm{Hom}_A(\Omega_{A/k}, A)$.

We can globalize / sheafify all of these definitions to obtain sheaves of vector fields and differentials. Thus, in the following, we let X be any smooth variety.

3.2 Distributions

Let $n = \dim X$, and let $\Omega_X^{\mathrm{top}} = \wedge^n \Omega_X$. Given a vector field ξ and a top form ω, we define the *Lie derivative* $\mathrm{Lie}(\xi)(\omega)$ by

$$\mathrm{Lie}(\xi)(\omega)(\xi_1, \ldots, \xi_n) = \xi\big(\omega(\xi_1, \ldots, \xi_n)\big) - \sum_{i=1}^{n} \omega(\xi_1, \ldots, [\xi, \xi_i], \ldots).$$

This satisfies

$$\begin{aligned}
(\mathrm{Lie}(f\xi))(\omega) &= (\mathrm{Lie}(\xi))(f\omega) \\
(\mathrm{Lie}(\xi))(f\omega) &= (\xi(f))\omega + f\,\mathrm{Lie}(\xi)(\omega) \\
(\mathrm{Lie}([\xi_1, \xi_2]))(\omega) &= -[\mathrm{Lie}(\xi_1), \mathrm{Lie}(\xi_2)]\omega.
\end{aligned}$$

Thus we obtain a *right* action of $\mathcal{D}_X$ on Ω_X^{top}.

Remark 3.1. David explains this as follows. In calculus, functions $f(t)$ have a left action by differential operators. Distributions $g(t)dt$ can be paired with functions via integration. Because functions have a left action by differential operators, the dual of functions (i.e. distributions) must admit a right action. Ansuman pointed out that the presence of minus signs in the above equations can be understood from this perspective: it's just the usual integration by parts!

3.3 Passing between left and right

There is a pair of equivalences

$$\Omega_X^{\mathrm{top}} \otimes_{\mathcal{O}_X} (-) : \mathcal{D}_X\text{-}\mathsf{Mod} \leftrightarrow \mathcal{D}_X^{\mathrm{op}}\text{-}\mathsf{Mod} : (-) \otimes_{\mathcal{O}_X} (\Omega_X^{\mathrm{top}})^{-1}$$

Here $\mathcal{D}_X$ acts on $M \otimes_{\mathcal{O}_X} \Omega_X^{\mathrm{top}}$ (on the right) via something like $\xi \cdot (m\omega) = -\xi(m)\omega + m\xi(\omega)$.

Remark 3.2. David explained that we can consider $\mathcal{D}\mathrm{iff}(\mathcal{O}_X, \Omega_X^{\mathrm{top}})$, the sheaf of "differential operators from $\mathcal{O}_X$ to Ω_X^{top}." The left action of $\mathcal{D}_X$ on $\mathcal{O}_X$ and the right action of $\mathcal{D}_X$ on Ω_X^{top} gives us a right $\mathcal{D}_X \otimes \mathcal{D}_X$-action on $\mathcal{D}\mathrm{iff}(\mathcal{O}_X, \Omega_X^{\mathrm{top}})$ (so we may view this as a $(\mathcal{D}_X^{\mathrm{op}}, \mathcal{D}_X)$-bimodule). The functor $\mathcal{D}_X\text{-}\mathsf{Mod} \to \mathcal{D}_X^{\mathrm{op}}\text{-}\mathsf{Mod}$ can also be described as $\mathcal{D}\mathrm{iff}(\mathcal{O}_X, \Omega_X^{\mathrm{top}}) \otimes_{\mathcal{D}_X} (-)$. The inverse functor admits an analogous description.

3.4 Functoriality and convolution

Let's focus on the affine case, so everything is a module. Given $f : X \to Y$, let $\mathcal{D}_{X \to Y} = \mathcal{O}_X \otimes_{\mathcal{O}_Y} \mathcal{D}_Y$. We may define a pullback of $\mathcal{D}$-modules via

$$f^* M = \mathcal{D}_{X \to Y} \otimes_{\mathcal{D}_Y} M.$$

This can also be interpreted as tensoring with the bimodule $\mathcal{D}\mathrm{iff}(f^*\mathcal{O}_Y, \mathcal{O}_X)$.

To define a pushforward of $\mathcal{D}$-modules, we need to pass from left to right $\mathcal{D}$-modules. We define

$$f_*(M) = (\Omega_X^{\mathrm{top}} \otimes_{\mathcal{O}_X} M) \otimes_{\mathcal{D}_X} \mathcal{D}_{X \to Y}) \otimes_{\mathcal{O}_Y} (\Omega_X^{\mathrm{top}})^{-1}$$

As above, this can be interpreted as tensoring with a bimodule (in this case $\mathcal{D}\mathrm{iff}(f^*\Omega_Y^{\mathrm{top}}, \Omega_X^{\mathrm{top}})$).

We may use these to define convolution products.

