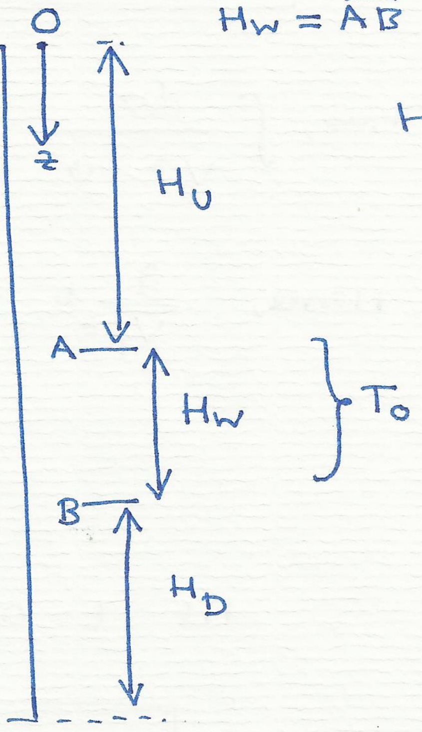
$H_W = \overline{AB}$ is window height.

$H_U = ?$

(1.) Oz is vertically downward.



Object released from O at $t = 0$.

$\therefore$ At time t, $z = \frac{1}{2} g t^2$.

Lets assume the object reaches A at $t = T$, and B at $t = T + T_0$.

$\therefore$ $H_U = \frac{1}{2} g T^2$.

& $H_U + H_W = \frac{1}{2} g (T + T_0)^2$.

$$\therefore \quad H_W = \frac{1}{2} g (T+T_0)^2 - \frac{1}{2} g T^2 = \frac{1}{2} g (2 T T_0 + T_0^2).$$

$$\text{Thus,} \quad T = \frac{H_W}{g T_0} - \frac{T_0}{2}.$$

$$\therefore \quad H_U = \frac{1}{2} g \underbrace{\left(\frac{H_W}{g T_0} - \frac{T_0}{2} \right)}_{\frac{1}{2 g T_0}(2H_W - g T_0^2)}^2 = \frac{1}{8 g T_0^2} (2 H_W - g T_0^2)^2.$$

(2.) Assume that the motion starts from the origin & takes place along the z-axis which is pointing vertically downwards. Following notation used in class,

E.O.M. is, $m \frac{dv}{dt} = mg - mkv^2$.

Here, $v = \dot{z}$.

When the body attains terminal velocity, $\frac{dv}{dt} = 0$.

$$\Rightarrow \quad mg - mkv^2 \Big|_{v = V_T} = 0.$$

$$\therefore \quad V_T^2 = \frac{g}{k}. \qquad \text{i.e., } k = \frac{g}{V_T^2}.$$

$$\Rightarrow \boxed{\frac{dv}{dt} = g\left(1-\frac{v^2}{V_T^2}\right)}.$$

Thus, $\displaystyle\int \frac{dv}{V_T^2 - v^2} = \frac{g}{V_T^2}\int dt.$

Hence, $\displaystyle\frac{g}{V_T^2}t = \frac{1}{2V_T}\int\left(\frac{1}{(V_T-v)} + \frac{1}{(V_T+v)}\right)dv.$

$$= \frac{1}{2V_T}\ln\left(\frac{V_T+v}{V_T-v}\right) + C.$$

At $t=0$ we have, $v=0$. $\Rightarrow C=0$.

$$\therefore \boxed{t = \frac{V_T}{2g}\ln\left(\frac{V_T+v}{V_T-v}\right)}$$

$\Downarrow$ inverted

$$\boxed{v = V_T \tanh\left(\frac{gt}{V}\right).}$$

$$\therefore \quad z = V_T\int \tanh\left(\frac{gt}{V}\right) dt.$$

$$= \frac{V_T^2}{g}\ln\left(\cosh\left(\frac{gt}{V}\right)\right) + D.$$

At $t=0$, $z=0$ $\Rightarrow D=0$.

$$\therefore \quad z = \frac{V_T^2}{g}\ln\left(\cosh\left(\frac{gt}{V}\right)\right).$$

If the body falls through a height $z=H$ in time $t=T$, then,

$$T = \left(\frac{V_T}{g}\right)\cosh^{-1}\left(e^{gH/V_T^2}\right).$$

3. a. $\vec{F} = m\vec{a}$

$$\Rightarrow \begin{cases} \ddot{x} = -\alpha \dot{x} \\ \ddot{y} = -g - \alpha \dot{y} \end{cases}$$

Let $\dot{x} = u_x$

$\Rightarrow \dot{u}_x = -\alpha u_x$

$\therefore u_x = Ae^{-\alpha t} = \dot{x}$.

