Complex Analysis

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The number system as we know it today is a result of gradual development as indicated in the following list:

Natural numbers:

- The numbers $1, 2, 3, 4, ...$ also called *positive integers*, were first used in counting.
- The set of all naturals are denoted by $\mathbb{N}$.
- If a and $b \in N$, the sum $a + b$ and product ab also $\in \mathbb{N}$.
- So, the set of natural numbers is said to be closed under the operations of **addition** and **multiplication** or to satisfy the **closure property** with respect to these operations.

Negative integers and zero

- The negative numbers are denoted by $-1, -2, -3, \ldots$.
- The negative numbers and 0 permit solutions of equations such as $x + b = a$ where a and b are any natural numbers. This leads to the operation of subtraction, or inverse of addition, and we write $x = a - b$.
- The set of positive and negative integers and zero is called the set of **integers** and is closed under the operations of addition, multiplication, and subtraction.
- The set of all naturals are denoted by $\mathbb{Z}$.

Rational numbers or fractions

- Rational numbers are numbers such as $\frac{3}{4}, -\frac{8}{3}$ etc.
- They permit solutions of equations such as $bx = a$ for all integers a and b where $b \neq 0$. This leads to the operation of division or inverse of multiplication, and we write $x = a/b$.
- The set of all rationals are denoted by $\mathbb{Q}$.
- The set $\mathbb{Z}$ is a subset of $\mathbb{Q}$, since integers correspond to rational numbers $\frac{a}{b}$ where $b = 1$.
- The set $\mathbb{Q}$ is closed under the operations of addition, subtraction, multiplication, and division, so long as division by zero is excluded.

Irrational numbers

- Numbers such as π, $\sqrt{2}$ are numbers that cannot be expressed as a/b where a and b are integers and $b \neq 0$.
- The set of rational and irrational numbers together is called the set of real numbers.
- The set of all real numbers are denoted by $\mathbb{R}$.

Real numbers can be represented by points on a line called the real axis, as indicated in the figure below. The point corresponding to zero is called the origin.

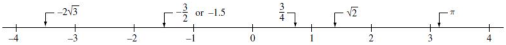


Fig. 1-1

Conversely, to each point on the line there is one and only one real number.

- If a point A corresponding to a real number a lies to the right of a point B corresponding to a real number b, we say that a is greater than b or b is less than a and write $a > b$ or $b < a$, respectively.
- The set of all values of x such that $a < x < b$ is called an **open interval** on the real axis while $a \leq x \leq b$, which also includes the endpoints a and b, is called a **closed interval**. The symbol x, which can stand for any real number, is called a real variable.
- The **absolute value** of a real number x, denoted by $|x|$, is defined as:

$$|x| = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -x & \text{if } x < 0 \end{cases}$$

- The **distance between two points** a and b on the real axis is $|a - b|$.

There is no real number x that satisfies the polynomial equation:

$$x^2 + 1 = 0$$

To permit solutions of this and similar equations, the set of complex numbers is introduced. We can consider a complex number as having the form $a + bi$ where a and b are real numbers and i, which is called the imaginary unit, has the property that:

$$\boxed{i^2 = -1}$$

If $z = a + bi$, then a is called the real part of z and b is called the imaginary part of z and are denoted by Re (z) and Im(z), respectively. The symbol z, which can stand for any complex number, is called a complex variable.

- Two complex numbers $a + bi$ and $c + di$ are **equal** if and only if:

$$a = c \text{ and } b = d.$$

- We can consider real numbers as a subset of the set of complex numbers with $b = 0$. Accordingly the complex numbers $0 + 0i$ and $3 + 0i$ represent the real numbers 0 and 3, respectively. If $a = 0$, the complex number bi is called a **purely imaginary number**.

- The **complex conjugate**, or briefly **conjugate**, of a complex number $a + bi$ is $a - bi$. The complex conjugate of a complex number z is often indicated by $\bar{z}$ or z^*.

Example: Find real numbers x and y such that:

$$3x + 2iy - ix + 5y = 7 + 5i.$$

Solution: The given equation can be written as:

$$3x + 5y + i(2y - x) = 7 + 5i.$$

Then equating real and imaginary parts, we get:

$$3x + 5y = 7, \text{ and, } 2y - x = 5$$

Solving simultaneously, we get:

$$x = -1, y = 2.$$

In performing operations with complex numbers, we can proceed as in the algebra of real numbers, replacing i^2 by -1 when it occurs.

Operations

- **Addition:**

$$(a+bi)+(c+di)=a+bi+c+di=(a+c)+(b+d)i$$

- **Subtraction:**

$$(a+bi)-(c+di)=a+bi-c-di=(a-c)+(b-d)i$$

- **Multiplication:**

$$(a+bi)(c+di)=ac+adi+bci+bdi^2=(ac-bd)+(ad+bc)i$$

Operations:

- **Division:** If $c \neq 0$ and $d \neq 0$, then:

$$\begin{aligned}
\frac{a+bi}{c+di} &= \frac{(a+bi)(c-di)}{(c+di)(c-di)} \\
&= \frac{ac-adi+bci-bdi^2}{c^2-d^2i^2} \\
&= \frac{ac+bd+(bc-ad)i}{c^2+d^2} \\
&= \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}i
\end{aligned}$$

The absolute value or modulus of a complex number $a + bi$ is defined as:

$$|a + bi| = \sqrt{a^2 + b^2}$$

Example

Question: Find the modulus of the complex number $-4 + 2i$.

