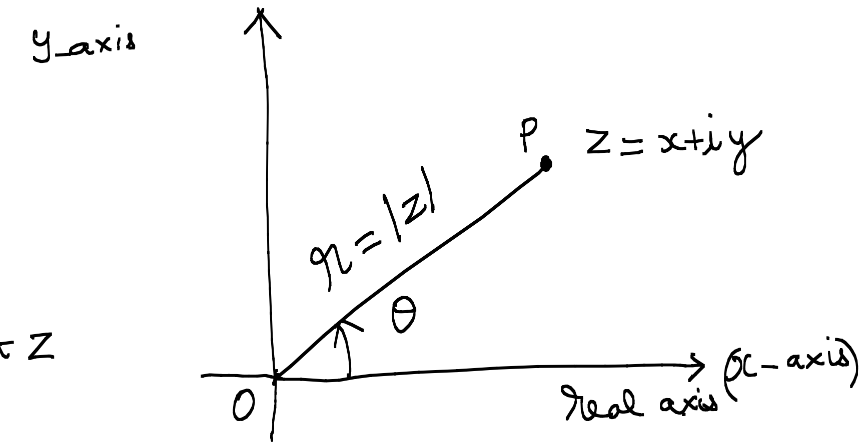
Division - I

Lecture - 2 | Wednesday | Date: 27-07-2011

Polar form of complex numbers:

Let $z = x + iy \neq 0$.



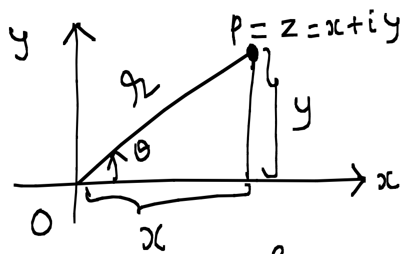
P = point z

$\overrightarrow{OP}$ = Vector joining the origin O and the point $P (=z)$

$|\overrightarrow{OP}|$ = Magnitude of $\overrightarrow{OP} = r = \sqrt{x^2 + y^2} = |z|$

θ = argument of z = amplitude of the vector $\overrightarrow{OP}$.

θ = Oriented angle from the positive real axis to the vector $\overrightarrow{OP}$ measured in radians.



$r = |z| = \sqrt{x^2 + y^2}$

$\cos\theta = \frac{x}{r}$, $\sin\theta = \frac{y}{r}$

$\tan\theta = y/x$

The number θ is determined only upto the multiples of 2π.

θ = argument of z.

$\boxed{\text{Polar form of } z = (r, \theta)}$ where r = Modulus of $z = |z|$, θ = argument of z

Example:

$$\arg(1+i) = \frac{\pi}{4} \quad \text{and} \quad \frac{\pi}{4} + 2k\pi \quad \text{where } k \in \mathbb{Z}$$

$$\arg(5) = 0 + 2k\pi = 2k\pi \quad \text{where } k \in \mathbb{Z}$$

$$\arg(1-i) = \frac{-\pi}{4} + 2k\pi \quad \text{where } k \in \mathbb{Z}$$

$$\arg(-i) = \frac{-\pi}{2} + 2k\pi \quad \left(\text{or } \frac{3\pi}{2} + 2k\pi\right)$$

For each non-zero z,

$z = (r, \theta)$ polar form

$= r(\cos\theta + i\sin\theta)$ Trigonometric form

$= r e^{i\theta}$ exponential form.

For the complex number $z = 0$, the modulus is 0, but the argument is undefined.

Note: If a complex number z is written in the polar form then it is understood that z is a non-zero complex number.

Principal value of argument of z

For each $z \neq 0$, there is only one value of argument of z, say Θ satisfying $-\pi < \Theta \leq \pi$. This value of $\arg z$ will be denoted by $\operatorname{Arg} z = \Theta$ (Capital theta) and is called the principal value of $\arg z$.

Example:

$$\arg(5) = \{\ldots\ldots, -4\pi, -2\pi, 0, 2\pi, 4\pi, 6\pi, \ldots\ldots\}$$

$$\mathrm{Arg}(5) = 0$$

$$\arg(i) = \left\{\ldots\ldots, \frac{\pi}{2} - 2\pi, \frac{\pi}{2}, \frac{\pi}{2} + 2\pi, \ldots\ldots\right\}$$

$$\mathrm{Arg}(i) = \pi/2$$

$$\arg(-2) = \{\ldots\ldots -3\pi, -\pi, \pi, 3\pi, \ldots\ldots\}$$

$$\mathrm{Arg}(-2) = \pi$$

Thus, for a non-zero complex number,

$$-\pi < \mathrm{Arg}(z) \leq \pi.$$

It is unique number in this interval $(-\pi, \pi]$.

Finding $\theta = \arg z$, where $z = x + iy \neq 0$.

θ can be computed by $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ and also with the information of signs of x and y. (ie), (x,y) is lying which quadrant)

Set $\phi = \tan^{-1}\left(\frac{y}{x}\right)$ = Principal value of $\tan^{-1}$

It lies in $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

$$\mathrm{Arg}(z) = \begin{cases} \phi & \text{if } x > 0 \\ \phi + \pi & \text{if } x < 0 \text{ and } y \geq 0 \\ \phi - \pi & \text{if } x < 0 \text{ and } y < 0 \\ \frac{\pi}{2} \text{ if } x = 0 \text{ and } y > 0, & \frac{-\pi}{2} \text{ if } x = 0 \text{ and } y < 0. \end{cases}$$

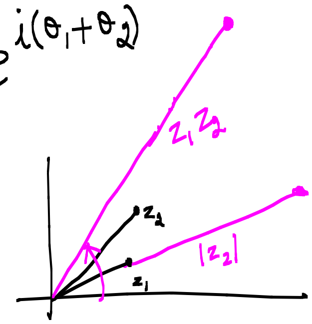
Relation between arg z and Arg z

$$\boxed{\arg z = \operatorname{Arg} z + 2\pi k} \quad \text{where } k \text{ is any integer}$$

Geometrical Interpretation of multiplication.

$$z_1 = r_1 e^{i\theta_1} \quad \text{and} \quad z_2 = r_2\, e^{i\theta_2}$$

$$z_1 z_2 = \left(r_1 e^{i\theta_1}\right)\left(r_2\, e^{i\theta_2}\right) = r_1 r_2\, e^{i(\theta_1+\theta_2)}$$



$$|z_1 z_2| = r_1 r_2$$

$$\arg(z_1 z_2) = \theta_1 + \theta_2$$

The vector z_1 $\longrightarrow$ Multiply its length r_1 by the factor r_2 and then rotate the resulting vector counter clockwise through the angle $\arg(z_2) = \theta_2$

$$\boxed{\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)} \longrightarrow (*)$$

Interpretation: If we know the values of any two quantities in the above equation $(*)$, we can compute a value of the third quantity.

Similarly $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2\, e^{i\theta_2}$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\, e^{i(\theta_1-\theta_2)} \implies \arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2 .$$

Note: In general $\boxed{Arg(z_1 z_2) \neq Arg\, z_1 + Arg\, z_2}$

(For principal value of $\arg z = Arg\, z$)

Find such z_1 and z_2.

Let $z = r e^{i\theta}$

Let n be a natural number.

$$z^n = (r e^{i\theta})^n = r^n e^{in\theta}$$

$$= r^n (\cos(n\theta) + i \sin(n\theta))$$

$$\Rightarrow \boxed{|z^n| = |z|^n} \text{ where } n \in \mathbb{N}.$$

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