$V = \mathbb{R}^4, (\mathbb{R})$

$W_1 = \{(x_1, x_2, x_3, x_4) \mid x_1 + x_2 - 2x_3 = 0\}$

$W_2 = \{(x_1, x_2, x_3, x_4) \mid x_1 - x_2 + x_3 = 0\}$

$\Rightarrow W_1$ & W_2 are subspaces of V [Easy to prove]

$\Rightarrow$ Find a basis of $W_1 \cap W_2$.

Ans: $W_1 \cap W_2 = \left\{ (x_1, x_2, x_3, x_4) \;\middle|\; \begin{matrix} x_1 + x_2 - 2x_3 = 0 \\ x_1 - x_2 + x_3 = 0 \end{matrix} \right\}$

$$= \left\{ (x_1, x_2, x_3, x_4) \;\middle|\; \begin{bmatrix} 1 & 1 & -2 & 0 \\ 1 & -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0 \right\}$$

$= \{ x \mid Ax = 0 \}$ = Null Space of A

where $A = \begin{bmatrix} 1 & 1 & -2 & 0 \\ 1 & -1 & 1 & 0 \end{bmatrix}$

Convert the given matrix A into RREF.

$$\begin{bmatrix} 1 & 1 & -2 & 0 \\ 1 & -1 & 1 & 0 \end{bmatrix} \sim \underbrace{\begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & -2 & 3 & 0 \end{bmatrix}}_{\text{REF}} \sim \underbrace{\begin{bmatrix} 1 & 0 & -1/2 & 0 \\ 0 & 1 & -3/2 & 0 \end{bmatrix}}_{\text{RREF}}$$

Null space of A = $\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \;\middle|\; \begin{bmatrix} 1 & 0 & -1/2 & 0 \\ 0 & 1 & \underbrace{-3/2 \quad 0}_{\text{Free Variables}} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0 \right\}$

$\Downarrow$

$x_1 = \frac{x_3}{2}$

$x_2 = \frac{3x_3}{2}$

$\Downarrow$

$$N \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1/2\, x_3 \\ 3/2\, x_3 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 1/2 \\ 3/2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$\Rightarrow$ Basis of Null Space of $A = \left\{ \begin{bmatrix} 1/2 \\ 3/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

$\therefore$ Basis of $W_1 \cap W_2 = \{(1/2, 3/2, 1, 0), (0, 0, 0, 1)\}$

• How to find a basis for $W_1 + W_2$?

$$W_1 + W_2 = \{\alpha + \beta \mid \alpha \in W_1 \text{ \& } \beta \in W_2\}$$

$W_1 \rightarrow B1$ (Basis), $W_2 \rightarrow B2$ (Basis)

$$\boxed{\Rightarrow L(B1 \cup B2) = W_1 + W_2}$$

Ex: $V = \mathbb{R}^3$

$W_1 = \{(x_1, x_2, x_3) \mid x_1 + x_2 - x_3 = 0\}$

$W_2 = \{(x_1, x_2, x_3) \mid x_1 + 2x_2 = 0\}$

Find Basis for $W_1 + W_2$.

Ans:

• $W_1 = \left\{ (x_1, x_2, x_3) \;\middle|\; \underbrace{\begin{bmatrix} 1 & 1 & -1 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = [0] \right\}$

$=$ Null Space of A

$\underline{x_1 = x_3 - x_2}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 - x_2 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Basis of $W_1 = \{(-1, 1, 0), (1, 0, 1)\}$

• Basis of $W_2 = \{(-2, 1, 0), (0, 0, 1)\}$

• $S = (B_1 \cup B_2) = \{(-1,1,0), (1,0,1), (-2,1,0), (0,0,1)\}$

$L(S) = W_1 + W_2$

Aim: L.I vectors in S

Let $A = \begin{bmatrix} -1 & 1 & -2 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$ where $L(S) =$ Column Space of A,

$\Rightarrow$ Basis of Column Space of A = Basis of $W_1 + W_2$

$$\underbrace{\begin{bmatrix} -1 & 1 & -2 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}}_{A} \sim \begin{bmatrix} 1 & -1 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$$\sim \underbrace{\begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}}_{REF(A)}$$

$\Rightarrow$ Colum Space $(A) = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right\}$

$\boxed{\text{Result:}}$ Suppose V is a vector space over F.
S is L.I subset of V.
If $v \notin L(S)$, then $S \cup \{v\}$ is L.I.

Proof: Given S is L.I subset of V.

Aim $\rightarrow$ $S \cup \{v\}$ is L.I where $v \notin L(S)$

Suppose $\exists\ a, c_1, c_2 \dots c_n \in F$ such that:

$c_1\alpha_1 + c_2\alpha_2 + \dots + c_n\alpha_n + av = 0$ where $\alpha_i \in S$

i) If $a = 0$, $\sum c_i\alpha_i = 0$

All c_i's $= 0$ is the only sol^n as S is L.I

ii) If $a \neq 0$, $\exists\ a^{-1} \in F$ s.t $aa^{-1} = 1 = a^{-1}a$

$av = -\sum c_i\alpha_i$

$\Rightarrow v = \sum \left(-\frac{c_i}{a}\right)\alpha_i$ = Linear Comb^n of α_i = Part of $L(S)$ $\times$

Not possible
As given $v \notin L(S)$

$\therefore$ Only sol^n for a is $a = 0$.

& $S \cup \{v\}$ is L.I !!

Co-ordinate

V: A vector space over F.
$\hookrightarrow$ finite dimensional

$\dim(V) = n$ (say)

Let $B = \{\alpha_1, \alpha_2 \dots \alpha_n\}$ an ordered basis of V.

Note: $V = \mathbb{R}^2$
$B_1 = \{(0,1), (1,0)\}$
$B_2 = \{(1,0), (0,1)\}$

If B_1 & B_2 are defined as ordered, then $B_1 \neq B_2$

For $\alpha \in V$, $\exists$ $c_1, c_2 \dots c_n \in F$ s.t

$$\alpha = c_1\alpha_1 + c_2\alpha_2 + \dots + c_n\alpha_n.$$

The co-ordinate of α w.r.t ordered basis B is denoted as follows :-

$$[\alpha]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

Note: Co-ordinate of α w.r.t ordered basis B is unique
[Proof is easy]. [Contradiction]

Result:
Consider B_1, B_2 (ordered basis)

$B_1 = \{\alpha_1, \alpha_2 \dots, \alpha_n\}$ $\quad$ $B_2 = \{\beta_1, \beta_2 \dots \beta_n\}$

Let $\alpha \in V$. Then $\exists$ an invertible matrix P s.t :

$$[\alpha]_{B_1} = P[\alpha]_{B_2}$$

(Co-ordinate of α w.r.t B_1 = P · Co-ordinate of α w.r.t B_2)

where J^{th} term of $P = [\beta_J]_{B_1}$

Proof: $[\alpha]_{B_2} = \begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{bmatrix}$

$$\alpha = x_1'\beta_1 + x_2'\beta_2 + \dots x_n'\beta_n$$

$$= \sum_{j=1}^{n} x_j'\beta_j \quad -(1)$$

Note that:

$\beta_j \in V \Rightarrow \exists$ unique scalar P_{ij} such that β_j can be written as linear comb^n of α_i
↳ Basis.

