1 Determine the flux of $\vec{F} = \rho^2 \cos^2 \phi \hat{e}_\rho + z \sin \phi \hat{e}_\phi$ over the closed cylinder $0 \le z \le 1$, ρ=4. Show that $\iiint_V \vec{\nabla}.\vec{F} dV = \iint_S \vec{F}.\vec{dS}$. All relevant steps carry marks

4 marks

$$\vec{\nabla}.\vec{F} = \left(\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho F_\rho \right) + \frac{1}{\rho} \frac{\partial F_\phi}{\partial \phi} + \frac{\partial F_z}{\partial z} \right)$$

$$F_\rho = \rho^2 \cos^2 \phi$$

$$F_\phi = z \sin \phi$$

$$\vec{\nabla}.\vec{F} = 3\rho \cos^2 \phi + \frac{1}{\rho} z \cos \phi$$

1 mark

$$\iiint_V \vec{\nabla}.\vec{F}\,dV = \iiint_V \left[3\rho\cos^2\phi + \frac{1}{\rho} z\cos\phi \right] \rho\, d\rho\, d\phi\, dz$$

Limits are ρ=0 to 4, ϕ=0 to 2π, z = 0 to 1

$=64\pi$

1 mark

$$\iint_S \vec{F}.d\vec{S} = \iint_S (\rho^2 \cos^2 \phi \hat{e}_\rho + z\sin\phi \hat{e}_\phi).\rho d\rho d\phi \hat{e}_z +$$ **TOP**

$$\iint_S (\rho^2 \cos^2 \phi \hat{e}_\rho + z\sin\phi \hat{e}_\phi).\rho d\rho d\phi(-\hat{e}_z) +$$ **BOTTOM**

$$\iint_S (\rho^2 \cos^2 \phi \hat{e}_\rho + z\sin\phi \hat{e}_\phi).\rho d\phi dz \hat{e}_\rho$$ **SIDE**

1 mark

$$\iint_S \vec{F}.d\vec{S} = \iint_S (\rho^2 \cos^2 \phi\, \hat{e}_\rho + z \sin \phi\, \hat{e}_\phi).\rho\, d\phi\, dz\, \hat{e}_\rho$$

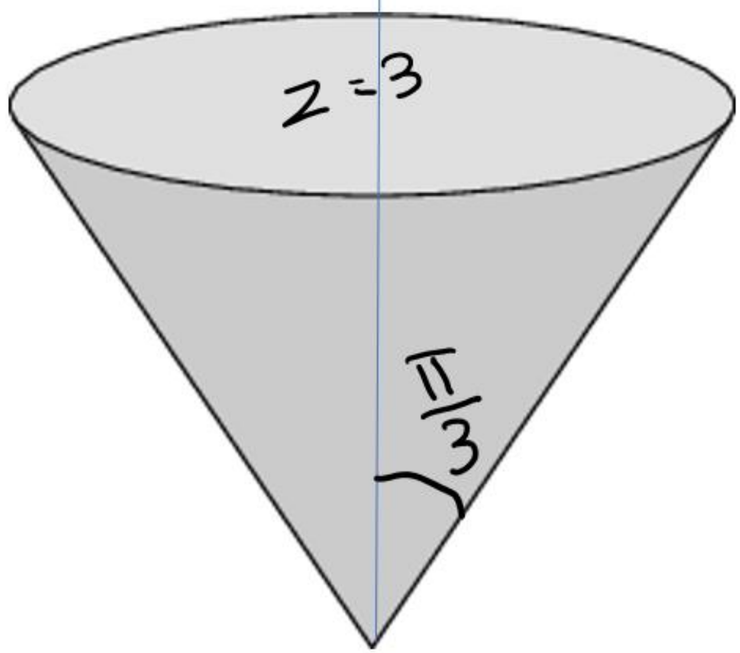
Limits ϕ =0 to 2π and z=0 to 1

=64π

1 mark

2

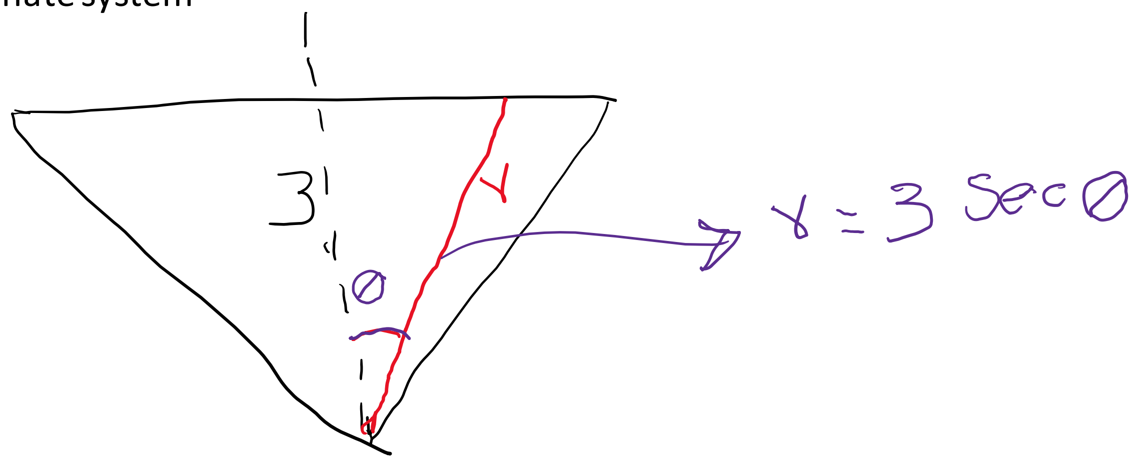
Find the volume of a cone whose angle is $\frac{\pi}{3}$ and below the plane $z = 3$ using the **spherical polar coordinate system.**