Let G be an algebraic group (with multiplication $\mathfrak{m} : \mathsf{G}^2 \to \mathsf{G}$) over $\mathbb{C}$. At closed points $a \in \mathsf{G}$, we may construct δ-modules $\delta_a = \mathcal{D}_\mathsf{G}/\mathcal{D}_\mathsf{G} \cdot \mathfrak{m}_{\mathsf{G},a}$. The convolution product of δ_a and δ_b is

$$m_*\big(\pi_1^*\delta_a \otimes_{\mathcal{O}_{\mathsf{G}\times\mathsf{G}}} \pi_2^*\delta_b\big) = \delta_{m(a,b)}$$

In general, the convolution product defines a monoidal structure $\star$ on $\mathcal{D}_\mathsf{G}$-Mod. The monoidal category $(\mathcal{D}_\mathsf{G}\text{-}\mathsf{Mod}, \star)$ is the *smooth categorical group algebra.* Actions of G on a space (e.g. G/B) give rise to module categories for the smooth categorical group algebra.

4 9/26 (Brian Yang) – Beilinson-Bernstein Localization

Let G be a semisimple Lie group with Lie algebra $\mathfrak{g}$. Let $\mathsf{B} \subset \mathsf{G}$ be a Borel subgroup, so G/B is the flag variety.

Beilinson-Bernstein localization gives an adjunction between:

- Representations of $\mathfrak{g}$ with central character 0 (this will be defined later).
- D-modules on G/B.

We can go from D-modules to representations by taking global sections. The left adjoint of this is Beilinson-Bernstein localization.

Really, we should think of our representations as living in category $\mathcal{O}$, which includes the finite dimensional representations and the Verma modules. This essentially consists of "highest weight" representations which can possibly be infinite-dimensional. We'd like to connect this with D-modules on the flag varieties.

4.1 Review of $\mathfrak{sl}_2$

Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$, with basis

$$e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \qquad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Let V be a representation of $\mathfrak{g}$. We say that $v \in V$ is a *heighest weight* vector of weight $n \geqslant 0$ if, letting $v_k = \frac{1}{k!} f^k v$ for $k \geqslant 0$, we have

$$hv_k = (n-2k)v_k, \qquad fv_k = (k+1)v_{k+1}, \qquad ev_k = (n-k+1)v_{k-1} \text{ with } ev_0 = 0.$$

In particular, the weight n is the eigenvalue of h corresponding to the eigenvector v.

For fixed n, we define the *Verma module* $M(n) = \mathrm{Span}_\mathbb{C}\{v_0, v_1, \ldots\}$ as the free (infinite-dimensional) $\mathfrak{g}$-representation with a highest weight vector of weight n. Letting $\mathfrak{b} = \mathrm{Span}_\mathbb{C}\{e, h\}$, we can also define $M(n)$ as $\mathcal{U}\mathfrak{g} \otimes_{\mathcal{U}\mathfrak{b}} \mathbb{C}_n \cdot v_0$.

We may view $M(-n-2)$ as $\mathrm{Span}_\mathbb{C}\{v_{n+1}, v_{n+2}, \ldots\} \subset M(n)$. The quotient $V(n) = M(n)/M(-n-2)$ is the unique irrep of dimension $n+1$. We call the resolution

$$0 \longrightarrow M(-n-2) \longrightarrow M(n) \longrightarrow V(n) \longrightarrow 0$$

the *Bernstein-Gelfand-Gelfand resolution.*

4.2 D-modules on $\mathbb{P}^1$ and representations

Let $\mathsf{G} = \mathrm{SL}_2(\mathbb{C})$. The flag variety of G is $X = \mathbb{P}^1$. We may write $X = U_\infty \cup U_0$ where $U_\infty = \{[w : 1]\}$ and $U_0 = \{[1 : z]\}$. The sheaf $\mathcal{D}_X$ satisfies $\mathcal{D}_X(U_0) = \mathbb{C}\langle z, \partial_z\rangle$ and $\mathcal{D}_X(U_\infty) = \mathbb{C}\langle w, \partial_w\rangle$.

We have an action of G on X by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} [x : y] = [ax + by : cx + dy].$$

This induces a Lie algebra homomorphism $\mathfrak{g} \to \mathrm{Vect}(X)$.

In the U_0 chart, we can compute

$$\begin{aligned} e \cdot f(z) &= \left.\frac{d}{dt}\right|_{t=0} \exp(te) f(z) \\ &= \left.\frac{d}{dt}\right|_{t=0} f(\exp(-te)z) \\ &= \left.\frac{d}{dt}\right|_{t=0} f\left(\begin{bmatrix} 1 & -t \\ 0 & 1 \end{bmatrix} [1:z]\right) \\ &= z^2 f'(z). \end{aligned}$$

That is, e acts by $z^2\partial_z$.

We can make similar computations for other elements / charts:

- In U_0, we have $e \mapsto z^2\partial_z$, $f \mapsto -\partial_z$, and $h \mapsto 2z\partial_z$.
- In U_∞, we have $e \mapsto -\partial_w$, $f \mapsto w^2\partial_w$, and $h \mapsto -2w\partial_w$.

These can all be interpreted geometrically: e is a vector field with a zero of order two at 0, f is a vector field with a zero of order two at ∞, and h generates the scaling action of $\mathbb{C}^\times$.

Using the homomorphism $\mathfrak{g} \to \mathrm{Vect}(X)$, we see that the global sections of any D-module on $\mathbb{P}^1$ give a representation of $\mathfrak{g}$!

Example 4.1. Letting $\mathcal{O}_X$ be the structure sheaf with the usual action of $\mathcal{D}_X$, we see that $\Gamma(X, \mathcal{O}_X) = \mathbb{C}$ consists of constant functions. Because $\mathrm{Vect}(X)$ acts trivially on constant functions, we get $\Gamma(X, \mathcal{O}_X) \cong V(0)$ as $\mathfrak{g}$-representations.

