

# STUDENT ARITHMETIC GEOMETRY ON THE BIRATIONAL GEOMETRY OF STACKS

ABSTRACT. These are my notes from the Fall 2024 student arithmetic geometry seminar. This is a papers seminar focused on birational geometry and stacks. I make no promises about the quality of the notes, but feel free to bring to my attention anything that could be improved.

## 1. 8/30 (MARTIN OLSSON) – ???

I missed this day. If you have notes you would like to share, please send them to me and I will T<sub>E</sub>X them up. (Alternatively, feel free to do so yourself and submit a pull request!)

## 2. 9/6 (MARTIN OLSSON) – ALGEBRAIC STACKS THROUGH EXAMPLES

Our goal this time is to give a more precise discussion of stacks. Throughout we fix a base scheme  $S$ .

### 2.1. Working definition of algebraic stacks.

**Definition 2.1.** An *algebraic stack* is a functor  $\mathfrak{X} : (\text{Sch}/S)^{\text{op}} \rightarrow \text{Grpd}$  such that:

- (1)  $\mathfrak{X}$  is a sheaf for the étale topology (i.e. satisfies descent),
- (2) the diagonal  $\Delta_{\mathfrak{X}}$  is representable, and
- (3)  $\mathfrak{X}$  admits a smooth cover by a scheme.

Making this precise takes some work – it is often technically easier to work with groupoids rather than fibered categories. The first condition gives the notion of a *stack* – algebraicity corresponds to the second and third conditions.

**Example 2.2.** Let  $U$  be a scheme over  $S$  and let  $G$  be a flat affine  $S$ -group scheme. Then we can construct a quotient stack  $[U/G]$ , defined below.

### 2.2. Principal homogeneous spaces and torsors.

Let  $G$  be an affine group scheme over  $S$ .

**Definition 2.3.** A *principal homogeneous space* under  $G$  is a flat surjective  $S$ -scheme  $P$  with left  $G$ -action such that the map

$$\begin{aligned} G \times_S P &\rightarrow P \times_S P \\ (g, x) &\mapsto (gx, x) \end{aligned}$$

is an isomorphism.

**Example 2.4.** There is an equivalence between the groupoid of invertible sheaves on  $S$  and the groupoid of  $\mathbb{G}_m$ -principal homogeneous spaces on  $S$ . Given a line bundle  $\mathcal{L}$ , the corresponding homogeneous space is  $\text{Isom}(\mathcal{L}, \mathcal{O}) = \text{Spec}_S \bigoplus_{n \in \mathbb{Z}} \mathcal{L}^{\otimes n}$ . Here  $u \in \mathbb{G}_m$  acts on  $\bigoplus_n \mathcal{L}^{\otimes n}$  via  $u \cdot \ell^{\otimes n} = u^n \ell^{\otimes n}$ .

Let  $\mathfrak{X} = [U/G]$ . Then  $\mathfrak{X}(T)$  is defined to be the groupoid of commutative diagrams

$$\begin{array}{ccc} P & \xrightarrow{\rho} & U \\ \downarrow & & \downarrow \\ T & \longrightarrow & S \end{array}$$

where  $P \rightarrow T$  is a principal  $G$ -homogeneous space and  $\rho$  is  $G$ -equivariant. This is an algebraic stack!

Descent follows from faithfully flat descent for affine schemes. For the second and third conditions, let  $(P, \rho), (P', \rho') \in \mathfrak{X}(T)$ . Define  $\text{Isom}_{\mathfrak{X}}((P, \rho), (P', \rho')) : (\text{Sch}/T)^{\text{op}} \rightarrow \text{Set}$  by

$$\text{Isom}_{\mathfrak{X}}((P, \rho), (P', \rho'))(V) = \{ \sigma : P_V \rightarrow P'_V \mid \sigma \text{ is an isomorphism over } V \text{ and } \rho' \circ \sigma = \rho \}.$$

The second condition is saying that  $\text{Isom}_{\mathfrak{X}}((P, \rho), (P', \rho'))$  is representable by a scheme. If  $G$  is smooth, then the third statement is saying that for the universal homogeneous space  $(P_0, \rho_0) \in \mathfrak{X}(\mathcal{U})$ , if we are given  $(P, \rho)$  on  $T$ , then the map  $\text{Isom}_{\mathfrak{X}}((P_0, \rho_0), (P, \rho)) \rightarrow T$  is a smooth surjection.