Solution:

$$\begin{aligned} |-4 + 2i| &= \sqrt{(-4)^2 + 2^2} \\ &= \sqrt{20} \\ &= 2\sqrt{5} \end{aligned}$$

If $z_1, z_2, z_3, ..., z_m$ are complex numbers, the following properties hold:

Properties

- $$|z_1 z_2 \cdots z_m| = |z_1||z_2| \cdots |z_m|$$
- $$|\frac{z_1}{z_2}| = \frac{|z_1|}{|z_2|}, \quad \text{if } z_2 \neq 0$$
- $$|z_1 + z_2 + \cdots + z_m| \leq |z_1| + |z_2| + \cdots + |z_m|$$
- $$|z_1 - z_2| \geq |z_1| - |z_2|$$

Since, $|z_1 + z_2|^2 = (z_1 + z_2)\left(\overline{z_1 + z_2}\right)$, we have

$$\begin{aligned}
|z_1 + z_2|^2 &= (z_1 + z_2)\left(\overline{z_1 + z_2}\right), \\
&= (z_1 + z_2)\left(\overline{z_1} + \overline{z_2}\right) \\
&= \left(z_1\overline{z_1} + z_1\overline{z_2} + z_2\overline{z_1} + z_2\overline{z_2}\right) \\
&= |z_1|^2 + z_1\overline{z_2} + \overline{z_1\overline{z_2}} + |z_2|^2 \\
&= |z_1|^2 + |z_2|^2 + 2 \cdot \mathrm{Re}\left(z_1\overline{z_2}\right) \\
&\leq |z_1|^2 + |z_2|^2 + 2\left|z_1\overline{z_2}\right| \\
&\leq |z_1|^2 + |z_2|^2 + 2|z_1||z_2| \\
&\leq (|z_1| + |z_2|)^2.
\end{aligned}$$

Taking positive square root on both sides,

$$|z_1 + z_2| \leq |z_1| + |z_2|.$$

Prove: $|z_1 - z_2| \geq |z_1| - |z_2|$.

We may begin with triangle inequality:

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

Replacing z_1 by $z_1 - z_2$, we have

$$|(z_1 - z_2) + z_2| \leq |z_1 - z_2| + |z_2|$$
$$\text{or, } |z_1 - z_2| \geq |z_1| - |z_2|.$$

Prove: A better form : $|z_1 - z_2| \geq ||z_1| - |z_2||$

We may begin with triangle inequality:

$$|z_1 + z_2| \leq |z_1| + |z_2|.$$

Replacing z_1 by $z_1 - z_2$, we have

$$\begin{aligned} |(z_1 - z_2) + z_2| &\leq |z_1 - z_2| + |z_2| \\ |z_1 - z_2| &\geq |z_1| - |z_2|. \end{aligned} \tag{1}$$

Interchanging z_1 and z_2, we have $|z_2 - z_1| \geq |z_2| - |z_1|$, or

$$|z_1 - z_2| \geq |z_2| - |z_1|. \tag{2}$$

Combining (1) and (2), we can get the desired result.

From a strictly logical point of view, it is desirable to define a complex number as an ordered pair (a, b) of real numbers a and b subject to certain operational definitions, which turn out to be equivalent to those above. These definitions are as follows, where all letters represent real numbers:

Definitions

- **Equality**

$$(a, b) = (c, d) \text{ if and only if } a = c, b = d.$$

- **Sum**

$$(a, b) + (c, d) = (a + c, b + d)$$

- **Product**

$$(a, b).(c, d) = (ac - bd, ad + bc)$$
$$m(a, b) = (ma, mb)$$

Suppose z_1, z_2, z_3 belong to the set S of complex numbers. Then:

- **Closure law:**

$$z_1 + z_2 \text{ and } z_1.z_2 \in S \quad \forall z_1, z_2 \in S.$$

- **Commutative law of addition:**

$$z_1 + z_2 = z_2 + z_1 \quad \forall z_1, z_2 \in S.$$

- **Associative law of addition:**

$$z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3 \quad \forall z_1, z_2, z_3 \in S.$$

- **Commutative law of multiplication:**

$$z_1.z_2 = z_2.z_1 \quad \forall z_1, z_2 \in S.$$

- **Associative law of multiplication:**

$$z_1.(z_2.z_3) = (z_1.z_2).z_3 \quad \forall z_1, z_2, z_3 \in S.$$

- **Distributive law:**

$$z_1.(z_2 + z_3) = (z_1.z_2) + (z_1.z_3) \quad \forall z_1, z_2, z_3 \in S.$$

- **Identity Property:** 0 is called the identity with respect to addition, 1 is called the identity with respect to multiplication, as:

$$\begin{aligned} z_1 + 0 = 0 + z_1 \quad &= z_1 \\ z_1.1 = 1.z_1 \quad &= z_1 \end{aligned}$$

- **Inverse Property (a):** For any complex number z_1 there is a unique number z in S such that:

$$z + z_1 = z_1 + z = 0$$

Then, z is called the *inverse of* z_1 with respect to addition and is denoted by $-z_1$.