Example 4.2. Let

$$N = \left\{ \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix} \right\}$$

act on X by restricting the G-action to N. This has two orbits: $\{0\}$ and U_∞. We can pushforward the structure sheaves of the orbits to X and obtain D-modules on X.

Let $\mathcal{M}_{\mathrm{pt}} = i_{\mathrm{pt}*}\mathcal{O}_{\mathrm{pt}}$. To compute this, note that

$$\mathcal{D}_{\mathrm{pt}\to U_0}(U_0) = \mathbb{C}[z]/(z) \otimes_{\mathbb{C}[z]} \mathbb{C}\langle z, \partial_z \rangle = \oplus_{i \geqslant 0} \mathbb{C} \cdot \delta\partial_z^i$$

where δ denotes a (formal) δ-function at 0. Since $U_0 \cong \mathbb{A}^1$ has trivial canonical bundle, we also get $\mathcal{D}_{U_0 \leftarrow \mathrm{pt}}(U_0) = \oplus_{i \geqslant 0} \mathbb{C} \cdot \partial_z^i \delta$. It follows that

$$\Gamma(X, \mathcal{M}_{\mathrm{pt}}) = \mathcal{M}_{\mathrm{pt}}(U_0) = \oplus_{i \geqslant 0} \mathbb{C} \cdot \partial_z^i \delta = M(-2).$$

Here the Lie algebra action is computed by taking the basis $m_k = \frac{(-1)^k}{k!}\partial_z^k\delta$ and using the fact that $z\delta = 0$.

David made some comments about why we can think of δ as a δ-function: because $z\delta = 0$, any map $\mathcal{M}_{\mathrm{pt}} \to \mathcal{F}$ must send δ to some f satisfying $zf = 0$.

We'd also like to understand the pushforward of the structure sheaf of the open orbit. For this, it's useful to start with some general remarks. If $i : U \hookrightarrow X$ is an open immersion and $\mathcal{M}$ is a $\mathcal{D}_U$-module, then $i_*\mathcal{M}(V) = \mathcal{M}(U \cap V)$. The $\mathcal{D}_X$-module structure on $i_*\mathcal{M}$ agrees with the $\mathcal{D}_U$-module structure on $\mathcal{M}$. That is, pushforwards of D-modules along open immersions are just the usual pushforwards with the "obvious" D-module structure!

Example 4.3. Let $\mathcal{M}_{\mathrm{open}} = i_{U_\infty *}\mathcal{O}_{U_\infty}$. Then $\Gamma(X, \mathcal{M}_{\mathrm{open}}) = \Gamma(U_\infty, \mathcal{O}_{U_\infty}) = \mathbb{C}[w]$ with its usual $\mathcal{D}_{U_\infty}$-module structure. Choosing the basis $n_k = (-1)^k w^k$, we can compute the Lie algebra action and see that $\Gamma(X, \mathcal{M}_{\mathrm{open}}) = M(0)^\vee$, the dual Verma module.

5 10/3 (Brian Yang) – Category $\mathcal{O}$

Let $\mathfrak{g} = \mathfrak{sl}_2$, with basis

$$e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad h = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

5.1 Introducing category $\mathcal{O}$

Definition 5.1. Let $\mathcal{O}$ be the category of $\mathcal{U}(\mathfrak{g})$-modules M such that

1. M is finitely generated over $\mathcal{U}(\mathfrak{g})$.
2. M has integral highest weight decomposition, i.e. $M = \oplus_{\mu \in \Lambda} M_\mu$ (where $\Lambda = \mathbb{Z}$ is the weight lattice) and there is λ such that $M_\mu \neq \{0\}$ only if $\mu \leqslant \lambda$.

For $\lambda \in \Lambda$, let $M(\lambda)$ be the *Verma module* from last week. This is indecomposable and has unique simple quotient $L(\lambda)$.

Remark 5.2. David points out that, for $\lambda \geqslant 0$, the $L(\lambda)$ are the usual finite-dimensional highest weight representations. For $\lambda < 0$, the $L(\lambda)$ are the same as $M(\lambda)$, and in particular are infinite-dimensional.

Theorem 5.3. *The map $\lambda \mapsto L(\lambda)$ gives a bijection between weights and simple modules of $\mathcal{O}$.*

5.2 The principal block

The center $Z(\mathfrak{g}) \subset \mathcal{U}(\mathfrak{g})$ is freely generated by the *Casimir element*

$$c = 2fe + h + \frac{h^2}{2}$$

This can also be interpreted as a quantization of the Killing form.

For $M \in \mathcal{O}$ with highest weight λ, we note that c acts as a scalar $\chi_\lambda(c)$ on the highest weight space M_λ (because c commutes with h). The same holds for any element of the center $Z(\mathfrak{g})$, so we obtain an algebra homomorphism $\chi_\lambda : Z(\mathfrak{g}) \to \mathbb{C}$, called the *central character*.

Example 5.4. The element c acts on $v \in M_\lambda$ by $\lambda + \lambda^2/2$, giving $\chi_{-2} = \chi_0 = 0$.

Definition 5.5. The *principal block* $\mathcal{O}_0 \subset \mathcal{O}$ is the subcategory of modules with central character zero.

