System of first order ODEs

ind variable $\rightarrow t$
dep. variables $\rightarrow x, y, z, \ldots$ etc
OR
$x_1, x_2, x_3 \ldots$ etc.

Notation.

Standard form for system of 2 first order ODEs

$$\dot{x} = f(x, y)$$
$$\dot{y} = g(x, y)$$

Solve for both x and y simulaniously $\left[\dot{x} \equiv x' \equiv \frac{dx}{dt}\right]$ (coupled system).

Linear system of first order ODEs

$$\dot{x} = ax + by + r_1$$
$$\dot{y} = cx + dy + r_2 \quad (*)$$

a, b, r_1, c, d, and r_2 are functions of t.

The above system is called homogenous if $r_1 \equiv r_2 \equiv 0$

The above system is called a system with constant coefficients (time invariant system OR autonomus) if a, b, c, d are constants. [famous keyword LTI systems]

Existence and uniqueness result

Any LTI system $(*)$ has a unique solution passing thr. initial data

$$x(0) = x_0$$
$$y(0) = y_0$$

Initial condition for $(*)$

First, we learn techniques to solve LTI homogeneous system

$$\left.\begin{aligned}\dot{x} &= ax + by \\ \dot{y} &= cx + dy\end{aligned}\right\} \text{ — } (*)$$

$\equiv$

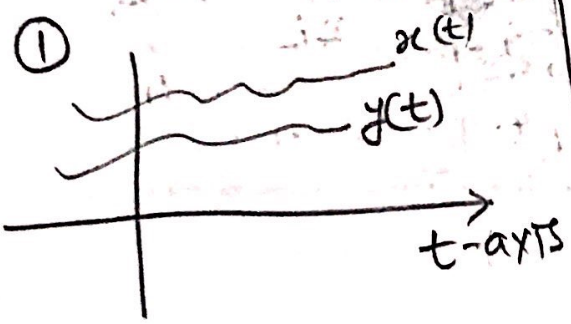
$\dot{X} = AX$; where $X = \begin{bmatrix} x \\ y \end{bmatrix}$; $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Let solution to $(*)$ be

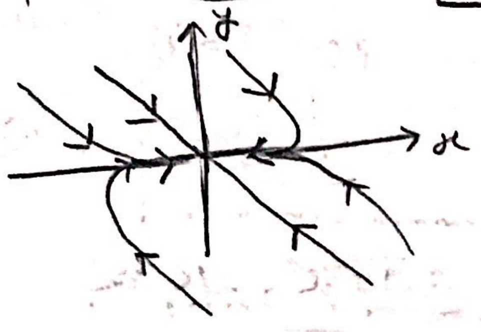
$$\left.\begin{aligned}x &= f(t) \\ y &= g(t)\end{aligned}\right\} \text{ — } (\text{Sol}_1)$$

Then (Sol_1) defines a parametric curve in x-y plane, which is called a path, orbit or trajectory of $(*)$.

There are two ways to plot solution of $(*)$

① 

② plot (Sol_1) in x-y plane (phase plan)



Matrix / eigenvalue - eigenvector method

for constant coefficient homogeneous linear system

$$\dot{X} = AX \quad \text{—— } (*)$$

$\dot{X}$ OR $X' \equiv \dfrac{dX}{dt}$

$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ $\quad \dfrac{dX}{dt} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix}$; $A \equiv n \times n$ matrix with real entries.

$\boxed{\text{Assume } n=2}$ → for simplicity only.

Trial solution; $\quad X(t) = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{\lambda t}$ $\quad \boxed{\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}}$

Then $\quad \lambda \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^{\lambda t} = A \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^{\lambda t}$

$$\Rightarrow \quad A \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \lambda \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \quad [e^{\lambda t} \neq 0]$$

Thus if λ is an eigenvalue of A and $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ is its corresponding eigenvector, Then $X(t)$ solves $(*)$

Remember! we used trial sol. $y = e^{mt}$ for $y'' + py' + qy = 0$. But there is no harm if we take $y = a e^{mt}$ $(a \neq 0)$ as trial solution. Then we obtain $a[m^2 + pm + q]e^{mt} = 0$. Therefore, we get the same AE $m^2 + pm + q = 0$

Now there are three cases.

case (1) $\quad \lambda_1$ and λ_2 are real and distinct

and $\begin{bmatrix} \alpha_1 \\ \phi \end{bmatrix}$ $\begin{bmatrix} \alpha_2 \\ \phi \end{bmatrix}$ are corresponding eigenvectors.

Then

$$\boxed{X(t) = c_1 \begin{bmatrix} \alpha_1 \\ \phi \end{bmatrix} e^{\lambda_1 t} + c_2 \begin{bmatrix} \alpha_2 \\ \phi \end{bmatrix} e^{\lambda_2 t}}$$

is general sol. to $(*)$, where c_1 and c_2 are two arbitrary constants.

