Lecture - 7 Division - 1 Tuesday 09-08-2011

Recall:

There are functions that merely satisfy the CR equations at z_0, but fail to be differentiable at z_0.

Example:
$$f(z) = \begin{cases} \dfrac{(\overline{z})^2}{z} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$$

Observe that,
$$f(z) = \frac{(\overline{z})^2}{z} = \left(\frac{x^3 - 3xy^2}{x^2+y^2}\right) + i\left(\frac{y^3 - 3x^2y}{x^2+y^2}\right) \quad \text{if } z \neq 0$$

Examining the differentiability of f at $z=0$:
$$\lim_{z \to 0} \frac{f(z) - f(0)}{z - 0}$$

Path I: Let z approach 0 along x-axis.
$$\lim_{\substack{y=0 \\ x \to 0}} \frac{f(z) - f(0)}{z - 0} = \lim_{\substack{y=0 \\ x \to 0}} \frac{x - 0}{x - 0} = 1$$

Path II: Let z approach 0 along $y=x$ and $x \to 0$.
$$\lim_{\substack{y=x \\ x \to 0}} \frac{f(z) - f(0)}{z - 0} = \lim_{\substack{x=0 \\ y \to 0}} \frac{-x - ix}{x + ix} = -1$$

Since the limiting values are distinct, we conclude that f is NOT differentiable at $z=0$.

Examining whether f satisfies CR equations at $z=0$.

$$u_x(0,0) = \lim_{x\to 0} \frac{u(x,0)-u(0,0)}{x} = \lim_{x\to 0} \frac{x-0}{x} = 1$$

$$u_y(0,0) = \lim_{y\to 0} \frac{u(0,y)-u(0,0)}{y} = \lim_{y\to 0} \frac{0-0}{y} = 0$$

$$v_x(0,0) = \lim_{x\to 0} \frac{v(x,0)-v(0,0)}{x} = \lim_{x\to 0} \frac{0-0}{x} = 0$$

$$v_y(0,0) = \lim_{y\to 0} \frac{v(0,y)-v(0,0)}{y} = \lim_{y\to 0} \frac{y-0}{y} = 1$$

$$\boxed{\begin{aligned} u_x(0,0) &= 1 = v_y(0,0) \\ u_y(0,0) &= 0 = -v_x(0,0) \end{aligned}}$$

Therefore, f satisfies the CR equations at $z=0$

Cauchy - Riemann Equations in POLAR FORM

Let $f(z) = f(re^{i\theta}) = u(r,\theta) + i\, v(r,\theta)$.

The polar form of the Cauchy - Riemann equations can be obtained as

$$\boxed{\begin{aligned} u_r &= \frac{1}{r} v_\theta \\ v_r &= \frac{-1}{r} u_\theta \end{aligned}}$$

For derivation,

See Section 22 of

Brown & Churchill Book.

Exercise: Verify that $f(z) = \frac{1}{z}$ for $z \neq 0$ satisfies the CR equations in polar form in $\mathbb{C} \setminus \{0\}$.

See: Example - 1, Page 67 in Section - 22.

$$f'(z_0) = e^{-i\theta}(u_r + i v_r) = \frac{-i}{z_0}(u_\theta + i v_\theta)$$

where all the partial derivatives are evaluated at $z_0 = (r_0, \theta_0)$.

Cauchy - Riemann Equations in COMPLEX FORM.

$$\boxed{\frac{\partial f}{\partial \bar{z}} = 0}$$

See: Section 22, Exercise 10 on pages 69 and 70. of Brown & Churchill.

ANALYTIC FUNCTIONS

Definition: Let $f(z)$ be a function defined on an open set S in $\mathbb{C}$. Then, the function $f(z)$ is said to be analytic on S if $f(z)$ is differentiable at each point of S.

Examples:

$f(z) = z$ is analytic in $\mathbb{C}$.

$f(z) = z^2$ is analytic in $\mathbb{C}$.

$P(z) = a_0 + a_1 z + \cdots + a_n z^n$ is analytic in $\mathbb{C}$.

$f(z) = \bar{z}$ is NOT analytic in $\mathbb{C}$.

$f(z) = |z|^2$ is NOT analytic in $\mathbb{C}$.

Other terminologies for analytic are
holomorphic or regular.

NOTE:

Analyticity is a property defined over open sets, while differentiability could conceivable hold at a point only.

