MA 101 Real Analysis
Limit and continuity for function of several variables

1. Find the following limits, if they exist:

$$(i)\ \lim_{(x,y)\to(-1,2)} \frac{xy^3}{x+y} \qquad (ii)\ \lim_{(x,y)\to(0,0)} \frac{x-y}{x^2+y^2} \qquad (iii)\ \lim_{(x,y)\to(0,0)} \frac{x^3y}{x^6+y^2}$$

$$(iv)\ \lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} \qquad (v)\ \lim_{(x,y,z)\to(0,0,0)} \frac{xyz}{x^2+y^4+z^4} \qquad (vi)\ \lim_{(x,y,z)\to(0,0,0)} \frac{xyz}{x^3+y^3+z^3}$$

$$(vii)\ \lim_{t\to 0} F(t) \text{ where } F(t) = (2\cos t, \frac{\sin t}{t}, t^2)$$

2. Consider the function $f(x,y) = \frac{x+y}{x-y}$ for $(x,y) \in R^2$ with $x+y \neq 0$. What can you say about the existence of $\lim_{(x,y)\to(0,0)} f(x,y)$.

3. Let $f(x,y) = \frac{x^2y^2}{x^2y^2+(x-y)^2}$ whenever $x^2y^2+(x-y)^2 \neq 0$. Compute the limit $\lim_{(x,y)\to(0,0)} f(x,y)$, if it exists.

4. Show that $\lim_{(x,y)\to(0,0)} \frac{xy(x^2-y^2)}{x^2+y^2} = 0$.

5. Show that $\lim_{(x,y)\to(0,0)} \frac{\sqrt{(x^2y^2+1)}-1}{x^2+y^2} = 0$.

6. Show that $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ does not exist.

7. Show that $\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x-y}$ does not exist.

8. Suppose

$$f(x,y) = \begin{cases} \dfrac{x^2-y^2}{x^2+y^2}, & \text{for } (x,y) \neq (0,0) \\ 0, & \text{for } (x,y) = (0,0) \end{cases}$$

Is f continuous at (0,0)?

9. Suppose

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & \text{for } (x,y) \neq (0,0) \\ 0, & \text{for } (x,y) = (0,0) \end{cases}$$

Is f continuous at (0,0)?

10. Suppose

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2}, & \text{for } (x,y) \neq (0,0) \\ 0, & \text{for } (x,y) = (0,0) \end{cases}$$

Show that $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

11. Suppose

$$f(x,y) = \begin{cases} x \sin \dfrac{1}{y} + y \sin \dfrac{1}{x}, & \text{for } \ (x,y) \neq (0,0) \\ 0, & \text{for } \ (x,y) = (0,0) \end{cases}$$

Then show that $\lim_{(x,y)\to(0,0)} f(x,y)$ exist.

12. Examine the following functions for continuity at $(0,0)$. The expression below give the value at $(x,y) \neq (0,0)$. At $(0,0)$, the function value should be taken as zero.

(i) $\frac{x^2}{x^2+y^2}$

(ii) $x^p y^q \frac{x^2-y^2}{x^2+y^2}, \quad p > 0, \quad q > 0$

(iii) $xy \log(x^2+y^2)$

(iv) $|x| + |y|$

13. Explain why the function $F : [0,\infty] \to R^2$ defined by

$$\begin{aligned} F(t) &= (t+1, 2t-3), & &\text{if } \ 0 \leq t < 1 \\ &= (t^2+1, t) & &\text{if } \ 1 \leq t < 2 \\ &= (7-t, |t-4|) & &\text{if } \ 2 \leq t \end{aligned}$$

is discontinuous. At which points, it is discontinuous?