Line integrals / Contour Integrals

Definition:

Suppose that $z = z(t)$ for $t \in [a, b]$ represent a contour C (that is, piecewise smooth curve), extending from a point $z_1 = z(a)$ to a point $z_2 = z(b)$. Let the function $f(z)$ be defined on C.

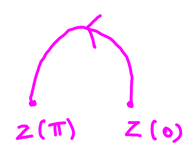
We define the line integral or contour integral of f along the curve C as follows:

$$\int_C f(z)\,dz = \int_a^b f(z(t))\, z'(t)\, dt.$$

Example: Let $C : z(t) = e^{it}$ for $t \in [0, \pi]$ and $f(z) = \bar{z}$ for $z \in \mathbb{C}$.

$$z(t) = e^{it} \text{ for } t \in [0, \pi] \Rightarrow z'(t) = i\, e^{it} \text{ for } t \in [0, \pi]$$

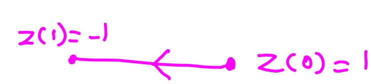
$$\int_C f(z)\,dz = \int_{t=0}^{\pi} f(z(t))\, z'(t)\, dt = \int_{t=0}^{\pi} \left(\overline{e^{it}}\right) \left(i\, e^{it}\right) dt$$



$$= \int_{t=0}^{\pi} e^{-it}\, i\, e^{it}\, dt = \int_{t=0}^{\pi} i\, dt = i\pi.$$

Example: Let $C : z(t) = 1 - 2t$ for $t \in [0, 1]$ be the straight line segment from $z(0) = 1$ to $z(1) = -1$. Let $f(z) = \bar{z}$ for $z \in \mathbb{C}$.

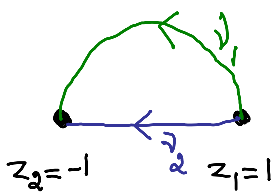
$$\int_C f(z)\,dz = \int_{t=0}^{1} f(z(t))\, z'(t)\, dt = \int_{t=0}^{1} (1 - 2t)(-2)\, dt$$



$$= \int_{t=0}^{1} (-2+4t)\,dt = \Big[-2t + 2t^2\Big]_{t=0}^{1} = 0$$

Observation: Let $z_1 = 1$ and $z_2 = -1$.

Let $\gamma_1(t) = e^{it}$ for $t \in [0, \pi]$ and $\gamma_2(t) = 1 - 2t$ for $t \in [0, 1]$



$$\int_{\gamma_1} \bar{z}\, dz = \pi i \quad \text{and} \quad \int_{\gamma_2} \bar{z}\, dz = 0$$

Thus, the line integral of $f(z) = \bar{z}$ over the curves γ_1 and γ_2 joining from $z_1 = 1$ to $z_2 = -1$ depends on the curves/paths.

Question: When a line integral of f does not depend on the paths?

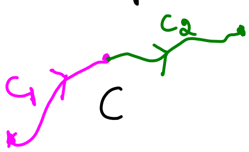
Answer: if f is analytic on the curve.

(or if f is a conservative field / gradient field)

Properties of Line Integrals:

1. If α and β are complex constants and if $f(z)$ and $g(z)$ are (piecewise) continuous complex valued functions defined on a contour C, then $$\int_C (\alpha f(z) + \beta g(z))\,dz = \alpha \int_C f(z)\,dz + \beta \int_C g(z)\,dz.$$

② Let C be a contour consists of a contour C_1 followed by a contour C_2 where the initial point of C_2 is the final point of C_1. It is denoted by the notation $C = C_1 + C_2$. If $f(z)$ is a (piecewise) continuous complex valued function on C then



$$\int_C f(z)\,dz = \int_{C_1+C_2} f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz .$$

③ If $f(z)$ is a (piecewise) continuous complex valued function defined on a contour C and if $-C$ is the opposite curve to C then

$$\int_{-C} f(z)\,dz = (-1)\int_C f(z)\,dz .$$

④ If $f(z)$ is a (piecewise) continuous complex valued function defined on a contour $C: z(t), t \in [a,b]$, then

$$\left| \int_C f(z)\,dz \right| \leq \int_C |f(z)|\,dz \leq \int_a^b |f(z(t))|\,|z'(t)|\,dt \leq ML$$

where $M = \text{Max}\left\{ |f(z)| \;\middle|\; z \text{ lies on the curve } C \right\}$

L = Length of the curve from $z(a)$ to $z(b)$.

