STUDENT ARITHMETIC GEOMETRY SEMINAR: ON THE BIRATIONAL GEOMETRY OF STACKS

Abstract. These are my notes from the Fall 2024 student arithmetic geometry seminar. This is a papers seminar focused on birational geometry and stacks. I make no promises about the quality of the notes, but feel free to bring to my attention anything that could be improved.

1. 8/30 (Martin Olsson) – ???

I missed this day. If you have notes you would like to share, please send them to me and I will TEX them up. (Alternatively, feel free to do so yourself and submit a pull request!)

2. 9/6 (Martin Olsson) – Algebraic Stacks through Examples

Our goal this time is to give a more precise discussion of stacks. Throughout we fix a base scheme S.

2.1. Working definition of algebraic stacks.

Definition 2.1. An *algebraic stack* is a functor $\mathfrak{X} : (\mathsf{Sch}/\mathsf{S})^{\mathrm{op}} \to \mathsf{Grpd}$ such that:

1. $\mathfrak{X}$ is a sheaf for the étale topology (i.e. satisfies descent),
2. the diagonal $\Delta_{\mathfrak{X}}$ is representable, and
3. $\mathfrak{X}$ admits a smooth cover by a scheme.

Making this precise takes some work – it is often technically easier to work with groupoids rather than fibered categories. The first condition gives the notion of a *stack* – algebraicity corresponds to the second and third conditions.

Example 2.2. Let U be a scheme over S and let G be a flat affine S-group scheme. Then we can construct a quotient stack $[\mathsf{U}/\mathsf{G}]$, defined below.

2.2. **Principal homogeneous spaces and torsors.** Let G be an affine group scheme over S.

Definition 2.3. A *principal homogeneous space* under G is a flat surjective S-scheme P with left G-action such that the map

$$\begin{aligned} \mathsf{G} \times_{\mathsf{S}} \mathsf{P} &\to \mathsf{P} \times_{\mathsf{S}} \mathsf{P} \\ (g, x) &\mapsto (gx, x) \end{aligned}$$

is an isomorphism.

Example 2.4. There is an equivalence between the groupoid of invertible sheaves on S and the groupoid of $\mathbb{G}_m$-principal homogeneous spaces on S. Given a line bundle $\mathcal{L}$, the corresponding homogeneous space is $\mathrm{Isom}(\mathcal{L}, \mathcal{O}) = \mathrm{Spec}_{\mathsf{S}} \oplus_{n\in\mathbb{Z}} \mathcal{L}^{\otimes n}$. Here $u \in \mathbb{G}_m$ acts on $\oplus_n \mathcal{L}^{\otimes n}$ via $u \cdot \ell^{\otimes n} = u^n \ell^{\otimes n}$.

Let $\mathfrak{X} = [\mathsf{U}/\mathsf{G}]$. Then $\mathfrak{X}(\mathsf{T})$ is defined to be the groupoid of commutative diagrams

$$\begin{tikzcd} \mathsf{P} \arrow[r, "\rho"] \arrow[d] & \mathsf{U} \arrow[d] \\ \mathsf{T} \arrow[r] & \mathsf{S} \end{tikzcd}$$

where $\mathsf{P} \to \mathsf{T}$ is a principal G-homogeneous space and ρ is G-equivariant. This is an algebraic stack!

Descent follows from faithfully flat descent for affine schemes. For the second and third conditions, let $(\mathsf{P}, \rho), (\mathsf{P}', \rho') \in \mathfrak{X}(\mathsf{T})$. Define $\mathrm{Isom}_{\mathfrak{X}}((\mathsf{P}, \rho), (\mathsf{P}', \rho')) : (\mathsf{Sch}_{/\mathsf{T}})^{\mathrm{op}} \to \mathsf{Set}$ by

$$\mathrm{Isom}_{\mathfrak{X}}((\mathsf{P}, \rho), (\mathsf{P}', \rho'))(\mathsf{V}) = \left\{ \sigma : \mathsf{P}_{\mathsf{V}} \to \mathsf{P}'_{\mathsf{V}} \middle| \sigma \text{ is an isomorphism over } \mathsf{V} \text{ and } \rho' \circ \sigma = \rho \right\}.$$

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The second condition is saying that $\mathrm{Isom}_{\mathfrak{X}}((P,\rho),(P',\rho'))$ is representable by a scheme. If G is smooth, then the third statement is saying that for the universal homogeneous space $(P_0,\rho_0) \in \mathfrak{X}(U)$, if we are given (P,ρ) on T, then the map $\mathrm{Isom}_{\mathfrak{X}}((P_0,\rho_0),(P,\rho)) \to T$ is a smooth surjection.

Example 2.5. Consider the case $U = S$ and $G = \mathbb{G}_m$. Here a principal $\mathbb{G}_m$-homogeneous space P corresponds to a line bundle $\mathcal{L}$. We have $\mathrm{Isom}_T(P,P') \simeq \mathrm{Isom}_T(\mathcal{L}',\mathcal{L}) \simeq \mathrm{Isom}_T(\mathcal{L}'\otimes\mathcal{L}^\vee, \mathcal{O})$. This is representable.

