CS 271

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11/10

IPP (Integer Programming Problem)

std: Max $z = C^T x$

s/t $Ax = b$

$x \geq 0$

All IPP ← all x_j's are int

Mixed IPP ← all x_j's are int

Max $z = c_1 x_1 + c_2 x_2$

s/t $x_1 + x_2 \leq 8$

$2x_1 + x_2 \leq 10$

$x_1, x_2 \geq 0$ & are integers

⇓

$x_1 + x_2 + x_3 = 8$

$2x_1 + x_2 + x_4 = 10$

all x_i's are integers & ≥ 0 → All IPP

incase, $\sqrt{2}\, x_1 + x_2 + x_4 = 10$ → x_4 may not be integer

↓

mixed IPP

Suppose you solve using normal simplex and then round off answer to integers

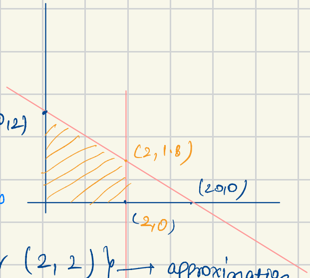
eg Max $z = x_1 + 5x_2$

s/t $x_1 + 10x_2 \leq 20$

$x_1 \leq 2$

$x_1, x_2 \geq 0$

⇒ solving using graphical method:



$z^* = 11$

$x_1^* = 2$

$x_2^* = 1.8$

are integers

⇓

(2,1) or (2,2) → approximation

(2,1): not optimal, as (0,2) fetch better result

(2,2): output of feasible region

---

can't find optimal sol^n

① Branch & bound method

can be applied to both kind (APP/MPP) → is graphical in method.

② Gomory cut

BRANCH AND BOUND METHOD

Problem

max $z = c^T x$

s/t $Ax = b$ # no. of x_j's = 2

$x \geq 0$

some or all x_j's are int

Step 1 : solve the given problem without considering any integer constraint. Say it is $(LP)_1$. Take Optimal solution x_1^*, x_2^*, z^*.

z^* is called upper bound for given problem.

Step 2 Select x_j which is constrained to be integer but x_j^* in step 1 has fractional value. And solve the following two problems.

$(LP)_2$	$(LP)_3$
Max $z = c^T x$ $Ax = b$ $x \geq 0$ $x_j \leq \text{floor}(x_j^*)$	Max $z = c^T x$ $Ax = b$ $x \geq 0$ $x_j \geq \text{ceil}(x_j^*)$

$\Rightarrow$ This is called branching of $(LP)_1$ into $(LP)_2$ & $(LP)_3$

Step 3 : $(LP)_K$ is called the kth node of the problem. Suppose $(LP)_K$ has optimal value z^* and it satisfies all requirement on x, then $(LP)_K$ is called saturated node and z_K^* is called current lower bound.

step 4: If all nodes are either saturated or having value less than current lower bound.

eg

max $z = 7x_1 + 9x_2$

s/t $-x_1 + 3x_2 \le 6$

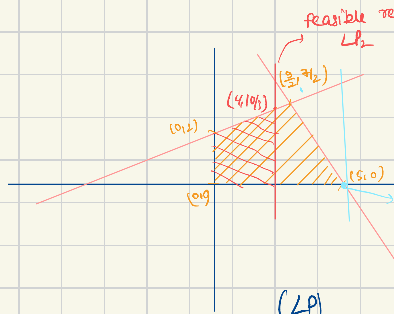
$7x_1 + x_2 \le 35$

$x_1 \le 7$

$x_2 \le 7$

$x_1, x_2 \ge 0$

$x_1, x_2 \rightarrow$ Integers



$x_1^* = 9/2$

$x_2^* = 7/2$

$z^* = 63$

$(LP)_1$

$x_1^* = 9/2$

$x_2^* = 7/2$ } $z^* = 63$

add $x_1 \le 4$ — add $x_1 \ge 5$

$(LP)_2$

$x_1^* = 4$

$x_2^* = 10/3$

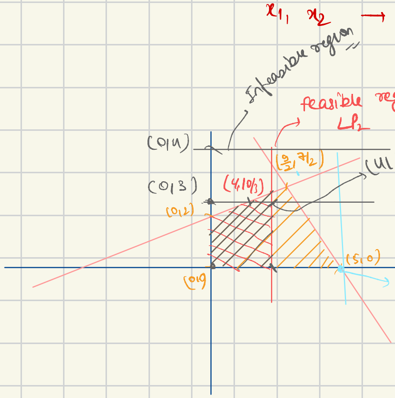
$z^* = 58$

$(LP)_3$

$x_1^* = 5$

$x_2^* = 0$

$z^* = 35$ } current lower bound



$x_1^* = 4$

$x_2^* = 3$

$z^* = 55$

$x_2 \le 3$ — $x_2 \ge 4$

$(LP)_4$

$x_1^* = 4$

$x_2^* = 3$

$z^* = 55$

$(LP)_5$

not a feasible region.