Definition: An antiderivative or primitive of a continuous function $f(z)$ in a domain D is a function $F(z)$ such that $\boxed{F'(z) = f(z)}$ for all z in D.

Example: An antiderivative of $\boxed{f(z) = \cos z}$ is $\boxed{F_\alpha(z) = \alpha + \sin z}$, α is any complex constant, since $F_\alpha'(z) = \cos z = f(z)$ for all $z \in \mathbb{C}$.

Results:

① If $F_1(z)$ and $F_2(z)$ are two antiderivatives of a function $f(z)$ in a domain D then $F_1(z) = F_2(z) + \alpha$ for all $z \in D$ where α is a complex constant. That is, antiderivative of a given function f is unique except for an additive complex constant.

② Suppose that $F(z)$ is an antiderivative of $f(z)$ in a domain D. If $F(z)$ is analytic in D then $f(z)$ is analytic in D.

Answer to: When the line integral of $f(z)$ joining two points z_1 and z_2 does not depend on the curves joining them?

Theorem: Suppose that f is a continuous function on a domain D. Then the following three statements are equivalent.

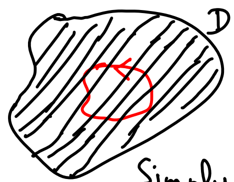
① f has an antiderivative F in D.

② The integrals of $f(z)$ along contours lying entirely in D and extending from any fixed point z_1 to any fixed point z_2 all have the SAME VALUE.

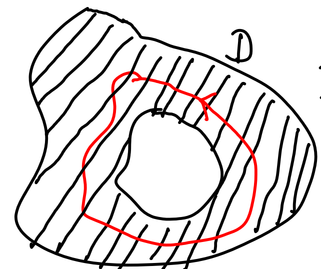
③ The integrals of $f(z)$ around closed contours lying entirely in D all have value ZERO.

Definition: (Simply connected domain)

A simply connected domain is a domain such that every simple closed contour within it encloses only points of D.



Simply connected



NOT Simply connected

(multiply connected)

Multiply connected domain: A domain that is not simply connected is called a multiply connected domain.

Examples of Simply connected Domains

$\{z \in \mathbb{C} \mid |z - z_0| < r\}$

$\{z \in \mathbb{C} \mid Re(z) > t_0\}$

$\{z \in \mathbb{C} \mid Re(z) < t_0\}$

$\mathbb{C}$,

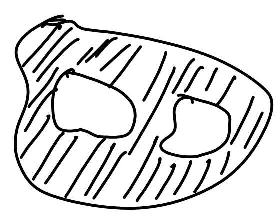
$\{z \in \mathbb{C} \mid Re(z) > t_0\}$

t_0 - fixed real number

Examples of Multiply connected domains

$\{z \in \mathbb{C} \mid r_1 < |z - z_0| < r_2\}$

$\{z \in \mathbb{C} \mid Re(z) < 0\} \setminus \{-1\}$



Any domain with holes

Result: Let $f(z) = u(x,y) + i\,v(x,y)$ be an analytic function on a domain D in $\mathbb{C}$. Then, the partial derivatives of all orders of the component functions $u(x,y)$ and $v(x,y)$ exist and continuous on D.

Further, the derivatives of all orders of f exist in D.

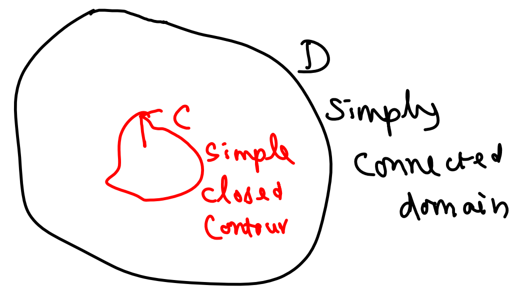
MAIN THEOREM.

Cauchy - Goursat Theorem / Cauchy's Integral Theorem / Cauchy's Integral theorem for simply connected domain :

Statement of the Theorem: (CAUCHY - GOURSAT THEOREM)

If a function f is analytic throughout a simply connected domain D, then for every closed contour C lying in D,

$$\boxed{\int_C f(z)\,dz = 0}$$



Proof of the theorem:

Let $f(z) = u(x,y) + i\,v(x,y)$ for $z = x+iy \in D$.

