1. Physical Interpretation of Curl

$\vec{\Omega} = \omega_1 \hat{i} + \omega_2 \hat{j} + \omega_3 \hat{k}$ and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Then $\vec{v} = \vec{\Omega} \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix}$

$= \hat{i}(\omega_2 z - \omega_3 y) + \hat{j}(x\omega_3 - z\omega_1) + \hat{k}(y\omega_1 - x\omega_2)$

Then $\vec{\nabla} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (\omega_2 z - \omega_3 y) & (x\omega_3 - z\omega_1) & (y\omega_1 - x\omega_2) \end{vmatrix}$

$= \hat{i}\left[\frac{\partial}{\partial y}(y\omega_1 - x\omega_2) - \frac{\partial}{\partial z}(x\omega_3 - z\omega_1)\right]$

$+ \hat{j}\left[\frac{\partial}{\partial z}(\omega_2 z - \omega_3 y) - \frac{\partial}{\partial x}(y\omega_1 - x\omega_2)\right]$

$+ \hat{k}\left[\frac{\partial}{\partial x}(x\omega_3 - z\omega_1) - \frac{\partial}{\partial y}(\omega_2 z - y\omega_3)\right]$

$= \hat{i}(\omega_1 + \omega_1) + \hat{j}(\omega_2 + \omega_2) + \hat{k}(\omega_3 + \omega_3)$

$= 2\vec{\Omega}$

$\therefore \vec{\Omega} = \frac{1}{2} \vec{\nabla} \times \vec{v}$

Physical Interpretation: Curl of any vector point function gives the measure of the angular velocity of any point of the vector field.

2. $\vec{F} = (x+y+1)\hat{i} + \hat{j} - (x+y)\hat{k}$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y+1 & 1 & -(x+y) \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}\{-(x+y)\} - \frac{\partial}{\partial z}(1)\right] - \hat{j}\left[\frac{\partial}{\partial x}\{-(x+y)\} - \frac{\partial}{\partial z}(x+y+1)\right]$$

$$+ \hat{k}\left[\frac{\partial}{\partial x}\{1\} - \frac{\partial}{\partial y}(x+y+1)\right]$$

$$= \hat{i}[-1] - \hat{j}[-1] + \hat{k}[-1]$$

$$= -\hat{i} + \hat{j} - \hat{k}$$

$$\vec{F}\cdot(\vec{\nabla}\times\vec{F}) = [(x+y+1)\hat{i} + \hat{j} - (x+y)\hat{k}]\cdot[-\hat{i} + \hat{j} - \hat{k}]$$

$$= -(x+y+1) + 1 + (x+y)$$

$$= 0$$

3. $W = \int_C \vec{F}\cdot\vec{dr}$

$\vec{F} = 3xy\,\hat{i} - y^2\,\hat{j}$ and $\vec{dr} = dx\,\hat{i} + dy\,\hat{j}$

$$\therefore W = \int_C (3xy\,\hat{i} - y^2\,\hat{j})\cdot(dx\,\hat{i} + dy\,\hat{j})$$

$$= \int_C (3xy\,dx - y^2\,dy)$$

Substitute $y = 2x^2$, then, also $dy = 4x\,dx$

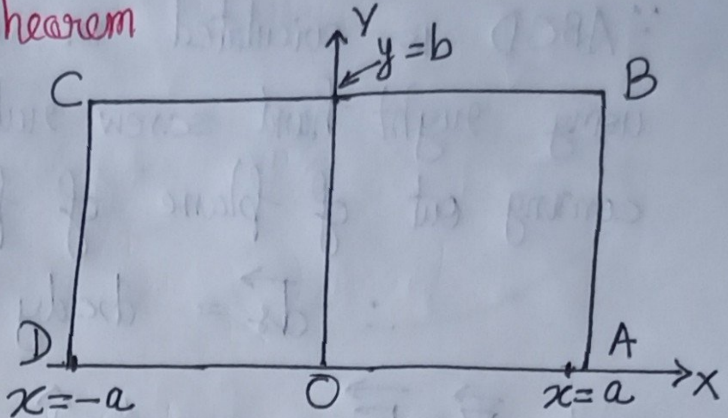
$$W = \int_{x=0}^{x=1} (3x(2x^2)\,dx - (2x^2)^2(4x\,dx))$$

