Lecture - 9 Division - I Thursday 11-08-2011

Define

$$\tan z = \frac{\sin z}{\cos z}, \quad \cot z = \frac{\cos z}{\sin z}$$

$$\sec z = \frac{1}{\cos z}, \quad \operatorname{cosec} z = \frac{1}{\sin z}$$

Hyperbolic Trigonometric Functions

Define

$$\boxed{\cosh z = \frac{e^{z}+e^{-z}}{2}} \quad \text{and} \quad \boxed{\sinh z = \frac{e^{z}-e^{-z}}{2}}$$

for all $z \in \mathbb{C}$.

Clearly, the functions $\sinh z$ and $\cosh z$ are analytic everywhere in $\mathbb{C}$.

Properties

1. $\cosh^2 z - \sinh^2 z = 1$
2. $\cosh(iz) = \cos z$
3. $\sinh(iz) = i \sin z$
4. $\cosh(z+2\pi i) = \cosh(z)$ and $\cosh(-z) = \cosh(z)$
5. $\sinh(z+2\pi i) = \sinh(z)$ and $\sinh(-z) = -\sinh(z)$.

Define

$$\tanh z = \frac{\sinh z}{\cosh z}, \quad \coth z = \frac{\cosh z}{\sinh z}, \quad \operatorname{sech} z = \frac{1}{\cosh z}, \quad \operatorname{cosech} z = \frac{1}{\sinh z}$$

Logarithm Function

The logarithm function can be defined as the inverse of the exponential function.

$$\text{Range of } e^z = \mathbb{C}\setminus\{0\}$$

So, logarithm can be defined only in $\mathbb{C}\setminus\{0\}$.

For any $z \neq 0$, the logarithm $\log z$ is defined as

$\log z = w$ = It is a root of the equation $e^w = z$.

Set $z = re^{i\theta}$ and $w = u + iv$

$$\Rightarrow e^u e^{iv} = re^{i\theta}$$

$$\Rightarrow e^u = r \text{ and } v = \theta + 2k\pi \quad \text{where } k \in \mathbb{Z}$$

$$\Rightarrow u = \ln r \text{ and } v = \theta + 2k\pi$$

where $\ln$ is the usual natural logarithm for positive real numbers

For any $z \neq 0$, define

$$\boxed{\log z = \ln|z| + i \arg(z)}$$

Examples:

$\log(1) = \ln|1| + i\arg(1) = \ln(1) + i(0 + 2k\pi)$ where $k \in \mathbb{Z}$

$= i\,2k\pi$ where $k \in \mathbb{Z}$

Image of 1

$\boxed{1} \longmapsto \boxed{\ldots\ldots -6\pi i, -4\pi i, -2\pi i, 0, 2\pi i, 4\pi i, \ldots\ldots}$

$$\log(1+i) = \ln|1+i| + i\arg(1+i)$$
$$= \ln\left(\sqrt{1^2+1^2}\right) + i\left(\frac{\pi}{4} + 2k\pi\right) \quad \text{where } k \in \mathbb{Z}$$
$$= \ln(\sqrt{2}) + i\left(\frac{\pi}{4} + 2k\pi\right)$$
$$= \frac{1}{2}\ln(2) + i\left(\frac{\pi}{4} + 2k\pi\right) \quad \text{where } k \in \mathbb{Z}$$

$$\boxed{1+i} \longmapsto \boxed{\ldots\ldots, \frac{1}{2}\ln(2) - i\frac{7\pi}{4}, \frac{1}{2}\ln(2) + i\frac{\pi}{4}, \frac{1}{2}\ln(2) + i\frac{9\pi}{4}, \ldots\ldots}$$

Each point z in $\mathbb{C}\setminus\{0\}$ is mapped to a set consisting of infinite number of values by the function $\log z$ which differ from each other by multiples of $2\pi i$.

Therefore,

$$\boxed{\log z \text{ is a multiple valued function}}$$

$\log z$ satisfies

$$\log(z_1 z_2) = \log z_1 + \log z_2$$
$$\log\left(\frac{z_1}{z_2}\right) = \log z_1 - \log z_2.$$

We must interpret the above two identities to mean if particular values are assigned to any two of their terms, then one can find a value of the third term so that the equation is satisfied.

Example: $z_1 = -1$ and $z_2 = -1$ and Given $\log(z_1) = \pi i$ and $\log(z_2) = \pi i$

$$\log(z_1 z_2) = \log(1) = 2\pi i = \pi i + \pi i = \log z_1 + \log z_2$$

Making multipled valued function into single valued function by suitable restriction.

For each z in the domain of the function $\} \longmapsto \{$ choose only one value w (say) from the set of values

$$z \longmapsto w$$

(w is picked from the set consisting the values $f(z)$ of z.)