$$\beta_j = \sum_{i=1}^{n} P_{ij}\alpha_i$$

Putting in eq^n (1),

$$\alpha = \sum_{j=1}^{n} x_j' \sum_{i=1}^{n} P_{ij}\alpha_i$$

$$= \sum_{j=1}^{n}\sum_{i=1}^{n} P_{ij} x_j' \alpha_i$$

$$= \sum_{i=1}^{n}\left(\sum_{j=1}^{n} P_{ij} x_j'\right)\alpha_i$$

$\Rightarrow$ Co-ordinate in α in B_1

$$= [\alpha]_{B_1} = \begin{bmatrix} \sum_{j=1}^{n} P_{1j} x_j' \\ \sum_{j=1}^{n} P_{2j} x_j' \\ \vdots \\ \sum_{j=1}^{n} P_{nj} x_j' \end{bmatrix}$$

$$= \begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1n} \\ P_{21} & P_{22} & \cdots & P_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ P_{n1} & P_{n2} & \cdots & P_{nn} \end{bmatrix} \begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{bmatrix}$$

Co-ordinate of β_1 w.r.t B_1 Coordinate of β_2 w.r.t B_1 ...

$$[\alpha]_{B_1} = P[\alpha]_{B_2}$$

where P : j^{th} column of P $= [\beta_j]_{B_1}$

Take $[\alpha]_{B_1} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = X$, $[\alpha]_{B_2} = \begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{bmatrix} = X'$

$$X = PX'$$

If $X = 0 \Rightarrow \alpha = 0 \Rightarrow X'$ has to be 0.

$\Rightarrow$ If $P = 0$, there can be non-zero X' possible which should not be possible.

$\therefore$ $P \neq 0$
& P is invertible.

Ex 1. Let $V = \mathbb{R}^3(\mathbb{R})$

$$B1 = \{ \underset{\alpha_1}{(1,0,1)}, \underset{\alpha_2}{(1,1,1)}, \underset{\alpha_3}{(0,0,1)} \}$$

$$B2 = \{ \underset{\beta_2}{(-1,1,1)}, \underset{\beta_2}{(0,1,1)}, \underset{\beta_3}{(0,0,1)} \}$$

Let $\alpha = (1,2,3) \in V$. Find the co-ordinates :

i) $[\alpha]_{B_1}$

ii) $[\alpha]_{B_2}$

iii) P s.t $[\alpha]_{B_1} = P[\alpha]_{B_2}$

$\cdot)$ $\alpha = (1,2,3) = c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3$

$= (c_1 + c_2 \;,\; c_2 \;,\; c_1 + c_2 + c_3)$

$\Rightarrow \boxed{\begin{matrix} c_2 = 2 \\ c_1 = -1 \\ c_3 = 2 \end{matrix}}$ & $\boxed{[\alpha]_{B_1} = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}}$

$\cdot)$ $\alpha = (1,2,3) = c_1\beta_1 + c_2\beta_2 + c_3\beta_3$

$= (-c_1 \;,\; c_1 + c_2 \;,\; c_1 + c_2 + c_3)$

$\boxed{\Rightarrow \quad [\alpha]_{B_2} = \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}}$

$\underline{\underline{P:}}$

$\cdot)$ $P_{1st \text{ column}} = [\beta_1]_{B_1}$

$(-1,1,1) = \sum c_i\alpha_i$

$= (c_1 + c_2 \;,\; c_2 \;,\; c_1 + c_2 + c_3)$

$[\beta_1]_{B_1} = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$

$\cdot)$ $(0,1,1) = (c_1 + c_2 \;,\; c_2 \;,\; c_1 + c_2 + c_3)$

$[\beta_2]_{B_1} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

$\cdot)\quad (0,0,1) = (c_1+c_2,\ c_2,\ c_1+c_2+c_3)$

$$[\beta_3]_{B1} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\boxed{P = \begin{bmatrix} -2 & -1 & 0 \\ 1 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}}$$

Verify: $[\alpha]_{B1} = P[\alpha]_{B2}$

$$\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 & -1 & 0 \\ 1 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 - 3 + 0 \\ -1 + 3 + 0 \\ -2 + 3 + 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

- If $[\alpha]_B$, $[\beta]_B$ is given:

$$[\alpha+\beta]_B = [\alpha]_B + [\beta]_B$$

- $\alpha \in V$, $c\alpha \in V$

$$[c\alpha]_B = c[\alpha]_B$$

$$[\alpha]_B = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Linear Transformation

Let V & W be vector spaces over "same field F"
& $T : V \to W$ be a mapping.
Then T is called a linear transformation from $V \to W$ if :→

$$T(c\alpha+\beta) = c\,T(\alpha) + T(\beta)$$
$$\forall\ \alpha, \beta \in V$$
$$\forall\ c \in F$$

Ex: V & W be vector spaces over F.

i) $O : V \to W$ s.t $O(\alpha) = 0$ $\forall \alpha \in V$ } Easy to Verify it is Linear Transformation

ii) $I : V \to V$
$I(\alpha) = \alpha \quad \forall\ \alpha \in V$
Then V is a linear transformation. } Easy to prove

Ex: $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined as :

$$T(x_1, x_2) = (x_1 + x_2,\ x_1 - x_2,\ 2x_1)$$

$\alpha = (x_1, x_2)$

$\beta = (y_1, y_2)$

$T(\alpha) = (x_1 + x_2,\ x_1 - x_2,\ 2x_1)$

$T(\beta) = (y_1 + y_2,\ y_1 - y_2,\ 2y_1)$

$\boxed{LHS}$

$$T(c\alpha + \beta) = \Big((cx_1 + y_1) + (cx_2 + y_2),\ (cx_1 + y_1) - (cx_2 + y_2),\ 2(cx_1 + y_1)\Big)$$
$$= \Big(c(x_1 + x_2) + (y_1 + y_2),\ c(x_1 - x_2) + (y_1 - y_2),\ 2(cx_1 + y_1)\Big)$$

$\boxed{RHS}$

$$cT(\alpha) + T(\beta) = (cx_1 + cx_2,\ cx_1 - cx_2,\ 2cx_1) + (y_1 + y_2,\ y_1 - y_2,\ 2y_1)$$
$$= (cx_1 + cx_2 + y_1 + y_2,\ cx_1 - cx_2 + y_1 - y_2,\ 2cx_1 + 2y_1)$$
$$= \Big(c(x_1 + x_2) + (y_1 + y_2),\ c(x_1 - x_2) + (y_1 - y_2),\ 2(cx_1 + y_1)\Big)$$

$\boxed{LHS = RHS}$ $\therefore$ T is Linear Transformation.

Q: $V = \{ \underbrace{a_0 + a_1 x + \dots + a_n x^n}_{= f(x)} \mid a_i \in \mathbb{R} \}$

$D : V \to V$ s.t

$D(f(x)) = f'(x)$

Prove that D is a Linear Operator / Transformation.