Upon integration,

$\dot{x} = Ae^{-\alpha t}$. At $t=0$, $\dot{x} = v_0 \cos\theta$.

$\therefore \dot{x} = v_0 \cos\theta \, e^{-\alpha t}$.

Integrating once more,

$$x = -\left(\frac{v_0 \cos\theta}{\alpha}\right) e^{-\alpha t} + B.$$

At $t=0, x=0$ $\cancel{\Rightarrow B = 0 + 1 = 1}$.

$\Rightarrow B = \frac{v_0 \cos\theta}{\alpha}$.

$$\therefore \boxed{x(t) = \left(\frac{v_0 \cos\theta}{\alpha}\right)(1 - e^{-\alpha t}).}$$

$\ddot{y} = -g - \alpha\dot{y} = -\alpha\left(\dot{y} + \frac{g}{\alpha}\right)$

$\therefore \frac{d}{dt}\left(\dot{y} + \frac{g}{\alpha}\right) = -\alpha\left(\dot{y} + \frac{g}{\alpha}\right)$.

$\Rightarrow \dot{y} + \frac{g}{\alpha} = Ce^{-\alpha t}$.

At $t=0$, $\dot{y} = v_0 \sin\theta$.

$\Rightarrow C = v_0 \sin\theta + \frac{g}{\alpha}$.

$$\therefore \boxed{\dot{y} = \left(v_0 \sin\theta + \frac{g}{\alpha}\right) e^{-\alpha t} - \frac{g}{\alpha}}$$

Integrating once more,

$$y = -\frac{1}{\alpha}\left(v_0 \sin\theta + \frac{g}{\alpha}\right) e^{-\alpha t} - \frac{gt}{\alpha} + D.$$

At $t=0$, $y=0 \Rightarrow D = \frac{1}{\alpha}\left(v_0 \sin\theta + \frac{g}{\alpha}\right)$.

$$\therefore \boxed{y(t) = \frac{1}{\alpha}\left(v_0 \sin\theta + \frac{g}{\alpha}\right)\left(1 - e^{-\alpha t}\right) - \frac{gt}{\alpha}}$$

(b) Given $m\alpha v_0 = mg$

$$\Rightarrow \frac{g}{\alpha} = v_0 .$$

$$\therefore \dot{y} = (v_0 \sin\theta + v_0) e^{-\alpha t} - v_0 .$$

At the highest point $\dot{y} = 0$

Let this happen at $t = t_p$

$$\Rightarrow e^{-\alpha t_p} = \frac{1}{(1+\sin\theta)} .$$

The value of x at time $t = t_p$ is,

$$x_p = \frac{v_0 \cos\theta}{g/v_0}\left(1 - \frac{1}{(1+\sin\theta)}\right)$$

$$\therefore x_p = \frac{v_0^2 \cos\theta \sin\theta}{g(1+\sin\theta)} .$$

$$\frac{dx_p}{d\theta} = 0 \Rightarrow \sin^3\theta + 2\sin^2\theta - 1 = 0 .$$

$$\therefore (\sin\theta + 1)(\sin^2\theta + \sin\theta - 1) = 0 .$$

Roots,

$$\left.\begin{aligned} \sin\theta &= -1 \\ \sin\theta &= -\frac{(\sqrt{5}+1)}{2} \end{aligned}\right\} \text{unphysical for present case.}$$

$$\sin\theta = \frac{\sqrt{5} - 1}{2} .$$

$$\therefore \theta = \sin^{-1}\left(\frac{\sqrt{5}-1}{2}\right) \approx 38.2^\circ .$$

Relative to the fixed reference frame!