So that this single valued function is Continuous/Analytic

Example:

Point z	Image of z under log
$\boxed{1}$	$\{\ldots\ldots, -6\pi i, -4\pi i, -2\pi i, \boxed{0}, 2\pi i, \ldots\ldots\}$
$\boxed{1.0001}$	$\{\ldots\ldots, (-2\pi i + \ln\|1.0001\|), \boxed{(0 + \ln\|1.0001\|)}, (2\pi i + \ln\|1.0001\|), \ldots\ldots\}$
$\boxed{2}$	$\{\ldots\ldots(-2\pi i + \ln\|2\|), \boxed{0 + \ln\|2\|}, (2\pi i + \ln\|2\|)\ldots\ldots\}$

Continuous means: Near by points of z should get mapped near by $f(z)$.

$$\left.\begin{aligned} 1 &\longmapsto \ln|1| + i0 \\ 1.0001 &\longmapsto \ln|1.0001| + i0 \\ 1.0002 &\longmapsto \ln|1.0002| + i0 \end{aligned}\right\}$$

Single valued and the resulting function is continuous and analytic.

$\log z$ is multiple valued, due to $\arg(z)$ is multiple valued.

So, we try to make $\arg(z)$ as single valued & continuous.

Principal value of logarithm: Log (First letter is written in capital letter)

For $z \neq 0$, define

$$\text{Log}\, z = \ln|z| + i\,\text{Arg}(z)$$

where $\text{Arg}(z)$ = Principal argument of z and $-\pi < \text{Arg}(z) \leq \pi$.

Note that, this function Log z is single valued.

But it fails to be continuous at any point lying on the ray $\theta = \pi$.

Rest of the domain, it is continuous.

So,

Let $D^* = \{ z = re^{i\theta} \mid r > 0 \text{ and } -\pi < \theta < \pi \}$

For $z \in D^*$, define

$$F(z) = \text{Log}\, z = \ln|z| + i\theta = \ln(r) + i\theta,$$

where $\theta = \text{Arg}(z)$.

The function $F(z) = \text{Log}\, z$ is single valued and analytic in D^*.

It is called the principal branch of the multiple valued function $f(z) = \log z$.

NOTE: In some book, the convention is: Log z is defined & single valued in the domain $\boxed{D = \{ z = re^{i\theta} \mid r > 0, -\pi < \theta \leq \pi \}}$ but it is analytic in $D^* = \{ z = re^{i\theta} \mid r > 0, -\pi < \theta < \pi \}$

Examples: $\log(i) = i\left(\frac{\pi}{2} + 2k\pi\right)$ where $k \in \mathbb{Z}$

$\mathrm{Log}(i) = i\frac{\pi}{2}$

$\log(-1) = i(\pi + 2k\pi)$ where $k \in \mathbb{Z}$

$\mathrm{Log}(-1) = i\pi$

$\log(-1-i) = \ln(\sqrt{2}) + i\left(\frac{-3\pi}{4} + 2k\pi\right)$ where $k \in \mathbb{Z}$

$\mathrm{Log}(-1-i) = \ln(\sqrt{2}) + i\left(\frac{-3\pi}{4}\right)$.

Why principal value of argument? Let us take some other restriction of interval for argument of z.

Let α be a fixed real number.

Let

$$D_\alpha^* = \left\{ z = re^{i\theta} \;\middle|\; r > 0 \text{ and } \alpha < \theta = \arg(z) < (\alpha + 2\pi) \right\}$$

For $z \in D_\alpha^*$,

$$F_\alpha(z) = \ln|z| + i\theta \quad \text{where } \alpha < \theta = \arg(z) < (\alpha + 2\pi).$$

For each $z \in D_\alpha^*$, $F_\alpha(z)$ is single valued and it is one of the values of $\log(z)$ and $F_\alpha(z)$ is analytic in D_α^*.

$F_\alpha(z)$ is called a branch of the logarithm function $\log(z)$.

Definition: (BRANCH)

Let $f(z)$ be a multiple valued function defined on a domain D in $\mathbb{C}$. A function $F(z)$ is said to be a branch of the multiple valued function $f(z)$ in a domain $D^* \subseteq D$ if

(i) $F(z)$ is single valued and analytic in D^*

(ii) For each $z \in D^*$, the value $F(z)$ is one of the values of $f(z)$.

Definition (BRANCH CUT):

A portion of line or curve that is introduced/omitted/removed in orde to define a branch $F(z)$ is called a branch cut.

Note that the branch $F(z)$ is not analytic at all points on the branch cut. For a multiple valued function $f(z)$, there will be more than one branch (several branches). To define each branch, there is a branch cut.

