METROECONOMICA

Vol. XXV Gennaio-Aprile 1973 Fasc. I

THE NOTION OF VERTICAL INTEGRATION IN ECONOMIC ANALYSIS (*)

by Luigi L. Pasinetti

Very few notions in economic analysis are so seldom explicitly mentioned as the notion of vertical integration and are at the same time so widely used, implicitly or without full awareness [1]. I came to this conviction during the discussions on a multi-sector model of economic growth which I presented a few years ago [2]. The synthetic

(*) This paper was presented for discussion at the « Gruppo per lo studio dei problemi della distribuzione del progresso tecnico e dello sviluppo economico » C.N.R. Rome (Italian National Research Council), on December 13, 1972. I am grateful for useful comments to Piercarlo Nicola, Paolo Varri, Sergio Parrinello, Antonio Gay.

(1) The notion of vertical integration is implicit in all discussions on the theory of value of the Classical economists. The same thing can be said of the marginalist economists. When, for example, Léon Walras adopted the device of eliminating intermediate commodities from his analysis of production, he was making use of the logical process of vertical integration. (See « Elements of Pure Economics », W. Jaffé, ed., pp. 241 and ff.). Keynesian macro-economic analysis is also generally carried out in terms of vertically integrated magnitudes (net national income, net savings, new investments, consumption, etc.). Very rarely, however, is the logical process of vertical integration explicitly discussed. Generally it is simply taken for granted.

(2) *A New Theoretical Approach to the Problems of Economic Growth*, in « The Econometric Approach to Development Planning »; « Pontificiae Academiae Scientiarum Scripta Varia », no. 28, Vatican City 1965 (republished by North Holland Publishing Co., Amsterdam 1965), pp. 571-696; to be referred to, in the following pages, simply as *New Theoretical Approach*.

notion of a « vertically integrated sector » is used explicitly in that model, but within the simplified context of an economic system in which capital goods are made by labour alone; and I have always been faced with questions (3).

An explicit and more general investigation of the meaning and relevance of vertical integration in economic analysis may therefore prove of some usefulness. Instead of starting from the synthetic notions and going back to their elementary components, I shall start here from these elementary components — i.e., from the now familiar schemes of interindustry analysis — and go on to the synthetic notions. The crucial role played by vertical integration in the theories of value, income distribution and economic growth should emerge clearly as the investigation develops. The whole analysis will be carried out with reference to the general case of production of all commodities by means of commodities. The simplified case of capital goods produced by labour alone will be shown at the end as a particular case.

1. Production of commodities by means of commodities

An economic system will be considered in which all commodities are produced by means of commodities, used as capital goods. Commodities enter the process of production at the beginning of each « year » as inputs, jointly with labour services, and commodities come out at the end of the year as outputs. The economic system is supposed to be *viable*, in the sense that it is capable of producing larger quantities of commodities than those required to replace used-up capital goods.

The following notation will be used throughout (4):

i) column vector $\mathbf{X}(t) \equiv [X_i(t)]$, $i = 1, 2, \ldots m$, to denote the physical quantities of the m commodities that are produced in year t;

ii) column vector $\mathbf{Y}(t) \equiv [Y_i(t)]$, $i = 1, 2, \ldots m$, to denote the physical net product of the economic system, i.e. what is available for consumption and new investments after deducting replacements from $\mathbf{X}(t)$. Of course $\mathbf{Y}(t)$ may further be regarded as a sum of commodities devoted to consumption and of commodities devoted to investments, to be denoted by column vectors $\mathbf{C}(t) \equiv [C_i(t)]$ and $\mathbf{J}^{(d)}(t) \equiv [J_i^{(d)}(t)]$, $i = 1, 2, \ldots m$, respectively. By definition, $\mathbf{C}(t) + \mathbf{J}^{(d)}(t) \equiv \mathbf{Y}(t)$;

(3) These questions have normally been concerned with the problem of how to construct the vertically integrated sectors in the general case. Some indications are given in Chapter VI of *New Theoretical Approach*, but in a brief and incomplete way.

(4) Letters in bold type will be used to denote vectors and matrices. Symbol $\equiv$ will be used to denote a definitional equality and in particular, as in all cases in this section, an equality of two different notations for the same thing.

iii) column vector $\mathbf{S}(t) \equiv [S_i(t)]$, $i = 1, 2, \ldots m$, to denote the physical quantities of commodities that are required as capital goods (capital stocks), at the beginning of year t, in order to obtain quantities $\mathbf{X}(t)$ at the end of year t;

iv) row vector $\mathbf{p}(t) \equiv [p_i(t)]$, $i = 1, 2, \ldots m$, to denote prices of commodities $1, 2, \ldots m$;

v) scalar $L(t)$, to denote the labour force required by the economic system in year t, measured, let us say, in man-years;

vi) scalar π to denote the (uniform) rate of profit;

vii) scalar $w(t)$ to denote the (uniform) wage rate.

As far as technology is concerned, two successive analytical stages will be taken — a procedure which is by now customary in this type of analysis. Usually, as is well known, one begins by considering, first, production with circulating capital goods; then one goes on to production with fixed capital goods. The advantage of this procedure is that almost all basic concepts can be singled out at the first stage, where relatively few analytical complications arise. The second stage can then be devoted to pointing out which conclusions still hold and which conclusions are affected by generalization. A slightly more general approach is taken in the present work. Fixed capital goods are introduced immediately at the first stage, but with a simplifying assumption on how they depreciate. The case of production with fixed capital goods in general will, of course, be considered at a second stage.

2. Fixed capital goods with a simplifying assumption

We shall begin by considering a technology which requires both circulating capital goods (which are used up within one year) and fixed capital goods (which last for more than one year). The simplifying assumption will be made that in each industry j a constant proportion δ_j of all fixed capital goods drops out of the production process each year $(j = 1, 2, \ldots m)$. Moreover, for the time being, all technical coefficients will be supposed to be constant through time.

The technique of the whole economic system will be represented by:

i) a row vector $\mathbf{a}_{[n]} \equiv [a_{nj}]$, $n = m + 1$, $j = \ , 2, \ldots m$, all $a_{nj} \geq 0$, where each a_{nj} denotes the annual input of labour required by one physical unit of the commodity produced in industry j;

ii) a square matrix $\mathbf{A} \equiv [a_{ij}]$; $i, j = 1, 2, \ldots m$, all $a_{ij} \geq 0$, in which each column j represents the physical stocks of capital goods (both circulating and fixed) required for the production of one physical unit of the commodity produced in industry j. Square matrix $\mathbf{A}$ may be regarded as the sum of two non-negative square matrices:

$\mathbf{A}^{(C)} \equiv [a_{ij}{}^{(C)}]$ and $\mathbf{A}^{(F)} \equiv [a_{ij}{}^{(F)}]$; $i, j = 1, 2, \ldots m$, as representing the stocks of circulating capital goods and the stocks of fixed capital goods respectively. Therefore $\mathbf{A} \equiv \mathbf{A}^{(C)} + \mathbf{A}^{(F)}$, by definition. Of course, each year in each industry j, the economic system has to replace all circulating capital goods and a fraction δ_j of the fixed capital goods; $j = 1, 2, \ldots m$. If we call $\hat{\delta}$ a diagonal matrix with alla the δ_j's on the main diagonal, we may therefore define another square matrix $\mathbf{A}^{\ominus} \equiv [a_{ij}{}^{\ominus}] \equiv \mathbf{A}^{(C)} + \mathbf{A}^{(F)}\hat{\delta}$, as representing that part of the initial stocks of capital goods that are actually used up each year by the production process. By definition, $\mathbf{A}^{\ominus} + \mathbf{A}^{(F)}(\mathbf{I} - \hat{\delta}) \equiv \mathbf{A}$. The particular case in which all capital goods are circulating capital goods is represented by $\mathbf{A}^{(F)} = \mathbf{O}$ and, therefore, $\mathbf{A}^{\ominus} = \mathbf{A}$.

With this notation, the physical economic system may be represented by the following system of equations:

$$(\mathbf{I} - \mathbf{A}^{\ominus})\, \mathbf{X}(t) = \mathbf{Y}(t), \tag{2.1}$$

$$\mathbf{a}_{[n]}\, \mathbf{X}(t) = L(t), \tag{2.2}$$

$$\mathbf{A}\ \mathbf{X}(t) = \mathbf{S}(t), \tag{2.3}$$

where (2.1), (2.2) represent the flows of commodities and labour services required in year t to produce net product $\mathbf{Y}(t)$, and (2.3) represents the stocks of capital goods required at the beginning of year t for production to be effected. At the same time, equilibrium prices are represented by the following system of equations:

$$\mathbf{p} = \mathbf{a}_{[n]}\, w + \mathbf{p}\mathbf{A}^{\ominus} + \mathbf{p}\mathbf{A}\pi, \tag{2.4}$$

which determines all prices if one of these and the wage rate (or alternatively the rate of profit) are fixed exogenously.

3. An « industry »

On the assumption just made concerning fixed capital goods, each industry j ($j = 1, 2, \ldots m$) produces only one good: commodity j; and in order to produce one physical unit of such a commodity, it needs a quantity of labour represented by the j^{th} coefficient of vector $\mathbf{a}_{[n]}$ and a series of heterogeneous stocks of capital goods, represented by the j^{th} column of matrix $\mathbf{A}$. Industry j may therefore be synthetically represented by a « direct labour coefficient » — the j^{th} component of vector $\mathbf{a}_{[n]}$ — and by what may be called a « unit of direct productive capacity » — a composite commodity defined by the j^{th} column of matrix $\mathbf{A}$.

In equations (2.1) — (2.3), the physical quantities of an economic system are classified precisely in this way, i.e., according to the criterion of the « industry ». This classification has the advantage of being immediately observable; but it maintains our attention at a rather superficial level. A re-classification of the same physical quantities may be obtained on the basis of a conceptually more complex, but analytically far more powerful criterion, which we are now going to consider.

4. A « VERTICALLY INTEGRATED SECTOR »

We may define a new vector $\mathbf{Y}_i(t)$ as a column vector the components of which are all zeros except the i^{th} one, defined here as $Y_i(t)$ — i.e., the i^{th} component of vector $\mathbf{Y}(t)$. Moreover we shall use: scalar $L^{(i)}(t)$ to denote the quantity of labour required, column vector $\mathbf{X}^{(i)}(t)$ to denote the physical quantities of commodities to be produced, and column vector $\mathbf{S}^{(i)}(t)$ to denote the stocks of capital goods required, in the whole economic system, in order to obtain physical quantity $Y_i(t)$ of final good i $(i = 1, 2, \ldots m)$.

