Estimating change in a specific direction :-

The change in the value of a differentiable function f, when we move a small distance ds from a point P_0 in a particular direction u, is given by

$$df = \underbrace{\left(\nabla f\Big|_{P_0} \cdot u \right)}_{\text{Direction derivative}} \cdot ds$$

Example:- How much value $f(x,y,z) = y\sin x + 2yz$ will change if the point $P(x,y,z)$ moves 0.1 unit from $P_0(0,1,0)$ straight towards $P_1(2,2,-2)$.

Sol:-

$$u = \frac{\overrightarrow{P_0P_1}}{|\overrightarrow{P_0P_1}|} = \frac{2}{3}i + \frac{1}{3}j - \frac{2}{3}k$$

$$\nabla f\Big|_{P_0} = i + 2k \quad \& \quad \nabla f\Big|_{P_0} \cdot u = \frac{-2}{3}$$

Thus change $df = -\frac{2}{3}(0.1) \simeq 0.067$ unit

Differentials:- For function of single variable $y = f(x)$ the change in f is defined (as x changes from a to $a+\Delta x$) $\Delta f = f(a+\Delta x) - f(a)$ & differential of f is

$$df = f'(a)\,\Delta x.$$

Total differential:- If we move from (x_0, y_0) to a point (x_0+dx, y_0+dy), the resulting change

$$df = f_x(x_0, y_0)\,dx + f_y(x_0, y_0)\,dy$$

is called the total differential of f.

Extreme values & Saddle points:-

Continuous functions of two variables assume extreme values on closed & bounded domains.

(Fig 14.36 Thomas Calculus).

In one variable:- $y=f(x)$ — Points where the graph has a horizontal tangent line

In two variable:- $z=f(x,y)$ — Points where the surface $z=f(x,y)$ has a horizontal tangent plane.

Def: Let $f(x,y)$ be defined on a region R containing the point (a,b). Then

1. If $f(a,b) \geq f(x,y)$ for all (x,y) in an open disk centered at (a,b), then $f(a,b)$ is a local maximum.
2. If $f(a,b) \leq f(x,y)$ for all (x,y) in an open disk centered at (a,b), then $f(a,b)$ is a local minimum.

(Fig. 14.38)

Theorem:- If $f(x,y)$ has a local maximum or minimum value at an interior point (a,b) of its domain & if the first partial derivatives exist there, then $f_x(a,b) = f_y(a,b) = 0$.

(This is known as first derivative test for local extrem values)

Def:- Critical point:- An interior point of the domain of a function $f(x,y)$ where both f_x & f_y are zero or where one or both of f_x & f_y do not exist is a critical point of f.

Remark:- We have following observations:-

(a) The function $f(x,y)$ can assume extreme values only at critical points & boundary points.

(b) Not every critical point give rise to a local extremum. (Similar to one variable case)

(c) A differentiable function of single variable might have point of inflection & a differentiable function of two variables might have saddle points

Def:- A differentiable function $f(x,y)$ has a saddle point at a critical point (a,b) if in every open disk centered at (a,b) there are domain points (x,y) where $f(x,y) > f(a,b)$ & domain points (x,y) where $f(x,y) < f(a,b)$. The corresponding point $(a,b,f(a,b))$ on the surface $z = f(x,y)$ is called a saddle point of the surface. (Fig 14-40)

Ex:- Find local extreme values of $f(x,y) = x^2+y^2$

Sol:- The domain of f is the entire plane (there are no boundary points). Local extreme values can occur only at $f_x = 2x = 0$ & $f_y = 2y = 0$, i.e. at origin.

Since f is never (−)ve, we can see from figure 14.41 that origin gives a local minimum.

Ex:- Find local extreme values (if any) of

$$f(x,y) = y^2 - x^2.$$

Sol:- The domain of f is the entire plane & $f_x = -2x$ & $f_y = 2y$ exist everywhere.

Therefore, local extrema can occur only at the origin $(0,0)$.

Observe that, f has the value $f(x,0) = -x^2 < 0$ along the (+)ve x axis

& f has the value $f(0,y) = y^2 > 0$ along the (+)ve y axis.

$\Rightarrow$ Every open disk in the xy plane centered at (0,0) contains points where the function is (+)ve & the points where function is (−)ve.

$\Rightarrow$ The function has a saddle point at the origin instead of a local extreme value.

$\Rightarrow$ The conclusion is that f has no local extreme values. (Fig. 14.42)

<u>Remark</u>:- The $f_x = f_y = 0$ at an interior point (a,b) of R does not guarantee that f has a local extreme value there.

<u>Qu?</u> Can we take help of second partial derivatives to conclude above?