$$= \int_0^1 (6x^3 - 16x^5)\,dx = -\frac{7}{6}$$

4. Verification of Stoke's Theorem

We need to prove,

$$\int_{ABCD} \vec{F}\cdot\vec{dr} = \int_{\text{area } ABCD} (\vec{\nabla}\times\vec{F})\cdot\vec{ds}$$



$$\int_{ABCD} \vec{F}\cdot\vec{dr} = \int_{AB} \vec{F}\cdot\vec{dr} + \int_{BC} \vec{F}\cdot\vec{dr} + \int_{CD} \vec{F}\cdot\vec{dr} + \int_{DA} \vec{F}\cdot\vec{dr}$$

$$\int \vec{F}\cdot\vec{dr} = \int \{(x^2+y^2)\hat{i} - 2xy\,\hat{j}\}\cdot(dx\,\hat{i} + dy\,\hat{j})$$

$$= \int (x^2+y^2)\,dx - \int 2xy\,dy$$

AB : $y : 0$ to b & $x = a$ & $dx = 0$

$$\int_{AB} \vec{F}\cdot\vec{dr} = \cancel{\int_{y=0}^{b} (a^2+y^2)} - 2a\int_0^b y\,dy = -2a\frac{b^2}{2} = -ab^2$$

$$\int_{BC} \vec{F}\cdot\vec{dr} = \int_{x=a}^{-a} (x^2+b^2)\,dx = \left.\frac{x^3}{3} + b^2x\right|_a^{-a} = -\frac{a^3}{3} - ab^2 - \frac{a^3}{3} - ab^2$$

$$= -\frac{2a^3}{3} - 2ab^2$$

$$\int_{CD} \vec{F}\cdot\vec{dr} = -2\int_{y=b}^{0} (-a)\,y\,dy = 2a\int_b^0 y\,dy = 2a\left.\frac{y^2}{2}\right|_b^0 = -ab^2$$

$$\int_{DA} \vec{F}\cdot\vec{dr} = \int_{x=-a}^{a} x^2\,dx = \left.\frac{x^3}{3}\right|_{-a}^{a} = \frac{a^3}{3} - \left(-\frac{a^3}{3}\right) = \frac{2a^3}{3}$$

$$\therefore \int_{ABCD} \vec{F}\cdot\vec{dr} = -ab^2 - \cancel{\frac{2a^3}{3}} - 2ab^2 - ab^2 + \cancel{\frac{2a^3}{3}} = -4ab^2.$$

$\because$ ABCD is circulated in counter-clockwise direction, so using right hand screw rule, $\vec{ds}$ is directed along $\hat{k}$, coming out of plane of paper.

$$\therefore \vec{ds} = dx\,dy\,\hat{k}$$

and $\vec{\nabla}\times\vec{F} = -4y\hat{k}$

$$\therefore \int_{ABCD} (\vec{\nabla}\times\vec{F})\cdot\vec{ds} = \int_{y=0}^{b}\int_{x=-a}^{a} (-4y)\hat{k}\cdot dx\,dy\,\hat{k}$$

$$= \int_{y=0}^{b}\int_{x=-a}^{a} -4y\,dx\,dy$$

$$= -4\int_{y=0}^{b} y\,dy \int_{x=-a}^{a} dx = -\frac{4}{2}y^2\Big|_0^b \times 2a$$

$$= -4ab^2$$

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5. More on elliptical orbit

In plane polar coordinates, conic section eqn. is,

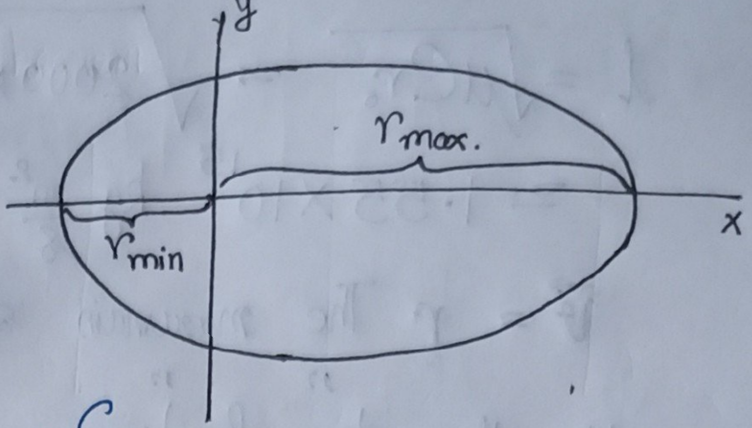
$$r = \frac{r_0}{1-\epsilon\cos\theta} \quad \text{(derived in class)}$$

