Basic Optimization Problems

1. One Dimensional Problem
2. Multi Dimensional Unconstrained Problem
3. Multi Dimensional Constrained Problem

- search methods

- search methods
  - Dichotomous search

- search methods
  - Dichotomous search
  - Fibonacci search

- search methods
  - Dichotomous search
  - Fibonacci search
  - Golden-section search

- search methods
  - Dichotomous search
  - Fibonacci search
  - Golden-section search
- approximation methods

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  - Quadratic interpolation method

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  - Dichotomous search
  - Fibonacci search
  - Golden-section search
- approximation methods
  - Quadratic interpolation method
  - Cubic interpolation method
- combination of a search method with an approximation method

Input: A unimodal function $f(x)$ which is known to have a minimum in the interval $[x_L, x_U]$.

- Definition: This interval is said to be the range of uncertainty.

iteration: The minimizer x^* of $f(x)$ can be located by reducing progressively the range of uncertainty until a sufficiently small range is obtained.

Search methods can be applied to any function and differentiability of $f(x)$ is not required.

In search methods, goal is achieved by using the values of $f(x)$ at suitable points in the current range of uncertainty. Reduction in the range of uncertainty is possible if the value of $f(x)$ is known at two points, say, x_a and x_b.

- Case 1: $f(x_a) < f(x_b)$

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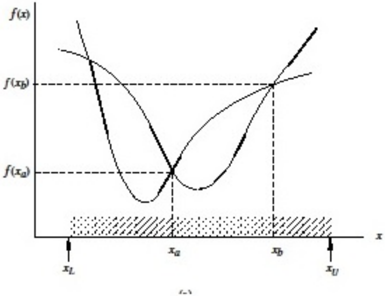

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- Case 1: $f(x_a) < f(x_b)$
- Conclusion:
  $x_L < x^* < x_b$

Search methods: Idea

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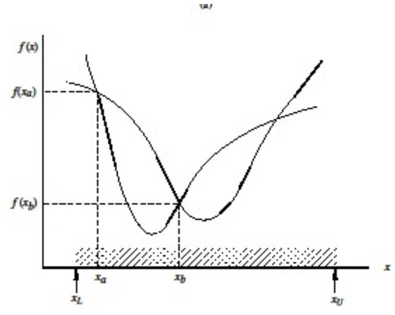

Search methods: Idea

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- Conclusion:
  $x_a < x^* < x_U$

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- Case 1: $f(x_a) < f(x_b)$
- Conclusion:
  $x_L < x^* < x_b$
- Case 2: $f(x_a) > f(x_b)$
- Conclusion:
  $x_a < x^* < x_U$
- Case 3: $f(x_a) = f(x_b)$

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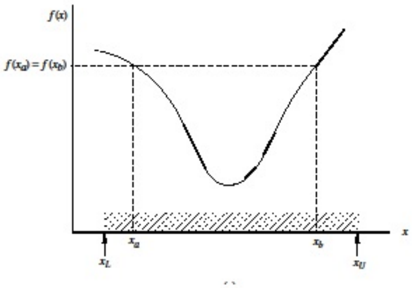
- Case 3: $f(x_a) = f(x_b)$



Search methods: Idea

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- Case 1: $f(x_a) < f(x_b)$
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- Conclusion:
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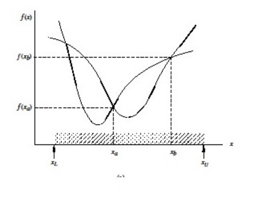
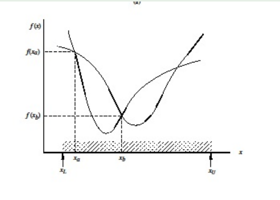
- Case 3: $f(x_a) = f(x_b)$
- Conclusion:
  $x_a < x^* < x_b$

Case 1: $f(x_a) < f(x_b)$



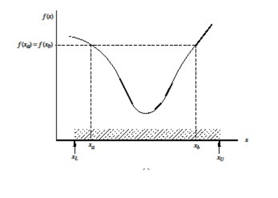
Conclusion:
$x_L < x^* < x_b$

Case 2: $f(x_a) > f(x_b)$



Conclusion:
$x_a < x^* < x_U$

Case 3: $f(x_a) = f(x_b)$



Conclusion:
$x_a < x^* < x_b$

In case 3: Both of the following statements are correct:
$x_L < x^* < x_b$ and $x_a < x^* < x_U$

1. $x_{L,k}$: lower limit of the range of uncertainty at k^{th} iteration
2. $x_{U,k}$: upper limit of the range of uncertainty at k^{th} iteration
3. $x_{a,k}$: first trial point at k^{th} iteration
4. $x_{b,k}$: second trial point at k^{th} iteration
5. $E_{a,k} = f(x_{a,k})$
6. $E_{b,k} = f(x_{b,k})$

