Division - I

MA201 - Lecture - 1   Tuesday   Date: 26-07-2011

Complex Numbers:

A Complex number Z is defined to be an ordered pair of real numbers x and y as

$$\boxed{Z = (x, y)}.$$

The set of complex numbers is denoted by $\mathbb{C}$ and is given by

$$\boxed{\mathbb{C} = \{ z = (x, y) \mid x \text{ and } y \text{ are real numbers} \}}$$

Ordered pair means → Order in which we write x and y in defining z matters. For example, $(1, 5)$ is not the same as $(5, 1)$.

In the complex number $Z = (x, y)$

x = Real part of $Z = \mathrm{Re}(z)$ or $\mathcal{R}(z)$

y = Imaginary part of $Z = \mathrm{Im}(z)$ or $\mathcal{I}(z)$

Real number x is identified as $(x, 0)$ in $\mathbb{C}$

$$\mathbb{R} \subseteq \mathbb{C}.$$

Numbers of the form $(x, 0)$ are called real numbers.

Numbers of the form $(0, y)$ are called pure imaginary numbers.

Two complex numbers $Z_1 = (x_1, y_1)$ and $Z_2 = (x_2, y_2)$ are equal if and only if $x_1 = x_2$ and $y_1 = y_2$.

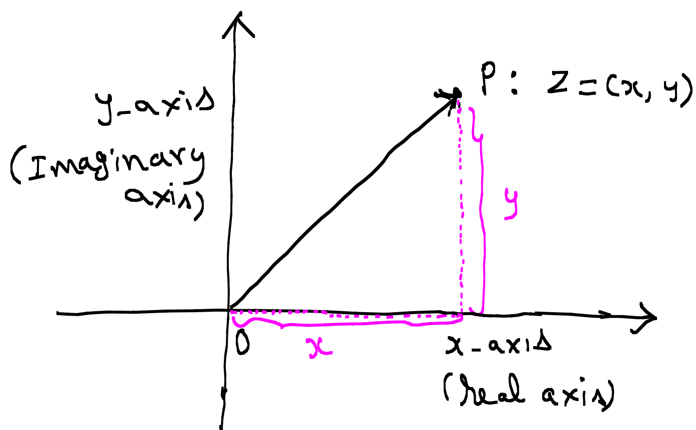
Example: $(1, 2) = (1, 2)$.

$(2, 3) \neq (3, 2)$.

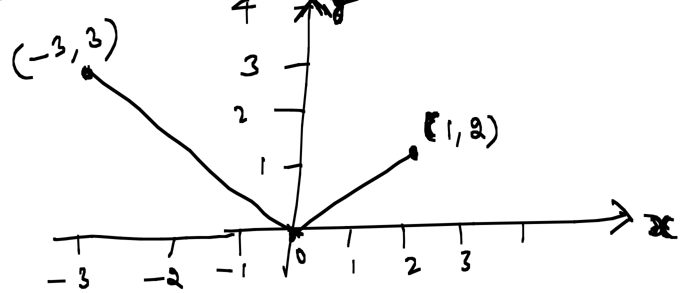
Geometric Interpretation of complex numbers:

$\mathbb{C} = \mathbb{R}^2$ (Two dimensional plane)

$\boxed{\text{Point } P = (x, y) \text{ in plane}} \longleftrightarrow \boxed{Z = (x, y) \text{ in } \mathbb{C}}$



Complex plane or Z-plane or the Argand plane / Gauss plane.



Addition Operation:

$$Z_1 = (x_1, y_1), \quad Z_2 = (x_2, y_2)$$

$$\boxed{Z_1 + Z_2 = (x_1 + x_2, \; y_1 + y_2)}$$

Subtraction Operation

$$Z_1 - Z_2 = (x_1 - x_2, \; y_1 - y_2)$$

Multiplication operation: $Z_1 = (x_1, y_1)$ and $Z_2 = (x_2, y_2)$

$$\boxed{Z_1 Z_2 = (x_1 x_2 - y_1 y_2, \; x_1 y_2 + x_2 y_1)}$$

Example: $(2, 3) \times (-1, 2) = (2 \times (-1) - 3 \times 2, \; 2 \times 2 + 3 \times (-1))$
$= (-8, 1)$

Division operation: $Z_1 = (x_1, y_1)$ and $Z_2 = (x_2, y_2) \neq 0$.

$$\frac{Z_1}{Z_2} = \left(\frac{1}{x_2^2 + y_2^2}\right)\left((x_1 x_2 + y_1 y_2), \; (x_2 y_1 - x_1 y_2)\right)$$

$$\Big(\mathbb{C}, \underset{\text{usual addition}}{+}, \underset{\text{usual multiplication}}{\times}\Big) \text{ is a field}$$

$(\mathbb{R}, +, \times)$ is a subfield of $(\mathbb{C}, +, \times)$

(Recall: Definition of field)

Bilinear form or $x+iy$ Notation:

Units : $(1,0)$ and $(0,1)$

Let us denote the number $(0,1)$ by i.

$$\boxed{i=(0,1)}$$

(iota)

Electrical engineers use j instead of i

$$Z=(x,y)=(x,0)(1,0)+(0,1)(y,0)=x\times 1+i\times y$$

$$\boxed{Z=x+iy}$$

$$i^2=(0,1)\times(0,1)=(-1,0)=-1.$$

Conjugate of complex number

The complex conjugate or simply Conjugate of a complex number $Z=x+iy$ is denoted by $\overline{Z}$ and is defined by

$$\boxed{\overline{Z}=x-iy}$$

Geometrically, $\overline{Z}$ is the reflection of the point $Z=x+iy$ on the real axis.