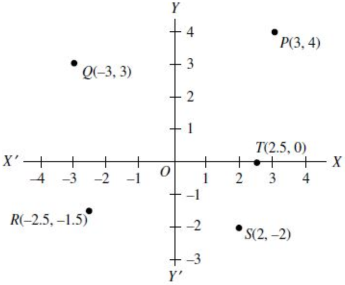
- **Inverse Property (b):** For any $z_1 \neq 0$ there is a unique number z in S such that:

$$z_1.z = z.z_1 = 1$$

Then, z is called the inverse of z_1 with respect to multiplication and is denoted by z^{-1} or $\frac{1}{z}$.

Note : In general, any set such as S, whose members satisfy the above, is called a **field**.

Suppose real scales are chosen on two mutually perpendicular axes $X'OX$ and $Y'OY$. We can locate any point in the plane determined by these lines by the ordered pair of real numbers (x, y) called rectangular coordinates of the point. Examples of the location of such points are indicated by P, Q, R, S, and T in Fig 1.



- Since a complex number $x + iy$ can be considered as an ordered pair of real numbers, we can represent such numbers by points in an xy-plane called the *complex plane* or **Argand diagram**.

- The complex number represented by P, for example, in the fig 1, could then be read as either $(3, 4)$ or $3 + 4i$.

- To each complex number there corresponds one and only one point in the plane, and conversely to each point in the plane there corresponds one and only one complex number.

- Because of this we often refer to the complex number z as the point z. Sometimes, we refer to the x and y axes as the real and imaginary axes, respectively, and to the complex plane as the z plane.

Distance formula

The distance between two points, $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, in the complex plane is:

$$|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Let P be a point in the complex plane corresponding to the complex number (x, y) or $x + iy$. Then we see that:

$$x = r\cos\theta, y = r\sin\theta$$

where $r = \sqrt{x^2 + y^2} = |x + iy|$ is called the *modulus* or *absolute value* of z and θ is called the *amplitude* or *argument* of z (denoted by arg z), is the angle that line OP makes with the positive x axis.

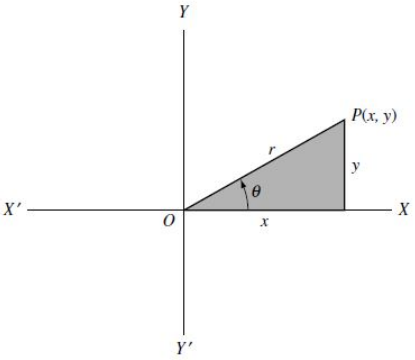
It follows that:

$$z = x + iy = r(\cos\theta + i\sin\theta)$$

which is called the **polar form** of the complex number, and r and θ are called **polar coordinates**. It is sometimes convenient to write the abbreviation cis θ for $\cos\theta + i\sin\theta$.

- For any complex number $z \neq 0$ there corresponds only one value of θ in $0 \leq \theta < 2\pi$.
- However, any other interval of length 2π, for example $-\pi < \theta \leq \pi$, can be used.
- Any particular choice, decided upon in advance, is called the **principal range**, and the value of θ is called its **principal value**.

Polar Representation of Complex Numbers



- The notation $\arg z$ is used to designate an arbitrary argument of z, which means that $\arg z$ is a set rather than a number. In particular, the relation

$$arg(z_1) = arg(z_2)$$

  is not an equation, but expresses equality of two sets.
- Therefore, two non-zero complex numbers $r_1(\cos\theta_1 + i\sin\theta_1)$ and $r_2(\cos\theta_2 + i\sin\theta_2)$ are equal if and only if

$$r_1 = r_2, \quad \theta_1 = \theta_2 + 2k\pi$$

  where $k \in \mathbb{Z}$.
- In order to make the argument of z a well-defined number, it is sometimes restricted to the interval $(-\pi, \pi]$. This special choice is called the principal value or the main branch of the argument and is written as Arg z.
- Note that there is no general convention about the definition of the principal value, sometimes its values are supposed to be in the interval $[0, 2\pi)$. This ambiguity is a perpetual source of misunderstandings and errors.

The principal value Arg z of a complex number $z = x + iy$ is normally given by

$$\theta = \arctan \frac{y}{x}$$

where $\frac{y}{x}$ is the slope, and arctan converts slope to angle. But this is correct only when $x > 0$, so the quotient is defined and the angle lies between $\frac{-\pi}{2}$ and $\frac{\pi}{2}$. We need to extend this definition to cases where x is not positive, considering the principal value of the argument separately on the four quadrants.