Algorithm for Dichotomous search method

√ Input $x_{L,1}$, $x_{U,1}$, $f(x)$, and ε

√ Calculate: $x_{a,1} = \frac{x_{L,1}+x_{U,1}}{2} - \varepsilon$

$$x_{b,1} = \frac{x_{L,1}+x_{U,1}}{2} + \varepsilon$$

√ **Until Convergence Do**: for $k = 1, 2, 3, \cdots$

Step 1 $E_{a,k} = f(x_{a,k})$ and $E_{b,k} = f(x_{b,k})$

Step 2 Check $E_{a,k} < E_{b,k}$

Yes

- $x_{L,k+1} = x_{L,k}$
- $x_{U,k+1} = x_{b,k}$
- $x_{a,k+1} = \frac{x_{L,k+1}+x_{U,k+1}}{2} - \varepsilon$
- $x_{b,k+1} = \frac{x_{L,k+1}+x_{U,k+1}}{2} + \varepsilon$

No

- $x_{L,k+1} = x_{a,k}$
- $x_{U,k+1} = x_{U,k}$
- $x_{a,k+1} = \frac{x_{L,k+1}+x_{U,k+1}}{2} - \varepsilon$
- $x_{b,k+1} = \frac{x_{L,k+1}+x_{U,k+1}}{2} + \varepsilon$

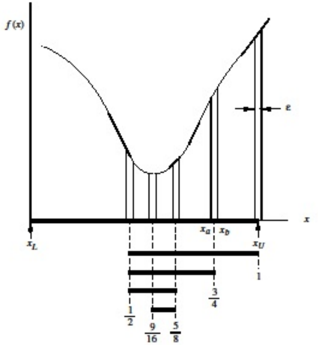
EndDo

Stoping Criteria

- $l_k < \varepsilon$ or
- $x_{a,k} > x_{b,k}$

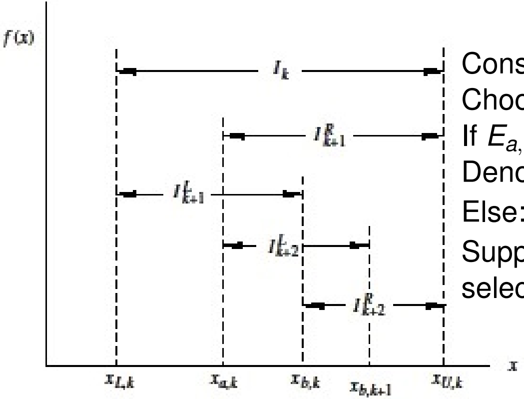
$x_{a,k} > x_{b,k}$ implies that $x_{a,k} \approx x_{b,k}$ within the precision of the computer used or that there is an error in the program.
Output: $x^* = \frac{1}{2}(x_{a,k+1} + x_{b,k+1})$ and $f^* = f(x^*)$

Important Conclusions on Dichotomous search method



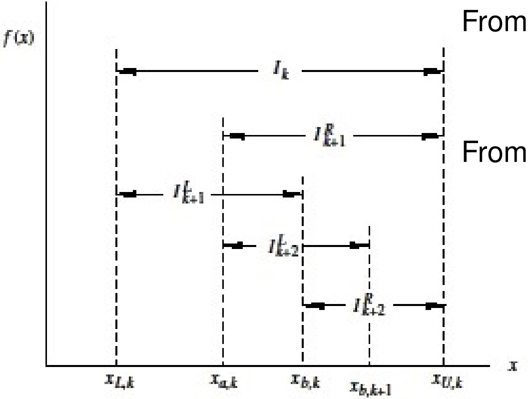
Conclusion: if the dichotomous search is applied to the function of as shown in the figure, the range of uncertainty will be reduced from $0 < x^* < 1$ to $9/16 + \varepsilon < x^* < 5/8 - \varepsilon$ in four iterations.

1. Each iteration reduces the range of uncertainty almost by half and, therefore, after k iterations, the interval of uncertainty reduces to $I_k = (\frac{1}{2})^k I_0$ where $I_0 = x_U - x_L$.
2. After 7 iterations the range of uncertainty would be reduced to less than 1% of the initial interval.
3. The corresponding computational effort would be 14 function evaluations since two evaluations are required for each iteration.



Consider: $I_k = [x_{L,k}, x_{U,k}]$
Choose $x_{a,k}$, $x_{b,k}$ as shown.
If $E_{a,k} < E_{b,k}$: $x_{L,k} < x^* < x_{b,k}$.
Denote: $I^L_{k+1} = [x_{L,k}, x_{b,k}]$
Else: $I^R_{k+1} = [x_{a,k}, x_{U,k}]$
Suppose the right interval I^R_{k+1} is selected.

