STUDENT ARITHMETIC GEOMETRY SEMINAR: ON THE BIRATIONAL GEOMETRY OF STACKS

Abstract. These are my notes from the Fall 2024 student arithmetic geometry seminar. This is a papers seminar focused on birational geometry and stacks. I make no promises about the quality of the notes, but feel free to bring to my attention anything that could be improved.

1. 8/30 (Martin Olsson) – ???

I missed this day. If you have notes you would like to share, please send them to me and I will TEX them up. (Alternatively, feel free to do so yourself and submit a pull request!)

2. 9/6 (Martin Olsson) – Algebraic Stacks through Examples

Our goal this time is to give a more precise discussion of stacks. Throughout we fix a base scheme S.

2.1. Working definition of algebraic stacks.

Definition 2.1. An *algebraic stack* is a functor $\mathfrak{X} : (\mathsf{Sch}/S)^{\mathrm{op}} \to \mathsf{Grpd}$ such that:

1. $\mathfrak{X}$ is a sheaf for the étale topology (i.e. satisfies descent),
2. the diagonal $\Delta_{\mathfrak{X}}$ is representable, and
3. $\mathfrak{X}$ admits a smooth cover by a scheme.

Making this precise takes some work – it is often technically easier to work with groupoids rather than fibered categories. The first condition gives the notion of a *stack* – algebraicity corresponds to the second and third conditions.

Example 2.2. Let U be a scheme over S and let G be a flat affine S-group scheme. Then we can construct a quotient stack $[U/G]$, defined below.

2.2. Principal homogeneous spaces and torsors.

Let G be an affine group scheme over S.

Definition 2.3. A *principal homogeneous space* under G is a flat surjective S-scheme P with left G-action such that the map

$$\begin{aligned} G \times_S P &\to P \times_S P \\ (g, x) &\mapsto (gx, x) \end{aligned}$$

is an isomorphism.

Example 2.4. There is an equivalence between the groupoid of invertible sheaves on S and the groupoid of $\mathbb{G}_m$-principal homogeneous spaces on S. Given a line bundle $\mathcal{L}$, the corresponding homogeneous space is $\mathrm{Isom}(\mathcal{L}, \mathcal{O}) = \mathrm{Spec}_S \oplus_{n\in\mathbb{Z}} \mathcal{L}^{\otimes n}$. Here $u \in \mathbb{G}_m$ acts on $\oplus_n \mathcal{L}^{\otimes n}$ via $u \cdot \ell^{\otimes n} = u^n \ell^{\otimes n}$.

Let $\mathfrak{X} = [U/G]$. Then $\mathfrak{X}(T)$ is defined to be the groupoid of commutative diagrams

$$\begin{tikzcd} P \arrow[r, "\rho"] \arrow[d] & U \arrow[d] \\ T \arrow[r] & S \end{tikzcd}$$

where $P \to T$ is a principal G-homogeneous space and ρ is G-equivariant. This is an algebraic stack!

Descent follows from faithfully flat descent for affine schemes. For the second and third conditions, let $(P, \rho), (P', \rho') \in \mathfrak{X}(T)$. Define $\mathrm{Isom}_{\mathfrak{X}}((P, \rho), (P', \rho')) : (\mathsf{Sch}_{/T})^{\mathrm{op}} \to \mathsf{Set}$ by

$$\mathrm{Isom}_{\mathfrak{X}}((P, \rho), (P', \rho'))(V) = \left\{ \sigma : P_V \to P'_V \middle| \sigma \text{ is an isomorphism over } V \text{ and } \rho' \circ \sigma = \rho \right\}.$$

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The second condition is saying that $\mathrm{Isom}_{\mathfrak{X}}((P,\rho),(P',\rho'))$ is representable by a scheme. If G is smooth, then the third statement is saying that for the universal homogeneous space $(P_0,\rho_0)\in\mathfrak{X}(U)$, if we are given (P,ρ) on T, then the map $\mathrm{Isom}_{\mathfrak{X}}((P_0,\rho_0),(P,\rho))\to T$ is a smooth surjection.

Example 2.5. Consider the case $U=S$ and $G=\mathbb{G}_m$. Here a principal $\mathbb{G}_m$-homogeneous space P corresponds to a line bundle $\mathcal{L}$. We have $\mathrm{Isom}_T(P,P')\simeq\mathrm{Isom}_T(\mathcal{L}',\mathcal{L})\simeq\mathrm{Isom}_T(\mathcal{L}'\otimes\mathcal{L}^\vee,\mathcal{O})$. This is representable.

Example 2.6. Let k be a field of characteristic p, and let $G=\mu_p$. Note that μ_p is flat but not smooth. Write $B\mu_p=[\operatorname{Spec}k/\mu_p]$. From the short exact sequence

$$0\longrightarrow\mu_p\longrightarrow\mathbb{G}_m\longrightarrow\mathbb{G}_m\longrightarrow 0$$

we see that the groupoid of μ_p-principal homogeneous spaces on a test scheme T is equivalent to the category of pairs $(\mathcal{L},\lambda)$ where $\lambda:\mathcal{L}^{\otimes p}\xrightarrow{\sim}\mathcal{O}$. Specifically, the principal homogeneous space corresponding to $(\mathcal{L},\lambda)$ is $\operatorname{Spec}_T\oplus_{i=0}^{p-1}\mathcal{L}^{\otimes i}$. It follows that $B\mu_p\simeq[\mathbb{G}_m/_p\mathbb{G}_m]$, where the $/_p$ indicates $\mathbb{G}_m$ acting on itself via the pth power map. We have taken a "flat algebraic stack" and replaced it by a genuine (smooth) algebraic stack! A theorem of Artin allows us to do this more generally.

Example 2.7. Let $\mathfrak{X}$ be a "stacky $\mathbb{P}^1$" with stabilizer μ_2 at $z=0$ and μ_3 at $z=\infty$. We may construct this as a stack over $\mathbb{P}^1$ with T-points (for $g:T\to\mathbb{P}^1$)

$$\mathfrak{X}(T)=\{(\mathcal{L}_0,\alpha_0:\mathcal{L}_0\to\mathcal{O}_T,\lambda_0:\mathcal{L}_0^{\otimes 2}\xrightarrow{\sim}g^*\mathcal{I}_0,\ \text{likewise at }\infty)\,|\,\text{relevant diagrams commute}\}.$$

Here the diagram for 0 asserts that α_0^2 equals the composite of λ_0 and the inclusion $g^*\mathcal{I}_0\hookrightarrow\mathcal{O}_T$. The diagram for ∞ is similar but involves α_∞^3.

This is not obviously a quotient stack $[U/G]$. In fact, it is impossible to write $\mathfrak{X}=[U/G]$ for a finite discrete group G: away from 0 and ∞, the map $U\to\mathfrak{X}$ would be a finite étale cover, which would have to be an n-fold multiplication map $\mathbb{G}_m\to\mathbb{G}_m$ for some n. But there's no possible choice of n that works (we have a two-fold cover at 0 and a three-fold cover at ∞).