Case 2 $\quad \lambda_1 = a+ib \quad \lambda_2 = a-ib$

Let $\begin{pmatrix}\alpha_1\\ \vdots\end{pmatrix}$ be eigenvector wrt λ_1

Then
$$X(t) = \begin{pmatrix}\alpha_1\\ \vdots\end{pmatrix} e^{(a+ib)t}$$
$$= \begin{pmatrix}u\\ \vdots\end{pmatrix} + i\begin{pmatrix}v\\ \vdots\end{pmatrix} \quad \text{is a solution to } \circledast$$

Thus general solution to $\circledast$ is

$$X(t) = c_1\begin{bmatrix}u\\ \vdots\end{bmatrix} + c_2\begin{bmatrix}v\\ \vdots\end{bmatrix}.$$

Case 3 $\quad \lambda_1 = \lambda_2 = \lambda$

Subcase 1 $\quad \boxed{a.m. = g.m} = 2$

then we obtain 2 LI eigenvectors say $\begin{pmatrix}\alpha_1\\ \vdots\end{pmatrix}$ and $\begin{pmatrix}\alpha_2\\ \vdots\end{pmatrix}$ for λ and in this case general solution to $\circledast$ is

$$\boxed{X(t) = c_1\begin{pmatrix}\alpha_1\\ \vdots\end{pmatrix} e^{\lambda t} + c_2\begin{pmatrix}\alpha_2\\ \vdots\end{pmatrix} e^{\lambda t}}$$

Subcase 2 $\quad \boxed{g.m = 1}$ such λ's are called defective eigenvalues.

Say $\begin{pmatrix}\alpha_1\\ \vdots\end{pmatrix}$ is an eigenvector wrt λ, then $X(t) = \begin{bmatrix}\alpha_1\\ \vdots\end{bmatrix} e^{\lambda t}$ is a solution to $\circledast$. For second solution take trial solution

$$X(t) = \left(\begin{bmatrix}\beta_2\\ \vdots\end{bmatrix} t + \begin{bmatrix}\beta_1\\ \vdots\end{bmatrix}\right) e^{\lambda t} \quad \text{for some } \begin{bmatrix}\beta_1\\ \vdots\end{bmatrix} \text{ \& } \begin{bmatrix}\beta_2\\ \vdots\end{bmatrix}$$

Thus, we obtain

$$\begin{bmatrix}\beta_2\\ \phi\end{bmatrix}e^{\lambda t} + \left(\begin{bmatrix}\beta_2\\ \phi\end{bmatrix}t + \begin{bmatrix}\beta_1\\ \phi\end{bmatrix}\right)\lambda e^{\lambda t}$$

$$= A\left(\begin{bmatrix}\beta_2\\ \phi\end{bmatrix}t + \begin{bmatrix}\beta_1\\ \phi\end{bmatrix}\right)e^{\lambda t}$$

$$\Rightarrow \quad [A-\lambda I]\left(\begin{bmatrix}\beta_2\\ \phi\end{bmatrix}t + \begin{bmatrix}\beta_1\\ \phi\end{bmatrix}\right)e^{\lambda t} = \begin{bmatrix}\beta_2\\ \phi\end{bmatrix}e^{\lambda t}$$

The above equation is satisfied if we take

$[A-\lambda I]\begin{bmatrix}\beta_2\\ \phi\end{bmatrix} = 0$ & $(A-\lambda I)\begin{bmatrix}\beta_1\\ \phi\end{bmatrix} = \begin{bmatrix}\beta_2\\ \phi\end{bmatrix}$.

Thus $\boxed{\begin{bmatrix}\beta_2\\ \phi\end{bmatrix} = \begin{bmatrix}\alpha_1\\ \phi\end{bmatrix}}$

$\boxed{\text{\& for } \begin{bmatrix}\beta_1\\ \phi\end{bmatrix}; \text{ solve } [A-\lambda I]\begin{bmatrix}\beta_1\\ \phi\end{bmatrix} = \begin{bmatrix}\alpha_1\\ \phi\end{bmatrix}}$

The matrix/ eigenvalues- eigenvectors method can be extended easily to a system of n- equations [except the subcase 2 of case 3]

☞ If $A \equiv n \times n$ then $\dot{X} = AX \equiv$ n^{th} order linear homogeneous ODE of constant coefficients.

2×2 system's general solution has two arbitrary constants.

Ex1 Solve:-

$$\dot{x} = -x + 2y$$
$$\dot{y} = \quad -3y$$

Learnt about competition models

x = defense budget of Country A

y = defense budget of Country B

$$\dot{x} = -x + by$$
$$\dot{y} = cx - 3y$$

$$\dot{X} = AX \quad ; \text{ where } X = \begin{bmatrix} x \\ y \end{bmatrix} \text{ and } A = \begin{bmatrix} -1 & 2 \\ 0 & -3 \end{bmatrix}$$

First, we find eigenvalues of A.