Then note: value of $f(x)$ is known at one interior point of I^R_{k+1}, namely, at point $x_{b,k}$. If $f(x)$ is evaluated at one more interior point, say, at point $x_{b,k+1}$, sufficient information is available to allow a further reduction in the region of uncertainty. This cycle of events can be repeated. Thus, only one function evaluation is required per iter, and the amount of computation will be reduced relative to that required in the dichotomous search.



From Figure:

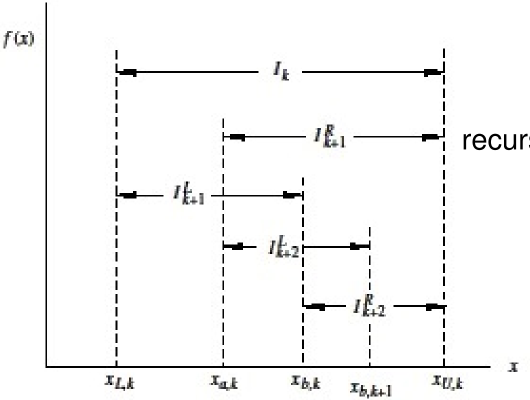
$$I_k = I_{k+1}^L + I_{k+2}^R$$

From convenience Assume:

$$I_{k+1}^L = I_{k+1}^R = I_{k+1}$$

$$I_{k+2}^L = I_{k+2}^R = I_{k+2}$$

Thus, we have the recursive relation: $I_k = I_{k+1} + I_{k+2}$

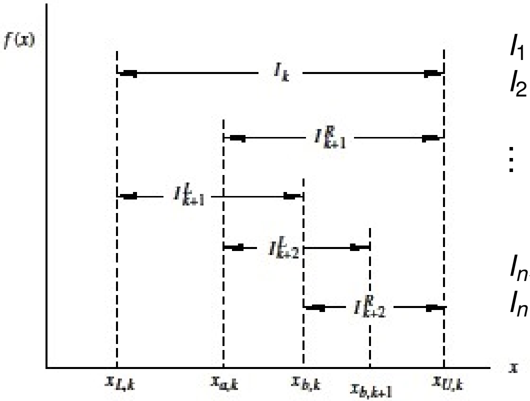


recursive relation:

$$I_k = I_{k+1} + I_{k+2}$$

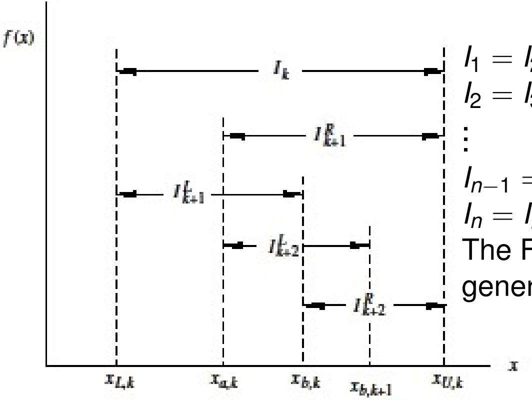
If the above procedure is repeated a number of times, a sequence of intervals $\{I_1, I_2, \cdots, I_n\}$ will be generated as follows:

$I_1 = I_2 + I_3, \qquad I_2 = I_3 + I_4, \cdots \qquad I_n = I_{n+1} + I_{n+2}$



$$
\begin{aligned}
I_1 &= I_2 + I_3 \\
I_2 &= I_3 + I_4 \\
&\vdots \\
I_{n-1} &= I_n + I_{n+1} \\
I_n &= I_{n+1} + I_{n+2}
\end{aligned}
$$

In the above set of n equations, there are $n+2$ variables and if I_1 is the given initial interval, $n+1$ variables remain. Therefore, an infinite set of sequences can be generated by specifying some additional rule. Two specific sequences of particular interest are the Fibonacci sequence and the golden-section sequence.



$$I_1 = I_2 + I_3$$
$$I_2 = I_3 + I_4$$
$$\vdots$$
$$I_{n-1} = I_n + I_{n+1}$$
$$I_n = I_{n+1} + I_{n+2}$$

The Fibonacci sequence is generated by assuming $I_{n+2} = 0$.

