MA 101

Solution for Assignment 4

Question 1. Find the value of α such that

$$\lim_{x\to -1} \frac{2x^2 - \alpha x - 14}{x^2 - 2x - 3}$$

exists. Find the limit.
sol.

$$x^2 - 2x - 3 = (x+1)(x-3)$$

So the denominator contains exactly one factor $(x+1)$
So in order that $\dfrac{2x^2 - \alpha x - 14}{x^2 - 2x - 3}$ has a limit as $x \to -1$, the only requirement is that:
$2x^2 - \alpha x - 14$ is divisible by $(x+1)$
Let $f(x) = 2x^2 - \alpha x - 14$ This is divisible by $(x+1)$ if and only if $f(-1) = 0$ Substituting $x = -1$ we have:

$$\begin{aligned} f(-1) &= 2(-1)^2 - \alpha(-1) - 14 \\ &= 2 + \alpha - 14 \\ &= \alpha - 12 \end{aligned}$$

So we require $\alpha = 12$
With this value of α :

$$f(x) = 2x^2 - 12x - 14 = 2\left(x^2 - 6x + 7\right) = 2(x+1)(x-7)$$
$$\frac{2x^2 - 12x + 14}{x^2 - 2x - 3} = \frac{2(x+1)(x-7)}{(x+1)(x-3)} = \frac{2(x-7)}{x-3}$$

So:

$$\lim_{x\to -1} \frac{2x^2 - 12x + 14}{x^2 - 2x - 3} = \frac{2(x+1)(x-7)}{(x+1)(x-3)} = \frac{2(x-7)}{x-3} = 4$$

Question 2. Let $\lim_{x\to 0} \frac{f(x)}{x^2} = 5$. Show that $\lim_{x\to 0} \frac{f(x)}{x} = 0$.
sol. we know

$$\begin{aligned} \lim_{x\to x_0} F(x) &= A \\ \lim_{x\to x_0} G(x) &= B \\ \lim_{x\to x_0} F(x) \lim_{x\to x_0} G(x) &= A \cdot B \\ \lim_{x\to x_0} F(x)G(x) &= AB. \end{aligned}$$

Let $F(x) = \frac{f(x)}{x^2}$ and $G(x) = x$

$$\lim_{x\to 0} \frac{f(x)}{x^2} x = 5.0$$
$$\implies \lim_{x\to 0} \frac{f(x)}{x} = 0.$$

Question 3. Let $f(x) = x$ if $x \in Q$ and $f(x) = 0$ if $x \in R\backslash Q$. Show that f is continuous at $x_o = 0$. Also show that it is discontinuous at any other point.
Solution: Let x be any real number. For each $n \in \mathrm{N}, \exists$ a rational number a_n and an irrational number b_n such that

$$\begin{aligned} & x - \frac{1}{n} < a_n < x + \frac{1}{n} \text{ and } x - \frac{1}{n} < b_n < x + \frac{1}{n} \\ \Rightarrow \quad & |a_n - x| < \frac{1}{n} \text{ and } |b_n - x| < \frac{1}{n}\ \forall n \\ \Rightarrow \quad & \lim_{n\to\infty} a_n = x = \lim_{n\to\infty} b_n \end{aligned}$$

If f is continuous at x, we must have

$$\begin{aligned} \lim_{n\to\infty} f(a_n) &= f(x) = \lim_{n\to\infty} f(b_n) \\ \text{But} \quad f(a_n) &= a_n \text{ and } f(b_n) = 0 \\ \therefore \quad \lim_{n\to\infty} a_n &= f(x) = \lim_{n\to\infty} 0 \\ &\Rightarrow x = f(x) = 0 \\ &\Rightarrow x = 0 \end{aligned}$$

Thus 0 is the only possible point of continuity.
Now, we shall show that f is actually continuous at 0.
Let $\varepsilon > 0$ be given. Also $f(0) = 0$. For a rational number x, we have

$$|f(x) - f(0)| = |x - 0| = |x|$$

For an irrational number x, we have $|f(x) - f(0)| = |0 - 0| = |0|$
In either case, $|f(x) - f(0)| = |x| < \varepsilon$ whenever $|x| < \varepsilon$ Choose $\delta = \varepsilon$, then $|f(x) - f(0)| < \varepsilon$ whenever $|x| < \delta$
$\Rightarrow f$ is continuous at 0.