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} -1-\lambda & 2 \\ 0 & -3-\lambda \end{vmatrix} = 0 \Rightarrow \lambda = -1, -3.$$

Now, we find eigenvector of A corresponding to $\lambda = -3$

$$[A - \lambda I] = [A + 3I] = \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 1 \\ 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

find its null space

eigenvector of A corresponding to $\lambda = -1$

$$[A - \lambda I] = \begin{bmatrix} 0 & \boxed{2} \\ 0 & -2 \end{bmatrix} \sim \begin{bmatrix} 0 & \boxed{1} \\ 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Thus general solution

$$X(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-3t} + c_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-t}$$

$$= \begin{bmatrix} -c_1 e^{-3t} + c_2 e^{-t} \\ c_1 e^{-3t} \end{bmatrix} ✓$$

$$\boxed{\begin{aligned} x(t) &= -c_1 e^{-3t} + c_2 e^{-t} \\ y(t) &= c_1 e^{-3t} \end{aligned}}$$

EX 2

$$\dot{x} = 3x + 2y \qquad \dot{X} = AX ; \quad X = \begin{bmatrix} x \\ y \end{bmatrix}$$
$$\dot{y} = -5x + y \qquad A = \begin{bmatrix} 3 & 2 \\ -5 & 1 \end{bmatrix}$$

eigenvalues of A

$|A - \lambda I| = 0 \Rightarrow \lambda^2 - 4\lambda + 13 = 0$

$\Rightarrow \lambda = \dfrac{4 \pm \sqrt{16-52}}{2} = \dfrac{4 \pm 6i}{2}$

$= 2 \pm 3i$

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

then

$|A - \lambda I| = 0$

$\Rightarrow \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0$

$\Rightarrow (a-\lambda)(d-\lambda) - bc = 0$

$\Rightarrow \lambda^2 - (a+d)\lambda + (ad-bc) = 0$

$\Rightarrow \lambda^2 - (\text{trace } A)\lambda + \det(A) = 0$

eigenvector of A wrt $\lambda = 2+3i$

$[A - (2+3i)I] = \begin{bmatrix} 3-2-3i & 2 \\ -5 & 1-2-3i \end{bmatrix}$

$= \begin{bmatrix} \boxed{1-3i} & 2 \\ -5 & -1-3i \end{bmatrix} \sim \begin{bmatrix} \boxed{1-3i} & 2 \\ 0 & 0 \end{bmatrix}$

$\sim \begin{bmatrix} \boxed{1} & \frac{1+3i}{5} \\ 0 & 0 \end{bmatrix}$

$\begin{bmatrix} -\frac{1}{5}(1+3i) \\ 1 \end{bmatrix}$

$-1-3i + (\text{multiplier})\,2$

$= -1-3i + \dfrac{5}{1-3i}\,2$

$= -1-3i + \dfrac{10\;\,1+3i}{1+9}$

$= -1-3i+1+3i = 0$

$\dfrac{2}{1-3i} = \dfrac{2(1+3i)}{10}$

$= \frac{1}{5}(1+3i)$

$X(t) = \begin{pmatrix} -\frac{1}{5} - \frac{3}{5}i \\ 1 \end{pmatrix} e^{(2+3i)t}$

$= e^{2t}\left\{\begin{bmatrix} -1/5 \\ 1 \end{bmatrix} + i\begin{bmatrix} -3/5 \\ 0 \end{bmatrix}\right\}(\cos 3t + i\sin t)$

Thus

$X(t) = e^{2t}\left[c_1\left\{\begin{bmatrix} -1/5 \\ 1 \end{bmatrix}\cos 3t - \begin{bmatrix} -3/5 \\ 0 \end{bmatrix}\sin t\right\} + c_2\left\{\begin{bmatrix} -1/5 \\ 1 \end{bmatrix}\sin t + \begin{bmatrix} -3/5 \\ 0 \end{bmatrix}\cos 3t\right\}\right]$

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Ex 3

$\dot{X} = AX$

$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$A = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$

x_i : temp in tank i

partition is made of the same material.

$$\begin{bmatrix} \dot{x}_1 = a(x_2 - x_1) + a(x_3 - x_1) \\ \dot{x}_2 = a(x_3 - x_2) + a(x_1 - x_2) \\ \dot{x}_3 = a(x_1 - x_3) + a(x_2 - x_3) \end{bmatrix}$$

$$\begin{bmatrix} \dot{x}_1 = -2ax_1 + ax_2 + ax_3 \\ \dot{x}_2 = ax_1 - 2ax_2 + ax_3 \\ \dot{x}_3 = ax_1 + ax_2 - 2ax_3 \end{bmatrix}$$

Law : rate of change of temp. $= \alpha$ [external temp − internal temp]

First find eigenvalues of A

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} -2-\lambda & 1 & 1 \\ 1 & -2-\lambda & 1 \\ 1 & 1 & -2-\lambda \end{vmatrix} = 0 \Rightarrow \lambda = 0, -3, -3$$

eigenvector of A corresponding to $\lambda = 0$

$$[A - \lambda I]_{\lambda=0} = (A) = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} -2 & 1 & 1 \\ 0 & -3/2 & 3/2 \\ 0 & 3/2 & -3/2 \end{bmatrix} \sim \begin{bmatrix} \boxed{-2} & 1 & 1 \\ 0 & \boxed{-3/2} & 3/2 \\ 0 & 0 & 0 \end{bmatrix}$$

S

$$\sim \begin{bmatrix} \boxed{1} & -1/2 & -1/2 \\ 0 & \boxed{1} & -1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 0 & -1 \\ 0 & \boxed{1} & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

eigenvector is

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

eigenvector of A wrt $\lambda = -3$

$$(A - 3I) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Thus general solution is

$$X(t) = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} e^{-3t} + c_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{-3t}$$

Ex 4 Solve! $\dot{X} = AX$ $X = \begin{bmatrix} x \\ y \end{bmatrix}$

$A = \begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix}$

Sol:- $|A - \lambda I| = 0 \Rightarrow \lambda^2 - 6\lambda + 9 = 0 \Rightarrow (\lambda - 3)^2 = 0 \Rightarrow \lambda = 3, 3$

$[A - 3I] = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & -1 \\ 0 & 0 \end{bmatrix}$ eigenvector is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Thus $\lambda = 3$ is a defective eigenvalue (its $am = 2$, $gm = 1$).

