Lecture 13, Division - I Wednesday, 24-08-2011

Maximum - Modulus Theorem/Principle: (Version - 1):

If a function f is analytic and non-constant in a domain (= open, connected set) D, then

$|f(z)|$ has NO MAXIMUM value in D.

That is, there is NO POINT z_0 in the domain D such that
$$|f(z)| \leq |f(z_0)| \text{ for all } z \in D.$$

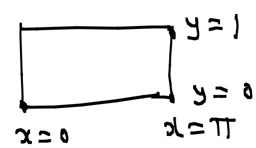
Version - 2: (Max. Modulus Theorem).

Suppose that a function f is continuous in a closed, bounded (= compact) region S and that $f(z)$ is analytic and non-constant in the interior of S. Then, the maximum value of $|f(z)|$ in S which is always reached, occurs somewhere on the boundary of S and never in the interior of S.

Example: $R = \{ z = x+iy \mid 0 \leq x \leq \pi,\ 0 \leq y \leq 1 \}$, $f(z) = \sin(z)$

$$|f(z)| = \sqrt{\sin^2 x + \sinh^2 y}$$

By the maximum - modulus theorem, the maximum value of $|f(z)|$ will attain only on the boundary ∂R of R.

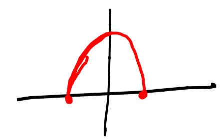


Maximum is reached at the point $\left(\frac{\pi}{2}, 1\right)$.

Note: For a real valued function $f: [a,b] \subseteq \mathbb{R} \to \mathbb{R}$, $|f(x)|$ may attain its maximum value at the interior of $[a,b]$.

Example: $f(x) = 1 - x^2$ for $x \in [-1, 1]$

Max $|f(x)| = 1$ and it is reached at $x = 0$.



Minimum Modulus theorem:

Let f be a continuous function in a closed, bounded region S and let f be analytic and non-constant in the interior of S. Further $f(z) \neq 0$ for all z in S. Then, $|f(z)|$ has a minimum value in S which occurs on the boundary of S and never in the interior of S.

SEQUENCES and SERIES

Now, Recall: Sequence of real/Complex numbers; Series of real/complex numbers — from MA 101

Also Recall: Sequences of functions; Series of functions; Power Series

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Sequence $\{a_n\}$ where a_n's are complex numbers

Sequence is a function from $\mathbb{N}$ to $\mathbb{C}$ (or $\mathbb{R}$) defined by $a(n) = a_n$.

Question: As $n \to \infty$, what is the behaviour of a_n?
(Long term behaviour)

Does it approach any values, as $n \to \infty$? (CONVERGENCE).

Definition:

Let $\{a_n\}$ be a sequence of complex numbers.

If there exists a complex number a^* such that for each $\varepsilon > 0$,

there is a natural number N_0 such that

$$|a_n - a^*| < \varepsilon \quad \text{for all } n \geq N_0, \quad \text{then}$$

we say that $\{a_n\}$ converges to a^*.

a^* is called the limit of the sequence $\{a_n\}$.

We write it as

$$\boxed{\{a_n\} \to a^* \text{ as } n \to \infty} \quad (\text{or}) \quad \boxed{\lim_{n \to \infty} a_n = a}$$

Properties: ① If $\{a_n\}$ converges then the limit of $\{a_n\}$ is unique.

② If $\{a_n\}$ converges then the set $\{a_n \mid n \in \mathbb{N}\}$ is bounded.

③ If $\{a_n\}$ converges then $\{|a_n|\}$ converges. But converse is not true. For example, $a_n = (-1)^n$.

Examples: $\left\{\frac{1}{n}\right\}$ is convergent. $\left\{\frac{1}{n}\right\} \to 0$ as $n \to \infty$.

$\{2n\}$ is not convergent. (diverges) $\{(-1)^n\}$ is not convergent. (diverges)

Let $a_n \in \mathbb{C}$ for $n \in \mathbb{N}$.

$\{a_n\} \to a^*$ iff $\{Re(a_n)\} \to Re(a^*)$ and
$\{Im(a_n)\} \to Im(a^*)$.

Recall: Let $\{a_n\}$ be a sequence of real numbers.

Define

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty}\left[\inf\{a_n, a_{n+1}, \ldots\}\right]$$

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty}\left[\sup\{a_n, a_{n+1}, \ldots\}\right]$$

Alternate notation: $\liminf \equiv \underline{\lim}$, $\limsup \equiv \overline{\lim}$

The concepts of limit inferior and limit superior are defined ONLY for sequence of real numbers.

Result: ① $\{a_n\}$ is a sequence of real numbers.
$\limsup a_n$ and $\liminf a_n$ always exist.
It may be $-\infty$ or $+\infty$ also.

② $\boxed{\liminf a_n \leq \limsup a_n}$

③ If $\lim\limits_{n\to\infty} a_n$ exists then

$$\boxed{\liminf_{n\to\infty} a_n = \lim_{n\to\infty} a_n = \limsup_{n\to\infty} a_n}$$

Series of Numbers:

Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of complex numbers. Then,

$$\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + \cdots\cdots$$ is called an (infinite) series of complex numbers.