However, we can realize $\mathfrak{X}=[(\mathbb{A}^2\setminus(0,0))/\mathbb{G}_m]$ as a weighted projective space (where $\mathbb{G}_m$ acts on the first coordinate by weight two and on the third coordinate by weight three).

3. 9/13 (Martin Olsson) – More Examples

3.1. **Stacky projective lines.** We begin with the example from last time.

Example 3.1. Consider again the stacky projective line $\mathfrak{X}$ with notation as above. One thinks of $\mathcal{L}_0$ as a "square root" of the ideal sheaf $\mathcal{I}_0$ and $\mathcal{L}_\infty$ as a "square root" of $\mathcal{I}_\infty$. Away from the points $\{0,\infty\}$, $\mathfrak{X}$ looks like $\mathbb{P}^1$. The fiber of $\mathfrak{X}$ over $0\in\mathbb{P}^1$ is $(\operatorname{Spec}k[\epsilon]/(\epsilon^2))/\mu_2$, where μ_2 acts by $\epsilon\mapsto-\epsilon$.

We can give multiple presentations of $\mathfrak{X}$. The typical approach to do so is to "rigidify" $\mathfrak{X}$ by choosing additional data and then quotienting by the choice of additional data.

(1) Let's rigidify $\mathfrak{X}$ by fixing isomorphisms $\mathcal{L}_0\simeq\mathcal{O}$ and $\mathcal{L}_\infty\simeq\mathcal{O}$. Then $\alpha_0,\alpha_\infty\in\Gamma(T,\mathcal{O}_T)$, so we may view these as functions f_0,f_∞. The λ's are replaced by γ_0, γ_∞, where γ_i trivializes $g^*\mathcal{I}_i$. We may then define a scheme U over $\mathbb{P}^1$ by

$$U(T)=\{(g:T\to\mathbb{P}^1,f_0,f_\infty,\gamma_0,\gamma_\infty)\,|\ \text{the necessary diagrams commute}\}.$$

This U is a scheme: we may view it explicitly as a closed subscheme of $(\mathbb{P}^1\times\mathbb{A}^2)\times_{\mathbb{P}^1}\mathrm{Isom}(\mathcal{O},\mathcal{I}_0)\times_{\mathbb{P}^1}\mathrm{Isom}(\mathcal{O},\mathcal{I}_\infty)$ (cut out by the condition that the diagrams commute). We have an action of $\mathbb{G}_m^2$ on U:

$$(u_0,u_\infty)\cdot(g,f_0,f_\infty,\gamma_0,\gamma_\infty)=(g,u_0f_0,u_\infty f_\infty,u_0^2\gamma_0,u_\infty^3\gamma_\infty)$$

Thus we may construct the quotient $[U/\mathbb{G}_m^2]$.

We claim that $\mathfrak{X}\simeq[U/\mathbb{G}_m^2]$. In fact, the point $(g,(\mathcal{L}_0,\alpha_0),(\mathcal{L}_0,\alpha_\infty),\lambda_0,\lambda_\infty)$ corresponds to the $\mathbb{G}_m^2$-torsor $\mathrm{Isom}(\mathcal{O},\mathcal{L}_0)\times_T\mathrm{Isom}(\mathcal{O},\mathcal{L}_\infty)$ together with its natural map to U.

(2) There are other presentations! This is an advantage of stacks (as opposed to just considering group actions). We can write $\mathfrak{X}\simeq[(\mathbb{A}^2\setminus\{0\})/\mathbb{G}_m]$ where $u\in\mathbb{G}_m$ acts by $u\cdot(s,t)=(u^2s,u^3t)$. Note that T-points of $[(\mathbb{A}^2\setminus\{0\})/\mathbb{G}_m]$ are

$$\{(\mathcal{L},a\in H^0(T,\mathcal{L}^{\otimes 2}),b\in H^0(T,\mathcal{L}^{\otimes 3}))\,|\,a,\ b\text{ not simultaneously }0\}.$$

To see the equivalence between this and $\mathfrak{X}$, fix an isomorphism $\mathcal{I}_0 \simeq \mathcal{I}_\infty$. This forces $\mathcal{L}_0^2 \simeq \mathcal{L}_\infty^3$, and we can compute $(\mathcal{L}_0 \otimes \mathcal{L}_\infty^{-1})^{\otimes 2} \simeq \mathcal{L}_\infty$ and $(\mathcal{L}_0 \otimes \mathcal{L}_\infty^{-1})^{\otimes 3} \simeq \mathcal{L}_0$. Letting $\mathcal{M} = \mathcal{L}_0^{-1} \otimes \mathcal{L}_\infty$, we see that $\alpha_0 \in \mathcal{M}^{\otimes 2}$ and $\alpha_\infty \in \mathcal{M}^{\otimes 3}$. Using this gives the equivalence.

3.2. **Weighted projective spaces.** We can generalize the latter construction.

Example 3.2. Let $\mathbb{G}_m$ act on $\mathbb{A}^{n+1}$ by $u \cdot (x_0, \dots, x_n) = (u^{\alpha_0} x_0, \dots, u^{\alpha_n} x_n)$. Removing the origin gives the weighted projective stack $\mathbb{P}(\alpha_0, \dots, \alpha_n) = [(\mathbb{A}^{n+1} \setminus \{0\})/\mathbb{G}_m]$. There are many interesting sub-examples:

(1) $\mathbb{P}^n = \mathbb{P}(1, \dots, 1)$
(2) We have $\overline{\mathfrak{M}}_{1,1} \simeq \mathbb{P}(4,6)$ (though we will not show this here).
(3) $\mathbb{P}(1,-1)$ has T-points
$$\{(\mathcal{L}, \alpha_+ \in H^0(T, \mathcal{L}), \alpha_- \in H^0(T, \mathcal{L}^{-1})) \mid a_+,\ a_- \text{ not simultaneously zero }\}.$$
This is covered by U_+ and U_- where
$$U_\pm(T) = \{(\mathcal{L}, a_+, a_-) \mid a_\pm \text{ generates } \mathcal{L}^{\otimes \pm 1}\}.$$
The intersection is $\mathbb{G}_m$, and $\mathbb{P}(1,-1)$ is the affine line with two origins.

This has some interesting applications.

Example 3.3. Consider the polynomial $x^2 + 29y^2 - 3z^3 \in \mathbb{Z}[x,y,z]$. We might like to consider a "projective variety" corresponding to this, but attempting to do so runs into an obvious issue: the polynomial is not homogeneous! We can obtain instead a "weighted projective variety"
$$[(\operatorname{Spec} \mathbb{Z}[x,y,z]/(x^2 + 29y^2 - 3z^3) \setminus \{0\})/\mathbb{G}_m] \subset \mathbb{P}(3,3,2).$$
This is proper over $\operatorname{Spec} \mathbb{Z}$. See a paper of Bhargava and Poonen for more discussion of this in the context of finding rational points on curves. A classic result of Darmon and Granville shows that this fails local-to-global (rational points exist locally but not globally).

