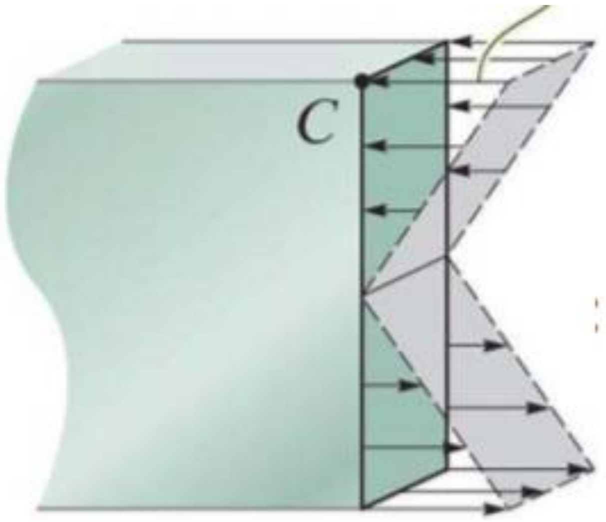
Engineering Mechanics (ME102)

Moments of Inertia

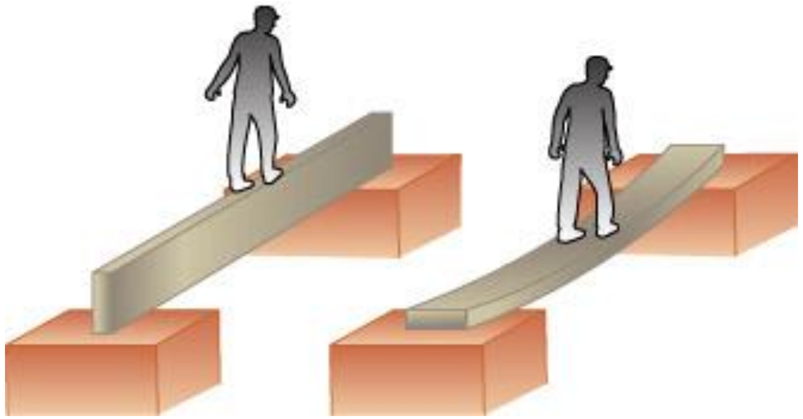
Introduction

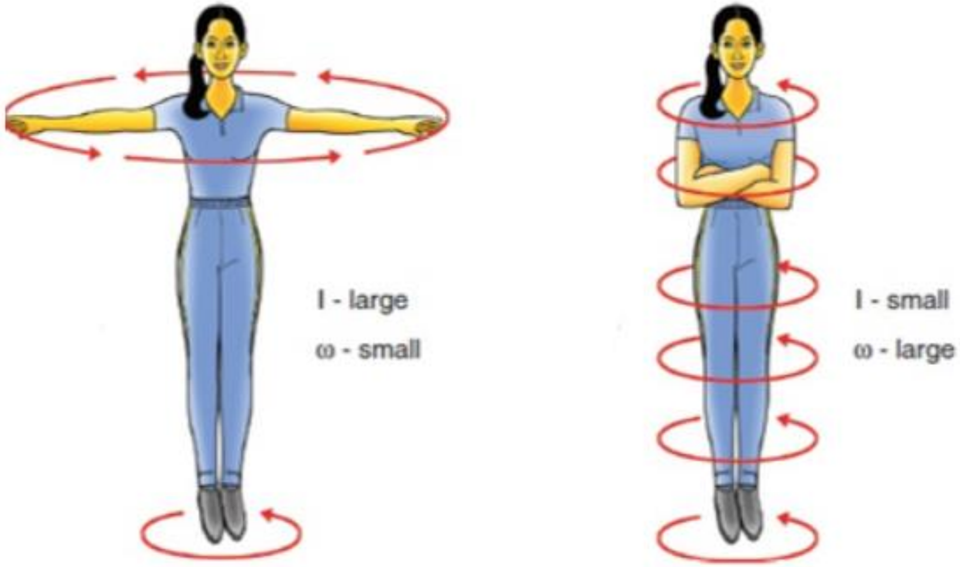
- Previously considered *distributed forces* which were *proportional to the area* over which they act.
  - The *resultant* was obtained by *summing or integrating over the areas*.
  - The *moment of the resultant* about any axis was determined by computing the *first moments of the areas* about that axis.

- We will now consider ***forces which are proportional to the area over which they act but also vary linearly with distance from a given axis***.
  - It will be shown that the *magnitude of the resultant* depends on the *first moment of the force distribution* with respect to the axis.
  - The *point of application of the resultant* depends on the *second moment of the distribution* with respect to the axis.

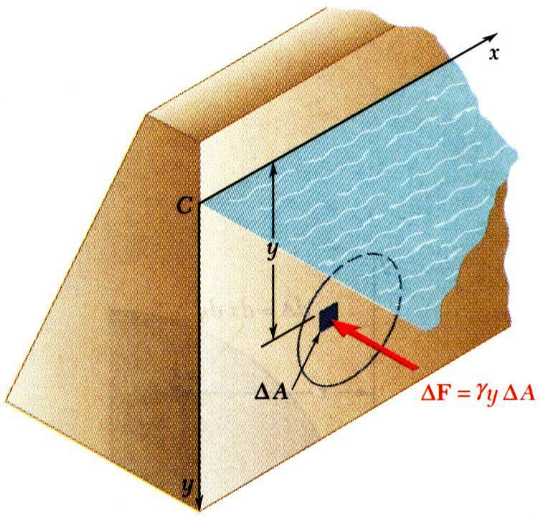


Introduction





Moment of Inertia of an Area



- Consider distributed forces $\Delta \mathbf{F}$ whose magnitudes are proportional to the elemental areas on ΔA which they act and also vary linearly with the distance of ΔA from a given axis.

- Example: Consider the net hydrostatic force on a submerged circular gate.

$$\Delta F = p\Delta A$$

The pressure, p, linearly increases with depth

$$p = \gamma y\text{, so}$$

$$\Delta F = \gamma y \Delta A\text{, and the resultant force is}$$

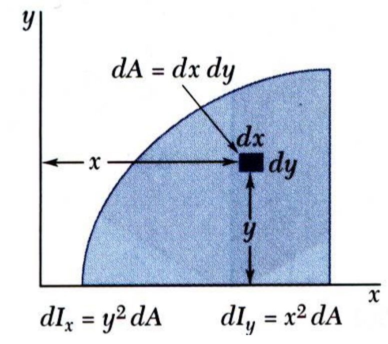
$$R = \sum_{\text{all } \Delta A} \Delta F = \gamma \int y\,dA\text{, while the moment produced is}$$

$$M_x = \gamma \int y^2 dA$$

- The integral $\int y\, dA$ is already familiar from our study of centroids.