$(z^*)_{LP_4} \le (z^*)_{LP_2} \le (z^*)_{LP_1}$

current lower bound = 55

Ans = 55

Oct 16

Gomory cut method for AIPP

max $z = c^T x$

s/t $Ax = b$

$x \ge 0$ & Int.

→ Only for AIPP

Max $z = 7x_1 + 9x_2$

s/t $-x_1 + 3x_2 \le 6$

$7x_1 + x_2 \le 35$

$x_1, x_2 \ge 0$ & Int.

x_3, x_4 are slack integer variable.

$\Rightarrow -x_1 + 3x_2 + x_3 = 6$

$7x_1 + x_2 + x_4 = 35$

	$c_j \rightarrow$	7	9	0	0
C_B	x_B	y_1	y_2	y_3	y_4
0 ←	$x_3 = 6$	-1	3	1	0
0	$x_4 = 35$	7	1	0	1
	$z = 0$				
	$z_j - c_j$	-7	-9	0	0
			↑		

Iteration-1

C_B	$c_j \rightarrow$ / X_B	7 / y_1	9 / y_2	0 / y_3	0 / y_4
9	$x_2 = 2$	$-1/3$	1	$1/3$	0
← 0	$x_4 = 33$	$24/3$	0	$-1/3$	1
	$Z = 18$				
	$Z_j - c_j$	-18 ↑	0	3	0

C_B	$c_j \rightarrow$ / X_B	7 / y_1	9 / y_2	0 / y_3	0 / y_4
9	$x_2 = 7/2$	0	1	$7/22$	$1/22$
7	$x_1 = 9/2$	1	0	$-1/22$	$3/22$
	$Z = 63$				
	$Z_j - c_j$	0	0	$\frac{28}{11}$	$15/11$

$N =$ non-basic

Suppose in optimal simplex table: $x_K + \sum_{j \in N} y_K^j x_j = b_K$

Then gomori cut for AIPP:

$$\sum_{j \in N} f_K^j x_j \geq f_i \quad \text{where}$$

valid constraint ←
$$f_K^j = y_K^j - \text{floor}(y_K^j)$$
$$f = b_K - \text{floor}(b_K)$$

$$x_1 - \frac{1}{22}x_3 + \frac{3}{22}x_4 = \frac{9}{2}$$

$$x_2 + \frac{7}{22}x_3 + \frac{1}{22}x_4 = 7/2$$

$$\frac{21}{22}x_3 + \frac{3}{22}x_4 \geq 1/2 \;=\; \frac{-21}{22}x_3 - \frac{3}{22}x_4 + x_6 = -1/2$$

$$\frac{7}{22}x_3 + \frac{1}{22}x_4 \geq 1/2 \;=\; \frac{-7}{22}x_3 - \frac{1}{22}x_4 + x_5 = -1/2$$

We need x_5 & x_6 to be slack variable.

C_B	$c_j \rightarrow$ / X_B	7 / y_1	9 / y_2	0 / y_3	0 / y_4	0 / y_5
9	$x_2 = 7/2$	0	1	$7/22$	$1/22$	0
7	$x_1 = 9/2$	1	0	$-1/22$	$3/22$	0
← 0	$x_5 = -1/2$	0	0	$-7/22$	$-1/22$	1
	$Z = 63$			↑		
	$Z_j - c_j$	0	0	$\frac{28}{11}$	$15/11$	

Use dual simplex now

	$C_j \to$	7	9	0	0	0	0
C_B	X_B	y_1	y_2	y_3	y_4	y_5	y_6
9	$x_2 = 3$	0	1	0	0	1	0
7	$x_1 = 32/7$	1	0	0	$1/7$	$-1/7$	0
0	$x_3 = 11/7$	0	0	1	$1/7$	$-22/7$	0
0	$x_6 = -4/7$	0	0	0	$-1/7$	$-6/7$	1
	$z = 59$						
	$z_j - c_j$	0	0	0	1	8	0

gomori cut corresponding : $\frac{1}{7}x_4 + \frac{6}{7}x_5 \geq \frac{4}{7}$

standard form : $-\frac{1}{7}x_4 - \frac{6}{7}x_5 + x_6 = -4/7$

$\Rightarrow$

Integer solution $\leftarrow$
$$\begin{cases} x_2 = 3 \\ x_1 = 4 \\ x_3 = 1 \\ x_4 = 4 \end{cases}$$

All positive.

