Analytic function: A function f is said to be analytic at z_0 if it has a derivative at each point in some nbh. of z_0.

Singular point: A point z_0 is called singular point of f if f fails to be analytic at z_0 but is analytic at some point in every nbh. of z_0.

Isolated singular point: A singular point z_0 is called isolated singular point if there is a deleted ε-nbh. $0 < |z-z_0| < \varepsilon$ of z_0 throughout f is analytic.

Example:

$$f(z) = \frac{z-1}{z^3(z^2+9)}$$

$z = 0$, $z = \pm 3i$ are singular point.

$\Rightarrow$ isolated singular points.

Residues:

If z_0 is an isolated singular point of f, then there is a positive number R s.t. f is analytic in.

$$0 < |z-z_0| < R.$$

Then $f(z)$ has Laurent series representation

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \dots + \frac{b_n}{(z-z_0)^n} + \dots$$

$$0 < |z-z_0| < R$$

$$b_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{-n+1}} \, dz \quad , n = 1, 2, \dots$$

C: positively oriented simple closed contour around z_0 that lies in~~side~~ $0 < |z-z_0| < R$

For $n = 1$,

$$b_1 = \frac{1}{2\pi i}\int_C f(z)\,dz \Rightarrow 2\pi i\, b_1 = \int_C f(z)\,dz. \quad (1)$$

The complex number b_1 is called residue of f at z_0.

$$\boxed{b_1 = \operatorname*{Res}_{z=z_0} f(z)} = \text{Coefficient of } \frac{1}{z-z_0}.$$

Example: ~~Suppose~~ Evaluate $\int_C \frac{e^z - 1}{z^4}\,dz$, where C is the ~~positively~~ oriented unit circle $|z| = 1$.

Sol: Since $z_0 = 0$ is an isolated singular point.

So in $0 < |z - z_0| < \infty$, f is analytic.

$\Rightarrow f$ is analytic in $0 < |z| < \infty$.

Now

$$\frac{e^z - 1}{z^4} = \frac{1}{z^4}\left[\sum_{n=0}^{\infty} \frac{z^n}{n!} - 1\right]$$

$$= \frac{1}{z^4}\left[z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots\right]$$

$$= \frac{1}{z^3} + \frac{1}{2z^2} + \frac{1}{6z} + \frac{1}{4!} + \frac{z}{5!} + \cdots$$

So the residue of f at $z = 0$ is $\frac{1}{6}$.

Now the value of the integral,

$$\int_C \frac{e^z - 1}{z^4}\,dz = 2\pi i\, b_1 = 2\pi i \cdot \frac{1}{6}$$

$$= \frac{\pi i}{3}$$

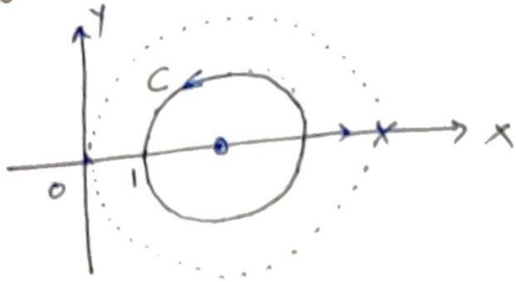
$$\boxed{\int_C \frac{e^z - 1}{z^4}\,dz = \frac{\pi i}{3}}$$

Evaluate the integral $\int_C \frac{dz}{z(z-2)^5}$, where C is a positively oriented circle $|z-2| = 1$.

Sol:

$$f(z) = \frac{1}{z(z-2)^5}.$$

$f(z)$ has isolated singular points $z = 0, 2$.



$\Rightarrow$ $f(z)$ is analytic in $0 < |z-2| < 2$.