Example 2.6. Let k be a field of characteristic p, and let $G = \mu_p$. Note that μ_p is flat but not smooth. Write $B\mu_p = [\operatorname{Spec} k/\mu_p]$. From the short exact sequence

$$0 \longrightarrow \mu_p \longrightarrow \mathbb{G}_m \longrightarrow \mathbb{G}_m \longrightarrow 0$$

we see that the groupoid of μ_p-principal homogeneous spaces on a test scheme T is equivalent to the category of pairs $(\mathcal{L},\lambda)$ where $\lambda : \mathcal{L}^{\otimes p} \xrightarrow{\sim} \mathcal{O}$. Specifically, the principal homogeneous space corresponding to $(\mathcal{L},\lambda)$ is $\operatorname{Spec}_T \oplus_{i=0}^{p-1} \mathcal{L}^{\otimes i}$. It follows that $B\mu_p \simeq [\mathbb{G}_m/_p\mathbb{G}_m]$, where the $/_p$ indicates $\mathbb{G}_m$ acting on itself via the pth power map. We have taken a "flat algebraic stack" and replaced it by a genuine (smooth) algebraic stack! A theorem of Artin allows us to do this more generally.

Example 2.7. Let $\mathfrak{X}$ be a "stacky $\mathbb{P}^1$" with stabilizer μ_2 at $z = 0$ and μ_3 at $z = \infty$. We may construct this as a stack over $\mathbb{P}^1$ with T-points (for $g : T \to \mathbb{P}^1$)

$$\mathfrak{X}(T) = \{(\mathcal{L}_0, \alpha_0 : \mathcal{L}_0 \to \mathcal{O}_T, \lambda_0 : \mathcal{L}_0^{\otimes 2} \xrightarrow{\sim} g^*\mathcal{I}_0, \text{ likewise at } \infty) \mid \text{relevant diagrams commute}\}.$$

Here the diagram for 0 asserts that α_0^2 equals the composite of λ_0 and the inclusion $g^*\mathcal{I}_0 \hookrightarrow \mathcal{O}_T$. The diagram for ∞ is similar but involves α_∞^3.

This is not obviously a quotient stack $[U/G]$. In fact, it is impossible to write $\mathfrak{X} = [U/G]$ for a finite discrete group G: away from 0 and ∞, the map $U \to \mathfrak{X}$ would be a finite étale cover, which would have to be an n-fold multiplication map $\mathbb{G}_m \to \mathbb{G}_m$ for some n. But there's no possible choice of n that works (we have a two-fold cover at 0 and a three-fold cover at ∞).

However, we can realize $\mathfrak{X} = [(\mathbb{A}^2 \setminus (0,0))/\mathbb{G}_m]$ as a weighted projective space (where $\mathbb{G}_m$ acts on the first coordinate by weight two and on the third coordinate by weight three).

3. 9/13 (Martin Olsson) – More Examples

3.1. **Stacky projective lines.** We begin with the example from last time.

Example 3.1. Consider again the stacky projective line $\mathfrak{X}$ with notation as above. One thinks of $\mathcal{L}_0$ as a "square root" of the ideal sheaf $\mathcal{I}_0$ and $\mathcal{L}_\infty$ as a "square root" of $\mathcal{I}_\infty$. Away from the points $\{0,\infty\}$, $\mathfrak{X}$ looks like $\mathbb{P}^1$. The fiber of $\mathfrak{X}$ over $0 \in \mathbb{P}^1$ is $(\operatorname{Spec} k[\epsilon]/(\epsilon^2))/\mu_2$, where μ_2 acts by $\epsilon \mapsto -\epsilon$.

We can give multiple presentations of $\mathfrak{X}$. The typical approach to do so is to "rigidify" $\mathfrak{X}$ by choosing additional data and then quotienting by the choice of additional data.

(1) Let's rigidify $\mathfrak{X}$ by fixing isomorphisms $\mathcal{L}_0 \simeq \mathcal{O}$ and $\mathcal{L}_\infty \simeq \mathcal{O}$. Then $\alpha_0, \alpha_\infty \in \Gamma(T, \mathcal{O}_T)$, so we may view these as functions f_0, f_∞. The λ's are replaced by γ_0, γ_∞, where γ_i trivializes $g^*\mathcal{I}_i$. We may then define a scheme U over $\mathbb{P}^1$ by

$$U(T) = \{(g : T \to \mathbb{P}^1, f_0, f_\infty, \gamma_0, \gamma_\infty) \mid \text{ the necessary diagrams commute}\}.$$

This U is a scheme: we may view it explicitly as a closed subscheme of $(\mathbb{P}^1 \times \mathbb{A}^2) \times_{\mathbb{P}^1} \mathrm{Isom}(\mathcal{O},\mathcal{I}_0) \times_{\mathbb{P}^1} \mathrm{Isom}(\mathcal{O},\mathcal{I}_\infty)$ (cut out by the condition that the diagrams commute). We have an action of $\mathbb{G}_m^2$ on U:

$$(u_0,u_\infty) \cdot (g, f_0, f_\infty, \gamma_0, \gamma_\infty) = (g, u_0 f_0, u_\infty f_\infty, u_0^2\gamma_0, u_\infty^3\gamma_\infty)$$

Thus we may construct the quotient $[U/\mathbb{G}_m^2]$.