4. 9/20 (Rose Lopez) – Birational Geometry of Stacks

References for today's talk are:

- Kresch and Tschinkel (2023) – *Birational Geometry of Deligne-Mumford Stacks*
- Kresch and Tschinkel (2019) – *Birational Types of Algebraic Orbifolds*
- Kontsevich and Tschinkel (2017) – *Specialization of Birational Types*
- Bergh and Rydh (2019) – *Functorial Destackification*

4.1. **Classical Birational Geometry.**

Definition 4.1. Let X and Y be varieties. A *rational map* $f : X \dashrightarrow Y$ is a morphism $f : U \to Y$ for some dense open $U \subset X$. We identify two such maps $f : U \to Y$, $f' : U' \to Y$ if $f|_{U \cap U'} = f'|_{U \cap U'}$.

We say that $f : X \dashrightarrow Y$ is *birational* if there exists an inverse map $Y \dashrightarrow X$. This induces an isomorphism on dense opens ($U \xrightarrow{\sim} V$) as well as an isomorphism on function fields.

Theorem 4.2 (Weak Factorization)**.** *Let $f : X \to Y$ be a birational map between smooth complete varieties over an algebraically closed field k of characteristic zero. Let $U \subset X$ be an open such that $f|_U$ is an isomorphism onto its image. Then f factors as a zigzag of blowups at smooth centers inducing isomorphisms on the image of U (??).*

4.2. **The Burnside Ring.** Fix an algebraically closed field k of characteristic zero.

Definition 4.3. Let Burn_n be the free abelian group generated by isomorphism classes of finitely generated fields of transcendence degree n. For a smooth projective irreducible variety X of dimension n, we define $[X] := [k(X)] \subset \mathrm{Burn}_n$. The *Burnside ring of varieties* is $\mathrm{Burn} = \oplus_n \mathrm{Burn}_n$ with ring structure given by $[X] \cdot [Y] = [X \times Y]$.

For $U = X \setminus D$ where $D = D_1 \cup \dots D_\ell$ is a simple normal crossing divisor, we define
$$[U] := [X] - \sum_i [D_i \times \mathbb{P}^1] + \sum_{i<j} [(D_i \cap D_j) \times \mathbb{P}^2]$$
We also let $[X \cup Y] = [X] + [Y]$. Note that $[U] = [U']$ if there exists a quasiprojective variety V and a pair of birational projective morphisms $V \to U$ and $V \to U'$.

Proposition 4.4. *The group* Burn_n *is generated by classes* $[U]$ *where* U *is* n*-dimensional and quasiprojective, modulo the* modified scissor relation

$$[U] = [V \times \mathbb{P}^{r-d}] + [U \setminus V]$$

for any smooth closed $V \subset U$ *of dimension* $< n$*.*

Proposition 4.5 (Specialization)**.** *Let* $\pi : X \to B$ *and* $\pi' : X' \to B$ *be smooth proper morphisms to a smooth connected curve* B*. If the generic fibers of* π *and* π' *are birational over* $k(B)$*, then for any closed* $b \in B$*, the fibers* $\pi^{-1}(b)$ *and* $(\pi')^{-1}(b)$ *are birational over* $k(b)$*.*

4.3. **Extension to stacks.**

Definition 4.6. Let $\mathfrak{X}$ and $\mathfrak{Y}$ be stacks. A rational map $f : \mathfrak{X} \dashrightarrow \mathfrak{Y}$ is a morphism defined on a dense open $\mathfrak{U} \subset \mathfrak{X}$. A 2-isomorphism $\alpha : (U, f) \to (U', f')$ is an isomorphism $f|_{U \cap U'} \xrightarrow{\sim} f'|_{U \cap U'}$. We say that f is birational if it induces an equivalence on dense opens.

Definition 4.7. Let $\mathfrak{X}$ and $\mathfrak{Y}$ be stacks. We say that $\mathfrak{X}$ and $\mathfrak{Y}$ are *birationally equivalent* if there exists a representable proper birational map $\mathfrak{X} \dashrightarrow \mathfrak{Y}$, or equivalently a span $\mathfrak{X} \leftarrow \mathfrak{X}' \to \mathfrak{Y}$ consisting of representable proper birational morphisms.

Example 4.8. Let $\mathfrak{X}$ be a stack and $\mathfrak{Z}$ a closed substack. We can define a *blowup* $\mathrm{B}\ell_{\mathfrak{Z}}\,\mathfrak{X} \to \mathfrak{X}$ by performing the usual scheme-theoretic blowup locally and gluing.

Example 4.9. Let $\mathfrak{X}$ be a smooth separated DM stack and $\mathfrak{D} \subset \mathfrak{X}$ a divisor. We can define a *root stack* $\sqrt[r]{(\mathfrak{X}, \mathfrak{D})} \to \mathfrak{X}$ by

$$\sqrt[r]{(\mathfrak{X}, \mathfrak{D})}(T) = \left\{(g : T \to \mathfrak{X}, \alpha : \mathcal{L} \to \mathcal{O}_T, \lambda : \mathcal{L}^{\otimes r} \xrightarrow{\sim} g^*\mathcal{I}_{\mathfrak{D}}) \,\middle|\, \text{the induced maps } \mathcal{L}^{\otimes r} \to \mathcal{O}_T \text{ agree}\right\}$$

The morphism $\sqrt[r]{(\mathfrak{X}, \mathfrak{D})} \to \mathfrak{X}$ is proper but not representable: the fibers over points in $\mathfrak{D}$ are stacky.

Example 4.10. Let $\mathfrak{X}$ be a stack and $\mathcal{L}$ a line bundle on $\mathfrak{X}$. The *gerbe of* n*th roots of* $\mathcal{L}$ is $\sqrt[n]{\mathcal{L}/\mathfrak{X}} \to \mathfrak{X}$ with

$$\sqrt[n]{\mathcal{L}/\mathfrak{X}}(T) = \left\{(g : T \to \mathfrak{X}, \mathcal{M} \in \mathsf{Pic}T, \epsilon : \mathcal{M}^{\otimes n} \xrightarrow{\sim} g^*\mathcal{L})\right\}$$

We can contrast this with the above example: essentially, the above example rigidifies things by considering $\mathcal{I}_{\mathfrak{D}} \to \mathcal{O}$. Over $\mathfrak{U} = \mathfrak{X} \setminus \mathfrak{D}$, the gerbe $\sqrt[n]{\mathcal{L}/\mathfrak{X}}|_{\mathfrak{U}}$ is trivializable.

There are two weak factorization results for stacks. One factorizes a representable proper birational map into blowups and can be used to define the "correct" (?) Burnside ring. Another factorizes a proper birational map into stacky blowups and can be used to define a weaker Burnside ring related to "G-equivariant Burnside rings."

Example 4.11. Let $C = \mathbb{A}^1$, $C_0 = \mathbb{A}^1 \setminus \{0\}$, E an elliptic curve, and $p \in E$. Define $\mathfrak{X}_0 = C_0 \times E \times B(\mathbb{Z}/2) \subset \mathfrak{X}_1 = C \times E \times B(\mathbb{Z}/2)$. Let $B = \mathrm{B}\ell_{(0,p)}(C \times E)$ and $\mathfrak{X}_2 = \sqrt{\mathcal{O}_B(D)/B}$. We get an inclusion $\mathfrak{X}_0 \to \mathfrak{X}_2$. This induces an equivalence on generic fibers but not on special fibers (over $(0, p)$??). The "specialization" result from earlier fails for stacks!