$\Rightarrow$ $f(z)$ has Laurent series expression in $0 < |z-2| < 2$.

$$\Rightarrow \int_C f(z)\,dz = 2\pi i \operatorname*{Res}_{z=2} f(z). \quad \text{——— } ①$$

Since $$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad (|z|<1)$$

So $$\frac{1}{z(z-2)^5} = \frac{1}{(z-2)^5} \cdot \frac{1}{z} = \frac{1}{(z-2)^5}\left[\frac{1}{2+z-2}\right]$$

$$= \frac{1}{(z-2)^5}\left[\frac{1}{2\left(1+\frac{z-2}{2}\right)}\right]$$

$$= \frac{1}{2(z-2)^5}\left[\frac{1}{1-\left(-\frac{z-2}{2}\right)}\right]$$

$$= \frac{1}{2(z-2)^5}\sum_{n=0}^{\infty}\left(-\frac{z-2}{2}\right)^n = \frac{1}{2(z-2)^5}\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} (z-2)^{n-5} \qquad 0 < |z-2| < 2.$$

$\operatorname{Res}_{z=2} f(z) = \frac{1}{2^5} = \frac{1}{32}$. Then from ①,

$$\boxed{\int_C \frac{dz}{z(z-2)^5} = 2\pi i \cdot \frac{1}{32} = \frac{\pi i}{16}}$$

<u>Cauchy Residue Theorem:</u> Let C be a simple closed contour, in the positive sense. If a function f is analytic and on C except a finite number of singular points z_K $(K=1,2,\dots,n)$ inside C, then

$$\boxed{\int_C f(z)\,dz = 2\pi i \sum_{K=1}^{n} \operatorname{Res}_{z=z_K} f(z)}$$

<u>Example:</u> Evaluate the integral

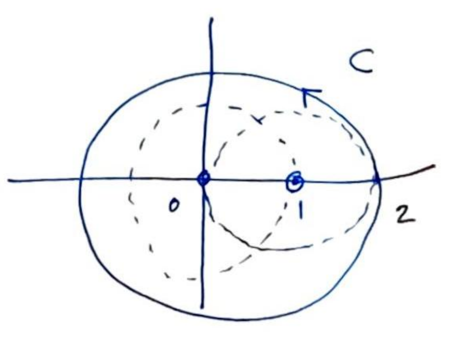
$$\int_C \frac{4z-5}{z(z-1)}\,dz, \quad C: |z| = 2 \text{ positively oriented.}$$

<u>Sol:</u> $f(z) = \frac{4z-5}{z(z-1)}$ has two isolated singular points

$z = 0$ and $z = 1$.

Then by Cauchy - Risidue thm,

$$\int_C f(z)\,dz = 2\pi i \left[\operatorname{Res}_{z=0} f(z) + \operatorname{Res}_{z=1} f(z) \right] \quad \text{—— ①}$$



Since $\frac{4z-5}{z(z-1)} = \frac{4z-5}{z} \cdot \frac{1}{(z-1)}$

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad (|z| < 1)$$

$$\frac{4z-5}{z}\cdot\left(-\frac{1}{1-z}\right) = \left(\frac{4z-5}{z}\right)\left[-1-z-z^2-z^3-\dots\right] \quad 0<|z|<1$$

$$= \left(4-\frac{5}{z}\right)\left(-1-z-z^2-z^3-\dots\right)$$

$$\operatorname*{Res}_{z=0} f(z) = \text{coefficient of } \frac{1}{z}$$

$$\boxed{\operatorname*{Res}_{z=0} f(z) = 5}$$

Again, $$\frac{4z-5}{z(z-1)} = \frac{4(z-1)-1}{z-1}\cdot\frac{1}{z} = \frac{4(z-1)-1}{z-1}\left[\frac{1}{1+(z-1)}\right]$$

$$= \left(4-\frac{1}{z-1}\right)\left[1-(z-1)+(z-1)^2-(z-1)^3+\dots\right]$$

$$(0<|z-1|<1).$$

$$\operatorname*{Res}_{z=1} f(z) = \text{coefficient of } \frac{1}{z-1}$$

$$\boxed{\operatorname*{Res}_{z=1} f(z) = -1}$$

Then from equ^n 1,

$$\int_C \frac{4z-5}{z(z-1)}\,dz = 2\pi i\,(5-1) = 8\pi i$$

Ans:

Types of Isolated Singularpoints:

If z_0 is an isolated singular point, then $f(z)$ has a Laurent series representation

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \dots + \frac{b_n}{(z-z_0)^n} + \dots \qquad (1)$$

in $0<|z-z_0|<R$.

The portion

$$\frac{b_1}{(z-z_0)} + \frac{b_2}{(z-z_0)^2} + \cdots + \frac{b_n}{(z-z_0)^n} + \cdots$$

of the series, is called the principal part of f at z_0.

Removable Singular point:

If $b_n = 0$ then

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \qquad 0 < |z-z_0| < R.$$

z_0 is called removable singular point.