**Example 2.5.** Consider the case  $\mathcal{U} = S$  and  $G = \mathbb{G}_m$ . Here a principal  $\mathbb{G}_m$ -homogeneous space  $P$  corresponds to a line bundle  $\mathcal{L}$ . We have  $\text{Isom}_T(P, P') \simeq \text{Isom}_T(\mathcal{L}', \mathcal{L}) \simeq \text{Isom}_T(\mathcal{L}' \otimes \mathcal{L}^\vee, \mathcal{O})$ . This is representable.

**Example 2.6.** Let  $k$  be a field of characteristic  $p$ , and let  $G = \mu_p$ . Note that  $\mu_p$  is flat but not smooth. Write  $B\mu_p = [\text{Spec } k/\mu_p]$ . From the short exact sequence

$$0 \longrightarrow \mu_p \longrightarrow \mathbb{G}_m \longrightarrow \mathbb{G}_m \longrightarrow 0$$

we see that the groupoid of  $\mu_p$ -principal homogeneous spaces on a test scheme  $T$  is equivalent to the category of pairs  $(\mathcal{L}, \lambda)$  where  $\lambda : \mathcal{L}^{\otimes p} \xrightarrow{\sim} \mathcal{O}$ . Specifically, the principal homogeneous space corresponding to  $(\mathcal{L}, \lambda)$  is  $\text{Spec}_T \oplus_{i=0}^{p-1} \mathcal{L}^{\otimes i}$ . It follows that  $B\mu_p \simeq [\mathbb{G}_m/p\mathbb{G}_m]$ , where the  $/p$  indicates  $\mathbb{G}_m$  acting on itself via the  $p$ th power map. We have taken a “flat algebraic stack” and replaced it by a genuine (smooth) algebraic stack! A theorem of Artin allows us to do this more generally.

**Example 2.7.** Let  $\mathfrak{X}$  be a “stacky  $\mathbb{P}^1$ ” with stabilizer  $\mu_2$  at  $z = 0$  and  $\mu_3$  at  $z = \infty$ . We may construct this as a stack over  $\mathbb{P}^1$  with  $T$ -points (for  $g : T \rightarrow \mathbb{P}^1$ )

$$\mathfrak{X}(T) = \{(\mathcal{L}_0, \alpha_0 : \mathcal{L}_0 \rightarrow \mathcal{O}_T, \lambda_0 : \mathcal{L}_0^{\otimes 2} \xrightarrow{\sim} g^*J_0, \text{ likewise at } \infty) \mid \text{relevant diagrams commute}\}.$$

Here the diagram for 0 asserts that  $\alpha_0^2$  equals the composite of  $\lambda_0$  and the inclusion  $g^*J_0 \hookrightarrow \mathcal{O}_T$ . The diagram for  $\infty$  is similar but involves  $\alpha_\infty^3$ .

This is not obviously a quotient stack  $[\mathcal{U}/G]$ . In fact, it is impossible to write  $\mathfrak{X} = [\mathcal{U}/G]$  for a finite discrete group  $G$ : away from 0 and  $\infty$ , the map  $\mathcal{U} \rightarrow \mathfrak{X}$  would be a finite étale cover, which would have to be an  $n$ -fold multiplication map  $\mathbb{G}_m \rightarrow \mathbb{G}_m$  for some  $n$ . But there’s no possible choice of  $n$  that works (we have a two-fold cover at 0 and a three-fold cover at  $\infty$ ).

