

Continuity

Let  $f: D \subseteq \mathbb{C} \rightarrow \mathbb{C}$ . Let  $z_0 \in D$ .

We say that  $f$  is continuous at  $z_0$  if

For every  $\epsilon > 0$ , there exists a  $\delta > 0$  such that

$$|z - z_0| < \delta \text{ implies that } |f(z) - f(z_0)| < \epsilon.$$

Examples:

$f(z) = z$  is continuous in  $\mathbb{C}$

$f(z) = \overline{z}$  is continuous in  $\mathbb{C}$

$f(z) = |z|$  is continuous in  $\mathbb{C}$

$f(z) = a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n$  is continuous in  $\mathbb{C}$ .

Continuity: Equivalent definitions.

①

$f$  is continuous at  $z_0$  iff  $\lim_{z \rightarrow z_0} f(z) = f(z_0)$

②

$f$  is continuous at  $z_0$  iff For every sequence  $\{z_n\} \rightarrow z_0$ , the sequence  $\{f(z_n)\} \rightarrow f(z_0)$ .

Continuous on the set  $D$ .

$f$  is continuous on the set  $D$  if  $f$  is continuous at each point of  $D$ .

### Examples of Discontinuous functions:

$$f(z) = \begin{cases} \frac{\operatorname{Re}(z)}{|z|} & \text{for } z \neq 0 \\ 0 & \text{for } z = 0 \end{cases} \quad \text{discontinuous at } z=0$$

$$g(z) = \begin{cases} \frac{\operatorname{Re}(z)}{|1+z|} & \text{for } z \neq 0 \\ 1 & \text{for } z = 0 \end{cases} \quad \begin{array}{l} \lim_{z \rightarrow 0} g(z) = 0 \\ \neq g(0) \\ \text{discontinuous} \\ \text{at } z=0 \text{ and} \\ z=-1. \end{array}$$

### Results:

① Let  $f: D \subseteq \mathbb{C} \rightarrow \mathbb{C}$  and let  $z_0 \in D$ .

$f$  is continuous at  $z_0$  iff  $u(x,y) = \operatorname{Re}(f(z))$  and  $v(x,y) = \operatorname{Im}(f(z))$  are continuous at  $z_0$ .

② Let  $f: D \subseteq \mathbb{C} \rightarrow \mathbb{C}$  and let  $z_0 \in D$ .

If  $f$  is continuous at  $z_0$  then  $\overline{f(z)}$  and  $|f(z)|$  are continuous at  $z_0$ .

③ Let  $f$  and  $g$  be continuous at  $z_0$ . Then

(i)  $f+g$  (ii)  $f-g$  (iii)  $kf$  ( $k=\text{constant}$ ) (iv)  $fg$   
are continuous at  $z_0$ .

(V)  $\frac{f}{g}$  is continuous at  $z_0$ , provided  $g(z_0) \neq 0$ .

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### Composition of two continuous functions

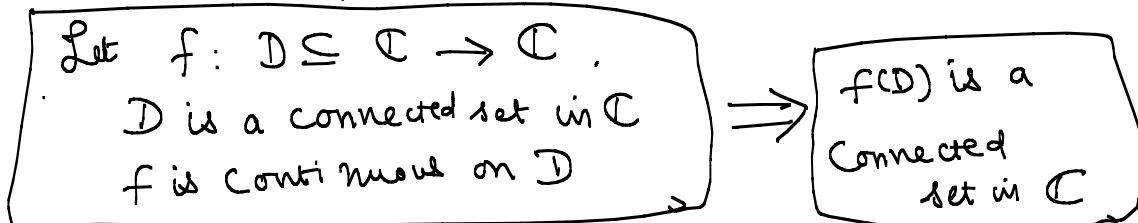
Suppose  $f$  is continuous at  $z_0$  and  $g$  is continuous at  $f(z_0)$ . Then

the function  $h(z) = g(f(z))$  is continuous at  $z_0$ .

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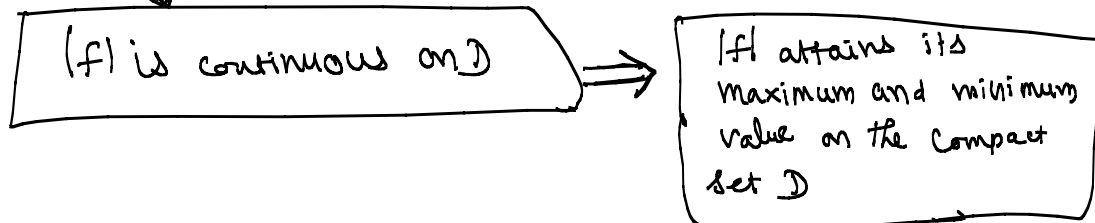
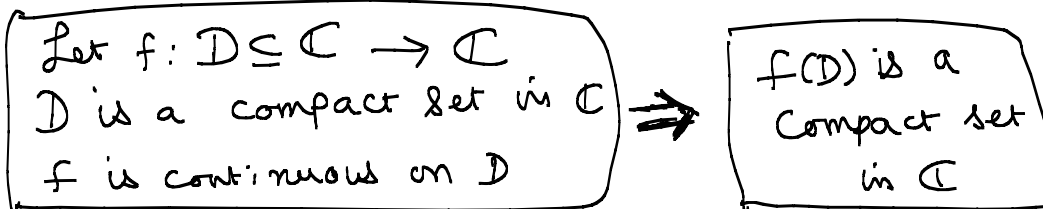
Result: Continuous image of connected set is connected.