Question 4. Let $f(x) = 1$ if $x \in Q$ and $f(x) = -1$ if $x \in R\backslash Q$. Show that f is discontinuous at every point.
Solution: Let x be any real number, then either x is rational or x is irrational.
Case (i) When x is a rational number.
Since in any interval there lie infinitely many rationals as well as infinitely many irrationals, therefore, for each $n \in \mathrm{N}, \exists$ an irrational number x_n such that

$$x - \frac{1}{n} < x_n < x + \frac{1}{n} \quad \Rightarrow \quad |x_n - x| < \frac{1}{n} \quad \forall n$$

$\Rightarrow$ The sequence $\langle x_n \rangle$ converges to x.
But $f(x_n) = -1$ for all n and $f(x) = 1$, so that $\lim_{n\to\infty} f(x_n) = -1 \neq f(x)$

$\therefore f$ is discontinuous at x, any rational number.

Case (ii) When x is an irrational number.
Since in any interval there lie infinitely many rationals as well as infinitely many irrationals, therefore, for each $n \in \mathrm{N}, \exists$ a rational number x_n such that

$$x-\frac{1}{n}<x_n<x+\frac{1}{n} \quad \Rightarrow \quad |x_n-x|<\frac{1}{n} \quad \forall n$$

$\Rightarrow$ The sequence $< x_n >$ converges to x.
But $f(x_n)=1$ for all n and $f(x)=-1$, so that $\lim_{n\to\infty} f(x_n)=1\neq f(x)$
$\therefore f$ is discontinuous at x, any irrational number.
Hence f is discontinuous for every real x.

Question 5. Give an example of a bounded function on $[-1,1]$ which does not have a maximum or a minimum.
Solution:

$$f(x)=\left\{\begin{array}{ll} x & x\in(-1,1) \\ 0 & x=1,-1 \end{array}\right\}$$

This function is bounded on $[-1,1]$ does not have a maximum or a minimum.
Question 6. Let $f:\mathbb{R}\to\mathbb{R}$ satisfy $f(x+y)=f(x)+f(y)$ for all $x,y\in R$. If f is continuous at 0 then show that f is continuous at every point $c\in R$.
Solution: Since f is continuous at 0, $\lim\limits_{h\to 0} f(0+h)=f(0)$.
This gives $\lim\limits_{h\to 0}[f(0)+f(h)]=f(0)$, i.e., $\lim\limits_{h\to 0} f(h)=0$.
Let $c\in\mathbb{R}$.
Then $\lim\limits_{h\to 0} f(c+h)=\lim\limits_{h\to 0}[f(c)+f(h)]=f(c)+\lim\limits_{h\to 0} f(h)=f(c)$.
This proves that f is continuous at c. Since c is arbitrary, f is continuous at $c\in\mathbb{R}$.
Question 7. Let $f:\mathbb{R}\to\mathbb{R}$ be such that for every $x,y\in R, |f(x)-f(y)|\leq|x-y|$. Show that f is continuous for all $x\in R$.
Solution: Let x_0 be a real number and take $\epsilon>0$. Then for $\delta=\epsilon$,

$$|x-x_0|<\delta \implies |f(x)-f(x_0)|<|x-x_0|<\delta=\epsilon$$

Implies $f(x)$ is continuous at $x=x_0$. Since x_0 is arbitrary, $f(x)$ is continuous on $\mathbb{R}$.