Note: If Domain = Codomain (e.g $V \to V$), then Linear Transformation is called Linear Operator

$T(f(x)) = f'(x)$

$T(g(x)) = g'(x)$

$T(c\alpha + \beta) = T(cf(x) + g(x))$

$= [cf(x) + g(x)]' = cf'(x) + g'(x)$

$= c\, T(f(x)) + T(g(x)) = \underline{\underline{c\,T(\alpha) + T(\beta)}}$

Ex: Let $A_{m\times n}$ - fixed, $a_{ij} \in \mathbb{R}$

$T: R^{n\times 1} \longrightarrow R^{m\times 1}$ s.t

$T(X) = AX$

Ans: $T(c$

If $T: V \to W$ is a linear transformation then $T(0_V) = 0_W$ (& vice-versa as well)

Proof:

$$0_V + 0_V = 0_V$$

$$T(0_V + 0_V) = T(0_V)$$

$$\Rightarrow T(0_V) + T(0_V) = T(0_V) \quad -(1)$$

Since $T(0_V) \in W$ $\exists$ $-T(0_V)$ s.t

$$T(0_V) + (-T(0_V)) = 0_W$$

By adding $-T(0_V)$ in (1),

$$T(0_V) + T(0_V) + (-T(0_V)) = T(0_V) + (-T(0_V))$$

$$\Rightarrow \underbrace{T(0_V) + 0_W}_{= T(0_V)} = 0_W$$

$$\boxed{\therefore T(0_V) = 0_W}$$

Q: Is T a linear transformation from $\mathbb{R}^3 \to \mathbb{R}^4$ defined by:

$$T(x_1, x_2, x_3) = (x_1, x_1 - x_2, x_2, x_1 + 1) \, ?$$

Ans: No!

$$T(\underset{\mathbb{R}^3}{0_V}) = T(0,0,0) = (0, 0-0, 0, 0+1)$$

$$= (0,0,0,1)$$

$$\neq 0_W = (0,0,0,0)$$

As $T(0_V) \neq 0_W$, T is not a linear trans.

If T is a linear transformation from V to W, then

$$T(c_1\alpha_1 + c_2\alpha_2 + \dots + c_n\alpha_n) = c_1T(\alpha_1) + c_2T(\alpha_2) + \dots + c_nT(\alpha_n)$$

Proof:
- Induction ✓
- Using $T(c\alpha+\beta) = cT(\alpha) + T(\beta)$ ✓

V: a finite dim. vector space over field F.

$\dim(V) = n$

$\Rightarrow$ Basis $= \mathcal{B} = \underbrace{\{\alpha_1, \alpha_2 \dots \alpha_n\}}_{\substack{n \text{ terms as} \\ \dim(V)=n}}$

Also $T: V \to W$ (linear transformation)

Given that $T(\alpha_i) = \beta_i$, $i \to 1$ to n.

Let $\alpha \in V$ be an arbitrary vector:

$T(\alpha) = ?$

As $\mathcal{B} \to$ Basis, we can write α as:

$\alpha = \sum_{i=1}^{n} a_i\alpha_i$ where $a_i \in F$

& $\{a_i\}$ set is unique [Co-ordinate of α w.r.t basis $\mathcal{B}$]

$$T(\alpha) = T\left(\sum a_i\alpha_i\right) = \sum_{i=1}^{n} a_i T(\alpha_i)$$

[Using $T(c_1\alpha_1 + c_2\alpha_2 + \dots + c_n\alpha_n) = c_1T(\alpha_1) + \dots c_nT(\alpha_n)$]

$$\Rightarrow T(\alpha) = \sum_{i=1}^{n} a_i T(\alpha_i) = \sum_{i=1}^{n} a_i\beta_i$$

Ex: $T: \mathbb{R}^2 \to \mathbb{R}^3$ s.t

$T(1,2) = (5,0,0)$

$T(1,1) = (4,0,0)$

Then find T.

Ans: $B = \{ \underset{\alpha_1}{(1,2)}, \underset{\alpha_2}{(1,1)} \} \Rightarrow B$ is ordered basis.

Let $\alpha = (x_1, x_2) \in \mathbb{R}^2$

then $T(x_1, x_2) = ?$

Now

$(x_1, x_2) \in \mathbb{R}^2$ & B is a basis

$\therefore \exists\, c_1, c_2$ s.t

$(x_1, x_2) = c_1\alpha_1 + c_2\alpha_2 = (c_1 + c_2, 2c_1 + c_2)$

$\Rightarrow c_1 + c_2 = x_1$ & $2c_1 + c_2 = x_2$

$\Rightarrow c_1 = x_2 - x_1$

$c_2 = 2x_1 - x_2$

$\therefore (x_1, x_2) = (x_2 - x_1)(1,2) + (2x_1 - x_2)(1,1)$

$\Rightarrow T(x_1, x_2) = T((x_2 - x_1)\alpha_1) + T((2x_1 - x_2)\alpha_2)$

$= (x_2 - x_1)(T(1,2)) + (2x_1 - x_2)T(1,1)$

$= (x_2 - x_1)[(5,0,0)] + (2x_1 - x_2)[(4,0,0)]$

$= [(5x_2 - 5x_1) + (8x_1 - 4x_2), 0, 0]$

Ans

$$\boxed{\therefore T(x_1, x_2) = (3x_1 + x_2, 0, 0)}$$

This is our linear transformation.

Null Space of a Linear Transformation

Let $T: V \to W$ be a linear transformation

Then null space of T is denoted by $\Rightarrow$

$$N(T) = \{ \alpha \in V \mid T(\alpha) = 0 \}$$

• Prove that $N(T)$ is a subspace of V

$\rightarrow$ i) $0 \in N(T) \Rightarrow N(T) \neq \phi$

ii) Let $\alpha \in N(T) \Rightarrow T(\alpha) = 0$
$\beta \in N(T) \Rightarrow T(\beta) = 0$

$$\Rightarrow T(c\alpha + \beta) = cT(\alpha) + T(\beta)$$
$$= c(0) + (0)$$
$$= 0$$

As $T(c\alpha + \beta) = 0$

$\Rightarrow c\alpha + \beta \in N(T)$

$\boxed{\therefore N(T) \text{ is a subspace of } V}$

Range Space of T

$T: V \to W$ linear transformation

then $R(T) = \{ T(\alpha) \mid \alpha \in V \}$

$\downarrow$

Range Space

$\Rightarrow R(T)$ is range of linear transformation!

• Prove that $R(T)$ is subspace of W

$\rightarrow$ i) $0_W \in R(T)$ [As $T(0_V) = 0_W$]

$\Rightarrow R(T) \neq \phi$

ii) Let $\beta_1 \in R(T) \Rightarrow \exists \alpha_1 \in V$ s.t $T(\alpha_1) = \beta_1$
$\beta_2 \in R(T) \Rightarrow \exists \alpha_2 \in V$ s.t $T(\alpha_2) = \beta_2$

then $\underline{\underline{c\beta_1 + \beta_2}} = cT(\alpha_1) + T(\alpha_2) = T(\underbrace{c\alpha_1 + \alpha_2}_{\in V})$

[$\alpha_1, \alpha_2 \in V$ & V is vectorspace $\Rightarrow c\alpha_1 + \alpha_2 \in V$]

$\Rightarrow \quad c\beta_1 + \beta_2 \in R(T)$

$\therefore$ $R(T)$ is subspace of V!

Rank - Nullity Theorem

Let V & W be vector space over the same field F & $T: V \to W$ be a linear transformation. If dim (V) is finite (say n) then:

$$\boxed{\text{rank}(T) + \text{nullity}(T) = \dim(V)}$$

Note:

If dim (V) is finite, then

- dim $N(T)$ is called nullity of T
- dim $R(T)$ is called rank of T

Q: Can you give an onto linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^4$?

