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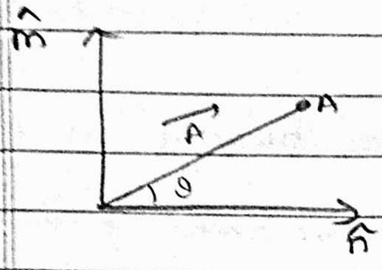
PH-101 Physics Tutorial - 2

① Que: let $\vec{A}$ be an arbitrary vector and let $\hat{n}$ be a unit vector in fixed direction. Show that

$$\vec{A} = (\vec{A} \cdot \hat{n})\,\hat{n} + (\hat{n} \times \vec{A}) \times \hat{n}$$

$\hat{m} \perp$ to $\hat{n}$



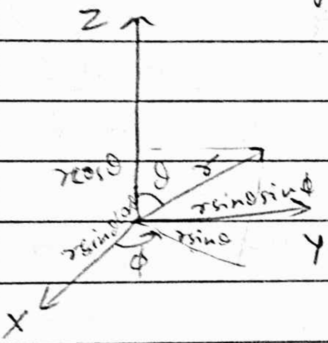
$$\vec{A} = |\vec{A}|\cos\theta\,\hat{n} + |\vec{A}|\sin\theta\,\hat{m}$$

$$(\vec{A} \cdot \hat{n})\,\hat{n} + (\hat{n} \times \vec{A}) \times \hat{n} = |\vec{A}|\cos\theta\,\hat{n} + |\vec{A}|\sin\theta\,(\hat{n} \times \hat{m}) \times \hat{n}$$

$$= |\vec{A}|\cos\theta\,\hat{n} + |\vec{A}|\sin\theta\,\hat{m}$$

$$= \vec{A}$$ Hence proved

Que ② Evaluate the integral $\iiint (x^2 + y^2 + z^2)\,dx\,dy\,dz$ over the volume V of a sphere having centre at the origin and radius equal to a



$$\iiint \underbrace{(x^2 + y^2 + z^2)}_{r^2} \underbrace{dx\,dy\,dx}_{dV}$$

$$= \iiint r^2 \left(r^2 \sin\theta \, d\theta \, d\phi \, dr\right)$$

$$= \int_0^a r^4\,dr \int_0^{\pi} \sin\theta\,d\theta \int_0^{2\pi} d\phi$$

$$= \left[\frac{r^5}{5}\right]_0^a \left[(-\cos\theta)\right]_0^{\pi} \left[\phi\right]_0^{2\pi}$$

$$= \left(\frac{a^5}{5}\right)\left(1+1\right)\left(2\pi - 0\right)$$

$$= \frac{2 \times 2\pi a^5}{5} = \frac{4\pi a^5}{5}$$

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(3) A particle moves in a plane with constant radial velocity 4m/s. The angular velocity is constant and has a magnitude of 2 rad/s. When the particle is 3 m from origin, find the magnitude of (a) velocity (b) acceleration.