Why constraint is meaningful ?

what we want, x_k must be integers

Given : $x_k + \sum_{j \in N} y_k^j x_j = b_k$

$$\Leftrightarrow x_k + \sum_{j \in N} (f_k^j + \text{floor}(y_k^j)) x_j = f + \text{floor}(b_k)$$

$$\Leftrightarrow \sum_{j \in N} f_k^j x_j - f = \underbrace{\text{floor}(b_k) - x_k - \sum_{j \in N} (\text{floor}(y_k^j)) x_j}_{\text{integers if all var are integer}}$$

≥ 0 $\qquad$ $\in [0,1)$

$\Downarrow$

this also need to be zero

$\rightarrow$ So this need to be ≥ 0

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Max $z = c^T x$ 　 [MIPP] 　 — MIPP

s/t $Ax = b$

$x \geq 0$ and some x_i's are integer

corresponding LPP: Max $z = c^T x$

S/t $Ax = b$

$x \geq 0$ — LPP

In optimal table of LPP: $x_k + \sum_{j \in N} y_k^j x_j = b_k$

Gomory cut for LPP: $\sum_{j \in N} f_k^j x_j \geq f$

Gomory cut for MIPP: $\sum_{j \in N_1} f_k^j x_j + \sum_{j \in N_2^+} y_k^j x_j + \sum_{j \in N_2^-} \frac{f}{1-f} |y_k^j| x_j \geq f$

where N_1 = index set of integer constraint non-basic variable

N_2 = index set of remaining non-basic variable

N_2^+ = variable in N_2 for which $y_k^j > 0$

N_2^- = variable in N_2 for which $y_k^j < 0$

Notation

$y_k^j = \text{floor}(y_k^j) + f_k^j$

$b_k = \text{floor}(b_k) + f$

Given: $x_k + \sum_{j \in N} y_k^j x_j = b_k$

$\Rightarrow \sum_{j \in N_1} y_k^j x_j + \sum_{j \in N_2} y_k^j x_j = \text{floor}(b_k) + f - x_k$

$\Rightarrow \underbrace{\sum_{j \in N_1} f_k^j x_j}_{\geq 0} + \underbrace{\sum_{j \in N_2^+} y_k^j x_j}_{\geq 0} + \sum_{j \in N_2^-} y_k^j x_j \underbrace{- f}_{\geq 0} = \text{floor}(b_k) - x_k - \sum_{j \in N_1} \text{floor}(y_k^j) x_j$ } an integer

So

LHS ≥ 0 {C-1}, ≤ -1 {C-2} and RHS is an integer

Case 1

$\sum_{j \in N_1} f_k^j x_j + \sum_{j \in N_2^+} y_k^j x_j - f \geq -\sum_{j \in N_2^-} y_k^j x_j \geq 0$

Case 2:

$$\sum_{j\in N_2^-} y_k^j x_j - f \le -1 - \underbrace{\sum_{j\in N_1} f_k^j x_j}\ - \underbrace{\sum_{j\in N_2^+} y_k^j x_j}$$ # $\Big\} \rightarrow$ both are positive

$$\Rightarrow \sum_{j\in N_2^-} y_k^j x_j - f \le -1$$

$$\Rightarrow \sum_{j\in N_2^-} (-y_k^j) x_j \geq 1 - f \quad \Rightarrow \sum \frac{1}{1-f} |y_k^j| x_j \geq 1$$

$$\boxed{\Rightarrow \sum_{j\in N_2^-} \frac{f}{1-f} |y_k^j| x_j \geq f}$$

either of case 1 or case 2 is correct.

Hence;

$$\sum_{j\in N_1} f_k^j x_j + \sum_{j\in N_2^+} y_k^j x_j + \sum_{j\in N_2^-} \frac{f}{1-f} |y_k^j| x_j \geq f$$

either
$x \geq y$
$z \geq y$
claim: $x+z \geq y$

example

max $z = 2x_1 + x_2$
s/t $x_1 + x_2 \le 5$
$6x_1 + 2x_2 \le 21$
$x_1, x_2 \geq 0$ x_1 is int.

x_3 & x_4 slack variable may not be integer as x_2 is not integer.