Ans: For Onto T, $\dim R(T) = \dim(\mathbb{R}^4)$
$= 4 = $ rank

Let nullity $= x$

Also $\dim V = \dim(\mathbb{R}^3) = 3$

$\Rightarrow 4 + x = 3$

$x \to$ No sol^n

$\therefore$ Ans $= \boxed{\text{No!}}$

Proof:

Given $\dim(V) = n$

$N(T)$ is subspace of V

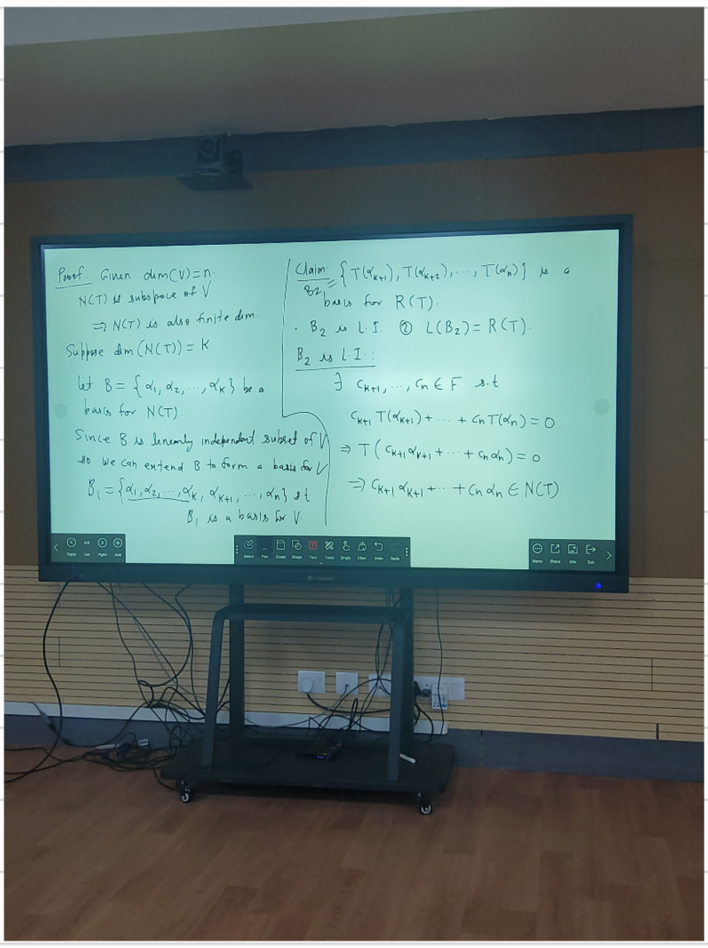
$\Rightarrow N(T)$ is also finite dimension.

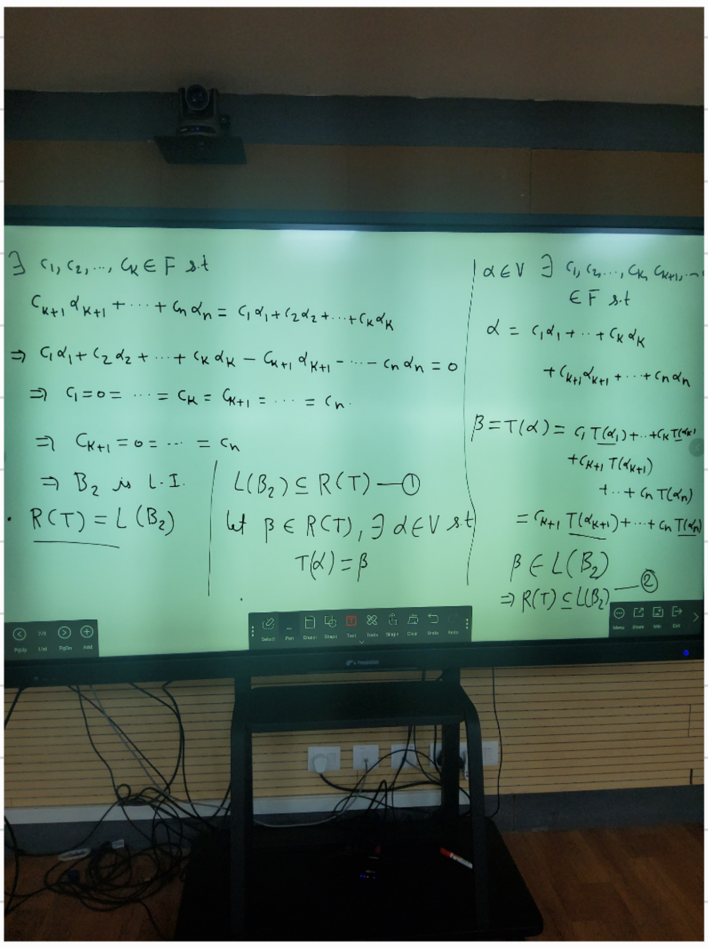
Suppose $\dim[N(T)] = k$,

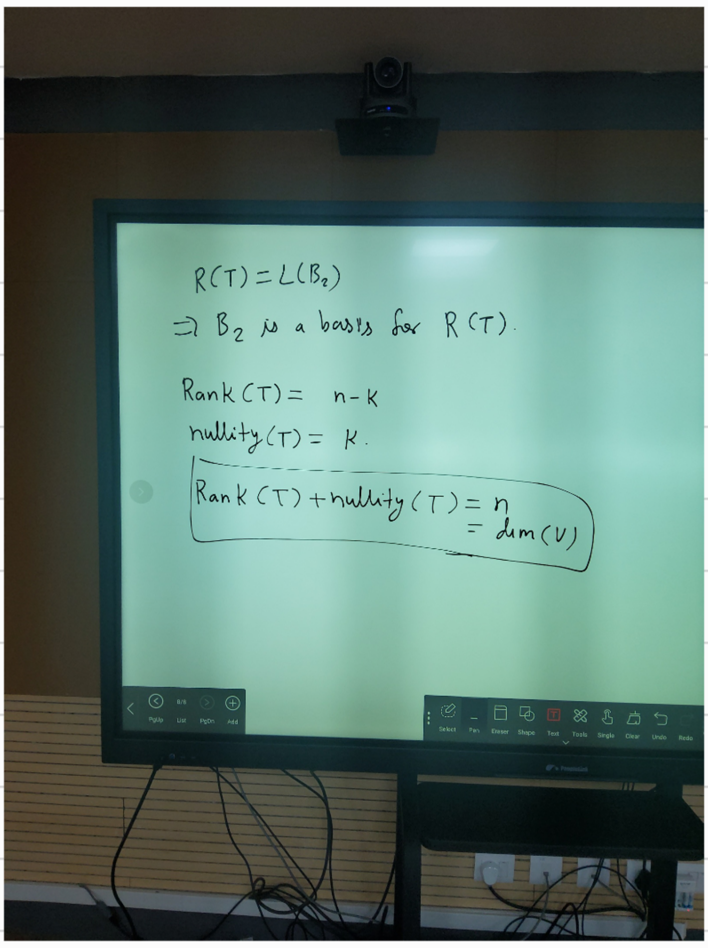
Let $B = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a basis for $N(T)$