Question 8. Use properties of limit to evaluate $\lim_{x\to 0}\left(\frac{\sin x}{\sqrt{1-\cos x}}\right)$.
Solution:

$$\begin{aligned}
\lim_{x\to 0}\left(\frac{\sin x}{\sqrt{1-\cos x}}\right) &= \lim_{x\to 0}\left(\frac{\sin x\sqrt{1+\cos x}}{\sqrt{1-\cos x}\sqrt{1+\cos x}}\right) \\
&= \lim_{x\to 0}\left(\frac{\sin x\sqrt{1+\cos x}}{\sin x}\right) \\
&= \sqrt{2}.
\end{aligned}$$

Question 9. Use the definition to establish the continuity of the following functions: (i) $f(x)=x^2$ at $x=3, x\in[0,7]$ (ii) $f(x)=\frac{1}{x}$ at $x=1/2, x\in[0,1]$ (iii) $f(x)=\sqrt{x}, x\geq 0$

Solution : (i) Take $\epsilon > 0$. Now,

$$\begin{aligned} |f(x) - f(3)| &< \epsilon \\ if\ |x^2 - 9| &< \epsilon \\ i.e.\ if\ |x-3||x+3| &< \epsilon \\ i.e.\ if\ 10|x-3| &< \epsilon \qquad \text{(Since } |x+3| < 10\ \forall\ x \in [0,7] \text{)} \\ i.e.\ if\ |x-3| &< \frac{\epsilon}{10}. \end{aligned}$$

Denote $\delta = \frac{\epsilon}{10}$. Then we have

$$|f(x) - f(3)| < \epsilon \text{ whenever } |x-3| < \delta.$$

Therefore, the function $f(x)$ is continuous at $x = 3$.

(ii) Take $\epsilon > 0$. Now,

$$\begin{aligned} |f(x) - f(1/2)| &< \epsilon \\ if\ |1/x - 2| &< \epsilon \\ i.e.\ if\ \frac{2}{x}|x - 1/2| &< \epsilon \end{aligned}$$

Take $\delta_1 > 0$ such that $|x - \frac{1}{2}| < \delta_1$ and $0 < \frac{1}{2} - \delta_1 < x < \frac{1}{2} + \delta_1$. Then we have $\frac{2}{x} < \frac{2}{\frac{1}{2} - \delta_1} = \frac{4}{1 - 2\delta_1}$ and so

$$\begin{aligned} |f(x) - f(1/2)| &< \epsilon \\ if\ |x - 1/2| &< \frac{\epsilon(1 - 2\delta_1)}{4} \end{aligned}$$

Denote $\delta_2 = \frac{\epsilon(1-2\delta_1)}{4}$ and let $\delta = min\{\delta_1, \delta_2\}$. Hence

$$|f(x) - f(1/2)| < \epsilon \text{ whenever } |x - 1/2| < \delta.$$

Therefore, the function $f(x)$ is continuous at $x = 1/2$.

(iii) Take $\epsilon > 0$ and $c > 0$. We show that $f(x)$ is continuous at $x = c$. Now,

$$\begin{aligned} |f(x) - f(c)| &< \epsilon \\ if\ |\sqrt{x} - \sqrt{c}| &< \epsilon \\ i.e.\ if\ \left|\frac{x - c}{\sqrt{x} + \sqrt{c}}\right| &< \epsilon \\ i.e.\ if\ \frac{1}{\sqrt{c}}|x - c| &< \epsilon \\ i.e.\ if\ |x - c| &< \epsilon\sqrt{c}. \end{aligned}$$

Denote $\delta = \epsilon\sqrt{c}$. Then we have

$$|f(x) - f(c)| < \epsilon \text{ whenever } |x - c| < \delta.$$

Thus, the function $f(x)$ is continuous at $x = c$. Also,

$$\lim_{x \to 0} f(x) = 0 = f(0)$$

which implies that f is continuous at 0 and therefore f is continuous at all points $c \geq 0$.

Question 10. Let $f(x) = \frac{2^2+2-6}{x-2}, x \neq 2$. Define $f(x)$ in a way such that it becomes continuous at $x = 2$.

Solution : $f(x)$ is continuous at $x = 2$ if

$$\lim_{x \to 2^-} f(x) = f(2) = \lim_{x \to 2^+} f(x).$$

Now for $x \neq 2$, $f(x) = \frac{x^2+x-6}{x-2} = x + 3$ and so $\lim_{x \to 2^-} f(x) = 5 = \lim_{x \to 2^+} f(x)$. Thus if we define $f(x) = \frac{x^2+x-6}{x-2}$ for $x \neq 2$ and $f(x) = 5$ for $x = 2$ then $f(x)$ becomes continuous at $x = 2$.