We say that $f(z)$ is analytic at a point z_0 if there exists an open neighborhood $N(z_0)$ of the point z_0 such that $f(z)$ is differentiable at each point of $N(z_0)$.

NOTE: If we say $f(z)$ is analytic in a set S which is not open in $\mathbb{C}$, then it actually means that $f(z)$ is analytic in an open set D which contains S.

Analyticity $\Longrightarrow$ Differentiability

Analyticity $\not\Longleftarrow$ Differentiability

Theorem: If f is analytic in an open set D then f is differentiable in D.

Converse is NOT true.

Counter example: $f(z) = |z|^2$ is differentiable at $z=0$,
but it is NOT analytic at $z=0$.

Reason: $f(z)=|z|^2$ is differentiable at only one point $z=0$ in $\mathbb{C}$.

Necessary condition for Analyticity:

Let $f(z)$ be analytic in an open set D in $\mathbb{C}$ then $f(z)$ satisfies the Cauchy - Riemann equations at each point of D.

Sufficient conditions for Analyticity:

Let $f(z) = u(x,y) + i\, v(x,y)$ be defined in an open set D.

If all first order partial derivatives of $u(x,y)$ and $v(x,y)$ exist, continuous and satisfy the Cauchy - Riemann equations at all points of D then f is analytic in D

Result: Let $f(z)$ and $g(z)$ be analytic in an open set D in $\mathbb{C}$.

Then, the following functions are analytic in D.

(i) $f(z) + g(z)$ (sum)

(ii) $K f(z)$ (Scalar multiplication) where K is a complex constant.

(iii) $f(z)\, g(z)$ (product)

(iv) If $g(z) \neq 0$ for all $z \in D$ then $\dfrac{f(z)}{g(z)}$ (Quotient)

Result (For composition): Let $f(z)$ be analytic in an open set D and $g(z)$ be analytic in an open set containing $f(D)$. Then the composition function $h(z) = g(f(z))$ is analytic in D.

Result: If $f(z)$ is analytic in a domain ($=$ Open, connected) D in $\mathbb{C}$ and if $f'(z) = 0$ for all $z \in D$ then $f(z)$ is a constant function in D.

Following result shows that NON-CONSTANT ANALYTIC FUNCTIONS can NOT take certain forms.

Result: Let $f(z) = u(x,y) + i\, v(x,y)$ be analytic in an open set D in $\mathbb{C}$. If any one of the following conditions hold in D, then the function $f(z)$ must be constant in D.

1. $f'(z) = 0$ for all $z \in D$.
2. $|f(z)|$ is constant in D.
3. $u(x,y)$ is constant in D.
4. $v(x,y)$ is constant in D.
5. $\arg(f(z))$ is constant in D.
6. $f(z)$ is real valued for all z in D.
7. $f(z)$ is pure imaginary valued for all z in D.

Hint for proof: First try to show that $u_x = 0$, $u_y = 0$, $v_x = 0$, $v_y = 0$. Then copy the arguments of $f'(z) = 0 \;\forall z \in D \Rightarrow f$ is constant in D.

Laplace Equation:

Let $\phi(x,y)$ be a real valued function of two real variables x and y. The partial differential equation

$$\boxed{\phi_{xx} + \phi_{yy} = 0} \quad (\text{or}) \quad \boxed{\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0}$$

is known as the Laplace equation (or sometimes referred to as potential equation).

Harmonic Functions:

A real valued function $\phi(x, y)$ is said to be harmonic in a domain D if all its second order partial derivatives are continuous in D and if at each point of D, $\phi(x,y)$ satisfies the Laplace equation
$\phi_{xx} + \phi_{yy} = 0$.

Examples of Harmonic functions:

$\phi(x,y) = x^2 - y^2$ is harmonic in $\mathbb{C}$

$\phi(x,y) = 2xy$ is harmonic in $\mathbb{C}$.

$\phi(x,y) = e^x \sin(y)$ is harmonic in $\mathbb{C}$.

Connection between Harmonic functions and Analytic functions

Theorem: If a function $f(z) = u(x,y) + i\,v(x,y)$ is analytic in a domain D then its component functions $u(x,y)$ and $v(x,y)$ are harmonic in D.

Proof: Since f is analytic in D, f satisfies CR equations in D.

$$u_x = v_y \quad \text{and} \quad u_y = -v_x \quad \text{in } D$$

$$\Rightarrow \quad u_{xx} = v_{yx} \quad \text{and} \quad u_{yy} = -v_{xy} \quad \text{in } D$$

$$\Rightarrow \quad u_{xx} + u_{yy} = v_{yx} - v_{xy} = 0 \quad \text{in } D.$$

Here, $v_{xy} = v_{yx}$ since v_{xy} and v_{yx} are continuous in D.

