Lecture - 6 Division - I Thursday 4-8-2011

Recall:

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

$$= \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

Example: $f(z) = z^2$.

Let z_0 be an arbitrary point in $\mathbb{C}$.

Examine the differentiability of $f(z) = z^2$ at z_0.

$$\lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z} = \lim_{\Delta z \to 0} \frac{(z_0+\Delta z)^2 - z_0^2}{\Delta z} = \lim_{\Delta z \to 0} \frac{2z_0\Delta z + (\Delta z)^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} (2z_0 + \Delta z) = 2z_0$$

$$\boxed{f'(z_0) = 2z_0} \quad \text{That is, } \boxed{\frac{d}{dz}(z^2) = 2z}$$

Example: $f(z) = |z|^2$

Case I: $z_0 \in \mathbb{C}$ and $z_0 \neq 0$

$$\lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z} = \lim_{\Delta z \to 0} \frac{|z_0+\Delta z|^2 - |z_0|^2}{\Delta z} = \lim_{\Delta z \to 0} \frac{(z_0+\Delta z)(\overline{z_0}+\overline{\Delta z}) - z_0\overline{z_0}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \overline{z_0} + \overline{\Delta z} + z_0 \frac{\overline{\Delta z}}{\Delta z}$$

We try to find limiting value of the above expression along two different paths

Path – I: $\Delta z \to 0$ along the path $\Delta y = 0$ and $\Delta x \to 0$

$$\lim_{\substack{\Delta y = 0 \\ \Delta x \to 0}} \left(\overline{z_0} + \Delta x + z_0 \frac{\Delta x}{\Delta x} \right) = \overline{z_0} + z_0 \longrightarrow (*)$$

Path – II $\Delta z \to 0$ along the path $\Delta x = 0$ and $\Delta y \to 0$

$$\lim_{\substack{\Delta x = 0 \\ \Delta y \to 0}} \left(\overline{z_0} - i\Delta y + z_0 \left(\frac{-i\Delta y}{i\Delta y} \right) \right) = \overline{z_0} - z_0 \longrightarrow (**)$$

From $(*)$ and $(**)$

$$\overline{z_0} + z_0 = \overline{z_0} - z_0 \Rightarrow z_0 = -z_0 \Rightarrow z_0 \text{ must be zero.}$$

If $z_0 \neq 0$ then $\overline{z_0} + z_0 \neq \overline{z_0} - z_0$

Therefore $\displaystyle\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$ does NOT exist, if $z_0 \neq 0$.

$\Rightarrow$ f is NOT differentiable at non-zero points z_0 in $\mathbb{C}$.

Case – II: $z_0 = 0$

$$\lim_{\Delta z \to 0} \frac{f(0 + \Delta z) - f(0)}{\Delta z} = \lim_{\Delta z \to 0} \frac{|\Delta z|^2 - 0}{\Delta z} = \lim_{\Delta z \to 0} \frac{\overline{\Delta z}\, \Delta z}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \overline{\Delta z} = 0$$

$$\Rightarrow f'(0) = 0.$$

f is differentiable only at $z = 0$.

MORAL of this example:

$f : \mathbb{R} \to \mathbb{R}$

$f(x) = |x|^2 = x^2$ differentiable on $\mathbb{R}$

$f : \mathbb{C} \to \mathbb{C}$

$f(z) = |z|^2$ is differentiable only at the origin in $\mathbb{C}$.

Theorem: Let $f : D \subseteq \mathbb{C} \to \mathbb{C}$. D is open. Let $z_0 \in D$. If f is differentiable at z_0 then f is continuous at z_0.

Proof: To show: f is continuous at z_0

i.e.) To show: $\lim_{z \to z_0} f(z) = f(z_0)$

i.e., To show: $\lim_{z \to z_0} \left(f(z) - f(z_0)\right) = 0$.

Consider,

$$f(z) - f(z_0) = \frac{\left(f(z) - f(z_0)\right)}{(z - z_0)} (z - z_0)$$

$$\lim_{z \to z_0} \left(f(z) - f(z_0)\right) = \lim_{z \to z_0} \frac{\left(f(z) - f(z_0)\right)}{(z - z_0)} (z - z_0)$$

$$= \left(\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}\right) \left(\lim_{z \to z_0} (z - z_0)\right)$$

$$= f'(z_0) \times 0 = 0$$

Result: Let f and g be differentiable at z_0.
Then, (i) $f+g$, kf, fg are differentiable at z_0
and (ii) $\frac{f}{g}$ is differentiable at z_0, provided $g(z_0) \neq 0$.

