**Question 12:** Let $\{x_n\}$ be a sequence in $\mathbb{R}$. Prove or disprove the following statements:

**(i)** Let $x_n = (-1)^n \forall n \in \mathbb{N}$. The sequence $(x_n)$ does not converge.

**(ii)** If $x_n \to l (l \neq 0)$ and $\{y_n\}$ is a bounded sequence, then $(x_n y_n)$ converges.

**(iii)** If the sequence $(x_n^2 + \frac{1}{n} x_n)$ converges then $(x_n)$ converges.

**Answer- 12.(i).** If possible let $(-1)^n$ converges to $l \in R$. Then by the defination of convergence we know that for any $\epsilon > 0$ there exists a natural number $m > 0$ such that

$$|(-1)^n - l| < \epsilon, \forall n \geq m \tag{5}$$

. choose, $\epsilon = \frac{1}{2}$, and let $k$ be an integer such that $k$ is even and $k > m$. Then by (i) we have

$$|(-1)^k - l| = |1 - l| < \frac{1}{2}$$

which implies

$$-\frac{1}{2} < 1 - l < \frac{1}{2}$$

. Now putting $n = k + 1$ in (i) we have

$$|(-1)^{k+1} - l| = |-1 - l| = |1 + l| < \frac{1}{2}$$

which implies

$$-\frac{1}{2} < 1 + l < \frac{1}{2}$$



. Now $2 = (1 - l) + (1 + l) < \frac{1}{2} + \frac{1}{2} = 1$ which is not possible therefore there does not exists any $l \in R$ such that the sequence converges.

which implies

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. Now putting $n = k+1$ in (i) we have

$$|(-1)^{k+1} - l| = |-1-l| = |1+l| < \frac{1}{2}$$

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. Now $2 = (1-l) + (1+l) < \frac{1}{2} + \frac{1}{2} = 1$ which is not possible therefore there does not exists any $l \in \mathbb{R}$ such that the sequence converges.

**Answer 12.(ii).Let** $x_n = 1, \forall n$ **and** $y_n = (-1)^n$**. Then** $x_n$ **being a constant sequence allways convergent and** $|y_n| = 1 < 2$ **which is bounded. But** $\{x_n y_n\} = \{(-1)^n\}$ **does not converges.**

Answer 12.(iii). Let $x_n = (-1)^n$. Then $x_n^2 + \frac{x_n}{n} = 1 + \frac{(-1)^n}{n}$. Let $a_n = 1 + \frac{(-1)^n}{n}$. Then

$$|a_n - 1| = |\frac{(-1)^n}{n}| = |\frac{1}{n}|$$

. Now let $\epsilon > 0$ be arbitary now $\frac{1}{n} < \epsilon$ implies $n > \frac{1}{\epsilon}$. Let $\lfloor \frac{1}{\epsilon} \rfloor = m$, therefore for any $\epsilon > 0$ there exists a natural number $m$ such that

. Now $2 = (1-l) + (1+l) < \frac{1}{2} + \frac{1}{2} = 1$ which is not possible therefore there does not exists any $l \in \mathbb{R}$ such that the sequence converges.

**Answer 12.(ii).**Let $x_n = 1, \forall n$ and $y_n = (-1)^n$. Then $x_n$ being a constant sequence allways convergent and $|y_n| = 1 < 2$ which is bounded. But $\{x_n y_n\} = \{(-1)^n\}$ does not converges.

**Answer 12.(iii).** Let $x_n = (-1)^n$. Then $x_n^2 + \frac{x_n}{n} = 1 + \frac{(-1)^n}{n}$. Let $a_n = 1 + \frac{(-1)^n}{n}$. Then

$$|a_n - 1| = \left|\frac{(-1)^n}{n}\right| = \left|\frac{1}{n}\right|$$

. Now let $\epsilon > 0$ be arbitary now $\frac{1}{n} < \epsilon$ implies $n > \frac{1}{\epsilon}$. Let $\lfloor \frac{1}{\epsilon} \rfloor = m$, therefore for any $\epsilon > 0$ there exists a natural number $m$ such that

. Now $2 = (1-l) + (1+l) < \frac{1}{2} + \frac{1}{2} = 1$ which is not possible therefore there does not exists any $l \in \mathrm{R}$ such that the sequence converges.