5. 9/27 (Xiangru Zeng) – Valuative Criteria for the Existence of Moduli Spaces

Let k be an algebraically closed field of characteristic zero. We may skip some less-than-essential details. References are:

- Alper – Good Moduli Spaces for Artin Stacks
- Alper, Halpern-Leistner, Heinloth – Existence of Moduli Spaces for Algebraic Stacks
- Alper – Notes on Stacks and Moduli

5.1. **Precursors.**

Theorem 5.1 (Keel-Mori)**.** *Let* $\mathfrak{X}$ *be a separated Deligne-Mumford stack of finite type over a base* S*. Then there exists coarse moduli space* $\pi : \mathfrak{X} \to X$*, i.e. a universal algebraic space* X *among algebraic spaces equipped with a map from* X *and such that* π *is bijective on geometric points. This satisfies* $\pi_*\mathcal{O}_{\mathfrak{X}} = \mathcal{O}_X$*. In characteristic zero, we end up with a "tame moduli space:"*

(1) If $\mathfrak{X} \to S$ *is flat, then* $X \to \mathrm{Spec}\,k$ *is flat.*

(2) *The forgetful functor* $\pi_* : \mathsf{QCoh}(\mathfrak{X}) \to \mathsf{QCoh}(X)$ *is exact.*

This requires $\mathfrak{X}$ to be separated – in particular, the stabilizer groups of $\mathfrak{X}$ must be affine. This is a relatively strict condition!

When considering problems of GIT, we often let $U \subset \mathbb{P}^n$ be a projective variety with linearized G-action (for some linearly reductive group G). The GIT quotient is then

$$U /\!\!/ G = \operatorname{Proj} \oplus_{n \geqslant 0} \Gamma(U, \mathcal{O}_U(n))^G.$$

We have a map $\pi : [U^{ss}/G] \to U /\!\!/ G$, where U^{ss} is the semistable locus. This satisfies:

(1) $\mathcal{O}(U /\!\!/ G) = \pi_*(\mathcal{O}(U^{ss}))^G$, and
(2) (In characteristic zero) π is affine.

Seshadri called a map satisfying the above conditions a *good quotient*.

5.2. Good moduli spaces.

Definition 5.2. A map $\pi : \mathfrak{X} \to X$, where $\mathfrak{X}$ is a (qcqs?) algebraic stack and X is a (qcqs?) algebraic space, is a *good moduli space* if

(1) $\pi_* \mathcal{O}_{\mathfrak{X}} \cong \mathcal{O}_X$, and
(2) $\pi_* : \mathsf{QCoh}(\mathfrak{X}) \to \mathsf{QCoh}(X)$ is exact.

Theorem 5.3. *Let* $\pi : \mathfrak{X} \to X$ *be a good moduli space. Then:*

(1) π *is surjective and universally closed.*
(2) For $x_1, x_2 \in \mathfrak{X}(k)$, *then* $\pi(x_1) = \pi(x_2)$ *if and only if* $\overline{\{x_1\}} \cap \overline{\{x_2\}} \neq \varnothing$.
(3) If $\mathfrak{X}$ *is noetherian, then* π *is universal among maps from* $\mathfrak{X}$ *to algebraic spaces.*

Example 5.4. Let G be a linearly reductive group acting on an affine scheme $\operatorname{Spec} A$. Then $[A/G] \to \operatorname{Spec} A^G$ is a good moduli space. In particular, $[\mathbb{A}^n/\mathbb{G}_m] \to \operatorname{Spec} k$ is a good moduli space. However, if we remove the origin, then $[(\mathbb{A}^n \setminus \{0\})/\mathbb{G}_m] \to \mathbb{P}^{n-1}$ is a good moduli space (in fact, the map is an equivalence).

In positive characteristic, the notion of a good moduli space is too restrictive. People consider "adequate moduli spaces" instead. We won't discuss these in detail.

5.3. Existence of good moduli spaces.

Theorem 5.5 (Alper, Halpern-Leistner, Heinloth)**.** *Let* $\mathfrak{X}$ *be an algebraic stack of finite type with affine diagonal. Then* $\mathfrak{X}$ *admits a separated good moduli space if and only if*

(1) $\mathfrak{X}$ *is* Θ*-reductive, and*
(2) $\mathfrak{X}$ *is* S*-complete.*

Let's explain what these mean. Let $\Theta = [\mathbb{A}^1/\mathbb{G}_m]$, and for any ring A, we write $\Theta_A = \Theta \times_{\operatorname{Spec} k} \operatorname{Spec} A$. For a DVR R with fraction field K, let $\Theta_R \setminus \{0\} = \operatorname{Spec} R \cup_{\operatorname{Spec} K} \Theta_K$.

Definition 5.6. Say $\mathfrak{X}$ is Θ-reductive if, for all DVRs R and all $\Theta_R \setminus \{0\}$, there exists a unique extension

$$\begin{tikzcd} \Theta_R \setminus \{0\} \arrow[r] \arrow[d] & \mathfrak{X}. \\ \Theta_R \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

For S-completeness, if R is a DVR with uniformizer π, let

$$\phi_R = [\operatorname{Spec} R[s,t]/(st - \pi)/\mathbb{G}_m]$$

where $\mathbb{G}_m$ acts by weight 1 on s and weight -1 on t.

Definition 5.7. Say $\mathfrak{X}$ is S-complete if, for all DVRs R, there exists a unique extension

$$\begin{tikzcd} \phi_R \setminus \{0\} \arrow[r] \arrow[d] & \mathfrak{X}. \\ \phi_R \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

Example 5.8. If $U = \operatorname{Spec} A$ is acted on by G, then $[U/G]$ is Θ-reductive if, whenever $\lambda : \mathbb{G}_m \to G$ and $\pi : \operatorname{Spec} R \to U$ are maps (with R a DVR) such that $\lim_{t\to 0} \lambda(t)\pi(\operatorname{Spec} K)$ exists, then $\lim_{t\to 0} \lambda(t)\pi(\operatorname{Spec} R)$ also exists. We can interpret S-completeness similarly.

The theorem is proved by first using the existence of good moduli spaces for quotient stacks $\operatorname{Spec} A/G$ In extending this result to more general $\mathfrak{X}$, we use the local structure theorem to write $\mathfrak{X}$ in an étale neighborhood of a point x as $[\operatorname{Spec} A/G_x]$, where G_x is the automorphism group of x. For gluing, we consider a diagram

$$\begin{tikzcd} {[\operatorname{Spec} B/G_x]} \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & {[\operatorname{Spec} A/G_x]} \arrow[r] \arrow[d] & \mathfrak{X} \\ \operatorname{Spec} B^{G_x} \arrow[r, shift left] \arrow[r, shift right] & \operatorname{Spec} A^{G_x} \arrow[r] & X. \end{tikzcd}$$

We can use Luna's fundamental theorem and "Θ-surjectivity" to glue things.