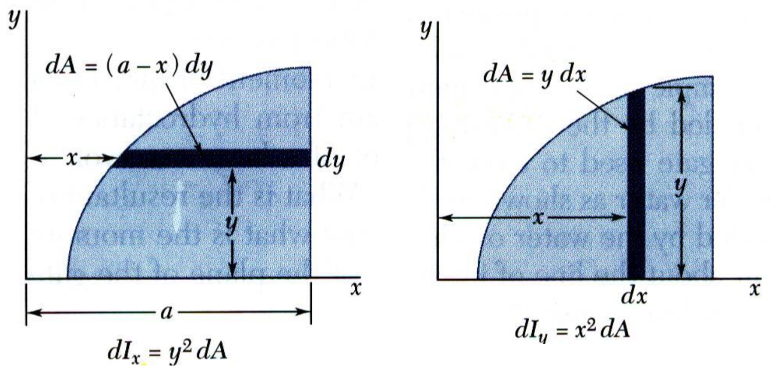
- The integral $\int y^2\, dA$ is one subject of this chapter, and is known as the *area moment of inertia*, or more precisely, the *second moment of the area.*

Moment of Inertia of an Area by Integration

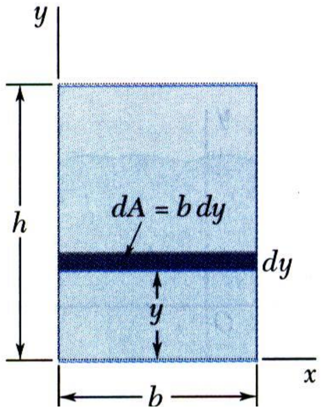


- *Second moments* or *moments of inertia* of an area with respect to the x and y axes,

$$I_x = \int y^2 dA \qquad I_y = \int x^2 dA$$



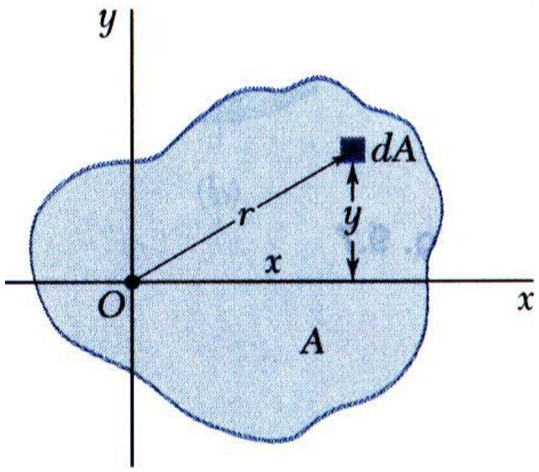
- Evaluation of the integrals is *simplified* by choosing dA to be a thin strip parallel to one of the coordinate axes.



- For a rectangular area,

$$I_x = \int y^2 dA = \int_0^h y^2 b dy = \tfrac{1}{3} b h^3$$

Polar Moment of Inertia



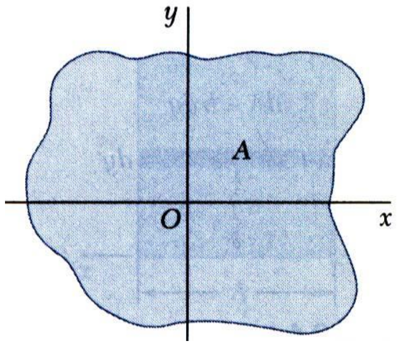
- The *polar moment of inertia* is an important parameter in problems involving torsion of cylindrical shafts and rotations of slabs.

$$J_0 = \int r^2 dA$$

- The polar moment of inertia is related to the rectangular moments of inertia,

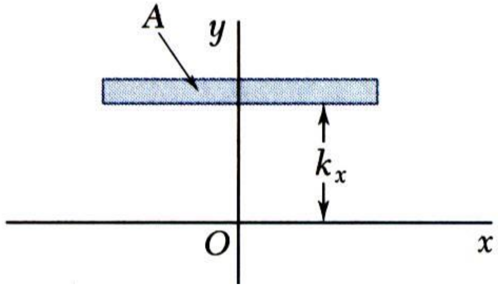
$$J_0 = \int r^2 dA = \int \left(x^2 + y^2\right) dA = \int x^2 dA + \int y^2 dA$$
$$= I_y + I_x$$

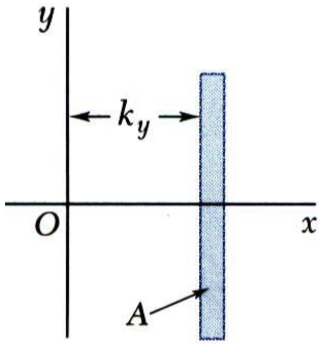
Radius of Gyration of an Area



- Consider area A with moment of inertia I_x. Imagine that the area is concentrated in a thin strip parallel to the x axis with equivalent I_x.

$$I_x = k_x^2 A \qquad k_x = \sqrt{\frac{I_x}{A}}$$





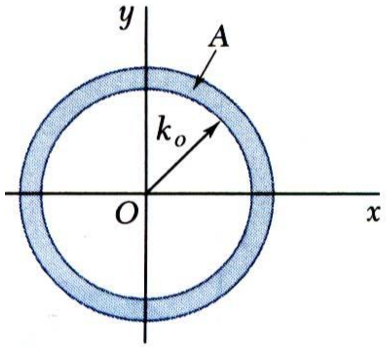
$k_x =$ *radius of gyration* with respect to the x axis

- Similarly,

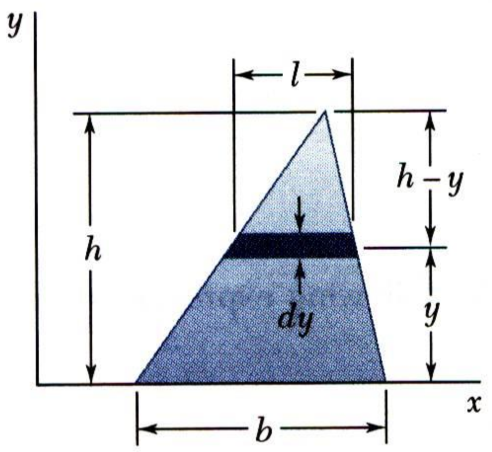
$$I_y = k_y^2 A \qquad k_y = \sqrt{\frac{I_y}{A}}$$

$$J_O = k_O^2 A \qquad k_O = \sqrt{\frac{J_O}{A}}$$

$$k_O^2 = k_x^2 + k_y^2$$



Sample Problems 9.1:



Determine the moment of inertia of a triangle with respect to its base.

SOLUTION:

- A differential strip parallel to the x axis is chosen for dA.

$$dI_x = y^2 dA \qquad dA = l\,dy$$

- For similar triangles,

$$\frac{l}{b} = \frac{h-y}{h} \qquad l = b\frac{h-y}{h} \qquad dA = b\frac{h-y}{h}dy$$

- Integrating dI_x from $y = 0$ to $y = h$,

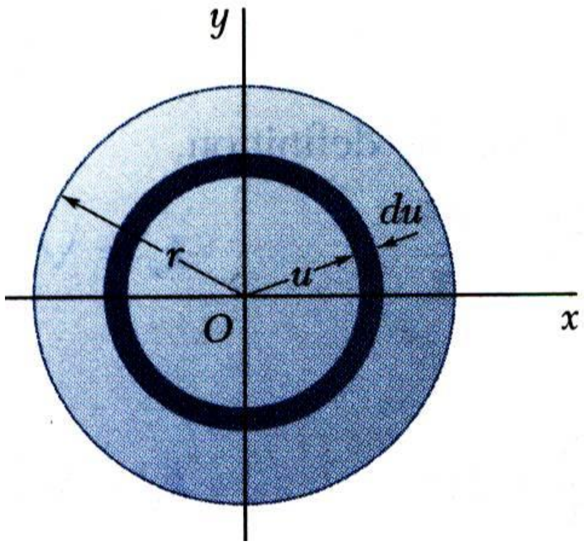
$$I_x = \int y^2 dA = \int_0^h y^2 b\frac{h-y}{h}dy = \frac{b}{h}\int_0^h \left(hy^2 - y^3\right)dy$$

$$= \frac{b}{h}\left[h\frac{y^3}{3} - \frac{y^4}{4}\right]_0^h$$

$$\boxed{I_x = \frac{bh^3}{12}}$$

Radius of Gyration of an Area

Sample Problems 9.2:



a) Determine the centroidal polar moment of inertia of a circular area by direct integration.

b) Using the result of part *a*, determine the moment of inertia of a circular area with respect to a diameter of the area.