(4) Polar co-ordinates of the particle at time t

are
$$\begin{cases} r = b\cosh \Omega t . \\ \theta = \Omega t . \end{cases}$$

The velocity is,

$$\vec{v} = \dot{r}\,\hat{r} + (r\dot{\theta})\,\hat{\theta} .$$

$$= (\Omega b \sinh \Omega t)\,\hat{r} + (\Omega b \cosh \Omega t)\,\hat{\theta} .$$

Speed is,

$$v = \sqrt{|\vec{v}|^2} = \sqrt{\Omega^2 b^2 \sinh^2 \Omega t + \Omega^2 b^2 \cosh^2 \Omega t}$$

$$= \Omega b \sqrt{\cosh 2\Omega t} .$$

The accleration is,

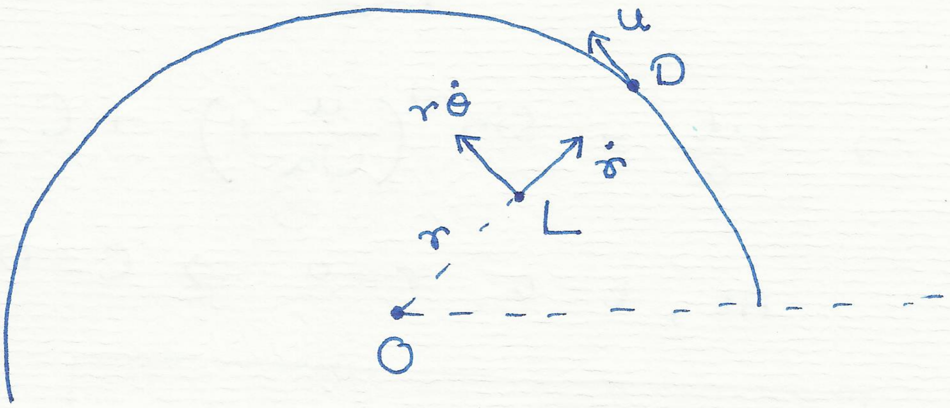
$$\bar{a} = (\ddot{r} - r\dot{\theta}^2)\,\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\hat{\theta} .$$

$$= (\Omega^2 b \cosh \Omega t - \Omega^2 b \cosh \Omega t)\,\hat{r}$$

$$+ (0 + 2\Omega^2 b \sinh \Omega t)\,\hat{\theta} .$$

$$\therefore \bar{a} = (2\Omega^2 b \sinh \Omega t)\,\hat{\theta} .$$

5.



(i.) Let the lion (L) have polar coordinates (r, θ) as shown above.

The velocity vector of L is

$$\vec{v} = \dot{r}\,\hat{r} + (r\dot{\theta})\,\hat{\theta}.$$

$$= \dot{r}\,\hat{r} + \left(\frac{ur}{a}\right)\hat{\theta}.$$

($\because$ the L stays on the radius $\overrightarrow{OD}$ which is rotating with angular velocity $\dot{\theta} = \frac{u}{a}$.)

Since, speed of L is U,

$$\Rightarrow \dot{r}^2 + \left(\frac{ur}{a}\right)^2 = U^2.$$

$$\boxed{\therefore \dot{r}^2 = \frac{u^2}{a^2}\left(\frac{U^2a^2}{u^2} - r^2\right).}$$

$\hookrightarrow$ Eqn satisfied by radial coordinate.

(ii.)

Thus, $\dot{r} = \left(\frac{u}{a}\right)\sqrt{\frac{U^2a^2}{u^2} - r^2}$. (Keeping +ve root).

Thus, $\frac{u}{a}\int dt = \int \frac{dr}{\sqrt{\frac{U^2a^2}{u^2} - r^2}}$.

$$\Rightarrow \quad \frac{ut}{a} = \sin^{-1}\left(\frac{u}{Ua}r\right) + C.$$

At, $t = 0$, $r = 0 \Rightarrow C = 0$.

$$\boxed{\therefore\ r = \left(\frac{Ua}{u}\right)\sin\left(\frac{ut}{a}\right).}$$

(iii) Daniel will get caught when $r = a$.

i.e., when $\sin\left(\frac{ut}{a}\right) = \frac{u}{U}$.

If $U \geq u$, this eqn. has a real solution, $t = \left(\frac{a}{u}\right)\sin^{-1}\left(\frac{u}{U}\right)$.