The function Arg $z : \mathbb{C} \setminus \{0\} \to (-\pi, \pi]$ is defined as follows:

$$\text{Arg(z)}=\begin{cases} \arctan \frac{y}{x} & \text{if } x > 0, y \in \mathbb{R} \\ \arctan \frac{y}{x} + \pi & \text{if } x < 0, y \geq 0 \\ \arctan \frac{y}{x} - \pi & \text{if } x < 0, y < 0 \\ \frac{\pi}{2} & \text{if } x = 0, y > 0 \\ -\frac{\pi}{2} & \text{if } x = 0, y < 0 \\ \text{undefined} & \text{if } x = 0, y = 0 \end{cases}$$

Thus, if $z = r(\cos\theta + i\sin\theta)$, with $r > 0$ and $-\pi < \theta \leq \pi$, then

$$\arg(z) = Arg(z) + 2n\pi, \quad n \in \mathbb{Z}.$$

EXAMPLE: The complex number $-1-i$, which lies in the third quadrant, has principal argument $-\frac{3\pi}{4}$. That is,

$$Arg(-1-i) = -\frac{3\pi}{4}$$

It must be emphasized that, because of the restriction $-\pi < \theta \leq \pi$ of the principle argument θ, it is not true that

$$Arg(-1-i) = \frac{5\pi}{4}$$

Since, we have

$$\arg(z) = Arg(z) + 2n\pi \quad (n = 0, \pm 1, \pm 2, ...).$$

Thus,

$$arg(-1-i) = -\frac{3\pi}{4} + 2n\pi \quad (n = 0, \pm 1, \pm 2, ...).$$

Note that the term $Arg(z)$ on the right-hand side of the previous equation can be replaced by any particular value of $\arg(z)$ and that one can write, for instance,

$$arg(-1-i) = \frac{5\pi}{4} + 2n\pi \quad (n = 0, \pm 1, \pm 2, ...).$$

Example: Express each of the following complex numbers in polar form.

$$2 + 2\sqrt{3}i$$

Solution: The modulus or absolute value of the complex number is:

$$r = \sqrt{4 + 12} = 4$$

The amplitude or argument of the complex number is:

$$\theta = \tan^{-1} \frac{y}{x} = \tan^{-1} \sqrt{3} = \frac{\pi}{3} \text{ radians}$$

Then, we have,

$$2 + 2\sqrt{3}i = r(\cos\theta + i\sin\theta) = 4(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$$

The result can also be written as $4cis\frac{\pi}{3}$.

Let us consider that:

$$z_1 = x_1 + iy_1 = r_1(\cos\theta_1 + i\sin\theta_1), \text{ and,}$$

$$z_2 = x_2 + iy_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

then we can show that :

$$z_1.z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$$

A generalization of above equation leads to:

$$z_1.z_2....z_n = r_1 r_2...r_n[\cos(\theta_1 + \theta_2 + ... + \theta_n) + i\sin(\theta_1 + \theta_2 + ... + \theta_n)]$$

and if $z_1 = z_2 = ... = z_n = z$, this becomes:

$$z^n = r^n(\cos n\theta + i\sin n\theta)$$

which is often called **De Moivre's theorem**.

Statement

For any x (complex number or real number), if n is a fraction / rational number (positive or negative), then one of the values of $(\cos x + i \sin x)^n$ is $\cos(nx) + i \sin(nx)$.

Proof: Let $n = p/q$ in its lowest form, where p is an integer positive or negative and q is a positive integer. Then,

$$\left(\cos p\frac{\theta}{q} + i \sin p\frac{\theta}{q}\right)^q = \cos p\theta + i \sin p\theta.$$

Therefore taking the qth root of both sides, we get $\left(\cos p\frac{\theta}{q} + i\sin p\frac{\theta}{q}\right)$ is one of the qth root of $(\cos p\theta + i\sin p\theta)$. But $(\cos p\theta + i\sin p\theta) = (\cos\theta + i\sin\theta)^p$. Hence $\left(\cos p\frac{\theta}{q} + i\sin p\frac{\theta}{q}\right)$ is one of the qth roots of

$$(\cos\theta + i\sin\theta)^p$$

i.e., $\left(\cos p\frac{\theta}{q} + i\sin p\frac{\theta}{q}\right)$ is one of the values of $(\cos\theta + i\sin\theta)^{p/q}$. Therefore one of the values of $(\cos\theta + i\sin\theta)^n$ is $(\cos n\theta + i\sin n\theta)$. This completes the proof.

By assuming that the infinite series expansion:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$$

holds when $x = i\theta$, we can arrive at the result:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

which is called **Euler's formula**. In general, we define:

$$e^z = e^{x+iy} = e^x.e^{iy} = e^x(\cos y + i\sin y)$$

Note:

- In the special case where $y = 0$ above equation reduces to e^x.
- De Moivre's theorem now reduces to $(e^{i\theta})^n = e^{in\theta}$.

EXAMPLE: The number $-1-i$ has exponential form:

$$-1-i = \sqrt{2}\exp\left[i\left(-\frac{3\pi}{4}\right)\right]. \tag{i}$$

With the agreement that $e^{-i\theta} = e^{i(-\theta)}$, this can also be written

$$-1-i = \sqrt{2}e^{-i(\frac{3\pi}{4})}.$$

Expression (i) is, of course, only one of an infinite number of possibilities for the exponential form of

$$-1-i = \sqrt{2}\exp\left[i\left(-\frac{3\pi}{4}+2n\pi\right)\right], \quad (n \in \mathbb{Z}). \tag{i}$$