SOLUTION:

- An annular differential area element is chosen,

$$dJ_O = u^2 dA \qquad dA = 2\pi u\, du$$

$$J_O = \int dJ_O = \int_0^r u^2 (2\pi u\, du) = 2\pi \int_0^r u^3 du$$

$$\boxed{J_O = \frac{\pi}{2} r^4}$$

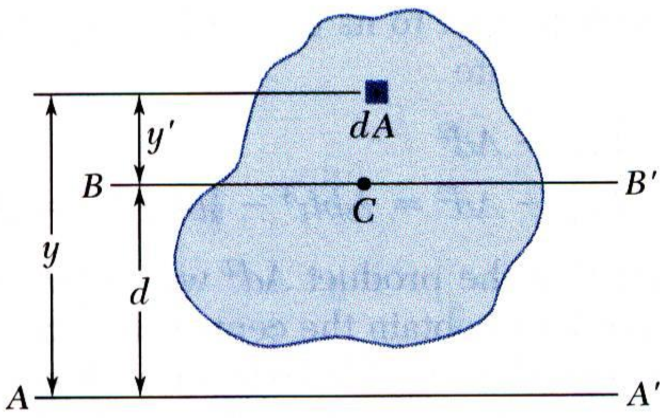
Why was an annular differential area chosen?

- From symmetry, $I_x = I_y$,

$$J_O = I_x + I_y = 2I_x \qquad \frac{\pi}{2} r^4 = 2I_x$$

$$\boxed{I_{diameter} = I_x = \frac{\pi}{4} r^4}$$

Parallel Axis Theorem



- Consider moment of inertia I of an area A with respect to the axis AA'

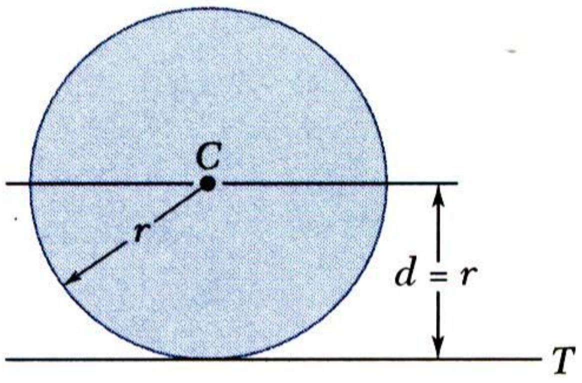
$$I = \int y^2 dA$$

- The axis BB' passes through the area centroid and is called a *centroidal axis.*

$$I = \int y^2 dA = \int (y' + d)^2 dA$$
$$= \int y'^2 dA + 2d \int y' dA + d^2 \int dA$$

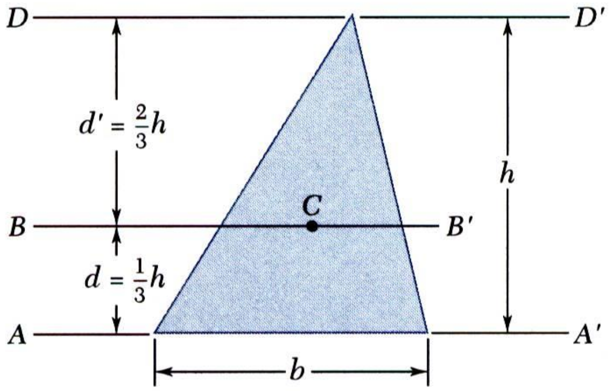
$$I = \bar{I} + Ad^2$$ *parallel axis theorem*

Parallel Axis Theorem



- Moment of inertia I_T of a circular area with respect to a tangent to the circle,

$$I_T = \bar{I} + Ad^2 = \tfrac{1}{4}\pi r^4 + \left(\pi r^2\right) r^2$$
$$= \tfrac{5}{4}\pi r^4$$



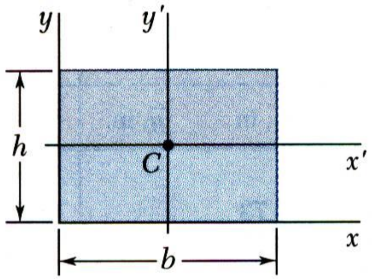
- Moment of inertia of a triangle with respect to a centroidal axis,

$$I_{AA'} = \bar{I}_{BB'} + Ad^2$$
$$\bar{I}_{BB'} = I_{AA'} - Ad^2 = \tfrac{1}{12} bh^3 - \tfrac{1}{2} bh\left(\tfrac{1}{3} h\right)^2$$
$$= \tfrac{1}{36} bh^3$$

Moment of Inertia of Composite Areas

- The moment of inertia of a composite area A about a given axis is obtained by adding the moments of inertia of the component areas A_1, A_2, A_3, ... , with respect to the same axis.