However, we can realize  $\mathfrak{X} = [(\mathbb{A}^2 \setminus (0, 0))/\mathbb{G}_m]$  as a weighted projective space (where  $\mathbb{G}_m$  acts on the first coordinate by weight two and on the third coordinate by weight three).

### 3. 9/13 (MARTIN OLSSON) – MORE EXAMPLES

**3.1. Stacky projective lines.** We begin with the example from last time.

**Example 3.1.** Consider again the stacky projective line  $\mathfrak{X}$  with notation as above. One thinks of  $\mathcal{L}_0$  as a “square root” of the ideal sheaf  $J_0$  and  $\mathcal{L}_\infty$  as a “square root” of  $J_\infty$ . Away from the points  $\{0, \infty\}$ ,  $\mathfrak{X}$  looks like  $\mathbb{P}^1$ . The fiber of  $\mathfrak{X}$  over  $0 \in \mathbb{P}^1$  is  $(\text{Spec } k[\epsilon]/(\epsilon^2))/\mu_2$ , where  $\mu_2$  acts by  $\epsilon \mapsto -\epsilon$ .

We can give multiple presentations of  $\mathfrak{X}$ . The typical approach to do so is to “rigidify”  $\mathfrak{X}$  by choosing additional data and then quotienting by the choice of additional data.

- (1) Let’s rigidify  $\mathfrak{X}$  by fixing isomorphisms  $\mathcal{L}_0 \simeq \mathcal{O}$  and  $\mathcal{L}_\infty \simeq \mathcal{O}$ . Then  $\alpha_0, \alpha_\infty \in \Gamma(T, \mathcal{O}_T)$ , so we may view these as functions  $f_0, f_\infty$ . The  $\lambda$ ’s are replaced by  $\gamma_0, \gamma_\infty$ , where  $\gamma_i$  trivializes  $g^*J_i$ . We may then define a scheme  $\mathcal{U}$  over  $\mathbb{P}^1$  by

$$\mathcal{U}(T) = \{(g : T \rightarrow \mathbb{P}^1, f_0, f_\infty, \gamma_0, \gamma_\infty) \mid \text{the necessary diagrams commute}\}.$$

This  $\mathcal{U}$  is a scheme: we may view it explicitly as a closed subscheme of  $(\mathbb{P}^1 \times \mathbb{A}^2) \times_{\mathbb{P}^1} \text{Isom}(\mathcal{O}, J_0) \times_{\mathbb{P}^1} \text{Isom}(\mathcal{O}, J_\infty)$  (cut out by the condition that the diagrams commute). We have an action of  $\mathbb{G}_m^2$  on  $\mathcal{U}$ :

$$(u_0, u_\infty) \cdot (g, f_0, f_\infty, \gamma_0, \gamma_\infty) = (g, u_0 f_0, u_\infty f_\infty, u_0^2 \gamma_0, u_\infty^3 \gamma_\infty)$$

Thus we may construct the quotient  $[\mathcal{U}/\mathbb{G}_m^2]$ .

We claim that  $\mathfrak{X} \simeq [\mathcal{U}/\mathbb{G}_m^2]$ . In fact, the point  $(g, (\mathcal{L}_0, \alpha_0), (\mathcal{L}_\infty, \alpha_\infty), \lambda_0, \lambda_\infty)$  corresponds to the  $\mathbb{G}_m^2$ -torsor  $\text{Isom}(\mathcal{O}, \mathcal{L}_0) \times_T \text{Isom}(\mathcal{O}, \mathcal{L}_\infty)$  together with its natural map to  $\mathcal{U}$ .