Then $dz = dx + i\,dy$.

$$\int_C f(z)\,dz = \int_C \big(u(x,y) + i\,v(x,y)\big)\,(dx + i\,dy)$$

$$= \int_C \Big((u\,dx - v\,dy) + i\,(v\,dx + u\,dy) \Big)$$

$$= \int_C (u\,dx - v\,dy) + i\int_C (v\,dx + u\,dy) \longrightarrow (*)$$

Since $f(z) = u(x,y) + i\,v(x,y)$ is analytic in D, the partial derivatives of all orders of the component functions $u(x,y)$ and $v(x,y)$ exist and they are continuous in D.

RECALL: Green's Theorem: Let C be a simple closed contour in $\mathbb{C} = \mathbb{R}^2$. Let R be a region enclosed by the contour C. Suppose that two real valued functions $P(x,y)$ and $Q(x,y)$, together with their first order partial derivatives, are continuous on and inside the contour C. Then

$$\int_C P\,dx + Q\,dy = \iint_R (Q_x - P_y)\,dA.$$

Apply Green's theorem to each of the integrals in ⊛, we get

$$\int_C f(z)\,dz = \iint_R (-v_x - u_y)\,dA + i\iint_R (u_x - v_y)\,dA$$

where R is the region enclosed by C.

Since f is analytic in D, f satisfies the Cauchy-Riemann equations $u_x = v_y$ and $u_y = -v_x$ in D. Therefore,

$$\int_C f(z)\,dz = \iint_R \left(-v_x - (-v_x)\right) dA + i\iint_R (v_y - v_y)\,dA$$

$$= \iint_R 0\,dA + i\iint_R 0\,dA = 0$$

Example: Let C be a positively oriented simple closed contour.

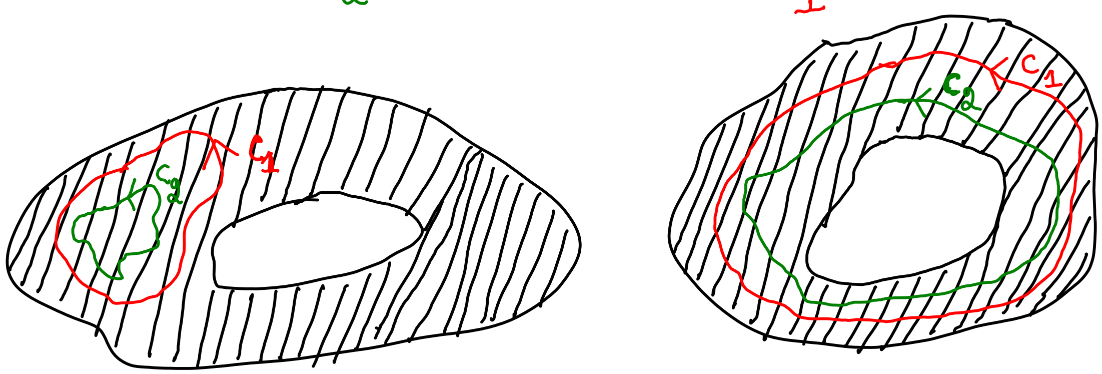
Then, by the Cauchy-Goursat theorem, it follows that

$$\int_C z^n \, dz = 0 \quad \text{where } n \text{ is a fixed natural number}$$

$$\int_C (\sin z + \cos z)\, dz = 0 , \qquad \int_C e^z \, dz = 0$$

Principles of Deformation of Contours:

Let C_1 and C_2 be two simple, closed, positively oriented contours such that C_2 lies interior to C_1.

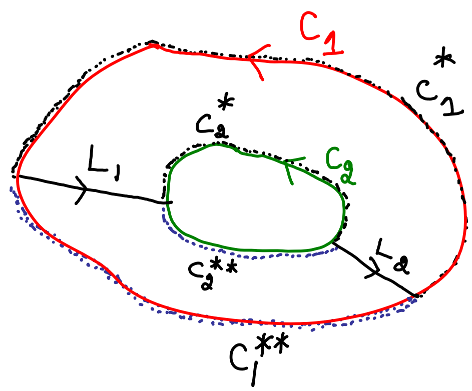


D = Domain as shaded

If $f(z)$ is analytic in a domain D that contains both C_1 and C_2 and region between them, then

$$\boxed{\int_{C_1} f(z)\, dz = \int_{C_2} f(z)\, dz}$$

Idea for the Proof:



Introduce two lines L_1 and L_2 as shown above.