When the sum of series can be computed?

Equivalent to say: When the series converges?

Convergence of Series:

Let $\sum_{n=0}^{\infty} a_n$ be a series of complex numbers. Define the sequence of partial sums as follows.

$$s_1 = a_0$$
$$s_2 = a_0 + a_1$$
$$s_3 = a_0 + a_1 + a_2$$
$$\cdots\cdots$$
$$s_n = a_0 + a_1 + a_2 + \cdots + a_{n-1},$$ and so on.

If there exists a complex number s such that the sequence $\{s_n\}$ of partial sums converges to s then we say that the series $\sum_{n=0}^{\infty} a_n$ converges and its sum is equal to s. In this case, we write it as

$$\boxed{\sum_{n=0}^{\infty} a_n = s}$$ Here $s = \lim_{n\to\infty} s_n$.

Note: If the sequence of partial sums does not converge then we say the series $\sum a_n$ diverges (= does not converge).

Examples:

$\sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n$ converges & $\sum_{n=0}^{\infty} \left(\frac{1}{2} + i\frac{1}{2}\right)^n$ converges

$\sum_{n=1}^{\infty} \frac{1}{n}$ does not converge.

Result: If $\sum_{n=0} a_n$ converges then $\boxed{a_n \to 0}$ as $n \to \infty$.

(That is, n^{th} term tends to zero)

Absolute Convergence:

We say that the series $\sum a_n$ converges absolutely if $\sum |a_n|$ converges.

Example: $\sum \frac{1}{n^2}$ converges absolutely.

$\sum \frac{(-1)^n}{n}$ converges, but not absolutely.

Result: If $\sum a_n$ converges absolutely then $\sum a_n$ converges.

But converse of this result is not true. For example, $\sum \frac{(-1)^n}{n}$.

Sequences of Functions:

Let $f_n : D \subseteq \mathbb{C} \to \mathbb{C}$ for $n = 1, 2, 3, \ldots\ldots$.

Let $z_0 \in D$.

Consider the sequence $\{f_n(z_0)\}$. It is just a sequence of numbers.

Suppose $\{f_n(z_0)\}$ converges to a number w_0.

Define a new function g at z_0 by

$$g(z_0) = w_0.$$

If $\{f_n(z_0)\}$ does not converge, then leave it. Take another point.
Similarly, vary the point z_0 in D and repeat the above process.

Formulating the above idea into a mathematical definition.

(Pointwise) Convergence: Let $f_n : D \subseteq \mathbb{C} \to \mathbb{C}$ where $n \in \mathbb{N}$.

We say that the sequence $\{f_n(z)\}$ of functions converges (pointwise) to a function $g(z)$, say, in D, if for each point $z_0 \in D$ and for each $\varepsilon > 0$, there exists a natural number N_0 (that depends on ε and may depend on the point z_0 also) such that $|f_n(z_0) - g(z_0)| < \varepsilon$ for $n \geq N_0$.

In this case, we write it as

$$\lim_{n\to\infty} f_n(z) = g(z) \quad \text{for } z \in D$$

(or)

$$\{f_n\} \to g \text{ on } D \qquad \text{(or)} \quad \{f_n(z)\} \to g(z) \text{ for } z \in D.$$

Example: $f_n : [0,1] \subseteq \mathbb{R} \to \mathbb{R}$ $\qquad n = 1, 2, 3, \ldots$

$$f_n(x) = x^n \quad \text{for } x \in [0,1]$$

Then,

$$f_n(x) \longrightarrow g(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1 \end{cases}$$

$$f_n : [-1, 1] \subseteq \mathbb{R} \to \mathbb{R}$$

$$f_n(x) = x^n \text{ for } x \in [-1, 1]$$

Then, $f_n(x) \longrightarrow h(x) = \begin{cases} 0 & \text{if } -1 < x < 1 \\ 1 & \text{if } x = 1 \end{cases}$ as $n \to \infty$.

Note that $\{f_n\}$ does not converge at the point $x = -1$.

Note: In $\{f_n\} \to g$ on D, It may happen that for a given $\varepsilon > 0$, $\exists$ N_0 which depends only on ε and not on the points z_0 such that $|f_n(z_0) - g(z_0)| < \varepsilon$ for $n \geq N_0$ and for all points z_0 in D

That is, for all points z_0, the same N_0 will do.

This situation of convergence on the set D is described as UNIFORM CONVERGENCE on the set D.

Uniform Convergence: Let $f_n : D \subseteq \mathbb{C} \to \mathbb{C}$ where $n \in \mathbb{N}$.