4 Marks

$$dV = r^2 \sin\theta \, dr \, d\theta \, d\phi$$

Finding the limits of ***r*** in spherical polar coordinate system



The limits of the r is from **0 to 3sec θ**

---(1 Marks)

$$\text{Limits:}\begin{cases} 0 \le r \le 3\sec\theta \\ 0 \le \theta \le \dfrac{\pi}{3} \\ 0 \le \phi \le 2\pi \end{cases}$$

Therefore, the volume of the cone is $$\int_0^{2\pi}\int_0^{\frac{\pi}{3}}\int_0^{3\sec\theta} r^2 \sin\theta \, dr\, d\theta\, d\phi$$

---(0.5 Marks)

$$V = \int_0^{2\pi} \int_0^{\frac{\pi}{3}} \int_0^{3\sec\theta} r^2 \sin\theta dr d\theta d\phi$$

$$V = \int_0^{2\pi} \int_0^{\frac{\pi}{3}} \left[\frac{r^3}{3} \sin\theta \right]_0^{3\sec\theta} d\theta d\phi$$

$$V = \int_0^{2\pi} \int_0^{\frac{\pi}{3}} 9\sec^3\theta \sin\theta d\theta d\phi$$

$$V = 9\int_0^{2\pi} \int_0^{\frac{\pi}{3}} \sec^2\theta \tan\theta d\theta d\phi$$

$$V = \frac{27}{2} \int_0^{2\pi} d\phi$$

---(1.5 Marks)

$$V = \frac{27}{2}\int_{0}^{2\pi} d\phi = 27\pi$$

--(1 Marks)

(3) Express the below mentioned integral into cylindrical polar co-ordinate system. Note: No need to work out the integral, but simply express the integral.

$$\int_{y=-1}^{y=1}\int_{x=0}^{x=\sqrt{1-y^2}}\int_{z=x^2+y^2}^{z=\sqrt{x^2+y^2}} xyz\,dz\,dx\,dy$$

The range of limits are

$$-1 \leq y \leq 1$$

$$0 \leq x \leq \sqrt{1-y^2}$$

$$x^2 + y^2 \leq z \leq \sqrt{x^2+y^2}$$

4 Marks

The ranges of φ ρ and Z are:

$$\text{Limits:}\begin{cases} 0 \le \rho \le 1 \text{ (0.5 mark for this limit)} \\ -\dfrac{\pi}{2} \le \varphi \le \dfrac{\pi}{2} \text{ (1 mark for this limit)} \\ r^2 \le z \le r \text{ (0.5 mark for this limit)} \end{cases}$$

---(2 Marks for getting all the limits correctly)

$$\int_{y=-1}^{y=1}\int_{x=0}^{x=\sqrt{1-y^2}}\int_{z=x^2+y^2}^{z=\sqrt{x^2+y^2}} xyz\,dz\,dx\,dy =$$

$$\int_{\rho=0}^{\rho=1}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{z=\rho^2}^{z=\rho} \rho^3 \cos\theta \sin\theta\, z\,dz\,d\theta\,d\rho$$

---(2 Marks for writing the integral correctly)

4 A force is described by

$$\vec{F} = -\hat{e}_x \frac{y}{x^2+y^2} + \hat{e}_y \frac{x}{x^2+y^2}$$

(a) Express $\vec{F}$ in cylindrical polar co-ordinates

(b) Calculate curl of $\vec{F}$ in cylindrical polar co-ordinates

3 marks

$$x = \rho\cos(\phi)$$
$$y = \rho\sin(\phi)$$
$$z = z$$

$$\hat{e}_x = \hat{e}_\rho \cos(\phi) - \hat{e}_\phi \sin(\phi) + 0\hat{e}_z$$
$$\hat{e}_y = \hat{e}_\rho \sin(\phi) + \hat{e}_\phi \cos(\phi) + 0\hat{e}_z$$
$$\hat{e}_z = \hat{e}_z$$

$$\vec{F} = \frac{\sin\phi}{\rho}\left(-\hat{e}_x\right) + \frac{\cos\phi}{\rho}\left(\hat{e}_y\right)$$

$$\vec{F} = \frac{\sin\phi}{\rho}\left(-\hat{e}_\rho \cos(\phi) + \hat{e}_\phi \sin(\phi)\right) + \frac{\cos\phi}{\rho}\left(\hat{e}_\rho \sin(\phi) + \hat{e}_\phi \cos(\phi)\right)$$

$$\vec{F} = \frac{1}{\rho}\hat{e}_\phi$$

1.5 mark

$$\vec{\nabla} \times \vec{A} = \left(\frac{1}{\rho} \frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z} \right) \hat{e}_\rho + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho} \right) \hat{e}_\phi + \frac{1}{\rho} \left(\frac{\partial \left(\rho A_\phi \right)_z}{\partial \rho} - \frac{\partial A_\rho}{\partial z} \right) \hat{e}_z$$

$$\vec{F} = \frac{1}{\rho} \hat{e}_\phi$$

$$\vec{\nabla} \times \vec{F} = \frac{1}{\rho} \left(\frac{\partial \left(\rho A_\phi \right)_z}{\partial \rho} \right) \hat{e}_z = 0$$

1.5 mark