Thus from above relations we have:

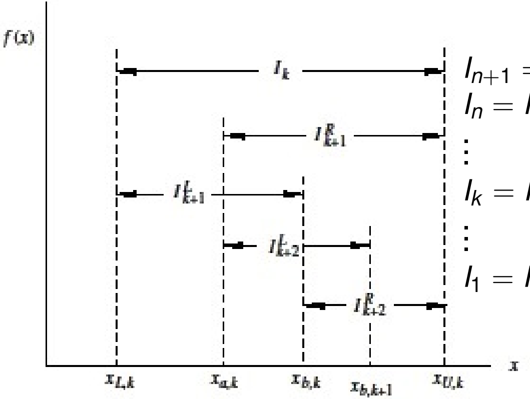
$I_{n+1} = I_n - I_{n+2} = I_n \equiv F_0 I_n, \qquad I_n = I_{n+1} + I_{n+2} = I_n \equiv F_1 I_n$

$I_{n-1} = I_n + I_{n+1} = 2I_n \equiv F_2 I_n, \qquad I_{n-2} = I_{n-1} + I_n = 3I_n \equiv F_3 I_n$

$I_{n-3} = I_{n-2} + I_{n-1} = 5I_n \equiv F_4 I_n, \qquad I_{n-4} = I_{n-3} + I_{n-2} = 8I_n \equiv F_5 I_n$

$\cdots \; \cdots \; \cdots \; \cdots \; \cdots \; \cdots$

$I_k = I_{k+1} + I_{k+2} \equiv F_{n-k+1} I_n, \cdots \qquad I_1 = I_2 + I_3 \equiv F_n I_n$

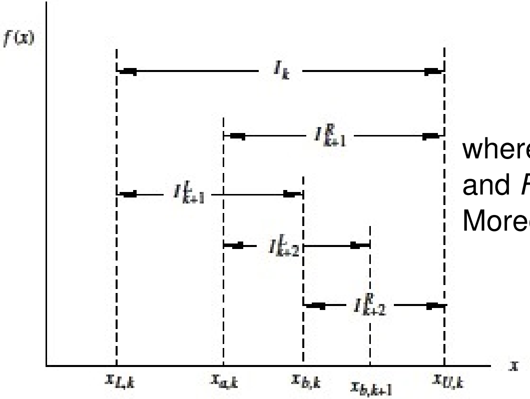


$$I_{n+1} = I_n - I_{n+2} = I_n \equiv F_0 I_n$$
$$I_n = I_{n+1} + I_{n+2} = I_n \equiv F_1 I_n$$
$$\vdots$$
$$I_k = I_{k+1} + I_{k+2} \equiv F_{n-k+1} I_n$$
$$\vdots$$
$$I_1 = I_2 + I_3 \equiv F_n I_n$$

Thus we have:
$I_{k+1} = F_{n-k} I_n, \quad I_{k+2} = F_{n-k-1} I_n$ i.e $I_{k+2} = \frac{F_{n-k-1}}{F_{n-k}} I_{k+1}$ Where, the sequence generated, namely,

$$\{1, 1, 2, 3, 5, 8, 13, \cdots\} = \{F_0, F_1, F_2, F_3, F_4, F_5, F_6 \cdots\}$$

is the well-known Fibonacci sequence.



$$I_{k+2} = \frac{F_{n-k-1}}{F_{n-k}} I_{k+1} \qquad (1)$$

where $F_k = F_{k-1} + F_{k-2} \forall k \geq 2$
and $F_0 = F_1 = 1$.
Moreover

$$I_1 = F_n I_n \qquad (2)$$

Note that: $x_{L,1}$, $x_{U,1}$, I_1 are known. So, $I_2 = \frac{F_{n-1}}{F_n} I_1$.
And from figure:
$x_{a,1} = x_{U,1} - I_2$ and $x_{b,1} = x_{L,1} + I_2$.
If the target range of uncertainty is I_n, then by Eqn. (2)

$$I_n = \frac{I_1}{F_n}$$

Fibonacci Search Method

- $\surd$ Input: $x_{L,1}$, $x_{U,1}$, $f(x)$ and n.
- $\surd$ Calculate: $\{F_0, F_1, F_2, F_3, F_4, F_5, F_6, \cdots F_n\}$.
- $\surd$ Calculate: $I_1 = x_{U,1} - x_{L,1}$, $\quad I_2 = \frac{F_{n-1}}{F_n} I_1$

  $x_{a,1} = x_{U,1} - I_2 \quad x_{b,1} = x_{L,1} + I_2$
- $\surd$ Calculate: $E_{a,1} = f(x_{a,1})$ and $E_{b,1} = f(x_{b,1})$
- $\surd$ **Until Convergence Do**: for $k = 1, 2, 3, \cdots$

Step 1 Calculate: $I_{k+2} = \frac{F_{n-k-1}}{F_{n-k}} I_{k+1}$

Step 2 Check $E_{a,k} < E_{b,k}$

Y: Then Updates

- $x_{L,k+1}$
- $x_{U,k+1}$
- $x_{a,k+1}$
- $x_{b,k+1}$
- $E_{a,k+1}$
- $E_{b,k+1}$