Problem 1: Calculate the principle argument of

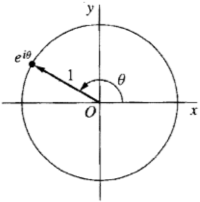
$$\frac{i}{-2-2i}.$$

Problem 2: Show that if Re $z_1 > 0$ and Re $z_2 > 0$, then

$$Arg(z_1 z_2) = Arg(z_1) + Arg(z_2),$$

where, $Arg(z_1 z_2)$ denotes the principal value of $\arg(z_l z_2), etc.$

Note how expression $z = re^{i\theta}$ with $r = 1$ tells us that the numbers $e^{i\theta}$ lie on the circle centered at the origin with radius unity, as shown in figure. Values of $e^{i\theta}$ are, then, immediate from that figure, without reference to Euler's formula. It is, for instance,



Example: Show that:

$$(a)\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}, \text{ and, } \quad (b)\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$$

Solution: We know that,

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{i}$$

and,

$$e^{-i\theta} = \cos\theta - i\sin\theta \tag{ii}$$

Adding equations (i) and (ii) and dividing by 2, we get:

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$$

Subtracting equation (ii) from (i) and dividing by $2i$, we get:

$$\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$$

The solutions of the equation $z^n = 1$ where n is a positive integer are called the nth roots of unity and are given by:

$$z = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n} = e^{\frac{2ki\pi}{n}}, \quad k = 0, 1, 2, \cdots, n-1.$$

If we let,

$$\omega = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n} = e^{\frac{2i\pi}{n}}$$

the n roots are given by:

$$1, \omega, \omega^2, \omega^3, \cdots, \omega^{n-1}$$

Geometrically, they represent the n vertices of a regular polygon of n sides inscribed in a circle of radius one with center at the origin. This circle has the equation $|z| = 1$ and is often called the unit circle.

Problem: Find all values of z for which:

$$z^5 = -32$$

and locate these values in the complex plane.

Solution: In polar form:

$$-32 = 32\cos(\pi + 2k\pi) + i\sin(\pi + 2k\pi), \quad k = 0, \pm 1, \pm 2, \ldots$$

Let us consider that:

$$z = r(\cos\theta + i\sin\theta)$$

Then, by De Moivre's theorem,

$$z^5 = r^5(\cos 5\theta + i\sin 5\theta) = 32\cos(\pi + 2k\pi) + i\sin(\pi + 2k\pi)$$

and so, we have,

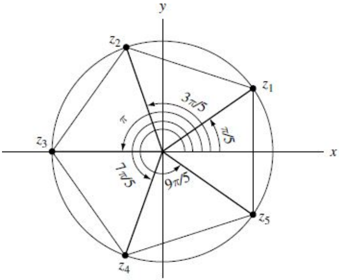
$$r^5 = 32, \quad 5\theta = \pi + 2k\pi. \text{ So, } r = 2, \quad \theta = \frac{\pi + 2k\pi}{5}$$

Hence, we get,

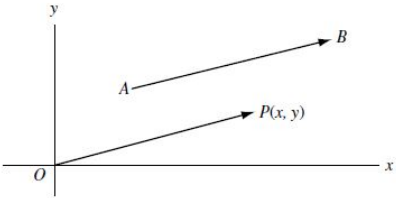
$$z = 2\left[\cos\frac{\pi + 2k\pi}{5} + i\sin\frac{\pi + 2k\pi}{5}\right]$$

If $k = 0$, $z = z_1 = 2(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5})$.
If $k = 1$, $z = z_2 = 2(\cos\frac{3\pi}{5} + isin\frac{3\pi}{5})$.
If $k = 2$, $z = z_3 = 2(\cos\frac{5\pi}{5} + isin\frac{5\pi}{5}) = -2$.
If $k = 3$, $z = z_4 = 2(\cos\frac{7\pi}{5} + isin\frac{7\pi}{5})$.
If $k = 4$, $z = z_5 = 2(\cos\frac{9\pi}{5} + isin\frac{9\pi}{5})$.

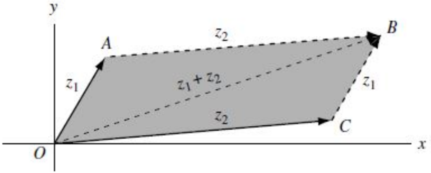
By considering $k = 5, 6, ...$ as well as negative values, $-1, -2, ...$, repetitions of the above five values of z are obtained. Hence, these are the only solutions or roots of the given equation. These five roots are called the fifth roots of -32. The values of z are indicated in figure below. Note that they are equally spaced along the circumference of a circle with center at the origin and radius 2. Another way of saying this is that the roots are represented by the vertices of a regular polygon.



A complex number $z = x + iy$ can be considered as a vector OP whose initial point is the origin O and whose terminal point P is the point (x, y) as in Fig 3. We sometimes call $OP = x + iy$ the *position vector* of P. Two vectors having the same length or magnitude and direction but different initial points, such as OP and AB are considered equal. Hence we write $OP = AB = x + iy$.



Addition of complex numbers corresponds to the parallelogram law for addition of vectors (see Fig 4). Thus to add the complex numbers z_1 and z_2, we complete the parallelogram OABC whose sides OA and OC correspond to z_1 and z_2. The diagonal OB of this parallelogram corresponds to $z_1 + z_2$.