For ellipse, $0 \le \epsilon < 1$ corresponding to $E<0$

Then $r_{max} = \dfrac{r_0}{1-\epsilon}$ [i.e., when $\cos\theta = 1 \Rightarrow \theta = 0$] (1)

& $r_{min} = \dfrac{r_0}{1+\epsilon}$ [i.e., when $\cos\theta = -1 \Rightarrow \theta = \pi$] (2)

Then length of major axis,



$$A = r_{min} + r_{max.}$$

$$= \frac{2r_0}{1-\epsilon^2}$$

$$= \frac{2\,(l^2/\mu C)}{1-\left[1+\frac{2El^2}{\mu C^2}\right]} = -\frac{C}{E}$$

or $A = \frac{C}{-E} \Rightarrow E = -\frac{C}{A}.$

Now for Earth's satellite, $r_{min.} = 6000$ km

$r_{max.} = 10000$ km.

~~Now,~~ ~~From r_{max}, r_{min} eqns~~

$$\cancel{1+\epsilon = \frac{r_0}{r_{min}}}$$

$(1) \div (2)$: $\frac{r_{max}}{r_{min}} = \frac{1+\epsilon}{1-\epsilon} \Rightarrow (1-\epsilon)\left(\frac{r_{max}}{r_{min}}\right) = 1+\epsilon$

$$\Rightarrow \frac{r_{max}}{r_{min}} - 1 = \epsilon\left(1 + \frac{r_{max}}{r_{min.}}\right)$$

$$\Rightarrow \epsilon = \frac{(r_{max.}/r_{min}) - 1}{(r_{max}/r_{min}) + 1} = \frac{(10000/6000) - 1}{(10000/6000) + 1} = 0.25$$

Then $r_0 = (1+\epsilon)\, r_{min.} = 1.25 \times 6000 = 1500$ km.

Now $\mu = \frac{M_{earth}\, M_{sat.}}{M_{Earth} + M_{sat.}} = \frac{5.97219\times10^{24} \times 2000}{5.97219\times10^{24} + 2000} \approxeq 2000$ kg

$C = G\, M_{Earth}\, M_{sat} = 7.97\times10^{17}$ Nm^2

$E = -\frac{C}{A} = -\frac{7.97\times10^{17}}{16000\times10^3} \simeq -4.98\times10^{10}$ J

$$l = \sqrt{\mu C r_0} = \sqrt{2000\times10^{3} \times 7.97\times10^{17} \times 1500\times10^{3}}$$

$$\simeq 1.55\times10^{15} \; kg\,\frac{m^2}{s}$$

The maximum speed occurs at $r = r_{min}$
" " " " " $r = r_{max}$.

At these two positions, $\vec{v}$ has only $\hat{\theta}$ component

$$v_{\theta} = r\dot{\theta}$$

Now $l = \mu r^2 \dot{\theta} \Rightarrow \dot{\theta} = \frac{l}{\mu r^2}$

$$\therefore v_{\theta} = r\frac{l}{\mu r^2} = \frac{l}{\mu r}$$

So, $v_{max} = \frac{l}{\mu r_{min}} = \frac{1.55\times10^{15}}{2000\times10^{3}\times6000\times10^{3}} = 129.2\,\frac{m}{s}$

& $v_{min} = \frac{l}{\mu r_{max}} = \frac{1.55\times10^{15}}{2000\times10^{3}\times10000\times10^{3}} = 77.5\,\frac{m}{s}$.

6. Motion of particle under $U(r) = k/r^2$

$$\frac{d\theta}{dr} = \frac{1}{\mu r^2}\frac{1}{\sqrt{\frac{2}{\mu}(E - U_{eff})}}$$