Essential Singular point:

If an infinite no. of coefficients b_n in the principal part are non zero, then z_0 is said to be an essential singular point of f.

i.e. $b_n \neq 0$ for infinite values of n.

Poles of order m: If the principal part of f at z_0 contains atleast one nonzero but number of such terms is only finite, then there exists a positive integer $m \geq 1$ s.t.

$$b_m \neq 0 \quad \text{and} \quad b_{m+1} = 0 = b_{m+2} = b_{m+3} = \cdots = 0$$

Then expression ①

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots + \frac{b_m}{(z-z_0)^m}.$$

where $b_m \neq 0$. $\qquad (0 < |z-z_0| < R)$.

Then z_0 is called a pole of order m.

* If $m = 1$, then z_0 is called a simple pole.

mple: $f(z) = \dfrac{1-\cosh z}{z^2}$. Discuss about singular point.

Sol: Since

$$\cosh z = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!} + \cdots$$

Then $z=0$ is an isolated singular point.

$$f(z) = \frac{1}{z^2}\left[1 - \left(1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!} + \cdots\right)\right]$$

$$= -\frac{1}{z^2}\left[\frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!} + \cdots\right]$$

$$= -\left[\frac{1}{2!} + \frac{z^2}{4!} + \frac{z^4}{6!} + \cdots\right] \quad (0 < |z| < \infty)$$

$z=0$ is a removable singular point

$f(0) = -\frac{1}{2}$.

Example: Discuss about the singular point of $f(z)$, $z=0$.

$$f(z) = \frac{1}{z^2(1-z)}.$$

Sol: $z=0$ and $z=1$ are isolated singular points.

$$f(z) = \frac{1}{z^2}\cdot\left[\frac{1}{1-z}\right],$$

$$= \frac{1}{z^2}\left[1 + z + z^2 + z^3 + \cdots\right] \quad (0 < |z| < 1)$$

$$= \frac{1}{z^2} + \frac{1}{z} + 1 + z + \cdots$$

$\Rightarrow z=0$ is a pole of order 2.

$$\operatorname*{Res}_{z=0} f(z) = 1.$$

Discuss about $z=1$: (exercise)

Q: Discuss about the singularity of $f(z)$ and find residue

$$f(z) = \frac{z^2+z-2}{z+1}$$

Sol:

$$f(z) = \frac{z^2+z-2}{(z+1)}$$

$$= \frac{z(z+1)-2}{(z+1)} = z - \frac{2}{z+1}$$

$$= -1 + (z+1) - \frac{2}{z+1}$$

$z=-1$ is an isolated singular point. It is a simple pole.

$$\operatorname*{Res}_{z=-1} f(z) = -2.$$

□

Residues at Poles:

Theorem: Let z_0 be an isolated singular point of a function f.
The following two statements are equivalent:

(i) z_0 is a pole of order m of f;

(ii) $f(z)$ can be written in the form

$$f(z) = \frac{\phi(z)}{(z-z_0)^m}, \text{ where } \phi(z) \text{ is analytic}$$

and nonzero at z_0.

Moreover, if (i) and (ii) are true,

$$\operatorname*{Res}_{z=z_0} f(z) = \phi(z_0) \quad \text{when } m=1$$

and

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!} \quad (m = 2, 3, \dots)$$

$$\phi(z) = \phi(z_0) + \frac{\phi'(z_0)}{1!}(z-z_0) + \frac{\phi''(z_0)}{2!}(z-z_0)^2 + \dots + \frac{\phi^{(m-1)}(z_0)}{(m-1)!}(z-z_0)^{m-1} + \sum_{n=m}^{\infty} \frac{\phi^{(n)}(z_0)}{n!}(z-z_0)^n$$

$|z-z_0| < \varepsilon$

Proof: Suppose (i) is held. Then

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \frac{b_1}{z-z_0} + \dots + \frac{b_m}{(z-z_0)^m}$$

$b_m \neq 0$

$0 < |z-z_0| < R$

$$\phi(z) = b_m + b_{m-1}(z-z_0) + \dots + b_1 (z-z_0)^{m-1} + \sum_{n=0}^{\infty} a_n (z-z_0)^{m+n}$$

$|z-z_0| < R$

$\therefore \phi(z)$ is analytic, $\phi(z_0) = b_m \neq 0$