Thus one solution to $\dot{X} = AX$ is $\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t}$

and another is $\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} t + \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} \right) e^{3t}$ where

$$[A - 3I] \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Thus general solution is

$$X(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + c_2 \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) e^{3t}$$

$$\left.\begin{aligned} x(t) &= c_1 e^{3t} + c_2 (t+1) e^{3t} \\ y(t) &= c_1 e^{3t} + c_2 t e^{3t} \end{aligned}\right] \text{ Ans.}$$

Exponential matrix formula.

Relook at solution of

$$\boxed{\begin{array}{l}\dot{X} = AX \\ A = 2\times 2\end{array}}$$ with initial condition $X(0) = \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$

\+ a.m. = g.m $\forall$ eigenvalues. Then

$$X(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} \alpha_1 \\ \downarrow \end{bmatrix} e^{\lambda_1 t} + c_2 \begin{bmatrix} \alpha_2 \\ \downarrow \end{bmatrix} e^{\lambda_2 t}$$

$$\Rightarrow \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \alpha_1 & \alpha_2 \\ \downarrow & \downarrow \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

By putting initial data, we obtain

$$\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} \alpha_1 & \alpha_2 \\ \downarrow & \downarrow \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \alpha_1 & \alpha_2 \\ \downarrow & \downarrow \end{bmatrix}^{-1} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

Therefore, solution is

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \alpha_1 & \alpha_2 \\ \downarrow & \downarrow \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \begin{bmatrix} \alpha_1 & \alpha_2 \\ \downarrow & \downarrow \end{bmatrix}^{-1} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

$$= P e^{Dt} P^{-1} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

$$= e^{At} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

$$\left[\; P = \begin{bmatrix} \alpha_1 & \alpha_2 \\ \downarrow & \downarrow \end{bmatrix}, \quad D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \quad e^{Dt} = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \right.$$

Thus

$$\boxed{X(t) = e^{At} X(0)}$$ → formula works in all case of a.m $\neq$ g.m also [Just verify!].

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Example

$$\dot{X} = AX \qquad A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$
$$X(0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \qquad X = \begin{bmatrix} x \\ y \end{bmatrix}$$

eigenvalues - eigenvecter method

$\lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1$

eigenvecter wrt $\lambda = -1$

$$(A+I) = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & -1 \\ 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

eigenvecter wrt $\lambda = 1$

$$(A-I) = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 1 \\ 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

solution

$$X(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{t}$$

at $t=0$, we have

$$c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left. \begin{aligned} c_1 - c_2 &= 2 \\ c_1 + c_2 &= 0 \end{aligned} \right) \Rightarrow \begin{aligned} c_1 &= 1 \\ c_2 &= -1 \end{aligned}$$

Thus

$$X(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} - \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{t}$$
$$= \begin{bmatrix} e^{-t} + e^{t} \\ e^{-t} - e^{t} \end{bmatrix}$$

exponential matrix formula:

a.m = g.m case hence

$$e^{At} = P e^{Dt} P^{-1}$$
$$= \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{t} \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$
$$= \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$
$$= \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{-t} & e^{-t} \\ -e^{t} & e^{t} \end{bmatrix}$$
$$= \frac{1}{2} \begin{bmatrix} e^{-t} + e^{t} & e^{-t} - e^{t} \\ e^{-t} - e^{t} & e^{-t} + e^{t} \end{bmatrix}$$

$$X(t) = e^{At} X(0)$$
$$= \frac{1}{2} \begin{bmatrix} e^{-t} + e^{t} & e^{-t} - e^{t} \\ e^{-t} - e^{t} & e^{-t} + e^{t} \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$
$$= \begin{bmatrix} e^{-t} + e^{t} \\ e^{-t} - e^{t} \end{bmatrix}$$

Recall ; by definition

$$e^{At} = \sum_{n=0}^{\infty} \frac{(At)^n}{n!}$$

$$= I + (At) + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \cdots$$

This series in RHS is UC (uniformly convergent) to e^{At}. Thus we can do term by term differentiation and obtain

$$\frac{d}{dt}\left(e^{At}\right) = A + A^2 t + \frac{1}{2!} A^3 t^2 + \cdots$$

$$= A\left[I + (At) + \frac{(At)^2}{2!} + \cdots\right]$$

$$= Ae^{At}$$

One Important property of exponential matrix

$$e^{A+B} = e^A \cdot e^B \iff AB = BA$$

holds when

(i) $A = kI$ (k is constant)

(ii) $A = -B$

(iii) $A = B^{-1}$

provides that

$$e^{A-A} = e^A e^{-A}$$

$$\| $$

$$e^0 = I$$

Thus, $\boxed{e^{-A} = (e^A)^{-1}}$

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$$\boxed{\dot{X} = AX + f(t)} \quad \text{—} \quad (\#)$$