**Answer 12.(ii).** Let $x_n = 1, \forall n$ and $y_n = (-1)^n$. Then $x_n$ being a constant sequence allways convergent and $|y_n| = 1 < 2$ which is bounded. But $\{x_n y_n\} = \{(-1)^n\}$ does not converges.

**Answer 12.(iii).** Let $x_n = (-1)^n$. Then $x_n^2 + \frac{x_n}{n} = 1 + \frac{(-1)^n}{n}$. Let $a_n = 1 + \frac{(-1)^n}{n}$. Then

$$|a_n - 1| = |\frac{(-1)^n}{n}| = |\frac{1}{n}|$$

. Now let $\epsilon > 0$ be arbitary now $\frac{1}{n} < \epsilon$ implies $n > \frac{1}{\epsilon}$. Let $\lfloor \frac{1}{\epsilon} \rfloor = m$, therefore for any $\epsilon > 0$ there exists a natural number $m$ such that

$$|a_n - 1| = |\frac{1}{n}| < \epsilon, \forall n \geq m$$

. hence $a_n$ converges to $1$. But $x_n$ does not converges.

**Question 13:** Let $(x_n)$ be a sequence defined by

$$x_n = (1+\alpha)^{-n} n^{\beta} \cos n.$$

for all $n \in \mathbb{N}$ where $\alpha$ and $\beta$ are fixed positive real numbers. Show that $(x_n)$

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Question 13: Let $(x_n)$ be a sequence defined by

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for all $n \in \mathbb{N}$ where $\alpha$ and $\beta$ are fixed positive real numbers. Show that $(x_n)$ converges.

Answer 13. By the ratio test of convergence we know that if $a_n$ is a sequence of positive real numbers such that $lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$ and if $L < 1$, then $a_n$ converges to 0. Now let $y_n = (1+\alpha)^{-n} n^{\beta}$. Now $lim_{n\to\infty} \frac{y_{n+1}}{y_n} = \frac{1}{1+\alpha} < 1$, hence $y_n$ converges to 0. Let $M > 1$ be such that $|cos\ n| \leq M$. Since $y_n$ is convergent given $\epsilon > 0$ for the positive number $\frac{\epsilon}{M}$, there exists some $N$ such that $|y_n| < \frac{\epsilon}{M}$ for all $n \geq N$.So

$$|(1+\alpha)^{-n} n^{\beta} cos\ n - 0| = |y_n cos\ n| \leq M|y_n| < M\frac{\epsilon}{M} = \epsilon, \forall n \geq N$$

. This shows that for all $\epsilon > 0$ there exists a natural number $N$ such that

$$|(1+\alpha)^{-n} n^{\beta} cos\ n| < \epsilon, \forall n \geq N$$

. Hence the sequence $x_n = (1+\alpha)^{-n} n^{\beta} cos\ n$ converges to 0.

Question 14: Show directly from the definition that if $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences, then $\{x_n + y_n\}$ and $\{x_n y_n\}$ are Cauchy sequences.

Solution: Let $\epsilon > 0$ be given.

Since $\{x_n\}$ is a Cauchy sequence, for $\epsilon_1 = \frac{\epsilon}{2} > 0$ there exists an $N_1 \in \mathbb{N}$ such that

13) $x_n = (1+\alpha)^n n^P \cos n$

$$= \frac{n^\beta}{(1+\alpha)^n} \cos n$$

$n^\beta < (1+\alpha)^n$

for fixed $\alpha$ and $P$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$$

if $L < 1$ then $a_n \to 0$

$$y_n = \frac{n^\beta}{(1+\alpha)^{+n}} = \lim_{n\to\infty} \frac{y_{n+1}}{y_n}$$

$(n+1)^\beta \qquad (1+\alpha)^n$

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$$\boxed{\text{if } L < 1 \text{ then } a_n \to 0}$$

$$y_n = \frac{n^\beta}{(1+\alpha)^{+n}} = \lim_{n\to\infty} \frac{y_{n+1}}{y_n}$$

$$= \lim_{n\to\infty} \frac{(n+1)^\beta}{(1+\alpha)^{n+1}} \cdot \frac{(1+\alpha)^n}{n^\beta}$$

$$= \lim_{n\to\infty} \frac{1}{1+\alpha} \left(1+\frac{1}{n}\right)^\beta$$

$$= \lim_{n\to\infty} \frac{1}{1+\alpha} < 1$$

hence $y_n \to 0$

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$$|x_m - x_n| < \epsilon_1 = \frac{\epsilon}{2} \quad (*)$$