We claim that $\mathfrak{X} \simeq [U/\mathbb{G}_m^2]$. In fact, the point $(g, (\mathcal{L}_0,\alpha_0), (\mathcal{L}_0,\alpha_\infty), \lambda_0, \lambda_\infty)$ corresponds to the $\mathbb{G}_m^2$-torsor $\mathrm{Isom}(\mathcal{O},\mathcal{L}_0) \times_T \mathrm{Isom}(\mathcal{O},\mathcal{L}_\infty)$ together with its natural map to U.

(2) There are other presentations! This is an advantage of stacks (as opposed to just considering group actions). We can write $\mathfrak{X} \simeq [(\mathbb{A}^2 \setminus \{0\})/\mathbb{G}_m]$ where $u \in \mathbb{G}_m$ acts by $u \cdot (s,t) = (u^2 s, u^3 t)$. Note that T-points of $[(\mathbb{A}^2 \setminus \{0\})/\mathbb{G}_m]$ are

$$\{(\mathcal{L}, a \in H^0(T, \mathcal{L}^{\otimes 2}), b \in H^0(T, \mathcal{L}^{\otimes 3})) \mid a, b \text{ not simultaneously } 0\}.$$

To see the equivalence between this and $\mathfrak{X}$, fix an isomorphism $\mathcal{I}_0 \simeq \mathcal{I}_\infty$. This forces $\mathcal{L}_0^2 \simeq \mathcal{L}_\infty^3$, and we can compute $(\mathcal{L}_0 \otimes \mathcal{L}_\infty^{-1})^{\otimes 2} \simeq \mathcal{L}_\infty$ and $(\mathcal{L}_0 \otimes \mathcal{L}_\infty^{-1})^{\otimes 3} \simeq \mathcal{L}_0$. Letting $\mathcal{M} = \mathcal{L}_0^{-1} \otimes \mathcal{L}_\infty$, we see that $\alpha_0 \in \mathcal{M}^{\otimes 2}$ and $\alpha_\infty \in \mathcal{M}^{\otimes 3}$. Using this gives the equivalence.

3.2. **Weighted projective spaces.** We can generalize the latter construction.

Example 3.2. Let $\mathbb{G}_m$ act on $\mathbb{A}^{n+1}$ by $u \cdot (x_0, \dots, x_n) = (u^{\alpha_0} x_0, \dots, u^{\alpha_n} x_n)$. Removing the origin gives the weighted projective stack $\mathbb{P}(\alpha_0, \dots, \alpha_n) = [(\mathbb{A}^{n+1} \setminus \{0\})/\mathbb{G}_m]$. There are many interesting sub-examples:

(1) $\mathbb{P}^n = \mathbb{P}(1, \dots, 1)$
(2) We have $\overline{\mathfrak{M}}_{1,1} \simeq \mathbb{P}(4, 6)$ (though we will not show this here).
(3) $\mathbb{P}(1, -1)$ has T-points
$$\{(\mathcal{L}, \alpha_+ \in H^0(T, \mathcal{L}), \alpha_- \in H^0(T, \mathcal{L}^{-1})) \mid a_+,\ a_- \text{ not simultaneously zero }\}.$$
This is covered by U_+ and U_- where
$$U_\pm(T) = \{(\mathcal{L}, a_+, a_-) \mid a_\pm \text{ generates } \mathcal{L}^{\otimes \pm 1}\}.$$
The intersection is $\mathbb{G}_m$, and $\mathbb{P}(1, -1)$ is the affine line with two origins.

This has some interesting applications.

Example 3.3. Consider the polynomial $x^2 + 29y^2 - 3z^3 \in \mathbb{Z}[x, y, z]$. We might like to consider a "projective variety" corresponding to this, but attempting to do so runs into an obvious issue: the polynomial is not homogeneous! We can obtain instead a "weighted projective variety"
$$[(\operatorname{Spec} \mathbb{Z}[x, y, z]/(x^2 + 29y^2 - 3z^3) \setminus \{0\})/\mathbb{G}_m] \subset \mathbb{P}(3, 3, 2).$$
This is proper over $\operatorname{Spec} \mathbb{Z}$. See a paper of Bhargava and Poonen for more discussion of this in the context of finding rational points on curves. A classic result of Darmon and Granville shows that this fails local-to-global (rational points exist locally but not globally).