Rectangle



$\bar{I}_{x'} = \frac{1}{12}bh^3$

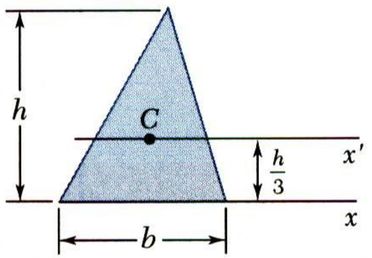
$\bar{I}_{y'} = \frac{1}{12}b^3h$

$I_x = \frac{1}{3}bh^3$

$I_y = \frac{1}{3}b^3h$

$J_C = \frac{1}{12}bh(b^2 + h^2)$

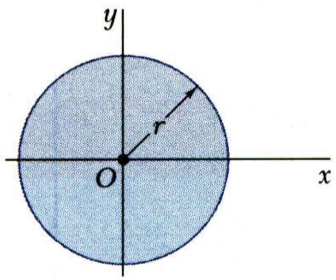
Triangle



$\bar{I}_{x'} = \frac{1}{36}bh^3$

$I_x = \frac{1}{12}bh^3$

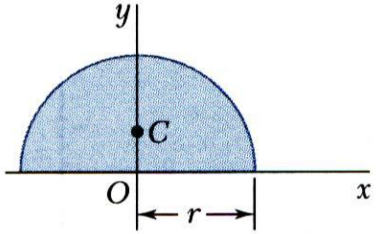
Circle



$\bar{I}_x = \bar{I}_y = \frac{1}{4}\pi r^4$

$J_O = \frac{1}{2}\pi r^4$

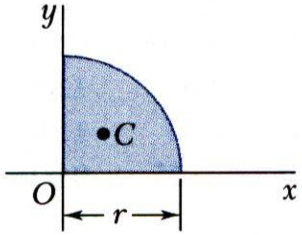
Semicircle



$I_x = I_y = \frac{1}{8}\pi r^4$

$J_O = \frac{1}{4}\pi r^4$

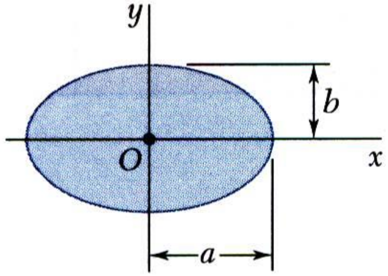
Quarter circle



$I_x = I_y = \frac{1}{16}\pi r^4$

$J_O = \frac{1}{8}\pi r^4$

Ellipse

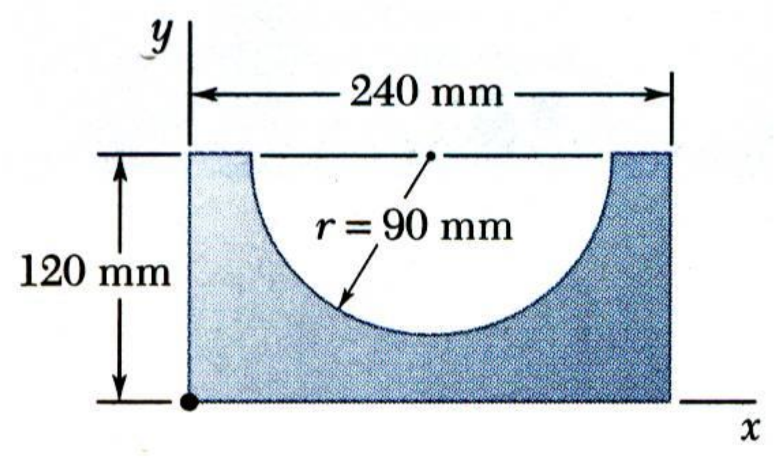


$\bar{I}_x = \frac{1}{4}\pi ab^3$

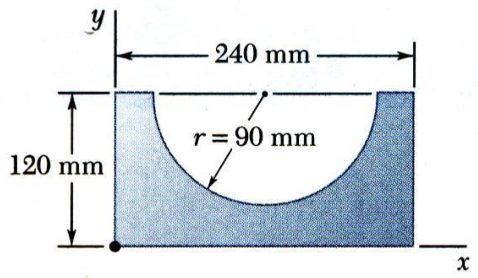
$\bar{I}_y = \frac{1}{4}\pi a^3b$

$J_O = \frac{1}{4}\pi ab(a^2 + b^2)$

Sample Problems 9.5:



Determine the moment of inertia of the shaded area with respect to the ***x*** axis.

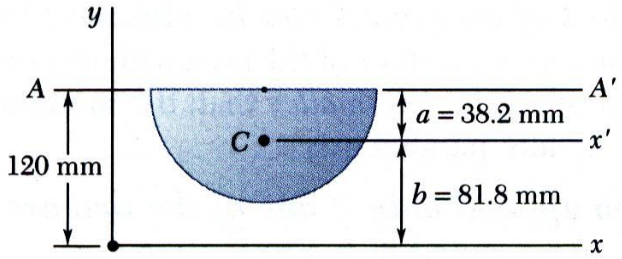


SOLUTION:

- Compute the moments of inertia of the bounding rectangle and half-circle with respect to the *x* axis.

Rectangle:

$$I_x = \tfrac{1}{3}bh^3 = \tfrac{1}{3}(240)(120)^3 = 138.2\times10^6\,\text{mm}^4$$



Half-circle:

Moment of inertia with respect to *AA'*,

$$I_{AA'} = \tfrac{1}{8}\pi r^4 = \tfrac{1}{8}\pi(90)^4 = 25.76\times10^6\,\text{mm}^4$$

$$a = \frac{4r}{3\pi} = \frac{(4)(90)}{3\pi} = 38.2\text{ mm}$$

$$\text{b} = 120 - \text{a} = 81.8\text{ mm}$$

$$A = \tfrac{1}{2}\pi r^2 = \tfrac{1}{2}\pi(90)^2$$
$$= 12.72\times10^3\,\text{mm}^2$$

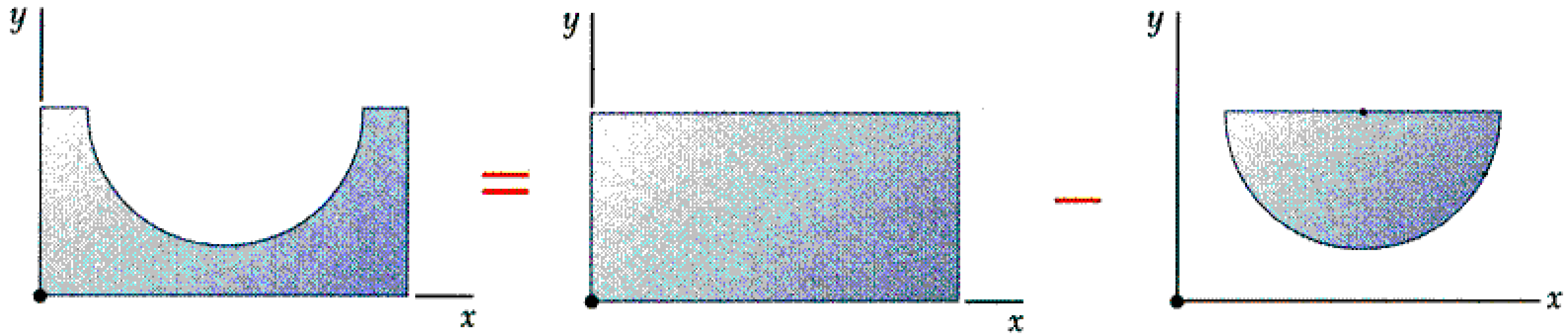
Moment of inertia with respect to *x'*,

$$\bar{I}_{x'} = I_{AA'} - Aa^2 = 25.76\times10^6 - \left(12.72\times10^3\right)(38.2)^2$$
$$= 7.20\times10^6\,\text{mm}^4$$

Moment of inertia with respect to *x*,

$$I_x = \bar{I}_{x'} + Ab^2 = 7.20\times10^6 + \left(12.72\times10^3\right)(81.8)^2$$
$$= 92.3\times10^6\,\text{mm}^4$$

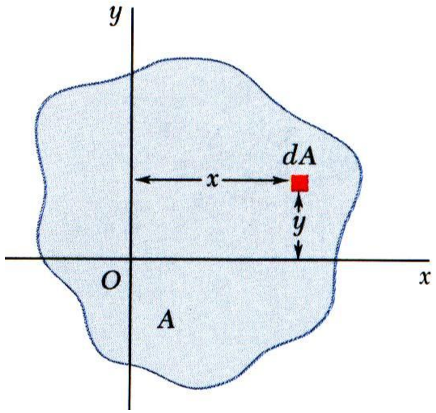
- The moment of inertia of the shaded area is obtained by subtracting the moment of inertia of the half-circle from the moment of inertia of the rectangle.



$$I_x \quad = \quad 138.2\times10^6\,\text{mm}^4 \quad - \quad 92.3\times10^6\,\text{mm}^4$$

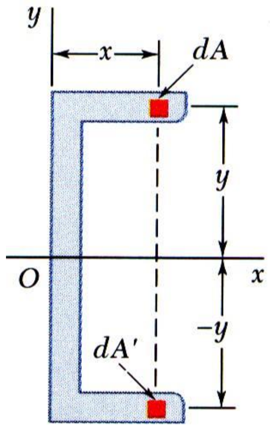
$$\boxed{I_x = 45.9\times10^6\,\text{mm}^4}$$