The simple objects of $\mathcal{O}_0$ are $L(0)$ and $L(-2)$. We also have $M(0)$ and $M(0)^\vee$ (defined below) in $\mathcal{O}_0$. Recall that $L(-2) = M(-2)$, so $\mathcal{O}_0$ is "symmetric about -1."

Remark 5.6. The symmetry about -1 is natural from the perspective of Beilinson-Bernstein localization. "Functions lie on one side of -1, distributions lie on the other, and the universe doesn't prefer functions to distributions."

Example 5.7. The action of $\mathfrak{g}$ on the simple objects of $\mathcal{O}_0$ can be computed directly using the formulas from last time.

1. $L(0)$ is concentrated in weight 0, with all
2. $M(0)$ is generated in weight 0, with the f vectors moving us to negative weights.
3. $M(-2)$ is similar but generated in weight -2. It fits into a short exact sequence

$$0 \longrightarrow M(-2) \longrightarrow M(0) \longrightarrow L(0) \longrightarrow 0 \ .$$

4. $M(0)^\vee$ is obtained by reversing the scalars corresponding to e and f in $M(0)$. It fits into a short exact sequence

$$0 \longrightarrow L(0) \longrightarrow M(0)^\vee \longrightarrow M(-2) \longrightarrow 0 \ .$$

These are *not* all of the indecomposable objects of $\mathcal{O}_0$.

5.3 Duality

Let $M \in \mathcal{O}$, and say $M = \oplus_{\mu \in \Lambda} M_\mu$. The dual of M is $M^\vee = \oplus_{\mu \in \Lambda} M_\mu^*$, which is a $\mathfrak{g}$-module via

$$\langle x \cdot n, m \rangle = \langle n, \tau(x) \cdot m \rangle$$

for $n \in M^\vee$, $m \in M$, and $x \in \mathfrak{g}$, where τ is the *Cartan involution*

$$\tau(h) = -h, \qquad \tau(e) = f, \qquad \tau(f) = e.$$

This is a bit weird – we need to account for the fact that the "obvious" definition of duality would turn a highest weight representation into a lowest weight representation.

Example 5.8. We have $M(-2) \cong M(-2)^\vee$ and $L(0) \cong L(0)^\vee$.

The operation $M \mapsto M^\vee$ does satisfy $(M^\vee)^\vee \cong M$, but it is not the duality for an obvious "tensor product."

5.4 Projectives and Injectives

Proposition 5.9. *The module* $M(0)$ *is projective in* $\mathcal{O}_0$.

Proof. We have

$$\mathrm{Hom}(M(0), N) = \{n \in N_0 \mid en = 0\} = N_0. \qquad \square$$

Corollary 5.10. *The module* $M(0)$ *is the projective cover of* $L(0)$. *Dually, the module* $M(0)^\vee$ *is the injective hull of* $L(0)$.

The analogous statement for $M(-2) = L(-2)$ is not true. Trying to find projective / injective hulls leads to the following.

Definition 5.11. The *tilting module* is $T = L(1) \otimes M(-1)$ (where the $\mathfrak{g}$-action on a tensor product is given by $x \cdot (v \otimes w) = (x \cdot v) \otimes w + v \otimes (x \cdot w)$).

Example 5.12. We can describe the $\mathfrak{g}$-action on the relevant modules:

1. The module $L(1)$ is concentrated in weights 1 and -1, with e and f going between the two weight spaces.
2. As with the other M's, $M(-1)$ is generated in weight -1, with f moving from higher to lower weight and e moving from lower to higher weight.
3. The tilting module T is generated in weights 0 and -2, and the nonzero weight spaces for $\lambda < 0$ are two-dimensional! This contrasts with the previous examples, where all weight spaces were at most one-dimensional.

Proposition 5.13. *The tilting module* T *is indecomposable, projective, injective, and self-dual as an object of* $\mathcal{O}_0$.

Proposition 5.14. *The tilting module* T *admits a filtration by Verma modules – in fact, there is a short exact sequence*

$$0 \longrightarrow M(0) \longrightarrow T \longrightarrow M(-2) \longrightarrow 0\ .$$

Proof. Letting $v_i \otimes w_j$ denote the basis vectors for $L(1) \otimes M(-1)$, we have

$$e(v_1 \otimes w_k + v_0 \otimes w_{k+1}) = -k(v_1 \otimes w_{k-1} + v_0 \otimes w_k)$$

and

$$f(v_1 \otimes w_k + v_0 \otimes w_{k+1}) = (k+2)(v_1 \otimes w_{k+1} + v_0 \otimes w_{k+2}).$$

This gives $M(0) \hookrightarrow T$, and we can compute $T/M(0) \cong M(-2)$. $\square$

6 10/10 (Multiple Speakers) – More on Beilinson-Bernstein

6.1 (Brian) – Conclusion

As before, we work with $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$. Last week we introduced category $\mathcal{O}$ and the principal block

Let $X = \mathbb{P}^1$, and let $G = \mathrm{SL}_2(\mathbb{C})$ act on $X = G/B$ in the usual way. There is a corresnpondence between $\mathfrak{g}$-modules and $\mathcal{D}_X$-modules:

$\mathfrak{g}$-modules	$\mathcal{D}_X$-modules (described over the U_0 chart)
$L(0)$	$\mathcal{D}_{U_0}/\mathcal{D}_{U_0}\partial_z$ (aka $\mathbb{C}[z]$)
$M(-2)$	$\mathcal{D}_{U_0}/\mathcal{D}_{U_0}z$ (aka $\mathbb{C}[\delta_z]$, aka δ_0)
$M(0)^\vee$	$\mathcal{D}_{U_0}/\mathcal{D}_{U_0}\partial_z z$
$M(0)$	$\mathcal{D}_{U_0}/\mathcal{D}_{U_0}z\partial_z$
$T = M(-1) \otimes L(1)$	$\mathcal{D}_{U_0}/\mathcal{D}_{U_0}z\partial_z z$

Remark 6.1. David commented that the $(-)^\vee$ operation on $\mathfrak{g}$-modules corresponds to a "Fourier transform" on $\mathcal{D}_X$-modules.

Let's see how this works in some examples:

Example 6.2. The $\mathfrak{g}$-module $M(0)^\vee$ arises from $i_*\mathcal{O}_{U_\infty}$. The sections of this $\mathcal{D}_X$-module over U_0 are given by $\mathcal{O}_X(U_0 \cap U_\infty) = \mathbb{C}[z, z^{-1}]$. Here $\mathcal{D}_X(U_0) = \mathbb{C}\langle z, \partial_z\rangle$, and ∂_z acts by differentiation. To see that this corresponds to $\mathcal{D}_{U_0}/\mathcal{D}_{U_0}\partial_z z$, we simply observe that $\mathcal{D}_{U_0}\partial_z z$ is the kernel of the surjection

$$\begin{aligned} \mathcal{D}_X(U_0) &\to \mathbb{C}[z, z^{-1}] \\ \beta &\mapsto \beta \cdot z^{-1}. \end{aligned}$$

Similar methods work for $L(0)$ and $M(-2)$.

The behavior of the sheaf corresponding to the tilting object is more complicated: it relies on a notion of "maximal extension."

6.2 (David) – More comments on Beilinson-Bernstein

Ultimately, all of the $\mathcal{D}$-modules we are considering here have no jumps at ∞, which explains why it suffices to consider the expressions in U_0. These correspond to highest weight representations for our fixed Borel (itself corresponding to $0 \in \mathbb{P}^1$). This explains why "jumps" in the $\mathcal{D}$-modules we consider can only occur at 0.

The $\mathcal{D}_X$-modules Brian wrote down are also important because they are *indecomposable*.

Example 6.3. One might ask what $\mathcal{D}_X/\mathcal{D}_X z^2$ corresponds to. But $\mathcal{D}_X/\mathcal{D}_X z^2 \cong \mathcal{D}_X/\mathcal{D}_X z \oplus \mathcal{D}_X/\mathcal{D}_X z$, so it would have to correspond to $M(-2)^{\oplus 2}$.

Example 6.4. Other $\mathcal{D}_X$-modules, like $\mathcal{D}_X/\mathcal{D}_X(z^2 - 2) = \delta_0 \oplus \delta_1$, correspond to representations that are not highest weight for our fixed Borel.

Example 6.5. The $\mathcal{D}$-module $\mathcal{D}_X/\mathcal{D}_X(\partial_z z \partial_z)$ can be built from two copies of $\mathcal{O}$ and one copy of δ_0. Swapnil pointed out that $h = \pm 2z\partial_z$ does not act semisimply (in particular, it acts nilpotently on 1). Thus the global sections of $\mathcal{D}_X/\mathcal{D}_X(\partial_z z \partial_z)$ are not in category $\mathcal{O}$. This corresponds to monodromy of the $\mathcal{D}$-module around ∞: on the subspace $\mathrm{span}\{1, 2z\partial_z\}$, we have monodromy

$$\exp \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

In particular, we can't extend solutions across ∞ without a jump.

In general, jumps correspond to failure of the $\mathcal{D}$-module to be a vector bundle with flat connection (which is equivalent to the $\mathcal{D}$-module being coherent as an $\mathcal{O}$-module).

One can also give a quiver description of category $\mathcal{O}$: it corresponds to representations of

$$\bullet \underset{p}{\overset{q}{\rightleftarrows}} \bullet$$

satisfying $pq = 0$. Here the vector space at the left node is the -2 weight space and the vector space at the right node is the 0 weight space. We can use this to see why the most recent example is not in category $\mathcal{O}$.

6.3 (Swapnil Garg) – Beilinson-Bernstein in general

Let's recall the general statement of Beilinson-Bernstein. Let G be a semisimple algebraic group and $K \subset G$ a subgroup. Then we have a diagram

$$\begin{tikzcd} \mathfrak{g}\text{-Mod}_0 \arrow[r, "\sim"] & \mathcal{D}_{G/B}\text{-Mod} \\ \mathfrak{g}\text{-Mod}_0^K \arrow[u] \arrow[r, "\sim"] & \mathcal{D}_{G/B}\text{-Mod}^K \arrow[u] \end{tikzcd}$$

Here $\mathcal{D}_{G/B}\text{-Mod}^K$ consists of "strongly K-equivariant $\mathcal{D}$-modules."