Therefore, $u(x,y)$ is harmonic in D. Similarly, we can show that $v(x,y)$ is harmonic in D.

Harmonic Conjugate:

Definition: Let $u(x,y)$ be a harmonic function on the domain D in $\mathbb{C}$. If there exists another harmonic function $v(x,y)$ in D such that the partial derivatives for $u(x,y)$ and $v(x,y)$ satisfy the Cauchy-Riemann equations $u_x = v_y$ & $u_y = -v_x$ throughout D then we say that $v(x,y)$ is a harmonic conjugate of $u(x,y)$ in D.

Note on the above definition:

1. We are <u>NOT</u> saying $u(x,y)$ is a harmonic conjugate of $v(x,y)$.
2. The meaning of the word conjugate here is <u>different from</u> the concept of complex conjugate of a complex number.

Example: Let $u(x,y) = x^2 - y^2$ for $z = (x,y) \in \mathbb{C}$.

Then, Set $v(x,y) = 2xy + K$ where K is a fixed real constant.

$$v_{xx} + v_{yy} = 0 \text{ in } \mathbb{C} \Rightarrow v(x,y) \text{ is harmonic in } \mathbb{C}.$$

Further, $u_x = 2x = v_y$ and $u_y = -2y = -v_x$ in $\mathbb{C}$.

Therefore, $v(x,y) = 2xy + K$ is a harmonic conjugate of $u(x,y)$ in $\mathbb{C}$.

<u>Note:</u> $u(x,y)$ is <u>NOT</u> a harmonic conjugate of $v(x,y)$.

$$\left.\begin{aligned} v_x &= 2y \neq u_y = -2y \\ v_y &= 2x \neq -u_x = -2x \end{aligned}\right\} \text{ if } z = (x,y) \neq (0,0).$$

Result: A function $f(z) = u(x,y) + i\,v(x,y)$ is analytic in a domain D if and only if $v(x,y)$ is a harmonic conjugate of $u(x,y)$.

Result: Let $f(z) = u(x,y) + i\,v(x,y)$ be analytic in a domain D.

Then, $g(z) = i\,f(z) = -v(x,y) + i\,u(x,y)$ is analytic in D.

This shows that,

$u(x,y)$ is a harmonic conjugate of $-v(x,y)$ in D.

Result: In a domain D,

v is a harmonic conjugate of u iff u is a harmonic conjugate of $-v$.

Existence of Harmonic Conjugate

Question ①: Given a harmonic function $u(x,y)$ in a domain D,

Does a harmonic conjugate $v(x,y)$ of $u(x,y)$ always exist in D?

Answer: NO.

Example: $u(x,y) = \log\left(\sqrt{x^2+y^2}\right)$ is harmonic in $D = \mathbb{C}\setminus\{0\}$

and it has no harmonic conjugate in D.

Question ②: Given a harmonic function $u(x,y)$ in a domain D,

Suppose that a harmonic conjugate $v(x,y)$ exists in D.

Is $v(x,y)$ unique?

Answer: YES, It is unique upto an additive constant.

That is, If $v_1(x,y)$ and $v_2(x,y)$ are two harmonic conjugates of $u(x,y)$ in D then

$v_1(x,y) - v_2(x,y) = K$ (real constant) for all (x,y) in D.

Proof: Let $f = u + i v_1$ and $g = u + i v_2$. Then $f - g = (u + i v_1) - (u + i v_2) = i(v_1 - v_2)$ analytic, pure imaginary

$\Rightarrow f - g = \text{Constant} \Rightarrow i(v_1 - v_2) = iK \Rightarrow v_1 - v_2 = K$

There are some domains, at which every harmonic functions have a harmonic conjugate.

Result:

Let D be either the whole plane $\mathbb{C}$ or some open disk.
If $u : D \to \mathbb{R}$ is a harmonic function in D then u has a harmonic conjugate in D.

How to find / compute a harmonic conjugate?

(OR)

Given $u(x,y) = \mathrm{Re}(f(z))$, How to find $v(x,y) = \mathrm{Im}(f(z))$?

or Given $v(x,y) = \mathrm{Im}(f(z))$, How to find $u(x,y) = \mathrm{Re}(f(z))$?