4. 9/20 (Rose Lopez) – Birational Geometry of Stacks

References for today's talk are:

- Kresch and Tschinkel (2023) – *Birational Geometry of Deligne-Mumford Stacks*
- Kresch and Tschinkel (2019) – *Birational Types of Algebraic Orbifolds*
- Kontsevich and Tschinkel (2017) – *Specialization of Birational Types*
- Bergh and Rydh (2019) – *Functorial Destackification*

4.1. **Classical Birational Geometry.**

Definition 4.1. Let X and Y be varieties. A *rational map* $f : X \dashrightarrow Y$ is a morphism $f : U \to Y$ for some dense open $U \subset X$. We identify two such maps $f : U \to Y$, $f' : U' \to Y$ if $f|_{U \cap U'} = f'|_{U \cap U'}$.

We say that $f : X \dashrightarrow Y$ is *birational* if there exists an inverse map $Y \dashrightarrow X$. This induces an isomorphism on dense opens ($U \xrightarrow{\sim} V$) as well as an isomorphism on function fields.

Theorem 4.2 (Weak Factorization). *Let $f : X \to Y$ be a birational map between smooth complete varieties over an algebraically closed field k of characteristic zero. Let $U \subset X$ be an open such that $f|_U$ is an isomorphism onto its image. Then f factors as a zigzag of blowups at smooth centers inducing isomorphisms on the image of U (??).*

4.2. **The Burnside Ring.** Fix an algebraically closed field k of characteristic zero.

Definition 4.3. Let Burn_n be the free abelian group generated by isomorphism classes of finitely generated fields of transcendence degree n. For a smooth projective irreducible variety X of dimension n, we define $[X] := [k(X)] \subset \mathrm{Burn}_n$. The *Burnside ring of varieties* is $\mathrm{Burn} = \oplus_n \mathrm{Burn}_n$ with ring structure given by $[X] \cdot [Y] = [X \times Y]$.

For $U = X \setminus D$ where $D = D_1 \cup \dots D_\ell$ is a simple normal crossing divisor, we define
$$[U] := [X] - \sum_i [D_i \times \mathbb{P}^1] + \sum_{i<j} [(D_i \cap D_j) \times \mathbb{P}^2]$$
We also let $[X \cup Y] = [X] + [Y]$. Note that $[U] = [U']$ if there exists a quasiprojective variety V and a pair of birational projective morphisms $V \to U$ and $V \to U'$.

Proposition 4.4. *The group* Burn_n *is generated by classes* $[U]$ *where* U *is* n*-dimensional and quasiprojective, modulo the* modified scissor relation

$$[U] = [V \times \mathbb{P}^{r-d}] + [U \setminus V]$$

for any smooth closed $V \subset U$ *of dimension* $< n$.

Proposition 4.5 (Specialization)**.** *Let* $\pi : X \to B$ *and* $\pi' : X' \to B$ *be smooth proper morphisms to a smooth connected curve* B*. If the generic fibers of* π *and* π' *are birational over* $k(B)$*, then for any closed* $b \in B$*, the fibers* $\pi^{-1}(b)$ *and* $(\pi')^{-1}(b)$ *are birational over* $k(b)$*.*

4.3. **Extension to stacks.**

Definition 4.6. Let $\mathfrak{X}$ and $\mathfrak{Y}$ be stacks. A rational map $f : \mathfrak{X} \dashrightarrow \mathfrak{Y}$ is a morphism defined on a dense open $\mathfrak{U} \subset \mathfrak{X}$. A 2-isomorphism $\alpha : (U, f) \to (U', f')$ is an isomorphism $f|_{U \cap U'} \xrightarrow{\sim} f'|_{U \cap U'}$. We say that f is birational if it induces an equivalence on dense opens.

Definition 4.7. Let $\mathfrak{X}$ and $\mathfrak{Y}$ be stacks. We say that $\mathfrak{X}$ and $\mathfrak{Y}$ are *birationally equivalent* if there exists a representable proper birational map $\mathfrak{X} \dashrightarrow \mathfrak{Y}$, or equivalently a span $\mathfrak{X} \leftarrow \mathfrak{X}' \to \mathfrak{Y}$ consisting of representable proper birational morphisms.

Example 4.8. Let $\mathfrak{X}$ be a stack and $\mathfrak{Z}$ a closed substack. We can define a *blowup* $\mathrm{B}\ell_{\mathfrak{Z}}\, \mathfrak{X} \to \mathfrak{X}$ by performing the usual scheme-theoretic blowup locally and gluing.

