Division - I

Lecture - 3, Thursday, Date: 28-07-2011

Recall from MA101: Single variable calculus:

Let x_0 be a positive real number.

Let n be a fixed natural number.

Find the solutions of the equation $x = x_0^{1/n}$

That is, Find all values of x such that $\boxed{x^n = x_0}$.

That is, Find n^{th} root of the positive real number x_0.

Example:

$\sqrt{1} = 1$ and $(1)^{1/n}$ is 1.

$x^2 = 2 \Rightarrow x = \sqrt{2} = 1.414\ldots\ldots$

Find all x such that $x^7 = 5$.

$5^{1/7}$ = Seventh root of 5.

What result you know in this regard?

Result on $x_0^{1/n}$ where $x_0 > 0$ (Existence of n^{th} root of x_0).

Let n be a fixed natural number. For each positive real number x_0, there exists a unique positive real number x such that $x^n = x_0$. That is, $x = x_0^{1/n}$.

Now, Question is: Existence of n^{th} root of any non-zero Complex number.

Square root of 1 are $+1$ and -1.

What about the square root of $(1+i)$? Think.

Finding n^{th} root of non-zero complex number z_0; where n is a fixed natural number.

Set $z_0 = r_0 e^{i(\theta_0 + 2k\pi)}$ where $r_0 = |z_0|$, $\theta_0 = \text{Arg}(z_0)$.

$r_0 > 0$; $k \in \mathbb{Z}$.

Find all z such that $z^n = z_0$.

Observe that z can not be zero, since $z_0 \neq 0$.

Let $z = r e^{i\theta}$

$$\boxed{z^n = z_0} \Rightarrow (r e^{i\theta})^n = r_0 e^{i(\theta_0 + 2k\pi)}$$

$$\Rightarrow \boxed{r^n e^{in\theta} = r_0 e^{i(\theta_0 + 2k\pi)}}$$

$$\Rightarrow \boxed{r^n = r_0} \text{ and } \boxed{n\theta = \theta_0 + 2k\pi}$$

$$\Rightarrow \boxed{r = r_0^{1/n}} \text{ and } \boxed{\theta = \frac{\theta_0 + 2k\pi}{n}}$$

$$\boxed{z = r_0^{1/n}\left(\cos\left(\frac{\theta_0 + 2k\pi}{n}\right) + i \sin\left(\frac{\theta_0 + 2k\pi}{n}\right)\right) \quad \text{where } k = 0, 1, 2, \ldots, (n-1)}$$

For $k = n$, we get the same value that of $k = 0$, and so on.

There are n distinct complex numbers z such that $z^n = z_0$.

" " " " $z = z_0^{1/n}$.

n^{th} root of unity $(1)^{1/n}$.

$$1 = 1\, e^{i(0+2k\pi)} \quad \text{where } k \in \mathbb{Z}$$

n^{th} roots of 1 are

$$w_k = 1\, e^{i\left(\frac{2k\pi}{n}\right)} \quad \text{where } k = 0, 1, 2, \ldots, (n-1).$$

For $k=1$,

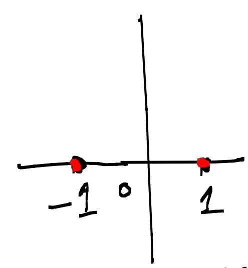
$$w_1 = e^{i\left(\frac{2\pi}{n}\right)} = \cos\left(\frac{2\pi}{n}\right) + i \sin\left(\frac{2\pi}{n}\right)$$

$$= \omega \text{ or } \omega_n \quad \text{(say)} \quad \text{Omega.}$$

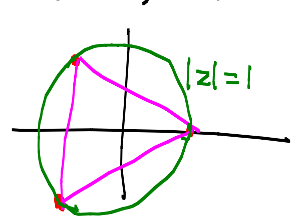
$$\boxed{w_0 = 1,\ w_1 = \omega,\ w_2 = \omega^2,\ w_3 = \omega^3, \ldots, w_{n-1} = \omega^{n-1}}$$

$$\boxed{\omega^n = 1}$$

$n=2$, $(1)^{1/2}$ are $w_0 = 1$, $w_1 = -1$.



$n=3$, $(1)^{1/3}$ are $w_0 = 1$, $w_1 = e^{i\left(\frac{2\pi}{3}\right)}$, $w_2 = e^{i\left(\frac{4\pi}{3}\right)}$



→ Cube root of unity lie on $|z|=1$ and form vertices of equilateral triangle.

→ n^{th} root of unity lie on $|z|=1$ and form the vertices of n-sided regular polygon.