Product of Inertia

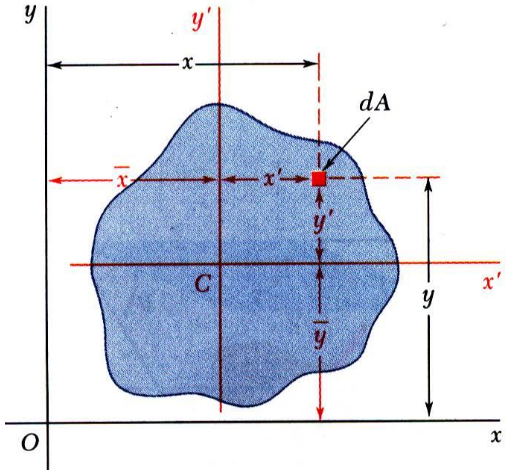


- *Product of Inertia:*

$$I_{xy} = \int xy\, dA$$



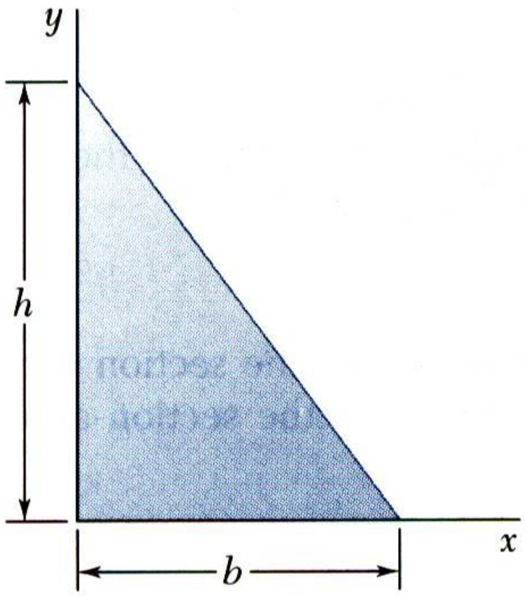
- When the x axis, the y axis, or both are an axis of symmetry, the product of inertia is zero.



- Parallel axis theorem for products of inertia:

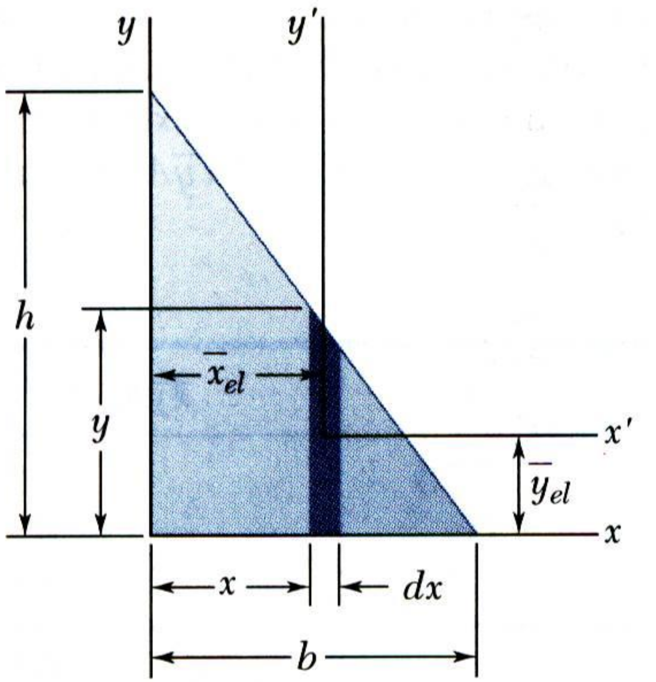
$$I_{x'y'} = \bar{I}_{xy} + \bar{x}\bar{y}A$$

Sample Problems 9.6:



Determine the product of inertia of the right triangle **(a)** with respect to the *x* and *y* axes and **(b)** with respect to centroidal axes parallel to the *x* and *y* axes.

Sample Problems 9.6:



SOLUTION:

- Determine the product of inertia using direct integration with the parallel axis theorem on vertical differential area strips

$$y = h\left(1 - \frac{x}{b}\right) \qquad dA = y\,dx = h\left(1 - \frac{x}{b}\right)dx$$

$$\bar{x}_{el} = x \qquad \bar{y}_{el} = \tfrac{1}{2}y = \tfrac{1}{2}h\left(1 - \frac{x}{b}\right)$$

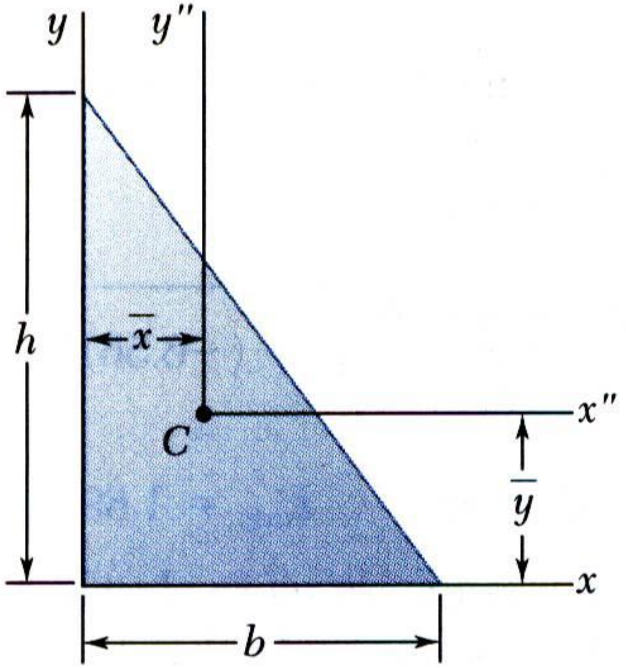
Integrating dI_x from $x = 0$ to $x = b$,

$$I_{xy} = \int dI_{xy} = \int \bar{x}_{el}\bar{y}_{el}\,dA = \int_0^b x\left(\tfrac{1}{2}\right)h^2\left(1 - \frac{x}{b}\right)^2 dx$$

$$= h^2\int_0^b\left(\frac{x}{2} - \frac{x^2}{b} + \frac{x^3}{2b^2}\right)dx = h^2\left[\frac{x^2}{4} - \frac{x^3}{3b} + \frac{x^4}{8b^2}\right]_0^b$$

$$\boxed{I_{xy} = \tfrac{1}{24}b^2h^2}$$

Sample Problems 9.6:



- Apply the parallel axis theorem to evaluate the product of inertia with respect to the centroidal axes.

$$\bar{x} = \tfrac{1}{3} b \qquad \bar{y} = \tfrac{1}{3} h$$

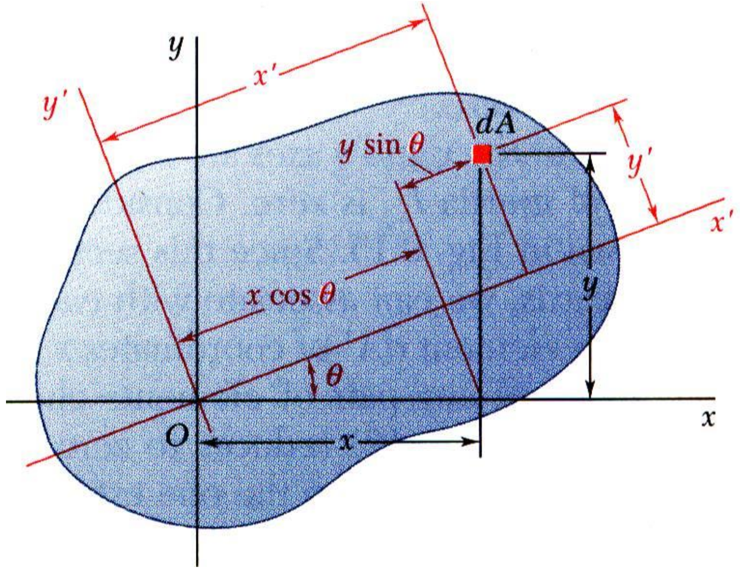
With the results from part *a*,

$$I_{xy} = \bar{I}_{x''y''} + \bar{x}\bar{y}A$$

$$\bar{I}_{x''y''} = \tfrac{1}{24} b^2 h^2 - \left(\tfrac{1}{3} b\right)\left(\tfrac{1}{3} h\right)\left(\tfrac{1}{2} bh\right)$$

$$\boxed{\bar{I}_{x''y''} = -\tfrac{1}{72} b^2 h^2}$$

Principal Axes and Principal Moments of Inertia



Given $I_x = \int y^2 dA \quad I_y = \int x^2 dA$

$$I_{xy} = \int xy\, dA$$

we wish to determine moments and product of inertia with respect to new axes x' and y'.