Question 11 (a) Show that the functions $x^2, \frac{1}{x}, \frac{1}{x^2}, x > 0$ are continuous at any point $c \in \mathbb{R}$ but not uniformly.

Solution: Clearly, x^2 is a polynomial function and is continuous everwhere. The only points of discontinuity of the functions $\frac{1}{x}, \frac{1}{x^2}$ is the point 0. Thus, the functions $x^2, \frac{1}{x}, \frac{1}{x^2}, x > 0$ are continuous at any point $c \in \mathbb{R}$.

Let us consider $f(x) = x^2$. Let $\epsilon = 2$ and choose an arbitrary $\delta > 0$. Let n be a natural number such that $\frac{1}{n} < \delta$. Further, let $x = n + \frac{1}{n}$ and $y = n$. Then

$$|x - y| = \frac{1}{n} < \delta$$

while

$$|f(x) - f(y)| = (n + \frac{1}{n})^2 - n^2 = 2 + \frac{1}{n^2} > \delta$$

We conclude that $f(x) = x^2$ is not uniformly continuous on $[0, \infty)$. We know that, a uniformly continuous function maps a Cauchy sequence to another Cauchy sequence.

Now suppose $f : (0, \infty] \to \mathbb{R}$ given by $f(x) = \frac{1}{x}$ is uniformly continuous on $(0, \infty]$. Then, consider the Cauchy sequence $S_n = \frac{1}{n}$ (we know this sequence is Cauchy since all convergent sequences are Cauchy and S_n converges to 0).

The theorem we just proved would imply that $f(S_n) = n$ is a Cauchy sequence in real line. This is a contradiction since the sequence of natural numbers is clearly not a Cauchy sequence. This completes the proof.

To show $f(x) = \frac{1}{x^2}$ is not uniformly continuous on $(0, 1]$, we use the Sequential criterion for absence of Uniform Continuity. Let

$$a_n = \frac{1}{n}, b_n = \frac{1}{2n}$$

Then,

$$|a_n - b_n| = \frac{3}{4n^2} \to 0.$$

However,

$$|f(a_n) - f(b_n)| = |n^2 - 4n^2| = 3n^2 \geq 3.$$

Hence, f is not uniformly continuous on $(0, 1]$.

(b) Show that the function $x^2, x \in [-a, a], a > 0$ and functions $\frac{1}{x}, \frac{1}{x^2}, x \geq b > 0$ are uniformly continuous on respective domain.

Solution: Let $x, y \in [-a, a]$. Note that

$$|x^2 - y^2| \leq |x + y||x - y| \leq 2a|x - y|.$$

So for preassigned $\epsilon > 0$, one can choose $\delta < \frac{\epsilon}{2a}$, such that,

$$|x - y| < \delta = \frac{\epsilon}{2a} \implies |x^2 - y^2| \leq \epsilon$$

Thus, $f(x) = x^2$ is uniformly continuous on $[-a, a]$.

A function f is said to be a Lipschitz function on an interval I, if there exists a positive real number k such that

$$|f(x_2) - f(x_1)| \leq k|x_2 - x_1|$$

for all $x_, x_1 \in I$.

We know that if for an interval I, $f : I \to \mathbb{R}$ be a Lipschitz function on I, then f is uniformly continuous on I. Let $I = [b, \infty)$. Then,

$$|\frac{1}{x} - \frac{1}{y}| = |\frac{y - x}{xy}| = \frac{|x - y|}{|xy|} \leq \frac{1}{b^2}|x - y|$$

and,

$$|\frac{1}{x^2} - \frac{1}{y^2}| = |\frac{y^2 - x^2}{x^2y^2}| = \frac{|x^2 - y^2|}{|x^2y^2|} \leq \frac{1}{|xy^2 + yx^2|}|x - y| \leq \frac{1}{2b^3}|x - y|$$

(Since, $x \geq b \implies \frac{1}{x} \leq \frac{1}{b}$.) Thus, both $\frac{1}{x}$ and $\frac{1}{x^2}$ are Lipschitz functions on $[b, \infty)$, and hence uniformly continuous.