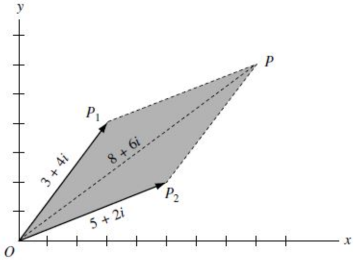


Problem: Perform the indicated operations both analytically and graphically:

$$(3+4i)+(5+2i)$$

Solution: **Analytically,** we have:

$$(3+4i)+(5+2i)=3+5+4i+2i=8+6i$$



Graphically:

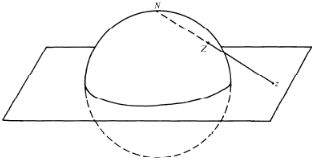
Represent the two complex numbers by points P_1 and P_2, respectively, as in Figure. Complete the parallelogram with OP_1 and OP_2 as adjacent sides. Point P represents the sum, $8 + 6i$, of the two given complex numbers.

Note the similarity with the parallelogram law for addition of vectors OP_1 and OP_2 to obtain vector OP. For this reason it is often convenient to consider a complex number $a + bi$ as a vector having components a and b in the directions of the positive x and y axes, respectively.

- Often in complex analysis we will be concerned with functions that become infinite as the variable approaches a given point. To discuss this situation we introduce the extended plane which is $\mathbb{C} \cup \{\infty\} = \mathbb{C}_\infty$. We also wish to introduce a distance function on $\mathbb{C}_\infty$.
- In order to discuss continuity properties of functions assuming the value infinity. To accomplish this and to give a concrete picture of $\mathbb{C}$ we represent $\mathbb{C}$ as the unit sphere in $\mathbb{R}^3$,

$$S = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}.$$

  Let $N = (0, 0, 1)$; that is, N is the north pole on S. Also, identify $\mathbb{C}$ with $\{(x_1, x_2, 0) : x_1, x_2 \in \mathbb{R}\}$, so that $\mathbb{C}$ cuts S along the equator. Now for each point $z \in \mathbb{C}$ consider the straight line in $\mathbb{R}^3$ through z and N. This intersects the sphere in exactly one point $Z \neq N$.



- If $|z| > 1$ then Z is in the northern hemisphere and if $|z| < 1$ then Z is in the southern hemisphere
- Also, for $|z| = 1$, $Z = z$.

Question: What happens to Z as $|z| \to \infty$?

Note

- Was the elimination of $-\infty$ worth all this effort? Not really.
- In fact, it is actually useful for $-\infty$ to mean "less than any real number". The set $\mathbb{R} \cup \{\infty\}$ was introduced in order to properly motivate our study of the extended complex plane.
- Consider the complex sequence $\{z_n\}$ defined by $z_n = n(\cos\theta + i\sin\theta)$, where $0 \leq \theta \leq 2\pi$.
- For each different value of θ, $\{z_n\}$ approaches ∞ along a different ray.
- Furthermore, since the complex numbers are not ordered, the symbol $-\infty$ would have no more meaning than the symbol $i\infty$.

Clearly Z approaches N; hence, we identify N and the point ∞ in $\mathbb{C}_\infty$. Thus $\mathbb{C}_\infty$ is represented as the sphere S. Let us explore this representation. Put $z = x + iy$ and let $Z = (x_1, x_2, x_3)$ be the corresponding point on S. We will find equations expressing x_1, x_2 and x_3 in terms of x and y. The line in $\mathbb{R}^3$ through z and N is given by

$$\{tN + (1-t)z, -\infty < t < \infty\},$$

or by

$$\{((1-t)x, (1-t)y, t) : -\infty < t < \infty\}. \tag{i}$$

Hence, we can find the coordinates of Z if we can find the value of t at which this line intersects S. If t is this value then

$$\begin{aligned} 1 &= (1-t)^2 x^2 + (1-t)^2 y^2 + t^2 \\ &= (1-t)^2 |z|^2 + t^2. \end{aligned}$$

From which we get

$$1 - t^2 = (1-t)^2 |z|^2$$

Since $t \neq 1$ $(z \neq \infty)$ we arrive at

$$t = \frac{|z|^2 - 1}{|z|^2 + 1}$$

Thus

$$x_1 = \frac{2x}{|z|^2+1}, \quad x_2 = \frac{2y}{|z|^2+1}, \quad x_3 = \frac{|z|^2-1}{|z|^2+1}.$$

But this gives

$$x_1 = \frac{z+\bar{z}}{|z|^2+1}, \quad x_2 = \frac{-i(z-\bar{z})}{|z|^2+1}, \quad x_3 = \frac{|z|^2-1}{|z|^2+1}. \tag{ii}$$

If the point Z is given $(Z \neq N)$ and we wish to find z then by setting $t = x_3$, and using (i), we arrive at

$$z = \frac{x_1 + ix_2}{1 - x_3}.$$

- Now let us define a distance function between points in the extended plane in the following manner: for $z, z' \in \mathbb{C}_\infty$, define the distance from z to z', $d(z, z')$, to be the distance between the corresponding points Z and Z' in $\mathbb{R}^3$.