- (2) There are other presentations! This is an advantage of stacks (as opposed to just considering group actions). We can write  $\mathfrak{X} \simeq [(\mathbb{A}^2 \setminus \{0\})/\mathbb{G}_m]$  where  $u \in \mathbb{G}_m$  acts by  $u \cdot (s, t) = (u^2 s, u^3 t)$ . Note that  $T$ -points of  $[(\mathbb{A}^2 \setminus \{0\})/\mathbb{G}_m]$  are

$$\{(\mathcal{L}, a \in H^0(T, \mathcal{L}^{\otimes 2}), b \in H^0(T, \mathcal{L}^{\otimes 3})) \mid a, b \text{ not simultaneously } 0\}.$$

To see the equivalence between this and  $\mathfrak{X}$ , fix an isomorphism  $\mathcal{I}_0 \simeq \mathcal{I}_\infty$ . This forces  $\mathcal{L}_0^2 \simeq \mathcal{L}_\infty^3$ , and we can compute  $(\mathcal{L}_0 \otimes \mathcal{L}_\infty^{-1})^{\otimes 2} \simeq \mathcal{L}_\infty$  and  $(\mathcal{L}_0 \otimes \mathcal{L}_\infty^{-1})^{\otimes 3} \simeq \mathcal{L}_0$ . Letting  $\mathcal{M} = \mathcal{L}_0^{-1} \otimes \mathcal{L}_\infty$ , we see that  $\alpha_0 \in \mathcal{M}^{\otimes 2}$  and  $\alpha_\infty \in \mathcal{M}^{\otimes 3}$ . Using this gives the equivalence.

**3.2. Weighted projective spaces.** We can generalize the latter construction.

**Example 3.2.** Let  $\mathbb{G}_m$  act on  $\mathbb{A}^{n+1}$  by  $\mathbf{u} \cdot (x_0, \dots, x_n) = (\mathbf{u}^{\alpha_0} x_0, \dots, \mathbf{u}^{\alpha_n} x_n)$ . Removing the origin gives the weighted projective stack  $\mathbb{P}(\alpha_0, \dots, \alpha_n) = [(\mathbb{A}^{n+1} \setminus \{0\})/\mathbb{G}_m]$ . There are many interesting sub-examples:

- (1)  $\mathbb{P}^n = \mathbb{P}(1, \dots, 1)$
- (2) We have  $\overline{\mathcal{M}}_{1,1} \simeq \mathbb{P}(4, 6)$  (though we will not show this here).
- (3)  $\mathbb{P}(1, -1)$  has T-points

$$\{(\mathcal{L}, \alpha_+ \in H^0(\mathcal{T}, \mathcal{L}), \alpha_- \in H^0(\mathcal{T}, \mathcal{L}^{-1})) \mid \mathbf{a}_+, \mathbf{a}_- \text{ not simultaneously zero}\}.$$

This is covered by  $\mathcal{U}_+$  and  $\mathcal{U}_-$  where

$$\mathcal{U}_\pm(\mathcal{T}) = \{(\mathcal{L}, \mathbf{a}_+, \mathbf{a}_-) \mid \mathbf{a}_\pm \text{ generates } \mathcal{L}^{\otimes \pm 1}\}.$$

The intersection is  $\mathbb{G}_m$ , and  $\mathbb{P}(1, -1)$  is the affine line with two origins.

This has some interesting applications.

**Example 3.3.** Consider the polynomial  $x^2 + 29y^2 - 3z^3 \in \mathbb{Z}[x, y, z]$ . We might like to consider a “projective variety” corresponding to this, but attempting to do so runs into an obvious issue: the polynomial is not homogeneous! We can obtain instead a “weighted projective variety”

$$[(\text{Spec } \mathbb{Z}[x, y, z]/(x^2 + 29y^2 - 3z^3) \setminus \{0\})/\mathbb{G}_m] \subset \mathbb{P}(3, 3, 2).$$

This is proper over  $\text{Spec } \mathbb{Z}$ . See a paper of Bhargava and Poonen for more discussion of this in the context of finding rational points on curves. A classic result of Darmon and Granville shows that this fails local-to-global (rational points exist locally but not globally).