(c) Show that $\sin x, \cos x, |x|$ are continuous at every point $c \in \mathbb{R}$

Solution: Let $\epsilon > 0$ and $x, y \in \mathbb{R}$. Now,

$$\begin{aligned} &|f(x) - f(y)| && < \epsilon \\ &\implies |\sin x - \sin y| && < \epsilon \\ &\implies \left|2\cos\frac{x + y}{2}\sin\frac{x - y}{2}\right| && < \epsilon \end{aligned}$$

But,

$$\left|2\cos\frac{x + y}{2}\sin\frac{x - y}{2}\right| \leq \left|2\sin\frac{x - y}{2}\right|$$

It suffices to show that

$$\left|2\sin\frac{x - y}{2}\right| \leq \epsilon, \text{when } |x - y| < \delta,$$

for some $\delta > 0$.

Now,

$$|x - y| < \delta \implies \left|\frac{x - y}{2}\right| < \delta.$$

Now,

$$\left|2\sin\frac{x-y}{2}\right| \leq \left|2\frac{x-y}{2}\right| < 2\delta$$

Choosing $\delta = \frac{\epsilon}{2}$, we have:

$$|x-y| < \delta \implies |\sin x - \sin y| < \epsilon.$$

This proves that $\sin x$ is continuous on $\mathbb{R}$. Similarly, we have:

$$\begin{aligned}
|\cos x - \cos y| &= \left|2\sin\frac{x+y}{2}\sin\frac{x-y}{2}\right| \\
&\leq 2\left|\sin\frac{x-y}{2}\right| \\
&\leq 2\left|\frac{x-y}{2}\right| \\
&\leq \Big|x-y| \\
&< \epsilon
\end{aligned}$$

if we choose $\epsilon = \delta$. This proves that $\cos x$ is continuous on $\mathbb{R}$.
For the function $f(x) = |x|$, let $\epsilon > 0$ and c be any arbitrary real number. Then,

$$|x-c| < \epsilon$$

for all

$$|x-c| < \delta = \epsilon.$$

Thus, $|x|$ is continuous at c. Since c is an arbitrary real number, hence, $|x|$ is continuous at all reals.
Question 12. Let $f : |0,1| \to \mathbb{R}$ be a continuous function. Show that $\exists x_0 \in [0,1]$ such that $f(x_0) = \frac{1}{3}\left(f\left(\frac{1}{4}\right) + f\left(\frac{1}{2}\right) + f\left(\frac{3}{4}\right)\right)$.
Question 13 Let $p(y)$ be a polynomial
$p(y) = a_n y^n + a_{n-1}y^{n-1} + \ldots + a_1 y + a_0$.
Suppose n is even $(n \neq 0), a_n = 1, a_0 = -1$. Show that $p(y)$ has at least two real roots.
Solution 13:$lim_{y\to\infty}p(y)$ is positive. Since a_n is positive and n is even, therefore the leading term will dominate. Similarly $lim_{y\to-\infty}p(y)$ is also positive. Hence there are two piont a, b such that $a < 0 < b$ with $p(a) > 0$ and $p(b) > 0$. Hence by the intermediate value theorem, there must be at least one root in each of the disjoint intervals $(a, 0)$ and $(0, b)$ making for at least two distinct roots.
Question 14. Compute the limit $\lim_{x\to\infty}\left(x^2 - x^3\sin\left(\frac{1}{x}\right)\right)$.
Solution: $\lim_{x\to\infty} x^2 - x^3\sin\left(\frac{1}{x}\right)$ put $x = \frac{1}{y}$ then $x \to \infty \Rightarrow y \to 0$ on substituting we get

$$\begin{aligned}
&\lim_{y\to 0}\frac{1}{y^2} - \frac{1}{y^3}\sin y \\
&= \lim_{y\to 0}\frac{y - \sin y}{y^3}\left[\frac{0}{0}\text{ form }\right]
\end{aligned}$$

on applying L'Hospital Rule

$$\lim_{y\to 0}\frac{1+\cos y}{3y^2}\left[\frac{0}{0}\text{ form }\right]$$

on applying L'Hospital Rule

$$= \lim_{y \to 0} \frac{\sin y}{6y} \quad \left[\frac{0}{0} \text{ ferm} \right]$$