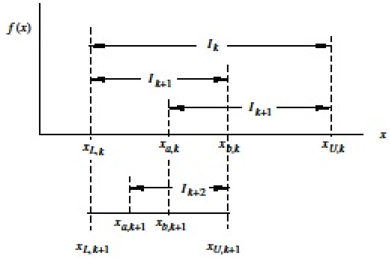
N: Then Updates

- $x_{L,k+1}$
- $x_{U,k+1}$
- $x_{a,k+1}$
- $x_{b,k+1}$
- $E_{a,k+1}$
- $E_{b,k+1}$

EndDo

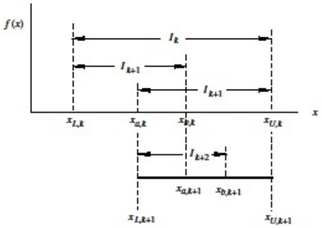
Fibonacci Search Method

$E_{a,k} < E_{b,k}$



- $x_{L,k+1} = x_{L,k}$
- $x_{U,k+1} = x_{b,k}$
- $x_{a,k+1} = x_{U,k+1} - I_{k+2}$
- $x_{b,k+1} = x_{a,k}$
- $E_{b,k+1} = E_{a,k}$
- $E_{a,k+1} = f(x_{a,k+1})$

$E_{a,k} \geq E_{b,k}$



- $x_{L,k+1} = x_{a,k}$
- $x_{U,k+1} = x_{U,k}$
- $x_{a,k+1} = x_{b,k}$
- $x_{b,k+1} = x_{L,k+1} + I_{k+2}$
- $E_{a,k+1} = E_{b,k}$
- $E_{b,k+1} = f(x_{b,k+1})$

Algorithm for Fibonacci search method

√ Input: $x_{L,1}$, $x_{U,1}$, $f(x)$ and n.

√ Calculate: $\{F_0, F_1, F_2, F_3, F_4, F_5, F_6, \cdots F_n\}$.

√ Calculate: $I_1 = x_{U,1} - x_{L,1}$, $\quad I_2 = \frac{F_{n-1}}{F_n} I_1$

$x_{a,1} = x_{U,1} - I_2 \quad x_{b,1} = x_{L,1} + I_2$

√ Calculate: $E_{a,1} = f(x_{a,1})$ and $E_{b,1} = f(x_{b,1})$

√ **Until Convergence Do**: for $k = 1, 2, 3, \cdots$

Step 1 Calculate: $I_{k+2} = \frac{F_{n-k-1}}{F_{n-k}} I_{k+1}$

Step 2 Check $E_{a,k} < E_{b,k}$

Yes

- $x_{L,k+1} = x_{L,k}$
- $x_{U,k+1} = x_{b,k}$
- $x_{a,k+1} = x_{U,k+1} - I_{k+2}$
- $x_{b,k+1} = x_{a,k}$
- $E_{b,k+1} = E_{a,k}$
- $E_{a,k+1} = f(x_{a,k+1})$

No

- $x_{L,k+1} = x_{a,k}$
- $x_{U,k+1} = x_{U,k}$
- $x_{a,k+1} = x_{b,k}$
- $x_{b,k+1} = x_{L,k+1} + I_{k+2}$
- $E_{a,k+1} = E_{b,k}$
- $E_{b,k+1} = f(x_{b,k+1})$

EndDo

Stoping Criteria for FSM

Yes

- $x_{L,k+1} = x_{L,k}$
- $x_{U,k+1} = x_{b,k}$
- $x_{a,k+1} = x_{U,k+1} - I_{k+2}$
- $x_{b,k+1} = x_{a,k}$
- $E_{b,k+1} = E_{a,k}$
- $E_{a,k+1} = f(x_{a,k+1})$

No

- $x_{L,k+1} = x_{a,k}$
- $x_{U,k+1} = x_{U,k}$
- $x_{a,k+1} = x_{b,k}$
- $x_{b,k+1} = x_{L,k+1} + I_{k+2}$
- $E_{a,k+1} = E_{b,k}$
- $E_{b,k+1} = f(x_{b,k+1})$