<u>general solution</u>

$$X(t) = X_c + X_p$$

where X_c solve corresponding homogeneous system $\dot{X} = AX$ i.e. $X_c = e^{At}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

(VoP technique:) Take $X_p = e^{At}\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$

By putting X_p in $(\#)$, we obtain

$$Ae^{At}\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} + e^{At}\begin{pmatrix} \dot{f}_1 \\ \dot{f}_2 \end{pmatrix} = Ae^{At}\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} + f(t)$$

$$\Rightarrow \quad e^{At}\begin{pmatrix} \dot{f}_1 \\ \dot{f}_2 \end{pmatrix} = f(t)$$

$$\Rightarrow \quad \begin{bmatrix} \dot{f}_1 \\ \dot{f}_2 \end{bmatrix} = e^{-At} f(t) \Rightarrow \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \int_0^t e^{-As} f(s)\,ds$$

Hence $\quad X_p = e^{At}\int_0^t e^{-As} f(s)\,ds = \int_0^t e^{A(t-s)} f(s)\,ds$

and $\quad \boxed{X(t) = e^{At}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \int_0^t e^{A(t-s)} f(s)\,ds}$

If we solve $(\#)$ with Initial condition $X(0) = X_0$, then

$$\boxed{X(t) = e^{At}X_0 + \int_0^t e^{A(t-s)} f(s)\,ds}$$

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Example

Solve IVP

$$\left.\begin{array}{l} \dfrac{dx}{dt} = -y + 3e^{2t} \\ \dfrac{dy}{dt} = -x \\ x(0) = 2 \\ y(0) = 0 \end{array}\right) \equiv \begin{array}{l} X' = AX + f \\ X = \begin{bmatrix} x \\ y \end{bmatrix};\ A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix};\ f = \begin{bmatrix} 3e^{2t} \\ -x \end{bmatrix} \\ X(0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \end{array}$$

from previous example, we know that

$$e^{At} = \frac{1}{2}\begin{bmatrix} e^{-t}+e^{t} & e^{-t}-e^{t} \\ e^{-t}-e^{t} & e^{-t}+e^{t} \end{bmatrix}$$

By VoP formula, we obtain

$$X(t) = e^{At}X(0) + \int_0^t e^{A(t-s)} f(s)\,ds$$

$$= e^{At}X(0) + e^{At}\int_0^t e^{-As} f(s)\,ds$$

$$= \begin{bmatrix} e^{-t}+e^{t} \\ e^{-t}-e^{t} \end{bmatrix} + e^{At}\int_0^t \frac{1}{2}\begin{bmatrix} e^{s}+e^{-s} & e^{s}-e^{-s} \\ e^{s}-e^{-s} & e^{s}+e^{-s} \end{bmatrix}\begin{bmatrix} 3e^{2s} \\ 0 \end{bmatrix} ds$$

$$= \begin{bmatrix} e^{-t}+e^{t} \\ e^{-t}-e^{t} \end{bmatrix} + \frac{3}{2}e^{At}\int_0^t \begin{bmatrix} e^{3s}+e^{s} \\ e^{3s}-e^{s} \end{bmatrix} ds$$

$$= \text{''} + \frac{3}{2}e^{At}\begin{bmatrix} \frac{1}{3}e^{3s}+e^{s} \\ \frac{1}{3}e^{3s}-e^{s} \end{bmatrix}\Bigg|_0^t$$

$$= \text{''} + \frac{3}{2}e^{At}\begin{bmatrix} \frac{1}{3}e^{3t}+e^{t}-\frac{4}{3} \\ \frac{1}{3}e^{3t}-e^{t}+\frac{2}{3} \end{bmatrix}$$

!

Now simplify the answer.

$$\boxed{2^{nd} \text{ order ODE } \equiv 2\times 2 \text{ system } \dot{X} = AX}$$

$$\frac{d^2y}{dt^2} + a_1 \frac{dy}{dt} + a_2 y = 0 \quad \text{--- (1)}$$

Take $y = x_1$

$\Rightarrow \dfrac{dy}{dt} = \dfrac{dx_1}{dt}$

Take $x_2 = \dfrac{dx_1}{dt}$ ———— (*)

$\Rightarrow \dfrac{dx_2}{dt} = \dfrac{d^2x_1}{dt^2} = \dfrac{d^2y}{dt^2} = -a_1\dfrac{dy}{dt} - a_2 y$

$= -a_1 x_2 - a_2 x_1$ — (**)

From (*) & (**), we obtain

$$\left.\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -a_2x_1 - a_1x_2\end{aligned}\right) \equiv \underbrace{\begin{bmatrix}\dot{x}_1 \\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1 \\ -a_2 & -a_1\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix}}$$

$$\equiv$$

$$\dot{X} = AX; \quad A = \begin{bmatrix}0 & 1 \\ -a_2 & -a_1\end{bmatrix} \quad \text{— (2)}$$

<u>See:</u> AE for (1): $m^2 + a_1 m + a_2 = 0$

char. Eq for (2): $\lambda^2 - (-a_1)\lambda + (a_2) = 0$

$\left[\lambda^2 - [\text{trance}(A)]\lambda + (\det A) = 0\right]$

$= \lambda^2 + a_1\lambda + a_2 = 0$

AE for (1) and Char. Eq for (2) are same.