Remark 6.6. David commented that this can all be understood naturally. The group G acts on $\mathfrak{g}$-Mod, where g sends V to the representation V^{Ad_g}, given by the same vector space with Ad_g-twisted $\mathfrak{g}$-action. We can restrict this to any subgroup $K \subset G$. A K-equivariant $\mathfrak{g}$-representation is a $\mathfrak{g}$-representation V together with a coherent collection of isomorphisms $\alpha_k : V \xrightarrow{\sim} V^{\mathrm{Ad}_k}$ for all $k \in K$. The vertical functors here are exactly given by forgetting this equivariant structure.

7 10/17 (Swapnil Garg) – Continued

Fix Lie groups $H \subset G$. Recall the commutative diagram

$$\begin{tikzcd} \mathfrak{g}\text{-Mod}_0 \arrow[r] & D(G/B) \\ \mathfrak{g}\text{-Mod}_0^H \arrow[u] \arrow[r] & D(G/B)^H \end{tikzcd}$$

where $D(X)$ denotes the category of $\mathcal{D}_X$-modules on X.

Here $(-)^H$ indicates that we consider H-equivariant objects, or equivalently $D(H)$-invariants in a module category (where $D(H)$ is an algebra under convolution). If $D(H) \curvearrowright \mathcal{C}$, the category $\mathcal{C}^H$ can be informally described as $V \in \mathcal{C}$ together with equivalences $V \xrightarrow{\sim} \mathrm{Ad}^*_{\delta_h} V$ varying smoothly / algebraically with $h \in H$.

Remark 7.1. David pointed out that we can think of $\mathsf{Vect} = D(\mathrm{pt})$ as a trivial $D(H)$-module. The H-invariants in $\mathcal{C}$ can be understood as $\mathsf{Fun}_{D(H)}(D(\mathrm{pt}), \mathcal{C})$.

Example 7.2. If G is a finite group acting on a vector space G, this categorifies the classical statement that $V^G = \mathrm{Hom}_{\mathbb{C}[G]}(\mathbb{C}, V)$.

7.1 Harish-Chandra modules

Definition 7.3. A *(Harish-Chandra)* $(\mathfrak{g}, H)$*-module* is a $\mathfrak{g}$-module with H-action such that the action of $\mathfrak{h} \subset \mathfrak{g}$ agrees with the Lie algebra action coming from $\mathfrak{h} = \mathrm{Lie}(H)$.

If H is connected, then the $\mathfrak{h}$-action determines the H-action, so $(\mathfrak{g}, \mathsf{H})$-modules form a full subcategory of $\mathfrak{g}$-modules. The essential question we are asking is: can the $\mathfrak{h}$-action be integrated / exponentiated to an H-action?

Writing

$$\exp(h) = 1 + h + \frac{h^2}{2} + \frac{h^3}{6} + \dots,$$

we see that we need the $\mathfrak{h}$-action to be locally finite on V, i.e. for all $v \in V$, we want the image $\mathcal{U}(\mathfrak{h}) \cdot v$ to be finite-dimensional. If $\mathfrak{h}$ is nilpotent, this is a sufficient condition: locally finite actions of nilpotent Lie algebras can be integrated. If $\mathfrak{h}$ has non-nilpotent elements, we need to be a bit more careful.

Example 7.4. Suppose $\mathsf{H} = \mathbb{G}_m$. Let v be an eigenvector of a generator $h \in \mathfrak{h}$, say $h \cdot v = \lambda v$. If $\lambda = 1/2$ (or something else non-integral), we can't actually exponentiate this action: torus actions must have integral weights!

Let's consider the cases of $\mathsf{H} = \mathsf{N} \subset \mathsf{G}$ and $\mathsf{H} = \mathsf{B} \subset \mathsf{G}$.

1. Because $\mathfrak{n}$ is nilpotent, an $\mathfrak{n}$-action integrates to an N-action iff $\mathfrak{n}$ acts locally finitely.
2. Because $\mathfrak{b}$ is solvable and $[\mathsf{B}, \mathsf{B}]$ is unipotent (so $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$ is nilpotent), a B-action corresponds to an action of $\mathsf{T} = \mathsf{B}/\mathsf{N}$ together with a locally finite action of $\mathfrak{n}$. By one of Lie's / Engel's theorems, this is equivalent to $\mathfrak{n}$ acting locally nilpotently (??).

Example 7.5. For $\mathsf{G} = \mathrm{SL}_2$, we have $\mathfrak{b} = \mathbb{C}e \oplus \mathbb{C}h$. The Verma module $M(\lambda)$ always has a locally finite $\mathfrak{b}$-action. However, this gives an action of T if and only if the h-weights are integral, i.e. $\lambda \in \mathbb{Z}$. Thus $M(\lambda)$ is a $(\mathfrak{g}, \mathsf{N})$-module for any λ, but is a $(\mathfrak{g}, \mathsf{B})$-module if and only if $\lambda \in \mathbb{Z}$.

Example 7.6. Let $\mathsf{G} = \mathrm{SL}_2$ again. Fix $\eta, t \in \mathbb{C}$. We can define the Whittaker module as $\oplus_{i \in \mathbb{N}} v_i$, where:

1. $ev_i = iv_{i-1} + \eta v_i$
2. $hv_i = (t - 2i - 1)v_i - 2\eta v_{i+1}$
3. $fv_i = (t - i - 1)v_{i+1} - \eta v_{i+2}$.

Here $\mathfrak{h}$ does not act semisimply. The corresponding $\mathcal{D}$-module is the pushforward of the structure sheaf of the open orbit "twisted by the exponential." This gives something which is not in category $\mathcal{O}$.

