Trigonometric Functions on the Perimeter of any Regular Polygon

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1 Introduction

This paper derives and proves a set of functions that serve to find the point of intersection between an angle in standard position and an n-sided regular polygon through only algebra. Through these, the standard equation for any regular polygon can be derived.

2 Theroems

Theorem 1.1 $\arccos(\cos(ax))$ is periodic over $\frac{2\pi}{a*n}$.
$\cos(ax)$ is periodic over $\frac{2\pi}{a*n}$ thus $\arccos(\cos(ax))$ must also be.
Theorem 1.2 For $x \in \mathbb{R}$ $\arccos(\cos(ax)) \in [0, \pi]$.
$\cos(x)$ has a domain over $\mathbb{R}$ and a range of [-1,1], $\arccos(x)$ has a domain over [-1,1] and a range of $[0, \pi]$. Due to the range of cos being the domain of arccos, all real numbers can be an input of x, and all numbers on the interval [-1,1] can be an output.
Theorem 1.3 $\cos(n(a + \frac{2\pi}{\mathrm{n}})) = \cos(na)$
$\cos(n(a + \frac{2\pi}{\mathrm{n}}))$ expands to $\cos(na + 2\pi))$ which due to cosine's periodic nature evaluates to $\cos(na)$
Theorem 2.1 $\sin(\alpha) - \sin(\beta) = 2\sin(\frac{\alpha+\beta}{2})\cos(\frac{\alpha-\beta}{2})$
Theorem 2.2 $\cos(\alpha) - \cos(\beta) = -2\sin(\frac{\alpha+\beta}{2})\sin(\frac{\alpha-\beta}{2})$

3 Trigonometric Equations

Let $n = \{x : x \in \mathbb{Z}^+ \text{ and } x \geq 3\}$

$$\mathrm{d_n}(\theta) = \cos(\frac{\pi}{\mathrm{n}})\sec(\frac{\pi - \arccos(\cos(n\theta))}{n})$$
$$\cos_n(\theta) = \mathrm{d_n}(\theta)\cos(\theta)$$
$$\sin_n(\theta) = \mathrm{d_n}(\theta)\sin(\theta)$$

This is derived through the original functions:

$$\mathrm{h_n}(\theta) = \sin(\frac{\pi}{2} - \frac{\pi}{\mathrm{n}} + (\theta \mod \frac{2\pi}{\mathrm{n}})) - \sin(\frac{\pi}{2} - \frac{\pi}{\mathrm{n}})$$
$$\mathrm{d_n}(\theta) = \frac{\mathrm{h_n}(\theta)}{\sin(\frac{\pi}{2} - \frac{\pi}{\mathrm{n}} + (\theta \mod \frac{2\pi}{\mathrm{n}}))}$$
$$\cos_n(\theta) = (1 - \mathrm{d_n}(\theta))\cos(\theta)$$
$$\sin_n(\theta) = (1 - \mathrm{d_n}(\theta))\sin(\theta)$$

These functions follow the premise that any n-sided regular polygon (shape $\mathrm{s_n}$) centered at (0,0) can be circumscribed within a circle where each vertex touches the circumference of the circumcircle.

Arc $\mathrm{a_n}$ can be constructed which is $\frac{1}{\mathrm{n^{th}}}$ of a circle, its diameter will be the same length as any line which constructs $\mathrm{s_n}$, and it's circumference will equal to the circumference of the section of the circumcircle spanning from adjacent vertices on $\mathrm{s_n}$.
Thus, the height of $\mathrm{a_n}$ at a given point on a circle subtracted from the radius of the circle will result in the distance from the midpoint of $\mathrm{s_n}$ to the point on the perimeter of $\mathrm{s_n}$ which intersects the ray of angle θ drawn in standard position.
$\mathrm{h_n}(\theta)$ serves to find the height of $\mathrm{a_n}$ at a position by splitting the circumcircle into $\mathrm{n^{ths}}$ (resulting in a period of $\frac{2\pi}{\mathrm{n}}$), finding the height of a point on $\mathrm{a_n}$ through sin and changing from the "left" vertex to the "right" vertex over the period of $\mathrm{a_n}$. The use of mod makes the function repeat over each period. It then subtracts by the height of the "right" vertex to equal 0 when $\theta = 0$
$\mathrm{d_n}(\theta)$ divides the height ($\mathrm{h_n}(\theta)$) by the angle between the assumed intersection point on $\mathrm{s_n}$ and the intersection point on the circumcircle to get the actual distance.
These functions then simplify to:

$$\mathrm{d_n}(\theta) = \cos(\frac{\pi}{\mathrm{n}})\sec(\frac{\pi}{\mathrm{n}} - (\theta \mod \frac{2\pi}{\mathrm{n}}))$$
$$\cos_n(\theta) = \mathrm{d_n}(\theta)cos(\theta)$$
$$\sin_n(\theta) = \mathrm{d_n}(\theta)sin(\theta)$$