Similarly

$$\left(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_n\right) y = 0 \qquad \left(D \equiv \frac{d}{dt}\right)$$

take
$$x_1 = y$$
$$x_2 = \frac{dx_1}{dt} = \frac{dy}{dt}$$
$$x_3 = \frac{dx_2}{dt} = \frac{d^2y}{dt^2}$$
$$\vdots$$
$$x_n = \frac{dx_{n-1}}{dt} = \frac{d^{n-1}y}{dt^{n-1}}$$

$\dot{X} = AX$ where

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}; \quad A = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ & & & & \ddots & 1 \\ -a_n & \cdots & & & -a_2 & -a_1 \end{bmatrix}$$

Moreover:

Char. equation of A is $\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_n = 0$

Imp conclusion

\# arbitary constants (parameters) in general solution to $\dot{X} = AX$ is equal to sum of orders of ODEs in the system.

System to ODE

$$\text{①} - \frac{dT_1}{dt} = -2T_1 + 2T_2$$

$$\text{②} - \frac{dT_2}{dt} = 2T_1 - 5T_2$$

$\frac{dT}{dt} = AT$ where

$T = \begin{bmatrix} T_1 \\ T_2 \end{bmatrix}$; $A = \begin{bmatrix} -2 & 2 \\ 2 & -5 \end{bmatrix}$

From ①, obtain $T_2 = \frac{1}{2}\left[\frac{dT_1}{dt} + 2T_1\right]$

and by putting this value in ②, we obtain

$$\frac{1}{2}\left[\frac{d^2T_1}{dt^2} + 2\frac{dT_1}{dt}\right] = 2T_1 - \frac{5}{2}\left[\frac{dT_1}{dt} + 2T_1\right]$$

$$\Rightarrow \frac{d^2T_1}{dt^2} + 2\frac{dT_1}{dt} = 4T_1 - 5\frac{dT_1}{dt} - 10T_1$$

$$\Rightarrow \boxed{\frac{d^2T_1}{dt^2} + 7\frac{dT_1}{dt} + 6T_1 = 0}$$

Moreover: AE of the above 2nd order ODE is

$$\boxed{m^2 + 7m + 6 = 0}$$

Compare!

and char. equation of A is

$$\begin{vmatrix} \lambda+2 & -2 \\ -2 & \lambda+5 \end{vmatrix} = 0 \equiv \lambda^2 - (-7)\lambda + 6 = 0$$

$$\equiv \boxed{\lambda^2 + 7\lambda + 6 = 0}$$

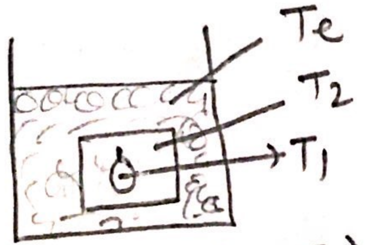
Elimination or operator method

Ex 1

$$\frac{dT_1}{dt} = -2T_1 + 2T_2 \quad \text{—(1)}$$

$$\frac{dT_2}{dt} = 2T_1 - 5T_2 \quad \text{—(2)}$$

Model description



$$\frac{dT_1}{dt} = a(T_2 - T_1)$$

$$\frac{dT_2}{dt} = a(T_1 - T_2) + b(T_e - T_2)$$

put $a=2$, $b=3$; $T_e = 0$.

From (1) put

$T_2 = \frac{1}{2}\left(\frac{dT_1}{dt} + 2T_1\right)$ in (2) and obtain

$$\frac{d^2T_1}{dt^2} + 7\frac{dT_1}{dt} + 6T_1 = 0 \quad \text{—(3)}$$

Solve (3):-

AE: $m^2 + 7m + 6 = 0$

$\Rightarrow m = -1, -6$

$$\boxed{T_1(t) = c_1 e^{-t} + c_2 e^{-6t}}$$

Hence

$$T_2(t) = \frac{1}{2}\left[-c_1 e^{-t} - 6c_2 e^{-6t} + 2c_1 e^{-t} + 2c_2 e^{-6t}\right]$$

$$= \frac{1}{2}\left[c_1 e^{-t} - 4c_2 e^{-6t}\right]$$

$$\boxed{T_2(t) = \frac{1}{2}c_1 e^{-t} - 2c_2 e^{-6t}}$$

Ex2

Solve

$$\left.\begin{array}{l} 2\frac{dx}{dt} - 2\frac{dy}{dt} - 3x = t \\ 2\frac{dx}{dt} + 2\frac{dy}{dt} + 3x + 8y = 2 \end{array}\right\} \equiv \begin{array}{ll} (2D-3)x - 2Dy = t & — (1) \\ (2D+3)x + (2D+8)y = 2 & — (2) \end{array}$$

from (1) $\times (2D+8)$ + (2) $\times 2D$, we obtain

$$(2D+8)(2D-3)x + 2D(2D+3)x = (2D+8)(t) + 2D(2)$$

$$\Rightarrow (2D+8)\left[2\frac{dx}{dt} - 3x\right] + 2D\left[2\frac{dx}{dt} + 3x\right] = \left(2\frac{d}{dt} + 8\right)(t) + 2\frac{d}{dt}(2)$$

$$\Rightarrow 4\frac{d^2x}{dt^2} - 6\frac{dx}{dt} + 16\frac{dx}{dt} - 24x + 4\frac{d^2x}{dt^2} + 6\frac{dx}{dt} = 2 + 8t$$

$$\Rightarrow 8\frac{d^2x}{dt^2} + 16\frac{dx}{dt} - 24x = 8t + 2$$

$$\Rightarrow \boxed{\frac{d^2x}{dt^2} + 2\frac{dx}{dt} - 3x = t + \frac{1}{4}} \quad — (3)$$

