MA101, Real Analysis
Riemann Integration

1. Show that $[x]$ is integrable on $[0, 3]$ and find $\int_0^2 [x]dx$.
2. Let $I = [a, b]$ be a closed and bounded interval and let P_1 and P_2 be partitions of I. Show that for any bounded function $f : [a, b] \to R$, we have
   (a) $L(P_1, f) \leq L(P_2, f)$ if $P_1 \leq P_2$.
   (b) $U(P_1, f) \geq U(P_2, f)$ if $P_1 \leq P_2$.
   (c) $L(P_1, f) \leq U(P_2, f)$ even if P_1 and P_2 are not comparable.
3. Prove that, a bounded function f on $[a, b]$ is Riemann integrable if and only if it is [Darboux] integrable, in which case the values of the integrals agree.
4. Prove that if f is integrable on $[a, b]$, then $|f|$ is integrable on $[a, b]$ and
$$\left| \int_a^b f(x)dx \right| \leq \int_a^b |f|dx.$$
5. Let $f : [-1, 1] \to \mathbb{R}$ be defined by $f(x) = 2x \sin \frac{1}{x^2} - \frac{2}{x} \cos \frac{1}{x^2}$ for $x \neq 0,\ f(0) = 0$. Show that $F' = f$ where $F(x) = x^2 \sin \frac{1}{x^2}$ for $x \neq 0$ and $F(0) = 0$ but $\int\limits_{-1}^{1} F'(t)\ dt$ does not exist.
6. Let f be continuous on $\mathbb{R}$ and $\alpha \neq 0$. If $g(x) = \frac{1}{\alpha} \int\limits_0^x f(t) \sin \alpha\ (x-t)\ dt$. Show that $f(x) = g''(x) + \alpha^2 g(x)$
7. Let $f : [0, 1] \to \mathbb{R}$ be continuous function such that $\int\limits_0^1 f(x)\ dx = 1$. Show that $\exists$ a point $c \in (0, 1)$ such that $f(c) = 3c^2$.
8. Let $f : [0, 1] \to (0, 1)$ be a continuous function. Show that the equation $2x - \int\limits_0^x f(t)\ dt = 1$ has exactly one solution in $(0, 1)$.
9. Let $f : [0, \frac{\pi}{4}] \to \mathbb{R}$ be continuous function. Show that $\exists c \in [0, \frac{\pi}{4}]$ such that $2 \cos 2c \int\limits_0^{\frac{\pi}{4}} f(t)\ dt = f(c)$.
10. Let $f : [a, b] \to \mathbb{R}$ be continuous and $\int\limits_a^b f(t)\ dt = \int\limits_x^b f(t)\ dt,\ x \in [a, b]$. Show that $f(x) = 0\ \forall x \in [a, b]$.
11. *Integration by parts :* Let $f, g : [a, b] \to \mathbb{R}$ be such that f' & g' are continuous on $[a, b]$, show that $\int\limits_a^b f(x)g'(x)\ dx = f(b)g(b) - f(a)g(a)$.
12. Let $f : [1, \infty) \to \mathbb{R}$ be defined by $f(x) = \int\limits_1^x \frac{\ln t}{1+t}\ dt$. Solve the equation $f(x) + f(\frac{1}{x}) = 2$.
13. Let $f(x) = x$ for rational x and $f(x) = 0$ for irrational x.

(a) Calculate the upper sum $U(f, P)$ and $L(f, P)$ where P is a partition on $[0, b]$.

(b) Is f-integrable on $[0, b]$?

14. Let f be continuous on $\mathbb{R}$ and define

$$G(x) = \int_0^{\sin x} f(t)dt \ for \ x \in \mathbb{R}.$$

Show that G is differentiable on $\mathbb{R}$ and compute G'.