on applying l Hospital Rive

$$= \lim_{y \to 0} \frac{\cos y}{6} = \frac{1}{6}$$

Question 15. Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous functions such that $f(a) \neq g(a)$ for some $a \in \mathbb{R}$. Show that $\exists$ a $\delta > 0$ such that $f(x) \neq g(x), \forall x$ such that $|x - a| < \delta$.
Solution: Let $h(x) = f(x) - g(x)$ since f and g are continuous and hence h is also continuous. Also,

$$f(a) \neq g(a) \Rightarrow h(a) \neq 0$$

Now, without loss of generality let us assume that $h(a) > 0$ and since h is continuous at a so for $\epsilon = \frac{\hbar(a)}{2} \quad \exists \quad \delta > 0$ such that
$|h(x) - h(a)| < \epsilon$ whenever $|x - a| < \delta$
$\Rightarrow -\epsilon < h(x) - ha(a) < \epsilon$
$\Rightarrow \frac{-h(a)}{2} + h(a) < h(x) < \frac{h(a)}{2} + h(a)$
$\Rightarrow \quad \frac{h(a)}{2} < h(x) < \frac{3h(a)}{2}$
$\Rightarrow h(x) > \frac{h(a)}{2} > 0$ whenever $|x - a| < \delta$
$\Rightarrow h(x) \neq 0$ whenever $|x - a| < \delta$

Question 16. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function with $f(0) = -2, f(1) = 3$.
Let $S = \{x \in [0, 1] | f(x) = 0\}$
(a) Slow that S is non empty.
(b) Let β be the supremum of the set S. Show that $\beta \in (0, 1)$.
(c) Show that $f(\beta) = 0$.
Solution: Since f is a continuous function and $f(0) = -2$, $f(1) = 3$. Therefore, there exists some points for which $f(x) = 0$, which implies S is a non-empty set. (**(a)** Proved)
Since S is subset of $(0, 1)$ (By definition of f, it is clear that $\{0, 1\}$ does not belongs to S. Since S is bounded and non-empty. Therefore, supremum of set S, $\beta \in (0, 1)$. Moreover infimum of set S also. (**(b)** Proved)
Since S is collection of all x such that $f(x) = 0$ and $x \in [0, 1]$. But it is also given that $f(0, 1) \neq 0$. Since f is a continuous function, therefore pre-image of a closed set is also closed. Therefore, S is a closed set. Hence $\beta \in S$. Therefore, $f(\beta) = 0$. (**(b)** Proved)
Question 17. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous at $c \in \mathbb{R}$. Then $|f|$ is continuous at c. Give an example to show that the reverse is not true.
Solution:

$|f| : \mathbb{R} \to \mathbb{R}$ is defined by $|f|(x) = |f(x)|, x \in \mathbb{R}$.
$||f|(x) - |f|(c)| = ||f(x)| - |f(c)| \leq |f(x) - f(c)|$
Let us choose $\epsilon > 0$.
Since f is continuous at c, there exists a positive δ such that
$|f(x) - f(c)| < \epsilon$ for all $x \in N(c, \delta) \cap \mathbb{R}$.
Therefore $||f|(x) - |f|(c)| < \epsilon$ for all $x \in N(c, \delta) \cap \mathbb{R}$.
This shows that $|f|$ is continuous at c.

Example: let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1, x \in \mathbb{Q}, \\ -1, x \in \mathbb{R} - \mathbb{Q}. \end{cases}$$

Then $|f| : \mathbb{R} \to \mathbb{R}$ is defined by $|f|(x) = 1, x \in \mathbb{R}$. Here $|f|$ is continuous on $\mathbb{R}$ but f is not continuous on $\mathbb{R}$.

Question 18. A real function f is continuous on $[0, 2]$ and $f(0) = f(2)$. Prove that there exists at least a point c in $[0, 1]$ such that $f(c) = f(c + 1)$.

Solution: *Case 1:* $f(0) = f(1)$ then $c = 0, 1$.

Case 2: If $f(0) \neq f(1)$, consider g on [0,1] defined by $g(x) = f(x) - f(x + 1)$.