MIPP $\sum_{j\in N_1} f_k^j x_j + \sum_{j\in N_2^+} y_k^j x_j + \sum_{j\in N_2^-} \frac{f}{1-f} |y_k^j| x_j \geq f$

Writing optimal table

		2	1	0	0
C_B	x_B	y_1	y_2	y_3	y_4
2	$x_1 = 11/4$	1	0	$-1/2$	$1/4$
1	$x_2 = 9/4$	0	1	$3/2$	$-1/4$
	$z = 31/4$ $z_j - c_j$	0	0	$1/2$	$1/4$

we are seeing in x_1 row

$N_1 = \phi$

$N_2 = \{3, 4\}$

$N_2^+ = \{4\}$

$N_2^- = \{3\}$

gomori cut :-

$$\frac{1}{4}x_4 + \frac{3/4}{1-3/4}\,\frac{1}{2}x_3 \geq 3/4$$

x_5 is as slack variable

$$-3/2\, x_3 - \frac{1}{4}x_4 + x_5 \leq -3/4$$

		2	1	0	0	0
C_B	X_B	y_1	y_2	y_3	y_4	y_5
2	$x_1 = 11/4$	1	0	$-1/2$	$1/4$	0
1	$x_2 = 9/4$	0	1	$3/2$	$-1/4$	0
0	$x_5 = -3/4$	0	0	$-3/2$	$-1/4$	1
	$31/4$	0	0	$1/2$	$1/4$	0

		2	1	0	0	0
C_B	X_B	y_1	y_2	y_3	y_4	y_5
2	$x_1 = 3$					
1	$x_2 = 3/2$					
0	$x_3 = 1/2$					
	$15/2$	0	0	0	$1/6$	$1/3$

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optimization

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How to solve LPP / MIPP using $\boxed{\text{MATLAB}}$

Atmost 9 arg u can give. ← linprog (f, A, b, Aeq, beq, lb, ub, x_0, opt)

opt ↓ dual-simplex / interior point

intlinprog (f, intcon, A, b, Aeq, beq, lb, ub, x_0, opt)

Stand. form

min $f^T x$

s/t $Ax \le b$

$A_{eq} x = b_{eq}$

$lb \le x \le ub$

→ these 7 matrix u need to define in MATLAB

standard form in matlab is min for

Problem: max $15x_1 + 25x_2$

s/t $3x_1 + 4x_2 \le 100$

$2x_1 + 3x_2 \le 70$

$x_1 + 2x_2 \le 30$

$x_2 \ge 3$

$x_1 \ge 0$

$f = [-15 \quad -25]$

$$A = \begin{pmatrix} 3 & 4 \\ 2 & 3 \\ 1 & 2 \\ 0 & -1 \end{pmatrix} \quad b = \begin{pmatrix} 100 \\ 70 \\ 30 \\ -3 \end{pmatrix}$$

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BASIC OPTIMIZATION PROBLEM

min/max $f(x)$ → Objective fn

s/t $x \in E$ → feasible region

→ we want to compute

3 kind of problem → Numerical optimization

① $f: [a, b] \to \mathbb{R}$ & f is unimodal.

↓ in this domain it has either maximal/minima

3 different techniques to find minimal/maxima w/o computing differentiation ←

① Dichotomous
② fibonacci
③ Golden section

other way of defining optimization
① Linear Problem
② Non linear Problem

② $f: \mathbb{R}^n \to \mathbb{R}$, $E = \mathbb{R}^n$ → unconstrained optimization problem

③ $f: \mathbb{R}^n \to \mathbb{R}$, $E \subset \mathbb{R}^n$ → constrained optimization problem

If $f \in C_1$ $\quad f: \mathbb{R}^n \to \mathbb{R}$ $\quad f(x_1, x_2 \dots x_n)$

Then $$\nabla f(x) = \begin{bmatrix} \partial f/\partial x_1 \\ \vdots \\ \partial f/\partial x_n \end{bmatrix}$$

⇓ this is gradient ↳ a column vector

eg $f(x) = x_1^2 + 2x_1x_2$

$$\nabla f = \begin{pmatrix} 2x_1 + 2x_2 \\ 2x_1 \end{pmatrix}$$

derivative is a linear op

$A: \mathbb{R}^n \to \mathbb{R}^m$

$A := m \times n$

Derivative $Df(x) = (\nabla f(x))^T$

2nd derivation → hessian matrix)

Second derivative : $H(x) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_2 \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_1} \\ \frac{\partial^2 f}{\partial x_1 \partial x_2} & & & \\ \vdots & & & \\ \frac{\partial^2 f}{\partial x_1 \partial x_n} & \cdots & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}_{n \times n \text{ matrix}}$

→ this two quantities are not always same, but

if $f \in C^2$, hessian is a $n \times n$ symmetric matrix

$f: \mathbb{R}^n \to \mathbb{R}$

$\boxed{f(x) = c}$ → level set of fn at level c

eg

$f: \mathbb{R}^2 \to \mathbb{R}$ $f(x) = 2$

↳ this is a curve

$f: \mathbb{R}^3 \to \mathbb{R}$, $f(x) = z$

↳ this is a surface.