Example 4.9. Let $\mathfrak{X}$ be a smooth separated DM stack and $\mathfrak{D} \subset \mathfrak{X}$ a divisor. We can define a *root stack* $\sqrt[r]{(\mathfrak{X}, \mathfrak{D})} \to \mathfrak{X}$ by

$$\sqrt[r]{(\mathfrak{X}, \mathfrak{D})}(T) = \left\{ (g : T \to \mathfrak{X}, \alpha : \mathcal{L} \to \mathcal{O}_T, \lambda : \mathcal{L}^{\otimes r} \xrightarrow{\sim} g^* \mathcal{I}_{\mathfrak{D}}) \;\middle|\; \text{the induced maps } \mathcal{L}^{\otimes r} \to \mathcal{O}_T \text{ agree} \right\}$$

The morphism $\sqrt[r]{(\mathfrak{X}, \mathfrak{D})} \to \mathfrak{X}$ is proper but not representable: the fibers over points in $\mathfrak{D}$ are stacky.

Example 4.10. Let $\mathfrak{X}$ be a stack and $\mathcal{L}$ a line bundle on $\mathfrak{X}$. The *gerbe of* n*th roots of* $\mathcal{L}$ is $\sqrt[n]{\mathcal{L}/\mathfrak{X}} \to \mathfrak{X}$ with

$$\sqrt[n]{\mathcal{L}/\mathfrak{X}}(T) = \left\{ (g : T \to \mathfrak{X}, \mathcal{M} \in \mathsf{Pic} T, \epsilon : \mathcal{M}^{\otimes n} \xrightarrow{\sim} g^* \mathcal{L}) \right\}$$

We can contrast this with the above example: essentially, the above example rigidifies things by considering $\mathcal{I}_{\mathfrak{D}} \to \mathcal{O}$. Over $\mathfrak{U} = \mathfrak{X} \setminus \mathfrak{D}$, the gerbe $\sqrt[n]{\mathcal{L}/\mathfrak{X}}|_{\mathfrak{U}}$ is trivializable.

There are two weak factorization results for stacks. One factorizes a representable proper birational map into blowups and can be used to define the "correct" (?) Burnside ring. Another factorizes a proper birational map into stacky blowups and can be used to define a weaker Burnside ring related to "G-equivariant Burnside rings."

Example 4.11. Let $C = \mathbb{A}^1$, $C_0 = \mathbb{A}^1 \setminus \{0\}$, E an elliptic curve, and $p \in E$. Define $\mathfrak{X}_0 = C_0 \times E \times B(\mathbb{Z}/2) \subset \mathfrak{X}_1 = C \times E \times B(\mathbb{Z}/2)$. Let $B = \mathrm{B}\ell_{(0,p)}(C \times E)$ and $\mathfrak{X}_2 = \sqrt{\mathcal{O}_B(D)/B}$. We get an inclusion $\mathfrak{X}_0 \to \mathfrak{X}_2$. This induces an equivalence on generic fibers but not on special fibers (over $(0, p)$??). The "specialization" result from earlier fails for stacks!

5. 9/27 (Xiangru Zeng) – Valuative Criteria for the Existence of Moduli Spaces

Let k be an algebraically closed field of characteristic zero. We may skip some less-than-essential details. References are:

- Alper – Good Moduli Spaces for Artin Stacks
- Alper, Halpern-Leistner, Heinloth – Existence of Moduli Spaces for Algebraic Stacks
- Alper – Notes on Stacks and Moduli

5.1. **Precursors.**

Theorem 5.1 (Keel-Mori)**.** *Let* $\mathfrak{X}$ *be a separated Deligne-Mumford stack of finite type over a base* S*. Then there exists coarse moduli space* $\pi : \mathfrak{X} \to X$*, i.e. a universal algebraic space* X *among algebraic spaces equipped with a map from* X *and such that* π *is bijective on geometric points. This satisfies* $\pi_* \mathcal{O}_{\mathfrak{X}} = \mathcal{O}_X$*. In characteristic zero, we end up with a "tame moduli space:"*

(1) If $\mathfrak{X} \to S$ *is flat, then* $X \to \mathrm{Spec}\, k$ *is flat.*

(2) *The forgetful functor* $\pi_* : \mathsf{QCoh}(\mathfrak{X}) \to \mathsf{QCoh}(X)$ *is exact.*

This requires $\mathfrak{X}$ to be separated – in particular, the stabilizer groups of $\mathfrak{X}$ must be affine. This is a relatively strict condition!

When considering problems of GIT, we often let $U \subset \mathbb{P}^n$ be a projective variety with linearized G-action (for some linearly reductive group G). The GIT quotient is then

$$U /\!\!/ G = \operatorname{Proj} \oplus_{n \geqslant 0} \Gamma(U, \mathcal{O}_U(n))^G.$$

We have a map $\pi : [U^{ss}/G] \to U /\!\!/ G$, where U^{ss} is the semistable locus. This satisfies:

(1) $\mathcal{O}(U /\!\!/ G) = \pi_*(\mathcal{O}(U^{ss}))^G$, and
(2) (In characteristic zero) π is affine.

Seshadri called a map satisfying the above conditions a *good quotient*.