Consider

Upper portion of Curves : $C_u = C_1^* + L_1 - C_2^* + L_2$

Lower portion of Curves : $C_l = C_1^{**} - L_2 - C_2^{**} - L_1$

Then, C_u and C_l are simple closed contours. Applying the Cauchy-Goursat theorem, we get

$$\int_{C_u} f(z)\,dz = 0 \quad \text{and} \quad \int_{C_l} f(z)\,dz = 0$$

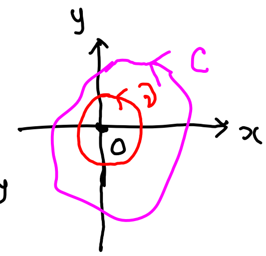
$$\Rightarrow \quad 0 = \int_{C_u} f(z)\,dz + \int_{C_l} f(z)\,dz = \int_{C_u + C_l} f(z)\,dz = \int_{C_1 - C_2} f(z)\,dz$$

$$= \int_{C_1} f(z)\,dz - \int_{C_2} f(z)\,dz$$

$$\Rightarrow \quad \boxed{\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz}$$

Use of the Principle of deformation of Contours:

Evaluate $\displaystyle\int_C \frac{dz}{z}$ where C is any positively oriented simple closed contour surrounding the origin.



Let γ be a circle $|z| = r$ where r is sufficiently small so that γ lies interior to C.

Then, by the principle of deformation of contours,

$$\int_C \frac{dz}{z} = \int_\gamma \frac{dz}{z}.$$

Now, we will evaluate $\displaystyle\int_\gamma \frac{dz}{z}$.

$$\gamma : z(t) = re^{it}, \quad t \in [0, 2\pi]$$

$$dz = z'(t) = r i e^{it}, \quad t \in [0, 2\pi].$$

Then

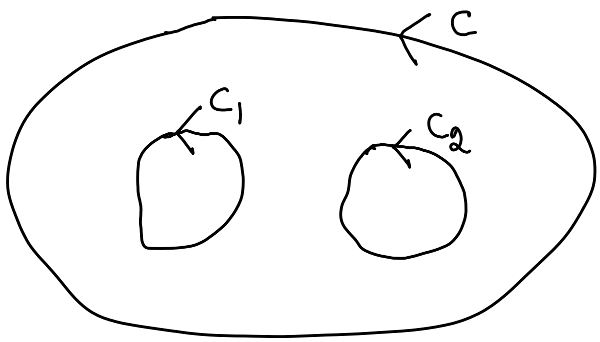
$$\int_\gamma \frac{dz}{z} = \int_0^{2\pi} \frac{r i e^{it}\, dt}{r e^{it}} = \int_0^{2\pi} i\, dt = 2\pi i.$$

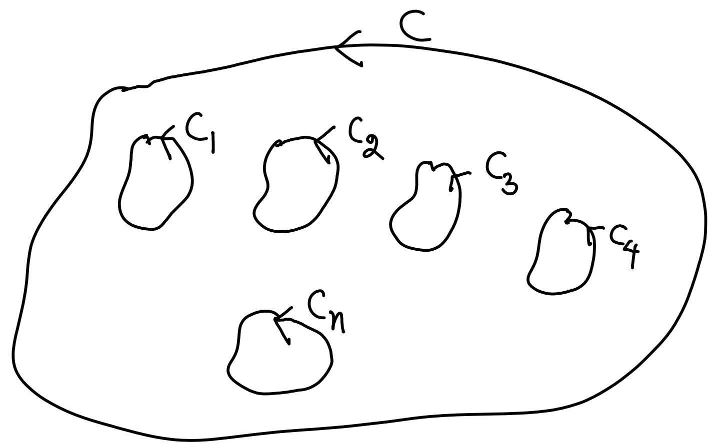
$$\int_C \frac{dz}{z} = 2\pi i$$ where C is any positively oriented simple closed contour surrounding the origin.