$\{x : f(x) = f(x_0)\}$

gradient is the dirn of maxm increase.

x^0 → initial guess

$x^0 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

x^1

x^2 → 2nd iteration

stop if $x^{k+1} - x^k < \epsilon$

or

change in objective fn $< \epsilon$

BASIC FRAMEWORK FOR PROBLEM 2 OF NUMERICAL OPTIMIZATION

min $f(x)$ s/t $x \in \mathbb{R}^n$ $f: \mathbb{R}^n \to \mathbb{R}$

Input f, x^0, ϵ

symmetric $n \times n$ matrix

gradient of f at x^k $= (n \times 1)$ matrix.

$$\boxed{\text{Iteration} \quad x^{k+1} = x^{k+1} - \alpha_k M_k (\nabla f(x^k))}$$

→ Until convergence we repeat

this

x^k → Approximation of minimizes in k^{th} operation (x^*)

α_k → step size

stop ① $\|x^{k+1} - x^k\| < \epsilon$

② $\|f(x^{k+1}) - f(x^k)\| < \epsilon$

} ultimate aim is to minimize $f(x)$

Output

$x^* = x^k$ if algorithm stops at kth operation.

$f^* = f(x^*)$

ultimate thing is choice of M_k

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$\min f(x)$. $f: \mathbb{R}^n \to \mathbb{R}$

Basic framework:

$x^{k+1} = x^k - \alpha_k M_k(\nabla f(x^k))$

chose this as g^k

for chosing α_k, take $\phi: \mathbb{R}^+ \to \mathbb{R}$ $[0,\infty)$

$\phi(\alpha) = f(x^k - \alpha M_k g^k)$

α_k is minimizer of $\phi(\alpha)$

4-Key words

① descent property : $f(x^{k+1}) < f(x^k)$

[needed for all algorithm]

② Quadratic termination property :

Apply Algo : $f(x) = \frac{1}{2}x^TAx + b^Tx + c$; $A = n \times n$ SPD.

$b, c \in \mathbb{R}^n$

super linear convergence if $a = 1, p = 0$

③ Globally convergent : $x^k \to x^*$ (local minima) for any x_0

④ Order of convergence : $\lim_{k\to\infty} \frac{\|x^{k+1} - x^*\|}{\|x^k - x^*\|^p} = a$

$(0 < a < \infty)$

SD

Prob: $\min f(x)$ $f: \mathbb{R}^n \to \mathbb{R}$

Input f x^0 ϵ

Until convergence , DO

① calculate $g^k = \nabla f(x^k)$

② find value of α that minimizes $f(x^k - \alpha g^k)$

③ $x^{k+1} = x^k - \alpha_k g^k$

END DO

dichotomous search method

min : $x_1^2 + x_1x_2 + 2x_2^2$

$f(x) = \frac{1}{2}x^TAx + b^Tx + c$

$A = \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}$ $b = 0 = c$

Proof that S.D. is a descent property

i.e. $f(x^{k+1}) < f(x^k)$

$\phi = f(x^k - \alpha g^k)$

$\frac{d\phi}{d\alpha}\Big|_{\alpha=0} = -Df(x^k - \alpha g^k) \cdot g^k \Big|_{\alpha=0}$

$= -Df(x^k) \cdot g^k$

$= -(g^k)^T \cdot (g^k) = -\|g^k\|^2 < 0$

$\phi = f(x^k - \alpha g^k)$

$\exists\ \bar{\alpha} > 0$ st

$\phi(\alpha) < \phi(0)$ $\forall \alpha \in [0, \bar{\alpha}]$

$\phi(\alpha_k)$ is minimizer, then $\phi(\alpha_k) = f(x^{k+1}) \le \phi(\alpha) < \phi(0)$

$= f(x^k)$

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Basic optimization Problem

① One dimensional problem

② Multi dimensional unconstrained problem

③ Multi dimensional constrained problem.

Coming back to 1st problem (Problem no I)

- Search method.
  - Dichotomous search
  - Fibonacci search
  - Golden section search.
- Approximation method

Problem:

Input: A unimodal fn $f(x)$ which is known to have a minm in interval $[x_L, x_U]$

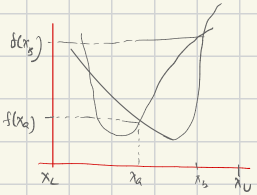
↳ this interval is called range of uncertainty and our aim is to reduce range of uncertainty.