5.2. Good moduli spaces.

Definition 5.2. A map $\pi : \mathfrak{X} \to X$, where $\mathfrak{X}$ is a (qcqs?) algebraic stack and X is a (qcqs?) algebraic space, is a *good moduli space* if

(1) $\pi_* \mathcal{O}_{\mathfrak{X}} \cong \mathcal{O}_X$, and
(2) $\pi_* : \mathsf{QCoh}(\mathfrak{X}) \to \mathsf{QCoh}(X)$ is exact.

Theorem 5.3. *Let* $\pi : \mathfrak{X} \to X$ *be a good moduli space. Then:*

(1) π *is surjective and universally closed.*
(2) For $x_1, x_2 \in \mathfrak{X}(k)$*, then* $\pi(x_1) = \pi(x_2)$ *if and only if* $\overline{\{x_1\}} \cap \overline{\{x_2\}} \neq \varnothing$.
(3) If $\mathfrak{X}$ *is noetherian, then* π *is universal among maps from* $\mathfrak{X}$ *to algebraic spaces.*

Example 5.4. Let G be a linearly reductive group acting on an affine scheme $\operatorname{Spec} A$. Then $[A/G] \to \operatorname{Spec} A^G$ is a good moduli space. In particular, $[\mathbb{A}^n/\mathbb{G}_m] \to \operatorname{Spec} k$ is a good moduli space. However, if we remove the origin, then $[(\mathbb{A}^n \setminus \{0\})/\mathbb{G}_m] \to \mathbb{P}^{n-1}$ is a good moduli space (in fact, the map is an equivalence).

In positive characteristic, the notion of a good moduli space is too restrictive. People consider "adequate moduli spaces" instead. We won't discuss these in detail.

5.3. Existence of good moduli spaces.

Theorem 5.5 (Alper, Halpern-Leistner, Heinloth)**.** *Let* $\mathfrak{X}$ *be an algebraic stack of finite type with affine diagonal. Then* $\mathfrak{X}$ *admits a separated good moduli space if and only if*

(1) $\mathfrak{X}$ *is* Θ*-reductive, and*
(2) $\mathfrak{X}$ *is* S*-complete.*

Let's explain what these mean. Let $\Theta = [\mathbb{A}^1/\mathbb{G}_m]$, and for any ring A, we write $\Theta_A = \Theta \times_{\operatorname{Spec} k} \operatorname{Spec} A$. For a DVR R with fraction field K, let $\Theta_R \setminus \{0\} = \operatorname{Spec} R \cup_{\operatorname{Spec} K} \Theta_K$.

Definition 5.6. Say $\mathfrak{X}$ is Θ-reductive if, for all DVRs R and all $\Theta_R \setminus \{0\}$, there exists a unique extension

$$\begin{tikzcd} \Theta_R \setminus \{0\} \arrow[r] \arrow[d] & \mathfrak{X}. \\ \Theta_R \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

For S-completeness, if R is a DVR with uniformizer π, let

$$\phi_R = [\operatorname{Spec} R[s,t]/(st - \pi)/\mathbb{G}_m]$$

where $\mathbb{G}_m$ acts by weight 1 on s and weight -1 on t.

Definition 5.7. Say $\mathfrak{X}$ is S-complete if, for all DVRs R, there exists a unique extension

$$\begin{tikzcd} \phi_R \setminus \{0\} \arrow[r] \arrow[d] & \mathfrak{X}. \\ \phi_R \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

Example 5.8. If $U = \operatorname{Spec} A$ is acted on by G, then $[U/G]$ is Θ-reductive if, whenever $\lambda : \mathbb{G}_m \to G$ and $\pi : \operatorname{Spec} R \to U$ are maps (with R a DVR) such that $\lim_{t\to 0} \lambda(t)\pi(\operatorname{Spec} K)$ exists, then $\lim_{t\to 0} \lambda(t)\pi(\operatorname{Spec} R)$ also exists. We can interpret S-completeness similarly.

The theorem is proved by first using the existence of good moduli spaces for quotient stacks $\operatorname{Spec} A/G$ In extending this result to more general $\mathfrak{X}$, we use the local structure theorem to write $\mathfrak{X}$ in an étale neighborhood of a point x as $[\operatorname{Spec} A/G_x]$, where G_x is the automorphism group of x. For gluing, we consider a diagram

$$\begin{tikzcd}
{[\operatorname{Spec} B/G_x]} \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & {[\operatorname{Spec} A/G_x]} \arrow[r] \arrow[d] & \mathfrak{X} \\
\operatorname{Spec} B^{G_x} \arrow[r, shift left] \arrow[r, shift right] & \operatorname{Spec} A^{G_x} \arrow[r] & X.
\end{tikzcd}$$

We can use Luna's fundamental theorem and "Θ-surjectivity" to glue things.

6. 10/4 (Will Fisher) – Smoothness of the Logarithmic Hodge Moduli Space

The reference for today is:

- De Cataldo, Herrero, Zhang – Geometry of the Logarithmic Hodge Moduli Space

We'll begin by reviewing this moduli space and why we should care about it.