If $Z = (x_1, x_2, x_3)$ and $Z' = (x_1', x_2', x_3')$ then

$$d(z, z') = [(x_1 - x_1')^2 + (x_2 - x_2')^2 + (x_3 - x_3')^2]$$

Using the fact that Z and Z' are on S, this gives:

$$\begin{aligned}
[d(z, z')]^2 &= 2 - 2(x_1x_1' + x_2x_2' + x_3x_3'). \\
&= 2 - 2\left(\frac{z + \bar{z}}{|z|^2 + 1}\right)\left(\frac{z' + \bar{z'}}{|z'|^2 + 1}\right) \\
&\quad + \left(\frac{-i(z - \bar{z})}{|z|^2 + 1}\right)\left(\frac{-i(z' - \bar{z'})}{|z'|^2 + 1}\right) + \left(\frac{|z|^2 - 1}{|z|^2 + 1}\right)\left(\frac{|z'|^2 - 1}{|z'|^2 + 1}\right) \\
&= \frac{4(|z - z'|)^2}{(1 + |z|^2)(1 + |z'|)^2}
\end{aligned}$$

Then, we get

$$d(z, z') = \frac{2(|z - z'|)}{\sqrt{(1 + |z|^2)(1 + |z'|)^2}}.$$

In a similar manner we get for z in $\mathbb{C}_\infty$,

$$d(z, \infty) = \frac{2}{\sqrt{(1 + |z|^2)}}.$$

This correspondence between points in S and $\mathbb{C}_\infty$ is called **Stereographic Projection**.

Dot Product

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be two complex numbers (vectors). The dot product (also called the scalar product) of z_1 and z_2 is defined as the real number:

$$z_1.z_2 = |z_1||z_2|\cos\theta$$

where θ is the angle between z_1 and z_2 which lies between 0 and π.

Note: Let z_1 and z_2 be non-zero. Then:

- A necessary and sufficient condition that z_1 and z_2 be perpendicular is that $z_1.z_2 = 0$.
- The magnitude of the projection of z_1 on z_2 is $\frac{|z_1.z_2|}{|z_2|}$.

Cross Product

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be two complex numbers (vectors). The cross product of z_1 and z_2 is defined as the vector:

$$z_1 \times z_2 = (0, 0, x_1.y_2 - y_1.x_2)$$

perpendicular to the complex plane having magnitude:

$$|z_1 \times z_2| = x_1.y_2 - y_1.x_2 = |z_1||z_2| \sin\theta$$

Note: Let z_1 and z_2 be non-zero. Then:

- A necessary and sufficient condition that z_1 and z_2 be parallel is that $|z_1 \times z_2| = 0$.
- The area of a parallelogram having sidesz_1 and z_2 is $|z_1 \times z_2|$.

A point in the complex plane can be located by rectangular coordinates (x, y) or polar coordinates (r, θ). Many other possibilities exist. One such possibility uses the fact that:

$$x = \frac{z + \bar{z}}{2}$$
$$y = \frac{z - \bar{z}}{2i}$$

where,

$$z = x + iy$$

The coordinates $(z, \bar{z})$ that locate a point are called *complex conjugate coordinates* or briefly *conjugate coordinates* of the point.

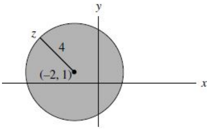
Problem: Find an equation for a circle of radius 4 with center at $(-2, 1)$.

Solution: The center can be represented by the complex number $-2+i$. If z is any point on the circle, the distance from z to $-2+i$ is:

$$|z-(-2+i)| = 4$$

Then $|z+2-i)| = 4$ is the required equation. In rectangular form, this is given by:

$$|(x+2)+i(y-1)| = 4, \quad \text{or, } (x+2)^2+(y-1)^2 = 16$$



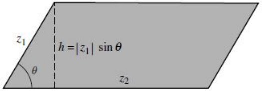
Problem: Prove that the area of a parallelogram having sides z_1 and z_2 is:

$$|z_1 \times z_2|.$$

Solution: From the diagram, we see:

Area of parallelogram

$$= (\text{base}).(\text{height}) \quad = (|z_2|)(|z_1| \sin\theta) = |z_1||z_2| \sin\theta \quad = |z_1 \times z_2|.$$



Problem: Prove that the equation of any circle or line in the $z-$ plane can be written as:

$$\alpha z\bar{z} + \beta z + \bar{\beta}\bar{z} + \gamma = 0$$

where α and γ are real constants while β may be a complex constant.

Solution: The general equation of a circle in the xy plane can be written:

$$A(x^2 + y^2) + Bx + Cy + D = 0$$

We know the following :

$$x = \frac{z + \bar{z}}{2}$$
$$y = \frac{z - \bar{z}}{2i}$$

which in conjugate coordinates becomes:

$$Az\bar{z} + B\left[\frac{z+\bar{z}}{2}\right] + C\left[\frac{z-\bar{z}}{2i}\right] + D = 0$$

which, on re-arranging becomes:

$$Az\bar{z} + z\left[\frac{B}{2} + \frac{C}{2i}\right] + \bar{z}\left[\frac{B}{2} - \frac{C}{2i}\right] + D = 0$$

Calling $A = \alpha$, $\frac{B}{2} + \frac{C}{2i} = \beta$ and $D = \gamma$, the required result follows.

Problem: Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ represent two non-collinear or non-parallel vectors. If a and b are real numbers (scalars) such that:

$$az_1 + bz_2 = 0.$$

Prove that: $a = 0$ and $b = 0$.