6. 10/4 (Will Fisher) – Smoothness of the Logarithmic Hodge Moduli Space

The reference for today is:

- De Cataldo, Herrero, Zhang – Geometry of the Logarithmic Hodge Moduli Space

We'll begin by reviewing this moduli space and why we should care about it.

6.1. Review of Hodge theory.

If X is a topological space and $\mathcal{F}$ is a sheaf of abelian groups on X, then we can define the sheaf cohomology $H^\bullet(X, \mathcal{F})$ by taking an injective resolution $0 \to \mathcal{F} \to \mathcal{I}^\bullet$ and setting

$$H^k(X, \mathcal{F}) := H^k(\Gamma(X, \mathcal{I}^\bullet)).$$

More generally, if $\mathcal{F}^\bullet$ is a chain complex, we define the *hypercohomology* $\mathbb{H}^\bullet(X, \mathcal{F})$ by choosing a quasi-isomorphism $\mathcal{F} \xrightarrow{\sim} \mathcal{I}^\bullet$ with $\mathcal{I}^\bullet$ a complex of injectives and setting

$$\mathbb{H}^k(X, \mathcal{F}^\bullet) := H^k(\Gamma(X, \mathcal{I}^\bullet)).$$

For Hodge theory, let X be a smooth projective variety over $\mathbb{C}$. By GAGA and the (p, q)-form decomposition, we have

$$\mathbb{H}^k(X, (\Omega_X^\bullet, d)) \cong H^k_{\mathrm{dR}}(X^{\mathrm{an}}, \mathbb{C}).$$

Formal manipulations give

$$\begin{aligned} \mathbb{H}^k(X, (\Omega_X^\bullet, 0)) &= \mathbb{H}^k\big(X, \oplus_i \Omega_X^i[-i]\big) \\ &= \oplus_i \mathbb{H}^k\big(X, \Omega_X^i[-i]\big) \\ &= \oplus_i \mathbb{H}^{k-i}(X, \Omega_X^i) \\ &= \oplus_{p+q=k} \mathbb{H}^q(X, \Omega_X^p). \end{aligned}$$

Thus the usual Hodge decomposition can be rewritten as

$$\mathbb{H}^k(X, (\Omega_X^\bullet, d)) \cong \mathbb{H}^k(X, (\Omega_X^\bullet, 0)).$$

This suggests that we should think of the Hodge decomposition as a statement about *sheaves* rather than as a statement just about X. Let's try to understand how to generalize this.

6.2. Non-abelian Hodge theory.

Let $\mathcal{F}$ be a vector bundle on X (still assumed to be a smooth complex projective variety). Suppose we have a connection $\nabla : \mathcal{F} \to \mathcal{F} \otimes \Omega_X^1$. Recall that this means ∇ satisfies the Leibniz rule

$$\nabla(fs) = s \otimes df + f\nabla s$$

for $f \in \mathcal{O}_X$, $s \in \mathcal{F}$.

If ∇ is flat, i.e. $\nabla \circ \nabla = 0$, then we get an associated complex

$$0 \longrightarrow \mathcal{F} \xrightarrow{\nabla} \mathcal{F} \otimes \Omega_X^1 \xrightarrow{\nabla} \mathcal{F} \otimes \Omega_X^2 \xrightarrow{\nabla} \dots$$

We can (and should) view $(\Omega_X^\bullet, d)$ as the complex associated to $\mathcal{O}_X$ with the flat connection d.

For the other side of the non-abelian Hodge theorem, we need a new definition.

Definition 6.1. Let $\mathcal{F}$ be a vector bundle on X. A *Higgs field* is an $\mathcal{O}_X$-linear map $\phi : \mathcal{F} \to \mathcal{F} \otimes \Omega_X^1$ satisfying $\phi \wedge \phi = 0$. We call the pair $(\mathcal{F}, \phi)$ a *Higgs bundle*.

The Higgs field condition can be understood locally: writing $\phi = \sum_i A_i dx_i$, we require

$$\phi \wedge \phi := \sum_{i<j} [A_i, A_j] dx_i \wedge dx_j = 0.$$

If $(\mathcal{F}, \phi)$ is a Higgs bundle, we get a complex

$$0 \longrightarrow \mathcal{F} \xrightarrow{\phi} \mathcal{F} \otimes \Omega^1_X \xrightarrow{\phi} \mathcal{F} \otimes \Omega^2_X \xrightarrow{\phi} \dots$$

In particular, $(\Omega^\bullet_X, 0)$ is the complex associated to the Higgs bundle $(\mathcal{O}_X, 0)$.

Remark 6.2. Arthur Ogus suggested an alternative viewpoint: we can think of the Higgs field structure as a suitable lift of $\mathcal{F}$ to the cotangent space T^*X.

We can now state the non-abelian Hodge theorem.

Theorem 6.3 (Non-abelian Hodge theorem)**.** *Let X be a smooth projective variety over $\mathbb{C}$. Then (up to stability conditions) there exists a "cohomology preserving" correspondence between:*

1. *The moduli space M_{flat} of flat bundles on X.*
2. *The moduli space M_{Higgs} of Higgs bundles on X.*

This correspondence sends $(\mathcal{O}_X, d)$ to $(\mathcal{O}_X, 0)$.

The non-abelian Hodge theorem is essentially analytic. However, we can hope for weaker algebraic statements: perhaps we can find relationships between the corresponding moduli spaces.

One result in this direction is a cohomology equivalence: $H^\bullet(M_{\text{flat}}) \cong H^\bullet(M_{\text{Higgs}})$. This can be shown by constructing a suitable geometric family interpolating between the two moduli spaces.

6.3. **Construction of the interpolation.** Let X be a smooth scheme over a base B.

Definition 6.4. Given a vector bundle $\mathcal{F}$ on X and $t \in \Gamma(B, \mathcal{O}_B)$, a t-*connection* on $\mathcal{F}$ is an $\mathcal{O}_B$-linear map $\nabla : \mathcal{F} \to \mathcal{F} \otimes \Omega^1_{X/B}$ satisfying the t-twisted Leibniz rule:

$$\nabla(fs) = ts \otimes df + f\nabla s$$

for $f \in \mathcal{O}_X$ and $s \in \mathcal{F}$. We can use the Leibniz rule to extend ∇ to $\mathcal{F}$-valued n-forms. This allows us to say that ∇ is *flat* if $\nabla \circ \nabla = 0$.

Example 6.5. If $t = 1$, a 1-connection is just a connection.

Example 6.6. If $t = 0$, the 0-twisted Leibniz rule is $\nabla(fs) = f\nabla(s)$, so a flat 0-connection is just a Higgs field.

The general goal is to use GIT to construct a moduli space $(M_{\text{Hodge}})_X$ of t-connections on X together with a smooth map

$$\begin{aligned} (M_{\text{Hodge}})_X &\to \mathbb{A}^1_B \\ (\mathcal{F}, \nabla_t) &\mapsto t. \end{aligned}$$

We'll restrict our hypotheses to make this tractable. In particular, we assume:

- B is a noetherian scheme.
- C is a smooth proper scheme over B with geometrically integral fibers of dimension 1.
- $D \hookrightarrow C$ is a relative Cartier divisor such that every geometric fiber over B is reduced and non-empty.
- The rank n and degree d are coprime.