6.1. **Review of Hodge theory.** If X is a topological space and $\mathcal{F}$ is a sheaf of abelian groups on X, then we can define the sheaf cohomology $H^\bullet(X, \mathcal{F})$ by taking an injective resolution $0 \to \mathcal{F} \to \mathcal{I}^\bullet$ and setting

$$H^k(X, \mathcal{F}) := H^k(\Gamma(X, \mathcal{I}^\bullet)).$$

More generally, if $\mathcal{F}^\bullet$ is a chain complex, we define the *hypercohomology* $\mathbb{H}^\bullet(X, \mathcal{F})$ by choosing a quasi-isomorphism $\mathcal{F} \xrightarrow{\sim} \mathcal{I}^\bullet$ with $\mathcal{I}^\bullet$ a complex of injectives and setting

$$\mathbb{H}^k(X, \mathcal{F}^\bullet) := H^k(\Gamma(X, \mathcal{I}^\bullet)).$$

For Hodge theory, let X be a smooth projective variety over $\mathbb{C}$. By GAGA and the (p, q)-form decomposition, we have

$$\mathbb{H}^k(X, (\Omega_X^\bullet, d)) \cong H^k_{\mathrm{dR}}(X^{\mathrm{an}}, \mathbb{C}).$$

Formal manipulations give

$$\begin{aligned}
\mathbb{H}^k(X, (\Omega_X^\bullet, 0)) &= \mathbb{H}^k\big(X, \oplus_i \Omega_X^i[-i]\big) \\
&= \oplus_i \mathbb{H}^k\big(X, \Omega_X^i[-i]\big) \\
&= \oplus_i \mathbb{H}^{k-i}(X, \Omega_X^i) \\
&= \oplus_{p+q=k} \mathbb{H}^q(X, \Omega_X^p).
\end{aligned}$$

Thus the usual Hodge decomposition can be rewritten as

$$\mathbb{H}^k(X, (\Omega_X^\bullet, d)) \cong \mathbb{H}^k(X, (\Omega_X^\bullet, 0)).$$

This suggests that we should think of the Hodge decomposition as a statement about *sheaves* rather than as a statement just about X. Let's try to understand how to generalize this.

6.2. **Non-abelian Hodge theory.** Let $\mathcal{F}$ be a vector bundle on X (still assumed to be a smooth complex projective variety). Suppose we have a connection $\nabla : \mathcal{F} \to \mathcal{F} \otimes \Omega_X^1$. Recall that this means ∇ satisfies the Leibniz rule

$$\nabla(fs) = s \otimes df + f\nabla s$$

for $f \in \mathcal{O}_X$, $s \in \mathcal{F}$.

If ∇ is flat, i.e. $\nabla \circ \nabla = 0$, then we get an associated complex

$$0 \longrightarrow \mathcal{F} \xrightarrow{\nabla} \mathcal{F} \otimes \Omega_X^1 \xrightarrow{\nabla} \mathcal{F} \otimes \Omega_X^2 \xrightarrow{\nabla} \cdots$$

We can (and should) view $(\Omega_X^\bullet, d)$ as the complex associated to $\mathcal{O}_X$ with the flat connection d.

For the other side of the non-abelian Hodge theorem, we need a new definition.

Definition 6.1. Let $\mathcal{F}$ be a vector bundle on X. A *Higgs field* is an $\mathcal{O}_X$-linear map $\phi : \mathcal{F} \to \mathcal{F} \otimes \Omega_X^1$ satisfying $\phi \wedge \phi = 0$. We call the pair $(\mathcal{F}, \phi)$ a *Higgs bundle*.

The Higgs field condition can be understood locally: writing $\phi = \sum_i A_i dx_i$, we require

$$\phi \wedge \phi := \sum_{i<j} [A_i, A_j] dx_i \wedge dx_j = 0.$$

If $(\mathcal{F}, \phi)$ is a Higgs bundle, we get a complex

$$0 \longrightarrow \mathcal{F} \xrightarrow{\phi} \mathcal{F} \otimes \Omega^1_X \xrightarrow{\phi} \mathcal{F} \otimes \Omega^2_X \xrightarrow{\phi} \dots$$

In particular, $(\Omega^\bullet_X, 0)$ is the complex associated to the Higgs bundle $(\mathcal{O}_X, 0)$.

Remark 6.2. Arthur Ogus suggested an alternative viewpoint: we can think of the Higgs field structure as a suitable lift of $\mathcal{F}$ to the cotangent space T^*X.

We can now state the non-abelian Hodge theorem.

Theorem 6.3 (Non-abelian Hodge theorem)**.** *Let X be a smooth projective variety over $\mathbb{C}$. Then (up to stability conditions) there exists a "cohomology preserving" correspondence between:*

(1) The moduli space M_{flat} of flat bundles on X.
(2) The moduli space M_{Higgs} of Higgs bundles on X.

This correspondence sends $(\mathcal{O}_X, d)$ to $(\mathcal{O}_X, 0)$.