$g(0) = f(0) - f(1)$

$g(1) = f(1) - f(2) = f(1) - f(0)$ (Since $f(0) = f(2)$)

So g changes sign over the interval [0,1] and g is continuous because f is continuous. Now by the **intermediate value theorem** there exists a number $c \in [0, 1]$ such that $g(c) = 0$, so $f(c) - f(c + 1) = 0$ and we have $f(c) = f(c + 1)$

Question. (19) (i)Give an example of a function f which satisfies the initial value problem I.V.P on a closed and bounded interval $[a, b]$, but is not continuous on $[a, b]$

Solution: Consider an I.V.P

$$\frac{dy}{dx} = y^2, y(0) = 1.$$

$$\text{Take } f = y = \frac{1}{1 - x} \text{ in } [0, 1]$$

which is no continuous on [0.1] but satisfies above I.V.P.

Question. (19) (ii) Give an example of a function f which is monotone increasing on a closed and bounded interval $[a, b]$ but does not satisfy the I.V.P on $[a, b]$.

Solution:- Consider an I.V.P

$$\frac{df}{dx} = e^x, f(0) = 2.$$

Take $f(x) = e^x$, which is monotonic increasing on $[0, 2]$ but does not satisfy above I.V.P on$[0, 2]$.

Solution 19 If we consider I.V.P as Intermediate value property, then

Intermediate value property: For any function f that's continuous over the interval $[a, b]$, the function will take any value between $f(a)$ and $f(b)$ over the interval. It means that for any value M between $f(a)$ and $f(b)$ there ' x' a value ' c ' in $[a, b]$ for which $f(c) = M$

(19)(i)

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$$f : [-1, 1] \to [-1, 1]$$

Discontinuous at $x = 0$, but take every value between [$-1, 1]$

(19)(ii)

$$f(x) = \begin{cases} x, & 0 \leqslant x < 1 \\ 4, & x = 1 \end{cases}$$

Which is monotonic increasing and bounded but does not satisfies intermediate value property as there is no point $x \in [0, 1]$ S.t $f(x) = 3$ (where $0 \leq 3 \leq 4$)

Question 20. Let $f : [0, \pi] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 0 & \text{if } x = 0, \\ x \sin \frac{1}{x} - \frac{1}{x} \cos \frac{1}{x} & \text{if } x \neq 0. \end{cases}$$

Is f continuous?
Solution:Here we use below theorem
Theorem: If $f(x)$ is continous at x if and only if $x_n \to 0 \implies f(x_n) \to f(0)$
We choose $x_n = \frac{1}{2n\pi + \frac{\pi}{4}}$
Then $x_n \to o$
$f(x_n) = (\frac{1}{2n\pi+\frac{\pi}{4}}) \sin 2n\pi + \frac{\pi}{4} - (2n\pi + \frac{\pi}{4}) \cos 2n\pi + \frac{\pi}{4}$
$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty}(\frac{1}{2n\pi+\frac{\pi}{4}} \frac{1}{\sqrt{2}} - \frac{1}{2n\pi+\frac{\pi}{4}} \frac{1}{\sqrt{2}})$
$= 0 - \lim_{n\to\infty}(2n\pi + \frac{\pi}{4} \frac{1}{\sqrt{2}})$
$= \infty$
that is limit not exist
$\implies f(x)$ is not continous at $x = 0$.

Question 21. Let $f : \mathbb{R} \to (0, \infty)$, satisfy $f(x + y) = f(x)f(y) \forall x \in \mathbb{R}$. Suppose f is continuous at $x = 0$. Show that f is continuous at all $x \in \mathbb{R}$.

Solution: Since $f(x)$ is continuous at 0, $\lim\limits_{h\to 0} f(0 + h) = f(0)$.
This gives $\lim\limits_{h\to 0}[f(0) + f(h)] = f(0)$ that is $\lim\limits_{h\to 0} f(h) = 0$
Let $c \in \mathbb{R}$.
Then $\lim\limits_{h\to 0} f(c + h) = \lim\limits_{h\to 0}[f(c) + f(h)] = f(c) + \lim\limits_{h\to 0} f(h) = f(c)$
This proves that f is continuous at c. Since c is arbitrary, f is continuous at every c in $\mathbb{R}$.