(a) $\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$

$= (4)\hat{r} + (3)(2)\hat{\theta}$

$|\vec{v}| = \sqrt{(16) + 36} = \sqrt{52} = 2\sqrt{13}\ m/s$

(b) $\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta}$

$= (0 - 3(2)^2)\hat{r} + (2(4)(2) + 0)\hat{\theta}$

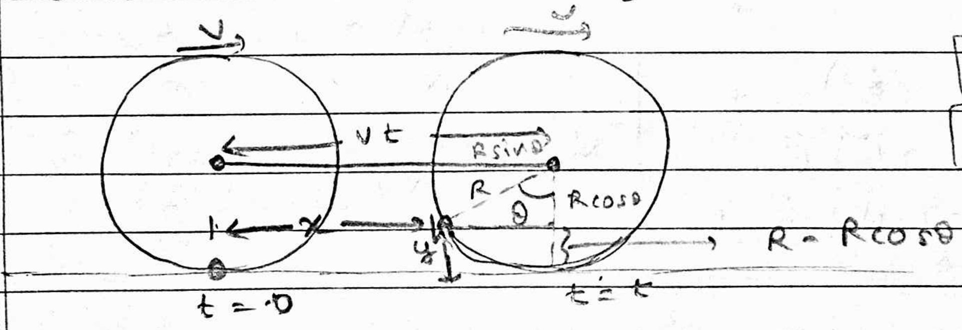
$= -12\hat{r} + 16\hat{\theta}$

$|\vec{a}| = \sqrt{144 + 256}$

$|\vec{a}| = \sqrt{400} = \boxed{20\ m/s^2}$

(4) A tire rolls in a straight line without slipping. Its centre moves with constant speed v. A small pebble lodged in the tread of the tire touches the road at $t=0$. Find the pebble's position, velocity & acceleration as function of time.

Sol^n



$\boxed{\theta = \omega t}$

$\boxed{v = R\omega}$

$vt = R\theta$

position

$x = vt - R\sin\theta = R\omega t - R\sin\theta$ or $R\theta - R\sin\theta$

$y = R - R\cos\theta$

$r(t) = (R\omega t - R\sin\overset{\omega t}{\theta})\hat{i} + R(1-\cos\theta)\hat{j}$

$v(t) = (R\omega - R\cos\omega t\,\omega)\hat{i} + R(\omega\sin\omega t)\hat{j}$

$= R\omega(1-\cos\omega t)\hat{i} + R\omega\sin(\omega t)\hat{j}$

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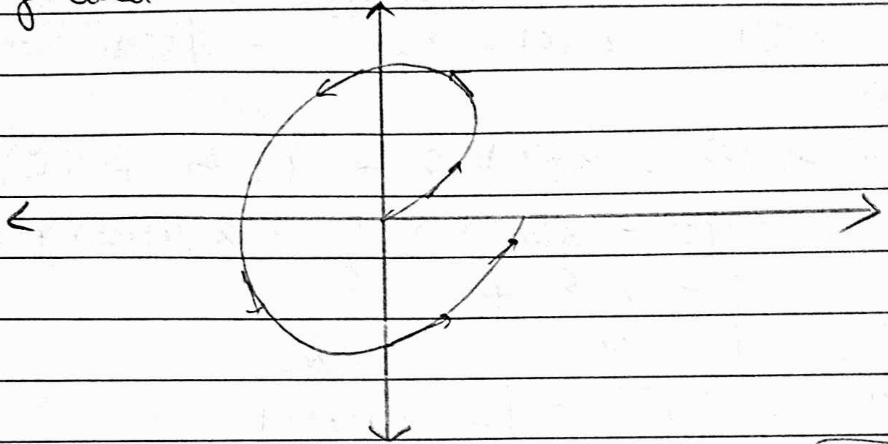
$a(t) = \omega^2 R \sin(\omega t)\,\hat{i} + \omega^2 R \cos(\omega t)\,\hat{j}$

Que: (5) A particle's trajectory is given by $r = A\theta$, where $A = (1/\pi)$ m/rad. Additionally, θ increases in time according to $\theta = \alpha t^2/2$ where α is a constant.

(a) Sketch the motion and indicate the approximate velocity and acceleration.



$r = A\theta = \frac{\theta}{\pi}$ $\quad r = \frac{\alpha t^2}{2\pi}$ $\quad r' = \frac{\alpha}{2\pi}(2t) = \boxed{\frac{\alpha t}{\pi}} = \dot{r}$

$\theta = \frac{\alpha t^2}{2}$ $\quad \boxed{r'' = \frac{\alpha}{\pi}} = \ddot{r}$

$\theta' = \boxed{\begin{matrix}\dot{\theta} = \alpha t \\ \ddot{\theta} = \alpha\end{matrix}}$

$\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$

$= \left(\frac{\alpha t}{\pi}\right)\hat{r} + \left(\frac{\alpha t^2}{2\pi}\alpha t\right)\hat{\theta}$

$\vec{v} = \left(\frac{\alpha t}{\pi}\right)\hat{r} + \left(\frac{\alpha^2 t^3}{2\pi}\right)\hat{\theta}$