Now $U_{eff} = \frac{1}{2}\frac{l^2}{\mu r^2} + U(r)$

$$= \frac{1}{2}\frac{l^2}{\mu r^2} + k/r^2 \quad (k>0)$$

$$\therefore E - U_{eff} = E - \frac{1}{2}\frac{l^2}{\mu r^2} + k/r^2$$

$$E - U_{eff} = E - \frac{1}{r^2}\left(\frac{l^2}{2\mu} - k\right)$$

$$= \frac{1}{r^2}\left[Er^2 - \left(\frac{l^2}{2\mu} - k\right)\right]$$

$$E - U_{eff} = \frac{E}{r^2}\left[r^2 - \frac{1}{E}\left(\frac{l^2}{2\mu} - k\right)\right]$$

$$\sqrt{\frac{2}{\mu}(E - U_{eff})} = \frac{1}{r}\left\{\frac{2E}{\mu}\left[r^2 - \frac{1}{E}\left(\frac{l^2}{2\mu} - k\right)\right]\right\}^{1/2}$$

$$\frac{d\theta}{dr} = \frac{1}{\mu r^{\cancel{2}}} \cdot \frac{\cancel{r}}{\sqrt{\frac{2E}{\mu}}\sqrt{r^2 - \frac{1}{E}\left(\frac{l^2}{2\mu} - k\right)}}$$

$$d\theta = \frac{1}{\sqrt{2E\mu}} \frac{dr}{r\sqrt{r^2 - \lambda^2}}$$

where $\lambda^2 = \frac{1}{E}\left(\frac{l^2}{2\mu} - k\right)$

Using the result : $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right|$

Then $\theta - \theta_0 = \frac{1}{\sqrt{2E\mu}} \frac{1}{\lambda}\left[\sec^{-1}\left|\frac{r}{\lambda}\right|\right]_{r_0}^{r}$

$$= \frac{1}{\sqrt{2E\mu}} \frac{1}{\sqrt{\frac{1}{E}\left(\frac{l^2}{2\mu} - k\right)}}\left[\sec^{-1}\left|\frac{r}{\sqrt{\frac{1}{E}\left(\frac{l^2}{2\mu} - k\right)}}\right| - \sec^{-1}\left|\frac{r_0}{\sqrt{\frac{1}{E}\left(\frac{l^2}{2\mu} - k\right)}}\right|\right.$$

$$\theta - \theta_0 = \frac{1}{2\mu\sqrt{\frac{l^2}{2\mu} - k}}\left[\sec^{-1}\left|\frac{r}{\sqrt{\frac{1}{E}\left(\frac{l^2}{2\mu} - k\right)}}\right| - \sec^{-1}\left|\frac{r_0}{\sqrt{\frac{1}{E}\left(\frac{l^2}{2\mu} - k\right)}}\right|\right]$$

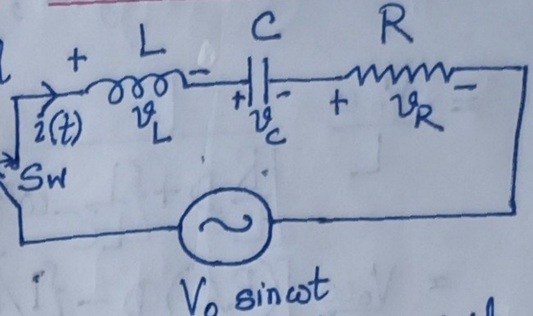
Certain points we can make about the trajectory.

(1) $\frac{l^2}{2\mu} - k > 0$ always, otherwise $\frac{1}{\sqrt{\frac{l^2}{2\mu} - k}}$ becomes imaginary

$\Rightarrow l^2 > 2\mu k$

(2) $E > 0$ always, otherwise $\frac{1}{\sqrt{\frac{1}{E}\underbrace{\left(\frac{l^2}{2\mu} - k\right)}_{>0}}}$ becomes imaginary

7. Series LCR Circuit: Forced Oscillations



Let the inductor L is totally demagnetized at time $t=0$

Let the capacitor C is totally ucharged at time $t=0$.

Let $i(t)$: current at any time t flowing through the circuit

$q(t)$: charge on capacitor plates at any time t.

Then using KVL : $v_L + v_C + v_R = V_0 \sin\omega t$ — (1)

Now, $v_L = L\dfrac{di(t)}{dt}$; $v_C = \dfrac{q(t)}{C}$; $v_R = i(t)R$

$\therefore$ Eqn.(1) becomes: $L\dfrac{di}{dt} + \dfrac{q}{C} + iR = V_0 \sin\omega t$

Differentiating w.r.t. t yields, $L\dfrac{d^2i}{dt^2} + \dfrac{1}{C}\dfrac{dq}{dt} + \dfrac{di}{dt}R = V_0\omega\cos\omega t$

But, $i(t) = \dfrac{dq(t)}{dt}$

$$\therefore L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{i}{C} = V_0\omega\cos\omega t$$

$$\Rightarrow \frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = \frac{V_0\omega}{L}\cos\omega t$$

$$\Rightarrow \left\{D^2 + \frac{R}{L}D + \frac{1}{LC}\right\} i(t) = \frac{V_0\omega}{L}\cos\omega t$$

