Division - 1

Lecture - 4 , Tuesday , 02-08-2011

Limit point of a set Let S be a set in $\mathbb{C}$.

A point $z_0 \in \mathbb{C}$ is said to be a limit point of the set S if every neighborhood $N(z_0)$ of z_0 contains atleast one point of S other than z_0.

Example: Let $S = \{z \in \mathbb{C} \mid |z| < 1\}$

Take any point z_0 with $|z_0| \leq 1$. Then, z_0 is a limit point of S.

Take any point z_0^* with $|z_0^*| > 1$. Then, z_0^* is not a limit point of S.

Let $S = \{z \in \mathbb{C} \mid |z| = 5\}$

Take any point z_0 with $|z_0| = 5$. Then, z_0 is a limit point of S.

Take any point z_0^* with $|z_0^*| \neq 5$. Then z_0^* is not a limit point of S.

Exercise: ① Find the limit points of $S = \{z = x + iy \mid 0 < x < 1, \ y \in \mathbb{R}\}$.

② Find the limit points of $S = \{z = x + iy \mid x = 1, \ y = 1, 2, 3, 4, 5\}$.

Result: ① A set S is closed iff S contains all its limit points.

② A set S is closed iff $S = \bar{S}$ = closure of S

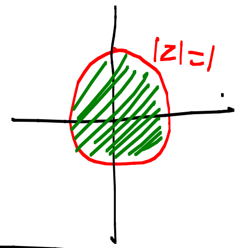
where $\bar{S}$ = closure of S = $S \cup \{\text{set of all limit points}\}$

Boundary point of a set: Let $S \subseteq \mathbb{C}$. A point $z_0 \in \mathbb{C}$ is said to be a boundary point of the set if every neighborhood $N(z_0)$ of z_0 contains atleast one point of S and at least one point not in S.

Example:

$$S = \{ z \in \mathbb{C} \mid |z| \leq 1 \}$$

Boundary points of S are $\{ z \in \mathbb{C} \mid |z| = 1 \}$



$$S = \{ z \in \mathbb{C} \mid |z| = 5 \}$$

Boundary points of S are $\{ z \in \mathbb{C} \mid |z| = 5 \} = S$ itself.

$$S = \{ z \in \mathbb{C} \mid 2 < |z| \leq 3 \}$$

Boundary points of S are $\{ z \in \mathbb{C} \mid |z| = 2 \} \cup \{ z \in \mathbb{C} \mid |z| = 3 \}$.

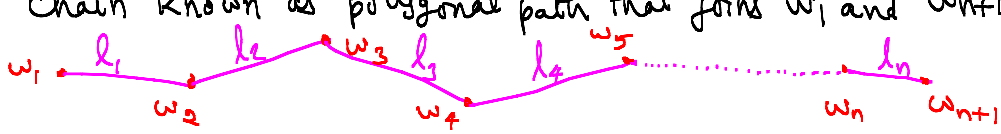
Results:

1. Arbitrary union of open sets is open.
2. Arbitrary intersection of closed sets is closed.
3. Finite intersection of open sets is open.
4. Finite union of closed sets is closed.

Polygonal path: Let $w_1, w_2, \ldots, w_{n+1}$ be $(n+1)$ points.

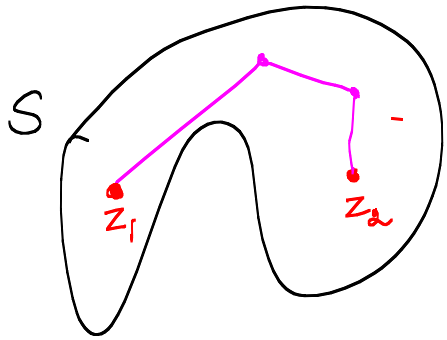
For each $k = 1, 2, \ldots, n$, let l_k denote the line segment joining w_k and w_{k+1}.

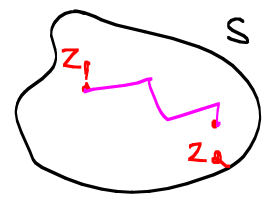
Then, the successive line segments $l_1, l_2, l_3, \ldots, l_n$ form a continuous chain known as polygonal path that joins w_1 and w_{n+1}.



Connected Set

A set S is said to be connected if every pair of points z_1, z_2 in S can be joined by a polygonal path (or by a continuous curve) that lies entirely in S.