#### 4. 9/20 (ROSE LOPEZ) – BIRATIONAL GEOMETRY OF STACKS

References for today’s talk are:

- Kresch and Tschinkel (2023) – *Birational Geometry of Deligne-Mumford Stacks*
- Kresch and Tschinkel (2019) – *Birational Types of Algebraic Orbifolds*
- Kontsevich and Tschinkel (2017) – *Specialization of Birational Types*
- Bergh and Rydh (2019) – *Functorial Destackification*

##### 4.1. Classical Birational Geometry.

**Definition 4.1.** Let  $X$  and  $Y$  be varieties. A *rational map*  $f : X \dashrightarrow Y$  is a morphism  $f : \mathcal{U} \rightarrow Y$  for some dense open  $\mathcal{U} \subset X$ . We identify two such maps  $f : \mathcal{U} \rightarrow Y$ ,  $f' : \mathcal{U}' \rightarrow Y$  if  $f|_{\mathcal{U} \cap \mathcal{U}'} = f'|_{\mathcal{U} \cap \mathcal{U}'}$ .

We say that  $f : X \dashrightarrow Y$  is *birational* if there exists an inverse map  $Y \dashrightarrow X$ . This induces an isomorphism on dense opens ( $\mathcal{U} \xrightarrow{\sim} \mathcal{V}$ ) as well as an isomorphism on function fields.

**Theorem 4.2** (Weak Factorization). *Let  $f : X \rightarrow Y$  be a birational map between smooth complete varieties over an algebraically closed field  $k$  of characteristic zero. Let  $\mathcal{U} \subset X$  be an open such that  $f|_{\mathcal{U}}$  is an isomorphism onto its image. Then  $f$  factors as a zigzag of blowups at smooth centers inducing isomorphisms on the image of  $\mathcal{U}$  (??).*

**4.2. The Burnside Ring.** Fix an algebraically closed field  $k$  of characteristic zero.

**Definition 4.3.** Let  $\text{Burn}_n$  be the free abelian group generated by isomorphism classes of finitely generated fields of transcendence degree  $n$ . For a smooth projective irreducible variety  $X$  of dimension  $n$ , we define  $[X] := [k(X)] \in \text{Burn}_n$ . The *Burnside ring of varieties* is  $\text{Burn} = \bigoplus_n \text{Burn}_n$  with ring structure given by  $[X] \cdot [Y] = [X \times Y]$ .

For  $\mathcal{U} = X \setminus D$  where  $D = D_1 \cup \dots \cup D_\ell$  is a simple normal crossing divisor, we define

$$[\mathcal{U}] := [X] - \sum_i [D_i \times \mathbb{P}^1] + \sum_{i < j} [(D_i \cap D_j) \times \mathbb{P}^2]$$

We also let  $[X \cup Y] = [X] + [Y]$ . Note that  $[\mathcal{U}] = [\mathcal{U}']$  if there exists a quasiprojective variety  $V$  and a pair of birational projective morphisms  $V \rightarrow \mathcal{U}$  and  $V \rightarrow \mathcal{U}'$ .

**Proposition 4.4.** *The group  $\text{Burn}_n$  is generated by classes  $[\mathcal{U}]$  where  $\mathcal{U}$  is  $n$ -dimensional and quasiprojective, modulo the modified scissor relation*

$$[\mathcal{U}] = [V \times \mathbb{P}^{r-d}] + [\mathcal{U} \setminus V]$$

for any smooth closed  $V \subset \mathcal{U}$  of dimension  $< n$ .

**Proposition 4.5** (Specialization). *Let  $\pi : X \rightarrow B$  and  $\pi' : X' \rightarrow B$  be smooth proper morphisms to a smooth connected curve  $B$ . If the generic fibers of  $\pi$  and  $\pi'$  are birational over  $k(B)$ , then for any closed  $b \in B$ , the fibers  $\pi^{-1}(b)$  and  $(\pi')^{-1}(b)$  are birational over  $k(b)$ .*

#### 4.3. Extension to stacks.

**Definition 4.6.** Let  $\mathcal{X}$  and  $\mathcal{Y}$  be stacks. A rational map  $f : \mathcal{X} \dashrightarrow \mathcal{Y}$  is a morphism defined on a dense open  $\mathcal{U} \subset \mathcal{X}$ . A 2-isomorphism  $\alpha : (\mathcal{U}, f) \rightarrow (\mathcal{U}', f')$  is an isomorphism  $f|_{\mathcal{U} \cap \mathcal{U}'} \xrightarrow{\sim} f'|_{\mathcal{U} \cap \mathcal{U}'}$ . We say that  $f$  is birational if it induces an equivalence on dense opens.