Find all the cube root of $64i$.

$$64i = 64\, e^{i\left(\frac{\pi}{2} + 2k\pi\right)} \quad \text{where } k \in \mathbb{Z}$$

$$W_k = (64)^{1/3}\, e^{i\left(\frac{\frac{\pi}{2} + 2k\pi}{3}\right)} \quad \text{where } k \in \mathbb{Z}$$

$$= 4\, e^{i\left(\frac{\frac{\pi}{2} + 2k\pi}{3}\right)} \quad \text{for } k = 0, 1, 2$$

$k=0$, $W_0 = 4\, e^{i\pi/6} = 2\sqrt{3} + i2$

$k=1$, $W_1 = 4\, e^{i\frac{5\pi}{6}} = -2\sqrt{3} + i2$

$k=2$, $W_2 = 4\, e^{i\frac{3\pi}{2}} = -4i$

W_0, W_1, W_2 lie on the equilateral triangle inscribed in the circle $|z| = 4$.

Let $\alpha = \frac{m}{n}$ with $\gcd(m, n) = 1$.

α = rational number in lowest terms.

Let Z_0 be a non-zero complex number.

Find all values of Z_0^{α} $\left(\text{That is, } Z_0^{\frac{m}{n}}\right)$.

If α is a positive integer (say, $\alpha = m$, here $n=1$); then Z_0^m will be a single complex number.

If α is a positive rational number $\frac{m}{n}$, with $n \neq 1$ and $\gcd(m,n)=1$, then

$$z_0^{\frac{m}{n}} = \left(z_0^m\right)^{1/n}$$

There are n complex numbers $w_0, w_1, \ldots, w_{n-1}$ such that

$$\left(w_k\right)^n = z_0^m \quad \text{for } k=0,1,\ldots,(n-1).$$

Example: Find all values of $(-8i)^{2/3}$.

$$(-8i)^{2/3} = \left((-8i)^2\right)^{1/3} = (-64)^{1/3}$$

Now, $-64 = 64\, e^{i(\pi + 2k\pi)}$ where $k \in \mathbb{Z}$.

$$(-64)^{1/3} = (64)^{1/3}\, e^{i\left(\frac{(2k+1)\pi}{3}\right)} \qquad k=0,1,2.$$

$$= \boxed{4\, e^{i\left(\frac{(2k+1)\pi}{3}\right)} \qquad k=0,1,2}$$

Note: From $(x_0)^{1/n}$ real function $\longrightarrow$ $(z_0)^{1/n}$ complex function

Domain of n^{th} root function: Extended from set of +ve real nos. to non-zero complex numbers.

Range of n^{th} root function: Expanded from +ve real numbers set to non-zero complex numbers.

Complex functions

$$f : D \subseteq \mathbb{C} \to \mathbb{C}$$

It is a rule that assigns a complex number w in $\mathbb{C}$ to each complex number z in D.

For each $z \in D$, Point z is assigned to the value w in $\mathbb{C}$.
" " mapped " " " " .

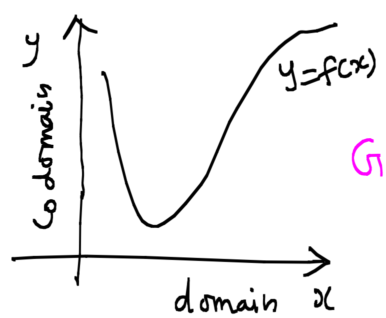
Notation: $\boxed{w = f(z) \text{ for } z \in D.}$

Usually z and w in f $(w = f(z))$ are written as $\boxed{z = x + iy}$ and $\boxed{w = u + iv}$

Visualizing function:

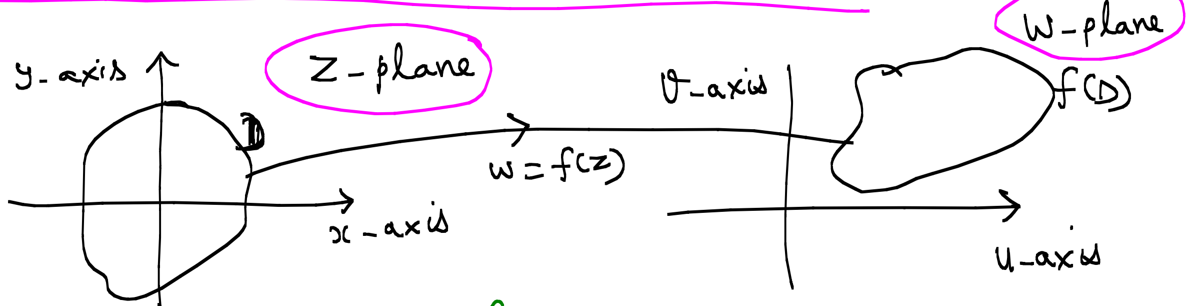
Real valued function of real variable

$$y = f(x) \text{ for } x \in S \subseteq \mathbb{R}$$



Graph of $f(x)$

complex valued function of complex variable



Take known shape/geometry in z-plane $\}\rightarrow$ Find its image and plot in the w-plane

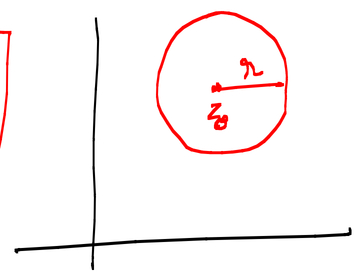
Known shapes/geometry := Straight lines, circle, rectangle, etc.

