Co-ordinate: Let V be a vector space over the field F with $\dim(V) = n$. Suppose that

$$W = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$$

is ordered basis for V. Let $\alpha \in V$, then $\exists\ c_1, c_2, \dots, c_n \in F$, s.t. $\alpha = c_1\alpha_1 + c_2\alpha_2 + \dots + c_n\alpha_n$; The co-ordinate of α w.r.to ordered basis W is denoted by

$$[\alpha]_W = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

co-ordinate of α w.r.t ordered basis W is unique.

Let $[\alpha]_W = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ and $[\alpha]_W = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$.

Then

$$[\alpha]_W = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \Rightarrow \alpha = c_1\alpha_1 + \dots + c_n\alpha_n \quad \text{— (i)}$$

and

$$[\alpha]_W = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \Rightarrow \alpha = b_1\alpha_1 + \dots + b_n\alpha_n \quad \text{— (ii)}$$

From (i) and (ii) we get

$$c_1\alpha_1 + \dots + c_n\alpha_n = b_1\alpha_1 + \dots + b_n\alpha_n$$

$$\Rightarrow (c_1 - b_1)\alpha_1 + \dots + (c_n - b_n)\alpha_n = 0$$

$\Rightarrow \quad c_1 - b_1 = 0, \quad c_2 - b_2 = 0, \quad \ldots, \quad c_n - b_n = 0 \quad (\because W \text{ is L.I})$

$\Rightarrow \quad c_i - b_i = 0 \quad \forall\, i = 1, 2, \ldots, n$

$\Rightarrow \quad c_i = b_i \quad \forall\, i = 1, 2, \ldots, n$

$\Rightarrow \quad [\alpha]_W = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ is unique.

<u>Example:</u> Let $V = \mathbb{R}^2$ be a vector space over the field $\mathbb{R}$. Suppose $B = \{(1,2), (3,0)\}$ is the ordered basis for V. Find the co-ordinate of $\alpha = (3,4)$ w.r.t. ordered basis B.

<u>Solution:</u> Given $B = \{\overset{\alpha_1}{(1,2)}, \overset{\alpha_2}{(3,0)}\}$ is an ordered basis for V. Suppose $\exists\ c_1, c_2 \in \mathbb{R}$ s.t.

$$\alpha = c_1 \alpha_1 + c_2 \alpha_2$$

$$\Rightarrow (3,4) = c_1 (1,2) + c_2 (3,0)$$

$$= (c_1, 2c_1) + (3c_2, 0)$$

$$= (c_1 + 3c_2, 2c_1)$$

$$\Rightarrow 2c_1 = 4 \Rightarrow c_1 = 2$$

and $\quad c_1 + 3c_2 = 3$

$$\Rightarrow 3c_2 = 3 - 2 = 1$$

$$\Rightarrow c_2 = \frac{1}{3}$$

Thus co-ordinate of $(3,4)$ w.r.t $B = \begin{bmatrix} 2 \\ \frac{1}{3} \end{bmatrix}$

$$\Rightarrow \big[(3,4)\big]_B = \begin{bmatrix} 2 \\ \frac{1}{3} \end{bmatrix}. \quad \square$$

Let $\overset{\dim(V)=n}{\overset{\shortparallel}{V}}$ be a vector space over the field F. Suppose that $B_1 = \{\alpha_1, \ldots, \alpha_n\}$ and $B_2 = \{\beta_1, \beta_2, \ldots, \beta_n\}$ are two ordered basis for V. Let $\alpha \in V$. Then

Q: Find a relation between $[\alpha]_B$ and $[\alpha]_{B'}$.

Ans: There exists an invertible matrix $P_{n\times n}$ s.t. (unique)

$$\boxed{[\alpha]_{B_1} = P\,[\alpha]_{B_2}}$$ where

j^{th} column of P i.e.

$$P_j = \underbrace{[\beta_j]_{B_1}}_{\text{co-ordinate of } \beta_j \text{ w.r.t ordered basis } B_1.}$$

$(j = 1, 2, \ldots, n)$.

Example: Let $V = \mathbb{R}^3$ be a vector space over the field $\mathbb{R}$. Suppose that $B_1 = \{\underset{\shortparallel \alpha_1}{(1,0,1)}, \underset{\shortparallel \alpha_2}{(1,1,1)}, \underset{\shortparallel \alpha_3}{(0,0,1)}\}$ and

$B_2 = \{\underset{\shortparallel \beta_1}{(-1,1,1)}, \underset{\shortparallel \beta_2}{(0,1,1)}, \underset{\shortparallel \beta_3}{(0,0,1)}\}$

are ordered basis for $\mathbb{R}^2$. Find the co-ordinate of $\alpha = (1,2,3)$ w.r.t ordered bases B_1 and B_2. Find a matrix P s.t.

$$[\alpha]_{B_1} = P\,[\alpha]_{B_2}.$$

Solution: First, we calculate $[\alpha]_{B_1}$.