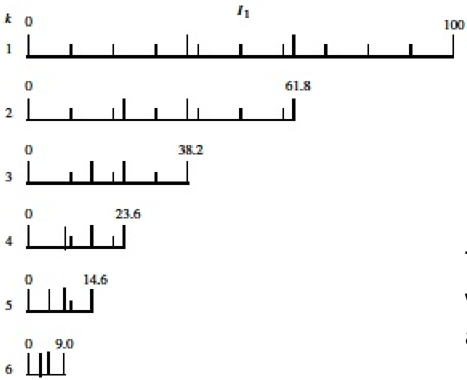
Stoping Criteria

- $k = n - 2$ or
- $x_{a,k+1} > x_{b,k+1}$

$x_{a,k+1} > x_{b,k+1}$ implies that $x_{a,k+1} \approx x_{b,k+1}$ within the precision of the computer used or that there is an error in the program.
Output: $x^* = x_{a,k+1}$ and $f^* = f(x^*)$

1. if $n = 11$ then $F_n = 144$ and so $I_n = \frac{I_1}{F_n}$ is reduced to a value less than 1% the value of I_1. This would entail 11 iterations and since one function evaluation is required per iteration, a total of 11 function evaluations would be required as opposed to the 14 required by the dichotomous search to achieve the same precision.
2. In effect, the Fibonacci search is more efficient than the dichotomous search. Indeed, it can be shown, that it achieves the largest interval reduction relative to the other search methods and it is, therefore, the most efficient in terms of computational effort required.
3. The disadvantage is the fact that the Fibonacci sequence of intervals can be generated only if n is known.

Golden-section Search Method



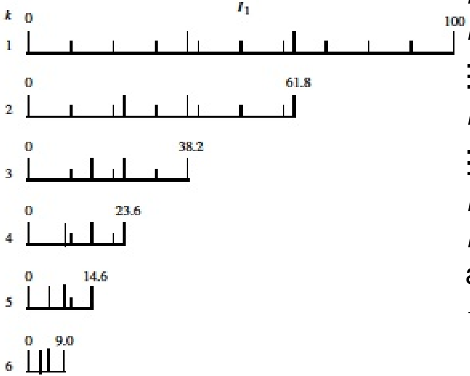
$I_1 = I_2 + I_3$
$I_2 = I_3 + I_4$
$\vdots$
$I_k = I_{k+1} + I_{k+2}$
$\vdots$
$I_{n-1} = I_n + I_{n+1}$
$I_n = I_{n+1} + I_{n+2}$

The Fibonacci sequence was generated by assuming $I_{n+2} = 0$.

The rule by which the lengths of successive intervals are generated is that the ratio of any two adjacent intervals is constant, that is

$$\frac{I_k}{I_{k+1}} = \frac{I_{k+1}}{I_{k+2}} = \frac{I_{k+2}}{I_{k+3}} = \cdots = \rho$$

Golden-section Search Method



$$I_1 = I_2 + I_3$$
$$I_2 = I_3 + I_4$$
$$\vdots$$
$$I_k = I_{k+1} + I_{k+2} \qquad (^{**})$$
$$\vdots$$
$$I_{n-1} = I_n + I_{n+1}$$
$$I_n = I_{n+1} + I_{n+2}$$

and

$$\frac{I_k}{I_{k+1}} = \frac{I_{k+1}}{I_{k+2}} = \cdots = \rho$$

Thus we have: $\frac{I_k}{I_{k+2}} = \rho^2$, $\frac{I_k}{I_{k+3}} = \rho^3$ and so on.

Divide (**) by I_{k+2}: $\frac{I_k}{I_{k+2}} = \frac{I_{k+1}}{I_{k+2}} + 1 \Rightarrow \rho^2 = \rho + 1 \Rightarrow \rho = \frac{1 \pm \sqrt{5}}{2}$.

The negative value of ρ is irrelevant and so $\rho = 1.618034$. This constant is known as the golden ratio. The term has arisen from the fact that in classical Greece, a rectangle with sides bearing a ratio $1 : \rho$ was considered the most pleasing rectangle and hence it came to be known as the golden rectangle.

Algorithm for Golden-section Search Method

- Input: $x_{L,1}$, $x_{U,1}$, $f(x)$ and ε.
- Take: $\rho = 1.618034$.
- Calculate: $I_1 = x_{U,1} - x_{L,1}$, $\quad I_2 = \frac{I_1}{\rho}$
  $x_{a,1} = x_{U,1} - I_2 \quad x_{b,1} = x_{L,1} + I_2$
- Calculate: $E_{a,1} = f(x_{a,1})$ and $E_{b,1} = f(x_{b,1})$
- **Until Convergence Do**: for $k = 1, 2, 3, \cdots$

Step 1 Calculate: $I_{k+2} = \frac{I_{k+1}}{\rho}$

Step 2 Check $E_{a,k} < E_{b,k}$

Yes

- $x_{L,k+1} = x_{L,k}$
- $x_{U,k+1} = x_{b,k}$
- $x_{a,k+1} = x_{U,k+1} - I_{k+2}$
- $x_{b,k+1} = x_{a,k}$
- $E_{b,k+1} = E_{a,k}$
- $E_{a,k+1} = f(x_{a,k+1})$