**Definition 4.7.** Let  $\mathcal{X}$  and  $\mathcal{Y}$  be stacks. We say that  $\mathcal{X}$  and  $\mathcal{Y}$  are *birationally equivalent* if there exists a representable proper birational map  $\mathcal{X} \dashrightarrow \mathcal{Y}$ , or equivalently a span  $\mathcal{X} \leftarrow \mathcal{X}' \rightarrow \mathcal{Y}$  consisting of representable proper birational morphisms.

**Example 4.8.** Let  $\mathcal{X}$  be a stack and  $\mathfrak{Z}$  a closed substack. We can define a *blowup*  $\text{Bl}_{\mathfrak{Z}} \mathcal{X} \rightarrow \mathcal{X}$  by performing the usual scheme-theoretic blowup locally and gluing.

**Example 4.9.** Let  $\mathcal{X}$  be a smooth separated DM stack and  $\mathcal{D} \subset \mathcal{X}$  a divisor. We can define a *root stack*  $\sqrt[n]{\mathcal{X}, \mathcal{D}} \rightarrow \mathcal{X}$  by

$$\sqrt[n]{\mathcal{X}, \mathcal{D}}(T) = \{(g : T \rightarrow \mathcal{X}, \alpha : \mathcal{L} \rightarrow \mathcal{O}_T, \lambda : \mathcal{L}^{\otimes n} \xrightarrow{\sim} g^* \mathcal{I}_{\mathcal{D}}) \mid \text{the induced maps } \mathcal{L}^{\otimes n} \rightarrow \mathcal{O}_T \text{ agree}\}$$

The morphism  $\sqrt[n]{\mathcal{X}, \mathcal{D}} \rightarrow \mathcal{X}$  is proper but not representable: the fibers over points in  $\mathcal{D}$  are stacky.

**Example 4.10.** Let  $\mathcal{X}$  be a stack and  $\mathcal{L}$  a line bundle on  $\mathcal{X}$ . The *gerbe of  $n$ th roots of  $\mathcal{L}$*  is  $\sqrt[n]{\mathcal{L}/\mathcal{X}} \rightarrow \mathcal{X}$  with

$$\sqrt[n]{\mathcal{L}/\mathcal{X}}(T) = \{(g : T \rightarrow \mathcal{X}, \mathcal{M} \in \text{Pic} T, \epsilon : \mathcal{M}^{\otimes n} \xrightarrow{\sim} g^* \mathcal{L})\}$$

We can contrast this with the above example: essentially, the above example rigidifies things by considering  $\mathcal{I}_{\mathcal{D}} \rightarrow \mathcal{O}$ . Over  $\mathcal{U} = \mathcal{X} \setminus \mathcal{D}$ , the gerbe  $\sqrt[n]{\mathcal{L}/\mathcal{X}}|_{\mathcal{U}}$  is trivializable.

There are two weak factorization results for stacks. One factorizes a representable proper birational map into blowups and can be used to define the “correct” (?) Burnside ring. Another factorizes a proper birational map into stacky blowups and can be used to define a weaker Burnside ring related to “G-equivariant Burnside rings.”

**Example 4.11.** Let  $C = \mathbb{A}^1$ ,  $C_0 = \mathbb{A}^1 \setminus \{0\}$ ,  $E$  an elliptic curve, and  $p \in E$ . Define  $\mathfrak{X}_0 = C_0 \times E \times B(\mathbb{Z}/2) \subset \mathfrak{X}_1 = C \times E \times B(\mathbb{Z}/2)$ . Let  $B = \text{Bl}_{(0,p)}(C \times E)$  and  $\mathfrak{X}_2 = \sqrt{\mathcal{O}_B(\mathcal{D})/B}$ . We get an inclusion  $\mathfrak{X}_0 \rightarrow \mathfrak{X}_2$ . This induces an equivalence on generic fibers but not on special fibers (over  $(0, p)$  ??). The “specialization” result from earlier fails for stacks!