$\therefore$ Daniel will get caught at

$$t = \left(\frac{a}{u}\right)\sin^{-1}\left(\frac{u}{U}\right).$$

(iv) Since $\theta = \frac{ut}{a}$, the polar equation of the path of lion is,

$$r = \frac{Ua}{u}\sin\theta.$$

Multiply both sides by r,

$$\Rightarrow \quad x^2 + y^2 = \left(\frac{Ua}{u}\right)y.$$

i.e., $\boxed{x^2 + \left(y - \frac{Ua}{2u}\right)^2 = \left(\frac{Ua}{2u}\right)^2}$

Center: $(0, Ua/2u)$
Radius: $\frac{Ua}{2u}$,

Eqn. of circle

<u>Note</u>: The Lion doesnot traverse full circle.

Daniel gets caught when the Lion has traversed an arc of length

$$\left(\frac{Ua}{u}, \sin^{-1}\left(\frac{u}{U}\right)\right).$$

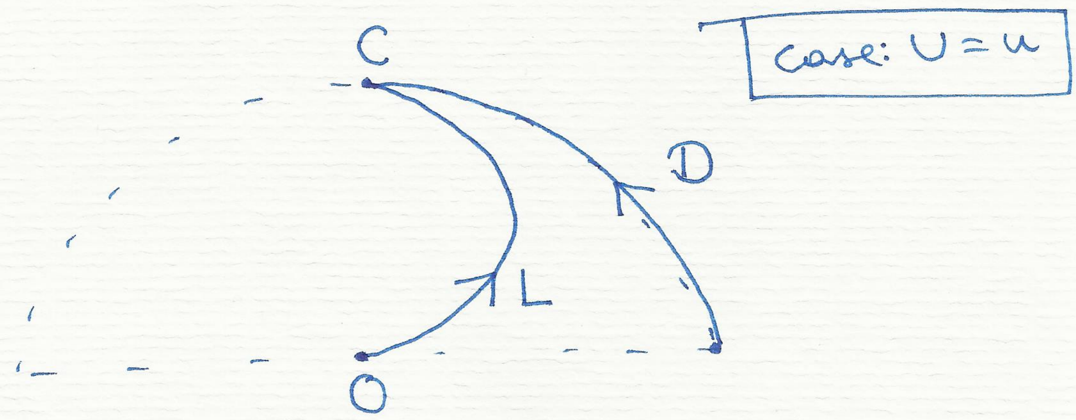
(v.) Special case: $U = u$.

Loci of Lion's path is

$$x^2 + \left(y - \frac{a}{2}\right)^2 = \left(\frac{a}{2}\right)^2.$$

Daniel will get caught when the Lion has traversed half of this circle.

The point of capture is $(0, a)$.



⑥ The velocity of bee is

$$\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} = \frac{2b}{\tau^2}(\tau - t)\hat{r} + \frac{bt}{\tau^3}(2\tau - t)\hat{\theta}.$$

$$\therefore\ |\vec{v}|^2 = \frac{b^2}{\tau^6}\left(t^4 - 4\tau t^3 + 8\tau^2 t^2 - 8\tau^3 t + 4\tau^4\right).$$

$$\frac{d}{dt}|\vec{v}|^2 = \frac{b^2}{\tau^6}\left(4t^3 - 12\tau t^2 + 16\tau^2 t - 8\tau^3\right)$$

$$= \frac{4b^2}{\tau^6}(t-\tau)\underbrace{(t^2 - 2\tau t + 2\tau^2)}_{\text{always positive}}.$$

$$\therefore\ \frac{d}{dt}|\vec{v}|^2 \begin{cases} < 0 & \text{for } t < \tau. \\ = 0 & \text{for } t = \tau. \\ > 0 & \text{for } t > \tau. \end{cases}$$

$\therefore$ $|\vec{v}|$ achieves its minimum value when $t = \tau$.

At this instant, $|\vec{v}| = \frac{b}{\tau}$.

(min. speed of bee).

The accl^{n} of the bee at time t is,

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}.$$

$$= \left(-\frac{2b}{\tau^2} - \frac{bt}{\tau^4}(2\tau - t)\right)\hat{r} + \left(0 + \frac{4b}{\tau^3}(\tau - t)\right)\hat{\theta}.$$

$$= -\frac{3b}{t^2}\hat{r} \quad \text{at } t = \tau.$$

$\therefore$ when speed of the bee is minimum, its acceleration is $-\frac{3b}{\tau^2}\hat{r}$.