Which is not algebraic due to mod (and the resultant floor function behind it), yet it can be noted that due to the nature of the original $\mathrm{h_n}(\theta)$, $\theta \mod \frac{2\pi}{\mathrm{n}}$ can be substituted for an equivalent function which has an equal period of $\frac{2\pi}{\mathrm{n}}$, increases lineraly over $[0, \frac{2\pi}{\mathrm{n}}]$ to $[0, \frac{\pi}{\mathrm{n}}]$ and decreases linearly at the same rate from $[\frac{\pi}{\mathrm{n}}, \frac{2\pi}{\mathrm{n}}]$.
These requirements can be met by $\frac{\arccos(\cos(n\theta))}{n}$, which due to **Theorem 1.1** is periodic over $\frac{2\pi}{\mathrm{n}}$ and due to **Theorem 1.2** has a domain of $n\theta \in \mathbb{R}$ and a range of $[0, \frac{\pi}{n}]$ thus changing the above equations to:

$$\mathrm{d_n}(\theta) = \cos(\frac{\pi}{\mathrm{n}})\sec(\frac{\pi - \arccos(\cos(n\theta))}{n})$$
$$\cos_n(\theta) = \mathrm{d_n}(\theta)cos(\theta)$$
$$\sin_n(\theta) = \mathrm{d_n}(\theta)sin(\theta)$$

3.1 Identities

Identity 1.1

$$\cos_n(\theta+\frac{2\pi}{\mathrm{n}})=\mathrm{d_n}(\theta)\cos(\theta+\frac{2\pi}{\mathrm{n}})$$
$$\sin_n(\theta+\frac{2\pi}{\mathrm{n}})=\mathrm{d_n}(\theta)\sin(\theta+\frac{2\pi}{\mathrm{n}})$$

This identity is true due to **Theorem 1.3**
Identity 1.2 When $0\le\theta\le\frac{2\pi}{\mathrm{n}}$ then $\mathrm{d_n}(\theta)=\cos(\frac{\pi}{\mathrm{n}})\sec(\frac{\pi}{\mathrm{n}}-\theta)$
When θ only spans one period, $\frac{\arccos(\cos(n\theta))}{n}$ is not required to make the function periodic.
Identity 1.3 Let $\mathrm{i}=\mathbb{Z}$ then:

$$\cos_n(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})=(1)\cos(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})=\cos(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})$$
$$\sin_n(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})=(1)\sin(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})=\sin(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})$$

This is true because $\arccos(\cos(n\theta))$ evaluates to 0 whenever $\theta=\mathrm{i}\,\frac{2\pi}{\mathrm{n}}$, which means $\sec(\frac{\pi}{\mathrm{n}}-\frac{\arccos(\cos(n\theta))}{n})$ simplifies to $\sec(\frac{\pi}{\mathrm{n}})$ which means $\mathrm{d_n}=\sec(\frac{\pi}{\mathrm{n}})\cos(\frac{\pi}{\mathrm{n}})=1$

4 Proof

Let $\mathrm{i}=\mathbb{Z}$. Assume $\mathrm{s_n}$ is made of n vertices and n line segments with the $\mathrm{i^{th}}$ vertex at the point $(\cos(\mathrm{i}\,\frac{2\pi}{\mathrm{n}}),\sin(\mathrm{i}\,\frac{2\pi}{\mathrm{n}}))$. The $\mathrm{i^{th}}$ line segment, $\mathrm{L_i}$, spans between v_i and v_{i+1} and is represented by the equation

$$\mathrm{L_i}=\frac{\sin((\mathrm{i}+1)\,\frac{2\pi}{\mathrm{n}})-\sin(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})}{\cos((\mathrm{i}+1)\,\frac{2\pi}{\mathrm{n}})-\cos(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})}x+\sin(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})-\frac{\sin((\mathrm{i}+1)\,\frac{2\pi}{\mathrm{n}})-\sin(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})}{\cos((\mathrm{i}+1)\,\frac{2\pi}{\mathrm{n}})-\cos(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})}\cos(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})$$

which due to **Theorem 2.1** and **Theorem 2.2** simplifies to

$$\mathrm{L_i}=-\cot((2\,\mathrm{i}+1)\,\frac{\pi}{\mathrm{n}})x+\sin(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})+\cot((2\,\mathrm{i}+1)\,\frac{\pi}{\mathrm{n}})x$$