Since B is linearly independent subset of V,

so we


Proof: Given $\dim(V) = n$.
$N(T)$ is subspace of V
$\Rightarrow N(T)$ is also finite dim.
Suppose $\dim(N(T)) = K$
Let $B = \{\alpha_1, \alpha_2, \dots, \alpha_K\}$ be a basis for $N(T)$.
Since B is linearly independent subset of V
so we can extend B to form a basis for V
$B_1 = \{\alpha_1, \alpha_2, \dots, \alpha_K, \alpha_{K+1}, \dots, \alpha_n\}$ s.t
B_1 is a basis for V.
Claim: $B_2 = \{T(\alpha_{K+1}), T(\alpha_{K+2}), \dots, T(\alpha_n)\}$ is a basis for $R(T)$.
① B_2 is L.I. ② $L(B_2) = R(T)$.
B_2 is L.I.:
$\exists\, c_{K+1}, \dots, c_n \in F$ s.t
$c_{K+1} T(\alpha_{K+1}) + \dots + c_n T(\alpha_n) = 0$
$\Rightarrow T(c_{K+1}\alpha_{K+1} + \dots + c_n\alpha_n) = 0$
$\Rightarrow c_{K+1}\alpha_{K+1} + \dots + c_n\alpha_n \in N(T)$


$\exists\, c_1, c_2, \dots, c_K \in F$ s.t
$c_{K+1}\alpha_{K+1} + \dots + c_n\alpha_n = c_1\alpha_1 + c_2\alpha_2 + \dots + c_K\alpha_K$
$\Rightarrow c_1\alpha_1 + c_2\alpha_2 + \dots + c_K\alpha_K - c_{K+1}\alpha_{K+1} - \dots - c_n\alpha_n = 0$
$\Rightarrow c_1 = 0 = \dots = c_K = c_{K+1} = \dots = c_n$.
$\Rightarrow c_{K+1} = 0 = \dots = c_n$
$\Rightarrow B_2$ is L.I.
• $R(T) = L(B_2)$
$L(B_2) \subseteq R(T)$ — ①
Let $\beta \in R(T)$, $\exists\, \alpha \in V$ s.t $T(\alpha) = \beta$
$\alpha \in V$ $\exists\, c_1, c_2, \dots, c_K, c_{K+1}, \dots \in F$ s.t
$\alpha = c_1\alpha_1 + \dots + c_K\alpha_K + c_{K+1}\alpha_{K+1} + \dots + c_n\alpha_n$
$\beta = T(\alpha) = c_1 T(\alpha_1) + \dots + c_K T(\alpha_K) + c_{K+1} T(\alpha_{K+1}) + \dots + c_n T(\alpha_n)$
$= c_{K+1} T(\alpha_{K+1}) + \dots + c_n T(\alpha_n)$
$\beta \in L(B_2)$
$\Rightarrow R(T) \subseteq L(B_2)$ — ②


$R(T) = L(B_2)$
$\Rightarrow B_2$ is a basis for $R(T)$.
$\text{Rank}(T) = n - K$
$\text{nullity}(T) = K$.
$\boxed{\text{Rank}(T) + \text{nullity}(T) = n = \dim(V)}$

Example: $T: R^2 \to R^2$, defined as:

$$T(x_1, x_2) = (0, x_1)$$

Find null space of T : $N(T)$
range space of T : $R(T)$
nullity of T
rank of T.

Ans: •) $N(T) = \{ \underline{(x_1, x_2)} \mid T(x_1, x_2) = (0,0) \}$

$$T(x_1, x_2) = (0, 0)$$

$$\Rightarrow (0, x_1) = (0, 0)$$

$$\Rightarrow \boxed{x_1 = 0}$$

$$\therefore N(T) = \{ (0, x_2) \mid x_2 \in R \}$$

$$= \{ x_2 (0, 1) \mid x_2 \in R \}$$

$\Rightarrow N(T) = L\{(0, 1)\}$ or span of $v = (0, 1)$

$\Rightarrow$ Basis of $N(T) = B = \{(0, 1)\}$,

$\Rightarrow \dim(N(T)) = 1 = \text{nullity}(T)$

•) $R(T) = \{ T(x_1, x_2) \mid (x_1, x_2) \in R^2 \}$

$$= \{ (0, x_1) \mid x_1 \in R \}$$

$$= \{ x_1 (0, 1) \mid x_1 \in R \}$$

Note: $R(T) = N(T) = L\{0, 1\}$

$\text{rank}(T) = 1$

∘) Also $\dim(V) = \dim(\mathbb{R}^2) = 2$

$\text{rank}(T) + \text{nullity}(T) = 1 + 1 = 2 = \dim(V)$

rank - nullity theorem verified !

Example : $T: \mathbb{R}^2 \to \mathbb{R}^3$

$$T(x_1, x_2) = (x_1 + 2x_2, 2x_1 + 4x_2, x_1 + 2x_2)$$

∘) $N(T) = \{ (x_1, x_2) \mid T(x_1, x_2) = (0,0,0) \}$

$\Rightarrow$
$$\left.\begin{array}{l} x_1 + 2x_2 = 0 \\ 2x_1 + 4x_2 = 0 \end{array}\right] \rightarrow \boxed{x_1 = -2x_2}$$
$$x_1 + 2x_2 = 0$$

$$N(T) = \{ (-2x_2, x_2) \mid T(x_1, x_2) = (0,0,0) \}$$
$$= \{ x_2(-2, 1) \mid T(x_1, x_2) = (0,0,0) \}$$

$\Rightarrow$ Basis of $N(T) = \{(-2, 1)\}$

$$\boxed{\dim(N(T)) = 1 = \text{nullity}(T)}$$

o) $R(T) = \{(x_1+2x_2,\ 2x_1+4x_2,\ x_1+2x_2) \mid (x_1, x_2, x_3) \in R^3\}$

$R(T) = L\{T(1,0),\ T(0,1)\}$

$= L\{\overset{\alpha_1}{(1,2,1)},\ \overset{\alpha_2}{(2,4,2)}\}$

$\alpha_2 = 2\alpha_1$

$= L\{(1,2,1)\}$

$\Rightarrow$ Basis of $R(T) = \{(1,2,1)\}$

$\boxed{\therefore\ \text{rank}(T) = \dim(R(T)) = 1}$

Let $T: V \to W$ be a linear transformation.

$N(T) = \{0\} \Longleftrightarrow T$ is one-one

Proof: i) $N(T) = \{0\}$

Then we have to show that T is one-one.

Let $T(\alpha_1) = T(\alpha_2)$

$\Rightarrow T(\alpha_1) - T(\alpha_2) = 0$

$\Rightarrow T(1\alpha_1 + (-1)\alpha_2) = 0$

$\Rightarrow \alpha_1 - \alpha_2 \in N(T) = \{0\}$

$\Rightarrow \alpha_1 - \alpha_2 = 0$

$\boxed{\therefore \alpha_1 = \alpha_2}$

ii) Converse

Suppose T is one-one

Then we have to show that $N(T) = \{0\}$

Let $\alpha \in N(T)$

$T(\alpha) = 0$

But we know $T(0_V) = 0_W$

$\Rightarrow T(0) = 0 = T(\alpha)$

$\therefore \boxed{\alpha = 0}$

Example: $T: R^2 \to R^2$

$T(x, y) = (y, x)$

$N(T) = \{ (x, y) \mid T(x, y) = (0, 0) \}$

$= \{(0, 0)\}$

As $N(T) = \{0_V\}$,

T is one-one!

nullity $(T) = 0$

rank $(T) = 2$

$R(T) = R^2$

Let $T: V \to W$ be a L·T
Then T is onto if $R(T) = W$

$T: V \to W$ be a L·T
Then T is invertible if & only if T is bijective
(one-one & onto)

Q: Let $T: R^3$ to R^3 is a linear operator.

$$T(x_1, x_2, x_3) = \{(x_1 + x_2, x_1 + 2x_2, x_2 + 3x_3\}$$

Is T invertible?
If Yes, find T^{-1}.