### 5. 9/27 (XIANGRU ZENG) – VALUATIVE CRITERIA FOR THE EXISTENCE OF MODULI SPACES

Let  $k$  be an algebraically closed field of characteristic zero. We may skip some less-than-essential details. References are:

- Alper – Good Moduli Spaces for Artin Stacks
- Alper, Halpern-Leistner, Heinloth – Existence of Moduli Spaces for Algebraic Stacks
- Alper – Notes on Stacks and Moduli

#### 5.1. Precursors.

**Theorem 5.1** (Keel-Mori). *Let  $\mathcal{X}$  be a separated Deligne-Mumford stack of finite type over a base  $S$ . Then there exists coarse moduli space  $\pi : \mathcal{X} \rightarrow X$ , i.e. a universal algebraic space  $X$  among algebraic spaces equipped with a map from  $\mathcal{X}$  and such that  $\pi$  is bijective on geometric points. This satisfies  $\pi_* \mathcal{O}_{\mathcal{X}} = \mathcal{O}_X$ . In characteristic zero, we end up with a “tame moduli space:”*

- (1) *If  $\mathcal{X} \rightarrow S$  is flat, then  $X \rightarrow \text{Spec } k$  is flat.*

(2) The forgetful functor  $\pi_* : \mathrm{QCoh}(\mathfrak{X}) \rightarrow \mathrm{QCoh}(X)$  is exact.

This requires  $\mathfrak{X}$  to be separated – in particular, the stabilizer groups of  $\mathfrak{X}$  must be affine. This is a relatively strict condition!

When considering problems of GIT, we often let  $U \subset \mathbb{P}^n$  be a projective variety with linearized  $G$ -action (for some linearly reductive group  $G$ ). The GIT quotient is then

$$U // G = \mathrm{Proj} \oplus_{n \geq 0} \Gamma(U, \mathcal{O}_U(n))^G.$$

We have a map  $\pi : [U^{ss}/G] \rightarrow U // G$ , where  $U^{ss}$  is the semistable locus. This satisfies:

- (1)  $\mathcal{O}(U // G) = \pi_*(\mathcal{O}(U^{ss}))^G$ , and
- (2) (In characteristic zero)  $\pi$  is affine.

Seshadri called a map satisfying the above conditions a *good quotient*.

## 5.2. Good moduli spaces.

**Definition 5.2.** A map  $\pi : \mathfrak{X} \rightarrow X$ , where  $\mathfrak{X}$  is a (qcqs?) algebraic stack and  $X$  is a (qcqs?) algebraic space, is a *good moduli space* if

- (1)  $\pi_* \mathcal{O}_{\mathfrak{X}} \cong \mathcal{O}_X$ , and
- (2)  $\pi_* : \mathrm{QCoh}(\mathfrak{X}) \rightarrow \mathrm{QCoh}(X)$  is exact.

**Theorem 5.3.** Let  $\pi : \mathfrak{X} \rightarrow X$  be a good moduli space. Then:

- (1)  $\pi$  is surjective and universally closed.
- (2) For  $x_1, x_2 \in \mathfrak{X}(k)$ , then  $\pi(x_1) = \pi(x_2)$  if and only if  $\overline{\{x_1\}} \cap \overline{\{x_2\}} \neq \emptyset$ .
- (3) If  $\mathfrak{X}$  is noetherian, then  $\pi$  is universal among maps from  $\mathfrak{X}$  to algebraic spaces.