Let $0\le a<b\le\frac{2\pi}{\mathrm{n}}$, a "modified" line can be drawn with the equation $\sin_n(\theta)$ and $\cos_n(\theta)$ which would be written as

$$y_m=\frac{\sin_n(b+\mathrm{i}\,\frac{2\pi}{\mathrm{n}})-\sin_n(a+\mathrm{i}\,\frac{2\pi}{\mathrm{n}})}{\cos_n(b+\mathrm{i}\,\frac{2\pi}{\mathrm{n}})-\cos_n(a+\mathrm{i}\,\frac{2\pi}{\mathrm{n}})}x+\sin_n(a)-\frac{\sin_n(b+\mathrm{i}\,\frac{2\pi}{\mathrm{n}})-\sin_n(a+\mathrm{i}\,\frac{2\pi}{\mathrm{n}})}{\cos_n(b+\mathrm{i}\,\frac{2\pi}{\mathrm{n}})-\cos_n(a+\mathrm{i}\,\frac{2\pi}{\mathrm{n}})}\cos_n(a)$$

the slope of y_m can be simplified through these steps:

$$\frac{\mathrm{d_n}(b+\mathrm{i}\,\frac{2\pi}{\mathrm{n}})\sin(b+\mathrm{i}\,\frac{2\pi}{\mathrm{n}})-\mathrm{d_n}(a+\mathrm{i}\,\frac{2\pi}{\mathrm{n}})\sin(a+\mathrm{i}\,\frac{2\pi}{\mathrm{n}})}{\mathrm{d_n}(b+\mathrm{i}\,\frac{2\pi}{\mathrm{n}})\cos(b+\mathrm{i}\,\frac{2\pi}{\mathrm{n}})-\mathrm{d_n}(a+\mathrm{i}\,\frac{2\pi}{\mathrm{n}})\cos(a+\mathrm{i}\,\frac{2\pi}{\mathrm{n}})}$$

$$\cancel{\frac{\cos(\frac{\pi}{\mathrm{n}})}{\cos(\frac{\pi}{\mathrm{n}})}}\,\frac{\sec(\frac{\pi}{\mathrm{n}}-b)\sin(b+\mathrm{i}\,\frac{2\pi}{\mathrm{n}})-\sec(\frac{\pi}{\mathrm{n}}-a)\sin(a+\mathrm{i}\,\frac{2\pi}{\mathrm{n}})}{\sec(\frac{\pi}{\mathrm{n}}-b)\cos(b+\mathrm{i}\,\frac{2\pi}{\mathrm{n}})-\sec(\frac{\pi}{\mathrm{n}}-a)\cos(a+\mathrm{i}\,\frac{2\pi}{\mathrm{n}})}*\frac{\cos(\frac{\pi}{\mathrm{n}}-a)\cos(\frac{\pi}{\mathrm{n}}-b)}{\cos(\frac{\pi}{\mathrm{n}}-a)\cos(\frac{\pi}{\mathrm{n}}-b)}$$

$$\frac{\sin(b+\mathrm{i}\,\frac{2\pi}{\mathrm{n}})\cos(\frac{\pi}{\mathrm{n}}-a)-\sin(a+\mathrm{i}\,\frac{2\pi}{\mathrm{n}})\cos(\frac{\pi}{\mathrm{n}}-b)}{\cos(b+\mathrm{i}\,\frac{2\pi}{\mathrm{n}})\cos(\frac{\pi}{\mathrm{n}}-a)-\cos(a+\mathrm{i}\,\frac{2\pi}{\mathrm{n}})\cos(\frac{\pi}{\mathrm{n}}-b)}$$