We'll write C_B for C as a B-scheme and C_S for a base change $C_B \times_B S$.

Definition 6.7. Let $(\mathfrak{M}_{\text{Hodge}})_{C_B} \to \mathbb{A}^1_B$ be the moduli stack of rank n, degree d flat t-connections. The groupoid of S-points (for $S \to \mathbb{A}^1_B$) consists of pairs $(\mathcal{F}, \nabla)$ where:

1. $\mathcal{F}$ is a vector bundle of rank n on C_S such that the restriction to each geometric fiber has degree d.
2. $\nabla : \mathcal{F} \to \mathcal{F} \otimes \omega_{C_S/S}(D_S)$ is a flat t-connection.

Taking GIT quotients, we obtain a moduli space $(M_{\text{Hodge}})_{C_B} \to \mathbb{A}^1_B$.

6.4. **Smoothness.** Our goal in the remaining time is to show $(M_{\mathrm{Hodge}})_{C_B} \to \mathbb{A}^1_B$ is smooth. We proceed in a few steps:

(1) The map $(\mathfrak{M}_{\mathrm{Hodge}})_{C_B} \to (M_{\mathrm{Hodge}})_{C_B}$ is a smooth surjection, in fact a smooth good moduli space in the sense of Alper. This allows us to reduce to checking that $(\mathfrak{M}_{\mathrm{Hodge}})_{C_B} \to \mathbb{A}^1_B$ is smooth. This is where the coprimality hypothesis arises.
(2) Then we develop an obstruction theory for t-connections, i.e. an obstruction module $\mathcal{Q}_x$ for every $x : \operatorname{Spec} A \to (\mathfrak{M}_{\mathrm{Hodge}})_{C_B}$ which is compatible with base change. This module controls lifts along square-zero extensions.
(3) This allows us to reduce to showing that $\mathcal{Q}_x$ vanishes for every geometric point $x : \operatorname{Spec} k \to (\mathfrak{M}_{\mathrm{Hodge}})_{C_B}$.
(4) Given such a geometric point x, we extend this to a family $\mathbb{A}^1_k \to (\mathfrak{M}_{\mathrm{Hodge}})_{C_B}$ which is compatible with the rescaling of t-connections and is such that x corresponds to $1 \in \mathbb{A}^1_k$.
(5) Degenerating our t-connection along this family allows us to reduce to showing that $\mathcal{Q}_x = 0$ for geometric points x corresponding to Higgs bundles.
(6) In this case, we can check that the desired result holds – this part really requires the existence of poles / the use of a divisor D with the stated properties.

7. 10/11 (Noah Olander) – Root Stacks and Tame Ramification

The goal of this talk is to discuss a partial generalization of a theorem of Grothendieck on fundamental groups.

7.1. **A theorem of Grothendieck.** Let X be a scheme, $\text{Ét}(X)$ the category of schemes which are étale over X, and $\text{FÉt}(X)$ the category of schemes which are finite étale over X. If $(R, \mathfrak{m})$ is a local ring and X is an R-scheme, we will write $X_n = X \times_{\operatorname{Spec} R} \operatorname{Spec}(R/\mathfrak{m}^{n+1})$.

Theorem 7.1. *Let R be a complete noetherian local ring. Suppose X is proper over R. Then the pullback map $\text{FÉt}(X) \to \text{FÉt}(X_0)$ is an equivalence.*

As a consequence, if X and X_0 are connected and $\overline{x} : \operatorname{Spec} \overline{k} \to X_0$ is a geometric point, then $\pi_1(X_0, \overline{x}) = \pi_1(X_0, \overline{x})$. We'll try to generalize this statement, but first we'll talk about the proof.

Proof. There are two main ingredients:

(1) *Grothendieck's existence theorem*: in the situation above, the natural map $\mathsf{Coh}(X) \to \lim_n \mathsf{Coh}(X_n)$ is an equivalence. Here $\lim_n \mathsf{Coh}(X_n)$ is the category of systems $(\mathcal{F}_n \in \mathsf{Coh}(X_n), \alpha_n : \mathcal{F}_{n+1} \xrightarrow{\sim} \mathcal{F}_n)$. This really requires completeness of R and properness of X.
(2) If $i : Z \hookrightarrow X$ is a closed immersion with $\mathcal{I}_Z^2 = 0$, then $\text{Ét}(X) \to \text{Ét}(Z)$ is an equivalence. One can see that this functor is fully faithful by the formal criterion for étaleness. Once we know the functor is fully faithful, the question of essential surjectivity reduces to a local question, and we can do this explicitly by lifting presentations.

Given these, we obtain $\text{FÉt}(X) \xrightarrow{\sim} \lim_n \text{FÉt}(X_n) \xrightarrow{\sim} \text{FÉt}(X_0)$. □

The theorem and its proof generalize easily to algebraic stacks which are proper over R. We'd like find a partial generalization to the non-proper case.

7.2. **Root stacks.** Our generalization will involve root stacks.

Let's consider the local situation first: let X be a scheme, $f \in \mathcal{O}(X)$. Suppose $r \in \mathbb{Z}_{\geqslant 1}$ is invertible in $\mathcal{O}(X)$. Write $X_r = \underline{\operatorname{Spec}}(\mathcal{O}_X[t]/(t^r - f))$. If T is a scheme over X, then

$$X_r(t) = \{g \in \mathcal{O}(T) \mid g^r = f\}$$

has an action of $\mu_r(T)$ via $a \cdot g = ag \in \mathcal{O}(T)$. The *root stack* is $\mathfrak{X}_r = [X_r/\mu_r]$.

Globally, let $\mathcal{L} \in \mathsf{Pic}(X)$ and $s \in \Gamma(X, \mathcal{L})$. Continue to assume r is invertible in $\mathcal{O}(X)$. We define a *root stack* $\mathfrak{X}_r$ over X with T-points (for $f : T \to X$) given by

$$\mathfrak{X}_r(T) = \left\{ \left(\mathcal{M} \in \mathsf{Pic}(T), \alpha : f^*\mathcal{L} \xrightarrow{\sim} \mathcal{M}^{\otimes r}, t \in \Gamma(T, \mathcal{M})\right) \,\middle|\, \alpha(f^*s) = t^r \right\}$$

If $\mathcal{L}$ is trivial, then this agrees with the local construction above. In particular, it follows that $\mathfrak{X}_r$ is algebraic.

Note that $\mathfrak{X}_r$ has a tautological line bundles $\mathcal{M}$ and section t such that (writing $p : \mathfrak{X}_r \to X$ for the structure morphism) we have $\mathcal{M}^{\otimes r} \cong p^*\mathcal{L}$ and $t^r = s$.

Example 7.2. Let $X = \mathbb{A}^1_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x]$, and let $f = x$. Then $X_2 = \operatorname{Spec} k[x,t]/(t^2 - x)$, and $\mathfrak{X}_2 = [X_2/\mu_2]$ looks like $\mathbb{A}^1_{\mathbb{C}}$ with a copy of $B\mu_2$ at the origin.

