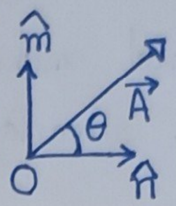
TUTORIAL 2 : SOLUTION

Problem 1: $\hat{m}$ and $\hat{n}$: unit vectors as shown.

$\vec{A}$ makes an angle θ with $\hat{n}$, as shown.



$$\therefore \; \vec{A} = A_n \hat{n} + A_m \hat{m}$$

where A_n : component of $\vec{A}$ along $\hat{n}$

& A_m : " " $\vec{A}$ " $\hat{m}$

Now, $A_n = |\vec{A}| \cos\theta$ & $A_m = |\vec{A}| \sin\theta$

$$\therefore \vec{A} = |\vec{A}| \cos\theta \, \hat{n} + |\vec{A}| \sin\theta \, \hat{m} \quad \text{—— (1)}$$

But $$\cos\theta = \frac{\vec{A} \cdot \hat{n}}{|\vec{A}|} \quad \text{— (2)}$$

Next is to obtain an expression for $\sin\theta$. For that imagine $\hat{k}$, a unit vector coming out of plane of paper and whose tail is at O.

Then, $\hat{n} \times \vec{A} = |\vec{A}| \sin\theta \, \hat{k} \Rightarrow \sin\theta \, \hat{k} = \dfrac{\hat{n} \times \vec{A}}{|\vec{A}|}$ —— (3)

We also observe that : $\hat{m} = \hat{k} \times \hat{n}$ —— (4)

Then, taking cross product by $\hat{n}$ to eqn. (3) and using eqn. (4),

$$\sin\theta \, \hat{m} = \frac{(\hat{n} \times \vec{A})}{|\vec{A}|} \times \hat{n} \quad \text{— (5)}$$

Using results of eqn. (5) & (2) in eqn. (1),

$$\vec{A} = (\vec{A} \cdot \hat{n}) \hat{n} + (\hat{n} \times \vec{A}) \times \hat{n}$$

QED.

Problem 2: Sometimes, rather most of times, inherent symmetry of a physical makes the solution a whole lot easier. In this problem, we take advantage of the spherical symmetry of the problem.

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta.$$

$$\therefore \underbrace{x^2+y^2+z^2}_{\text{Integrand}} = r^2$$

and $\underbrace{dx\,dy\,dz}_{\text{Volume element}} \equiv dV \equiv r^2\sin\theta\,dr\,d\theta\,d\phi$

$$\therefore \iiint_V (x^2+y^2+z^2)\,dx\,dy\,dz = \int_{r=0}^{a}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} r^2 . r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$= \int_{r=0}^{a} r^4\,dr \int_{\theta=0}^{\pi}\sin\theta\,d\theta \int_{\phi=0}^{2\pi} d\phi$$

$$= \left.\frac{r^5}{5}\right|_0^a \times \left.(-\cos\theta)\right|_0^\pi \times 2\pi$$

$$= \frac{2\pi a^5}{5} \times (-1)\,[-1-1]$$

$$= \frac{4\pi a^5}{5}$$

Problem 3: Here, we make use of the expressions for $\vec{v}$ and $\vec{a}$ derived in class.

$$\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

$$\text{and } \vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}$$

Given $\dot{r} = 4$ m/s and $\dot{\theta} = 2$ rad/s. Also $r = 3$ m

(a) $\vec{v} = 4\hat{r} + 3\times 2\,\hat{\theta} = 4\hat{r} + 6\hat{\theta}$

$\therefore |\vec{v}| = \sqrt{4^2+6^2} = \sqrt{16+36} = \sqrt{52}$ m/s.

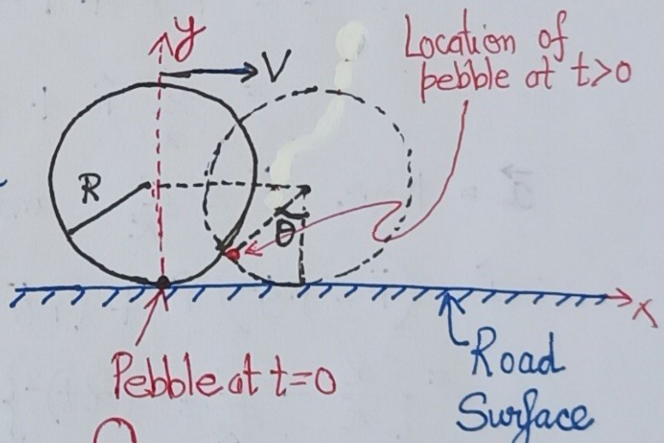
(b) $\vec{a} = (0 - 3\times 2^2)\hat{r} + (3\times 0 + 2\times 4\times 2)\hat{\theta}$ $[\ddot{r} = 0\ \&\ \ddot{\theta} = 0]$