The non-abelian Hodge theorem is essentially analytic. However, we can hope for weaker algebraic statements: perhaps we can find relationships between the corresponding moduli spaces.

One result in this direction is a cohomology equivalence: $H^\bullet(M_{\text{flat}}) \cong H^\bullet(M_{\text{Higgs}})$. This can be shown by constructing a suitable geometric family interpolating between the two moduli spaces.

6.3. **Construction of the interpolation.** Let X be a smooth scheme over a base B.

Definition 6.4. Given a vector bundle $\mathcal{F}$ on X and $t \in \Gamma(B, \mathcal{O}_B)$, a t-*connection* on $\mathcal{F}$ is an $\mathcal{O}_B$-linear map $\nabla : \mathcal{F} \to \mathcal{F} \otimes \Omega^1_{X/B}$ satisfying the t-twisted Leibniz rule:

$$\nabla(fs) = ts \otimes df + f\nabla s$$

for $f \in \mathcal{O}_X$ and $s \in \mathcal{F}$. We can use the Leibniz rule to extend ∇ to $\mathcal{F}$-valued n-forms. This allows us to say that ∇ is *flat* if $\nabla \circ \nabla = 0$.

Example 6.5. If $t = 1$, a 1-connection is just a connection.

Example 6.6. If $t = 0$, the 0-twisted Leibniz rule is $\nabla(fs) = f\nabla(s)$, so a flat 0-connection is just a Higgs field.

The general goal is to use GIT to construct a moduli space $(M_{\text{Hodge}})_X$ of t-connections on X together with a smooth map

$$\begin{aligned} (M_{\text{Hodge}})_X &\to \mathbb{A}^1_B \\ (\mathcal{F}, \nabla_t) &\mapsto t. \end{aligned}$$

We'll restrict our hypotheses to make this tractable. In particular, we assume:

- B is a noetherian scheme.
- C is a smooth proper scheme over B with geometrically integral fibers of dimension 1.
- $D \hookrightarrow C$ is a relative Cartier divisor such that every geometric fiber over B is reduced and non-empty.
- The rank n and degree d are coprime.

We'll write C_B for C as a B-scheme and C_S for a base change $C_B \times_B S$.

Definition 6.7. Let $(\mathfrak{M}_{\text{Hodge}})_{C_B} \to \mathbb{A}^1_B$ be the moduli stack of rank n, degree d flat t-connections. The groupoid of S-points (for $S \to \mathbb{A}^1_B$) consists of pairs $(\mathcal{F}, \nabla)$ where:

(1) $\mathcal{F}$ is a vector bundle of rank n on C_S such that the restriction to each geometric fiber has degree d.
(2) $\nabla : \mathcal{F} \to \mathcal{F} \otimes \omega_{C_S/S}(D_S)$ is a flat t-connection.

Taking GIT quotients, we obtain a moduli space $(M_{\text{Hodge}})_{C_B} \to \mathbb{A}^1_B$.

6.4. **Smoothness.** Our goal in the remaining time is to show $(\mathsf{M}_{\mathrm{Hodge}})_{\mathsf{C}_{\mathsf{B}}} \to \mathbb{A}^1_{\mathsf{B}}$ is smooth. We proceed in a few steps:

(1) The map $(\mathfrak{M}_{\mathrm{Hodge}})_{\mathsf{C}_{\mathsf{B}}} \to (\mathsf{M}_{\mathrm{Hodge}})_{\mathsf{C}_{\mathsf{B}}}$ is a smooth surjection, in fact a smooth good moduli space in the sense of Alper. This allows us to reduce to checking that $(\mathfrak{M}_{\mathrm{Hodge}})_{\mathsf{C}_{\mathsf{B}}} \to \mathbb{A}^1_{\mathsf{B}}$ is smooth. This is where the coprimality hypothesis arises.
(2) Then we develop an obstruction theory for t-connections, i.e. an obstruction module $\mathcal{Q}_x$ for every $x : \operatorname{Spec} A \to (\mathfrak{M}_{\mathrm{Hodge}})_{\mathsf{C}_{\mathsf{B}}}$ which is compatible with base change. This module controls lifts along square-zero extensions.
(3) This allows us to reduce to showing that $\mathcal{Q}_x$ vanishes for every geometric point $x : \operatorname{Spec} k \to (\mathfrak{M}_{\mathrm{Hodge}})_{\mathsf{C}_{\mathsf{B}}}$.
(4) Given such a geometric point x, we extend this to a family $\mathbb{A}^1_k \to (\mathfrak{M}_{\mathrm{Hodge}})_{\mathsf{C}_{\mathsf{B}}}$ which is compatible with the rescaling of t-connections and is such that x corresponds to $1 \in \mathbb{A}^1_k$.
(5) Degenerating our t-connection along this family allows us to reduce to showing that $\mathcal{Q}_x = 0$ for geometric points x corresponding to Higgs bundles.
(6) In this case, we can check that the desired result holds – this part really requires the existence of poles / a divisor D.