For each particular net product $Y_i(t)$, we obtain from (2.1) — (2.3):

$$\text{(4.1)} \qquad \mathbf{X}^{(i)}(t) = (\mathbf{I} - \mathbf{A}\Theta)^{-1}\,\mathbf{Y}_i(t),$$

$$\text{(4.2)} \qquad L^{(i)}(t) = \mathbf{a}_{[n]}\,(\mathbf{I} - \mathbf{A}\Theta)^{-1}\,\mathbf{Y}_i(t),$$

$$\text{(4.3)} \qquad \mathbf{S}^{(i)}(t) = \mathbf{A}\,(\mathbf{I} - \mathbf{A}\Theta)^{-1}\,\mathbf{Y}_i(t), \qquad i = 1, 2, \ldots, m,$$

.e., in fact, m sub-systems (as Piero Sraffa has called them) [5]. From (4.1) — (4.3) and the definition of $\mathbf{Y}_i(t)$ it follows that

$$\text{(4.4)} \qquad \sum_i^m \mathbf{Y}_i(t) = \mathbf{Y}(t) \quad ; \quad \sum_i^m \mathbf{X}^{(i)}(t) = \mathbf{X}(t)\,;$$

$$\text{(4.5)} \qquad \sum_i^m L^{(i)}(t) = L(t) \quad ; \quad \sum_i^m \mathbf{S}^{(i)}(t) = \mathbf{S}(t).$$

The m sub-systems add up to the original complete economic system.

The economic meaning of the coefficients appearing on the right hand side of (4.1) has been widely illustrated in the economic literature. Matrix $(\mathbf{I} - \mathbf{A}\Theta)^{-1}$ is known as the Leontief inverse matrix [6] — its i^{th} column $(i = 1, 2, \ldots m)$ contains the series of heterogeneous commodities that are directly and indirectly required in the whole economic system to obtain one physical unit of commodity i as a final good. On the other hand less attention has been paid to the economic meaning of the coefficients that appear on the right hand side of (4.2) and (4.3) [7]. More synthetically, we may define

---

(5) Piero Sraffa, « Production of Commodities by means of Commodities », Cambridge 1960, p. 89.

(6) After Wassily W. Leontief's work « The Structure of American Economy », New York 1941, and 1951.

(7) It is again Wassily W. Leontief who first applied these concepts, in a well known empirical investigation: *Domestic Production and Foreign Trade:*

$$(4.6) \qquad \mathbf{a}_{[n]} (\mathbf{I} - \mathbf{A}\Theta)^{-1} \equiv \mathbf{v} \equiv [v_i] ,$$
$$(4.7) \qquad \mathbf{A} (\mathbf{I} - \mathbf{A}\Theta)^{-1} \equiv \mathbf{H} \equiv [\mathbf{h}_i] , \qquad i = 1, 2, \dots m ,$$

where each $\mathbf{h}_i$ is a column vector, and thus re-write (4.2), (4.3) in a more compact way:

$$(4.2\text{ b}) \qquad L^{(i)}(t) = \mathbf{v}\ \mathbf{Y}_i(t) \equiv v_i\, Y_i ,$$
$$(4.3\text{ b}) \qquad \mathbf{S}^{(i)}(t) = \mathbf{H}\ \mathbf{Y}_i(t) \equiv \mathbf{h}_i\, Y_i , \qquad i = 1, 2, \dots m .$$

Each coefficient v_i in (4.2b) expresses in a consolidated way the quantity of labour directly and indirectly required in the whole economic system to obtain one physical unit of commodity i as a final good. We shall call it the *vertically integrated labour coefficient* for commodity i ($i = 1, 2, \dots m$). Likewise, each column vector $\mathbf{h}_i$ in (4.3 b) expresses in a consolidated way the series of heterogeneous physical quantities of commodities $1, 2, \dots m$, which are directly and indirectly required as stocks, in the whole economic system, in order to obtain one physical unit of commodity i as a final good ($i = 1, 2, \dots m$). This is another particular composite commodity which we shall call *a unit of vertically integrated productive capacity* for commodity i ($i = 1, 2, \dots m$).

Scalar v_i and column vector $\mathbf{h}_i$, together, represent what we may call the *vertically integrated sector* for the production of commodity i as a final good (whether for consumption of for investment); $i = 1, 2, \dots m$. A vertically integrated sector is therefore a compact way of representing a sub-system, as it synthesizes each sub-system into a single labour coefficient v_i and a single composite commodity $\mathbf{h}_i$. For an economic system with m commodities, we obviously obtain m labour coefficients (the m components of row vector $\mathbf{v}$) and m units of productive capacity (the m columns of matrix $\mathbf{H}$), i.e., m vertically integrated sectors for the production of the m commodities as final goods.

In comparison with the previous section, we may say that vector $\mathbf{a}_{[n]}$ and matrix $\mathbf{A}$ classify the total quantity of labour $L(t)$ and total quantities of stocks of capital goods $\mathbf{S}(t)$ according to the criterion of the industry in which they are required:

$$(4.8) \qquad L(t) = \mathbf{a}_{[n]}\ \mathbf{X}(t) ; \qquad \mathbf{S}(t) = \mathbf{A}\ \mathbf{X}(t) .$$

All these quantities are directly observable and directly quantifiable. Vector $\mathbf{v}$ and matrix $\mathbf{H}$ reclassify the same physical quantities according to the criterion of the vertically integrated sector for which they are directly and indirectly required:

The American Capital Position Re-examined, « Proceedings of the American Philosophical Society », vol. 97, no. 4, Sept. 1953, pp. 332-49; and *Factor Proportions and the Structure of American Trade: Further Theoretical and Empirical Analysis*, « The Review of Economics and Statistics », Nov. 1956, pp. 386-407.

$$(4.2\text{ b})\qquad L^{(i)}(t) = \mathbf{a}_{[n]}(\mathbf{I} - \mathbf{A}\Theta)^{-1}\,\mathbf{Y}_i(t) = \mathbf{v}\,\mathbf{Y}_i(t)\,,$$

$$(4.3\text{ b})\qquad \mathbf{S}^{(i)}(t) = \mathbf{A}\,(\mathbf{I} - \mathbf{A}\Theta)^{-1}\,\mathbf{Y}_i(t) = \mathbf{H}\,\mathbf{Y}_i(t)\,,$$

$$i = 1\,,\,2\,,\,\ldots\,m\,,$$

$$(4.9)\qquad L(t) = \sum_i^m L^{(i)}(t)\quad;\quad \mathbf{S}(t) = \sum_i^m \mathbf{S}^{(i)}(t)\,.$$

Neither $\mathbf{v}$ nor $\mathbf{H}$ are directly observable, but they can be obtained through post-multiplication by $(\mathbf{I} - \mathbf{A}\Theta)^{-1}$ from quantities $\mathbf{a}_{[n]}$ and $\mathbf{A}$, that are directly observable. They are therefore quantifiable in an indirect way.

To conclude, precisely the same physical quantities $L(t)$ and $\mathbf{S}(t)$ appear in both (4.8) and (4.9), but are classified according to two different criteria — the more immediate criterion of the « industry » in the former, the conceptually more complex criterion of the « vertically integrated sector » in the latter. Both classifications are empirically quantifiable — the former directly and the latter through the indirect logical process of vertical integration.

5. Vertical integration in the theory of value and income distribution

When Adam Smith put forward the proposition that every commodity finally resolves itself into wages, profits and rents (8), he rightly sensed that he had reached an important conclusion — he had (implicitly) grasped the basic concept of vertical integration.

In our analysis, neither the price system written in section 2 on the basis of directly observable magnitudes

$$(2.4)\qquad \mathbf{p} = \mathbf{a}_{[n]}\,w + \mathbf{p}\,\mathbf{A}\Theta + \mathbf{p}\,\mathbf{A}\,\pi\,,$$

nor what may be called its « solution »

$$(5.1)\qquad \mathbf{p} = \mathbf{a}_{[n]}(\mathbf{I} - \mathbf{A}\Theta - \mathbf{A}\,\pi)^{-1}\,w$$

can give us a clear idea of Adam Smith's theoretical insight.

But if we perform a few logical operations and re-write (2.4) as

$$(5.2)\qquad \mathbf{p}\,(\mathbf{I} - \mathbf{A}\Theta) = \mathbf{a}_{[n]}\,w + \mathbf{p}\,\mathbf{A}\,\pi\,,$$

$$(5.3)\qquad \mathbf{p} = \mathbf{a}_{[n]}(\mathbf{I} - \mathbf{A}\Theta)^{-1}\,w + \mathbf{p}\,\mathbf{A}\,(\mathbf{I} - \mathbf{A}\Theta)^{-1}\,\pi\,,$$

we can see the two notions characterizing a vertically integrated sector reappear. After substitution from definitions (4.6), (4.7), the (5.3) may be written

$$(5.4)\qquad \mathbf{p} = \mathbf{v}\,w + \mathbf{p}\,\mathbf{H}\,\pi\,.$$

(8) Adam Smith, « The Wealth of Nations », E. Cannan edition, pp. 49 and ff., especially p. 52.

This is a remarkable expression, as it explicitly shows that each price is ultimately made up of only two components: wages and profits [9]. It is precisely the logical operation of vertical integration that makes this evident by consolidating all the complex intermediate stages into one single labour coefficient and one single unit of productive capacity — the former being multiplied by the wage rate and the latter (after being evaluated at current prices) by the rate of profit. It may be noticed that vector $\mathbf{pH}$ is nothing but a vector of the m vertically integrated capital-output ratios multiplied by the price of the final commodity to which they refer. Hence, one alternative way of representing the m vertically integrated sectors might be that of using vector $\mathbf{v}$ (the m vertically integrated labour coefficients) and vector $\mathbf{pH}$ (the m vertically integrated capital-per-unit-of-output ratios) [10].

Another property of (5.4) is that it exposes the antagonism of wages and profits in income distribution. When $\pi = 0$, the second addendum vanishes and prices become

$$\mathbf{p} = \mathbf{v}\, w \,. \tag{5.5}$$

Wages are obviously at their maximum, as they absorb the whole purchasing power deriving from prices.