Since $\{y_n\}$ is a Cauchy sequence, for $\epsilon_2 = \frac{\epsilon}{2} > 0$ there exists an $N_2 \in \mathbb{N}$ such that if $n, m \geq N_2$ then:

$$|y_m - y_n| < \epsilon_2 = \frac{\epsilon}{2} \quad (**)$$



Let $N = \max\{N_1, N_2\}$. Then if $n, m \geq N$ we have that both $(*)$ and $(**)$ hold, so:

$$\begin{aligned} |\{x_m + y_m\} - \{x_n + y_n\}| &= |\{x_m - x_n\} + \{y_m - y_n\}| \\ &\leq |x_m - x_n| + |y_m - y_n| < \epsilon_1 + \epsilon_2 = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Therefore $\{x_n + y_n\}$ is a Cauchy sequence.
Similarly, we get

$$\begin{aligned} |x_n y_n - x_m y_m| &= |x_n y_n - x_m y_n + x_m y_n - x_m y_m| \\ &= |y_n (x_n - x_m) + x_m (y_n - y_m)| \\ &< |y_n| |x_n - x_m| + |x_m| |y_n - y_m| \end{aligned}$$

Since $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences and every Cauchy sequence is bounded, it follows that for $0 < M_1, M_2 \in \mathbb{R}$ that $|x_n| \leq M_1$ and $|y_n| \leq M_2$ for all $n \in \mathbb{N}$ and so:

**Question 19:** Let $\{x_n\}$ be a sequence of positive real numbers such that $L := \lim\left(\frac{x_{n+1}}{x_n}\right)$ exists. If $L < 1$, then $\{x_n\}$ converges and $\lim(x_n) = 0$.



**Solution:** Let us choose a positive $\epsilon$ such that $L + \epsilon < 1$.
Since $\lim\left(\frac{x_{n+1}}{x_n}\right) = L$, there exists a natural number $k$ such that

$$L - \epsilon < \left(\frac{x_{n+1}}{x_n}\right) < L + \epsilon \textit{ for all } n \geq k.$$

Let $L + \epsilon = r$. Then $0 < r < 1$.
Therefore $\left(\frac{x_{n+1}}{x_n}\right) < r$ for all $n \geq k$.
Hence we have

$$\left(\frac{x_{k+1}}{x_k}\right) < r, \ \left(\frac{x_{k+2}}{x_{k+1}}\right) < r, \ldots \left(\frac{x_n}{x_{n+1}}\right) < r, \ \textit{for } n \geq k+1.$$

Multiplying, $\left(\frac{x_n}{x_k}\right) < r^{n-k}$ for $n \geq k+1$
or, $x_n < \left(\frac{x_k}{r^k}\right) r^n$ for $n \geq k+1$.

**Solution:** Let us choose a positive $\epsilon$ such that $L + \epsilon < 1$.
Since $\lim\left(\frac{x_{n+1}}{x_n}\right) = L$, there exists a natural number $k$ such that

$$L - \epsilon < \left(\frac{x_{n+1}}{x_n}\right) < L + \epsilon \textit{ for all } n \geq k.$$

Let $L + \epsilon = r$. Then $0 < r < 1$.
Therefore $\left(\frac{x_{n+1}}{x_n}\right) < r$ for all $n \geq k$.
Hence we have

$$\left(\frac{x_{k+1}}{x_k}\right) < r, \ \left(\frac{x_{k+2}}{x_{k+1}}\right) < r, \ldots \left(\frac{x_n}{x_{n+1}}\right) < r, \textit{ for } n \geq k+1.$$

Multiplying, $\left(\frac{x_n}{x_k}\right) < r^{n-k}$ for $n \geq k+1$
or, $x_n < \left(\frac{x_k}{r^k}\right) r^n$ for $n \geq k+1$.
Now $\lim r^n = 0$ since $0 < r < 1$; and $\left(\frac{x_k}{r^k}\right)$ is a fixed positive number.
Therefore $\lim(x_n) = 0$.
**Question 20:** Show that if $p > 0$ and $\alpha$ is real, then $\lim_{n\to\infty} \frac{n^\alpha}{(1+p)^n} = 0$.