Lemma 7.3. *Let* $D = V(s)$*.*

1. *If* $r \in \mathcal{O}(X)^\times$*, then* $\mathfrak{X}_r$ *is a Deligne-Mumford stack.*
2. *Let* $U = X \setminus D$*. Then* $\mathfrak{X}_r \to X$ *is an isomorphism over* U*.*
3. *Let* $\mathfrak{D}_r = V(t)$*. Then* $\mathfrak{D}_r$ *is a* μ_r*-gerbe over* D*.*
4. *If* $s : \mathcal{O}_X \hookrightarrow \mathcal{L}$*, then* $t : \mathcal{O}_{\mathfrak{X}_r} \hookrightarrow \mathcal{M}$*.*
5. *If* $s : \mathcal{O}_X \hookrightarrow \mathcal{L}$ *and* X *and* D *are regular noetherian schemes, then* $\mathfrak{X}_r$ *and* $\mathfrak{D}_r$ *are regular.*

Proof. Everything is local, so we reduce to the case $\mathcal{L} = \mathcal{O}_X$.

1. The map $X_r \to \mathfrak{X}_r$ is an étale cover by a scheme.
2. The map $U \times_X X_r \to U$ is a μ_r-torsor. Now observe that, if R is a ring and $a \in R^\times$, then $\mu_r(R)$ acts simply transitively on the set of roots $\{t \mid t^r = a\}$.
3. This follows from a local computation together with the fact that D has trivial μ_r action.
4. Left as an exercise.
5. Left as an exercise. □

7.3. **Tame ramification.** Let A be a DVR and $K = \operatorname{Frac}(A)$. Let L/K be a finite separable extension, and let B be the integral closure of A in L. Let $\mathfrak{m}_1, \dots, \mathfrak{m}_s$ denote the maximal ideals of B, and let e_i be the ramification index of $A \to B_{\mathfrak{m}_i}$ (i.e. the integers such that $\pi_A \mapsto u\pi_B^{e_i}$ where $u \in B^\times$).

Definition 7.4. The extension L/K is:

1. *unramified* with respect to A if $e_i = 1$ for all i.
2. *tamely ramified* with respect to A if $e_i \in A^\times$ for all i. (Equivalently, $\operatorname{char} A/\mathfrak{m}_A$ does not divide e_i for any i.)

Lemma 7.5 (Abhyankar)**.** *If* L/K *is tamely ramified with respect to* A*,* $\pi \in A$ *is a uniformizer, and* $r \in \mathbb{Z}$ *is such that* $e_i | r$ *for all* i *and* $r \in A^\times$*, then* $L[\pi^{1/r}]/K[\pi^{1/r}]$ *is unramified with respect to* $A[T]/(T^r - \pi)$*.*

Let X be an integral regular noetherian scheme. Let D be an effective divisor on X with generic points $\eta_1, \dots, \eta_m$. Write $U = X \setminus D$. Let $f : Y \to U$ be finite étale with Y integral, and write L/K for the corresponding extension of function fields.

Definition 7.6. In the above setup, we say f is *unramified* (resp. *tamely ramified*) along D if K/L is so with respect to $\mathcal{O}_{X,\eta_i}$ for all i.

Proposition 7.7 (Generalized Abhyankar's Lemma)**.** *Let* X *be a regular scheme over a field* k*, and let* $D \subset X$ *be a regular divisor. Let* $f : Y \to X$ *be tamely ramified along* D*. Then there exists* $n \in \mathbb{Z}_{\geqslant 1}$ *invertible in* k *such that* f *extends to a finite étale morphism* $\mathfrak{Y} \to \mathfrak{X}_r$ *(where* $Y \subset \mathfrak{Y}$ *is a dense open).*

We're out of time, but we can at least state the theorem we've been building up to.

Theorem 7.8. *Let* R *be a regular noetherian complete local ring. Let* X/R *be smooth and proper, and let* $D \subset X$ *be a smooth effective Cartier divisor. Let* $U = X \setminus D$*. Then any finite étale morphism* $Y_0 \to U_0$ *which is tamely ramified along* D_0 *extends to a finite étale morphism* $Y \to U$ *which is tamely ramified along* D*.*

8. 10/18 (Daigo Ito) – Good Moduli Spaces of Objects in Abelian Categories

Throughout we let $k = \overline{k}$, $\operatorname{char} k = 0$. (Most results about abelian categories here work over any commutative ring. Some need excellence and $\operatorname{char} k = 0$.) We're following a paper by Alper on the existence of good moduli spaces.

8.1. **Background on semistability.** Let X be a projective variety over k. Can we construct a nice moduli space of coherent sheaves on X? The answer is *yes* if we restrict to a nice enough class of sheaves (e.g. *semistable* sheaves). Can we understand this more generally?

Definition 8.1. Fix an ample line bundle $\mathcal{O}(1)$ on X and a "numerical K-class" $v \in K^{\mathrm{num}}(X) := K(X)/\equiv^X$, where $\mathcal{F} \equiv^X 0$ if $\chi(\mathcal{F}, \mathcal{G}) = 0$ for all $\mathcal{G}$. Here

$$\chi(\mathcal{F}, \mathcal{G}) = \sum_i (-1)^i \dim \operatorname{Ext}^i(\mathcal{F}, \mathcal{G}).$$

Define the moduli problem $\mathcal{M}_\nu(\mathcal{O}(1)) : \mathsf{Sch}^{\mathrm{op}} \to \mathsf{Set}$ by letting $\mathcal{M}_\nu(\mathcal{O}(1))(\mathsf{T})$ consist of T-flat coherent sheaves on $\mathsf{X} \times \mathsf{T}$ such that, for all t, the sheaves $\mathcal{F}_\mathsf{t}$ are semistable (w/r/t $\mathcal{O}(1)$) and $\mathrm{ch}(\mathcal{F}_\mathsf{t}) = \nu$, where ch is the numerical Chern character.

The following can be found in Huybrechts and Lehn:

Theorem 8.2. *The moduli problem $\mathcal{M}_\nu(\mathcal{O}(1))$ admits a coarse moduli space $\mathsf{M}_\nu(\mathcal{O}(1))$ which is a projective variety.*

Idea of proof. One shows that there exists $\mathfrak{m} > 0$ such that, for all semistable sheaves $\mathcal{F}$ with $\mathrm{ch}(\mathcal{F}) = \nu$, there exists a vector space V and a surjection $\mathcal{O}(-\mathfrak{m}) \otimes \mathsf{V} \twoheadrightarrow \mathcal{F}$. (Specifically, we can take $\dim \mathsf{V} = \chi(\mathcal{F}(\mathfrak{m}))$.) Let R be the open subscheme of semistable quotients in $\mathrm{Quot}(\mathcal{O}(-\mathfrak{m}) \otimes \mathsf{V}, \nu)$. We take $\mathsf{M}_\nu(\mathcal{O}(1)) = \mathsf{R} /\!\!/ \mathrm{PGL}(\nu)$. To conclude, we use the fact that Gieseker semistability agrees with GIT semistability. □

8.2. Background on good moduli spaces.

Definition 8.3. A qcqs morphism $\pi : \mathfrak{X} \to \mathsf{X}$, where $\mathfrak{X}$ is an algebraic stack and X is an algebraic space, is a *good moduli space* if:

1. The pushforward $\pi_* : \mathsf{QCoh}(\mathfrak{X}) \to \mathsf{QCoh}(\mathsf{X})$ is exact, and
2. The natural map $\mathcal{O}_\mathsf{X} \to \pi_* \mathcal{O}_\mathfrak{X}$ is an isomorphism.