Conversely, when $w = 0$, profits are at their maximum and (5.4) becomes a linear and homogeneous system of equations

$$\mathbf{p}\,(\mathbf{I} - \Pi\, \mathbf{H}) = \mathbf{0} \,, \tag{5.6}$$

where Π stands for the rate of profit corresponding to $w = 0$, or maximum rate of profit. Since the economic system is viable *ex-hypothesis*, Π must be positive. Maximum rate of profit Π also emerges, from (5.6), as the reciprocal of the eigenvalue — which we may call λ — of matrix $\mathbf{H}$. Non-trivial solutions require, of course,

$$\det\,(\lambda\, \mathbf{I} - \mathbf{H}) = 0 \,, \tag{5.7}$$

an algebraic equation that yields m roots for λ. However, since $\mathbf{H}$ is a non-negative matrix, [11] we know on the basis of the Perron-Frobenius theorem [12] that its maximum eigenvalue λ_{max}: *a*) is a real and positive number; *b*) has a non-negative eigenvector (i.e. non-negative prices) associated with it; *c*) is also the eigenvalue which is

---

(9) No rents are considered in the present scheme.

(10) This is in fact the way suggested in Chapter VI of *New Theoretical Approach*, where the procedure to obtain them is also given (multiplication of direct labour coefficients and direct capital-output ratios by the inverse Leontief matrix).

(11) Both $\mathbf{A}^{\ominus}$ and $\mathbf{A}$ are non-negative *ex-hypothesis*, and moreover, the technique is supposed to be viable, which implies that $(\mathbf{I} - \mathbf{A}^{\ominus})^{-1}$ is non-negative. It follows that $\mathbf{H}$ is also non-negative.

(12) See, for example, F. R. Gantmacher, « The Theory of Matrices », New York 1959.

maximum in modulus (i.e. $\lambda_{max} = |\lambda|_{max}$). This is the only root of (5.7) that is economically relevant [13] and we shall therefore define straightaway:

$$\Pi = \frac{1}{\lambda_{max}} . \tag{5.8}$$

For any positive π lower than Π, we also know on the basis of the Perron-Frobenius theorem that $(\mathbf{I} - \pi\,\mathbf{H})^{-1}$ is non-negative, so that the general solution of (5.4),

$$\mathbf{p} = \mathbf{v}\,(\mathbf{I} - \pi\,\mathbf{H})^{-1}\,w , \tag{5.9}$$

yields non-negative prices and an inverse monotonic relation between π and w, whatever the standard in terms of which w is measured [14].

The same problems may be looked at in a more « classical » way if the wage rate itself is used as the *numeraire* of the price system, i.e., if we put $w = 1$. In this case all prices come to be expressed in terms of the wage rate, i.e., in terms of « labour commanded ». But the components of $\mathbf{v}$, the vertically integrated labour coefficients, express what classical economists called « labour embodied » (and Marx simply called « values »). Therefore, wages — by being distributed in proportion to « labour embodied » as appears from (5.4) — can « command » only part of the purchasing power deriving from prices. The difference

$$\mathbf{p} - \mathbf{v} = \mathbf{p}\,\mathbf{H}\,\pi \tag{5.10}$$

is absorbed by profits. « Solution » (5.9) becomes

$$\mathbf{p} = \mathbf{v}\,(\mathbf{I} - \pi\,\mathbf{H})^{-1} , \tag{5.11}$$

and may also be regarded as expressing the « transformation » of $\mathbf{v}$ into $\mathbf{p}$, i.e., of Marxian values into prices. The linear operator $(\mathbf{I} - \pi\,\mathbf{H})^{-1}$, where the m units of vertically integrated productive capacity are shown to play a crucial role, represents such « transformation » in logical terms. Only when $\pi = 0$, does « labour commanded » become equal to classical « labour embodied » (and prices to Marxian « values »), i.e.,

$$\mathbf{p} = \mathbf{v} , \tag{5.12}$$

while matrix $\mathbf{H}$ drops out of the picture altogether.

---

[13] The assumption is made, following Sraffa, that the internal rate of reproduction of non-basic commodities (if there are any) is higher than the internal rate of reproduction of basic commodities.

[14] Since no price can become negative in terms of *any* standard (within the interval, $0 \leq \pi \leq \Pi$), no price can fall faster than w as π is increased. It follows that π and w (in terms of *any* standard) must be inversely and monotonically related to each other. See the detailed proof given by Piero Sraffa, *op. cit.*, pp. 39-40.

6. A particular unit of measurement for capital goods

We may go back to the physical quantity system. So far in this analysis all commodities have been measured in terms of the physical units that are commonly used to measure them (e.g., tons, bushels, numbers, etc.). But expressions (4.7), (4.3 b) suggest the possibility of an alternative physical unit of measurement for capital goods. More precisely they suggest the possibility of measuring capital goods in terms of a particular composite commodity which we may call « physical unit of vertically integrated productive capacity ».

There clearly exists one such physical unit for each final good that is produced. If there are m final goods, there exist m physical units of vertically integrated productive capacity, represented by the columns of matrix $\mathbf{H}$, i.e. by,

$$\mathbf{h}_i = \mathbf{A}\,(\mathbf{I} - \mathbf{A}\Theta)^{-1}\,\mathbf{e}_i\,, \qquad i = 1, 2, \ldots m\,, \tag{6.1}$$

where $\mathbf{e}_i$ is the i^{th} unit column vector.

For the purpose of our analysis, a composite commodity does not present any conceptual difficulty. (As a matter of fact, any commodity — e.g., a pair of shoes — can always be considered as composed of various elementary commodities — i.e., leather, string, rubber, etc. — put together in fixed proportions). Therefore, when such units are used, the existing stocks of capital goods may be represented by an m component column vector,

$$\mathbf{K}(t) \equiv [K_i(t)]\,, \qquad i = 1, 2, \ldots m\,. \tag{6.2}$$

It follows from definition that, in equilibrium,

$$K_i(t) = Y_i(t)\,, \qquad i = 1, 2, \ldots m\,. \tag{6.3}$$

It is always possible to « translate » capital goods expressed in terms of vertically integrated productive capacities into capital goods expressed in ordinary physical units by the transformation

$$\mathbf{S}(t) = \mathbf{H}\,\mathbf{K}(t)\,. \tag{6.4}$$

Matrix $\mathbf{H}$ thereby appears as a linear operator which — when applied to a vector of physical quantities measured in terms of vertically integrated productive capacities — reclassifies them in terms of ordinary physical units. When $\mathbf{H}$ is a non-singular matrix, there even exists a unique inverse transformation

$$\mathbf{K}(t) = \mathbf{H}^{-1}\,\mathbf{S}(t)\,. \tag{6.5}$$

But of course $\mathbf{H}^{-1}$ need not necessarily exist. (That is: there may be more than one way, or there may be no exhaustive way, of forming vertically integrated units of productive capacity from arbitrarily given existing stocks of ordinary capital goods).

7. Vertically integrated sectors for investment goods expressed in physical units of vertically integrated productive capacity

When capital goods are measured in physical units of vertically integrated productive capacity, new investments (which are additions to the existing stocks of capital goods) must be measured in the same units. But new investments are considered to be final goods and we know that it is possible to conceptually construct a vertically integrated sector in correspondence to each final good. Such a logical construction has been obtained in section 4 for final goods measured in ordinary physical units. It now becomes possible to obtain similar logical constructions for investment goods measured in physical units of vertically integrated productive capacity.

We may denote by $\mathbf{J}^{(v)}(t) \equiv [J_i^{(v)}(t)]$, $i = 1, 2, \ldots m$, the column vector of new investments measured in units of vertically integrated productive capacity for the corresponding final goods $1, 2, \ldots m$. And by $\mathbf{J}_i^{(v)}(t)$, $i = 1, 2, \ldots m$, a column vector whose components are all zeros except the i^{th} one which is equal to $J_i^{(v)}(t)$. Obviously, $\sum_i^m \mathbf{J}_i^{(v)}(t) = \mathbf{J}^{(v)}(t)$. It follows from (6.4) that $\mathbf{J}^{(v)}(t)$ — new investments expressed in physical units of vertically integrated productive capacity — and $\mathbf{J}^{(d)}(t)$ — new investments expressed in ordinary (direct) physical units — are related by

$$\mathbf{J}^{(d)}(t) = \mathbf{H}\,\mathbf{J}^{(v)}(t)\,. \tag{7.1}$$

Similarly to what has been done in section 4, we may denote by $L^{(k_i)}(t)$ the labour services and by $\mathbf{X}^{(k_i)}(t)$, $\mathbf{S}^{(k_i)}(t)$, respectively, the column vectors of the physical quantities produced, and of the stocks of capital goods required, in the whole economic system, for the production of final good $J_i^{(v)}(t)$. Of course, $i = 1, 2, \ldots m$.

For each physical quantity $J_i^{(v)}(t)$ we may now write the corresponding sub-system:

$$\mathbf{X}^{(k_i)}(t) = (\mathbf{I} - \mathbf{A}\Theta)^{-1}\,\mathbf{H}\,\mathbf{J}_i^{(v)}(t)\,, \tag{7.2}$$

from which, after substitution into (2.2), (2.3), we obtain

$$L^{(k_i)}(t) = \mathbf{a}_{[n]}\,(\mathbf{I} - \mathbf{A}\Theta)^{-1}\,\mathbf{H}\,\mathbf{J}_i^{(v)}(t) \equiv \mathbf{v}\,\mathbf{H}\,\mathbf{J}_i^{(v)}(t)\,, \tag{7.3}$$

$$\mathbf{S}^{(k_i)}(t) = \mathbf{A}\,(\mathbf{I} - \mathbf{A}\Theta)^{-1}\,\mathbf{H}\,\mathbf{J}_i^{(v)}(t) \equiv \mathbf{H}^2\,\mathbf{J}_i^{(v)}(t)\,, \tag{7.4}$$

$$i = 1, 2, \ldots m\,.$$

Here again scalar $L^{(k_i)}(t)$ is the quantity of labour and vector $\mathbf{S}^{(k_i)}(t)$ is the series of stocks of capital goods directly and indirectly required in the whole economic system in order to produce quantity $J_i^{(v)}(t)$ of the investment good (measured in units of vertically integrated productive capacity) required for final good i. Therefore vector $\mathbf{vH}$ in (7.3), which we may call $\mathbf{v}_k$, i.e.,

$$(7.5) \qquad \mathbf{v}_k \equiv \mathbf{vH} \equiv \mathbf{a}_{[n]} (\mathbf{I} - \mathbf{A}\Theta)^{-1} \mathbf{A} (\mathbf{I} - \mathbf{A}\Theta)^{-1},$$

is a vector of vertically integrated labour coefficients and matrix

$$(7.6) \qquad \mathbf{H}^2 \equiv \mathbf{A} (\mathbf{I} - \mathbf{A}\Theta)^{-1} \mathbf{H} \equiv \mathbf{A} (\mathbf{I} - \mathbf{A}\Theta)^{-1} \mathbf{A} (\mathbf{I} - \mathbf{A}\Theta)^{-1},$$

in (7.4), is a matrix the columns of which represent units of vertically integrated productive capacity. Vector $\mathbf{v}_k$ and matrix $\mathbf{H}^2$ together represent the m vertically integrated sectors for the m investment goods expressed in units of vertically integrated productive capacity.