That is,



Result: Continuous image of compact set is compact.

That is,



## Differentiation

Recall from MATH 101:

Let  $f: D \subseteq \mathbb{R} \rightarrow \mathbb{R}$ .  $D$  is an open set in  $\mathbb{R}$ .  
Let  $x_0 \in D$ . We say that  $f$  is differentiable at  $x_0$  if  $\lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0}$  exists.

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In the SAME WAY, we generalize it to  $\mathbb{C}$

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Let  $f: D \subseteq \mathbb{C} \rightarrow \mathbb{C}$ . Let  $D$  be an open set in  $\mathbb{C}$  and let  $z_0 \in D$ .

We say that  $f$  is differentiable at  $z_0$  if

$$\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0} \text{ exists}$$

The value of this limit is denoted by  $f'(z_0)$

$$f'(z_0) = \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

 $\rightarrow (*)$

and it is called the derivative of  $f$  at  $z_0$ .

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Set  $\Delta z = z - z_0$

$$\Rightarrow z = z_0 + \Delta z.$$

$(*)$  can be rewritten as

$$f'(z_0) = \lim_{\Delta z \rightarrow 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

Example: Let  $f(z) = z$  for  $z \in \mathbb{C}$ .

Let  $z_0$  be an arbitrary point in  $\mathbb{C}$ .

Check whether  $f$  is differentiable at  $z_0$ .

$$\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \rightarrow z_0} \frac{z - z_0}{z - z_0} = \lim_{z \rightarrow z_0} 1 = 1$$

$$f'(z_0) = 1, \quad f \text{ is differentiable at } z_0.$$

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That is, the function  $f(z) = z$  is differentiable at each point  $z$  in  $\mathbb{C}$  and the derivative is given by

$$f'(z) = 1 \quad \text{for all } z \in \mathbb{C}$$

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Example: Let  $f(z) = \bar{z}$  for  $z \in \mathbb{C}$ .

Let  $z_0$  be an arbitrary point in  $\mathbb{C}$ .

Check whether  $f(z) = \bar{z}$  is differentiable at  $z_0$ ?

$$\begin{aligned} \lim_{\Delta z \rightarrow 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} &= \lim_{\Delta z \rightarrow 0} \frac{\overline{z_0 + \Delta z} - \bar{z}_0}{\Delta z} \\ &= \lim_{\Delta z \rightarrow 0} \frac{\overline{\Delta z}}{\Delta z} \end{aligned}$$

Path I:  $\Delta z \rightarrow 0$  along the real axis.  $\Delta y = 0$  and  $\Delta x \rightarrow 0$ .

$$\lim_{\substack{\Delta y = 0 \\ \Delta x \rightarrow 0}} \frac{\overline{\Delta z}}{\Delta z} = \lim_{\substack{\Delta y = 0 \\ \Delta x \rightarrow 0}} \frac{\Delta x}{\Delta x} = 1$$

Path II:  $\Delta z \rightarrow 0$  along the imaginary axis.  $\Delta x = 0$  and  $\Delta y \rightarrow 0$ .

$$\lim_{\substack{\Delta x=0 \\ \Delta y \rightarrow 0}} \frac{\overline{\Delta z}}{\Delta z} = \lim_{\substack{\Delta x=0 \\ \Delta y \rightarrow 0}} \frac{-i \Delta y}{i \Delta y} = -1$$

Since  $\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$  approaches two different values

as  $\Delta z \rightarrow 0$  along two different values, we conclude that

$\lim_{\Delta z \rightarrow 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$  does not exist, and hence  $f'(z_0)$

does not exist. Therefore,  $f$  is NOT differentiable at  $z_0$ .

$f(z) = \overline{z}$  is nowhere differentiable in  $\mathbb{C}$

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Additional Information to think:

Consider the same function  $f(z) = \overline{z}$  as

$$f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

$$f(x, y) = (x, -y) \text{ for } (x, y) \in \mathbb{R}^2 \text{ as}$$

vector valued function of vector variable.

Find out whether  $f(x, y) = (x, -y)$  is differentiable in  $\mathbb{R}^2$  or not?

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Exercise (for tomorrow's class)

Examine the differentiability of the following functions in  $\mathbb{C}$ .

$$\textcircled{1} f(z) = |z| \quad \textcircled{2} f(z) = |z|^2 \quad \textcircled{3} f(z) = z^2$$

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Lecture 5 ends

Division - I