**Solution:** Let us consider the series:

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n^\alpha}{(1+p)^n}$$

From D'Alembert's ratio test for a series of positive real numbers $\sum_{n=1}^{\infty} a_n$, we know that, if,

$$L = \lim\left(\frac{a_{n+1}}{}\right).$$

Now $\lim r^n = 0$ since $0 < r < 1$; and $(\frac{1}{r^k})$ is a fixed positive number.
Therefore $\lim(x_n) = 0$.
**Question 20: Show that if $p > 0$ and $\alpha$ is real, then $\lim_{n\to\infty} \frac{n^\alpha}{(1+p)^n} = 0$.**

**Solution: Let us consider the series:**

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n^\alpha}{(1+p)^n}$$

**From D'Alembert's ratio test for a series of positive real numbers $\sum_{n=1}^{\infty} a_n$, we know that, if,**

$$L = \lim_{n\to\infty} \left(\frac{a_{n+1}}{a_n}\right),$$

**then $\sum_{n=1}^{\infty} a_n$ is convergent if $L < 1$, $\sum_{n=1}^{\infty} a_n$ is divergent if $L > 1$. In this case, we have:**

$$\begin{aligned} L &= \lim_{n\to\infty} \left(\frac{a_{n+1}}{a_n}\right) \\ &= \lim_{n\to\infty} \left(\frac{(n+1)^\alpha}{(1+p)^{n+1}} \frac{(1+p)^n}{n^\alpha}\right) \\ &= \lim_{n\to\infty} \left[\left(1+\frac{1}{n}\right)^\alpha \frac{1}{1+p}\right] \\ &= \frac{1}{1+p} < 1 \end{aligned}$$

**As this limit is less than 1, the series is convergent, which is only possible if the sequence $\frac{n^\alpha}{(1+p)^n} \to 0$ as $n \to \infty$. This completes the proof.**

**Question 21: Let C be a sequence defined by the recursion formula : $x_{n+1} =$**

Question 21: Let C be a sequence defined by the recursion formula : $x_{n+1} = \sqrt{7 + x_n}, x_1 = \sqrt{7}$. Show that $\{x_n\}$ converges to the positive root of $x^2 - x - 7 = 0$

Solution: From the given recursion formula, we have:

$$x_{n+1}^2 = 7 + x_n, \quad x_n^2 = 7 + x_{n-1}$$



for all $n > 1$. Thus, we have:

$$x_{n+1}^2 - x_n^2 = x_n - x_{n-1} \quad \text{or, } (x_{n+1} + x_n)(x_{n+1} - x_n) = x_n - x_{n-1}$$

for all $n > 1$. Since $x_n > 0$ for all $n$, $x_{n+1} >$ or $< x_n$ according as $x_n >$ or $< x_{n-1}$. But $x_2 > x_1$ and consequently, $x_3 > x_2$, $x_4 > x_3$, ... and so on. Therefore, $\{x_n\}$ is a monotone increasing sequence.

Again we have:

$$x_n^2 < x_{n+1}^2 = 7 + x_n$$

for all naturals $n$. This implies that:

$$x_n^2 - x_n - 7 < 0, \quad \text{or, } (x_n - a)(x_n - b) < 0$$

for all $n > 1$. Thus, we have:

$$x_{n+1}^2 - x_n^2 = x_n - x_{n-1} \quad \text{or, } (x_{n+1} + x_n)(x_{n+1} - x_n) = x_n - x_{n-1}$$

for all $n > 1$. Since $x_n > 0$ for all $n$, $x_{n+1} >$ or $< x_n$ according as $x_n >$ or $< x_{n-1}$. But $x_2 > x_1$ and consequently, $x_3 > x_2$, $x_4 > x_3$, ... and so on. Therefore, $\{x_n\}$ is a monotone increasing sequence.

Again we have:

$$x_n^2 < x_{n+1}^2 = 7 + x_n$$

for all naturals $n$. This implies that:

$$x_n^2 - x_n - 7 < 0, \quad \text{or, } (x_n - a)(x_n - b) < 0$$

where $a, b$ are the roots of the equation $x^2 - x - 7 = 0$. One of the roots is negative and the other is positive. Without the loss of generality, let us assume that $a < 0$. Since $x_n > 0$ for all $n$, $x_n > a$, and hence, $x_n < b$ for all $n$. This proves that the sequence $\{x_n\}$ is bounded above and therefore the sequence is convergent.