As was to be expected, the vertically integrated sectors for investment goods, expressed in physical units of vertically integrated productive capacity, have been obtained through a logical operation of vertical integration performed twice.

8. Prices of investment goods expressed in units of vertically integrated productive capacity

When investment goods are expressed in ordinary physical units, their prices are those found in section 5 (i.e., the prices of the commodities of system (2.1) — (2.4), whether they are used for consumption or for investment). But when investment goods are expressed in physical units of vertically integrated productive capacity, their prices — which we may denote by row vector $\mathbf{p}_k \equiv [p_{k_i}]$, $i = 1, 2, \dots m$ — are a weighted average of the prices $\mathbf{p}$ of their elementary components, namely

$$(8.1) \qquad \mathbf{p}_k = \mathbf{pH}.$$

After substitution from (5.4) and (7.5), we obtain

$$(8.2) \qquad \mathbf{p}_k = \mathbf{v}_k w + \mathbf{p}_k \mathbf{H} \pi.$$

This is a new price system in which prices, instead of being referred to the m ordinary commodities as in system (5.4), are referred to m composite commodities obtained by reclassifying the m ordinary commodities of system (5.4) by the operation of vertical integration (i.e., by multiplication by $\mathbf{H}$). Of course, the price system (5.4) and the price system (8.2) are equivalent. They yield the same maximum rate of profit. (As may be seen, Π emerges here, as in (5.4), as the reciprocal of the maximum eigenvalue of $\mathbf{H}$). And they yield the same maximum wage rate in terms of any pre-assigned standard. If we put $\pi = 0$ and $w = 1$, the components of $\mathbf{p}_k$ again turn out to be equal to the corresponding vertically integrated labour coefficients ($\mathbf{v}_k$ in this case). For all intermediate cases in which $0 < \pi < \Pi$,

$$(8.3) \qquad \mathbf{p}_k = \mathbf{v}_k (\mathbf{I} - \pi \mathbf{H})^{-1} w,$$

which gives for $\mathbf{p}_k$ precisely the same general expression that (5.9) gives for $\mathbf{p}$. All remarks and elaborations made for $\mathbf{p}$ in section 5 could therefore be repeated for $\mathbf{p}_k$ here.

9. Vertically integrated sectors of higher order

After performing the logical operation of vertical integration twice, it is natural to ask oneself whether there is any meaning in performing it a third time. The answer is straightforward. The units of vertically integrated productive capacity for investment goods, expressed in units of vertically integrated productive capacity, are themselves composite commodities. We may therefore conceptually construct the vertically integrated sectors for these newly found composite commodities. Such vertically integrated sectors clearly require a logical process of vertical integration to be performed three times. For analytical convenience, we may call such sectors « vertically integrated sectors of the third order » and, therefore, we may now call vertically integrated sectors of the second order, and vertically integrated sectors of the first order, respectively, the logical constructions obtained in section 7 and in section 4.

After using subscript k to denote the vertically integrated labour coefficients of the second order, we shall for consistency use subscript k^2 to denote the vertically integrated labour coefficients of the third order, to be obtained from the second order vertically integrated labour coefficients through post-multiplication by $\mathbf{H}$, i.e.,

$$(9.1) \qquad \mathbf{v}_{k^2} \equiv \mathbf{v}_k \mathbf{H} \equiv \mathbf{a}_{(n)} (\mathbf{I} - \mathbf{A}\Theta)^{-1} \mathbf{H}\,\mathbf{H}\,.$$

No new notation is needed for the matrix of the vertically integrated productive capacities of the third order, which clearly is $\mathbf{H}^3$.

These definitions now allow us to generalize the logical process of vertical integration to any higher order we may like. We can proceed from the vertically integrated sectors of the third order to the vertically integrated sectors of the fourth order, and from those of the fourth order to those of the fifth order, of the sixth order... , and so on step by step to the vertically integrated sectors of the s^{th} order, where s is any natural number as high as we may choose. Analytically each step in this process to a higher and higher order of vertical integration is simply represented by post-multiplication by matrix $\mathbf{H}$. The m units of vertically integrated productive capacity thus play a crucial role in the whole process.

In other terms, we may characterize the m vertically integrated sectors of the s^{th} order by:

a) a vector

$$(9.2) \qquad \mathbf{v}_{k^{s-1}} \equiv \mathbf{a}_{(n)} (\mathbf{I} - \mathbf{A}\Theta)^{-1} \underbrace{\mathbf{H} \ldots \mathbf{H}}_{(s-1)\ \text{times}} \equiv \mathbf{v}\,\mathbf{H}^{s-1} \equiv \mathbf{v}_{k^{s-2}} \mathbf{H}\,,$$

the components of which are the m s^{th} order vertically integrated labour coefficients; and by

b) a matrix

$$\mathbf{A}\,(\mathbf{I} - \mathbf{A}\Theta)^{-1}\,\underbrace{\mathbf{H}\ldots\mathbf{H}}_{(s-1)\text{ times}} \equiv \mathbf{H}\,\mathbf{H}^{s-1} \equiv \mathbf{H}^{s}, \tag{9.3}$$

the columns of which represent the m s^{th} order vertically integrated physical units of productive capacity.

Each series of m physical units of the s^{th} order vertically integrated productive capacity has of course (associated with it) its own series of m prices, which for consistency we shall denote by row vector $\mathbf{p}_{k^{s-1}}$. We clearly have

$$\mathbf{p}_{k^{s-1}} = \mathbf{p}_{k^{s-2}}\,\mathbf{H} = (\mathbf{v}_{k^{s-2}}\,w + \mathbf{p}_{k^{s-2}}\,\mathbf{H}\,\pi)\,\mathbf{H} = \mathbf{v}_{k^{s-1}}\,w + \mathbf{p}_{k^{s-1}}\,\mathbf{H}\pi, \tag{9.4}$$

from which we obtain

$$\mathbf{p}_{k^{s-1}} = \mathbf{v}_{k^{s-1}}\,(\mathbf{I} - \pi\,\mathbf{H})^{-1}\,w\,, \tag{9.5}$$

a remarkable general expression, of which (8.3) and (5.9) may be regarded as particular cases. All the theoretical remarks and elaborations made for prices $\mathbf{p}$ in section 5 could now be referred to prices $\mathbf{p}_{k^{s-1}}$ in general.

10. Higher order vertical integration and reduction of prices to a sum of weighted quantities of labour

The notion of higher order vertical integration may at first appear to be a very highly abstract notion indeed, and one may wonder whether any application of it can be found at all. But let us analyse the price system more deeply.

By using first order vertical integration, we have been able in section 5 to split up each price into its two basic components — wages and profits. When the wage rate itself is used as the *numeraire* — i.e., when w is put equal to unity — (5.4) actually becomes

$$\mathbf{p} = \mathbf{v} + \mathbf{p}\mathbf{H}\,\pi\,, \tag{10.1}$$

which shows the two components of prices in yet another light. The total purchasing power of prices, in terms of « labour commanded », is shown to be equal to « labour embodied » plus a residual absorbed by profits. A solution for $\mathbf{p}$ may of course be obtained immediately:

$$\mathbf{p} = \mathbf{v}\,(\mathbf{I} - \pi\,\mathbf{H})^{-1}, \tag{5.11}$$

as was done already in section 5. But an alternative procedure may also be followed — a procedure of successive approximations, which is conceptually far more interesting from a theoretical point of view.

Residual $\mathbf{pH}\pi$ contains the same prices that appear on the left hand side of (10.1). It may therefore itself be further split up into two components. After substitution from (8.1) and (8.2) we obtain

$$(10.2) \qquad \mathbf{pH}\pi \equiv \mathbf{p}_k\pi \equiv \mathbf{v}_k\pi + \mathbf{p}_k\,\mathbf{H}\pi^2 \,.$$

Second order vertical integration has thereby come on to the scene. The two components of $\mathbf{pH}\pi$ are shown to be: profits on the second order vertically integrated labour coefficients and a second order residual, itself containing $\mathbf{p}_k$. A chain argument has been started. Residual $\mathbf{p}_k\,\mathbf{H}\pi^2$ may itself be split up into two further components by using the notion of third order vertical integration. After substitution from (9.4) we obtain

$$(10.3) \qquad \mathbf{p}_k\,\mathbf{H}\pi^2 \equiv \mathbf{p}_{k^2}\,\pi^2 \equiv \mathbf{v}_{k^2}\,\pi^2 + \mathbf{p}_{k^2}\,\mathbf{H}\pi^3 \,,$$

which in turn shows the second order residual as a sum of the rate of profit (at the second power) on third order vertically integrated labour coefficients plus a third order residual containing prices, and itself liable to be split up into two further components. This logical chain may be pursued, step by step, to whatever degree we may choose. By using the same recurring formula (9.4), we obtain

$$(10.4) \qquad \begin{aligned} \mathbf{p}_{k^2}\,\mathbf{H}\,\pi^3 &\equiv \mathbf{p}_{k^3}\,\pi^3 \equiv \mathbf{v}_{k^3}\,\pi^3 + \mathbf{p}_{k^3}\,\mathbf{H}\,\pi^4 \,, \\ &\;\;\vdots \\ \mathbf{p}_{k^{s-1}}\,\mathbf{H}\,\pi^s &\equiv \mathbf{p}_{k^s}\,\pi^s \equiv \mathbf{v}_{k^s}\,\pi^s + \mathbf{p}_{k^s}\,\mathbf{H}\,\pi^{s+1} \,, \end{aligned}$$

where s is a natural number as high as we may choose. Each step may now be substituted back into the previous one, in (10.4), and then in (10.3), (10.2), (10.1), so as to obtain

$$(10.5) \qquad \mathbf{p} = \mathbf{v} + \mathbf{v}_k\,\pi + \mathbf{v}_{k^2}\,\pi^2 + \ldots \mathbf{v}_{k^s}\,\pi^s + \mathbf{p}_{k^s}\,\mathbf{H}\,\pi^{s+1} \,.$$

