Lecture - 8, Division - I, Wednesday, 10-08-2011

Laplace Equation in Polar form

$$\boxed{r^2 \phi_{rr} + r \phi_r + \phi_{\theta\theta} = 0}$$

Laplace Equation in Complex form

$$\boxed{\frac{\partial^2 \phi}{\partial z \, \partial \bar{z}} = \frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial \bar{z}}\right) = 0}$$

Bounded function:

Let $f : D \subseteq \mathbb{C} \to \mathbb{C}$ be a function defined on a set D.

We say f is bounded on the set D if there exists a real number $M > 0$ such that

$$|f(z)| \leq M \text{ for all } z \in D.$$

Extended Complex Plane

$$\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$$

∞ is called the Point at Infinity.

It obeys, for any $z \in \mathbb{C}$

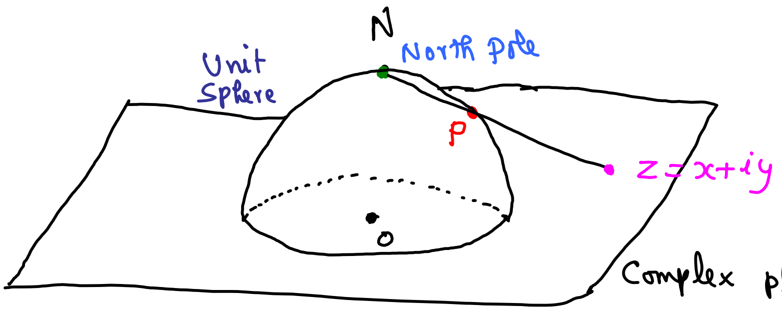
$$z + \infty = \infty = \infty + z.$$

If $z^* \neq 0$ and $z^* \in \mathbb{C}$ then $\dfrac{z^*}{0} = \infty$, $\dfrac{z^*}{\infty} = 0$

Note: $0 \cdot \infty$, $\dfrac{\infty}{\infty}$, $\dfrac{\infty}{0}$ are not defined.

Visualization of Extended Complex plane

Riemann Sphere & Stereographic Projection.



Complex plane $\mathbb{C}$
= Plane passing through equator

For each point $z = x + iy$ in the plane, connect z to the north pole N by a straight line. It will pierce the sphere at a point P (say).
Give a one-to-one correspondance between $\mathbb{C}$ and the unit sphere S in $\mathbb{R}^3$ by

$$z \longmapsto P \quad \text{for all } z \in \mathbb{C}$$

$$\infty \longmapsto N \text{ (North pole)}$$

$$\hat{\mathbb{C}} \longleftrightarrow S \text{ (Unit sphere)}$$

$\Big\}$ One-to-one & onto map

Stereographic projection

This Sphere S is known as the Riemann Sphere.

Limits involving the point at infinity

Theorem: If z_0 and w_0 are points in the z and w planes, respectively, then

$$\boxed{\lim_{z \to z_0} f(z) = \infty \quad \text{if and only if} \quad \lim_{z \to z_0} \frac{1}{f(z)} = 0}$$

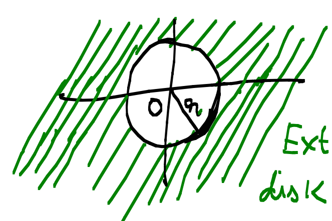
$$\boxed{\lim_{z \to \infty} f(z) = w_0 \quad \text{if and only if} \quad \lim_{z \to 0} f\left(\frac{1}{z}\right) = w_0}$$

$$\boxed{\lim_{z \to \infty} f(z) = \infty \quad \text{if and only if} \quad \lim_{z \to 0} \frac{1}{f\left(\frac{1}{z}\right)} = 0}$$

Exercise: Write the definitions for the above terms/limits in terms of ε and δ. Work out some examples of limits involving ∞.

(Open) Neighborhood of the point at infinity is given by

$$N(\infty) = \left\{ z \in \hat{\mathbb{C}} \;\middle|\; |z| > r \text{ for any } r > 0 \right\}$$



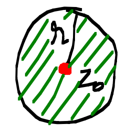
$N(\infty) =$ Exterior of disk

Deleted / punctured neighborhood

A deleted / punctured neighborhood of a point z_0 is given by

$$\left\{ z \in \mathbb{C} \;\middle|\; 0 < |z - z_0| < r \text{ for any } r > 0 \right\}$$

$$= N(z_0) \setminus \{z_0\}$$

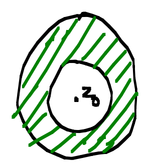


Centre z_0 is removed from the open disk.