Solution: The given condition $az_1 + bz_2 = 0$ is equivalent to:

$$a(x_1 + iy_1) + b(x_2 + iy_2) = 0.$$

The above equation can be written as:

$$ax_1 + bx_2 + i(ay_1 + by_2) = 0$$

Then, we have the following:

$$ax_1 + bx_2 = 0$$
$$ay_1 + by_2 = 0$$

These equations have the simultaneous solution:

$$a = 0 \quad , b = 0;$$

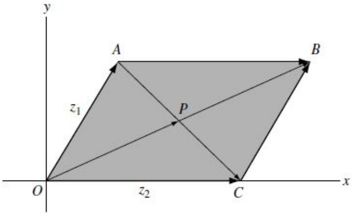
if the following condition is true:

$$\frac{y_1}{x_1} \neq \frac{y_2}{x_2}$$

that is, if the vectors are non-collinear or non-parallel vectors.

Problem: Prove that the diagonals of a parallelogram bisect each other.

Solution: Let us consider the diagram below:



Let $OABC$ be the given parallelogram with diagonals intersecting at P. Since $z_1 + AC = z_2$, $AC = z_2 - z_1$. Then:

$$AP = m(z_2 - z_1) \text{ where, } 0 \leq m \leq 1.$$

Since, $OB = z_1 + z_2$, hence:

$$OP = n(z_1 + z_2) \text{ where, } 0 \leq n \leq 1.$$

But, $OA + AP = OP$, i.e., $z_1 + m(z_2 - z_1) = n(z_1 + z_2)$. Hence,

$$(1 - m - n)z_1 + (m - n)z_2 = 0.$$

Hence, by previous problem, $1 - m - n = 0$ and $m - n = 0$. Thus $m = n = \frac{1}{2}$ and so P is the midpoint of both diagonals.

Problem: Prove that:

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

Also, give a graphical interpretation.

Solution: Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then we must show that:

$$\sqrt{(x_1 + x_2)^2 + (y_1 + y_2)^2} \leq \sqrt{x_1^2 + y_1^2} + \sqrt{x_2^2 + y_2^2}$$

Squaring both sides, this will be true if:

$$(x_1 + x_2)^2 + (y_1 + y_2)^2 \leq (x_1^2 + y_1^2) + (x_2^2 + y_2^2) + 2\sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)}$$

That is, if:

$$x_1x_2 + y_1y_2 \leq \sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)}$$

or if (squaring both sides again):

$$x_1^2x_2^2 + y_1^2y_2^2 + 2x_1x_2y_1y_2 \leq x_1^2x_2^2 + y_1^2y_2^2 + x_1^2y_2^2 + y_1^2x_2^2$$

From the last equation, we get:

$$2x_1x_2y_1y_2 \leq x_1^2y_2^2 + y_1^2x_2^2$$

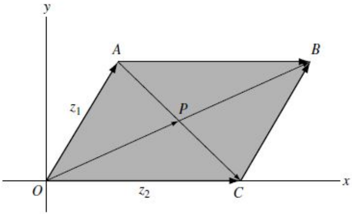
This implies that:

$$(x_1y_2 - x_2y_1)^2 \geq 0$$

which is true. Reversing the steps, which are reversible, proves the result.

Graphical Interpretation:

The result follows graphically from the fact that $|z_1|$, $|z_2|$, $|z_1 + z_2|$ represent the lengths of the sides of a triangle and that the sum of the lengths of two sides of a triangle is greater than or equal to the length of the third side.



- **Neighborhoods:** A delta, or $\delta-$neighborhood of a point z_0 is the set of all points z such that:

$$|z - z_0| < \delta$$

where δ is any given positive number. A deleted $\delta-$neighborhood of z_0 is a neighborhood of z_0 in which the point z_0 is omitted, that is:

$$0 < |z - z_0| < \delta$$

- **Limit Points:** point z_0 is called a limit point, cluster point, or point of accumulation of a point set S if every deleted $\delta-$neighborhood of z_0 contains points of S. Since δ can be any positive number, it follows that S must have infinitely many points. Note that z_0 may or may not belong to the set S.
- **Closed Sets:** A set S is said to be closed if every limit point of S belongs to S, i.e., if S contains all its limit points. For example, the set of all points z such that $|z| \leq 1$ is a closed set.

- **Bounded Sets:** A set S is called bounded if we can find a constant M such that:

$$|z| \leq M$$

for every point z in S. An unbounded set is one which is not bounded. A set which is both bounded and closed is called **compact**.

- **Interior, Exterior and Boundary Points:** A point z_0 is called an interior point of a set S if we can find a $\delta-$neighborhood of z_0 all of whose points belong to S. If every $\delta-$neighborhood of z_0 contains points belonging to S and also points not belonging to S, then z_0 is called a boundary point. If a point is not an interior or boundary point of a set S, it is an exterior point of S.

- **Open Sets:** An open set is a set which consists only of interior points. For example, the set of points z such that:

$$|z| \leq 1$$

is an open set.

- **Connected Sets:** An open set S is said to be connected if any two points of the set can be joined by a path consisting of straight line segments (i.e., a polygonal path) all points of which are in S.
- **Open Regions or Domains:** An open connected set is called an open region or domain.
- **Closure of a Set:** If to a set S we add all the limit points of S, the new set is called the closure of S and is a closed set.