$= -12\hat{r} + 16\hat{\theta}$

$\therefore |\vec{a}| = \sqrt{(-12)^2 + 16^2} = \sqrt{144+256} = \sqrt{400} = 20$ m/s^2

Problem 4: We are free to choose the origin of coordinate wherever we want but it would very convenient if we choose it at the location of pebble at $t=0$.

As the tyre rolls, the location of pebble changes. It subtends an angle θ with the verticle, as shown.



x & y coordinate of pebble at t:

$x = Vt - R\sin\theta$

But $Vt = R\theta$ $\quad \therefore$ $\left.\begin{aligned} x &= R\theta - R\sin\theta \\ \& \ y &= R - R\cos\theta \end{aligned}\right\}$ Position

$\therefore \dot{x} = R\dot{\theta} - R\cos\theta\,\dot{\theta}$ $\qquad$ But $R\dot{\theta} = V$

$\left.\begin{aligned} \therefore \dot{x} &= V(1-\cos\theta) \\ \& \ \dot{y} &= -R(-\sin\theta)\dot{\theta} = V\sin\theta. \end{aligned}\right\}$ Velocity components

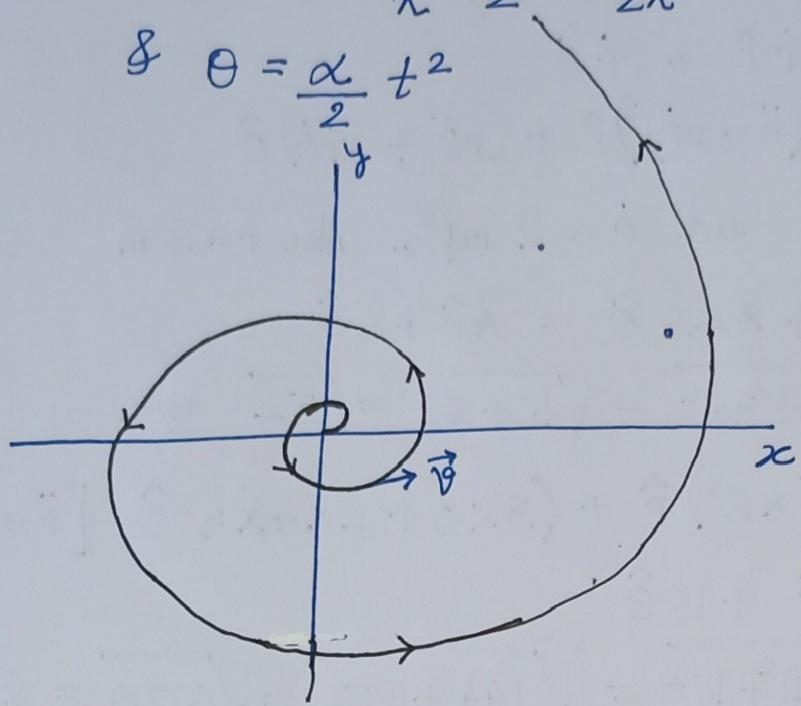
$\left.\begin{aligned} \ddot{x} &= V\sin\theta\,\dot{\theta} = \frac{V^2}{R}\sin\theta = \frac{V^2}{R}\sin\left(\frac{Vt}{R}\right) \\ \ddot{y} &= \frac{V^2}{R}\cos\left(\frac{Vt}{R}\right) \end{aligned}\right\}$ Acceleration components.

Note: Try to define the angular position of pebble in some other way and work out the problem.