$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta}$

$= \left(\frac{\alpha}{\pi} - \frac{\alpha t^2}{2\pi}(\alpha t)^2\right)\hat{r} +$

$\left(2\frac{\alpha t}{\pi}(\alpha t) + \frac{\alpha t^2 \alpha}{2\pi}\right)\hat{\theta}$

$= \frac{\alpha}{\pi}\left(1 - \frac{\alpha^2 t^4}{2}\right)\hat{r} + \frac{\alpha^2 t^2}{\pi}\left(2 + \frac{1}{2}\right)\hat{\theta}$

$= \frac{\alpha}{\pi}\left(\frac{2 - \alpha^2 t^4}{2}\right)\hat{r} + \frac{5}{2}\frac{\alpha^2 t^2}{\pi}\hat{\theta}$

(b) What value of θ makes the radial acceleration zero.

radial acceleration $\rightarrow \frac{\alpha}{\pi}\left(\frac{2-\alpha^2 t^4}{2}\right) = 0$

$$\frac{\alpha^2 t^4}{2} = 1 \Leftarrow 1 - \frac{\alpha^2 t^4}{2} = 0$$

$$\alpha^2 t^4 = 2$$

$$\boxed{\alpha t^2 = \sqrt{2}} \qquad \theta = \frac{\alpha t^2}{2} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$

$\theta = \frac{1}{\sqrt{2}}$ radial acceleration is zero.

(c) At what angles do the radial and tangential acce. have equal magnitudes.

radial acceleration = tangential acceleration

$$\cancel{\frac{\alpha}{\pi}}\left(\frac{2-\alpha^2 t^4}{\cancel{2}}\right) = \frac{5}{\cancel{2}}\,\frac{\alpha^2 t^2}{\cancel{\pi}}$$

$$2 - \underbrace{\alpha^2 t^4}_{x^2} = 5\,\underbrace{\alpha t^2}_{x}$$

$$x^2 - 2 + x = 0$$

$$\boxed{x^2 + x - 2 = 0}$$ got the value of x from here

$$x_1 = \frac{-5+\sqrt{33}}{2} \qquad x_2 = \frac{-5-\sqrt{33}}{2}$$

$$= 0.37228 \qquad x_2 = -5.37228$$

(6) The height of a certain hill is given by
$h(x,y) = 10(2xy - 3x^2 - 4y^2 - 18x + 28y + 12)$
where y is the distance north and x is the distance east of South Hadley.

(a) Where is top of hill located.

$$\vec{\nabla} h = 10(2y - 6x - 0 - 18 + 0 + 0)\hat{x} + 10(2x - 0 - 8y - 0 + 28 + 0)\hat{y} = 0$$

$$= 10(2y - 6x - 18 + 2x - 8y + 28) = 0$$

$$= 20(3y - 2x - 5) = 0 \Rightarrow 3y + 2x - 5 = 0$$

$10(\cancel{2}y - \cancel{6}^{3}x - \cancel{18}^{9}) = 0$

$\boxed{y - 3x - 9 = 0}$

$10(2x - 8y + 28) = 0$

$\boxed{x - 4y + 14 = 0}$

$\cancel{4y - 12x - 36 = 0}$

$-11x - 22 = 0$

$\boxed{x = -2}$

$y = 3x + 9 = -6 + 9$

$\boxed{y = 3}$

$(x, y) = (-2, 3)$

(b) How high is the hill?

$h(-2, 3) \Rightarrow 10(2(-2)(3) - 3(-2)^2 - 4(3)^2 - 18(-2) + 28(3) + 12)$

$= 10(-12 - \cancel{12} - 36 + \cancel{36} + 84 + \cancel{12})$

$= 720$

$= 720$ units.

(c) How steep is the slope at a point $(x, y) \equiv (1, 1)$? in what direction, is the slope steepest, at that point?