The point of this is to obtain $\mathfrak{g}$-modules which are N-equivariant but not B-equivariant, though the above example might not actually work.

Remark 7.7. David suggested a simpler example, looking at $X = \mathbb{A}^1$. Then $\mathbb{A}^1$-equivariant $\mathcal{D}$-modules on $\mathbb{A}^1$ are built from $\mathcal{O}_{\mathbb{A}^1} = D/D\partial$. We can consider the "exponential $\mathcal{D}$-module" $M_{\exp} = D/D(\partial - 1)$. As a quasicoherent sheaf, this looks like $\mathcal{O}$. However, the connection is different: the flat sections of $M_{\exp}$ look like exponentials, which are not algebraic! There are no homomorphisms between $D/D\partial$ and $M_{\exp}$. The two are *analytically* the same but not *algebraically* the same.

Next time, we will discuss category $\mathcal{O}$ in more detail.

Definition 7.8. For a general semisimple $\mathfrak{g}$, category $\mathcal{O}$ is the category of $\mathfrak{g}$-modules such that:

- $\mathfrak{h}$ acts semisimply,
- The modules are finitely generated,
- $\mathfrak{n}$ acts locally nilpotently, and
- The center $Z(\mathfrak{g}) \subset \mathcal{U}(\mathfrak{g})$ acts locally finitely.

Within this we can define subcategories $\mathcal{O}_0$ and $\hat{\mathcal{O}}_0^{-1}$.

8 10/24 (Guest lecture by Sanath Devalapurkar) – Trichotomies in Homotopy Theory and Geometric Langlands Duality

A good inspiration for the following is Arnold's paper on mathematical trinities. We have many "trichotomies" appearing in mathematics:

Name	Additive	Multiplicative	Elliptic
Group laws	$\mathbb{G}_a$, $x+y$, $f(x)=x$	$\mathbb{G}_m$, xy, $f(x)=\exp(x)$	Elliptic curve E, $f(x)=\wp(x)$
Representation theory	Lie algebra $\mathfrak{g}$	Algebraic group G	"Elliptic group"
Topology	Ordinary cohomology (classical rings)	(Complex) K-theory	Elliptic cohomology Ell_E
Geometry	Cuspidal curves	Nodal curves	Elliptic curves
Division algebras	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$

We'd like to use topology to understand the rest of the entries here.

8.1 Langlands duality

Suppose we have a reductive group G over $\mathbb{C}$. Then we can study various aspects of G:

1. The topology of G and related spaces.

2. The algebra / representation theory of G and related objects (e.g. $\tilde{N}$).

Langlands duality transforms the topology of G into the algebra of the dual group $\check{G}$.

Given such a G, we can define the *affine Grassmannian* $\mathrm{Gr}_G = G\big(\mathbb{C}((t))\big)/G\big(\mathbb{C}[[t]]\big)$.

Example 8.1. Let G be a torus T. Then $\mathrm{Gr}_T = \mathbb{X}_\bullet(T)$, the cocharacter lattice of T (at least at the level of underlying spaces). There exists a *dual torus* $\check{T}$ satisfying $\mathbb{X}^\bullet(T) = \mathbb{X}_\bullet(T)$. We can identify

$$\mathsf{Sh}(\mathrm{Gr}_T; k) \simeq \bigoplus_{\mathbb{X}_\bullet(T)} \mathsf{Sh}(\mathrm{pt}; k) \simeq \bigoplus_{\mathbb{X}_\bullet(T)} k\text{-}\mathsf{Mod} \simeq \bigoplus_{\mathbb{X}^\bullet(\check{T})} k\text{-}\mathsf{Mod} \simeq \mathsf{Rep}\check{T}.$$

Note here that "$\check{T}$ lives over k." Actually, if we account for T-equivariance (and let k_T denote the T-equivariant k-cohomology of a point), we get

$$\mathsf{Sh}_{T(\mathbb{C}[[t]])}(\mathrm{Gr}_T; k) \simeq \bigoplus_{\mathbb{X}_\bullet(T)} \mathsf{Sh}_{T(\mathbb{C})}(\mathrm{pt}; k) \simeq \bigoplus_{\mathbb{X}_\bullet(T)} k_T\text{-}\mathsf{Mod} \simeq \bigoplus_{\mathbb{X}^\bullet(\check{T})} k_T\text{-}\mathsf{Mod} \simeq \mathsf{QCoh}(\mathrm{Spec}\, k_T \times B\check{T}_k).$$

For ordinary cohomology, we have $H^*(BT; k) = \mathrm{Sym}\, \mathfrak{t}^*[-2]$, so we end up with

$$\mathsf{Sh}_{T(\mathbb{C}[[t]])}(\mathrm{Gr}_T; k) \simeq \mathsf{QCoh}(\mathfrak{t}[2] \times B\check{T}).$$

At the underived level, we have more generally:

Theorem 8.2 (Geometric Satake (Mirkovic-Vilonen, Ginzburg)). *For a reductive group G, we have*

$$\mathsf{Sh}_{G(\mathbb{C}[[t]])}(\mathrm{Gr}_G; \mathbb{Z})^\heartsuit \simeq \mathsf{Rep}(\check{G}_\mathbb{Z})$$

where $\heartsuit$ denotes the category of perverse sheaves.