Annular region = region between two concentric circles

$$\left\{ z \in \mathbb{C} \;\middle|\; r_1 < |z - z_0| < r_2 \right\}$$

$\boxed{\text{Less than or Less than or equal to}}$



CHAPTER 3 of Brown & Churchill Book

Elementary Functions

We study

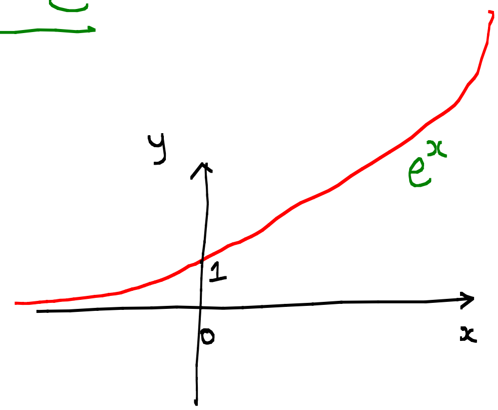
- Exponential Function e^z
- Trigonometric Functions $\sin z$, $\cos z$, ... etc
- Logarithm Function $\log z$
- Complex exponents / Powers Function z^w

Exponential Function: e^z

Recall:

Real exponential function

$e^x : \mathbb{R} \to \mathbb{R}$



Definition:

e^x is the unique function which satisfies the following conditions.

(i) $f(x)$ is a continuous real valued function for all $x \in \mathbb{R}$ with $f(0) = 1$

(ii) $f(x_1 + x_2) = f(x_1)\, f(x_2)$ for all x_1 and x_2 in $\mathbb{R}$

(iii) $f'(x) = f(x)$ for all $x \in \mathbb{R}$

In the same way, the complex exponential function e^z can be defined.

Exponential function is defined as the solution of the differential equation $f'(z) = f(z)$ for all $z \in \mathbb{C}$ with the initial value $f(0) = 1$.

Properties of Exponential Function:

① If $z = x + iy$ then $\boxed{e^z = e^x(\cos y + i \sin y)}$.

② $|e^z| = e^x = e^{\mathrm{Re}(z)}$ and $\arg(e^z) = y = \mathrm{Im}(z)$

(3) $$\frac{d}{dz} e^z = e^z \quad \text{for all } z \in \mathbb{C}.$$

e^z is differentiable at each point of $\mathbb{C}$.
e^z is analytic at each point of $\mathbb{C}$.

(4) $$e^{z_1 + z_2} = e^{z_1} e^{z_2} \quad \text{for all } z_1 \text{ and } z_2 \text{ in } \mathbb{C}.$$

(5) $$e^{z + 2\pi i} = e^z \quad \text{for all } z \in \mathbb{C}$$

That is, e^z is a periodic function with period $2\pi i$.

(6) $e^z \neq 0$ for any $z \in \mathbb{C}$. Reason: $|e^z| = e^x \neq 0 \ \forall z \in \mathbb{C}$

(7) $$\overline{(e^z)} = e^{(\bar{z})} \quad \text{for } z \in \mathbb{C}$$

(8) $|e^z| \leq e^{|z|}$ for $z \in \mathbb{C}$, Equality holds if $z \geq 0$.

(9) $$\boxed{e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}} \quad \text{for all } z \in \mathbb{C}$$