Note: $x' = x\cos\theta + y\sin\theta$

$$y' = y\cos\theta - x\sin\theta$$

- The change of axes yields

$$I_{x'} = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2}\cos 2\theta - I_{xy}\sin 2\theta$$

$$I_{y'} = \frac{I_x + I_y}{2} - \frac{I_x - I_y}{2}\cos 2\theta + I_{xy}\sin 2\theta$$

$$I_{x'y'} = \frac{I_x - I_y}{2}\sin 2\theta + I_{xy}\cos 2\theta$$

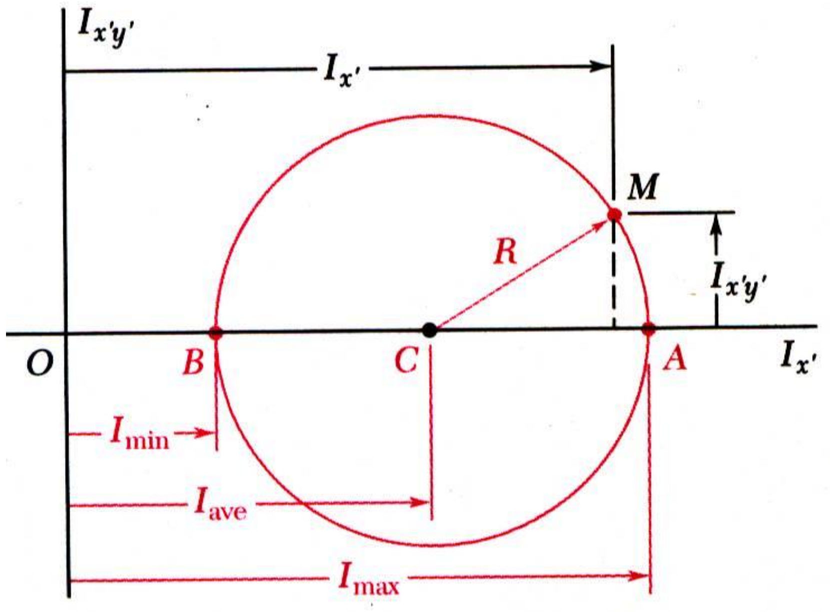
- The equations for $I_{x'}$ and $I_{x'y'}$ are the parametric equations for a circle,

$$\left(I_{x'} - I_{ave}\right)^2 + I_{x'y'}^2 = R^2$$

$$I_{ave} = \frac{I_x + I_y}{2} \qquad R = \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2}$$

- The equations for $I_{y'}$ and $I_{x'y'}$ lead to the same circle.

Principal Axes and Principal Moments of Inertia



- At the points A and B, $I_{x'y'} = 0$ and $I_{x'}$ is a maximum and minimum, respectively.

$$I_{\max,\min} = I_{ave} \pm R$$

$$\tan 2\theta_m = -\frac{2I_{xy}}{I_x - I_y}$$

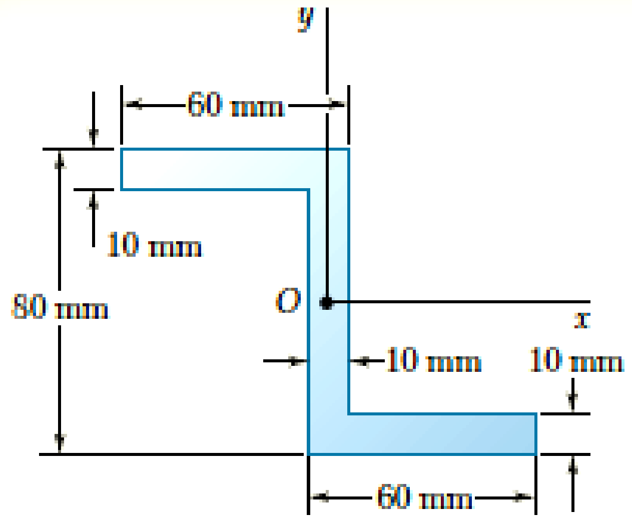
- The equation for θ_m defines two angles, 90° apart which correspond to the *principal axes* of the area about O.

- I_{max} and I_{min} are the *principal moments of inertia* of the area about O.

$$(I_{x'} - I_{ave})^2 + I_{x'y'}^2 = R^2$$

$$I_{ave} = \frac{I_x + I_y}{2} \qquad R = \sqrt{\left(\frac{I_x - I_y}{2}\right) + I_{xy}^2}$$

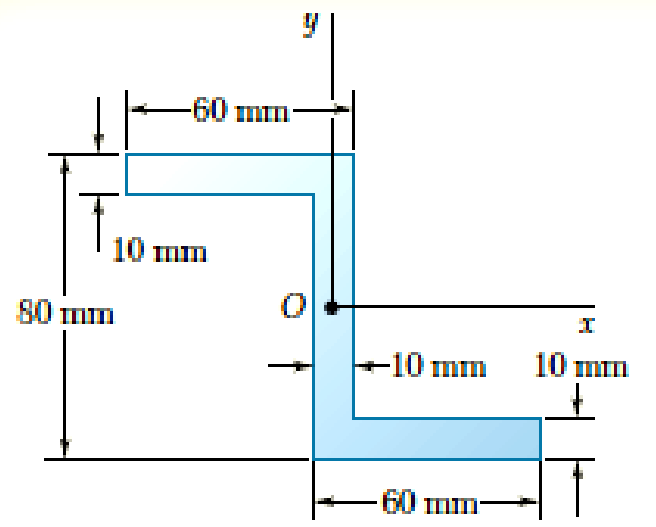
Sample Problems 9.7:

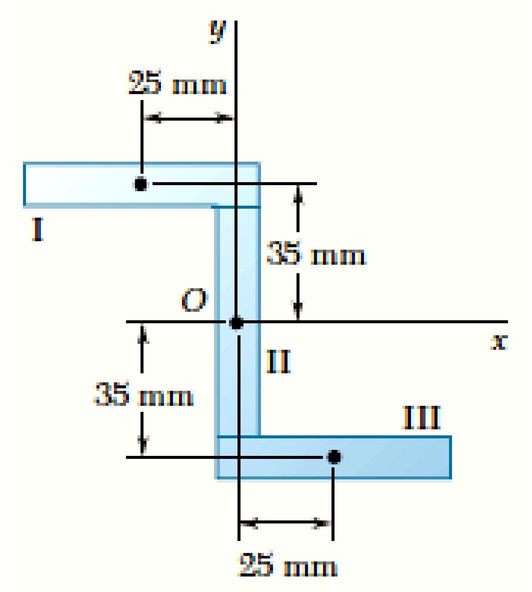


For the section shown, the moments of inertia with respect to the *x* and *y* axes are $I_x = 1.66 \times 10^6\ mm^4$ and $I_y = 1.12 \times 10^6\ mm^4$.