$$\frac{\sin(b+\mathrm{i}\,\frac{2\pi}{\mathrm{n}})(\cos(\frac{\pi}{\mathrm{n}})\cos(a)+\sin(\frac{\pi}{\mathrm{n}})\sin(a))-\sin(a+\mathrm{i}\,\frac{2\pi}{\mathrm{n}})(\cos(\frac{\pi}{\mathrm{n}})\cos(b)+\sin(\frac{\pi}{\mathrm{n}})\sin(b))}{\cos(b+\mathrm{i}\,\frac{2\pi}{\mathrm{n}})(\cos(\frac{\pi}{\mathrm{n}})\cos(a)+\sin(\frac{\pi}{\mathrm{n}})\sin(a))-\cos(a+\mathrm{i}\,\frac{2\pi}{\mathrm{n}})(\cos(\frac{\pi}{\mathrm{n}})\cos(b)+\sin(\frac{\pi}{\mathrm{n}})\sin(b))}*\frac{\sec(\frac{\pi}{\mathrm{n}})}{\sec(\frac{\pi}{\mathrm{n}})}$$

$$\frac{\sin(b+\mathrm{i}\,\frac{2\pi}{\mathrm{n}})(\cos(a)+\tan(\frac{\pi}{\mathrm{n}})\sin(a))-\sin(a+\mathrm{i}\,\frac{2\pi}{\mathrm{n}})(\cos(b)+\tan(\frac{\pi}{\mathrm{n}})\sin(b))}{\cos(b+\mathrm{i}\,\frac{2\pi}{\mathrm{n}})(\cos(a)+\tan(\frac{\pi}{\mathrm{n}})\sin(a))-\cos(a+\mathrm{i}\,\frac{2\pi}{\mathrm{n}})(\cos(b)+\tan(\frac{\pi}{\mathrm{n}})\sin(b))} \qquad \mathbf{(3.1)}$$

Splitting the equation up the numerator simplifies like so:

$$\sin(b+\mathrm{i}\,\frac{2\pi}{\mathrm{n}})(\cos(a)+\tan(\frac{\pi}{\mathrm{n}})\sin(a))-\sin(a+\mathrm{i}\,\frac{2\pi}{\mathrm{n}})(\cos(b)+\tan(\frac{\pi}{\mathrm{n}})\sin(b))$$

$$(\sin(b)\cos(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})+\cos(b)\sin(\mathrm{i}\,\frac{2\pi}{\mathrm{n}}))(\cos(a)+\tan(\frac{\pi}{\mathrm{n}})\sin(a))-\ldots$$
$$\ldots(\sin(a)\cos(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})+\cos(a)\sin(\mathrm{i}\,\frac{2\pi}{\mathrm{n}}))(\cos(b)+\tan(\frac{\pi}{\mathrm{n}})\sin(b))$$

$$\cos(a)\sin(b)\cos(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})+\cancel{\sin(a)\sin(b)\cos(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})\tan(\frac{\pi}{\mathrm{n}})}+\cancel{\cos(a)\cos(b)\sin(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})}+\ldots$$
$$\ldots\sin(a)\cos(b)\sin(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})\tan(\frac{\pi}{\mathrm{n}})-\sin(a)\cos(b)\cos(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})-\cancel{\sin(a)\sin(b)\cos(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})\tan(\frac{\pi}{\mathrm{n}})}-\ldots$$
$$\ldots\cancel{\cos(a)\cos(b)\sin(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})}-\cos(a)\sin(b)\sin(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})\tan(\frac{\pi}{\mathrm{n}})$$

$$\cos(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})(\cos(a)\sin(b)-\sin(a)\cos(b))+\sin(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})\tan(\frac{\pi}{\mathrm{n}})(\sin(a)\cos(b)-\cos(a)\sin(b))$$

$$\sin(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})\tan(\frac{\pi}{\mathrm{n}})(\sin(a)\cos(b)-\cos(a)\sin(b))-\cos(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})(\sin(a)\cos(b)-\cos(a)\sin(b))$$

$$(\sin(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})\tan(\frac{\pi}{\mathrm{n}})-\cos(\mathrm{i}\,\frac{2\pi}{\mathrm{n}}))(\sin(a)\cos(b)-\cos(a)\sin(b))$$

$$(\sin(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})\tan(\frac{\pi}{\mathrm{n}}) - \cos(\mathrm{i}\,\frac{2\pi}{\mathrm{n}}))\sin(a-b)$$