Mapping properties of e^z

Horizontal line $y = y_0$ in the z-plane

$$L = \{ z = x + iy \in \mathbb{C} \,/\, x \in \mathbb{R},\ y = y_0 \}$$

$$\omega = e^z$$

$$w = e^z = e^x e^{iy_0} \quad \text{for } x \in \mathbb{R}$$

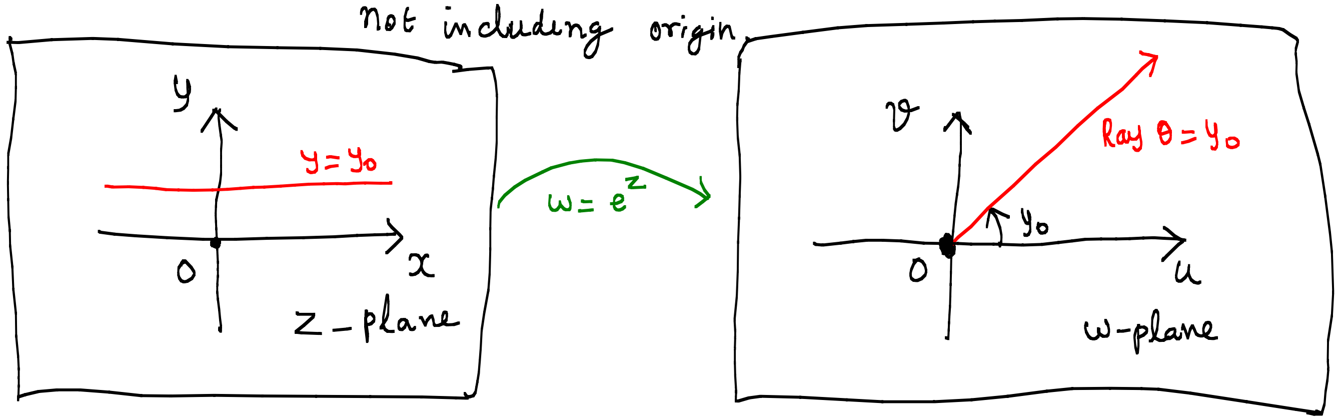
$$\arg(w) = y_0$$

$$|w| = e^x > 0 \quad \forall\ z = x+iy$$

Image set

$$\{ w = r e^{iy_0} \mid 0 < r < \infty \}$$

= Ray with $\theta = y_0$ emanating from the origin, but not including origin.



Vertical line $x = x_0$ in the z-plane

$$\{ z = x+iy \in \mathbb{C} \mid x = x_0 \text{ and } y \in \mathbb{R} \}$$

$$w = e^z$$

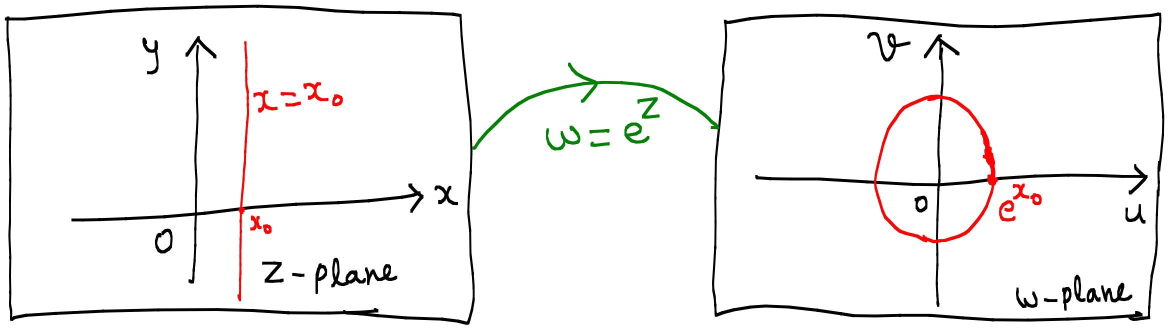
$$\Rightarrow \quad w = e^{x_0} e^{iy} \quad \text{for } y \in \mathbb{R}.$$

Image set

$$\{ w = e^{x_0} e^{iy} \in \mathbb{C} \mid y \in \mathbb{R} \}$$

= Circle centered at the origin and radius e^{x_0}.

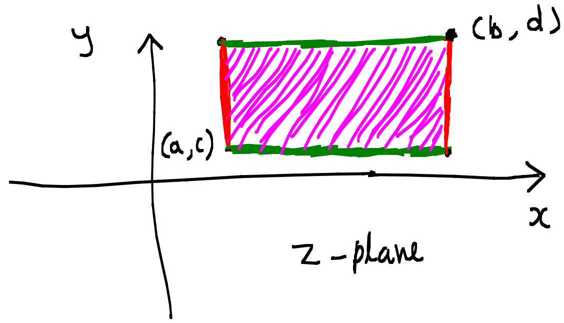
As y varies in an interval of length 2π, the image point traces the circle one round.

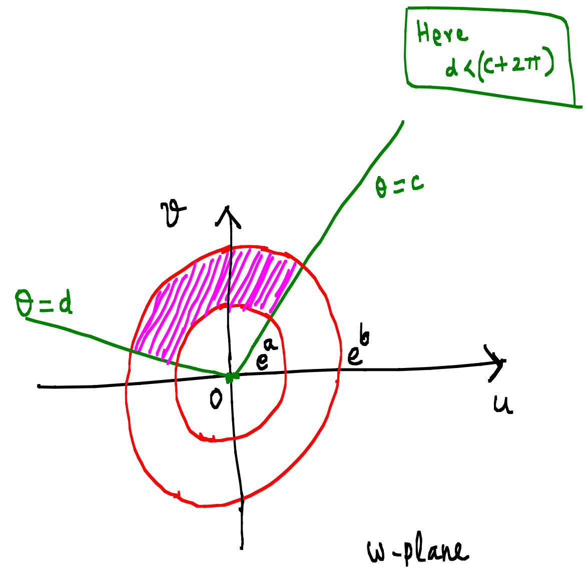


Rectangular region

$$\left\{ z = x + iy \in \mathbb{C} \;\middle|\; \begin{array}{l} a < x < b \\ c < y < d \end{array} \right\}$$

where $d < (c + 2\pi)$.