Ans: •) $N(T) = \{(0, 0, 0)\} \Rightarrow$ nullity $= 0$

$\Downarrow$

rank $(T) = 3 =$ dim $(R(T))$

$\therefore$ T is one-one

•) $R(T) \subseteq R^3$

dim $(R(T)) = 3$ dim $(R^3) = 3$

$\Rightarrow$ $R(T) = R^3$ $\Rightarrow$ $R(T)$ is onto

$\therefore$ T is invertible!

•) $T^{-1}(x_1, x_2, x_3) = ?$

$$T^{-1}(x_1, x_2, x_3) = (y_1, y_2, y_3)$$

$$\Rightarrow (x_1, x_2, x_3) = T(y_1, y_2, y_3)$$

$$= (y_1 + y_2, y_1 + 2y_2, y_2 + 3y_3)$$

$$\Rightarrow \begin{aligned} y_1 + y_2 &= x_1 \\ y_1 + 2y_2 &= x_2 \\ y_2 + 3y_3 &= x_3 \end{aligned}$$

$$\boxed{\begin{aligned} y_2 &= x_2 - x_1 \\ y_1 &= 2x_1 - x_2 \\ y_3 &= \tfrac{1}{3}(x_3 - x_2 + x_1) \end{aligned}}$$

$$\therefore T^{-1}(x_1, x_2, x_3) = \left\{ (x_2 - x_1), (2x_1 - x_2), \tfrac{1}{3}(x_3 - x_2 + x_1) \right\}$$

Q: Let $T: P_2 \to P_4$ be a L.T

$$P_n = \{ a_0 + a_1 x + \cdots a_n x^n \mid a_i \in R \}$$

$$T(P(x)) = \int_0^1 P(x)\, dx$$

Then find $N(T)$, $R(T)$, nullity (T), rank (T)

Ans:

$$N(T) = \{ P(x) \mid T(P(x)) = 0 \}$$

$$\Rightarrow \int_0^1 P(x)\, dx = 0$$

$$N(T) = \{ a_0 + a_1 x + a_2 x^2 \mid T(a_0 + a_1 x + a_2 x^2) = 0 \}$$

$$T(a_0 + a_1 x + a_2 x^2) = \int_0^1 (a_0 + a_1 x + a_2 x^2)\, dx = a_0 + \frac{a_1}{2} + \frac{a_2}{3}$$

$$T(a_0 + a_1 x + a_2 x^2) = 0 = a_0 + \frac{a_1}{2} + \frac{a_2}{3}$$

$$\Rightarrow a_0 = -\frac{a_1}{2} - \frac{a_2}{3}$$

$$N(T) = \{ -\frac{a_1}{2} - \frac{a_2}{3} + a_1 x + a_2 x^2 \}$$

$$= \{ a_1 (x - \frac{1}{2}) + a_2 (x^2 - \frac{1}{3}) \}$$

$$= L\{ (x - \frac{1}{2}), (x^2 - \frac{1}{3}) \}$$

= Basis of $N(T)$

Nullity $(T) = 2$

$\Rightarrow$ rank $(T) = 1$

Recall

$\Rightarrow$ If $\dim(V) =$ finite & T is L.T, $T: V \to W$

then $\boxed{\text{rank}(T) + \text{nullity}(T) = \dim(V)}$ Rank nullity theorem

Q: Find range & null space of

$T: \mathbb{R}^4 \to \mathbb{R}^3$,

$T(x_1, x_2, x_3, x_4) = (x_1 - x_4, x_2 + x_3, x_3 - x_4)$

Ans:

$N(T) = \{(x_1, x_2, x_3, x_4) \mid T(x_1, x_2, x_3, x_4) = (0,0,0)\}$, $x_i \in \mathbb{R}$

$\Rightarrow (x_1 - x_4, x_2 + x_3, x_3 - x_4) = (0,0,0)$

$x_1 = x_4$ $\qquad$ $x_4 = x_1$

$x_2 = -x_3$ $\Rightarrow$ $x_3 = x_1$

$x_3 = x_4$ $\qquad$ $x_2 = -x_1$

$N(T) = \{x_4(1,-1,1,1)\}$

$\Rightarrow N(T) = L\{(1,-1,1,1)\}$.

Basis of $N(T) = \{(1,-1,1,1)\}$

$\therefore \dim(N(T)) = 1 = \text{nullity}(T)$

By rank-nullity theorem, $\text{rank}(T) + 1 = \dim(V) = \dim(\mathbb{R}^4) = 4$

$\boxed{\therefore \text{rank}(T) = 3}$

$\text{rank}(T) = \dim(R(T)) = 3$

Also $R(T) \subseteq \mathbb{R}^3 \longrightarrow \dim(\mathbb{R}^3) = 3$

$\downarrow$

$\dim(R(T)) = 3$

$\therefore \boxed{R(T) = \mathbb{R}^3}$

Proof: Let $R(T) \neq \mathbb{R}^3$

$B = \{(\alpha_1, \alpha_2, \alpha_3)\}$ is basis of $R(T)$

If $R(T) \neq \mathbb{R}^3$, $\exists$ $x \in \mathbb{R}^3$ s.t $\beta \notin R(T)$

$\Rightarrow x$ is linearly independent subset of $\mathbb{R}^3$ & span (B)

$\Rightarrow$ $\underbrace{B \cup \{x\} = \{\alpha_1, \alpha_2, \alpha_3, x\}}_{\dim = 4}$ is L.I subset of $\underbrace{\mathbb{R}^3}_{\dim = 3}$

$\neq \dim(R(T)) = 3$

$\therefore$ Not Possible!

Q: Find a L.T $T: R^3 \to R^3$ whose range space is generated by $\{(1,2,3), (2,1,0)\}$

Ans: $R(T) = L\{(1,2,3), (2,1,0)\}$

$= L\{(1,2,3), (2,1,0), (0,0,0)\}$

$T: R^3 \to R^3$

Standard basis for R^3 $= \{(1,0,0), (0,1,0), (0,0,1)\}$

Now, $R(T) = L\{\text{Image of Basis}\}$

$= L\{\underset{(1,2,3)}{\underset{\parallel}{T(1,0,0)}}, \underset{(2,1,0)}{\underset{\parallel}{T(0,1,0)}}, \underset{(0,0,0)}{\underset{\parallel}{T(0,0,1)}}\}$

$T(1,0,0) = (1,2,3)$

$T(0,1,0) = (2,1,0)$

$T(0,0,1) = (0,0,0)$

$T(x_1, x_2, x_3) = ?$

$(x_1, x_2, x_3) = x_1(1,0,0) + x_2(0,1,0) + x_3(0,0,1)$

$\Rightarrow T(x_1, x_2, x_3) = T(x_1(1,0,0)) + T(x_2(0,1,0)) + T(x_3(0,0,1))$

$= x_1(1,2,3) + x_2(2,1,0) + x_3(0,0,0)$

$$\boxed{\therefore T(x_1, x_2, x_3) = (x_1 + 2x_2, 2x_1 + x_2, 3x_1)}$$

Q: Find a one-one L.T $T: R^4 \to R^7$

Ans: Let $B = \{\underset{\alpha_1}{(1,0,0,0)}, \underset{\alpha_2}{(0,1,0,0)}, \underset{\alpha_3}{(0,0,1,0)}, \underset{\alpha_4}{(0,0,0,1)}\}$

be basis for R^4

Let 4 L.I vectors from R^7 =

$\underset{\beta_1}{(1,0,0,0,0,0,0)}, \underset{\beta_2}{(0,1,0,0,0,0,0)}, \underset{\beta_3}{(0,0,1,0,0,0,0)}, \underset{\beta_4}{(0,0,0,1,0,0,0)}$