Let $\lim x_n = L$. By definition, $x_{n+1}^2 = 7 + x_n$ for all $n$. Taking limits on both sides as $n \to \infty$, we have:

$$L^2 = 7 + L$$

Therefore, $(L - a)(L - b) = 0$. But $L \neq a$, since each element of the sequence is positive and $a < 0$. Therefore $L = b$. That is, the sequence converges to the positive root of the equation $x^2 - x - 7 = 0$.

Question 22: Show that the sequence $\{x_n\} = 1 - \frac{1}{2} + \frac{1}{3} + \cdots + (-1)^{n+1}\frac{1}{n}$ is convergent.

**Solution:** We will approach this question by forming complementary subse-

Question 22: Show that the sequence $\{x_n\} = 1 - \frac{1}{2} + \frac{1}{3} + \cdots + (-1)^{n+1}\frac{1}{n}$ is convergent.

**Solution:** We will approach this question by forming complementary subsequences and check whether they are convergent or not. We have

$$\begin{aligned}
x_n &= 1 - \frac{1}{2} + \frac{1}{3} + \cdots + (-1)^{n+1}\frac{1}{n} \\
\implies x_{2n} &= 1 - \frac{1}{2} + \frac{1}{3} + \cdots + (-1)^{2n+1}\frac{1}{2n} \\
&= 1 - \frac{1}{2} + \frac{1}{3} + \cdots + \frac{(-1)}{2n} \\
&= 1 - \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2n-1} - \frac{1}{2n}
\end{aligned}$$

Let $\{y_n\} = \{x_{2n}\}$, then

$$\begin{aligned}
y_n - y_{n-1} &= \left(1 - \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2n-1} - \frac{1}{2n}\right) - \left(1 - \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2n-3} - \frac{1}{2n-2}\right) \\
&= \frac{1}{2n-1} - \frac{1}{2n} > 0
\end{aligned}$$

$\implies \{y_n\} > \{y_{n-1}\}$

$\implies \{y_n\}$ is monotonically increasing sequence.

$\implies \{x_{2n}\}$ is monotonically increasing sequence.

Now we show that $\{x_{2n}\}$ is bounded above.

$$
\begin{aligned}
x_{2n} &= 1-\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2n-1}-\frac{1}{2n}\\
&= 1-\left(\frac{1}{2}-\frac{1}{3}+\frac{1}{4}+\cdots-\frac{1}{2n-1}+\frac{1}{2n}\right)\\
&= 1-\left(\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)+\cdots+\left(\frac{1}{2n-2}-\frac{1}{2n-1}\right)+\frac{1}{2n}\right)\\
&= 1-\lambda, \quad \lambda>0 \text{ where}
\end{aligned}
$$

$$
\lambda=\left(\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)+\cdots+\left(\frac{1}{2n-2}-\frac{1}{2n-1}\right)+\frac{1}{2n}\right)
$$

$\implies \{x_{2n}\} \leq 1$

$\implies \{x_{2n}\}$ is convergent $\forall\ n \in \mathbb{N}$

Similarly, we can show that $\{x_{2n+1}\}$ is convergent.
Let $\lim\limits_{n\to\infty} x_{2n} = l_1$ and $\lim\limits_{n\to\infty} x_{2n+1} = l_2$

Now consider $x_{n+1}-x_n = (-1)^{n+1}\frac{1}{n}$

Taking $n\to\infty$, we have $l_1-l_2=0 \implies l_1=l_2$.
Since $\{x_{2n+1}\}$, $\{x_{2n}\}$ forms complementary sequences of $\{x_n\}$ and converges to same limit.
$\therefore \{x_n\}$ is convergent.

Question 23: Let $\lim_{n\to\infty} x_n = 0$, then prove that $\lim_{n\to\infty}\frac{x_1+x_2+\cdots+x_n}{n}=0$.
Solution: (Recall:) A sequence $\{x_n\}$ converges to a real number $A$ if and only if for each real number $\epsilon>0$, there exists a positive integer $n^*$ such that

$\therefore \{x_n\}$ is convergent.