Solve (3)

AE: $m^2 + 2m - 3 = 0$

$\Rightarrow (m+3)(m-1) = 0$

$\Rightarrow m = +1, -3$

$x(t) = c_1 e^{-3t} + c_2 e^{t}$

By VoP $y_p = f_1 e^{-3t} + f_2 e^{t}$

$$f_1 = \int \frac{W_1}{W} r\,dt = -\int \frac{e^{t}}{4e^{-2t}}\left(t + \frac{1}{4}\right)dt$$

$$= -\frac{1}{4}\int t e^{3t}\,dt - \frac{1}{16}\int e^{3t}\,dt$$

$$= -\frac{1}{4}\left[t\frac{e^{3t}}{3} - \frac{e^{3t}}{9} + \frac{1}{4}\frac{e^{3t}}{3}\right] = -\frac{1}{12}te^{3t} + \frac{1}{144}e^{3t}$$

$f_2 = \int \frac{W_2}{W} r\,dt = -\frac{1}{4}te^{-t} - \frac{5}{16}e^{-t}$. Thus $\boxed{y_p(t) = -\frac{1}{3}t - \frac{11}{36}}$

$$W = \begin{vmatrix} e^{-3t} & e^{t} \\ -3e^{-3t} & e^{t} \end{vmatrix} = e^{-2t} + 3e^{-2t} = 4e^{-2t}$$

$$W_1 = \begin{vmatrix} 0 & e^{t} \\ 1 & e^{t} \end{vmatrix} = -e^{t}$$

$$W_2 = \begin{vmatrix} e^{-3t} & 0 \\ -3e^{-3t} & 1 \end{vmatrix} = e^{-3t}$$

Therefore,

$$\boxed{x(t) = c_1 e^{-3t} + c_2 e^{t} - \frac{1}{3}t - \frac{11}{36}}$$

Now from ① & ②, we obtain

$$\boxed{4\frac{dx}{dt} + 8y = t+2} \quad \text{—— ④}$$

$$\Rightarrow y(t) = -\frac{1}{2}\frac{dx}{dt} + \frac{1}{8}(t+2)$$

$$= -\frac{1}{2}\left[-3c_1 e^{-3t} + c_2 e^{t} - \frac{1}{3}\right] + \frac{1}{8}(t+2)$$

$$= \frac{3}{2}c_1 e^{-3t} - \frac{1}{2}c_2 e^{t} + \frac{1}{6} + \frac{t}{8} + \frac{1}{4}$$

$$\boxed{y(t) = \frac{3}{2}c_1 e^{-3t} - \frac{1}{2}c_2 e^{t} + \frac{t}{8} + \frac{5}{12}}$$

See: Eq ④ is purely algebraic in y and <u>NOT</u> ODE, i.e. there is not derivative term of y.

Hence if we know $x(t)$, we obtain y from ④ without solving any more ODE and hence without introducing any arbitrary constant.

<u>Remember</u> # arbitrary constants in general solution to Ex 2 = 2.

But it is not always possible, we get and equation like ④ in all cases.

Ex 3 Solve:

$$\left.\begin{aligned}\frac{d^2x}{dt^2}+\frac{dy}{dt}-x+y&=1\\ \frac{d^2y}{dt^2}+\frac{dx}{dt}-x+y&=0\end{aligned}\right) \equiv \begin{aligned}(D^2-1)x+(D+1)y&=1 \quad -(1)\\ (D-1)x+(D^2+1)y&=0 \quad -(2)\end{aligned}$$

In this example, we cannot obtain an equation like eq. (4) in previous example.

i.e. we cannot eliminate either all derivative terms of x or all derivative terms of y from (1) and (2) easily.

From (1) $-(D+1)\times$(2), we obtain

$$(D+1)y-(D+1)(D^2+1)y=1$$

$$\Rightarrow \left(\frac{dy}{dt}+y\right)-(D+1)\left[\frac{d^2y}{dt^2}+y\right]=1$$

$$\Rightarrow \frac{dy}{dt}+y-\frac{d^3y}{dt^2}-\frac{dy}{dt}-\frac{d^2y}{dt^2}-y=1$$

$$\Rightarrow \boxed{\frac{d^3y}{dt^2}+\frac{d^2y}{dt^2}=-1} \quad -(3)$$

Solve (3)

AE $\quad m^3+m^2=0$

$\Rightarrow m=0,0,-1$

$y(t)=(c_1+c_2t)+c_3e^{-t}$

VoP:

$y_p(t)=f_1+f_2t+f_3e^{-t}$

$$f_1=\int\frac{W_1}{W}r\,dt=\int\frac{-te^{-t}-e^{-t}}{e^{-t}}(-1)\,dt$$

$$=\int(t+1)\,dt=\frac{1}{2}t^2+t$$

$$f_2=\int\frac{W_2}{W}r\,dt=\int\frac{e^{-t}}{e^{-t}}(-1)\,dt=-t;\quad f_3=\int\frac{W_3}{W}r\,dt=\int\frac{1}{e^{-t}}(-1)\,dt$$

$$=-e^{t}$$

Thus $y_p(t)=\left(\frac{1}{2}t^2+t\right)+(-t)t+(-e^{t})e^{-t}$

$$=\frac{1}{2}t^2+t-t^2-1 \Rightarrow \boxed{y_p=-1+t-\frac{1}{2}t^2}$$

$$W=\begin{vmatrix}1 & t & e^{-t}\\ 0 & 1 & -e^{-t}\\ 0 & 0 & e^{-t}\end{vmatrix}=e^{-t}$$

$$W_1=\begin{vmatrix}0 & t & e^{-t}\\ 0 & 1 & -e^{-t}\\ 1 & 0 & e^{-t}\end{vmatrix}=-te^{-t}-e^{-t}$$

$$W_2=\begin{vmatrix}1 & 0 & e^{-t}\\ 0 & 0 & -e^{-t}\\ 0 & 1 & e^{-t}\end{vmatrix}=e^{-t}$$

$$W_3=\begin{vmatrix}1 & t & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{vmatrix}=1$$

Therefore

$$\boxed{y(t) = c_1 + c_2 t + c_3 e^{-t} - 1 + t - \frac{1}{2}t^2} \quad \text{—} (4)$$

Now from

$(D^2+1) \times (1) - (D+1) \times (2)$, we obtain

$$(D^2+1)(D^2-1)x - (D+1)(D-1)x = (D^2+1)(1)$$

$$\Rightarrow (D^4-1)x - (D^2-1)x = 1$$

$$\Rightarrow \frac{d^4x}{dt^2} - x - \frac{d^2x}{dt} + x = 1$$

$$\Rightarrow \frac{d^4x}{dt^2} - \frac{d^2x}{dt} = 1 \quad \text{—} (5)$$

Solve (5)

AE $m^4 - m^2 = 0$

$\Rightarrow m = 0, 0, \pm 1$

$x(t) = k_1 + k_2 t + k_3 e^{-t} + k_4 e^{t}$

$(D^4 - D^2)x = 1 = e^{0t}$

Thus, by exponential rule

$$x(t) = \frac{t^2 e^{0t}}{L''(0)} \qquad L(\alpha) = \alpha^4 - \alpha^2$$

$$L'(\alpha) = 4\alpha^3 - 2\alpha$$

$$= -\frac{1}{2}t^2 \qquad L''(\alpha) = 12\alpha - 2$$

Thus $\boxed{x(t) = k_1 + k_2 t + k_3 e^{-t} + k_4 e^{t} - \frac{1}{2}t^2}$ —(6)

Thus solution is expressed by (4) & (6). But here we have 7 - arbitrary constants, we have to express any three variables in terms of other 4's.

Put values from (4) and (6) in (2), we obtain

$$\frac{d^2}{dt^2}\left[c_1+c_2t+c_3e^{-t}-1+t-\frac{1}{2}t^2\right]+\frac{d}{dt}\left[k_1+k_2t+k_3e^{-t}+k_4e^{t}-\frac{1}{2}t^2\right]$$
$$-\left(k_1+k_2t+k_3e^{-t}+k_4e^{t}-\frac{1}{2}t^2\right)+\left(c_1+c_2t+c_3e^{-t}-1+t-\frac{1}{2}t^2\right)=0$$

$$\Rightarrow \frac{d}{dt}\left[c_2-c_3e^{-t}+1-t\right]+\left[k_2-k_3e^{-t}+k_4e^{t}-t\right]$$
$$-\left(k_1+k_2t+k_3e^{-t}+k_4e^{t}-\frac{1}{2}t^2\right)+\left(c_1+c_2t+c_3e^{-t}-1+t-\frac{1}{2}t^2\right)=0$$

$$\Rightarrow c_3e^{-t}-1+k_2-k_3e^{-t}+k_4e^{t}-t-k_1-k_2t-k_3e^{-t}$$
$$-k_4e^{t}+\frac{1}{2}t^2+c_1+c_2t+c_3e^{-t}-1+t-\frac{1}{2}t^2=0$$

$$\Rightarrow (-2+k_2-k_1+c_1)+(-1-k_2+c_2+1)t$$
$$+(c_3-k_3-k_3+c_3)e^{-t}+(k_4-k_4)e^{t}=0$$

$$\Rightarrow \left.\begin{aligned} c_1-k_1+k_2&=2\\ c_2-k_2&=0\\ 2c_3-2k_3&=0\end{aligned}\right) \Rightarrow \begin{aligned} c_2&=k_2\\ c_3&=k_3\\ c_1&=k_1-k_2\end{aligned}$$

Thus

$$\left.\begin{aligned} x(t)&=k_1+k_2t+k_3e^{-t}+k_4e^{t}-\frac{1}{2}t^2\\ y(t)&=(k_1-k_2)+k_2t+k_3e^{-t}-1+t-\frac{1}{2}t^2\\ &=(k_1-k_2-1)+(k_2+1)t+k_3e^{-t}-\frac{1}{2}t^2\end{aligned}\right]$$ Answer