No

- $x_{L,k+1} = x_{a,k}$
- $x_{U,k+1} = x_{U,k}$
- $x_{a,k+1} = x_{b,k}$
- $x_{b,k+1} = x_{L,k+1} + I_{k+2}$
- $E_{a,k+1} = E_{b,k}$
- $E_{b,k+1} = f(x_{b,k+1})$

EndDo

Stoping Criteria

- $l_{k+2} < \varepsilon$ or
- $x_{a,k+1} > x_{b,k+1}$

$x_{a,k+1} > x_{b,k+1}$ implies that $x_{a,k+1} \approx x_{b,k+1}$ within the precision of the computer used or that there is an error in the program.

Output: x^* and $f^* = f(x^*)$ where

1. If $E_{a,k+1} < E_{b,k+1}$ then $x^* = \frac{1}{2}(x_{L,k+1} + x_{a,k+1})$
2. If $E_{a,k+1} = E_{b,k+1}$ then $x^* = \frac{1}{2}(x_{a,k+1} + x_{b,k+1})$
3. If $E_{a,k+1} > E_{b,k+1}$ then $x^* = \frac{1}{2}(x_{b,k+1} + x_{U,k+1})$

Important Conclusions on GSM

1. A known relation between F_n and ρ which is applicable for large values of n is $F_n = \frac{\rho^{n+1}}{\sqrt{5}}$ (e.g., if $n = 11$, $F_n = 144$ and $\frac{\rho^{n+1}}{\sqrt{5}} = 144.001$).
2. Since in the FSM $I_n = \frac{I_1}{F_n}$, after n iteration the region of uncertainty for the Fibonacci search is $\Lambda_{FSM} = I_n = \frac{I_1}{F_n} = \frac{\sqrt{5}}{\rho^{n+1}} I_1$. Similarly, for the golden-section search $\Lambda_{GSM} = I_n = \frac{I_1}{\rho^{n-1}}$. Hence $\frac{\Lambda_{GSM}}{\Lambda_{FSM}} = \frac{\rho^2}{\sqrt{5}} = 1.17$. Therefore, if the number of iterations is the same in the two methods, the region of uncertainty in the golden-section search is larger by about 17% relative to that in the Fibonacci search. Alternatively, the golden-section search will require more iterations to achieve the same precision as the Fibonacci search.
3. But above disadvantage of GSM is offset by the fact that the total number of iterations need not be supplied at the start of the optimization.

Input: A unimodal function $f(x)$ which is known to have a minimum in the interval $[x_1, x_3]$.

- Definition: This interval is said to be the range of uncertainty (bracket on x^*).

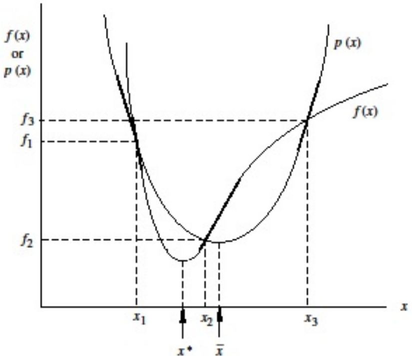
iteration: The minimizer x^* of $f(x)$ can be located by reducing progressively the range of uncertainty until a sufficiently small range is obtained by approximating the given function with low-order polynomial, usually a second or third order polynomial. (resp. quadratic or cubic interpolation method.)

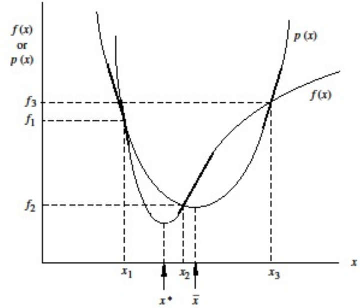
Search methods can be applied to any function and differentiability of $f(x)$ is not required. But in approximation methods $f(x)$ is required to be continuous and differentiable also.

Approximate given $f(x)$ by a second degree polynomial $p(x) = a_0 + a_1x + a_2x^2$ such that

$$p(x_i) = a_0 + a_1x_i + a_2x_i^2 = f(x_i) = f_i, \quad \forall i = 1, 2, 3$$

where $[x_1, x_3]$ is the initial range and x_2 is a point s.t. $x_1 < x_2 < x_3$. If f_i are known then the above set of equations can be solved for the values of a_0, a_1, a_2. Thus the plots of $f(x)$ and $p(x)$ will assume the following form:





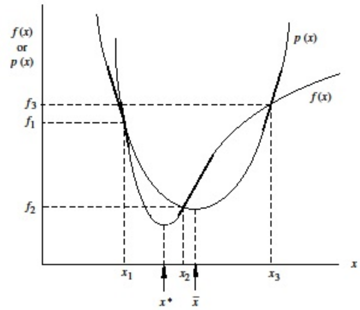
$p(x) = a_0 + a_1 x + a_2 x^2$
$p(x_i) = a_0 + a_1 x_i + a_2 x_i^2 = f_i$ (**)
Denote: $\overline{x}$ = argmin $p(x)$.