and the denominator simplifies to

$$\cos(b+\mathrm{i}\,\frac{2\pi}{\mathrm{n}})(\cos(a)+\tan(\frac{\pi}{\mathrm{n}})\sin(a)) - \cos(a+\mathrm{i}\,\frac{2\pi}{\mathrm{n}})(\cos(b)+\tan(\frac{\pi}{\mathrm{n}})\sin(b))$$

$$(\cos(b)\cos(\mathrm{i}\,\frac{2\pi}{\mathrm{n}}) - \sin(b)\sin(\mathrm{i}\,\frac{2\pi}{\mathrm{n}}))(\cos(a)+\tan(\frac{\pi}{\mathrm{n}})\sin(a)) - \ldots$$
$$\ldots(\cos(a)\cos(\mathrm{i}\,\frac{2\pi}{\mathrm{n}}) - \sin(a)\sin(\mathrm{i}\,\frac{2\pi}{\mathrm{n}}))(\cos(b)+\tan(\frac{\pi}{\mathrm{n}})\sin(b))$$

$$\cancel{\cos(a)\cos(b)\cos(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})} + \sin(a)\cos(b)\cos(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})\tan(\frac{\pi}{\mathrm{n}}) - \cos(a)\sin(b)\sin(\mathrm{i}\,\frac{2\pi}{\mathrm{n}}) - \ldots$$
$$\ldots\cancel{\sin(a)\sin(b)\sin(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})\tan(\frac{\pi}{\mathrm{n}})} - \cancel{\cos(a)\cos(b)\cos(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})} - \cos(a)\sin(b)\cos(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})\tan(\frac{\pi}{\mathrm{n}}) + \ldots$$
$$\ldots\sin(a)\cos(b)\sin(\mathrm{i}\,\frac{2\pi}{\mathrm{n}}) + \cancel{\sin(a)\sin(b)\sin(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})\tan(\frac{\pi}{\mathrm{n}})}$$

$$(\cos(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})\tan(\frac{\pi}{\mathrm{n}}))(\sin(a)\cos(b)-\cos(a)\sin(b)) + \sin(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})(\sin(a)\cos(b)-\cos(a)\sin(b))$$
$$(\cos(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})\tan(\frac{\pi}{\mathrm{n}}) + \sin(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})(\sin(a)\cos(b)-\cos(a)\sin(b))$$
$$(\cos(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})\tan(\frac{\pi}{\mathrm{n}}) + \sin(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})\sin(a-b)$$

meaning that we can now simplify the original fraction (denoted **3.1**) to become:

$$\frac{(\sin(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})\tan(\frac{\pi}{\mathrm{n}}) - \cos(\mathrm{i}\,\frac{2\pi}{\mathrm{n}}))\sin(a-b)}{(\cos(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})\tan(\frac{\pi}{\mathrm{n}}) + \sin(\mathrm{i}\,\frac{2\pi}{\mathrm{n}}))\sin(a-b)}$$
$$\frac{\sin(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})\tan(\frac{\pi}{\mathrm{n}}) - \cos(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})}{\cos(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})\tan(\frac{\pi}{\mathrm{n}}) + \sin(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})} * \frac{\cos(\frac{\pi}{\mathrm{n}})}{\cos(\frac{\pi}{\mathrm{n}})}$$
$$\frac{\sin(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})\sin(\frac{\pi}{\mathrm{n}}) - \cos(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})\cos(\frac{\pi}{\mathrm{n}})}{\cos(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})\sin(\frac{\pi}{\mathrm{n}}) + \sin(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})\cos(\frac{\pi}{\mathrm{n}})}$$
$$\frac{-(\cos(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})\cos(\frac{\pi}{\mathrm{n}}) - \sin(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})\sin(\frac{\pi}{\mathrm{n}}))}{\sin(\frac{\pi}{\mathrm{n}})\cos(\mathrm{i}\,\frac{2\pi}{\mathrm{n}}) + \cos(\frac{\pi}{\mathrm{n}})\sin(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})}$$
$$\frac{-\cos(\mathrm{i}\,\frac{2\pi}{\mathrm{n}} + \frac{\pi}{\mathrm{n}})}{\sin(\mathrm{i}\,\frac{2\pi}{\mathrm{n}} + \frac{\pi}{\mathrm{n}})}$$
$$-\cot(\mathrm{i}\,\frac{2\pi}{\mathrm{n}} + \frac{\pi}{\mathrm{n}})$$

$$-\cot((2\,\mathrm{i}+1)\,\frac{\pi}{\mathrm{n}})$$

meaning that the equation of the "modified" line becomes:

$$y_m = -\cot((2\,\mathrm{i}+1)\,\frac{\pi}{\mathrm{n}})x + \sin_n(a + \mathrm{i}\,\frac{2\pi}{\mathrm{n}}) + \cot((2\,\mathrm{i}+1)\,\frac{\pi}{\mathrm{n}})\cos_n(a + \mathrm{i}\,\frac{2\pi}{\mathrm{n}})$$