Examples

$\{z \in \mathbb{C} \mid |z - z_0| < r\}$, $\{z \in \mathbb{C} \mid |z - z_0| > r\}$

$\{z \in \mathbb{C} \mid |z - z_0| \leq r\}$, $\{z \in \mathbb{C} \mid |z - z_0| \geq r\}$

$\{z \in \mathbb{C} \mid |z - z_0| = r\}$, $\{z \in \mathbb{C} \mid Re(z) > t_0\}$

$\{z \in \mathbb{C} \mid Re(z) \geq t_0\}$, $\{z \in \mathbb{C} \mid |Im(z)| \leq 1\}$

Connected sets

$\{z \in \mathbb{C} \mid Re(z) \neq 1\}$ = Disconnected set (Not connected)

$\{z \in \mathbb{C} \mid |z| < 2 \text{ and } |z| > 3\}$ = Disconnected set.

Important Definition:

Domain: An open, connected set $S \subseteq \mathbb{C}$ is called a domain.

Examples: $S = \{z \in \mathbb{C} \mid |z - z_0| < r\}$ domain, $S = \{z \in \mathbb{C} \mid 1 < |z - z_0| < 2\}$ domain

Region: A domain, together with some, none, or all of its boundary points, is called a region.

Example:

$\{z \in \mathbb{C} \mid |z| \leq 1\}$, $\{z \in \mathbb{C} \mid 1 \leq |z| < 2\}$

These are called regions.

Compact set: A set $S \subseteq \mathbb{C}$ is said to be compact if it is closed and bounded.

Examples:

$\{z \in \mathbb{C} \mid |z - z_0| \leq r\}$ ← Compact set

$\{z \in \mathbb{C} \mid |z - z_0| = r\}$ ← compact set

$\{z \in \mathbb{C} \mid |z - z_0| \geq r\}$ ← NOT compact (Unbounded)

$\{z \in \mathbb{C} \mid |z - z_0| < r\}$ ← NOT compact (Not closed)

$\{z \in \mathbb{C} \mid 0 \leq r_1 \leq |z - z_0| \leq r_2\}$ ← compact.

$\{z \in \mathbb{C} \mid \mathrm{Im}(z) \geq 0\}$ ← NOT compact (Unbounded)

Straight Line segment joining two points z_1 and z_2:

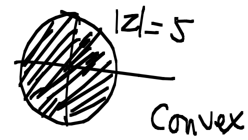
Equation of the line segment joining z_1 and z_2 is

$$\boxed{z = t z_2 + (1-t) z_1} \quad \text{for } t \in [0, 1]$$

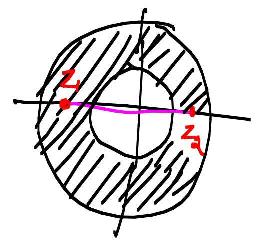
Convex Set A set $S \subseteq \mathbb{C}$ is said to be a convex set if for every pair of points z_1 and z_2 in S, the straight line segment joining z_1 and z_2 lies entirely inside S.

Examples:

$\{z \in \mathbb{C} \mid |z| \le 5\}$ Convex set



$\{z \in \mathbb{C} \mid 1 \le |z| \le 2\}$ Not convex



$\{z \in \mathbb{C} \mid Re(z) \geq 0\}$ Convex set.

Limits of functions:

Let $w = f(z)$ be a complex function of a complex variable z that is defined for all values of z in some neighborhood of z_0, except perhaps at the point z_0.

We say that f has the limit w_0 as z approaches z_0 if for each positive number $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$0 < |z - z_0| < \delta \text{ implies } |f(z) - w_0| < \varepsilon.$$

We write it as

$$\boxed{\lim_{z \to z_0} f(z) = w_0}.$$

Note: z can approach z_0 from any direction. For the $\lim_{z \to z_0} f(z)$ to exist, it is required that $f(z)$ must approach the same value w_0, no matter how z approaches z_0.

Example:

$$\lim_{z\to 0} \bar{z} = 0.$$

$$\lim_{z\to(1+2i)} z^2 = (1+2i)^2 = -3+4i.$$

$$\lim_{z\to 0} \frac{Re(z)}{|z|} = \lim_{z=(x,y)\to(0,0)} \frac{x}{\sqrt{x^2+y^2}}.$$

Path I: $z=(x,y)\to(0,0)$ along the line $y=x$ in the 1^{st} quadrant.

$$\lim_{\substack{y=x\\ x>0\\ x\to 0}} \frac{x}{\sqrt{x^2+y^2}} = \frac{1}{\sqrt{2}}$$

Path II: $z=(x,y)\to(0,0)$ along the line $y=-x$ in the 2^{nd} quadrant

$$\lim_{\substack{y=-x\\ x<0\\ x\to 0}} \frac{x}{\sqrt{x^2+y^2}} = \frac{-1}{\sqrt{2}}.$$

Since $\frac{Re(z)}{|z|}$ approaches two different values as $z\to 0$ along two different paths, we conclude that

$\lim_{z\to 0} \frac{Re(z)}{|z|}$ does NOT exist.