It is possible, but nontrivial, to extend this to a *derived* statement. For simplicity, we'll assume G is of ADE type.

Theorem 8.3 (Derived Satake (Bezrukavnikov-Finkelberg)). *With the above caveat, we have*

$$\mathsf{Sh}_{G(\mathbb{C}[[t]])}(\mathrm{Gr}_G; \mathbb{Q}) \simeq \mathsf{QCoh}(\mathfrak{g}_\mathbb{Q}[2]/\check{G}_\mathbb{Q})$$

where $\mathfrak{g} = \check{\mathfrak{g}}$ and $\check{G}$ acts by the adjoint action.

8.2 More general coefficients

How can we extend the derived Satake theorem to different coefficients? One general principle of Langlands is that everything should happen “motivically,” hence should be robust under changes of coefficients.

Modern homotopy theory enlarges classical algebra by replacing sets with spaces. Abelian groups are replaced by spectra, and commutative rings are replaced by “$\mathbb{E}_\infty$-rings,” which give multiplicative cohomology theories. In some cases, these can be described by simplicial abelian groups / rings. One should expect the Langlands program to work with these more general coefficients.

Example 8.4. Going back to our example of a torus T, we see that the only part of the computation that assumed k was ordinary cohomology was our identification of $\operatorname{Spec} \mathsf{k}_\mathsf{T}$. What is the correct description of k_T for more general k?

Let's consider $\mathsf{k} = \mathrm{KU}$. If X is compact, the group $\mathrm{KU}^0(\mathsf{X})$ is the Grothendieck group of complex vector bundles on X. Taking $\mathsf{T} = \mathbb{G}_m$, we get $\mathsf{k}_\mathsf{T} = \mathrm{KU}_{\mathsf{S}^1}$. For an S^1-space X, the group $\mathrm{KU}^0_{\mathsf{S}^1}(\mathsf{X})$ is the Grothendieck group of complex vector bundles on X with compatible S^1-action. In particular, $\mathrm{KU}_{\mathsf{S}^1}(\mathrm{pt}) = \mathbb{Z}[\mathsf{q}^\pm]$, the Grothendieck group of S^1-representations. We can write $\mathsf{q} = \mathsf{c}_1(\mathcal{O}(1))$, and the tensor product of line bundles gives

$$\begin{aligned} \mathbb{Z}[\mathsf{q}^\pm] &\to \mathbb{Z}[\mathsf{q}^\pm, (\mathsf{q}')^\pm] \\ \mathsf{q} &\mapsto \mathsf{q}\mathsf{q}'. \end{aligned}$$

We may identify $\operatorname{Spec} \mathrm{KU}_{\mathsf{S}^1} = \mathbb{G}_m$ as group schemes.

More generally, we have $\operatorname{Spec} \mathrm{KU}_\mathsf{T} = \mathsf{T}$. The corresponding geometric Satake theorem is

$$\mathsf{Sh}_{\mathsf{T}(\mathbb{C}[[t]])}(\mathrm{Gr}_\mathsf{T}; \mathrm{KU}) \simeq \mathsf{QCoh}(\mathsf{T} \times \mathsf{B}\check{\mathsf{T}}).$$

For non-abelian G, we have the following.

Theorem 8.5 (Devalapurkar)**.** *Let G be an ADE group with $\pi_1(\mathsf{G})$ torsion-free. Then*

$$\mathsf{Sh}_{\mathsf{G}(\mathbb{C}[[t]])}(\mathrm{Gr}_\mathsf{G}; \mathrm{KU}) \simeq \mathsf{QCoh}(\mathsf{G}_\mathbb{Z}/\check{\mathsf{G}}_\mathbb{Z}).$$

One can obtain a similar theorem in the non-ADE case by reducing to this. There isn't a reasonable “perverse t-structure” on the left-hand side, so we don't have an easier abelian case to rely on.

We can also try to understand elliptic cohomology. (There are further theories appearing beyond elliptic cohomology, but we won't consider these.)

1. For $\mathsf{k} = \mathbb{Q}, \mathbb{Z}, \ldots$, we can identify $\check{\mathfrak{g}}/\check{\mathsf{G}}$ with the moduli stack of semistable $\check{\mathsf{G}}$-bundles on a cuspidal curve. This looks like the Chevalley base $\mathfrak{t}/\!/W$.
2. For $\mathsf{k} = \mathrm{KU}$, we can identify $\check{\mathsf{G}}/\check{\mathsf{G}}$ with the moduli stack of semistable $\check{\mathsf{G}}$-bundles on a nodal curve. This looks like $\mathsf{T}/\!/W$.
3. For elliptic cohomology Ell_E, the moduli stack should parametrize semistable $\check{\mathsf{G}}$-bundles on E. This looks like $\mathrm{Bun}^0_\mathsf{T}(\mathsf{E})/\!/W$.

It's somewhat strange and interesting that a moduli stack of bundles appears on the spectral side! We can still prove a theorem of the desired form.

Theorem 8.6 (Devalapurkar)**.**

$$\mathsf{Sh}_{\mathsf{G}(\mathbb{C}[[t]])}(\mathrm{Gr}_\mathsf{G}; \mathrm{Ell}_\mathsf{E}) \simeq \mathsf{QCoh}(\mathrm{Bun}^{ss}_\mathsf{G}(\mathsf{E})).$$