Problem 5: $r = A\theta = \frac{1}{\pi}\frac{\alpha t^2}{2} = \frac{\alpha}{2\pi}t^2$

& $\theta = \frac{\alpha}{2}t^2$

(a)



(b) $\dot{r} = \frac{\alpha}{2\pi}.2t = \frac{\alpha}{\pi}t$ & $\dot{\theta} = \alpha t$

$\therefore \vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} = \left(\frac{\alpha t}{\pi}\right)\hat{r} + \frac{\alpha t^2}{2\pi}\times\alpha t\,\hat{\theta}$

$= \frac{\alpha t}{\pi}\hat{r} + \frac{\alpha^2 t^3}{2\pi}\hat{\theta}$

Now, $\ddot{r} = \frac{\alpha}{\pi}$ and $\ddot{\theta} = \alpha$

$\therefore \vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}$

$= \left\{\frac{\alpha}{\pi} - \frac{\alpha}{2\pi}t^2(\alpha t)^2\right\}\hat{r} + \left\{\frac{\alpha}{2\pi}t^2\times\alpha + 2\frac{\alpha t}{\pi}\times\alpha t\right\}\hat{\theta}$

$= \left(\frac{\alpha}{\pi} - \frac{\alpha^3 t^4}{2\pi}\right)\hat{r} + \left(\frac{\alpha^2 t^2}{2\pi} + \frac{2\alpha^2 t^2}{\pi}\right)\hat{\theta}$

$\vec{a} = \left(\frac{\alpha}{\pi} - \frac{\alpha^3 t^4}{2\pi}\right)\hat{r} + \left(\frac{5\alpha^2 t^2}{2\pi}\right)\hat{\theta} = a_r\hat{r} + a_\theta\hat{\theta}$

Radial acceleration $= 0 \Rightarrow \frac{\alpha}{\pi} - \frac{\alpha^3 t^4}{2\pi} = 0 \Rightarrow 1 - \frac{\alpha^2 t^4}{2} = 0$

$\Rightarrow t^4 = 2/\alpha^2 \Rightarrow t^2 = \pm\frac{\sqrt{2}}{\alpha}$. We choose +ve, since imaginary time is not possible.

$\therefore t^2 = \sqrt{2}/\alpha$

$\Rightarrow \theta = \frac{\alpha}{2}\frac{\sqrt{2}}{\alpha} = \frac{1}{\sqrt{2}}$

(c) $a_r = a_\theta \Rightarrow \frac{\alpha}{\pi} - \frac{\alpha^3 t^4}{2\pi} = \frac{5\alpha^2 t^2}{2\pi}$ —— (1)

But $\theta = \frac{\alpha t^2}{2} \Rightarrow t^2 = \frac{2\theta}{\alpha}$

$\therefore$ Eqn. (1) modifies as: $1 - \frac{\alpha^2}{2}\left(\frac{2\theta}{\alpha}\right)^2 = \frac{5\alpha}{2}\frac{2\theta}{\alpha}$

$\Rightarrow 1 - 2\theta^2 = 5\theta$

$\Rightarrow 2\theta^2 + 5\theta - 1 = 0. \quad \Rightarrow \theta = 0.186 \text{ rad.} = 10.66^\circ$
$\qquad\qquad\qquad\qquad\qquad\qquad -2.686 \text{ rad} = -153.90^\circ$

Since θ increase with time t, it can not take -ve values. Hence, $\theta = 10.66^\circ$ is when $a_r = a_\theta$.

Problem 6: Given: $h(x,y) = 10(2xy - 3x^2 - 4y^2 - 18x + 28y + 12) \rightarrow$ Height of hill.

(a) Top of the hill is located where $\vec{\nabla} h = 0$.

$\Rightarrow \vec{\nabla}\left[10(2xy - 3x^2 - 4y^2 - 18x + 28y + 12)\right] = 0$

$\Rightarrow \frac{\partial}{\partial x}\left[10(2xy - 3x^2 - 4y^2 - 18x + 28y + 12)\right]\hat{i}$
$\qquad + \frac{\partial}{\partial y}\left[10(2xy - 3x^2 - 4y^2 - 18x + 28y + 12)\right]\hat{j} = 0$

$\Rightarrow (2y - 6x - 18)\hat{i} + (2x - 8y + 28)\hat{j} = 0.$

$\Rightarrow \left\{\begin{array}{l} -6x + 2y = 18 \\ 2x - 8y = -28 \end{array}\right\} \Rightarrow \left.\begin{array}{l} x = -2 \\ y = 3 \end{array}\right\}$ Location of top of hill

(b) $h_{max}(x,y)\Big|_{\substack{x=-2\\y=3}} = 10\left[2(-2)(3) - 3(-2)^2 - 4(3)^2 - 18(-2) + 28(3) + 12\right]$

$= 10\left[\cancel{-12} - 12 - \cancel{36} + \cancel{36} + 84 \cancel{+12}\right]$

$= 720 \text{ m}.$

(c) Steepness of slope at (x,y) is given by $|\vec{\nabla} h|$ at (x,y).