$\in R^7$

Let $T(\alpha_1) = \beta_1$

$T(\alpha_2) = \beta_2$

$T(\alpha_3) = \beta_3$

$T(\alpha_4) = \beta_4$

$$T(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, x_4, 0, 0, 0)$$

Representation of a Transformation By Matrix

Let $T: V \to W$ be a linear transformation,

$\dim(V) = n$, $\dim(W) = m$

Suppose $B_1 = \{\alpha_1, \alpha_2 \dots \alpha_n\}$ is an ordered basis for V

$B_2 = \{\beta_1, \beta_2 \dots \beta_m\}$ " " for W

$T(\alpha_j) \in W$ $[j = 1, 2 \dots, n]$

There exist unique scalars $a_{ij} \in F$ s.t

$$T(\alpha_j) = \sum_{i=1}^{i=n} a_{ij} \beta_i \quad -(1), \quad \text{i.e} \quad [T(\alpha_j)]_{B_2} = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{bmatrix}$$

Let $\alpha \in V$. There exist scalars $x_1, x_2 \dots, x_n$ s.t

$$\alpha = x_1\alpha_1 + x_2\alpha_2 + \dots x_n\alpha_n$$

$$= \sum_{j=1}^{n} x_j \alpha_j \quad \text{i.e} \quad [\alpha]_{B_1} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = X$$

$$\Rightarrow T(\alpha) = T\left(\sum_{j=1}^{n} x_j \alpha_j\right) = \sum_{j=1}^{n} x_j T(\alpha_j)$$

Using eq^n (1)

$$T(\alpha) = \sum_{j=1}^{n} x_j T(\alpha_j) = \sum_{j=1}^{n} x_j \left(\sum_{i=1}^{n} a_{ij} \beta_i\right)$$

$$= \sum_{j=1}^{n} \left(\sum_{i=1}^{n} a_{ij} x_j\right) \beta_i$$

If $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is the co-ordinate of α w.r.t B_1 $\{$i.e $[\alpha]_{B_1} = X\}$

Then AX is the co-ordinate of β w.r.t B_2 !

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$ is called <u>Matrix of T</u>

T_B

Example : $T: P_3 \to P_3$ | $P_3 = \{ a_0 + a_1 x + a_2 x^2 + a_3 x^3 \mid a_i \in \mathbb{R} \}$

$T(p(x)) = p'(x)$

Find the matrix of T w.r.t $B = \{1, x, x^2, x^3\}$

Here $B_1 = B_2 = B$

Ans. $[T]_{B:B} = A$; j^{th} column of $A = [T(\alpha_j)]_{B_2 = B}$

$T(1) = 0$

$T(x) = 1$

$T(x^2) = 2x$

$T(x^3) = 3x^2$

$A_1 = [T(1)]_B = [0]_B$

$A_2 = [T(x)]_B = [1]_B$

$A_3 = [T(x^2)]_B = [2x]_B$

$A_4 = [T(x^3)]_B = [3x^2]_B$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Q: Suppose $T: \mathbb{R}^3 \to \mathbb{R}^3$ is a L.T, defined as

$$T(x_1, x_2, x_3) = (2x_1, 2x_1 - 5x_2, 2x_2 + x_3)$$

Find $\underbrace{[T]_{B_1:B_2}}_{=A}$, $B_1 = \{(1,1,1), (1,1,0), (1,0,0)\}$

$B_2 = \{(1,0,1), (1,1,0), (0,0,1)\}$

Ans:

1^{st} Column of A $= [T(1,1,1)]_{B_2}$

2^{nd} ———— $= [T(1,1,0)]_{B_2}$

3^{rd} ———— $= [T(1,0,0)]_{B_2}$

$T(1,1,1) = (2, -3, 3)$
$T(1,1,0) = (2, -3, 2)$
$T(1,0,0) = (2, 2, 0)$

• $(2,-3,3) = c_1(1,0,1) + c_2(1,1,0) + c_3(0,0,1)$

$= (c_1 + c_2, c_2, c_1 + c_3)$

$\boxed{\begin{aligned} c_1 &= 5 \\ c_2 &= -3 \\ c_3 &= -2 \end{aligned}}$

1^{st} Column $= \begin{bmatrix} 5 \\ -3 \\ 2 \end{bmatrix}$

• $(2,-3,2) = (c_1 + c_2, c_2, c_1 + c_3)$

$2^{nd} = \boxed{\begin{aligned} c_1 &= 5 \\ c_2 &= -3 \\ c_3 &= -3 \end{aligned}}$

• $(2,2,0) = (c_1 + c_2, c_2, c_1 + c_3)$

$3^{rd} = \boxed{\begin{aligned} c_1 &= 0 \\ c_2 &= 2 \\ c_3 &= 0 \end{aligned}}$

$$\therefore \quad A = \boxed{\begin{bmatrix} 5 & 5 & 0 \\ -3 & -3 & 2 \\ -2 & -3 & 0 \end{bmatrix}}$$

Q: Find a linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$, whose matrix representation is

$$[T]_B = \begin{bmatrix} 2 & 0 & 0 \\ 2 & -5 & 0 \\ 0 & 2 & 1 \end{bmatrix} \quad \text{w.r.t std. basis.}$$

Ans:

Std. basis of $\mathbb{R}^3$ = $((1,0,0), (0,1,0), (0,0,1))$

Given: $[T(1,0,0)]_B = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$ $\quad [T(0,1,0)]_B = \begin{bmatrix} 0 \\ -5 \\ 2 \end{bmatrix}$

$[T(0,0,1)]_B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\Rightarrow$

- $T(1,0,0) = 2(1,0,0) + 2(0,1,0) + 0(0,0,1)$

$\Rightarrow \boxed{T(1,0,0) = (2,2,0)}$

- $T(0,1,0) = 0(1,0,0) + (-5)(0,1,0) + 2(0,0,1)$

$\Rightarrow \boxed{T(0,1,0) = (0,-5,2)}$

- $T(0,0,1) = 0(1,0,0) + 0(0,1,0) + 1(0,0,1)$

$\Rightarrow \boxed{T(0,0,1) = (0,0,1)}$

$(x_1, x_2\ x_3) \in R^3$

$(x_1, x_2, x_3) = x_1(1,0,0) + x_2(0,1,0) + x_3(0,0,1)$

$\Rightarrow T(x_1, x_2, x_3) = x_1[T(1,0,0)] + x_2[T(0,1,0)] + x_3[T(0,0,1)]$

$= x_1(2,2,0) + x_2(0,-5,2) + x_3(0,0,1)$

$$\boxed{\therefore T(x_1, x_2, x_3) = (2x_1,\ 2x_1 - 5x_2,\ 2x_2 + x_3)}$$

Q: Find the L.T $T: P_2 \to R^3$ whose matrix representation is :

