Real Analysis (MA101), Tutorial Sheet-II (Sequence)

1. Let $\{u_n\}$ be a convergent sequence of real numbers converging to u. Then the sequence $\{|u_n|\}$ converges to $|u|$.

2. Let $\{u_n\}$ be a convergent sequence of real numbers and there exists a natural number m such that $u_n > 0$ for all $n \geq m$. Then $\lim u_n \geq 0$.

3. Let $\{u_n\}$ and $\{v_n\}$ be two convergent sequences that converge to u and v, respectively. Then

   (i) $\lim(u_n + v_n) = u + v$;

   (ii) If $c \in \mathbb{R}$ then $\lim(cu_n) = cu$;

   (iii) $\lim(u_n v_n) = uv$;

   (iv) $\lim \frac{u_n}{v_n} = \frac{u}{v}$, provided $\{v_n\}$ is a sequence of nonzero real numbers and $v \neq 0$.

4. Prove that $0 < \alpha < 1$, $\lim\limits_{n\to\infty} \alpha^n = 0$

5. Use the definition of limit of a sequence to establish the following limits:

   (i) $\lim\ (\ln\ (\frac{n}{n+1})) = 0$

   (ii) $\lim(\frac{3n+1}{2n+5}) = \frac{3}{2}$

   (iii) $\lim(\frac{n^2-1}{3n^2+5}) = \frac{1}{3}$

   (iv) If $c > 0$, then $\lim\ c^{\frac{1}{n}} = 1$.

6. A sequence $\{x_n\}$ is defined by $x_n = \frac{1}{2}(x_{n-1} + x_{n-2})$ for $n \geq 1$ and $x_1 = 0$ and $x_2 = 1$. Prove that the sequence $\{x_n\}$ converges and prove that the limit is $\frac{2}{3}$.

7. Let $x_n = (a^n + b^n)^{\frac{1}{n}}$ for all $n \in \mathbb{N}$ and $0 < a < b$, show that $\lim x_n = b$.

8. Prove that a monotone decreasing sequence, if bounded below, is convergent and it converges to the greatest lower bound.

9. Use Sandwich theorem to prove that

   (i) $\lim(2^n + 3^n)^{\frac{1}{n}} = 3$;

   (ii) $\lim \frac{1.3.5....(2n-1)}{2.4.6....2n} = 0$;

   (iii) $\lim(\frac{1}{1+n^2} + \frac{2}{2+n^2} + ... + \frac{n}{n+n^2}) = \frac{1}{2}$;

   (iv) $\lim((\sqrt{2} - 2^{\frac{1}{3}})(\sqrt{2} - 2^{\frac{1}{5}})...(\sqrt{2} - 2^{\frac{1}{2n+1}})) = 0$.

10. Show that the following sequence x_n is bounded and monotone for $n \in N$. Also, find the limit.

    (i) $x_1 = 2$ and $x_{n+1} = 2 - \frac{1}{x_n}$

    (ii) $x_1 = 1$ and $x_{n+1} = \sqrt{2 + x_n}$

    (iii) $x_1 \geq 2$ and $x_{n+1} = 1 + \sqrt{x_n - 1}$

    (iv) $x_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + ... + \frac{1}{n!}$

    (iv) $x_n = \frac{3n-1}{n+2}$

11. Let A be a nonempty bounded subset of $\mathbb{R}$ and $\alpha = inf A$. Show that there exists a sequence (a_n) such that $a_n \in A$ for all $n \in \mathbb{N}$ and $a_n \to \alpha$.

12. Let $\{x_n\}$ be a sequence in $\mathbb{R}$. Prove or disprove the following statements:

    (i) Let $x_n = (-1)^n\ \forall n \in \mathbb{N}$. The sequence (x_n) does not converge.

    (ii) If $x_n \to l (l \neq 0)$ and $\{y_n\}$ is a bounded sequence, then $(x_n y_n)$ converges.

    (iii) If the sequence $(x_n^2 + \frac{1}{n} x_n)$ converges then (x_n) converges.

13. Let (x_n) be a sequence defined by

$$x_n = (1 + \alpha)^{-n} n^{\beta} \cos n.$$

    for all $n \in \mathbb{N}$ where α and β are fixed positive real numbers. Show that (x_n) converges.

14. Show directly from definition that if $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences, then $\{x_n + y_n\}$ and $\{x_n y_n\}$ are Cauchy sequences.

15. If $x_n = \sqrt{n}$, show that $\{x_n\}$ satisfies $\lim |x_{n+1} - x_n| = 0$, but that is not a Cauchy Sequence.

16. If $0 < r < 1$ and $|x_{n+1} - x_n| < r^n$ for all $n \in \mathbb{N}$, show that $\{x_n\}$ is a Cauchy Sequence.

17. If $x_1 = 2$ and $x_{n+1} = 2 + \frac{1}{x_n}$ for $n \geq 1$. Show that $\{x_n\}$ is a contractive sequence. Find the limit.

18. The polynomial equation $x^3 - 5x + 1 = 0$ has root r with $0 < r < 1$. Use an appropriate contractive sequence to calculate r with in 10^{-4}.

19. Let $\{x_n\}$ be a sequence of positive real numbers such that $L := \lim \left(\frac{x_{n+1}}{x_n} \right)$ exists. If $L < 1$, then $\{x_n\}$ converges and $\lim(x_n) = 0$.

20. Show that if $p > 0$ and α is real, then $\lim_{n\to\infty} \frac{n^\alpha}{(1+p)^n} = 0$.

21. Let C be a sequence defined by the recursion formula : $x_{n+1} = \sqrt{7 + x_n}$, $x_1 = \sqrt{7}$. Show that $\{x_n\}$ converges to the positive root of $x^2 - x - 7 = 0$

22. Show that the sequence $\{x_n\} = 1 - \frac{1}{2} + \frac{1}{3} + \cdots + (-1)^{n+1}\frac{1}{n}$ is convergent.

23. Let $\lim_{n\to\infty} x_n = 0$, then prove that $\lim_{n\to\infty} \frac{x_1 + x_2 + \cdots + x_n}{n} = 0$.