Question 23: Let $\lim_{n\to\infty} x_n = 0$, then prove that $\lim_{n\to\infty} \frac{x_1+x_2+\cdots+x_n}{n} = 0$.
Solution: (Recall:) A sequence $\{x_n\}$ converges to a real number $A$ if and only if for each real number $\epsilon > 0$, there exists a positive integer $n^*$ such that

$$|x_n - A| < \epsilon \quad \text{for all} \quad n \geq n^*$$

Here, we are given that $\lim\limits_{n\to\infty} x_n = 0$

$$\implies |x_n - 0| < \frac{\epsilon}{2} \quad \forall n > n_0$$

$$\implies |x_n| < \frac{\epsilon}{2} \quad \forall n > n_0 \tag{6}$$

We can write

$$\frac{x_1+x_2+\cdots+x_n}{n} = \frac{x_1+x_2+\cdots+x_{n_0}}{n} + \frac{x_{n_0+1}+x_{n_0+2}+\cdots+x_n}{n}$$

$$\implies \left|\frac{x_1+x_2+\cdots+x_n}{n}\right| = \left|\frac{x_1+x_2+\cdots+x_{n_0}}{n} + \frac{x_{n_0+1}+x_{n_0+2}+\cdots+x_n}{n}\right|$$

$$\implies \left|\frac{x_1+x_2+\cdots+x_n}{n}\right| \leq \left|\frac{x_1+x_2+\cdots+x_{n_0}}{n}\right| + \left|\frac{x_{n_0+1}+x_{n_0+2}+\cdots+x_n}{n}\right| \tag{7}$$

Since, $\lim\limits_{n\to\infty} x_n = 0$ which implies $\{x_n\}$ is convergent. And, we know every convergent sequence sequence is bounded. So, $\{x_n\}$ is a bounded sequence.

$$\text{Let,} \quad k \leq x_n \leq K \quad \forall n \in \mathbb{N} \tag{8}$$

From Eq.(7), we have

$$\left|\frac{x_1 + x_2 + \cdots + x_{n_0}}{n}\right| \leq \frac{|x_1| + |x_2| + \cdots + |x_{n_0}|}{n}$$

$$\implies \left|\frac{x_1 + x_2 + \cdots + x_{n_0}}{n}\right| \leq \frac{K + K + \cdots + n_0 - \textbf{time}}{n} \quad \left(\textbf{From Eq.}(8)\right)$$

$$\implies \left|\frac{x_1 + x_2 + \cdots + x_{n_0}}{n}\right| \leq \tfrac{n_0 K}{n}$$

**Thus, if** $\dfrac{n_0 K}{n} < \dfrac{\epsilon}{2}$, **then** $\left|\dfrac{x_1 + x_2 + \cdots + x_{n_0}}{n}\right| \leq \dfrac{n_0 K}{n} < \dfrac{\epsilon}{2}$

$$\implies \quad \textbf{if } \frac{n}{n_0 K} > \frac{2}{\epsilon}, \textbf{ then } \left|\frac{x_1 + x_2 + \cdots + x_{n_0}}{n}\right| < \frac{\epsilon}{2}$$

$$\implies \quad \textbf{if } n > \frac{2 n_0 K}{\epsilon}, \textbf{ then } \left|\frac{x_1 + x_2 + \cdots + x_{n_0}}{n}\right| < \frac{\epsilon}{2}$$

$$\textbf{Thus,} \quad \left|\frac{x_1 + x_2 + \cdots + x_{n_0}}{n}\right| < \frac{\epsilon}{2} \quad \forall\, n > n_0{}' \left(= \frac{2 n_0 K}{\epsilon}\right) \tag{9}$$

**Again from Eq.**(7)**, we have**

$$\left|\frac{x_{n_0+1} + x_{n_0+2} + \cdots + x_n}{n}\right| \leq \frac{|x_{n_0+1}| + |x_{n_0+2}| + \cdots + |x_n|}{n}$$

**From Eq.**(6), $\left|\dfrac{x_{n_0+1} + x_{n_0+2} + \cdots + x_n}{n}\right| < \dfrac{\frac{\epsilon}{2} + \frac{\epsilon}{2} + \cdots + \frac{\epsilon}{2}}{n}$

$$\implies \left|\frac{x_{n_0+1} + x_{n_0+2} + \cdots + x_n}{n}\right| < \frac{(n - n_0)\epsilon}{2n}$$

$$\left|\frac{x_{n_0+1}+x_{n_0+2}+\cdots+x_n}{n}\right| \leq \frac{|x_{n_0+1}|+|x_{n_0+2}|+\cdots+|x_n|}{n}$$