There still remains an $(s+1)^{th}$ order residual, but this residual can be made as small as may suit one's purpose by making s as great as is necessary. In the limit, as $s \to \infty$, the residual vanishes [15]

---

[15] The $(s+1)^{th}$ order residual, after substitution from recurring formula (9.4), may be written as

$$\mathbf{p}_{k^s}\,\mathbf{H}\,\pi^{s+1} = \mathbf{p}\,(\pi\,\mathbf{H})^{s+1} \,.$$

Supposing $\mathbf{p} > \mathbf{o}$, a necessary and sufficient condition for this expression to vanish, as $s \to \infty$, is $\lim_{s\to\infty} (\pi\,\mathbf{H})^s = \mathbf{O}$. This is precisely the case if $\pi < \dfrac{1}{|\lambda|_{max}}$. A proof can be given by using the similarity transformation of matrix $\mathbf{H}$

and prices entirely resolve themeselves into an infinite sum of weighted quantities of labour:

$$\text{(10.6)} \qquad \mathbf{p} = \mathbf{v} + \mathbf{v}_{k}\,\pi + \mathbf{v}_{k^2}\,\pi + \mathbf{v}_{k^3}\,\pi^3 + \ldots$$

The remarkable upshot of this succession is that at the first round of approximation we find the first order vertically integrated labour coefficients, at the second round of approximation we find the second order vertically integrated labour coefficients, at the third round the third order vertically integrated labour coefficients, and so on. Since these rounds go on to infinity, *all* higher order vertically integrated labour coefficients contribute to the logical process of finding the final solution.

The condition under which the infinite series (10.6) is convergent can be seen immediately upon substitution from (9.2). We obtain

$$\text{(10.7)} \qquad \mathbf{p} = \mathbf{v}\,[\mathbf{I} + \pi\,\mathbf{H} + \pi^2\,\mathbf{H}^2 + \pi^3\,\mathbf{H}^3 + \ldots],$$

where within square brackets appear in succession all the higher order units of vertically integrated productive capacity, appropriately weighted with the powers of the rate of profit. It is not difficult to see that the series is convergent provided that $\pi < \dfrac{1}{|\lambda|_{max}} = \Pi$ [16]. Only when $\pi = \Pi$, i.e., when all the purchasing power of prices is absorbed by profits, is the series not convergent, and prices can « command » an infinite quantity of labour. To the opposite extreme is the case in which $\pi = 0$, which makes all profit-weighted addenda vanish; and prices (in terms of « labour commanded ») become equal to the only unweighted addendum in the series — classical « labour embodied ». In between these two extremes, i.e. for $0 < \pi < \Pi$, the series is infinite and convergent. As may be noticed, the series actually corresponds to the well known iterative numerical method for obtaining the inverse of matrix $(\mathbf{I} - \pi\,\mathbf{H})$ which appears in (5.11)

---

called its *Jordan canonical form*, i.e., $\mathbf{F} = \mathbf{V}\,\mathbf{H}\,\mathbf{V}^{-1}$, where $\mathbf{V}$ is a square non-singular matrix and $\mathbf{F}$ is a matrix with all eigenvalues of $\mathbf{H}$ on its main diagonal and either zeros or ones on the diagonal next to the main one. Clearly $\mathbf{F}^s = \mathbf{V}\,\mathbf{H}^s\,\mathbf{V}^{-1}$. It can now be seen that if $\pi < \dfrac{1}{|\lambda|_{max}}$, all elements of $(\pi\,\mathbf{F})^s$ tend to zero as s tends to infinity. This ensures the tendency of $(\pi\,\mathbf{H})^s$ to $\mathbf{O}$ as s tends to infinity.

(16) This result is an immediate consequence of what is shown in the previous footnote, the series being a geometric one. The convergence of the infinite series (10.7), when $\pi < \dfrac{1}{|\lambda|_{max}}$, is a particular case of a more general theorem concerning functions of matrices. For a rigorous proof of this more general theorem, see for example: C.C. MacDuffee, « The Theory of Matrices », New York 1946, pp. 97 and ff.; Salvatore Cherubino, « Calcolo delle Matrici », Roma 1957, ch. IV.

In other words, the sum of the infinite series converges to inverse matrix

$$(\mathbf{I} - \pi\,\mathbf{H})^{-1}, \tag{10.8}$$

which means that step-by-step solution (10.6) converges to « exact » solution (5.11).

The notions of higher order vertically integrated labour coefficients have therefore the remarkable property of conferring an economic meaning of high theoretical relevance on each round of approximation to be carried out in the search for the price solution. They resolve the price of every commodity into a sum of profit-weighted quantities of labour. (17)

11. A « DUAL » EXERCISE

Matrix **H** and all its powers have a dual conterpart which, though not essential to the arguments of the present paper, will here be evinced explicitly for the sake of completeness. The analytical framework of the previous pages enables us to proceed very quickly at this stage, as we can start directly with an application that brings out all the dual notions at once.

Suppose that in the economic system considered so far the labour force is growing in time at the steady percentage rate $g > 0$ per annum, i.e.,

$$L(t) = L(0)\,[1 + g]^t. \tag{11.1}$$

And suppose that average *pro-capite* consumption is also constant through time, so that we may write **c** for the column vector of average *pro-capite* consumption coefficients. We have

$$\mathbf{C}(t) = \mathbf{c}\,\mu\,L(t), \tag{11.2}$$

where μ is the (constant) proportion of active to total population.

We shall consider the problem of finding a solution for the equilibrium (full employement) composition of total production $\mathbf{X}(t)$ in each year t. Of course, $\mathbf{X}(t)$ must include: commodities for consumption, commodities for new investments (i.e., for the expansion at rate g of all fixed and circulating capital goods, whether used for the production of consumption or of investment goods) and commodities for the replacement of all used-up capital goods (whether used up by production of consumption or of investment goods), i.e.,

(17) As may be realized, expression (10.6), by being the iterative solution of (5.11), also represents an iterative solution of Marx's «transformation problem». So Marx was not off the track, after all, when he sensed he could start from « values » and calculate profits directly on them. But he tried to settle the problem in one step, while what is needed is a long iterative process.

(11.3) $$\mathbf{X}(t) = \mathbf{C}(t) + g\,\mathbf{A}\,\mathbf{X}(t) + \mathbf{A}^{\ominus}\,\mathbf{X}(t).$$

This system of equations may of course be solved immediately for $\mathbf{X}(t)$. If we follow a slightly round-about way, we obtain

$$\mathbf{X}(t) = (\mathbf{I} - \mathbf{A}^{\ominus})^{-1}\,\mathbf{C}(t) + g\,(\mathbf{I} - \mathbf{A}^{\ominus})^{-1}\,\mathbf{A}\,\mathbf{X}(t),$$

(11.4) $$\mathbf{X}(t) = [\mathbf{I} - g\,(\mathbf{I} - \mathbf{A}^{\ominus})^{-1}\,\mathbf{A}]^{-1}\,(\mathbf{I} - \mathbf{A}^{\ominus})^{-1}\,\mathbf{C}(t).$$

We may now define a new matrix $\mathbf{G}$, i.e.,

(11.5) $$\mathbf{G} \equiv (\mathbf{I} - \mathbf{A}^{\ominus})^{-1}\,\mathbf{A},$$

which immediately appears as dual to $\mathbf{H}$. After substitution into (11.4) we may write

(11.6) $$\mathbf{X}(t) = (\mathbf{I} - g\,\mathbf{G})^{-1}\,(\mathbf{I} - \mathbf{A}^{\ominus})^{-1}\,\mathbf{C}(t).$$

This expression concerning physical quantities is clearly dual to expression (5.11) concerning prices. In general, of course,

$$\mathbf{G} \equiv (\mathbf{I} - \mathbf{A}^{\ominus})^{-1}\,\mathbf{A} \neq \mathbf{A}\,(\mathbf{I} - \mathbf{A}^{\ominus})^{-1} \equiv \mathbf{H}.$$

But $\mathbf{G}$ and $\mathbf{H}$ have exactly the same eigenvalues. In particular, $\lambda_{max} = |\lambda|_{max}$ is the maximum eigenvalue of both of them.

We may now proceed, as in the previous section, to finding the solution of (11.3) through the alternative procedure of successive approximations. Total production $\mathbf{X}(t)$ must certainly contain a batch of commodities, which we may call $\mathbf{X}^{I}(t)$, that provide for consumption goods $\mathbf{C}(t)$ and for all commodities that go to replace the used-up means of production for producing $\mathbf{C}(t)$, i.e.,

$$\mathbf{X}^{I}(t) = \mathbf{C}(t) + \mathbf{A}^{\ominus}\,\mathbf{X}^{I}(t),$$

(11.7) $$\mathbf{X}^{I}(t) = (\mathbf{I} - \mathbf{A}^{\ominus})^{-1}\,\mathbf{C}(t).$$

If the economic system were stationary (i.e., if $g = 0$), that would be all that is needed; $\mathbf{X}^{I}(t)$ would simply coincide with $\mathbf{X}(t)$ and this would be the end of the story. But we are supposing $g > 0$. Therefore, another batch of commodities, which we may call $\mathbf{X}^{II}(t)$, is needed for *expansion* of the capital goods needed for the production of $\mathbf{X}^{I}(t)$ and also for replacement of the capital goods to be used up for $\mathbf{X}^{II}(t)$, i.e.,

$$\mathbf{X}^{II}(t) = g\,\mathbf{A}\,\mathbf{X}^{I}(t) + \mathbf{A}^{\ominus}\,\mathbf{X}^{II}(t),$$

(11.8) $$\mathbf{X}^{II}(t) = g\,(\mathbf{I} - \mathbf{A}^{\ominus})^{-1}\,\mathbf{A}\,\mathbf{X}^{I}(t) \equiv g\,\mathbf{G}\,\mathbf{X}^{I}(t).$$

A chain argument has now been started. What has been said for $\mathbf{X}^{I}(t)$ must be repeated for $\mathbf{X}^{II}(t)$. A third batch of commodities $\mathbf{X}^{III}(t)$ is needed for expansion at growth rate g of $\mathbf{X}^{II}(t)$ and replacement of the corresponding capital goods, i.e., (11.9) $$\mathbf{X}^{III}(t) = g\,\mathbf{G}\,\mathbf{X}^{II}(t) = g^2\,\mathbf{G}^2\,\mathbf{X}^{I}(t)\,.$$