Then auxiliary eqn.: $D^2 + \dfrac{R}{L}D + \dfrac{1}{LC} = 0$.

$$\Rightarrow m = \frac{-(R/L) \pm \sqrt{(R/L)^2 - 4/(LC)}}{2}$$

If $\dfrac{R}{L}$ is very small so that higher powers than 1 can be neglected,

$$m = \frac{-(R/L) \pm i\,2/\sqrt{LC}}{2} = -\frac{R}{2L} \pm i\frac{1}{\sqrt{LC}}$$

Then complimentary function, $C.F. = e^{-Rt/2L}\left(c_1 \cos\dfrac{t}{\sqrt{LC}} + c_2 \sin\dfrac{t}{\sqrt{LC}}\right)$

and particular integral, P.I. $= \dfrac{1}{\left\{D^2 + \frac{R}{L}D + \frac{1}{LC}\right\}} \dfrac{V_0 \omega}{L} \cos\omega t$

$$= \frac{V_0\omega}{L} \frac{1}{\left[\frac{R}{L}D + \left\{\frac{1}{LC} - \omega^2\right\}\right]} \cos\omega t$$

$$= \frac{V_0\omega}{L} \frac{\left[(R/L)D - \{1/LC - \omega^2\}\right]}{\left[\frac{R}{L}D + \left\{\frac{1}{LC} - \omega^2\right\}\right]\left[\frac{R}{L}D - \left\{\frac{1}{LC} - \omega^2\right\}\right]} \cos\omega t$$

$$= \frac{V_0\omega}{L} \frac{\left[(R/L)D - \{1/LC - \omega^2\}\right]}{\left(\frac{R}{L}\right)^2 D^2 - \left\{\frac{1}{LC} - \omega^2\right\}^2} \cos\omega t$$

$$= \frac{V_0\omega}{L} \frac{\left[(R/L)D - \{1/LC - \omega^2\}\right] \cos\omega t}{-\frac{\omega^2 R^2}{L^2} - \left\{\frac{1}{LC} - \omega^2\right\}^2}$$

$$= -\frac{V_0\omega}{L} \frac{1}{\frac{\omega^2 R^2}{L^2} + \left\{\frac{1}{LC} - \omega^2\right\}^2} \cdot \left[\frac{R}{L}D \cdot \cos\omega t - \left\{\frac{1}{LC} - \omega^2\right\} \cos\omega t\right]$$

$$= \frac{-V_0\omega}{L\left\{\left(\frac{\omega R}{L}\right)^2 + \left(\frac{1}{LC} - \omega^2\right)^2\right\}} \left[-\frac{R\omega}{L} \sin\omega t - \left\{\frac{1}{LC} - \omega^2\right\} \cos\omega t\right]$$

Thus the complete solution is,

$i(t) = C.F. + P.I$

$$= e^{-Rt/2L}\left(c_1 \cos\frac{t}{\sqrt{LC}} + c_2 \sin\frac{t}{\sqrt{LC}}\right) + \frac{\omega V_0}{L\left\{\left(\frac{\omega R}{L}\right)^2 + \left(\frac{1}{LC} - \omega^2\right)^2\right\}}$$

$$\times \left[\frac{\omega R}{L} \sin\omega t + \left\{\frac{1}{LC} - \omega^2\right\} \cos\omega t\right]$$

$$i(t) = e^{-Rt/2L}\left(c_1 \cos\frac{t}{\sqrt{LC}} + c_2 \sin\frac{t}{\sqrt{LC}}\right) + \frac{\omega V_0}{L\left\{\left(\frac{\omega R}{L}\right)^2 + \left(\frac{1}{LC} - \omega^2\right)^2\right\}}$$

$$\times \left[\frac{\omega R}{L}\sin\omega t + \left\{\frac{1}{LC} - \omega^2\right\}\cos\omega t\right]$$

As $t \to \infty$, first term $\to 0$ and second term gives the steady state solutions

Determination of $q(t)$:

$$L\frac{di}{dt} + \frac{q}{C} + iR = V_0 \sin\omega t \quad \text{[From KVL]} \text{ —— (A)}$$