Further,

$$\frac{d}{dz}(f+g) = \frac{d}{dz} f + \frac{d}{dz} g \Big|_{\text{at } z = z_0}$$

$$\frac{d}{dz}(fg) = f \frac{d}{dz}(g) + g \frac{d}{dz}(f) \Big|_{\text{at } z = z_0}$$

If $g(z_0) \neq 0$ then $\frac{d}{dz}\left(\frac{f}{g}\right) = \left.\frac{f'g - fg'}{(g)^2}\right|_{\text{at } z = z_0}$

<u>Examples</u>: Polynomials are differentiable at each point of $\mathbb{C}$.

<u>Partial Derivatives</u> of Component functions of f

Let D be an open set in $\mathbb{C}$ and let $z_0 \in D$.

Let $f(z) = u(x,y) + i\,v(x,y)$ be a function defined on D.

Here

$$\left.\begin{array}{l} u(x,y) = Re(f(z)) \\ v(x,y) = Im(f(z)) \end{array}\right\}$$

These are component / coordinate functions of f.

$u : D \subseteq \mathbb{C} \to \mathbb{R}$

$v : D \subseteq \mathbb{C} \to \mathbb{R}$

First order partial derivatives of the component functions $u(x,y)$ and $v(x,y)$ of f.

$$u_x\Big|_{z=z_0} = \frac{\partial u}{\partial x}\Big|_{z=z_0} = \lim_{\substack{h\to 0 \\ h\in\mathbb{R}}} \frac{u(x_0+h,\, y_0) - u(x_0, y_0)}{h}$$

$$u_y\Big|_{z=z_0} = \frac{\partial u}{\partial y}\Big|_{z=z_0} = \lim_{k\to 0} \frac{u(x_0,\, y_0+k) - u(x_0, y_0)}{k}$$

$$v_x\Big|_{z=z_0} = \frac{\partial v}{\partial x}\Big|_{z=z_0} = \lim_{h\to 0} \frac{v(x_0+h,\, y_0) - v(x_0, y_0)}{h}$$

$$v_y\Big|_{z=z_0} = \frac{\partial v}{\partial y}\Big|_{z=z_0} = \lim_{k\to 0} \frac{v(x_0, y_0+k) - v(x_0,y_0)}{k}$$

Relation between $\boxed{f'(z_0)}$ and $\boxed{u_x, u_y, v_x, v_y \text{ at } z_0}$

Results we are going to prove:

THEOREM

① $\boxed{f'(z_0) \text{ exists}} \implies$ u_x, u_y, v_x, v_y at z_0 exist. Further they satisfy the Cauchy-Riemann equations $u_x = v_y$ & $u_y = -v_x$ at z_0

NECESSARY CONDITION

Providing EXAMPLE

u_x, u_y, v_x, v_y exist in a neighborhood $N(z_0)$ of z_0

AND

u_x, u_y, v_x, v_y are continuous at z_0

$\not\Rightarrow$ DOES NOT IMPLY

f is differentiable at z_0

Example: $f(z) = |z|^2$

Providing EXAMPLE

u_x, u_y, v_x, v_y exist in a neighborhood $N(z_0)$ of z_0

AND

Satisfies CR equations at z_0: $u_x = v_y$ and $u_y = -v_x$ at z_0

$\not\Rightarrow$ DOES NOT IMPLY

f is differentiable at z_0

Example:
$$f(z) = \begin{cases} \dfrac{(\bar{z})^2}{z} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$$

THEOREM

u_x, u_y, v_x, v_y exists in a neighborhood $N(z_0)$ of z_0.

AND

u_x, u_y, v_x, v_y are continuous at z_0

AND

Satisfied Cauchy-Riemann equations at z_0

$$u_x = v_y \quad \text{at } z_0$$
$$u_y = -v_x$$

$\Longrightarrow$ Implies

f is differentiable at z_0

SUFFICIENT CONDITIONS

Necessary condition for differentiability: Let $f(z) = u(x,y) + i\, v(x,y)$ be a function defined in an open set D in $\mathbb{C}$. Let $z_0 = x_0 + i y_0 \in D$.

If f is differentiable at z_0 then the first order partial derivatives of u and v must exist at z_0 and they must satisfy the Cauchy-Riemann (CR) equations $\boxed{u_x = v_y, \quad u_y = -v_x}$ at z_0.