$\vec{\nabla} h\Big|_{(x,y)=(1,1)} = (2 - 6 - 18)\hat{i} + (2 - 8 + 28)\hat{j} = -22\hat{i} + 22\hat{j}.$

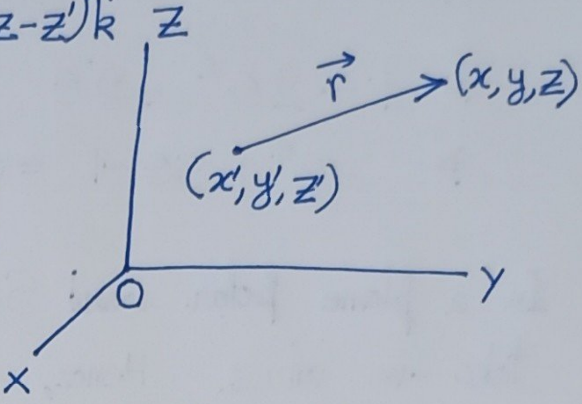
$\therefore |\vec{\nabla} h| = \sqrt{22^2 + 22^2} = 22\sqrt{2}$

The direction of steepest slope is the unit vector along $\vec{\nabla} h$: $\hat{n}$

$$\hat{n} = \frac{-22\hat{i} + 22\hat{j}}{22\sqrt{2}} = -\frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}.$$

<u>Problem 7</u>: $\vec{r} = (x-x')\hat{i} + (y-y')\hat{j} + (z-z')\hat{k}$

and $r = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$



(a) $\vec{\nabla}(r^2) = \vec{\nabla}\{(x-x')^2 + (y-y')^2 + (z-z')^2\}$

$= \frac{\partial}{\partial x}\{\quad\}\hat{i} + \frac{\partial}{\partial y}\{\quad\}\hat{j} + \frac{\partial}{\partial z}\{\quad\}\hat{k}$

$= 2(x-x')\hat{i} + 2(y-y')\hat{j} + 2(z-z')\hat{k}$

$= 2[(x-x')\hat{i} + (y-y')\hat{j} + (z-z')\hat{k}] = 2\vec{r}$ QED

(b) $\vec{\nabla}(1/r) = \vec{\nabla}\left[\{(x-x')^2 + (y-y')^2 + (z-z')^2\}^{-1/2}\right]$

$= \frac{\partial}{\partial x}[\quad]\hat{i} + \frac{\partial}{\partial y}[\quad]\hat{j} + \frac{\partial}{\partial z}[\quad]\hat{k}$

Now, $\frac{\partial}{\partial x}[\quad] = -\frac{1}{2}[\quad]^{-3/2} \cdot 2(x-x') = -[\quad]^{-3/2}(x-x')$

Similarly for y & z components.

$\therefore \vec{\nabla}(1/r) = -[\quad]^{-3/2} \cdot [(x-x')\hat{i} + (y-y')\hat{j} + (z-z')\hat{k}]$

$= -\frac{1}{r^3} \cdot \vec{r} = -\frac{1}{r^3} r\hat{r} = -\frac{\hat{r}}{r^2}$ QED.

(c) $\because r^n = [(x-x')^2 + (y-y')^2 + (z-z')^2]^{n/2}$

$\therefore \vec{\nabla}(r^n) = \frac{\partial}{\partial x}[\quad]^{n/2}\hat{i} + \frac{\partial}{\partial y}[\quad]^{n/2}\hat{j} + \frac{\partial}{\partial z}[\quad]^{n/2}\hat{k}$

Now, $\frac{\partial}{\partial x}[\quad]^{n/2} = \frac{n}{2}[\quad]^{(n/2)-1} \cdot 2(x-x') = n[\quad]^{n/2-1}(x-x')$

$= n r^{n-2}(x-x')$

Similarly for y and z components.

$\therefore \vec{\nabla}(r^n) = n r^{n-2}\vec{r} = n r^{n-1}\hat{r}.$