Recall: Sets and its topology (open, closed, connected, etc)

Open Ball / Open Neighborhood / Open disk:

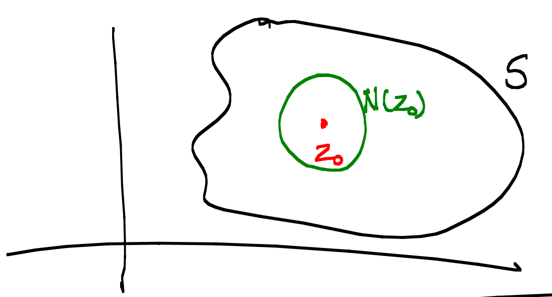
An open ball (open neighborhood) centered at the point z_0 with radius r is denoted by $B_r(z_0)$ (or $B(z_0, r)$ or $B(z_0)$ or $N(z_0)$ or $N_r(z_0)$ or $N(z_0, r)$) and is defined by

$$N(z_0,r) = \boxed{B(z_0, r) = \{ z \in \mathbb{C} \mid |z - z_0| < r \}}$$



Interior point of a set

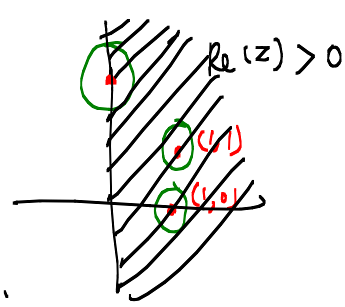
Let S be a set in $\mathbb{C}$. A point $z_0 \in \mathbb{C}$ is said to be an interior point of the set S if there exists an open neighborhood $N(z_0)$ of z_0 such that $\boxed{N(z_0) \subset S}$.



$N(z_0) \subset S$

z_0 is an interior point of S.

$$S = \{ z \in \mathbb{C} \mid Re(z) > 0 \}$$



Points $(1,0)$ and $(1,1)$ are interior points.

Point $(0,1)$ is not an interior point.

Open Sets

A set $S \subseteq \mathbb{C}$ is said to be an open set in $\mathbb{C}$ if every point of S is an interior point of S.

Example:

$\{z \in \mathbb{C} \mid |z| < 1\}$ = Open set (Not closed)

$\{z \in \mathbb{C} \mid |z| > 1\}$ = open set (Not closed)

$\{z \in \mathbb{C} \mid 2 < |z| < 3\}$ = open set (Not closed)

$\{z \in \mathbb{C} \mid |z| \leq 1\}$ = NOT open (Closed)

$\{z \in \mathbb{C} \mid \mathrm{Re}(z) > 0\}$ = open set (Not closed)

$\{z \in \mathbb{C} \mid \mathrm{Re}(z) \neq 4\}$ = open set (Not closed)

$\{z = x+iy \in \mathbb{C} \mid 0 < x < 1, y = 0\}$ = NOT open (NOT closed)

$\{z \in \mathbb{C} \mid |z-2| = 3\}$ = NOT open. (closed)

Closed set

A set $S \subseteq \mathbb{C}$ is said to be a closed set if the complement set $\mathbb{C} \setminus S$ is an open set in $\mathbb{C}$.

Examples: See above

Note: Empty set $\emptyset$ and the whole set $\mathbb{C}$ are both open and closed.

There are sets which are neither open nor closed.

$\{z = x+iy \mid 0 < x < 1, y = 0\}$ = Not open and Not closed.

Bounded Set Let S be a non-empty set in $\mathbb{C}$. Let $z_0 \in S$. If there exists an open ball $B(z_0, r)$ with centre at z_0 and radius r such that $\boxed{S \subset B(z_0, r)}$, then (That is, S can be put/embedded inside an open ball) we say that S is a bounded set in $\mathbb{C}$.

Examples:

$\{z \in \mathbb{C} \mid |z-2| < 5\}$ Bounded

$\{z \in \mathbb{C} \mid 2 < |z| < 7\}$ Bounded

$\{z \in \mathbb{C} \mid |z| \leq 5\}$ Bounded.

$\{z \in \mathbb{C} \mid |z| = 5\}$ Bounded

$\{z \in \mathbb{C} \mid Re(z) > 0\}$ = Not bounded (unbounded).

$\{z \in \mathbb{C} \mid |z-3| \geq 2\}$ = Not bounded (Unbounded).

$\{z \in \mathbb{C} \mid 1 \leq Re(z) \leq 2\}$ = Not bounded (unbounded).

$\{z \in \mathbb{C} \mid 1 \leq Re(z) \leq 2, \ -1 \leq Im(z) \leq 1\}$ = Bounded