Also, $f'(z_0)$ can be written

$$\boxed{f'(z_0) = (u_x + i\, v_x)\Big|_{z=z_0} = (v_y - i\, u_y)\Big|_{z=z_0}}$$

Proof:

$z_0 = x_0 + i y_0$. Set $\Delta z = \Delta x + i \Delta y$

$$f(z_0 + \Delta z) - f(z_0) = f(x_0 + \Delta x,\ y_0 + \Delta y) - f(x_0, y_0)$$
$$= \Big(u(x_0 + \Delta x,\ y_0 + \Delta y) + i\, v(x_0 + \Delta x,\ y_0 + \Delta y)\Big) - \Big(u(x_0, y_0) + i\, v(x_0, y_0)\Big)$$

$$= \left(u(x_0+\Delta x,\ y_0+\Delta y) - u(x_0,y_0)\right) + i\left(v(x_0+\Delta x,\ y_0+\Delta y) - v(x_0,y_0)\right)$$

Consider $\displaystyle \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$

$$= \lim_{\Delta z \to 0} \frac{\left(u(x_0+\Delta x,\ y_0+\Delta y) - u(x_0,y_0)\right) + i\left(v(x_0+\Delta x,\ y_0+\Delta y) - v(x_0,y_0)\right)}{(\Delta x + i\Delta y)}$$

Path I: $\Delta y = 0$ and $\Delta x \to 0$.

$$\lim_{\substack{\Delta y = 0 \\ \Delta x \to 0}} \frac{\left(u(x_0+\Delta x,\ y_0) - u(x_0,y_0)\right) + i\left(v(x_0+\Delta x,\ y_0) - v(x_0,y_0)\right)}{\Delta x}$$

$$= \left(\lim_{\Delta x \to 0} \frac{u(x_0+\Delta x,\ y_0) - u(x_0,y_0)}{\Delta x}\right) + i\left(\lim_{\Delta x \to 0} \frac{v(x_0+\Delta x,\ y_0) - v(x_0,y_0)}{\Delta x}\right)$$

$$= \boxed{\left(\frac{\partial u}{\partial x}\right)\Bigg|_{z=z_0} + i\left(\frac{\partial v}{\partial x}\right)\Bigg|_{z=z_0}} \longrightarrow \text{Equation } 1$$

Path II: $\Delta x = 0$ and $\Delta y \to 0$

$$\lim_{\substack{\Delta x = 0 \\ \Delta y \to 0}} \frac{\left(u(x_0,\ y_0+\Delta y) - u(x_0,y_0)\right) + i\left(v(x_0,\ y_0+\Delta y) - v(x_0,y_0)\right)}{i\Delta y}$$

$$= \left(\lim_{\Delta y \to 0} \frac{u(x_0,y_0+\Delta y) - u(x_0,y_0)}{i\Delta y}\right) + \left(\lim_{\Delta y \to 0} \frac{v(x_0,\ y_0+\Delta y) - v(x_0,y_0)}{\Delta y}\right)$$

$$= \boxed{-i\left(\frac{\partial u}{\partial y}\right)\bigg|_{z=z_0} + \left(\frac{\partial v}{\partial y}\right)\bigg|_{z=z_0}} \longrightarrow \text{Equation (2)}$$

Since f is differentiable at z_0, from the equations (1) and (2) we get

$$\boxed{\left(\frac{\partial u}{\partial x}\right)\bigg|_{z=z_0} + i\left(\frac{\partial v}{\partial x}\right)\bigg|_{z=z_0} = f'(z_0) = \left(\frac{\partial v}{\partial y}\right)\bigg|_{z=z_0} - i\left(\frac{\partial u}{\partial y}\right)\bigg|_{z=z_0}} \hookrightarrow (*)$$

It shows that u_x, u_y, v_x, v_x exist at z_0.

Equating the real and imginary parts both sides in $(*)$, we get

$$\boxed{\begin{aligned} \left(\frac{\partial u}{\partial x}\right)\bigg|_{z=z_0} &= \left(\frac{\partial v}{\partial y}\right)\bigg|_{z=z_0} \\ \left(\frac{\partial u}{\partial y}\right)\bigg|_{z=z_0} &= -\left(\frac{\partial v}{\partial x}\right)\bigg|_{z=z_0} \end{aligned}} \longrightarrow (A)$$

The above set of equations given in (A) are called the Cauchy - Riemann equations.

In brief, we say it as CR equations and is written as

$$\boxed{u_x = v_y \quad \text{and} \quad u_y = -v_x}$$

Note: There are functions which satisfy the Cauchy-Riemann equations at $z=z_0$, but fail to be differentiable at z_0.