And so the chain argument goes on. A fourth batch of commodities $\mathbf{X}^{IV}(t)$ is needed for the successive round, and then a fifth batch of commodities, a sixth batch, a seventh, and so on to infinity:

(11.10) $$\begin{aligned} \mathbf{X}^{IV}(t) &= g\,\mathbf{G}\,\mathbf{X}^{III}(t) \equiv g^3\,\mathbf{G}^3\,\mathbf{X}^{I}(t)\,, \\ &\vdots \\ \mathbf{X}^{s}(t) &= g\,\mathbf{G}\,\mathbf{X}^{s-1}(t) \equiv g^{s-1}\,\mathbf{G}^{s-1}\,\mathbf{X}^{I}(t)\,. \\ &\vdots \end{aligned}$$

Total production $\mathbf{X}(t)$ clearly consists of the conceptual sum of the infinite serie

(11.11) $$\mathbf{X}(t) = \mathbf{X}^{I}(t) + \mathbf{X}^{II}(t) + \mathbf{X}^{III}(t) + \dots$$

or, after substitution from (11.7) - (11.10),

(11.12) $$\mathbf{X}(t) = [\mathbf{I} + g\,\mathbf{G} + g^2\,\mathbf{G}^2 + \dots]\,(\mathbf{I} - \mathbf{A}\Theta)^{-1}\,\mathbf{C}(t)\,,$$

where in square brackets appears the series of all the powers of matrix $\mathbf{G}$, appropriately weighted with the powers of g. This series in clearly dual to the series in (10.7), while expression $(\mathbf{I} - \mathbf{A}\Theta)^{-1}\mathbf{C}(t)$ is dual to expression $\mathbf{a}_{[n]}\,(\mathbf{I} - \mathbf{A}\Theta)^{-1} \equiv \mathbf{v}$. Again it is not difficult to see that the present series converges to inverse matrix

(11.13) $$(\mathbf{I} - g\,\mathbf{G})^{-1}\,,$$

provided only that $g < \dfrac{1}{|\lambda|_{max}}$, (18) which is exactly the same condition required for convergence of the series in (10.7). As was to be expected, step-by-step solution (11.12) converges to exact solution (11.6).

The problem remains of giving matrix $\mathbf{G}$ an explicit economic interpretation. We have seen in the previous pages that the columns of matrix $\mathbf{H}$ represent the m units of vertically integrated productive capacity — each column i of $\mathbf{H}$ represents the series of heterogeneous commodities directly and indirectly required as *capital good stocks* in the whole economic system in order to produce one physical unit of final good i ($i = 1, 2, \dots m$). The economic meaning of $\mathbf{G}$ is the exact dual counterpart. Each column j of matrix $\mathbf{G}$ represents the series of heterogeneous commodities directly and indirectly required

(18) The proof may be given along the same lines as those indicated with reference to $\pi < \dfrac{1}{|\lambda|_{max}}$ and matrix $\mathbf{H}$ in footnotes 15) and 16) above.

as *flows* in the whole economic system in order to produce all the stocks of capital goods necessary for one physical unit of commodity j $(j = 1, 2, \ldots m)$. While $\mathbf{H}$ is a matrix of stocks for the production of flows, $\mathbf{G}$ is a matrix of flows for the production of stocks.

And, of course, the logical process that leads to matrix $\mathbf{G}$ can be applied all over again, in the same way as the logical process leading to matrix $\mathbf{H}$ has been applied all over again in section 9. The flows represented by $\mathbf{G}$ themselves require stocks of capital goods, and the production of these stocks requires (directly and indirectly) the flows represented by $(\mathbf{I} - \mathbf{A}\Theta)^{-1}\,\mathbf{A}\,(\mathbf{I} - \mathbf{A}\Theta)^{-1}\,\mathbf{A} \equiv \mathbf{G}^2$. A further step back yields $\mathbf{G}^3$ and a still further step back yields $\mathbf{G}^4$, and so on. This logical process may be pursued to any higher order as we may choose; each step requiring pre-multiplication by $\mathbf{G}$. All these higher order notions represented by the powers of matrix $\mathbf{G}$, appropriately weighted with the powers of g, then appear in the infinite series (11.12), where they confer a specific economic meaning on the successive rounds of approximation to be carried out in the search for the equilibrium growth solution.

12. Production with fixed capital goods in general

The whole analysis has been carried out so far on the simplifying assumption that a constant proportion of all fixed capital goods drops out of the production process each year. This assumption may now be relaxed. In general, the way in which fixed capital goods wear out may vary widely from one industry to another and from one type of equipment to another. But in principle there is no difficulty in representing analytically any pattern of capital-good wear and tear. All capital goods may be considered, at the beginning of each year, to be entering the production process as particular inputs. Then, when, at the end of the year, they come out of the production process one year older, they may be considered as different commodities jointly produced with the commodities they contribute to produce.

This procedure requires each industry to be decomposed into as many « activities » as there are « years » in which the capital goods are used, each activity representing the same process of production but with a fixed capital good of a different age. Since each activity except one (the final one in which the capital good concerned drops out of the production process) produces jointly with the good that is produced also a capital good of a different age (which is considered as a different commodity), equality is maintained between the number of activities and the number of commodities.

Analytically, if technical coefficients remain constant through time, the technique for the whole economic system may be represented by:

i) a non-negative row vector of direct labour coefficients $\mathbf{a}_{(n)} \equiv [a_{nj}]$, $n = m + 1$, $j = 1, 2, \ldots m$, where j stands now for the j^{th} activity and m for the number of activities (and of commodities);

ii) a non-negative square matrix of commodity-input coefficients $\mathbf{A} \equiv [a_{ij}]$, $i, j = 1, 2, \ldots m$. This matrix includes all capital goods, both circulating and fixed, since all of them are considered as entering the production process as inputs at the beginning of the year;

iii) a non-negative square matrix of commodity-output coefficients $\mathbf{B} \equiv [b_{ij}]$; $i, j = 1, 2, \ldots m$. This matrix represents all commodities existing at the end of each year — consumption goods, and capital goods of all types, new and old.

To complete the notation a convention must be chosen regarding the normalization of all technical coefficients (i.e., regarding the scale to which each unit-activity is referred). And the choice made here is to refer all the coefficients on each column (activity) j of $\mathbf{a}_{[n]}$, $\mathbf{A}$, $\mathbf{B}$, to the physical unit produced of commodity $i = j$ $(i, j = 1, 2, \ldots m)$. This procedure has the convenient property of making all elements on the main diagonal of $\mathbf{B}$ equal to unity, (after suitable re-arrangement of rows and columns), and therefore of allowing us to make use of all notation defined in section 2.

When production takes place with fixed capital goods in general, the physical economic system is thus represented by systems of equations

$$(\mathbf{B} - \mathbf{A})\, \mathbf{X}(t) = \mathbf{Y}(t), \tag{12.1}$$

$$\mathbf{a}_{[n]}\, \mathbf{X}(t) = L(t), \tag{12.2}$$

$$\mathbf{A}\, \mathbf{X}(t) = \mathbf{S}(t), \tag{12.3}$$

and prices by system of equations

$$\mathbf{p}\,\mathbf{B} = \mathbf{a}_{[n]}\, w + \mathbf{p}\,\mathbf{A} + \mathbf{p}\,\mathbf{A}\,\pi. \tag{12.4}$$

In the particular case of constant proportion depreciation, the number of activities reduces to one in each industry; output matrix $\mathbf{B}$ reduces to indentity matrix $\mathbf{I}$; and input matrix $\mathbf{A}$, which appears both in (12.1) and on the second addendum of (12.4), reduces to $\mathbf{A}^{\Theta}$. General system of production with fixed capital goods (12.1) - (12.4) reduces to the previously considered particular system (2.1) - (2.4).

13. Generalizations and restrictions

The complications of production with fixed capital goods in general make it no longer possible to give an unambiguous meaning to the notion of « industry ». (Each industry may be made up of many activities each of which has its own labour coefficient and its own unit of direct productive capacity). But the notion of vertically integrated sector remains unaffected by complications. The whole economic system remains susceptible to being conceptually decomposed into m sub-systems precisely in the same way as has been done in section 4. The only formal difference is that matrix $(\mathbf{B} - \mathbf{A})^{-1}$ takes the place of matrix $(\mathbf{I} - \mathbf{A}\Theta)^{-1}$ in expressions (4.1), (4.2), (4.3). Therefore, for production with fixed capital goods in general, the vertically integrated labour coefficients and the physical units of vertically integrated productive capacity come to be defined respectively by the components and by the columns of

$$(13.1) \qquad \mathbf{v} \equiv \mathbf{a}_{[n]} (\mathbf{B} - \mathbf{A})^{-1},$$

$$(13.2) \qquad \mathbf{H} \equiv \mathbf{A} (\mathbf{B} - \mathbf{A})^{-1},$$

which represent a generalization of (4.6), (4.7). Similarly the vertically integrated sectors of higher order come to be defined by

$$(13.3) \qquad \mathbf{v}_{k^{s-1}} \equiv \mathbf{a}_{[n]} (\mathbf{B} - \mathbf{A})^{-1} \mathbf{H}^{s-1},$$

$$(13.4) \qquad \mathbf{H}^{s} \equiv [\mathbf{A} (\mathbf{B} - \mathbf{A})^{-1}]^{s},$$

(where s is any positive natural number), which represent a generalization of (9.2), (9.3).

What becomes more difficult to do, in the case of production with fixed capital goods in general, is to devise a neat way of discriminating between the cases in which the above expressions have and the cases in which they do not have an economic meaning. In the simplified case of the previous pages the procedure is clear. Non-negativity of $\mathbf{A}\Theta$ is sufficient to ensure non-negativity of $(\mathbf{I} - \mathbf{A}\Theta)^{-1}$. But here the fact that both $\mathbf{B}$ and $\mathbf{A}$ are non-negative *ex-hypothesis* does not necessarily imply that $(\mathbf{B} - \mathbf{A})^{-1}$, and as a consequence $\mathbf{v}$, $\mathbf{H}$ and $\mathbf{G}$, should also be non-negative. Actually $\mathbf{v}$, $\mathbf{H}$ and $\mathbf{G}$ might indeed contain some negative elements and still make good economic sense. What we can say is that, since prices cannot be negative, $\mathbf{v}$ cannot be accepted as economically meaningful if it contains negative components when the rate of profit is zero. But there is nothing to prevent prices from all being positive, even if some components of $\mathbf{v}$ are negative, when the rate of profit is positive. And in this case a vector $\mathbf{v}$ with some negative components would make perfectly good economic sense. Similarly, we can say that if a particular column k of matrix $\mathbf{H}$ contains some negative components, the production of commodity k alone as a final good would require some activities to be run in reverse, and this would be impossible (and thus would have no economic sense). But commodity k might not be produced as a final good at all, or there might be no necessity to produce it alone (the sub-systems are only conceptual, not real, constructions). And in this case too a matrix $\mathbf{H}$ with some negative components would make perfectly good economic sense. A similar (but dual) argument can be developed for matrix $\mathbf{G}$.