Exercise: If C denotes a positively oriented circle $|z - z_0| = R$, then

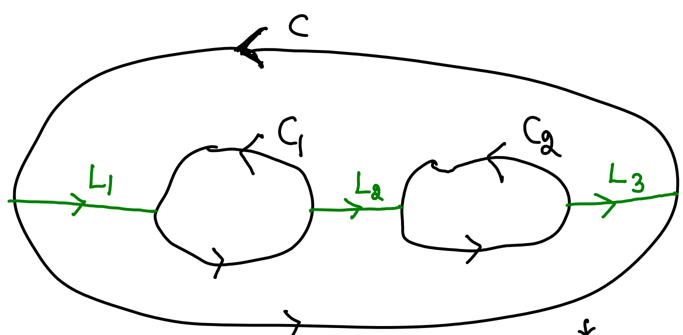
Show that $$\int_C (z - z_0)^{n-1}\, dz = \begin{cases} 0 & \text{when } n = \pm 1, \pm 2, \ldots\ldots \\ 2\pi i & \text{when } n = 0. \end{cases}$$

Extending the proof's Idea of principle of deformation of contour for the following domains as shown below.





Extending the Idea



Upper portion of the curves: $C_u = C^* + L_1 - C_1^* + L_2 - C_2^* + L_3$

Lower Portion of the Curves: $C_l = C^{**} - L_3 - C_2^{**} - L_2 - C_1^{**} - L_1$

Then $C_u + C_l = C - C_1 - C_2$

$$0 = \int_{C_u+C_l} f(z)\,dz = \int_{C-C_1-C_2} f(z)\,dz = \int_C f(z)\,dz - \int_{C_1} f(z)\,dz - \int_{C_2} f(z)\,dz$$

$$\Rightarrow \boxed{\int_C f(z)\,dz = \sum_{K=1}^{2}\left(\int_{C_K} f(z)\,dz\right)}$$

Now extend it to n-curves $C_1, C_2, \ldots, C_n$ and the curve C.

Cauchy Theorem for Multiply Connected Domains:

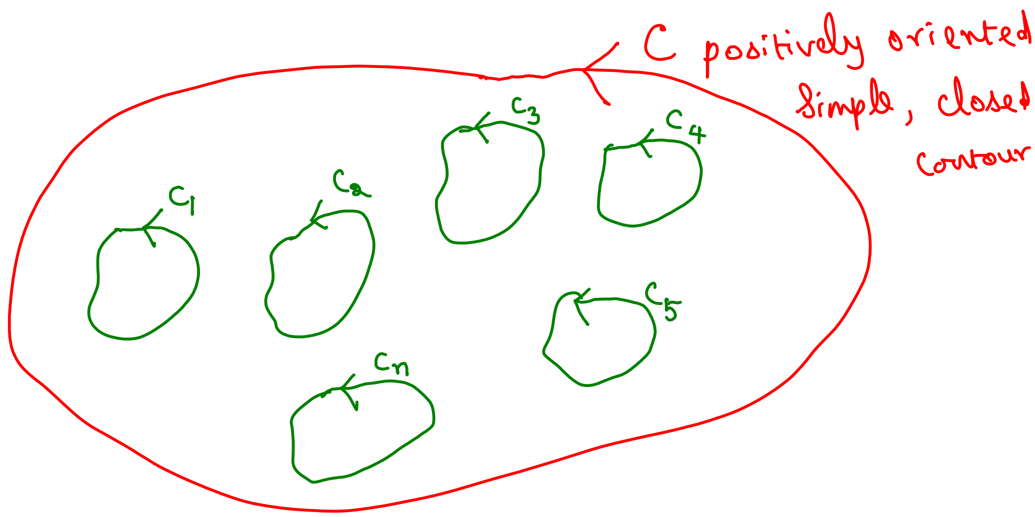
Suppose that

(i) C is a simple closed contour positively oriented,

(ii) C_k, $k = 1, 2, \ldots, n$ denotes a finite number of simple closed contours, all positively oriented, that are interior to C, disjoint, and whose interiors have no points in common.

If a function f is analytic on all of these contours and throughout the multiply connected domain consisting of all points inside C and exterior to each C_k then

$$\boxed{\int_C f(z)\,dz = \sum_{k=1}^{n} \left(\int_{C_k} f(z)\,dz \right)}.$$



C_k's are also positively oriented simple closed contours.

C_k's are disjoint. Their interiors have no points in common.