Iteration: minimize x^* can be find by reducing range of uncertainity until a sufficiently small range is obtained.

Idea: chose two trial point between range of uncertainity. (x_a & x_b)

C-I $f(x_a) < f(x_b)$

$\Rightarrow$ minimizer is between x_L & x_b.



C-2 $f(x_a) > f(x_b)$

$x_a < x^* < x_U$

C-3 $f(x_a) = f(x_b)$

$x_a < x^* < x_b$

} we include C-3 in either C-I or C-2

Algorithm for dichotomous search

- Input $x_{L,1}$, $x_{U,1}$, $f(x)$ and ϵ
- Calculate $x_{a,1} = \frac{x_{L,1} + x_{U,1}}{2} - \epsilon$

$$x_{b,1} = \frac{x_{c,1,d} + x_{u,1}}{2} + \epsilon$$

Until convergence do.

Prob $A = 2\times2$ D 10×10

check from slides

$y' = \hat{y}$ → Take NN

$\min (NN_1' - NN_1)^2$

↳ will give you wt.

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Game theory

① Game

② Players

③ Two person zero sum game.

④ Pay off matrix

Two players $\begin{cases} A \\ B \end{cases}$

Zero sum: gain of one = loss of o

$P \equiv m \times n$ → player A has m options to execute

B ... n ....

$$\begin{array}{c|c} & \text{'B'} \\ \text{'A'} \begin{array}{c} a_m \\ \vdots \\ a_2 \\ a_1 \end{array} & \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \\ & b_1 \; b_2 \; \cdots \; b_n \end{array}$$

P_{ij} = gain of A : loss of B.

if A execute option i & B go with option j.

The set $\{a_1, a_2, \ldots, a_m\}$ is called strategy of 'A'

where each $0 \le a_i \le 1$ & $\sum a_i = 1$

Here the set contains probabilities w.r.t options if we repeat the game many times.

Similar thing goes for set B.

Let's write a game {जो sir ने class me Kraya tha}

Always write Pay off matrix w.r.t player A.

$$\begin{array}{cc|cc} & & H & T \\ & & \tfrac{1}{2} & \tfrac{1}{2} \\ \hline H & \tfrac{1}{2} & +1 & -1 \\ T & \tfrac{1}{2} & -1 & +1 \end{array}$$

(rows: 'A', columns: 'B')

⑤ Expected payoff: If A's strategy is $x \in \mathbb{R}^m$

& B's strategy is $y \in \mathbb{R}^n$

then $E(x, y) = x^T P y$

By default

A is maximizing Player

B is minimizing Player.

$E(x,y)$ is a number $\begin{cases} \text{A expect to win per game} \\ \text{B expect to lose per game} \end{cases}$

⑥ Safe game

$\max_{x} \min_{y} E(x,y) \longrightarrow$ A's Aim

$\min_{y} \max_{x} E(x,y) \longrightarrow$ B's Aim

Ex

$$\begin{array}{c|cc} & 3/10 & 7/10 \\ \hline 1/5 & 5 & 0 \\ 4/5 & -1 & 2 \end{array}$$

$$E(X,Y) = \begin{pmatrix} \frac{1}{5} & \frac{4}{5} \end{pmatrix} \begin{pmatrix} 5 & 0 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 3/10 \\ 7/10 \end{pmatrix} = 1.18$$

7. Pure strategy:

$$\begin{array}{cc|cc} & & 1-q & q \\ \hline 1/2 & 3/4 & 8 & 10 \\ 1/2 & 1/4 & 9 & 5 \end{array}$$

$$E(X,Y) = \begin{pmatrix} 3/4 & 1/4 \end{pmatrix} \begin{pmatrix} 8 & 10 \\ 9 & 5 \end{pmatrix} \begin{pmatrix} 1-q \\ q \end{pmatrix}$$

$$= 8.25 + 0.5q$$

The best q should be $q=0$ → minimise for B.

$\Rightarrow$ strategy of $B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

In this case

$$E(X,Y) = \begin{pmatrix} 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 8 & 10 \\ 9 & 5 \end{pmatrix} \begin{pmatrix} 1-q \\ q \end{pmatrix}$$

$$= \begin{pmatrix} 8.5 & 7.5 \end{pmatrix} \begin{pmatrix} 1-q \\ q \end{pmatrix}$$

$$= 8.5 - 8.5q + 7.5q$$

$$= 8.5 - 1q$$ → in this case best value for B: $q = 1$

$$= \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Conclusion If one player of a game applies a fixed strategy (known to other player) then the other player always has a counter optimal pure strategy.