That is

$$\alpha = c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3$$

$$\Rightarrow (1,2,3) = c_1(1,0,1) + c_2(1,1,1) + c_3(0,0,1)$$

$$(1,2,3) = (c_1 + c_2,\ c_2,\ c_1 + c_2 + c_3)$$

$\Rightarrow c_1 + c_2 = 1, \quad c_2 = 2, \quad c_1 + c_2 + c_3 = 3.$

$\Rightarrow c_3 = 2, \quad c_2 = 2, \quad c_1 = -1.$ Thus

$$[\alpha]_{B_1} = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}. \quad \text{——— ①}$$

Similarly,

$$\alpha \cancel{[\alpha]_{B_2}} = b_1 \beta_1 + b_2 \beta_2 + b_3 \beta_3$$

$$(1,2,3) = b_1(-1,1,1) + b_2(0,1,1) + b_3(0,0,1)$$

$$(1,2,3) = (-b_1, \ b_1 + b_2, \ b_1 + b_2 + b_3)$$

$\Rightarrow -b_1 = 1, \quad b_1 + b_2 = 2, \quad b_1 + b_2 + b_3 = 3$

$\Rightarrow b_1 = -1, \quad b_2 = 3, \quad b_3 = 1$

$$[\alpha]_{B_2} = \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} \quad \text{——— ②}$$

To find P.

$$P_j = [\beta_j]_{B_1},$$

j^{th} column of P = co-ordinate of β_j w.r.t B_1.

So.
$$P_1 = [\beta_1]_{B_1}$$

$$(-1,1,1) = c_1(1,0,1) + c_2(1,1,1) + c_3(0,0,1)$$

$$= (c_1 + c_2, \ c_2, \ c_1 + c_2 + c_3)$$

$\Rightarrow c_1 + c_2 = -1, \quad c_2 = 1, \quad c_1 + c_2 + c_3 = 1$

$\Rightarrow c_1 = -2, \quad c_2 = 1, \quad c_3 = 2$

$$P_1 = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix},$$

Similarly,

$$(0,1,1) = (c_1 + c_2,\ c_2,\ c_1 + c_2 + c_3)$$

$$\Rightarrow c_1 + c_2 = 0, \quad c_2 = 1, \quad c_1 + c_2 + c_3 = 1 \Rightarrow c_3 = 1$$

$$\Rightarrow c_1 = -1, \quad c_2 = 1, \quad c_3 = 1$$

$$P_2 = [\beta_2]_{B_1} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}.$$

Again.

$$(0,0,1) = (c_1 + c_2,\ c_2,\ c_1 + c_2 + c_3)$$

$$\Rightarrow c_1 + c_2 = 0, \quad c_2 = 0, \quad c_1 + c_2 + c_3 = 1 \Rightarrow c_3 = 1$$

$$\Rightarrow c_1 = 0, \quad c_2 = 0, \quad c_3 = 1$$

$$P_3 = [\beta_3]_{B_1} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Thus

$$P = \begin{bmatrix} -2 & -1 & 0 \\ 1 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}.$$

$$\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 & -1 & 0 \\ 1 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}$$

$$[\alpha]_{B_1} = P \quad [\alpha]_{B_2} \qquad \square$$

Excercise.

Let V be a vector space over the field F with $\dim(V) = n$ and $B_1 = \{\alpha_1, \dots, \alpha_n\}$ be an ordered basis for V. Then show that

(i) $[\alpha+\beta]_{B_1} = [\alpha]_{B_1} + [\beta]_{B_1}$

(ii) $[c\alpha]_{B_1} = c\,[\alpha]_{B_1}$

$\square$

Linear transformation:

Let V and W be vector spaces over the same field F. A function $T : V \to W$ is called a linear transformation from V into W, if

$$T(\alpha+\beta) = T(\alpha) + T(\beta)$$
$$T(c\alpha) = c\,T(\alpha) \quad \forall\ \alpha, \beta \in V,\ c \in F.$$

Example: ① Let $V = \mathbb{R}^2$, $W = \mathbb{R}^3$ be vector spaces over $\mathbb{R}$. Define a map

$$T : V \to W$$

s.t. $T(x, y) = (x+y,\ x-y,\ x)$.

Show that T is a linear transformation.

Solution: Let $\alpha = (x_1, x_2)$, $\beta = (y_1, y_2) \in \mathbb{R}^2$.

Now $\alpha + \beta = (x_1 + y_1,\ x_2 + y_2)$

Then
$$T(\alpha+\beta) = (x_1 + y_1 + x_2 + y_2,\ x_1 + y_1 - x_2 - y_2,\ x_1 + y_1)$$
$$= (x_1 + x_2,\ x_1 - x_2,\ x_1) + (y_1 + y_2,\ y_1 - y_2,\ y_1)$$
$$= T(x_1, x_2) + T(y_1, y_2)$$
$$= T(\alpha) + T(\beta).$$

Again $c\alpha = (cx_1, cx_2)$

$$T(c\alpha) = T(cx_1, cx_2) = (cx_1 + cx_2,\ cx_1 - cx_2,\ cx_1)$$
$$= c\,(x_1 + x_2,\ x_1 - x_2,\ x_1)$$
$$= c\,T(\alpha)$$

$\Rightarrow$ T is a linear transformation.

Example 2: Let V be a space of polynomial functions f from F into F, given by

$$f(x) = c_0 + c_1 x + \cdots + c_n x^n.$$

Let $D: V \to V$ s.t. $D(f) = f'$.

Show that D is a linear transformation.

Example 3: Let A be a fixed $m \times n$ matrix with entries in the field F. Define $T: F^{n\times 1} \longrightarrow F^{m\times 1}$ s.t.

$$T(X) = AX.$$

Show that T is a Linear transformation.

Solution: Let X and Y $\in F^{n\times 1}$. Now

$$T(X+Y) = A(X+Y) = AX + AY$$

and $T(cX) = A(cX) = cAX$

$\Rightarrow$ T is a linear transformation.

Note: We can define a Linear transformation in one step. Let V and W be vector spaces over F. Then T is a linear transformation from V into W, if

$$T(c\alpha + \beta) = cT(\alpha) + T(\beta) \quad \forall\ \alpha, \beta \in V$$

and $c \in F$.