The idea is find $\frac{di}{dt}$ from $i(t)$; plug in the expressions for di/dt and $i(t)$, and then perform the algebra to find $q(t)$

$$\frac{di}{dt} = \left(-\frac{R}{2L}\right)e^{-Rt/2L}\left(c_1 \cos\frac{t}{\sqrt{LC}} + c_2 \sin\frac{t}{\sqrt{LC}}\right) + e^{-Rt/2L}\left(\frac{-c_1}{\sqrt{LC}}\sin\frac{t}{\sqrt{LC}} + \frac{c_2}{\sqrt{LC}}\cos\frac{t}{\sqrt{LC}}\right)$$

$$+ \frac{\omega V_0}{L\left\{\left(\frac{\omega R}{L}\right)^2 + \left(\frac{1}{LC} - \omega^2\right)^2\right\}}\left[\frac{R\omega^2}{L}\cos\omega t - \omega\left\{\frac{1}{LC} - \omega^2\right\}\sin\omega t\right]$$

or $\frac{di}{dt} = \left(\frac{c_2}{\sqrt{LC}} - \frac{R}{2L}c_1\right)e^{-Rt/2L}\cos\frac{t}{\sqrt{LC}} - \left(\frac{R}{2L}c_2 + \frac{c_1}{\sqrt{LC}}\right)e^{-Rt/2L}\sin\frac{t}{\sqrt{LC}}$

$$+ \frac{\omega V_0}{L\left\{\left(\frac{\omega R}{L}\right)^2 + \left(\frac{1}{LC} - \omega^2\right)^2\right\}}\left[\frac{R\omega^2}{L}\cos\omega t - \omega\left\{\frac{1}{LC} - \omega^2\right\}\sin\omega t\right]$$

From eqn(A) : $q(t) = C\left[V_0 \sin\omega t - iR - L\frac{di}{dt}\right]$

or, $q(t) = C\Big[V_0\sin\omega t - e^{-Rt/2L}\left(c_1 R\cos\frac{t}{\sqrt{LC}} + c_2 R\sin\frac{t}{\sqrt{LC}}\right)$

$$+ \frac{\omega V_0 R}{L\left\{\left(\frac{\omega R}{L}\right)^2 + \left(\frac{1}{LC} - \omega^2\right)^2\right\}}\left\{\frac{\omega R}{L}\sin\omega t + \left\{\frac{1}{LC} - \omega^2\right\}\cos\omega t\right\}$$

$$+ \left(\frac{R}{2}c_1 - \sqrt{\frac{L}{C}}c_2\right)e^{-Rt/2L}\cos\frac{t}{\sqrt{LC}} + \left(\frac{R}{2}c_2 + c_1\sqrt{\frac{L}{C}}\right)e^{-Rt/2L}\sin\frac{t}{\sqrt{LC}}$$

$$- \frac{\omega V_0}{\left\{\left(\frac{\omega R}{L}\right)^2 + \left(\frac{1}{LC} - \omega^2\right)^2\right\}}\left\{\frac{R\omega^2}{L}\cos\omega t - \omega\left\{\frac{1}{LC} - \omega^2\right\}\sin\omega t\right\}\Big]$$

or, $q(t) = C\Big[V_0\sin\omega t - \left(\frac{c_1 R}{2} + \sqrt{\frac{L}{C}}c_2\right)e^{-Rt/2L}\cos\frac{t}{\sqrt{LC}} - \left(\frac{c_2 R}{2} - c_1\sqrt{\frac{L}{C}}\right)e^{-Rt/2L}\sin\frac{t}{\sqrt{LC}}$

$$+ \frac{\omega V_0}{\left\{\left(\frac{\omega R}{L}\right)^2 + \left(\frac{1}{LC} - \omega^2\right)^2\right\}}\left\{\frac{R}{L}\left(\frac{1}{LC} - 2\omega^2\right)\cos\omega t + \omega\left\{\left(\frac{R}{L}\right)^2 + \left(\frac{1}{LC} - \omega^2\right)\right\}\sin\omega t\right\}\Big]$$

Here again, we observe that, as $t \to \infty$, 2nd and 3rd term dies down. The exponential terms in $i(t)$ & $q(t)$ are transients which die down with time, leaving behind the steady state solution.

The solution is still not complete, as c_1 & c_2 are to be determined. We have the initial condition $i(0) = 0$ & $q(0) = 0$. These two yields to linear eqn. in c_1 & c_2 which can be solved to obtain its values. $\longrightarrow$ Left as a homework exercise.