8. Value of game

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Important result

If one player of a game employs a fixed strategy, then the opponent has an optimal counterstrategy that is pure.

Aim: Graphical method to solve $2\times n$ / $m \times 2$ games

Proof for 2×2 game

	$1-q$	q
$1-p$	a	b
p	c	d

$$E(x,y) = \begin{bmatrix} 1-p & p \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1-q \\ q \end{bmatrix}$$

$$= \begin{bmatrix} (1-p)a + cp & (1-p)b + pd \end{bmatrix} \begin{bmatrix} 1-q \\ q \end{bmatrix} = (a - ap + cp) + q\underbrace{(-a + ap - cp + b - bp + dp)}_{e}$$

to minimize for B,

it will execute $q = \begin{cases} 0 & , e \geq 0 \\ 1 & , e < 0 \end{cases}$

Q) find an optimal strategy of A if it is known that B is committed to the strategy $\begin{bmatrix} 0.5 \\ 0.4 \\ 0.1 \end{bmatrix}$ in the following game

0.5	0.4	0.1
1	1	1
2	0	3
-4	5	10
3	-1	2

In this case A's strategy will be pure

$\Rightarrow$ if $X = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ $E(x,y) = 1$

✓ $= \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ $(x,y) = 1.3$

$= \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ $E(x,y) = 1$

✓ $= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ $E(x,y) = 1.3$

either of them A will apply

$\rightarrow$ but what if there is a mixed strategy

$\Rightarrow X = \begin{pmatrix} 0 \\ 0.3 \\ 0 \\ 0.7 \end{pmatrix}$ value of $E(x,y) = 1.3$ $\rightarrow$ same value

Graphical method for 2x2

$$\begin{array}{c|cc} 1-p & a & b \\ p & c & d \end{array}$$

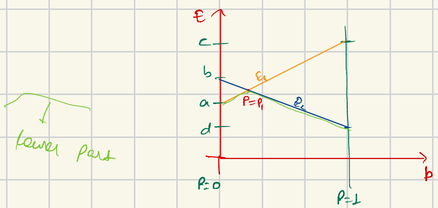
① If $x = \begin{pmatrix} 1-p \\ p \end{pmatrix}$ $y = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$E(x,y) = a + (c-a)p = E_1(p)$$

② $x = \begin{pmatrix} 1-p \\ p \end{pmatrix}$ $y = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$E(x,y) = b + (d-b)p = E_2(p)$$

Draw E_1 & E_2 on p-E plane



E_1 is a SL joining $(0,a)$ & $(1,c)$

E_2 is a SL joining $(0,b)$ & $(1,d)$

Observation: we want a number p such that pts $(p, E_1(p))$ & $(p_2, E_2(p))$ lies on lower part of graph

If $p \geq p_1$, then B's optimal strategy: $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

If $p \leq p_1$, then B's optimal strategy: $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

The safe strategy for A: maximize the minimum part of $E(x,y)$

$\Downarrow$

that is it will occur at $p = p_1$.

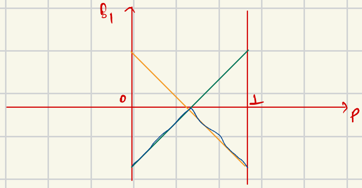
If line do not intersect, then situation will arise at any corner points.

Qs

		$1-p$	p
A	$1-p$	-1	1
	p	1	-1

$E_1(p) = (1-p \quad p)\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = -1+2p$

$E_2(p) = 1-2p$



Optimal p: $-1+2p = 1-2p$

$\Rightarrow p = 0.5$

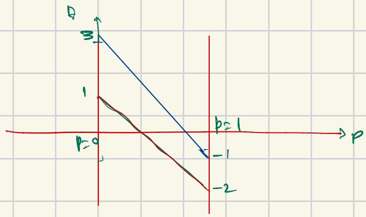
Optimal value of game $= 0$

$x^* = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}$

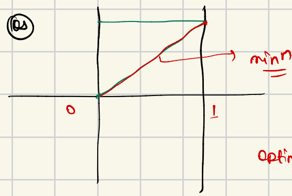
$y^* = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ or any combn $\begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}$

Qs

3	1
-1	-2



(Qs)



0	1
2	1

optimal : $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

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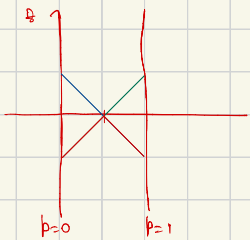
max min strategy for 'A' for $2 \times n$ games

If (p_1, E_p) is any highest point on the lower part of the graph of of $E_i(p)$ $(1 \le i \le n)$ then $[1-p_1, p_1]$ is called max min str. for 'A_1' and E_{p_1} is A's maximum expectation.