Example: Let $u(x,y) = x^2 - y^2$ for (x,y) in $\mathbb{C}$.

Find a harmonic conjugate $v(x,y)$ of $u(x,y)$ in $\mathbb{C}$.

(OR)

Construct an analytic function $f(z) = u(x,y) + i\,v(x,y)$ in $\mathbb{C}$, where $u(x,y) = x^2 - y^2$ in $\mathbb{C}$.

Solution:

Step-1: Verifying that u is harmonic.

Given that $u(x,y) = x^2 - y^2$

$$u_x = 2x, \quad u_{xx} = 2, \quad u_y = -2y, \quad u_{yy} = -2$$

$\Rightarrow u_{xx} + u_{yy} = 2 - 2 = 0$ at all points $z = x + iy$ in $\mathbb{C}$.

Therefore, $u(x,y) = x^2 - y^2$ is harmonic in $\mathbb{C}$.

Step-2:

Since f is analytic, f satisfies the CR equations.

$$u_x(x,y) = 2x = v_y(x,y).$$

Holding x fixed, and integrating both sides with respect to y,

$$\int v_y(x,y)\,dy = \int 2x\,dy$$

$$\boxed{v(x,y) = 2xy + \phi(x)} \longrightarrow (1)$$

where $\phi(x)$ is arbitrary function of x.

Step-3:

Differentiating $v(x,y)$ given by Equation (1) with respect to x partially, we get

$$\boxed{v_x(x,y) = 2y + \phi'(x)} \longrightarrow (2)$$

But the CR equation $u_y(x,y) = -v_x(x,y)$ gives that

$$\boxed{v_x(x,y) = -u_y(x,y) = 2y} \longrightarrow (3)$$

From equations (2) and (3), we get

$$\boxed{\phi'(x) = 0}$$

Integrating it with respect to x, we get

$$\int \phi'(x)\,dx = \int 0\,dx$$

$\Rightarrow \quad \phi(x) = c$ where c is an arbitrary real constant.

Substituting the value of $\phi(x)$ in Equation (1), we get

$$\boxed{v(x,y) = 2xy + c}$$ where c is a real constant

Step-4 Writing $f(z)$ or Expressing $f(z)$ in terms of z.

$$f(z) = f(x+iy) = (x^2 - y^2) + i\,(2xy + c)$$ where c is a real constant

Put $x = \dfrac{z+\bar{z}}{2}$ and $y = \dfrac{z-\bar{z}}{2i}$ in the expression of $f(z)$

and simplify to get the expression of f in terms of z.

Easier Method: If f is analytic, then easier method to express $f(x+iy)$ in terms of z is: Put $y=0$ and $x=z$ in the expression of $f(x+iy)$.

$$f(z) = f(x+iy) = (x^2 - y^2) + i\,(2xy + c)$$

$$\Rightarrow \boxed{f(z) = z^2 + ic}$$ where c is a real constant

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Note: If f is analytic in $\mathbb{C}$, then f can not contain any $\bar{z}$ terms.

Reason: If f is analytic in $\mathbb{C}$ then $\dfrac{\partial f}{\partial \bar{z}} = 0$ CR equations in Complex form.

Exercise: (1) Let $u(x,y)$ and $v(x,y)$ be two harmonic functions in a domain D.

(i) Is $u+v$ harmonic in D?

(ii) Is uv harmonic in D?

Exercise: (2) Suppose that, in a domain D, a function v is a harmonic conjugate of u and also that u is a harmonic conjugate of v. Then, show that $u(x,y)$ and $v(x,y)$ must be constant functions in D.

Note: Let $f(z) = u(x,y) + i\,v(x,y)$ be analytic in D.

Then, $\underline{i\,f(z) = -v(x,y) + u(x,y)}$ is analytic in D.

Add $\Rightarrow$ $(1+i)\,f(z) = \left[u(x,y) - v(x,y)\right] + i\left[u(x,y) + v(x,y)\right]$ is analytic in D

Set $g(z) = (1+i)\,f(z)$.

Real part of $g(z) = u - v = U$ (say)

Imaginary part of $g(z) = u + v = V$ (say)

Given expression of U, we can find V and hence we can compute u and v

Similarly,

Given expression of V, we can find U and hence we can compute u and v.

Exercise: Solve Exercises (5), (6) and (7) of Section (25) in page no. 79 of Brown & Churchill (7th edition)

Lecture 7 ends

Division - 1