Lecture - 5 Division - I Wednesday 3-8-2011

Continuity

Let $f : D \subseteq \mathbb{C} \to \mathbb{C}$. Let $z_0 \in D$.

We say that f is continuous at z_0 if

For every $\varepsilon > 0$, there exists a $\delta > 0$ such that $|z - z_0| < \delta$ implies that $|f(z) - f(z_0)| < \varepsilon$.

Examples: $f(z) = z$ is continuous in $\mathbb{C}$

$f(z) = \overline{z}$ is continuous in $\mathbb{C}$

$f(z) = |z|$ is continuous in $\mathbb{C}$

$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ is continuous in $\mathbb{C}$.

Continuity : Equivalent definitions.

(1) f is continuous at z_0 iff $\lim\limits_{z \to z_0} f(z) = f(z_0)$

(2) f is continuous at z_0 iff For every sequence $\{z_n\} \to z_0$, the sequence $\{f(z_n)\} \to f(z_0)$.

Continuous on the set D.

f is continuous on the set D if f is continuous at each point of D.

Examples of Discontinuous functions:

$$f(z) = \begin{cases} \dfrac{\mathrm{Re}(z)}{|z|} & \text{for } z \neq 0 \\ 0 & \text{for } z = 0 \end{cases}$$

discontinuous at $z=0$

$$g(z) = \begin{cases} \dfrac{\mathrm{Re}(z)}{|1+z|} & \text{for } z \neq 0 \\ 1 & \text{for } z = 0 \end{cases}$$

$\lim_{z \to 0} g(z) = 0 \neq g(0)$

discontinuous at $z=0$ and $z=-1$.

Results:

① Let $f : D \subseteq \mathbb{C} \to \mathbb{C}$ and let $z_0 \in D$.
f is continuous at z_0 iff $u(x,y) = \mathrm{Re}(f(z))$ and $v(x,y) = \mathrm{Im}(f(z))$ are continuous at z_0.

② Let $f : D \subseteq \mathbb{C} \to \mathbb{C}$ and let $z_0 \in \mathbb{C}$.
If f is continuous at z_0 then $\overline{f(z)}$ and $|f(z)|$ are continuous at z_0.

③ Let f and g be continuous at z_0. Then
(i) $f+g$ (ii) $f-g$ (iii) Kf (K = constant) (iv) fg
are continuous at z_0.

(v) $\frac{f}{g}$ is continuous at z_0, provided $g(z_0) \neq 0$.

Composition of two continuous functions

Suppose f is continuous at z_0 and g is continuous at $f(z_0)$. Then the function $h(z) = g(f(z))$ is continuous at z_0.

Result: Continuous image of connected set is connected.

That is, $\boxed{\begin{array}{l} \text{Let } f: D \subseteq \mathbb{C} \to \mathbb{C}. \\ D \text{ is a connected set in } \mathbb{C} \\ f \text{ is continuous on } D \end{array}} \Longrightarrow \boxed{\begin{array}{l} f(D) \text{ is a} \\ \text{connected set in } \mathbb{C} \end{array}}$

Result: Continuous image of compact set is compact.

That is, $\boxed{\begin{array}{l} \text{Let } f: D \subseteq \mathbb{C} \to \mathbb{C} \\ D \text{ is a compact set in } \mathbb{C} \\ f \text{ is continuous on } D \end{array}} \Longrightarrow \boxed{\begin{array}{l} f(D) \text{ is a} \\ \text{compact set} \\ \text{in } \mathbb{C} \end{array}}$

$\Downarrow$

$\boxed{|f| \text{ is continuous on } D} \Longrightarrow \boxed{\begin{array}{l} |f| \text{ attains its} \\ \text{maximum and minimum} \\ \text{value on the compact} \\ \text{set } D \end{array}}$

Differentiation

Recall from MA101:

Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$. D is an open set in $\mathbb{R}$.
Let $x_0 \in D$. We say that f is differentiable at x_0 if $\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$ exists.

In the SAME WAY, we generalize it to $\mathbb{C}$

Let $f : D \subseteq \mathbb{C} \to \mathbb{C}$. Let D be an open set in $\mathbb{C}$ and let $z_0 \in D$.

We say that f is differentiable at z_0 if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \text{ exists}$$

The value of this limit is denoted by $f'(z_0)$

$$\boxed{f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}} \longrightarrow *$$

and it is called the derivative of f at z_0.

Set $\Delta z = z - z_0$

$$\Rightarrow z = z_0 + \Delta z.$$

$*$ can be rewritten as

$$\boxed{f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}}$$

Example: Let $f(z) = z$ for $z \in \mathbb{C}$.

Let z_0 be an arbitrary point in $\mathbb{C}$.

Check whether f is differentiable at z_0.

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{z - z_0}{z - z_0} = \lim_{z \to z_0} 1 = 1$$

$f'(z_0) = 1$. f is differentiable at z_0.

That is, the function $f(z) = z$ differentiable at each point z in $\mathbb{C}$ and the derivative is given by

$$f'(z) = 1 \text{ for all } z \in \mathbb{C}$$

Example: Let $f(z) = \bar{z}$ for $z \in \mathbb{C}$.

Let z_0 be an arbitrary point in $\mathbb{C}$.

Check whether $f(z) = \bar{z}$ is differentiable at z_0?

$$\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\overline{z_0 + \Delta z} - \bar{z_0}}{\Delta z}$$

$$= \boxed{\lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z}}$$

Path I: $\Delta z \to 0$ along the real axis. $\Delta y = 0$ and $\Delta x \to 0$.

$$\lim_{\substack{\Delta y = 0 \\ \Delta x \to 0}} \frac{\overline{\Delta z}}{\Delta z} = \lim_{\substack{\Delta y = 0 \\ \Delta x \to 0}} \frac{\Delta x}{\Delta x} = 1$$

Path II: $\Delta z \to 0$ along the imaginary axis. $\Delta x = 0$ and $\Delta y \to 0$.

$$\lim_{\substack{\Delta x = 0 \\ \Delta y \to 0}} \frac{\overline{\Delta z}}{\Delta z} = \lim_{\substack{\Delta x = 0 \\ \Delta y \to 0}} \frac{-i\,\Delta y}{i\,\Delta y} = -1$$

Since $\dfrac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$ approaches two different values

as $\Delta z \to 0$ along two different values, we conclude that

$\displaystyle\lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$ does not exist, and hence $f'(z_0)$

does not exist. Therefore, f is NOT differentiable at z_0.

$\boxed{f(z) = \overline{z} \text{ is nowhere differentiable in } \mathbb{C}}$

Additional Information to think:

Consider the same function $f(z) = \overline{z}$ as

$f: \mathbb{R}^2 \to \mathbb{R}^2$

$f(x,y) = (x, -y)$ for $(x,y) \in \mathbb{R}^2$ as

vector valued function of vector variable.

Find out whether $f(x,y) = (x,-y)$ is differentiable in $\mathbb{R}^2$ or not?

Exercise (for tommorow's class)

Examine the differentiability of the following functions in $\mathbb{C}$.

(1) $f(z) = |z|$ (2) $f(z) = |z|^2$ (3) $f(z) = z^2$

Lecture 5 ends Division – I