Determine **(a)** the orientation of the principal axes of the section about *O*, and **(b)** the values of the principal moments of inertia about *O*.

Sample Problems 9.7:





SOLUTION:

- Compute the product of inertia with respect to the *xy* axes by dividing the section into three rectangles.

Apply the parallel axis theorem to each rectangle,

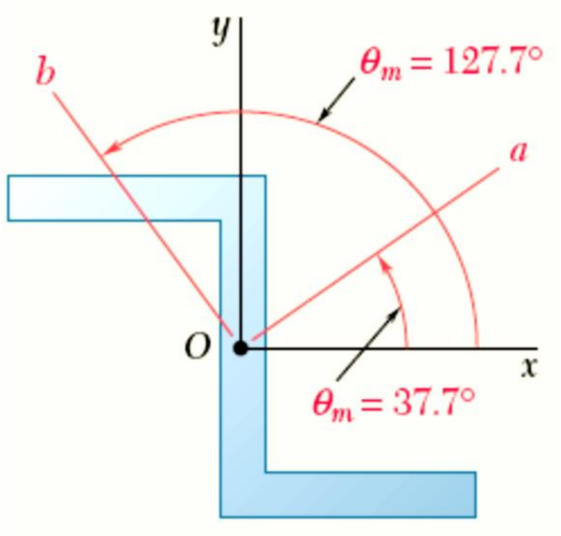
$$I_{xy} = \sum\left(\bar{I}_{x'y'} + \bar{x}\bar{y}A\right)$$

Note that the product of inertia with respect to centroidal axes parallel to the *xy* axes is zero for each rectangle.

Rectangle	Area, mm^2	$\bar{x}$, mm	$\bar{y}$, mm	$\bar{x}\bar{y}A$, mm^4
I	600	−25	+35	-5.25×10^5
II	600	0	0	0
III	600	+25	−35	-5.25×10^5
				$\Sigma \bar{x}\bar{y}A = -1.05 \times 10^6$

$$\boxed{I_{xy} = \sum \bar{x}\bar{y}A = -1.05 \times 10^6 \text{ mm}^4}$$

Sample Problems 9.7:



$$I_x = 1.66 \times 10^6 \text{ mm}^4$$
$$I_y = 1.22 \times 10^6 \text{ mm}^4$$
$$I_{xy} = -1.05 + 10^6 \text{ mm}^4$$

- Determine the orientation of the principal axes (Eq. 9.25) and the principal moments of inertia (Eq. 9. 27).

$$tan\, 2\theta_m = -\frac{2I_{xy}}{I_x - I_y} = -\frac{2\left(-1.05 \times 10^6\right)}{1.66 \times 10^6 - 1.22 \times 10^6} = +3.89$$

$$2\theta_m = 75.6° \text{ and } 255.6\,°$$

$$\boxed{\theta_m = 38.8° \text{ and } \theta_m = 127.8°}$$

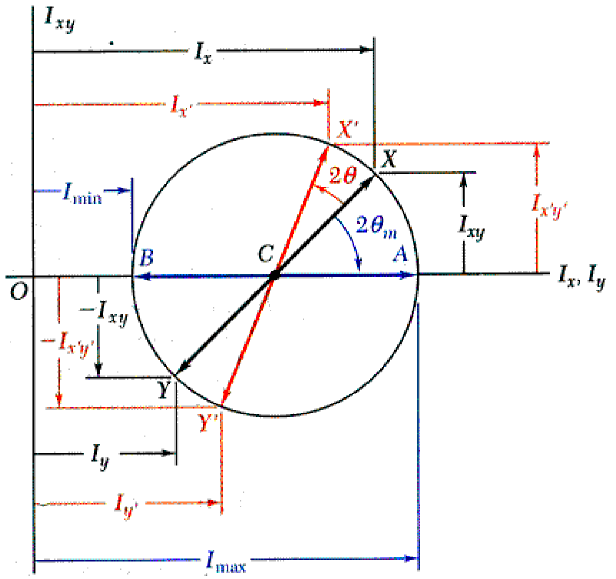
$$I_{max,min} = \frac{I_x + I_y}{2} \pm \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2}$$

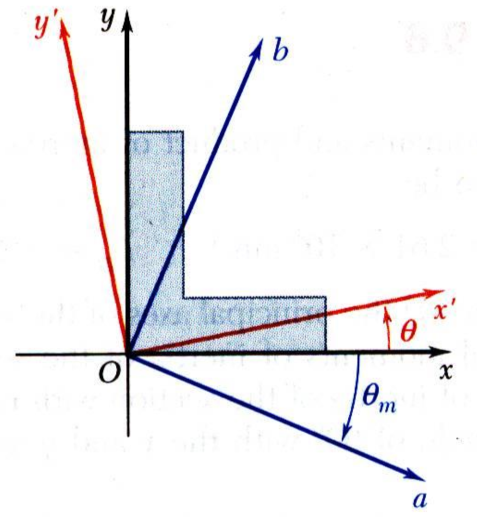
$$= \frac{1.66 \times 10^6 + 1.22 \times 10^6}{2}$$

$$\pm \sqrt{\left(\frac{1.66 \times 10^6 - 1.22 \times 10^6}{2}\right)^2 + \left(-1.05 \times 10^6\right)^2}$$

$$\boxed{\begin{aligned} I_a &= I_{max} = 2.47 \times 10^6 \text{ mm}^4 \\ I_b &= I_{min} = 0.306 \times 10^6 \text{ mm}^4 \end{aligned}}$$

Mohr's Circle for Moments and Products of Inertia



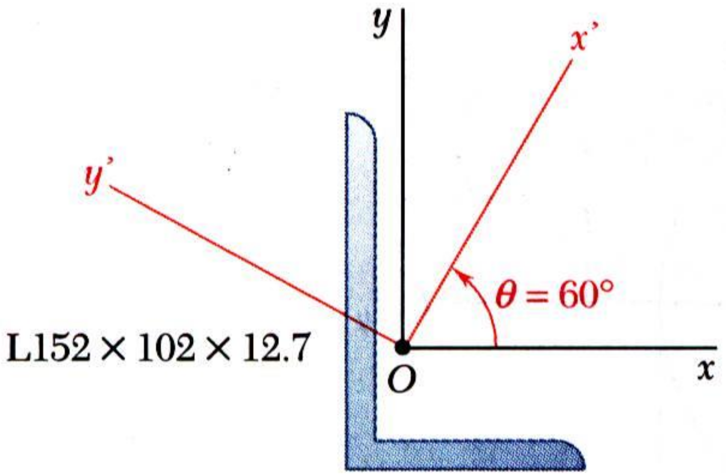


- The moments and product of inertia for an area are plotted as shown and used to construct *Mohr's circle*,

$$I_{ave} = \frac{I_x + I_y}{2} \qquad R = \sqrt{\left(\frac{I_x - I_y}{2}\right) + I_{xy}^2}$$

- Mohr's circle may be used to graphically or analytically determine the moments and product of inertia for any other rectangular axes including the principal axes and principal moments and products of inertia.

Sample Problems 9.8:



The moments and product of inertia with respect to the x and y axes are I_x = 7.20x10^6 mm^4, I_y = 2.59x10^6 mm^4, and I_{xy} = -2.54x10^6 mm^4.