$\vec{\nabla} h = 10(2y - 6x - 18)\hat{x} + 10(2x - 8y + 28)\hat{y}$

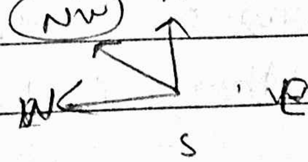
$\vec{\nabla} h = 10(2 - 6 - 18)\hat{x} + 10(2 - 8 + 28)\hat{y}$

$= -220\hat{x} + 220\hat{y}$ $\boxed{NW}$

$|\vec{\nabla} h| = \sqrt{(220)^2 + (220)^2} = 220\sqrt{2}$

$= 311.1269$

NW



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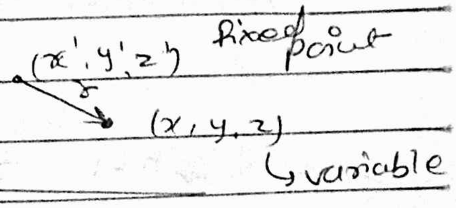
⑦ let $\vec{r}$ be the separation vector from a fixed point (x', y', z') to the point (x, y, z) and let r be its length show that

(a) $\vec{\nabla}(r^2) = 2\vec{r}$



$$\vec{r} = (x - x')\hat{i} + (y - y')\hat{j} + (z - z')\hat{k}$$

$$r^2 = (x - x')^2 + (y - y')^2 + (z - z')^2$$

variable → const

$$\vec{\nabla} r^2 = \left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}\right)\left((x - x')^2 + (y - y')^2 + (z - z')^2\right)$$

$$= 2(x - x')\hat{x} + 2(y - y')\hat{y} + 2(z - z')\hat{z}$$

$$= 2\left((x - x')\hat{x} + (y - y')\hat{y} + (z - z')\hat{z}\right)$$

$$= 2\vec{r}$$

(b) $\nabla\left(\frac{1}{r}\right) = -\frac{\hat{r}}{r^2}$

$$\nabla\left(\frac{1}{r}\right) = \left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}\right)\left((x - x')^2 + (y - y')^2 + (z - z')^2\right)^{-1/2}$$

$$= \frac{-1}{\cancel{2}}\left((x - x')^2 + (y - y')^2 + (z - z')^2\right)^{-3/2}\left[\cancel{2}(x - x')\hat{x} + \cancel{2}(y - y')\hat{y} + \cancel{2}(z - z')\hat{z}\right]$$

$$= -\frac{1}{(x - x')^2 + (y - y')^2 + (z - z')^2}\;\frac{(x - x')\hat{x} + (y - y')\hat{y} + (z - z')\hat{z}}{\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}}$$

$$= -\frac{1}{r^2}\left(\frac{r}{|r|}\right) = -\frac{1}{r^2}\hat{r}$$

$$= -\frac{\hat{r}}{r^2}$$

(c) general formula for $\nabla(r^n)$?

$$\vec{\nabla} r^n = \left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}\right)\left(\left((x - x')^2 + (y - y')^2 + (z - z')^2\right)^{\frac{n}{2}}\right)$$

$$= \frac{n}{\cancel{2}}\left((x - x')^2 + (y - y')^2 + (z - z')^2\right)^{\frac{n-2}{2}}\left(\cancel{2}(x - x')\hat{x} + \cancel{2}(y - y')\hat{y} + \cancel{2}(z - z')\hat{z}\right)$$

$$= n\left((x - x')^2 + (y - y')^2 + (z - z')^2\right)^{\frac{n-2}{2}}\left((x - x')\hat{x} + (y - y')\hat{y} + (z - z')\hat{z}\right)$$

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$$\vec{\nabla}(r^n) = n\left[\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}\right]^{n-2}\left((x-x')\hat{x}+(y-y')\hat{y}+(z-z')\hat{z}\right)$$

$$= n\left[\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}\right]^{n-1}\frac{\left((x-x')\hat{x}+(y-y')\hat{y}+(z-z')\hat{z}\right)}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}$$

$$= n \; r^{n-1} \; \frac{\vec{r}}{|r|}$$

$$\boxed{= \; n r^{n-1} \; \hat{r}}$$