In any case the prices of the m commodities, expressed in ordinary physical units, continue to be given by the formulations of section 5 above, again with the only difference that more general matrix $(\mathbf{B} - \mathbf{A})$ is to replace $(\mathbf{I} - \mathbf{A}^{\ominus})$, and more general matrix $\mathbf{A}$ to replace $\mathbf{A}^{\ominus}$. Similarly the prices of the *m* composite commodities, expressed in physical units of vertically integrated productive capacity of any order, continue to be expressed by (9.4), (9.5), with the more general definitions of $\mathbf{v}$ and $\mathbf{H}$ given by (13.1), (13.2). Actually, if both $\mathbf{v}$ and $\mathbf{H}$ happen to be non-negative, all remarks made in section 5 hold good in their entirety. In the case in which $\mathbf{v}$ and/or $\mathbf{H}$ do happen to contain some negative elements, what is no longer certain is that prices should remain all non-negative (i.e., economically meaningful) at *all* levels of the rate of profit; and as a consequence that the relation between w and π should always be inverse and monotonic in terms of *all* commodities. However, the remarks made in section 5 on the relationship between the classical notions of « labour commanded » and « labour embodied » continue to hold.

But the most interesting results of all refer to the elaborations of sections 10 and 11, which do continue to hold. The step-by-step solution for prices $\mathbf{p}$ continues to be represented by infinite series (10.6) or (10.7), with the vertically integrated units of productive capacity of all orders $\mathbf{H}$, $\mathbf{H}^2$, $\mathbf{H}^3$, ... and with the vertically integrated labour coefficients of all orders, $\mathbf{v}$, $\mathbf{v}_k$, $\mathbf{v}_{k^2}$, ..., being defined by more general expressions (13.1) - (13.4). The condition of convergence of the series is again the same, i.e. $\pi < \dfrac{1}{|\lambda|_{max}}$ (19). Similarly, the step-by-step solution for total production $\mathbf{X}(t)$ continues to be expressed by the infinite series (11.12), with more general matrix $(\mathbf{B} - \mathbf{A})^{-1}$ in the place of $(\mathbf{I} - \mathbf{A}^{\ominus})^{-1}$. The series is again convergent for $g < \dfrac{1}{|\lambda|_{max}}$.

What must be added here is that we can no longer be certain that the eigenvalue of $\mathbf{H}$ and $\mathbf{G}$ which is maximum in modulus — i.e. $|\lambda|_{max}$ — is also the eigenvalue which is economically relevant. If there exists a $|\lambda|_{max} > \lambda_e$, where λ_e represents the economically relevant eigenvalue, the series (10.7) and (11.12) converge for all π and g smaller than $\dfrac{1}{|\lambda|_{max}}$, but do not converge for π and $g \geq \dfrac{1}{|\lambda|_{max}}$. In other words (and with reference to π, for the sake of brevity, since the same thing can be repeated for g), if we define $\pi^{\bullet} = \dfrac{1}{|\lambda|_{max}}$, the series (10.7) converges for all rates of profit within the range $0 < \pi < \pi^{\bullet}$. In those cases in which λ in $|\lambda|_{max}$ is a real and positive number, $\pi^{\bullet}$ coincides with Π, and (10.7) converges, as before, for all economically significant rates of profit up to Π (but not at, or

(19) See footnotes 15) and 16) above.

beyond, II). In other more complicated cases in which $\pi^* < \pi_e$, where $\pi_e = \frac{1}{\lambda_e}$, π^* becomes the new critical level of the rate of profit. The series (10.7) converges for all rates of profit up to π^*, but not at or beyond, π^*. It is however important to realize that $|\lambda|_{max}$ is finite. Therefore π^* is in any case positive. This means that, from zero upwards (even in the most complicated cases of joint production!), there always exists a range of positive rates of profit within which the series (10.7) is convergent.

The reduction of prices to a sum of weighted quantities of labour is thereby revealed to be a result of great generality. The series

$$\mathbf{p} = \mathbf{v} + \mathbf{v}_k \pi + \mathbf{v}_{k^2} \pi^2 + \mathbf{v}_{k^3} \pi^3 + \ldots \tag{13.1}$$

where $\mathbf{v}_{k^s} \equiv \mathbf{v} [\mathbf{A} (\mathbf{B} - \mathbf{A})^{-1}]^s$, clearly represents a generalization of Piero Sraffa's reduction of prices to *dated* quantities of labour (which is only possible in the case of single-product industries). The logical process of infinite successive vertical integration is thus revealed to be more general, and to go much deeper, than the logical process of infinite chronological decomposition. A generalization of this type, with all its theoretical implications, is no doubt one of the most remarkable results of the present analysis.

14. Technical progress

So far in our analysis all technical coefficients have been supposed to be absolutely constant through time. But the notion of a vertically integrated sector is not only unaffected by technical change; it actually acquires greater relevance when technical change is present. In particular the notion of a physical unit of productive capacity, by being defined with reference to the commodity that is produced, continues to make sense, as a physical unit, whatever complications technical change may cause to its composition in terms of ordinary commodities.

If there is technical progress in the economic system, we may suppose, for consistency with our previous analysis, that changes take place at discontinuous points in time. Technical coefficients may be supposed to change at the beginning of each year; and then remain constant during the year. With this convention, the whole previous analysis may simply be re-interpreted as referring to a particular year t. This means that all magnitudes considered in the previous pages must be *dated*. Not only physical quantities $\mathbf{X}(t)$, $\mathbf{Y}(t)$, $\mathbf{S}(t)$, etc., but also prices $\mathbf{p}(t)$, technique $\mathbf{a}_{(n)}(t)$, $\mathbf{A}(t)$, $\mathbf{B}(t)$, and, as a consequence, vertically integrated sectors $\mathbf{v}(t)$, $\mathbf{H}(t)$; $\mathbf{v}_k(t)$, $\mathbf{H}^2(t)$; $\mathbf{v}_{k^2}(t)$, $\mathbf{H}^3(t)$; etc., must be written with a time suffix.

A distinction, however, has to be made at this point between two types of technical progress.

a) *Disembodied technical progress.* We may call 'disembodied' technical progress those improvements that do not affect the technical characteristics of capital goods, and simply enable production of larger physical quantities of commodities out of existing capital goods. Analytically this type of technical progress is expressed by the diminution of some technical coefficients and presents no difficulty. Capital goods, measured in ordinary physical units, remain the same, but their relations to capital goods measured in terms of vertically integrated productive capacities change as time goes on. This means that a particular matrix $\mathbf{H}(t)$ expressing the relation between the two types of units comes into existence for each particular year t; so that an appropriate $\mathbf{H}(t)$ has to be used in each year in order to go from the vertically integrated units to the ordinary ones.

b) *Embodied technical progress.* We may call 'embodied' technical progress those improvements that need to be embodied into specific (new) capital goods. In particular these improvements are supposed to be embodied into such new capital goods that render the old ones either entirely or partially obsolete, in the sense that the old capital goods, even if they continue to be used for the time being, will never be replaced by physical capital goods of the same type when they will be replaced. It must therefore be specified that notation $\mathbf{a}_{(n)}(t)$, $\mathbf{A}(t)$, $\mathbf{B}(t)$ is to be understood as denoting the latest technique for the whole economic system, as this is known at the beginning of year t, so that $\mathbf{v}(t) \equiv \mathbf{a}_{(n)}(t)\,[\mathbf{B}(t) - \mathbf{A}(t)]^{-1}$, $\mathbf{H}(t) \equiv \mathbf{A}(t)\,[\mathbf{B}(t) - \mathbf{A}(t)]^{-1}$ represent the corresponding vertically integrated sectors *as they would be* if the technique of time t had been known in the past and the composition of the capital goods had thereby become evenly balanced. This means that technique $\mathbf{a}_{(n)}(t)$, $\mathbf{A}(t)$, $\mathbf{B}(t)$, and corresponding vertically integrated sectors $\mathbf{v}(t)$, $\mathbf{H}(t)$ represent *hypothetical* magnitudes in this case. The *actual* economic system, if it is to be represented in the same way, requires a different notation; for example we may write $\bar{\mathbf{a}}_{(n)}(t)$, $\bar{\mathbf{A}}(t)$, $\bar{\mathbf{B}}(t)$, to denote the technique which is actually in operation in year t (a mixture of activities of different « vintages »). Then $\bar{\mathbf{v}}(t) \equiv \bar{\mathbf{a}}_{(n)}(t)\,[\bar{\mathbf{B}}(t) - \bar{\mathbf{A}}(t)]^{-1}$, $\bar{\mathbf{H}}(t) \equiv \bar{\mathbf{A}}(t)\,[\bar{\mathbf{B}}(t) - \bar{\mathbf{A}}(t)]^{-1}$ will represent the corresponding *actual* vertically integrated sectors. It goes without saying that all cases considered so far, including that of embodied technical progress, may be regarded as particular cases in which hypothetical and actual vertically integrated sectors happen to coincide.

Of course, both the hypothetical and the actual vertically integrated sectors are relevant — when they are distinct fom each other — but for different purposes. The *hypothetical* vertically integrated sectors are crucial to the determination of prices, as they express the latest technique. The *actual* vertically integrated sectors become relevant for the purpose of representing the physical economic system.

15. THE PARTICULAR CASE OF CAPITAL GOODS PRODUCED BY LABOUR ALONE

It becomes rather simple at this point to go back to the multi-sector model of economic growth of *New Theoretical Approach*, and view it as a particular case of the analysis of the previous pages. The assumption that all capital goods are made by labour alone, and that they wear out according to a constant proportion $\left[\frac{1}{T}\right]$, makes it a particular case of the analysis of the first sections of the present paper. And the assumption that technical progress takes place by diminution of all labour coefficients makes it a particular case of disembodied technical progress.