Using Mohr's circle, determine (a) the principal axes about O, (b) the values of the principal moments about O, and (c) the values of the moments and product of inertia about the x' and y' axes

- Based on the circle, evaluate the moments and product of inertia with respect to the $x'y'$ axes.

The points X' and Y' corresponding to the x' and y' axes are obtained by rotating CX and CY counterclockwise through an angle $2\theta = 2(60^o) = 120^o$. The angle that CX' forms with the x' axes is $\phi = 120^o - 47.8^o = 72.2^o$.

$$I_{x'} = OF = OC + CX' \cos\phi = I_{ave} + R\cos 72.2^o$$

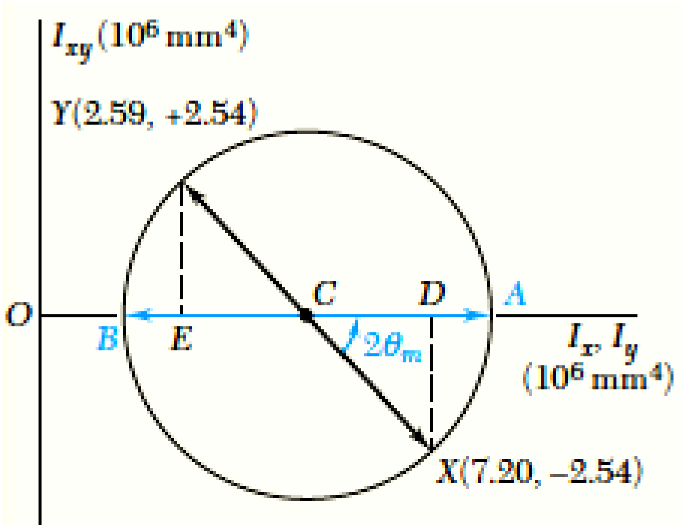
$$\boxed{I_{x'} = 5.94 \times 10^6 \text{mm}^4}$$

$$I_{y'} = OG = OC - CY' \cos\phi = I_{ave} - R\cos 72.2^o$$

$$\boxed{I_{y'} = 3.85 \times 10^6 \text{mm}^4}$$

$$I_{x'y'} = FX' = CY' \sin\phi = R\sin 72.2^o$$

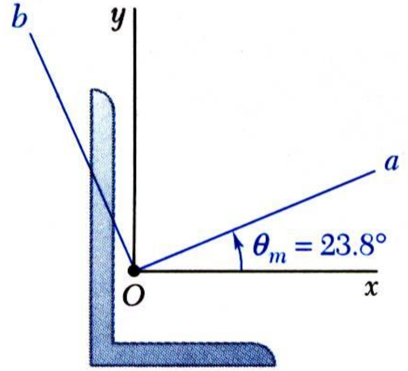
$$\boxed{I_{x'y'} = 3.27 \times 10^6 \text{mm}^4}$$



$$I_x = 7.20 \times 10^6 \,\text{mm}^4$$

$$I_y = 2.59 \times 10^6 \,\text{mm}^4$$

$$I_{xy} = -2.54 \times 10^6 \,\text{mm}^4$$



<u>SOLUTION</u>:

- Plot the points (I_x , I_{xy}) and (I_y ,-I_{xy}). Construct Mohr's circle based on the circle diameter between the points.

$$OC = I_{ave} = \tfrac{1}{2}\left(I_x + I_y\right) = 4.895 \times 10^6 \,\text{mm}^4$$

$$CD = \tfrac{1}{2}\left(I_x - I_y\right) = 2.305 \times 10^6 \,\text{mm}^4$$

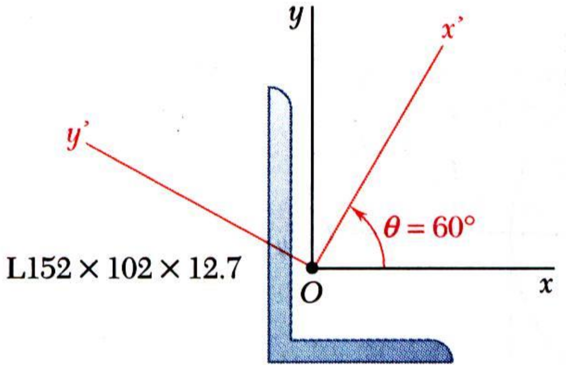
$$R = \sqrt{(CD)^2 + (DX)^2} = 3.43 \times 10^6 \,\text{mm}^4$$

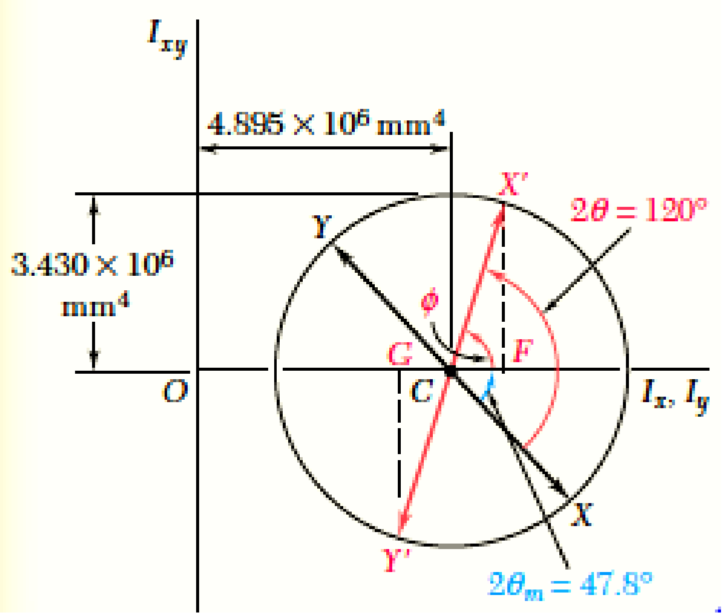
- Based on the circle, determine the orientation of the principal axes and the principal moments of inertia.

$$\tan 2\theta_m = \frac{DX}{CD} = 1.102 \qquad 2\theta_m = 47.8° \qquad \boxed{\theta_m = 23.9°}$$

$$I_{max} = OA = I_{ave} + R \qquad \boxed{I_{max} = 8.33 \times 10^6 \,\text{mm}^4}$$

$$I_{min} = OB = I_{ave} - R \qquad \boxed{I_{min} = 1.47 \times 10^6 \,\text{mm}^4}$$





$$OC = I_{ave} = 4.895\times10^6\,\text{mm}^4$$

$$R = 3.43\times10^6\,\text{mm}^4$$

- Based on the circle, evaluate the moments and product of inertia with respect to the *x'y'* axes.

The points *X'* and *Y'* corresponding to the *x'* and *y'* axes are obtained by rotating *CX* and *CY* counterclockwise through an angle $2\theta = 2(60^o) = 120^o$. The angle that *CX'* forms with the *x'* axes is $\phi = 120^o - 47.8^o = 72.2^o$.

$$I_{x'} = OF = OC + CX'\cos\phi = I_{ave} + R\cos 72.2^o$$

$$\boxed{I_{x'} = 5.94\times10^6\,\text{mm}^4}$$

$$I_{y'} = OG = OC - CY'\cos\phi = I_{ave} - R\cos 72.2^o$$

$$\boxed{I_{y'} = 3.85\times10^6\,\text{mm}^4}$$

$$I_{x'y'} = FX' = CY'\sin\phi = R\sin 72.2^o$$

$$\boxed{I_{x'y'} = 3.27\times10^6\,\text{mm}^4}$$