This leads to the conclusion that when $0 \leq a < b \leq \frac{2\pi}{\mathrm{n}}$ the slopes of y_m and y are equal. This leads to the fact that if $a = 0$, due to **Identity 1.3**, then for any applicable value of b y_m becomes:

$$y_m = -\cot((2\,\mathrm{i}+1)\,\frac{\pi}{\mathrm{n}})x + \sin(\mathrm{i}\,\frac{2\pi}{\mathrm{n}}) + \cot((2\,\mathrm{i}+1)\,\frac{\pi}{\mathrm{n}})\cos(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})$$

Meaning that $y_m = \mathrm{L_i}$ when $a = 0$ and due to **Identity 1.3** $\sin_n(a{+}\mathrm{i}\,\frac{2\pi}{\mathrm{n}}) = \sin(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})$ and $\cos_n(a{+}\mathrm{i}\,\frac{2\pi}{\mathrm{n}}) = \cos(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})$ thus the point at point $\mathrm{p_a}$ (located at $(\cos_n(a), \sin_n(a))$) is equal to point $\mathrm{v_i}$.
Because $y_m = L_i$, point $\mathrm{p_b}$ (located at $(\cos_n(b), \sin_n(b))$) is always on line segment $\mathrm{L_i}$, thus when $a \neq 0$, $\mathrm{p_a}$ must also always be on line segment $\mathrm{L_i}$.
This means that for any value x where $0 \leq x \leq \frac{2\pi}{\mathrm{n}}$, the point $\mathrm{p_x}$ (found at $(\cos_n(x + \mathrm{i}\,\frac{2\pi}{\mathrm{n}}), \sin_n(x + \mathrm{i}\,\frac{2\pi}{\mathrm{n}}))$) falls on line $\mathrm{L_i}$ of shape $\mathrm{s_n}$, and since $\mathrm{i} \in \mathbb{Z}$, any point p_θ (found at $(\cos_n(\theta), \sin_n(\theta))$) will always be located on the perimeter of shape $\mathrm{s_n}$.

5 Shape Equations

From these formulas, the equation for any regular polygon can be derived. The equation is found through a mutation of the ellipse equation by $\mathrm{d_n}(\theta)$:

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = \mathrm{d_n^2}(\theta) * 1$$

To derive theta, one would start with the assumption that $\theta = \arctan(\frac{(y-k)a}{(x-h)b})$, which would find the angle at the correct point offset from the center, but, due to the left side of the equation involving x^2, the shape is mirrored over $x = h$, which works perfectly for even sided polygons as they are symmetric over $x = h$. Yet, odd-sided polygons do not work due to their anti-symmetry over $x = h$. This can be fixed by firstly noting that the angle is calculated correctly in Quadrants I and IV, and π radians off from the expected angle in Quadrants II and III. Thus when $x < h$, π must be added to θ. Using the sign function, which is defined as

$$\mathrm{sgn}(x) = \frac{x}{|x|} = \frac{|x|}{x}$$

and modifying sgn to equal 1 when $x < h$ and 0 when $x \geq h$ results in the formula:

$$-\frac{1}{2}(\frac{x-h}{|x-h|}-1)$$

which when multiplied by π and added to θ, θ becomes equal to:

$$\arctan(\frac{(y-k)a}{(x-h)b})-\frac{\pi}{2}(\frac{x-h}{|x-h|}-1)$$

meaning that the standard equation, with the above equation substitued for θ becomes:

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=\mathrm{d}_{\mathrm{n}}^2(\arctan(\frac{(y-k)a}{(x-h)b})-\frac{\pi}{2}(\frac{x-h}{|x-h|}-1))$$

Interestingly, due to the existence of θ inside this equation, a constant value can be added to it to make any shape rotate, allowing for quick and efficient rotation calculations.