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 1 & 1 \\ 5 & 4 & 1 & -1 \end{bmatrix}_{3\times 4}$$

$$B_1 = \{ 1,\ 1+x^2,\ x+x^3,\ 1+x+x^2 \}$$

$$B_2 = \{ (1,0,1),\ (2,4,5),\ (0,0,1) \}$$

Ans:

- $[T(1)]_{B_2} = \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix}$

$\Rightarrow$

$$T(1) = 1(1,0,1) + 0(2,4,5) + 5(0,0,1)$$

$$T(1) = (1,0,6)$$

- $[T(1+x^2)]_{B_2} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$

$\Rightarrow$ $T(1+x^2) = 2(1,0,1) + 1(2,4,5) + 4(0,0,1)$

$$= (4,4,11)$$

- $T(x+x^3) = 3(1,0,1) + 1(2,4,5) + 1(0,0,1)$

$$= (5,4,9)$$

$\cdot)\quad T(1+x+x^2) = 0(1,0,1) + 1(2,4,5) + (-1)(0,0,1)$

$= (2,4,4)$

$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \in P_3$

$(a_0 + a_1 x + a_2 x^2 + a_3 x^3) = c_1(1) + c_2(1+x^2) + c_3(x+x^3)$
$+ c_4(1+x+x^2)$

$= 1(c_1 + c_2 + c_4) + x(c_3 + c_4)$

$+ x^2(c_2 + c_4) + x^3(c_3)$

$\Rightarrow\quad c_1 + c_2 + c_4 = a_0$

$c_3 + c_4 = a_1$

$c_2 + c_4 = a_2$

$c_3 = a_3$

$$\boxed{\begin{aligned} c_3 &= a_3 \\ c_4 &= a_1 - a_3 \\ c_2 &= a_2 - a_1 + a_3 \\ c_1 &= a_0 - a_2 \end{aligned}}$$

$\Rightarrow T(p(x)) = (a_0 - a_2)[T(1)]$

$+ (a_2 - a_1 + a_3)[T(1+x^2)]$

$+ (a_3)[T(x+x^3)]$

$+ (a_1 - a_3)[T(1+x+x^2)]$

$= (a_0 - a_2)[1,0,6] + (a_2 - a_1 + a_3)[4,4,11] + a_3[5,4,9]$

$+ (a_1 - a_3)\left[2, 4, 4\right]$

$$= \Big[\big[(a_0 - a_2)1 + 4(a_2 - a_1 + a_3) + 5(a_3) + 2(a_1 - a_3)\big],$$

$$\big[(a_0 - a_2)0 + 4(a_2 - a_1 + a_3) + 4(a_3) + 4(a_1 - a_3)\big],$$

$$\big[(a_0 - a_2)6 + 11(a_2 - a_1 + a_3) + 9(a_3) + 4(a_1 - a_3)\big]\Big]$$

$$= \Big[(a_0 + 3a_2 - 2a_1 + 7a_3),$$

$$(4a_2 + 4a_3),$$

$$(6a_0 + 5a_2 - 7a_1 + 10a_3)\Big]$$

Q: $V = \{ [a_{ij}]_{3 \times 3} \mid a_{ij} \in$

$AX = B$

Eg:

$$2x_1 + x_2 + x_3 = 5$$
$$4x_1 - 6x_2 \qquad = 2$$
$$-2x_1 + 7x_2 + 2x_3 = 9$$

$$\Rightarrow \underbrace{\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 5 \\ 2 \\ 9 \end{bmatrix}}_{B}$$

$$\Rightarrow \quad X = A^{-1}B$$

Gauss Elimination $[A|B]$

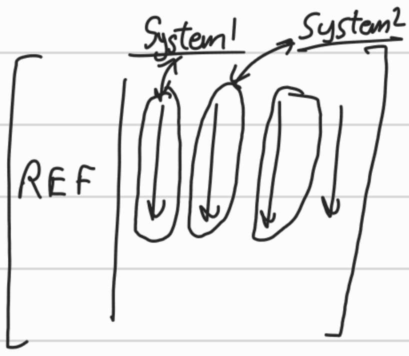
•) $$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 4 & -6 & 0 & -2 \\ -2 & 7 & 2 & 9 \end{array}\right] = \left[\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -12 \\ 0 & 8 & 3 & 14 \end{array}\right] = \left[\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -12 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

→ Back Substitution :

$$x_3 = 2 \qquad \Rightarrow \boxed{x_3 = 2}$$
$$-8x_2 - 2x_3 = -12 \qquad \Rightarrow \boxed{x_2 = 1}$$
$$2x_1 + x_2 + x_3 = 5 \qquad \Rightarrow \boxed{x_1 = 1}$$

$$x_1 = 1, \quad x_2 = 1, \quad x_3 = 2, \quad \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

•) $$\left[\begin{array}{c|ccc} & a_1 & a_2 & \cdots \\ A & b_1 & b_2 & \\ & \vdots & & \\ & \vdots & \cdots & \cdots \end{array}\right] \rightarrow$$



① Matrix Multiplication : Usual / Row - Column / Dot-product / Inner product way

Eg : $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix} = \begin{bmatrix} 1 & 2 \end{bmatrix}\begin{bmatrix} 5 \\ 7 \end{bmatrix} + \begin{bmatrix} 3 & 4 \end{bmatrix}\begin{bmatrix} 6 \\ 8 \end{bmatrix}$

② $AB = C \Rightarrow$ Columns of C are Linear Combination of columns of A !

Eg: $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \left[\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 \\ 7 \end{bmatrix} \middle| \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 6 \\ 8 \end{bmatrix} \right]$

$\underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}}_{A}\begin{bmatrix} 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 19 \\ 43 \end{bmatrix} = \underbrace{5\begin{bmatrix} 1 \\ 3 \end{bmatrix} + 7\begin{bmatrix} 2 \\ 4 \end{bmatrix}}_{\text{Part of Column Space of } A.}$

$\Rightarrow$ Sol^n of $AX = B$ exists if & only if :

•) B = Part of Column Space of A !

or •) Rank (A) = Rank (A|B)

$\Rightarrow$ Sol^n is unique if exists $\Leftrightarrow$

Rank (A) = # Variables in X

Sol^n exists uniquely

Rank (A) = Rank(A|B) = # Variables in X

(If A is square $\equiv A^2$.)

③ $AB = C$

Rows of C are L.C of rows of B

Eg: $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix}\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \\ \hline \begin{bmatrix} 3 & 4 \end{bmatrix}\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \end{bmatrix} =$

$$\begin{bmatrix} 1\begin{bmatrix}5 & 6\end{bmatrix} + 2\begin{bmatrix}7 & 8\end{bmatrix} \\ \hline 3\begin{bmatrix}5 & 6\end{bmatrix} + 4\begin{bmatrix}7 & 8\end{bmatrix} \end{bmatrix}$$

(4) Column-Row multiplication = Outer product Way

(= Opposite of 1^{st} way)

$$\begin{bmatrix}1 & 2 \\ 3 & 4\end{bmatrix}\begin{bmatrix}5 & 6 \\ 7 & 8\end{bmatrix} = \begin{bmatrix}1 \\ 3\end{bmatrix}\begin{bmatrix}5 & 6\end{bmatrix} + \begin{bmatrix}2 \\ 4\end{bmatrix}\begin{bmatrix}7 & 8\end{bmatrix}$$