We say that the sequence $\{f_n\}$ of functions converges uniformly to a function $g(z)$ on the set D, if for each $\varepsilon > 0$, there exists a natural number N^* that depends only on ε such that

$$|f_n(z) - g(z)| < \varepsilon \quad \text{for } n \geq N^* \text{ and for all } z \text{ in } D.$$

In this case, we write it as $\lim_{n\to\infty} f_n(z) = g(z)$ uniformly on D

$\{f_n\} \rightrightarrows g$ on D. (or) $\{f_n\} \xrightarrow{\text{uniformly}} g$ on D

(or) $\{f_n\} \to g$ (uniformly) on D.

Example: $f_n(x) = \frac{1}{x+n}$ for $x \in [0, 1]$ where $n \in \mathbb{N}$.

Set $g(x) = 0$ for all $x \in [0,1]$.

Let $\varepsilon > 0$ be given. Choose $N^* > \frac{1}{\varepsilon}$. For example, N^* = Integral part of $\left\{\left(\frac{1}{\varepsilon}\right)+1\right\} \geq \frac{1}{\varepsilon}$.

Note N^* does not depend on the points x.

Then, $\left|f_n(x) - g(x)\right| = \left|\frac{1}{x+n} - 0\right| = \left|\frac{1}{x+n}\right| < \varepsilon$ for $n \geq N^*$ and for all $x \in [0,1]$.

Reason: $n \geq N^* > \frac{1}{\varepsilon}$.

Since $x \geq 0$, $x+n \geq n > \frac{1}{\varepsilon}$ $\Rightarrow$ $\frac{1}{x+n} < \varepsilon$ for all $x \in [0,1]$.

Therefore, $\{f_n\} \to g$ uniformly on $[0,1]$.

Absolute Convergence: Let $f_n : D \subseteq \mathbb{C} \to \mathbb{C}$ where $n \in \mathbb{N}$.

We say that the sequence $\{f_n\}$ of functions converges absolutely in D if for each point z in D, the sequence $\{|f_n(z)|\}$ converges.

Example: $f_n(x) = x^n$ for $x \in (-1, 1)$ converges absolutely in $(-1, 1)$.
where $n \in \mathbb{N}$

Important Note:

Uniform convergence is needed to do the following:

$$\lim\left(\int f_n\right) = \int\left(\lim f_n\right)$$

$$\lim_{x \to x_0}\left(\lim_{n \to \infty} f_n(x)\right) = \lim_{n \to \infty}\left(\lim_{x \to x_0} f_n(x)\right)$$

If $\{f_n\}$ does not converge uniformly, the above identities need not be true.

For details: Read from any Real Analysis Book

Series of Functions:

Let $f_n : D \subseteq \mathbb{C} \to \mathbb{C}$ where $n = 0, 1, \ldots$

Consider $\sum_{n=0}^{\infty} f_n(z)$.

Define sequence of partial sums functions

$$s_1(z) = f_0(z)$$
$$s_2(z) = f_0(z) + f_1(z)$$
$$s_3(z) = f_0(z) + f_1(z) + f_2(z)$$

$s_n(z) = f_0(z) + f_1(z) + \cdots + f_{n-1}(z)$, and so on.

We say that $\sum f_n(z)$ converges at a point z_0 if the sequence $\{s_n(z_0)\}$ of partial sums at z_0 converges.

We say that $\sum f_n(z)$ converges on the set D if for each point z in D, the sequence $\{s_n(z)\}$ of partial sums converges.

Let $\lim_{n\to\infty} s_n(z) = s(z)$ for $z \in D$.

Then, we write it as

$$\boxed{\sum_{n=0}^{\infty} f_n(z) = s(z)} \text{ for } z \in D$$ and

the function $s(z)$ is called the sum function of the series $\sum f_n(z)$.

Example: $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for $x \in (-1, 1)$.

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z} \text{ for } |z| < 1$$

Let $\sum_{n=0}^{\infty} f_n(z)$ be a series of functions.

Let $\left\{ s_n(z) = \sum_{k=0}^{n-1} f_k(z) \right\}$ be its sequence of partial sums.

If $\{s_n(z)\}$ converges (pointwise) to $s(z)$ on D then we say that the series $\sum f_n(z)$ converges (pointwise) on D and we write it as $\sum_{n=0}^{\infty} f_n(z) = s(z)$ for $z \in D$.

If $\{s_n(z)\}$ converges uniformly to $s(z)$ on the set D then we say that the series $\sum f_n(z)$ converges uniformly on D and we write it as

$$\sum_{n=0}^{\infty} f_n(z) = s(z) \text{ for } z \in D \text{ (uniformly).}$$

If $\left\{ t_n(z) = \sum_{k=0}^{n-1} |f_k(z)| \right\}$ converges (pointwise) on D then we say that the series $\sum f_n(z)$ converges absolutely on D.

Brief Summary:

Sequence of real/complex Numbers/
Series of real/complex Numbers

(ordinary) Convergence

Absolute Convergence

Sequence of Functions/
Series of Functions (and hence for Power Series)

Ordinary/Pointwise Convergence

Uniform Convergence on the set D

Absolute Convergence
