(Vector Spaces, Subspaces and Linear Span)

1(i). Suppose we define addition on $\mathbb{R}^2$ by the rule $(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, 0)$. Show that additive identity does not exist in $\mathbb{R}^2$ w.r.t. above rule.

1(ii). Suppose we define addition on $\mathbb{R}^3$ by the rule $(a_1, a_2, a_3) + (b_1, b_2, b_3) = (a_1b_1, a_2b_2, a_3b_3)$. Show that we have an additive identity for this operation in $\mathbb{R}^3$ but inverse may not exist for some elements.

2. Let $\mathbb{R}^+$ be the set of all positive real numbers. Define operations of addition $\bigoplus$ and the scalar multiplication $\bigotimes$ as follows: $u \bigoplus v = uv$ for all $u, v \in \mathbb{R}^+$ and $\alpha \bigotimes u = u^\alpha$ for all $u \in \mathbb{R}^+$ and $\alpha \in \mathbb{R}$ (here $\mathbb{R}$ is the field of scalars). Prove that $(\mathbb{R}^+, \bigoplus, \bigotimes)$ is a real vector space.

3. Let $\mathrm{V} = \mathbb{R}^2$. Define operations of addition $\bigoplus$ and the scalar multiplication $\bigotimes$ as follows: $(a_1, a_2) \bigoplus (b_1, b_2) = (a_1 + b_2, a_2 + b_1)$ and $\alpha \bigotimes (a_1, a_2) = (\alpha a_1, \alpha a_2), \alpha \in \mathbb{R}$ (here $\mathbb{R}$ is the field of scalars). Does $(\mathrm{V}, \bigoplus, \bigotimes)$ form a real vector space? Give reasons for your assertion.

4. Elaborate: In any real vector space $(\mathrm{V}, \bigoplus, \bigotimes)$, we have
(i) $\alpha \bigotimes \mathbf{0} = \mathbf{0}$ for every scalar α.
(ii) $0 \bigotimes u = \mathbf{0}$ for every $u \in V$.
(iii) $(-1) \bigotimes u = -u$ for every $u \in V$.
(iv) $\alpha \bigotimes u = \mathbf{0} \Rightarrow \alpha = 0 \,\text{or}\, u = \mathbf{0}$, where u is vector and α is scalar.

5. Prove that a nonempty subset S of a vector space $(\mathrm{V}, \bigoplus, \bigotimes)$ is a subspace iff $(\alpha \bigotimes u) \bigoplus v \in \mathrm{S}$ for all scalars α and $u, v \in \mathrm{S}$.

6. Let $\mathrm{V} = C[0, 1]$ be the set of all real valued function defined and continuous on the closed interval $[0, 1]$. Prove that V is a real vector space with respect to pointwise addition and multiplication. Further, determine that which of the following subsets of V are subspaces
(i) $\{f \in \mathrm{V} : f(1/2) = 0\}$
(ii) $\{f \in \mathrm{V} : f(3/4) = 1\}$
(iii) $\{f \in \mathrm{V} : f(0) = f(1)\}$
(iv) $\{f \in \mathrm{V} : f(x) = 0$ only at a finite number of points$\}$

7. Determine whether each of the following set S form a subspace of $\mathbb{R}^4$, if addition and multiplication rules are defined in the usual way.
(i) $S = \{(a, b, c, d) : a = c + d\}$.
(ii) $S = \{(a, b, c, d) : b = c - d \text{ and } a = c + d\}$.
(iii) $S = \{(a, b, c, d) : c = d\}$.
(iv) $S = \{(-a + c, a - b, b + c, a + b) : a, b, c \in \mathbb{R}\}$.
(v) $S = \{(a, b, c, d) : a = 1\}$.
(vi) $S = \{(a, b, c, d) : a \leq b\}$.
(vii) $S = \{(a, b, c, d) : a = b = c = d\}$.
(viii) $S = \{(a, b, c, d) : a \text{ is an integer}\}$.
(ix) $S = \{(a, b, c, d) : a^2 - b^2 = 0\}$.

8. Which of the following subsets of $\mathcal{P}$ are subspaces. Where, $\mathcal{P}$ is the real vector space of all polynomials w.r.t. usual vector addition and scalar multiplication rules.
(i) $\{p \in \mathcal{P} : \deg. p \leq 4\}$ (ii) $\{p \in \mathcal{P} : \deg. p = 4\}$
(iii) $\{p \in \mathcal{P} : \deg. p \geq 4\}$ (iv) $\{p \in \mathcal{P} : p(1) = 0\}$
(v) $\{p \in \mathcal{P} : p(2) = 1\}$ (vi) $\{p \in \mathcal{P} : p'(1) = 0\}$

9. Which of the following subsets of $\mathbb{R}^{2\times 2}$ are subspaces. Note that, $\mathbb{R}^{m\times n}$ is the vector space over real field of all matrices of order $m \times n$ under usual definitions of addition and scalar multiplication of matrices.
(i) All diagonal matrices. (ii) All upper triangular matrices.

(iii) All symmetric matrices. (iv) All invertible matrices.

(v) All matrices which commute with a given matrix T.

(vi) All matrices with zero determinant.

10. Let W_1 and W_2 be subspaces of a vector space V such that $\mathrm{W}_1 \bigcup \mathrm{W}_2$ is also a subspace. Show that $\mathrm{W}_1 \subseteq \mathrm{W}_2$ or $\mathrm{W}_2 \subseteq \mathrm{W}_1$.

11. Let W_1 and W_2 be subspaces of a vector space V such that $\mathrm{W}_1 + \mathrm{W}_2 = \mathrm{V}$ and $\mathrm{W}_1 \bigcap \mathrm{W}_2 = \{0\}$. Show that for each vector u in V there are *unique* vectors $u_1 \in \mathrm{W}_1$ and $u_2 \in \mathrm{W}_2$ such that $u = u_1 + u_2$.

12. Let $\mathrm{S} = \{(1, 2, 3), (1, 1, -1), (3, 5, 5)\}$. Determine which of the following are in $L[\mathrm{S}]$
(i) $(0, 0, 0)$ (ii) $(1, 1, 0)$ (iii) $(4, 5, 0)$ (iv) $(1, -3, 8)$.

13. In the complex vector space $\mathbb{C}^2$, determine whether or not $(1 + i, 1 - i) \in L[(1 + i, 1), (1, 1 - i)]$.

14. Let M and N be subsets of the vector space $(\mathrm{V}, +, .)$. Define $M + N = \{m + n : m \in M \text{ and } n \in N\}$. Then
(i) $M \subset N \Rightarrow L[M] \subset L[N]$ (ii) M is a subspace of $V \Leftrightarrow L[M] = M$ (iii) $L[L[M]] = L[M]$.

Answers

3. Not a vector space. 6. (i) Yes (ii) No (iii) Yes (iv) No
7. (i) Yes (ii) Yes (iii) Yes (iv) Yes (v) No (vi) No (vii) Yes (viii) No (ix) No
8. (i) Yes (ii) No (iii) No (iv) Yes (v) No (vi) Yes
9. (i) Yes (ii) Yes (iii) Yes (iv) No (v) Yes (vi) No
12. (i) and (iii) are in $L[S]$. 13. Yes