If A employs max min strategy, then A can expect to win, on average at least E_{p_1} in each play of game.

(Ex)

$1-q$	q
1	-1
-1	1



$E_1 = 1-2p$

$E_2 = -1+2p$

$E_1 = E_2 \Rightarrow p = \frac{1}{2}$

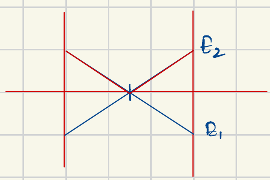
highest point: $(\frac{1}{2}, 0)$

maxmin st of A: $\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$

maxmin of $A = 0$

$A = \begin{pmatrix} 1 \\ 0 \end{pmatrix}_2$ $B_1 = 1-2q$

$A = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ K $B_2 = -1+2q$



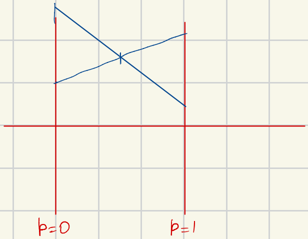
Lowest point: $(1/2, 0)$

min max st: $(1/2, 1/2)$

minmax exp of 'A' $= 0$.

ex

0.8	1
0.9	0.5



$E_1 = 0.8 + 0.1p$

$E_2 = 1 - 0.5p$

$E_1 = E_2 \Leftrightarrow p = 1/3$

max min st of A = $\begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix}$

max min exp of A = 0.833

$E_1 = 0.8 + 0.2q$

$E_2 = 0.9 - 0.4q$

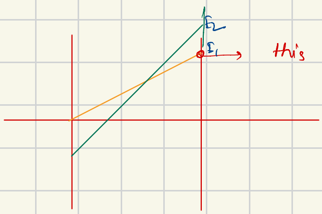
$E_1 = E_2 \Rightarrow q = 1/6$

min max st of B = $\begin{pmatrix} 5/6 \\ 1/6 \end{pmatrix}$

min max st of A = 0.833

ex

0	-1
2	3



this the maximum point

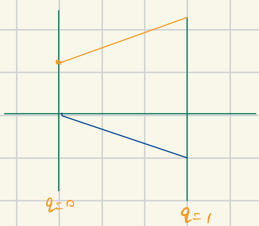
higher point : (1, 2)

max min strategy : $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

max min expectation : 2

If A will apply $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, B will apply pure strategy $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ &

then A's expectation = 2



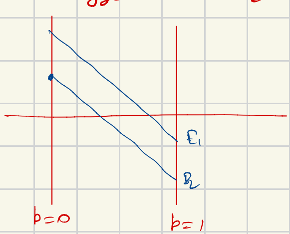
min max strateg of B = $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

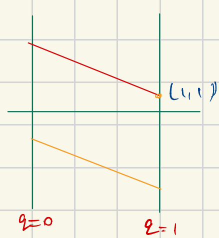
min max exp of A = 2

Rule of dominance : delete smaller rows & bigger column

Ex

3	1
-1	-2





min max strategy of B = $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

min max exp of A = 1

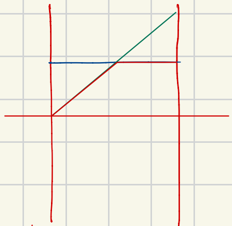
highest point : (0, 1)

maxmin st of A = $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

maxmin exp of A = 1

Ex

0	1
2	1

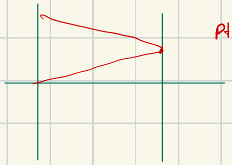


$p=0$ $p=1$

ht point $= (p, 1)$ $p \geq 0.5$

max min st $= \begin{pmatrix} 1-p \\ p \end{pmatrix}$

max min exp $= 1$



$q=0$ $q=1$

Pt: $(1,1)$

min max st of $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

exp of $A = 1$

min max st of B : {m×2 games}

If (q_1, f_{q_1}) is any lowest point on the upper part of graphs of $f_i(q)$ $(1 \leq i \leq m)$, then $[1-q_1\ q_1]$ is called min max strategy for 'B' and f_{q_1} is min max expectation of 'A'.

If B employs max min st, then A can ~~expect~~ win at most f_{q_1} in each play of game.