Example: If $\boxed{Z=3+4i}$ then $\boxed{\overline{Z}=3-4i}$

If $z=-3-4i$ then $\overline{z}=-3+4i$.

Properties of complex conjugation:

1. $z_1 = z_2$ iff $\overline{z_1} = \overline{z_2}$
2. $\overline{\overline{z}} = z$
3. $\overline{z} = z$ iff z is a real number
4. $Re(z) = \dfrac{z+\overline{z}}{2}$ and $Im(z) = \dfrac{z-\overline{z}}{2i}$
5. $\overline{z_1+z_2} = \overline{z_1} + \overline{z_2}$
6. $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$
7. $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\overline{z_1}}{\overline{z_2}}$ provided $z_2 \neq 0$.
8. $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$.

Modulus of a complex number:

The absolute value or modulus of a complex number $z = x+iy$ is denoted by $|z|$ and is given by

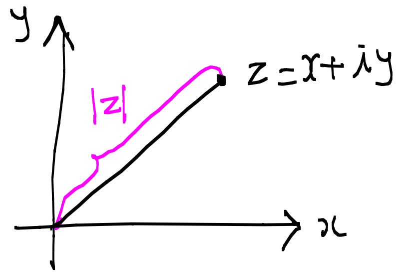
$$|z| = \sqrt{x^2+y^2}$$

(The principal (non-negative) square root of (x^2+y^2)).

Note that $|z| \geq 0$. $|z| = 0$ iff $z = 0$.

Example:

$$|4+3i| = \sqrt{4^2+3^2} = \sqrt{25} = 5.$$



Properties:

① $|\overline{z}| = |z| = |-z|$

② $\boxed{|z|^2 = z\overline{z}}$

③ If $z = x+iy$ then $|z| \le |x| + |y|$

④ If $z = x+iy$ then $x = Re(z) \le |Re(z)| \le |z|$

$y = Im(z) \le |Im(z)| \le |z|$

⑤ $|z_1 z_2| = |z_1|\,|z_2|$

⑥ $|z_1 + z_2| \le |z_1| + |z_2|$ Triangle Inequality.

Proof:

$$
\begin{aligned}
|z_1+z_2|^2 &= (z_1+z_2)\overline{(z_1+z_2)} \\
&= (z_1+z_2)(\overline{z_1}+\overline{z_2}) \\
&= z_1\overline{z_1} + z_1\overline{z_2} + z_2\overline{z_1} + z_2\overline{z_2} \\
&= |z_1|^2 + z_1\overline{z_2} + \overline{(z_1\overline{z_2})} + |z_2|^2 \\
&= |z_1|^2 + 2\,Re(z_1\overline{z_2}) + |z_2|^2 \\
&\le |z_1|^2 + 2\,|z_1\overline{z_2}| + |z_2|^2 \\
&= |z_1|^2 + 2\,|z_1|\,|z_2| + |z_2|^2 = (|z_1|+|z_2|)^2
\end{aligned}
$$

Since $Re(z) = \dfrac{z+\overline{z}}{2}$

Since $|Re(z)| \le |z|$ and $|z| = |\overline{z}|$

$$\Rightarrow \boxed{|z_1 + z_2| \leq |z_1| + |z_2|}$$

⊛ Here, equality occurs if three points O, z_1, z_2 are colinear.

Now,

$$|z_1| = |z_1 - z_2 + z_2| \leq |z_1 - z_2| + |z_2|$$

$$|z_1| - |z_2| \leq |z_1 - z_2| \longrightarrow (1)$$

$$|z_2| = |z_2 - z_1 + z_1| \leq |z_2 - z_1| + |z_1|$$

$$|z_2| - |z_1| \leq |z_2 - z_1| = |z_1 - z_2|$$

$$|z_2| - |z_1| \leq |z_1 - z_2| \longrightarrow (2)$$

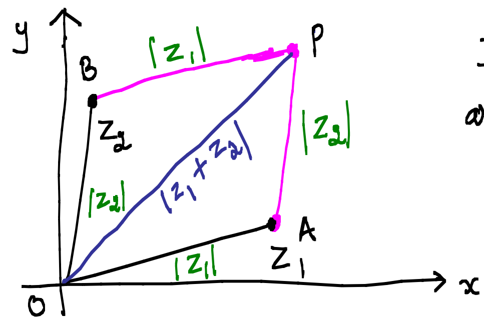
Combining (1) and (2), we get

$$\boxed{\big||z_1| - |z_2|\big| \leq |z_1 - z_2|}$$

Also, $\boxed{\big||z_1| - |z_2|\big| \leq |z_1 + z_2|}$

Geometrical Interpretation of Triangle Inequality.

$$|z_1 + z_2| \leq |z_1| + |z_2|$$



In the triangle whose sides are $|z_1|$ and $|z_2|$,

$$|\vec{OP}| \leq |\vec{OA}| + |\vec{AP}|$$

(Sum of lengths of other sides)

Generalizing Triangle Inequality for more than two points

$$|z_1 + z_2 + z_3 + \cdots\cdots + z_n| \leq |z_1| + |z_2| + |z_3| + \cdots\cdots + |z_n| .$$

Lecture - 1 ends. (Division - 1)