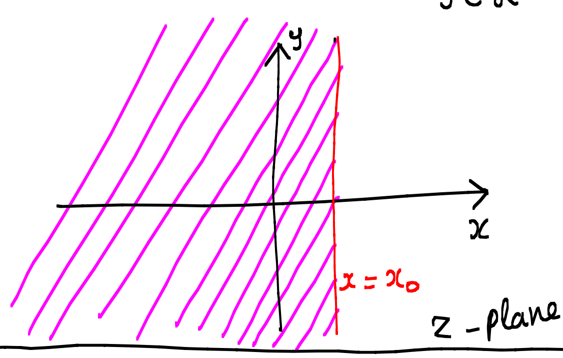
If $d > (c + 2\pi)$, then image set will be $\{ w \in \mathbb{C} \mid e^a < |w| < e^b \}$

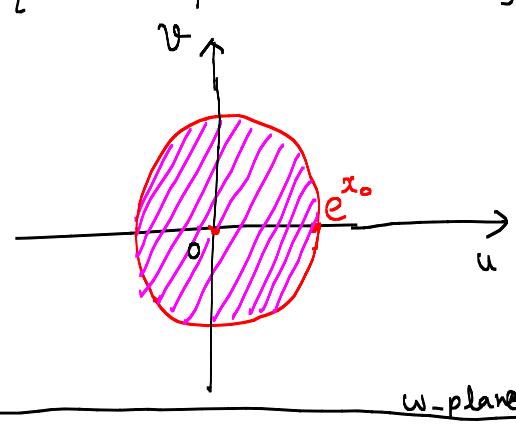
Left Half plane

$$\left\{ z = x + iy \in \mathbb{C} \;\middle|\; \begin{array}{l} \mathrm{Re}(z) = x < x_0, \\ y \in \mathbb{R} \end{array} \right\} \xrightarrow{w = e^z}$$

Image set

$$\{ w = re^{i\theta} \mid 0 < r < e^{x_0}, \ \theta \in \mathbb{R} \}$$





Trigonometric Functions

We know that

$$e^{ix} = \cos x + i \sin x$$
$$e^{-ix} = \cos x - i \sin x$$

$$\Rightarrow \boxed{\cos x = \frac{e^{ix} + e^{-ix}}{2}} \text{ and } \boxed{\sin x = \frac{e^{ix} - e^{-ix}}{2i}}$$

for any $x \in \mathbb{R}$.

We define complex cosine and sine function in similar way.

$$\boxed{\cos z = \frac{e^{iz} + e^{-iz}}{2}} \text{ and } \boxed{\sin z = \frac{e^{iz} - e^{-iz}}{2i}}$$

for any $z \in \mathbb{C}$

We can see that all the familiar trigonometric formulas/identities involving cosines and sines of real variables remain valid in complex domain.

Properties of Cosines and Sines.

1. $\sin^2 z + \cos^2 z = 1$ for all $z \in \mathbb{C}$
2. $\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$
3. $\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$.
4. $\sin(z + 2\pi) = \sin z$ and $\cos(z + 2\pi) = \cos z$ (periodic)

(5) $\sin(-z) = -\sin z$ (odd) and $\cos(-z) = \cos z$ (Even)

(6) $\sin z = 0$ iff $z = n\pi$ where $n \in \mathbb{Z}$

(7) $\cos z = 0$ iff $z = (n + \frac{1}{2})\pi$ where $n \in \mathbb{Z}$

(8) $\sinh(y) \le |\sinh(y)| \le |\sin z| \le \cosh(y)$

$\sinh(y) \le |\sinh(y)| \le |\cos z| \le \cosh(y)$.

$\boxed{\sin z \text{ and } \cos z \text{ are unbounded in } \mathbb{C}}$

Reason:
As $|y| \to \infty$, $|\sinh(y)| \to \infty$
$\Rightarrow |\sin z| \to \infty$

(9) $$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} \quad \text{for } z \in \mathbb{C}$$

(10) $$\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} \quad \text{for } z \in \mathbb{C}$$

(11) If $z = x + iy$ then $\boxed{\sin(z) = \sin(x)\cosh(y) + i\cos(x)\sinh(y)}$

$\boxed{\cos(z) = \cos(x)\cosh(y) - i\sin(x)\sinh(y)}$

(12) $$\frac{d}{dz}\sin(z) = \cos z \quad \text{for any } z \in \mathbb{C}$$

(13) $$\frac{d}{dz}\cos z = -\sin z \quad \text{for any } z \in \mathbb{C}.$$

Exercises: Explore the mapping properties of (1) $w = \sin z$ and (2) $w = \cos(z)$

Division-I: Lecture - 8 ends