Example 8.4. Let G be a reductive group scheme acting linearly on a projective scheme X. Then the GIT quotient map $[\mathsf{X}^{ss}/\mathsf{G}] \to \mathsf{X}^{ss} /\!\!/ \mathsf{G}$ is a good moduli space.

8.3. Moduli stacks of objects of abelian categories.

Let $\mathcal{A}$ be a k-linear Grothendieck abelian category, e.g. $\mathcal{A} = \mathsf{QCoh}(\mathsf{X})$ or $\mathcal{A} = \mathsf{Ind}(\mathcal{B})$ for some small abelian category $\mathcal{B}$. We'd like to construct a moduli stack $\mathcal{M}_\mathcal{A}$ of nice objects of $\mathcal{A}$.

Definition 8.5. An object $\mathsf{E} \in \mathcal{A}$ is

1. *finitely presented* (or *compact*) if $\mathrm{Hom}_\mathcal{A}(\mathsf{E}, -)$ commutes with arbitrary filtered colimits.
2. *noetherian* if every ascending chain of subobjects terminates.

We say $\mathcal{A}$ is *locally noetherian* if $\mathcal{A}$ has a generating set of noetherian objects.

Example 8.6. For $\mathcal{A} = \mathsf{R}\text{-}\mathsf{Mod}$, these agree with the usual notions.

Example 8.7. If X is a noetherian scheme, then $\mathsf{QCoh}(\mathsf{X})$ is locally noetherian.

We will assume from now on that $\mathcal{A}$ is locally noetherian. This implies that noetherian objects and compact objects agree in $\mathcal{A}$.

Definition 8.8. Let R be a (commutative) k-algebra. We define $\mathcal{A}_\mathsf{R}$ as the category of R-module objects in $\mathcal{A}$ (so objects of $\mathcal{A}_\mathsf{R}$ are pairs $(\mathsf{E} \in \mathcal{A}, \xi_\mathsf{E} : \mathsf{R} \to \mathrm{End}_\mathcal{A}(\mathsf{E}))$).

Even if $\mathcal{A}$ is noetherian, $\mathcal{A}_\mathsf{R}$ need not be locally noetherian. This will hold if R is of finite type, though. We will generally avoid this issue.

For a morphism of (commutative) k-algebras $\mathsf{f} : \mathsf{S} \to \mathsf{T}$, there is a forgetful functor $\mathsf{f}_* : \mathcal{A}_\mathsf{T} \to \mathcal{A}_\mathsf{S}$ which is faithful, faithfully exact, and commutes with filtered colimits. This functor admits a left adjoint $\mathsf{f}^* = \mathsf{T} \otimes_\mathsf{S} (-)$.

For $\mathsf{E} \in \mathcal{A}_\mathsf{R}$, we also have a functor $(-) \otimes_\mathsf{R} \mathsf{E} : \mathcal{A}_\mathsf{R} \to \mathcal{A}_\mathsf{R}$. We say E is R-*flat* if $(-) \otimes_\mathsf{R} \mathsf{E}$ is exact.

Definition 8.9. Let $\mathcal{A}$ be a locally noetherian category. We define the pseudofunctor $\mathcal{M}_\mathcal{A} : \mathsf{CAlg}_\mathsf{k} \to \mathsf{Grpd}$ by letting $\mathcal{M}_\mathcal{A}(\mathsf{R})$ be the groupoid of R-flat and finitely presented $\mathsf{E} \in \mathcal{A}_\mathsf{R}$. This sends a morphism f to f^*, which is well-defined in a suitable sense.

Theorem 8.10. *The pseudofunctor $\mathcal{M}_\mathcal{A}$ is a stack for the fppf topology, hence naturally extends to a fppf stack on Sch_k.*

8.4. Gieseker stability and Bridgeland stability.

Let $\mathcal{A} = \mathsf{QCoh}(\mathsf{X})$ for a projective k-scheme X.

Theorem 8.11 (Lieblich)**.** *The moduli stack $\mathcal{D}^\mathrm{b}_{\mathrm{pug}}(\mathsf{X})$ of universally gluable relatively perfect complexes on X is an algebraic stack, locally of finite type.*

Here "universally gluable" denotes vanishing of negative self-Ext groups.

Theorem 8.12 (Halpern-Leistner)**.** *Let $\mathcal{C}$ be the heart of a "nice" t-structure on $\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathsf{X})$. Then $\mathcal{M}_{\mathsf{Ind}(\mathcal{C})}$ is an open substack of $\mathcal{D}^{\mathrm{b}}_{\mathrm{pug}}(\mathsf{X})$.*

Example 8.13. The standard t-structure on $\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathsf{X})$ has $\mathsf{Coh}(\mathsf{X})$ as its heart. We have $\mathsf{QCoh}(\mathsf{X}) = \mathsf{IndCoh}(\mathsf{X})$ as abelian categories. It follows that $\mathcal{M}_{\mathsf{QCoh}(\mathsf{X})}$ embeds as an open substack of $\mathcal{D}^{\mathrm{b}}_{\mathrm{pug}}(\mathsf{X})$.

Theorem 8.14. *Let $\mathcal{A}$ be locally noetherian. Choose a locally constant function $\mathsf{p}_\nu : |\mathcal{M}_\mathcal{A}| \to \pi_0(\mathcal{M}_\mathcal{A}) \to \mathsf{V}$ (where V is a totally ordered abelian group) satisfying $\mathsf{p}_\nu(\mathcal{E}) = 0$ for all $\mathcal{E} \in \nu$ and $\mathsf{p}_\nu(\mathcal{E} \oplus \mathcal{F}) = \mathsf{p}_\nu(\mathcal{E}) + \mathsf{p}_\nu(\mathcal{F})$. Say that a k-point $\mathcal{E} \in \mathcal{M}_\mathcal{A}(\mathsf{k})$ is* semistable *if, for all $\mathcal{F} \subset \mathcal{E}$, $\mathsf{p}_\nu(\mathcal{F}) \leqslant 0$. Then the substack of semistable objects $\mathcal{M}^{\mathrm{ss},\nu}_\mathcal{A} \subset \mathcal{M}_\mathcal{A}$ admits a separated good moduli space. Furthermore, if $\mathcal{M}^{\mathrm{ss}}_\mathcal{A}$ agrees with the semistable locus in the sense of Θ-stratifications, then it admits a proper good moduli space.*

The first semistability condition here generalizes Gieseker semistability. The second semistability condition here generalizes GIT semistability. Thus we can recover the original construction of moduli of semistable sheaves from this theorem.