**From Eq.**(6), $\left|\frac{x_{n_0+1}+x_{n_0+2}+\cdots+x_n}{n}\right| < \frac{\frac{\epsilon}{2}+\frac{\epsilon}{2}+\cdots+\frac{\epsilon}{2}}{n}$

$$\implies \left|\frac{x_{n_0+1}+x_{n_0+2}+\cdots+x_n}{n}\right| < \frac{(n-n_0)\epsilon}{2n}$$

$$\implies \left|\frac{x_{n_0+1}+x_{n_0+2}+\cdots+x_n}{n}\right| < \frac{\epsilon}{2}-\frac{n_0\epsilon}{2n}$$

$$\implies \left|\frac{x_{n_0+1}+x_{n_0+2}+\cdots+x_n}{n}\right| < \frac{\epsilon}{2} \quad \forall n > n_0 \tag{10}$$

**Take** $N_0 =$ **Max** $(n_0, n_0')$

**Eq.**(7) $\implies \left|\frac{x_1+x_2+\cdots+x_n}{n}\right| < \frac{\epsilon}{2}+\frac{\epsilon}{2} \quad \forall n > N_0$

$$\implies \left|\frac{x_1+x_2+\cdots+x_n}{n}\right| < \epsilon \quad \forall n > N_0$$

$$\implies \lim_{n\to\infty}\frac{x_1+x_2+\cdots+x_n}{n} = 0$$

**Hence, proved**

$$(0,1) = \bigcup_{i=1}^{\infty} (\frac{1}{i}, 1).$$

**Question 24.** Prove that $e$ is an irrational number.
**Solution.** We know

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$$

$$e^1 = e = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} + \cdots \text{ Then } e - e_n > 0$$

$$\text{Let } e_n = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}$$

$$\begin{aligned} e - e_n &= \lim_{N\to\infty} \left[\frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \cdots + \frac{1}{(n+N)!}\right] \\ &= \frac{1}{(n+1)!} \lim_{N\to\infty} \left[ 1 + \tfrac{1}{n+2} + \cdots + \tfrac{1}{(n+2)(n+3)\cdots(n+N-1)(n+N)} \right] \\ &= \frac{1}{(n+1)!} \lim_{N\to\infty} \frac{1 - \frac{1}{(n+1)^N}}{1 - \frac{1}{n+1}} \\ &= \frac{1}{(n+1)!} \cdot \frac{n+1}{n} \end{aligned}$$

Thus, $0 < e - e_n < \frac{1}{n \cdot n!} \cdots (1)$
If $e$ is a rational number then $e = \frac{p}{q}$, where $p$ and $q$ are integers and $q \neq 0$.
Then we have from (1)

$$= \frac{1}{(n+1)!} \lim_{N\to\infty} \frac{1-\frac{1}{(n+1)^N}}{1-\frac{1}{n+1}}$$

$$= \frac{1}{(n+1)!} \cdot \frac{n+1}{n}$$

Thus, $0 < e - e_n < \frac{1}{n \cdot n!} \ldots (1)$
If $e$ is a rational number then $e = \frac{p}{q}$, where $p$ and $q$ are integers and $q \neq 0$.
Then we have from (1)
$0 < q!(e - e_q) < \frac{1}{q} \ldots (2)$
Now, $q! \times e$ is an integer, since $eq = p$ is an integer.

Also, $q! \times e_q = q!\left(\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} \cdots + \frac{1}{q!}\right)$ is an integer. Thus $q!(e - e_q)$ is an integer and from (2) we see that it lies strictly in between 0 and 1 which is absurd. Therefore, $e$ is an irrational number.

**Solution 23:** Take $x \in (0,1)$.

Then $0 < x < 1$.
By using Archimedean property on $x$ and $1$,
we have $nx > 1$, for some positive integer $n$,
i.e. $\frac{1}{n} < x < 1$
thus $x \in \left(\frac{1}{n}, 1\right) \subseteq \bigcup_{i=1}^{\infty} \left(\frac{1}{i}, 1\right)$

hence, $(0,1) \subseteq \bigcup_{i=1}^{\infty} \left(\frac{1}{i}, 1\right) \ldots (1)$
Again take,

$$y \in \bigcup_{i=1}^{\infty} \left(\frac{1}{i}, 1\right)$$

Then $y \in \left(\frac{1}{m}, 1\right)$, for some positive integer $m$.

$$\text{So, } 0 < \frac{1}{m} < y < 1 \text{ and hence } \bigcup_{i=1}^{\infty} \left(\frac{1}{i}, 1\right) \subseteq (0,1) \ldots (2)$$

Combining (1) and (2)

$$(0,1) = \bigcup_{i=1}^{\infty} \left(\frac{1}{i}, 1\right).$$

**Question 24.** Prove that $e$ is an irrational number.
**Solution.** We know

Combining all the inequalities, the result follows.