Definition (BRANCH POINTS):

Any point that is common to all branch cuts of f is called a branch point.

Example:

$$f(z) = \log z \quad \text{(Multiple valued function)}$$

Let α be a fixed real number.

$\boxed{\text{Ray } \theta = \alpha + 2\pi, \text{ including the origin}}$ ← Branch cut

$$D_\alpha^* = \{ z = re^{i\theta} \mid r > 0 \text{ and } \alpha < \theta < (\alpha + 2\pi) \}$$

Branch → $\boxed{F_\alpha(z) = \ln|z| + i\theta}$ where $\alpha < \theta = \arg(z) < (\alpha + 2\pi)$

for $z \in D_\alpha^*$

Branch point →

The origin $O = (0, 0)$ is common to all branch cuts and hence O is the branch point of $f(z) = \log z$

Complex Exponents / Powers Function: z^c

Let c be any fixed complex number.

For any $z \neq 0$, we define

$$\boxed{z^c = \exp\big(c \log(z)\big)} \qquad \text{for } z \neq 0.$$

where exp = exponential function e^z

log = the multiple valued logarithm function $\log z$.

Note: z^c is a multiple valued function, in general.

Example:

$$(i)^{(-2i)} = \exp\big((-2i)\log(i)\big) = \exp\big((-2i)\, i(2n + \tfrac{1}{2})\pi\big)$$

where $n \in \mathbb{Z}$

$$= \exp\big((4n+1)\pi\big) \quad \text{where } n \in \mathbb{Z}$$

Properties: $\displaystyle z^{-c} = \frac{1}{z^{c}}$

$$\frac{d}{dz}\left(z^{c}\right) = c\, z^{c-1} \quad \text{for } z \text{ in } \left\{ z = re^{i\theta} \,\middle|\, r>0,\ \alpha<\theta<\alpha+2\pi \right\}$$

α is a fixed real number.

Principal branch of $z^{c} = \exp\left(c\, \mathrm{Log}(z)\right)$ where $\mathrm{Log}(z)$ is the principal branch of $\log(z)$.

Case - I: When c is an integer. Analyze z^{c}.

Case - II: When $c = \frac{1}{n}$ where $n \in \mathbb{N}$. Analyze $z^{c} = z^{1/n}$

List out all Branches of $z^{1/n}$

Exercise.

Inverse Trigonometric Functions

Example: $\sin^{-1}(z)$.

We know that $\sin(z)$ is a periodic function with period 2π.

Domain of definition of $\sin(z) = \mathbb{C}$

Range of $\sin(z) = \mathbb{C}$ (Infinite number of copies)

Inverse of sine function

For each $z \in \mathbb{C}$, find all w such that $\boxed{\sin w = z}$.

There exists infinitely many solutions to $\sin w = z$.

Rewriting $\sin w = z$ by

$$\frac{e^{iw} - e^{-iw}}{2i} = z$$

$$\Rightarrow \quad e^{iw} - e^{-iw} - z\,2i = 0$$

Multiply by e^{iw}, we get

$$\boxed{e^{2iw} - 2iz\,e^{iw} - 1 = 0}$$

It is a quadratic equation in e^{iw}. Solving it,

$$e^{iw} = \frac{2iz + \sqrt{(-2iz)^2 - 4(1)(1)}}{2}$$

$$e^{iw} = iz + \sqrt{(1-z^2)}$$

$$\Rightarrow \quad iw = \log\left[iz + (1-z^2)^{1/2}\right]$$

$$\Rightarrow \quad w = (-i)\log\left[iz + (1-z^2)^{1/2}\right]$$

$$\boxed{\arcsin(z) = \sin^{-1}(z) = (-i)\log\left[iz + (1-z^2)^{1/2}\right]}$$

for $z \in \mathbb{C}$.

It is a multiple valued function.

Exercise: Show that $\sin^{-1}(-i) = n\pi + i(-1)^{n+1}\ln(1+\sqrt{2})$ where $n \in \mathbb{Z}$.

$$\arccos(z) = \cos^{-1}(z) = (-i)\log\left[z + i(1-z^2)^{1/2}\right]$$

$$\arctan(z) = \tan^{-1}(z) = \left(\frac{i}{2}\right)\log\left(\frac{i+z}{i-z}\right)$$

$$\frac{d}{dz}\sin^{-1}(z) = \frac{1}{(1-z^2)^{1/2}}$$

$$\frac{d}{dz}\cos^{-1}(z) = \frac{-1}{(1-z^2)^{1/2}}$$

depends on the values for the square roots.

$$\frac{d}{dz}\tan^{-1}(z) = \frac{1}{1+z^2}$$

Division-I: Lecture-9 ends