Note that: $\overline{x} = -\frac{a_1}{2a_2}$

By solving (**): $a_1 = \frac{(x_2^2 - x_3^2)f_1 + (x_3^2 - x_1^2)f_2 + (x_1^2 - x_2^2)f_3}{(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)}$

$a_2 = \frac{(x_2 - x_3)f_1 + (x_3 - x_1)f_2 + (x_1 - x_2)f_3}{(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)}$

Thus $\overline{x} = \frac{(x_2^2 - x_3^2)f_1 + (x_3^2 - x_1^2)f_2 + (x_1^2 - x_2^2)f_3}{2[(x_2 - x_3)f_1 + (x_3 - x_1)f_2 + (x_1 - x_2)f_3]}$



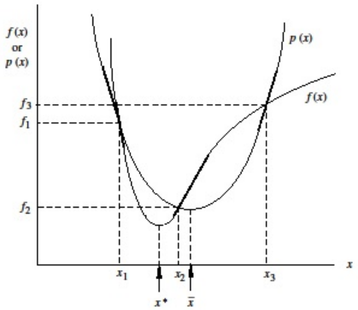
$$p(x) = a_0 + a_1 x + a_2 x^2$$

Denote: $\overline{x}$ = argmin $p(x)$.

$$\overline{x} = \frac{(x_2^2 - x_3^2)f_1 + (x_3^2 - x_1^2)f_2 + (x_1^2 - x_2^2)f_3}{2[(x_2 - x_3)f_1 + (x_3 - x_1)f_2 + (x_1 - x_2)f_3]}$$

If $f(x)$ cannot be represented accurately by a second-order polynomial, a number of such iterations can be performed. The appropriate strategy is to attempt to reduce the interval of uncertainty in each iteration as was done in the search methods. This can be achieved by rejecting either x_1 or x_3 and then using the two remaining points along with point $\overline{x}$ for a new interpolation.

Algorithm for Quadratic Interpolation Method



$$p(x) = a_0 + a_1 x + a_2 x^2$$

Denote: $\overline{x}$ = argmin $p(x)$.

$$\overline{x} = \frac{(x_2^2 - x_3^2) f_1 + (x_3^2 - x_1^2) f_2 + (x_1^2 - x_2^2) f_3}{2[(x_2 - x_3) f_1 + (x_3 - x_1) f_2 + (x_1 - x_2) f_3]}$$

If $x_2 < \overline{x} < x_3$, then there may be two cases:

- if $\overline{f} = f(\overline{x}) \leq f_2$ then $x_1 = x_2,\ f_1 = f_2,\ x_2 = \overline{x}$ and $f_2 = \overline{f}$ (rejection of x_1)
- otherwise if $\overline{f} > f_2$ then assign $x_3 = \overline{x}$ and $f_3 = \overline{f}$ (rejection of x_3)

Algorithm for Quadratic Interpolation Method

- √ Input x_1, x_3, $f(x)$, and ε
- √ Set $x_0 = 10^{99}$
- √ Compute $x_2 = \frac{x_1+x_3}{2}$ and $f_i = f(x_i)$ for $i = 1, 2, 3$
- √ Compute $\overline{x}$ by derived formula and $\overline{f} = f(\overline{x})$.
- √ **Until** $||\overline{x} - x_0|| \leq \varepsilon$ **DO:**

Step 1 IF

$x_1 < \overline{x} < x_2$

if $\overline{f} \leq f_2$

- $x_3 = x_2, \; f_3 = f_2$
- $x_2 = \overline{x}, \; f_2 = \overline{f}$

otherwise if $\overline{f} > f_2$

- $x_1 = \overline{x}, \; f_1 = \overline{f}$

$x_2 < \overline{x} < x_3$

if $\overline{f} \leq f_2$

- $x_1 = x_2, \; f_1 = f_2$
- $x_2 = \overline{x}, \; f_2 = \overline{f}$

otherwise if $\overline{f} > f_2$

- $x_3 = \overline{x}, \; f_3 = \overline{f}$

Step 2 Set $x_0 = \overline{x}$

Step 3 Compute $\overline{x}$ by derived formula and $\overline{f} = f(\overline{x})$.