(ii) Prove that $\sup(S \bigcup T) = \max\{\sup S, \sup T\}$.

**Solution 19:** Let $x \in S \bigcup T$

$\Rightarrow x \in S$ or $x \in T \Rightarrow x \leq \sup S$ or $x \leq \sup T \Rightarrow x \leq \max\{\sup S, \sup T\}$ . So $\max\{\sup S, \sup T\}$ is an upper bound of $S \bigcup T$.

Let $M$ is any other upper bound of $S \bigcup T$. Then $\forall x \in S \bigcup T$, $x \leq M$.

$\Rightarrow \forall x \in S \bigcup T$, $x \leq M$. $\Rightarrow \sup S \leq M$ and $\sup T \leq M$. $\Rightarrow \max\{\sup A, \sup B\} \leq M$.

So, $\max\{\sup A, \sup B\}$ is the least upper bound of $S \bigcup T$.

**Question 20:** If $y > 0$, show that there exist $n \in \mathbb{N}$ such that

$$\frac{1}{2^n} < y$$

**Solution 20:** If possible let us suppose that for all $n \in \mathbb{N}$, $\frac{1}{2^n} \geq y$.

Now, for $n \in \mathbb{N}$, $2^n > 0$ and we know that $2^n > n$ for all $n \in \mathbb{N}$. Hence $y \leq \frac{1}{2^n} < \frac{1}{n}$ for all $n \in \mathbb{N}$. This implies that $y$ will be the lower bound for $1/n$. (Argument 1)

But from the Archimedean property there exists a positive integer $m$ such that $my > 1$ i.e. $y > \frac{1}{m}$ which contradicts the Argument 1. This shows our supposition was wrong

$$\geq 1+(k+1)x \qquad\qquad [\text{since } kx^2 \geq 0].$$

Hence, result is true for $n = k+1$. Therefore result is true for all $n \in \mathbb{N}$.

**Question 18:**
Give an example of a set which is
(a)bounded above but not bounded below.
**Solution:** A=$\{x \in \mathbb{R} : x < 1\}$
(b) bounded below but not bounded above.
**Solution:** $\mathbb{N}$
(c) bounded above as well as below.

**Solution:** A=$\{x \in \mathbb{R} : x^2 < 1\}$
(d) neither bounded above nor bounded below.
**Solution:** $\mathbb{Z}$

**Question 19:**
Let $S$ and $T$ be nonempty bounded subsets of $\mathbb{R}$.
(i) Prove that if $S \subseteq T$, then $\inf T \leq \inf S \leq \sup S \leq \sup T$.
**Proof:** $\forall\ s \in S,\ s \geq \inf S$. Since $S \subseteq T$, $s \geq \inf T\ \forall\ s \in S$, i.e., $\inf T$ is a lower bound

In both the cases numerator and denominator are integers and none of the denominator is zero. Hence in both the cases $m+n$ is rational.

Similarly, we can prove that $mn$ is also rational.

**Question 17:** If $x > -1$, then show that $(1+x)^n \geq 1+nx$ for all $n \in \mathbb{N}$.

**Solution 17:** We prove this result by mathematical induction on $n$. If $n = 1$ then $(1+x)^n = 1+nx$ hence result is true for $n = 1$. Let us assume that the result is true for $n = k$ i.e.

$$(1+x)^k \geq 1+kx. \tag{9}$$

Now we prove that result is true for $n = k+1$ i.e. we need to prove

$$(1+x)^{(k+1)} \geq 1+(k+1)x. \tag{10}$$

Consider

$$\begin{aligned}
(1+x)^{(k+1)} &= (1+x)^k(1+x) \\
&\geq (1+kx)(1+x) && \text{[using (1)]} \\
&= 1+x+kx+kx^2 \\
&= 1+(k+1)x+kx^2 \\
&\geq 1+(k+1)x && [\text{since } kx^2 \geq 0].
\end{aligned}$$

Hence, result is true for $n = k+1$. Therefore result is true for all $n \in \mathbb{N}$.

**Question 18:**