**Example 5.4.** Let  $G$  be a linearly reductive group acting on an affine scheme  $\mathrm{Spec} A$ . Then  $[A/G] \rightarrow \mathrm{Spec} A^G$  is a good moduli space. In particular,  $[\mathbb{A}^n/G_m] \rightarrow \mathrm{Spec} k$  is a good moduli space. However, if we remove the origin, then  $[(\mathbb{A}^n \setminus \{0\})/G_m] \rightarrow \mathbb{P}^{n-1}$  is a good moduli space (in fact, the map is an equivalence).

In positive characteristic, the notion of a good moduli space is too restrictive. People consider “adequate moduli spaces” instead. We won’t discuss these in detail.

## 5.3. Existence of good moduli spaces.

**Theorem 5.5** (Alper, Halpern-Leistner, Heinloth). Let  $\mathfrak{X}$  be an algebraic stack of finite type with affine diagonal. Then  $\mathfrak{X}$  admits a separated good moduli space if and only if

- (1)  $\mathfrak{X}$  is  $\Theta$ -reductive, and
- (2)  $\mathfrak{X}$  is  $S$ -complete.

Let’s explain what these mean. Let  $\Theta = [\mathbb{A}^1/G_m]$ , and for any ring  $A$ , we write  $\Theta_A = \Theta \times_{\mathrm{Spec} k} \mathrm{Spec} A$ . For a DVR  $R$  with fraction field  $K$ , let  $\Theta_R \setminus \{0\} = \mathrm{Spec} R \cup_{\mathrm{Spec} K} \Theta_K$ .

**Definition 5.6.** Say  $\mathfrak{X}$  is  $\Theta$ -reductive if, for all DVRs  $R$  and all  $\Theta_R \setminus \{0\}$ , there exists a unique extension

$$\begin{array}{ccc} \Theta_R \setminus \{0\} & \longrightarrow & \mathfrak{X} \\ \downarrow & \nearrow \exists! & \\ \Theta_R & & \end{array}$$

For  $S$ -completeness, if  $R$  is a DVR with uniformizer  $\pi$ , let

$$\phi_R = [\mathrm{Spec} R[s, t]/(st - \pi)/G_m]$$

where  $G_m$  acts by weight 1 on  $s$  and weight  $-1$  on  $t$ .

**Definition 5.7.** Say  $\mathfrak{X}$  is  $S$ -complete if, for all DVRs  $R$ , there exists a unique extension

$$\begin{array}{ccc} \phi_R \setminus \{0\} & \longrightarrow & \mathfrak{X} \\ \downarrow & \nearrow \exists! & \\ \phi_R & & \end{array}$$

**Example 5.8.** If  $\mathcal{U} = \operatorname{Spec} A$  is acted on by  $G$ , then  $[\mathcal{U}/G]$  is  $\Theta$ -reductive if, whenever  $\lambda : \mathbb{G}_m \rightarrow G$  and  $\pi : \operatorname{Spec} R \rightarrow \mathcal{U}$  are maps (with  $R$  a DVR) such that  $\lim_{t \rightarrow 0} \lambda(t)\pi(\operatorname{Spec} K)$  exists, then  $\lim_{t \rightarrow 0} \lambda(t)\pi(\operatorname{Spec} R)$  also exists. We can interpret  $S$ -completeness similarly.

The theorem is proved by first using the existence of good moduli spaces for quotient stacks  $\operatorname{Spec} A/G$ . In extending this result to more general  $\mathfrak{X}$ , we use the local structure theorem to write  $\mathfrak{X}$  in an étale neighborhood of a point  $x$  as  $[\operatorname{Spec} A/G_x]$ , where  $G_x$  is the automorphism group of  $x$ . For gluing, we consider a diagram

$$\begin{array}{ccccc} [\operatorname{Spec} B/G_x] & \rightrightarrows & [\operatorname{Spec} A/G_x] & \longrightarrow & \mathfrak{X} \\ \downarrow & & \downarrow & & \\ \operatorname{Spec} B^{G_x} & \rightrightarrows & \operatorname{Spec} A^{G_x} & \longrightarrow & X. \end{array}$$

We can use Luna's fundamental theorem and “ $\Theta$ -surjectivity” to glue things.