Such a simplified economic system possesses many convenient properties. The vertically integrated labour coefficients for consumption goods are expressed by the sum of the direct labour coefficients plus the quantities of labour required by replacements [20], and the vertically integrated labour coefficients for investment goods are simply expressed by direct labour coefficients. The vertically integrated units of productive capacity for consumption goods are expressed by unit vectors, and those for investment goods by zero vectors. Second-order vertical integration is even simpler. In consumption good industries, the second order vertically integrated labour coefficients for the capital goods (measured in units of vertically integrated productive capacity) coincide with their direct labour coefficients; and in investment good industries the second order vertically integrated labour coefficients are all zero. Finally, the second order vertically integrated units of productive capacity are all represented by zero vectors. The vertically integrated sectors of any higher order are all zero.

In matrix notation, if we denote by $\mathbf{O}_i$ the null square matrix of the i^{th} order, and by $\mathbf{I}_i$ the identity matrix of the i^{th} order, the matrices defined in the previous pages (using the symbols adopted in *New Theoretical Approach*) reduce to the following:

$$\mathbf{A} = \begin{bmatrix} \mathbf{O}_{n-1} & \mathbf{O}_{n-1} \\ \mathbf{I}_{n-1} & \mathbf{O}_{n-1} \end{bmatrix} \quad ; \quad \mathbf{A}^{\Theta} = \begin{bmatrix} \mathbf{O}_{n-1} & \mathbf{O}_{n-1} \\ \frac{1}{T}\,\mathbf{I}_{n-1} & \mathbf{O}_{n-1} \end{bmatrix} \quad ;$$

(20) It may be useful, in this respect, to point out a misleading formulation that appears on p. 669 of *New Theoretical Approach*. The vector on the left hand side of equality (VI.3) is written with symbols a_{ni}, $i = 1, 2, \ldots n\text{-}1$, which — in the previous chapters — are used to indicate *direct* labour coefficients. What should have been done was to use a new symbol, for example — as in the present analysis — $v_1, v_2, \ldots v_{n-1}$.

$$(\mathbf{I} - \mathbf{A}^{\Theta})^{-1} = \begin{bmatrix} \mathbf{I}_{n-1} & \mathbf{O}_{n-1} \\ \frac{1}{T}\mathbf{I}_{n-1} & \mathbf{I}_{n-1} \end{bmatrix} ; \quad \mathbf{H} \equiv \mathbf{A}\,(\mathbf{I} - \mathbf{A}^{\Theta})^{-1} = \begin{bmatrix} \mathbf{O}_{n-1} & \mathbf{O}_{n-1} \\ \mathbf{I}_{n-1} & \mathbf{O}_{n-1} \end{bmatrix} ;$$

$$\mathbf{H}^2 = \mathbf{O} \quad ; \quad (\mathbf{I} - \pi\,\mathbf{H})^{-1} = \begin{bmatrix} \mathbf{I}_{n-1} & \mathbf{O}_{n-1} \\ \pi\,\mathbf{I}_{n-1} & \mathbf{I}_{n-1} \end{bmatrix} \quad ;$$

$$\mathbf{a}_{[n]}(t) = [a_{n1}(t)\; a_{n2}(t) \ldots a_{n,n-1}(t)\;\; a_{nk_1}(t) \ldots a_{nk_{n-1}}(t)] ;$$

$$\mathbf{v}(t) \equiv \mathbf{a}_{[n]}(t)\,(\mathbf{I} - \mathbf{A}^{\Theta})^{-1} = [(a_{n1}(t) + \frac{1}{T}\,a_{nk_1}(t)) \ldots (a_{n,n-1}(t) + $$
$$+ \frac{1}{T}\,a_{nk_{n-1}}(t))\;\; a_{nk_1}(t) \ldots a_{nk_{n-1}}(t)\,] ;$$

$$\mathbf{v}_k(t) \equiv \mathbf{v}(t)\,\mathbf{H} = [a_{nk_1}(t)\;\; a_{nk_2}(t) \ldots a_{nk_{n-1}}(t)\;\; 0\;\; 0 \ldots 0] ;$$

$$\mathbf{v}_{k_2}(t) \equiv \mathbf{v}_k(t)\,\mathbf{H} = [0\;\; 0 \ldots 0] .$$

By substituting these particular expressions into (5.9) or (10.6), we obtain

$$p_i(t) = \left[a_{ni}(t) + \left(\frac{1}{T} + \pi\right) a_{nk_i}(t)\right] w(t) ,$$

$$p_{k_i}(t) = a_{nk_i}(t)\; w(t) ,$$

which are precisely the « solutions » for prices given in the original formulation [21].

---

[21] See *New Theoretical Approach*, p. 597. A more complex case is considered on pp. 598-601, in which the capital goods produced by the investment good industries are supposed to be used both in the consumption good industries and in the investment good industries, given a proportion γ_i, $i = 1, 2, \ldots n\text{-}1$, between their productive capacities for the two types of industries. In this case, the matrices defined here (again by using the symbols of *New Theoretical Approach*, and supposing for notational simplicity that T_k, T, γ, π are all uniform) reduce to the following:

$$\mathbf{A} = \begin{bmatrix} \mathbf{O}_{n-1} & \mathbf{O}_{n-1} \\ \mathbf{I}_{n-1} & \gamma\mathbf{I}_{n-1} \end{bmatrix} \quad ; \quad \mathbf{A}^{\Theta} = \begin{bmatrix} \mathbf{O}_{n-1} & \mathbf{O}_{n-1} \\ \mathbf{I}_{n-1} & \gamma\frac{1}{T_k}\mathbf{I}_{n-1} \end{bmatrix} \quad ;$$

$$(\mathbf{I} - \mathbf{A}^{\Theta})^{-1} = \begin{bmatrix} \mathbf{I}_{n-1} & \mathbf{O}_{n-1} \\ \frac{1}{T}\,\frac{T_k}{T_k - \gamma}\,\mathbf{I}_{n-1} & \frac{T_k}{T_k - \gamma}\,\mathbf{I}_{n-1} \end{bmatrix} \quad ;$$

$$\mathbf{a}_{[n]}(t) = [a_{n1}(t)\; a_{n2}(t) \ldots a_{n,n-1}(t)\;\; a_{nk_1}(t) \ldots a_{nk_{n-1}}(t)] ;$$

$$\mathbf{A}\,(\mathbf{I} - \mathbf{A}^{\Theta})^{-1} \equiv \mathbf{H} = \begin{bmatrix} \mathbf{O}_{n-1} & \mathbf{O}_{n-1} \\ \left(1 + \frac{1}{T}\,\gamma\,\frac{T_k}{T_k - \gamma}\right)\mathbf{I}_{n-1} & \gamma\,\frac{T_k}{T_k - \gamma}\,\mathbf{I}_{n-1} \end{bmatrix} ;$$

16. New analytical possibilities for dynamic analysis

But the use made of vertically integrated sectors in *New Theoretical Approach* has also been aimed at the wider purpose of opening up new possibilities for dynamic analysis.

In the general case of production of all commodities by means of fixed capital goods and of technical progress of the most general type, the relation between ordinary physical capital goods and capital goods in units of productive capacity breaks down at the end of each period and the problem arises of what meaning one can give to the physical operation of replacement of the capital goods. Clearly « replacement » ceases to have any meaningful sense in terms of ordinary physical units. On the other hand, « replacement » does continue to make sense in terms of physical units of productive capacity. Even in the midst of a maze of physical and qualitative changes, we may indeed continue to say that replacement of used-up capital goods has taken place if, at the end of each period, the economic system has recovered the same productive capacities as it had at the beginning.

The analytical consequences of these remarks are far reaching. With technical progress, any relation in which capital goods are expressed in ordinary physical units becomes useless for dynamic analysis. But relations expressed in physical units of productive capacity continue to hold through time, and actually acquire an autonomy of their own, quite independently of their changing composition. At the same time the elaborations of the previous pages provide the way for a return to the ordinary physical units any time that this is necessary, within each period t. Of course a different result will be obtained for each single period.

This property seems to me to confer on the logical process of vertical integration an analytical relevance for dynamic investigations which perhaps has not been completely realized as yet. The vertically inte-

---

$$(\mathbf{I}-\pi\mathbf{H}) = \begin{bmatrix} \mathbf{I}_{n-1} & \mathbf{O}_{n-1} \\ -\pi\left(1+\frac{1}{T}\gamma\frac{T_k}{T_k-\gamma}\right)\mathbf{I}_{n-1} & \left(1-\pi\gamma\frac{T_k}{T_k-\gamma}\right)\mathbf{I}_{n-1} \end{bmatrix};$$

$$(\mathbf{I}-\pi\mathbf{H})^{-1} = \begin{bmatrix} \mathbf{I}_{n-1} & \mathbf{O}_{n-1} \\ \frac{\pi}{T}\,\frac{\gamma T_k+(T_k-\gamma)T}{T_k-\gamma-\pi\gamma T_k}\mathbf{I}_{n-1} & \frac{T_k-\gamma}{T_k-\gamma-\pi\gamma T_k}\mathbf{I}_{n-1} \end{bmatrix};$$

$$\mathbf{v} = \Big[\Big(a_{n1}(t)+\frac{1}{T}\,\frac{T_k}{T_k-\gamma}a_{nk_1}(t)\Big)\ldots\Big(a_{n,n-1}(t)+\frac{1}{T}\,\frac{T_k}{T_k-\gamma}a_{nk_{n-1}}(t)\Big)$$

$$\frac{T_k}{T_k-\gamma}a_{nk_1}(t)\ldots\frac{T_k}{T_k-\gamma}a_{nk_{n-1}}(t)\Big]$$

Here again, as can easily be checked, post-multiplication of $\mathbf{v}$ by $(\mathbf{I}-\pi\mathbf{H})^{-1}$, and by w, yields the expressions for prices given on p. 600 of *New Theoretical Approach*.

grated sectors seem to belong to that category of synthetic notions which, once obtained, contribute to reduce in many directions the very order of magnitude of the analytical difficulties. An example of this is given after all by the multi-sector model of economic growth, from which the present analysis has started, which has permitted the investigation of a whole series of structural dynamic relations — something which would have been impossible to do with any traditional growth model.

It may not be too unjustified to hope that a better understanding, and a more explicit utilization, of the logical process of vertical integration might help to overcome the widely recognized failure of modern economic theory to come to grips with the analytical difficulties of technical change.