Note: Concept of $f: D\subseteq \mathbb{R}^2 \to \mathbb{R}^2$, $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = (u_0, v_0)$ SAME $\equiv$ Equivalent Concept of $f: D\subseteq \mathbb{C}\to\mathbb{C}$, $\lim_{z\to z_0} f(z) = w_0$

$$f : N(z_0) \subseteq \mathbb{C} \to \mathbb{C}$$

$$w = f(z) = u(x,y) + i\, v(x,y).$$

Here $u(x,y) = \mathrm{Re}(f(z))$ and $v(x,y) = \mathrm{Im}(f(z))$ are component/coordinate functions of f.

Relation between $\lim\limits_{z \to z_0} f(z)$ and $\lim\limits_{z \to z_0} u(x,y)$ & $\lim\limits_{z \to z_0} v(x,y)$??

Result: Let $f(z) = u(x,y) + i\, v(x,y)$ be a complex function that is defined in some neighborhood of z_0, except perhaps at $z_0 = x_0 + i y_0$. Then

$$\boxed{\lim_{z \to z_0} f(z) = w_0 = u_0 + i v_0 \iff \lim_{(x,y) \to (x_0,y_0)} u(x,y) = u_0 \ \& \lim_{(x,y) \to (x_0,y_0)} v(x,y) = v_0 .}$$

Example: Let $f(z) = z^2 = (x^2 - y^2) + i\,(2xy)$.

$$\lim_{z \to (1+2i)} z^2 = -3 + 4i \iff \lim_{(x,y) \to (1,2)} x^2 - y^2 = -3 \text{ and } \lim_{(x,y) \to (1,2)} 2xy = 4$$

Example: Let $f(z) = \left(\frac{\mathrm{Re}(z)}{|z|}\right) + i\,(2xy)$.

Analyze $\lim\limits_{z \to 0} f(z)$.

$$\mathrm{Re}(f(z)) = u(x,y) = \frac{\mathrm{Re}(z)}{|z|} = \frac{x}{\sqrt{x^2+y^2}}$$

Since $\lim\limits_{(x,y) \to (0,0)} \frac{\mathrm{Re}(z)}{|z|} = \lim\limits_{(x,y) \to (0,0)} \frac{x}{\sqrt{x^2+y^2}}$ does not exist, we conclude that $\lim\limits_{z \to 0} f(z)$ does NOT exist.

Result: Let $\lim_{z \to z_0} f(z) = A$ and $\lim_{z \to z_0} g(z) = B$. Then

(i) $\lim_{z \to z_0} \left(f(z) + g(z) \right) = A + B$

(ii) $\lim_{z \to z_0} \left(k\, f(z) \right) = kA$ where k is a constant.

(iii) $\lim_{z \to z_0} \left(f(z)\, g(z) \right) = AB$

product

(iv) $\lim_{z \to z_0} \dfrac{f(z)}{g(z)} = \dfrac{A}{B}$ provided $B \neq 0$.

Limit of f: definition in terms of sequences:

$\lim_{z \to z_0} f(z) = w_0$ iff for every sequence $\{z_n\}$ converging to z_0, the sequence $\{f(z_n)\}$ converges to w_0

Note: METHOD to show $\lim_{z \to z_0} f(z)$ does not exist.

Try to find: Two different paths for $z \to z_0$, on which the function $f(z)$ approaches two different values.

See: Example $\lim_{z \to 0} \dfrac{\operatorname{Re}(z)}{|z|}$ does NOT exist.

Note: METHOD to show $\lim_{z \to z_0} f(z)$ exists and is equal to w_0.

Step ①: Guess the limiting value w_0.

Step ②: Compute $|f(z) - w_0|$

Try to find an upper bound expression for this in terms of simple functions/expression or $|z - z_0|$

$$|f(z) - w_0| \le \boxed{\text{Find simple expression in terms of } |z - z_0|} < \varepsilon$$

This is to be less than ε

What δ is to be chosen? Find it.

Step ③

$$0 < |z - z_0| < \delta \implies |f(z) - w_0| \le \text{Simple expression} < \varepsilon$$

Let $z_0 \neq 0$. Show that $\lim_{z \to z_0} \frac{1}{z} = \frac{1}{z_0}$ using ε-δ method (Follow the steps explained above).

Show that $\lim_{z \to z_0} \overline{z} = \overline{z_0}$.

Lecture 4 ends. Division - I