Example:

$$f(z) = \begin{cases} \dfrac{(\bar{z})^2}{z} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$$

f satisfies CR equations at $z=0$, but f is not differentiable at $z=0$.

Use / Application of Necessary condition for differentiability:

f is differentiable at $z_0 \Rightarrow$ u_x, u_y, v_x, v_y exist at z_0 and they satisfy CR equations at z_0.

$\boxed{\text{NOT satisfying CR equations at } z_0} \Rightarrow \boxed{f \text{ is NOT differentiable at } z_0}$

Example: $f(z) = |z|^2 = x^2+y^2$.

Here, $u(x,y) = x^2+y^2$, $v(x,y)=0$

$$u_x = 2x, \quad u_y = 2y, \quad v_x = 0, \quad v_y = 0.$$

When $(x,y) \neq (0,0)$, $u_x = 2x \neq 0 = v_y$

$$u_y = 2y \neq 0 = -v_x$$

$f(z)=|z|^2$ does not satisfy CR equations at $z \neq 0$. Therefore, we conclude that $f(z) = |z|^2$ is NOT differentiable at $z \neq 0$.

SUFFICIENT CONDITIONS FOR DIFFERENTIABILITY

Let $f(z) = u(x,y) + i\, v(x,y)$ be defined on an open set D in $\mathbb{C}$.
Let $z_0 \in \mathbb{C}$.

Suppose that

(i) The first order partial derivatives u_x, u_y, v_x and v_y exist at all points in some neighborhood $N(z_0)$ of the point z_0.

(ii) u_x, u_y, v_x, v_y are continuous at z_0.

(iii) CR equations $u_x = v_y$ and $u_y = -v_x$ are satisfied at z_0.

Then

f is differentiable at z_0.

Proof of the above result (sufficient conditions for differentiability) is omitted for this course. Interested students can read from Brown & churchill book – Section 21.

Exercise: Using sufficient conditions for differentiability, show that the following functions are differentiable in $\mathbb{C}$.

(i) $f(z) = z^2$ (ii) $g(z) = e^x \cos y + i\, e^x \sin y$.

Theorem: Let D be a domain ($=$ open, connected set) in $\mathbb{C}$.
Let $f: D \subseteq \mathbb{C} \to \mathbb{C}$.
If $f'(z) = 0$ for all z in D then f is a constant function in D.

Proof: We know that $f'(z) = u_x + i v_x = v_y - i u_y$.

Since $f'(z) = 0$ for all z in D, it follows that

$$u_x = 0, \quad u_y = 0, \quad v_x = 0, \quad v_y = 0 \text{ at all points } z \text{ in } D.$$

$u : D \subseteq \mathbb{C} \to \mathbb{R}$ (real valued)

$u_x = 0$, $u_y = 0$ in D

grad $u = \nabla u(x,y) = \vec{0}$ in D

Rate of change of $u(x,y)$ at the point z along the direction of unit vector $\vec{e}$ is

$$D_{\vec{e}}\, u \big|_z = \langle \nabla u|_z, \vec{e} \rangle = 0$$

$\Rightarrow$ u is a constant function in D

$v : D \subseteq \mathbb{C} \to \mathbb{R}$ (real valued)

$v_x = 0$, $v_y = 0$ in D

grad $v = \nabla v(x,y) = \vec{0}$ in D

Rate of change of $v(x,y)$ at the point z along the direction of unit vector $\vec{e}$ is

$$D_{\vec{e}}\, v \big|_z = \langle \nabla v|_z, \vec{e} \rangle = 0$$

$\Rightarrow$ v is a constant function in D

$\Rightarrow$ $f = u + iv$ is a constant function in D

Note: In the above result, we can NOT drop the condition of connectedness of D.

Counter example:
$$f(z) = \begin{cases} 2 & \text{if } \mathrm{Re}(z) < 4 \\ 7 & \text{if } \mathrm{Re}(z) > 4 \end{cases}$$

$D = \{ z \in \mathbb{C} \mid \mathrm{Re}(z) \neq 4 \}$ = Open, but not connected.

Observe that $f'(z) = 0$ for all points z in D, but f is not constant in D.

Chain rule: (Differentiability under composition)

Suppose that f is differentiable at z_0 and g is differentiable at $f(z_0)$.

Then, the composition function $h(z) = g(f(z))$ is differentiable at z_0 and $\boxed{h'(z_0) = g'(f(z_0))\, f'(z_0)}$

(Without proof)

Lecture 6 ends. Division 1