# Trigonometric Coefficients for Regular S-Sided Polygons

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# 1 Introduction

This paper creates and proves a function that finds the distance between the center of a regular s-sided polygon and a point on its perimeter at a given angle without knowing that point's position. This, when multiplied by sine or cosine allows finding the position of these points on the perimeter of the polygon. This allows for the creation of a general equation for any regular s-sided polygon similar to the equation of a circle. This equation can then be used to find areas of intersection between polygons and perform operations on the shapes. Following this, the modified sine and cosine functions are then proved to always generate a point on the perimeter of an s-sided polygon, given s and an angle.

# 2 Trigonometric Equations

Let $s = \{x : x \in \mathbb{Z}^+ \text{ and } x \geq 3\}$, and

$$\mathrm{d_s}(\theta) = \cos(\frac{\pi}{\mathrm{s}})\sec(\frac{\pi - \arccos(\cos(s\theta))}{s})$$
$$\cos_s(\theta) = \mathrm{d_s}(\theta)\cos(\theta)$$
$$\sin_s(\theta) = \mathrm{d_s}(\theta)\sin(\theta)$$

This is found by starting with the functions

$$\mathrm{h_s}(\theta) = \sin(\frac{\pi}{2} - \frac{\pi}{\mathrm{s}} + (\theta \mod \frac{2\pi}{\mathrm{s}})) - \sin(\frac{\pi}{2} - \frac{\pi}{\mathrm{s}})$$
$$\mathrm{d_s}(\theta) = \frac{\mathrm{h_s}(\theta)}{\sin(\frac{\pi}{2} - \frac{\pi}{\mathrm{s}} + (\theta \mod \frac{2\pi}{\mathrm{s}}))}$$
$$\cos_s(\theta) = (1 - \mathrm{d_s}(\theta))\cos(\theta)$$
$$\sin_s(\theta) = (1 - \mathrm{d_s}(\theta))\sin(\theta)$$

These functions follow the premise that the verticies of any s-sided regular polygon ($P_s$) centered at (0,0) always touch $P_s$'s circumcircle.
Arc $a$ can be constructed which is $\frac{1}{\mathrm{s^{th}}}$ of the circumcircle, its diameter is the same length as any line which constructs $P_s$, and it's circumference equals to the circumference of the section of the circumcircle spanning from adjacent vertices on $P_s$.
Thus, the height of $a$ at a given point on a circle subtracted from the radius of the circle will result in the distance from the center of $P_s$ to the point on the perimeter of $P_s$ which intersects the ray of angle $\theta$ drawn in standard position.
$\mathrm{h_s}(\theta)$ serves to find the height of $a$ at a position by splitting the circumcircle into $\mathrm{s^{ths}}$ (resulting in a period of $\frac{2\pi}{\mathrm{s}}$) and changing from the sine of the "left" vertex to the sine of the "right" vertex over the period of $\frac{2\pi}{\mathrm{s}}$ (the length of $a$). The use of mod makes the function periodic. It then subtracts by the height of a vertex so that $\mathrm{h_s}(\theta) = 0$ when $\theta = 0$

$\mathrm{d_s}(\theta)$ divides the height $\mathrm{h_s}(\theta)$ by the angle between the assumed intersection point on $P_s$ and the intersection point on the circumcircle to get the actual distance.
These functions then simplify to:

$$\mathrm{d_s}(\theta) = \cos(\frac{\pi}{\mathrm{s}}) \sec(\frac{\pi}{\mathrm{s}} - (\theta \mod \frac{2\pi}{\mathrm{s}}))$$
$$\cos_s(\theta) = \mathrm{d_s}(\theta) \cos(\theta)$$
$$\sin_s(\theta) = \mathrm{d_s}(\theta) \sin(\theta)$$

Which is not algebraic due to mod, yet it can be noted that due to the nature of $\mathrm{h_s}(\theta)$ being mirrored over each period, $\theta \mod \frac{2\pi}{\mathrm{s}}$ can be substituted for an equivalent function which has the same period of $\frac{2\pi}{\mathrm{s}}$, increases linearly from 0 to $\frac{\pi}{\mathrm{s}}$ over $[0, \frac{\pi}{\mathrm{s}}]$ and decreases linearly at the same rate over $[\frac{\pi}{\mathrm{s}}, \frac{2\pi}{\mathrm{s}}]$.
These requirements can be met by $\frac{\arccos(\cos(n\theta))}{n}$[1], which is periodic over $\frac{2\pi}{\mathrm{s}}$, has a domain of $\theta \in \mathbb{R}$, and a range of $[0, \frac{\pi}{s}]$, thus the above equations can be changed to:

$$\mathrm{d_s}(\theta) = \cos(\frac{\pi}{\mathrm{s}}) \sec(\frac{\pi - \arccos(\cos(s\theta))}{s})$$
$$\cos_s(\theta) = \mathrm{d_s}(\theta) \cos(\theta)$$
$$\sin_s(\theta) = \mathrm{d_s}(\theta) \sin(\theta)$$

## 2.1 Identities

**Identity 1.1** Let $\mathrm{n} = \mathbb{Z}$:

$$\cos_s(\theta + \mathrm{n}\frac{2\pi}{\mathrm{s}}) = \mathrm{d_s}(\theta) \cos(\theta + \mathrm{n}\frac{2\pi}{\mathrm{s}})$$
$$\sin_s(\theta + \mathrm{n}\frac{2\pi}{\mathrm{s}}) = \mathrm{d_s}(\theta) \sin(\theta + \mathrm{n}\frac{2\pi}{\mathrm{s}})$$

**Identity 1.2** When $0 \leq \theta \leq \frac{2\pi}{\mathrm{s}}$ then $\mathrm{d_s}(\theta) = \cos(\frac{\pi}{\mathrm{s}}) \sec(\frac{\pi}{\mathrm{s}} - \theta)$
When $\theta$ only spans one period, $\frac{\arccos(\cos(s\theta))}{s}$ is not required to make the function periodic.
**Identity 1.3** Let $\mathrm{n} = \mathbb{Z}$ then:

$$\cos_s(\mathrm{n}\frac{2\pi}{\mathrm{s}}) = (1) \cos(\mathrm{n}\frac{2\pi}{\mathrm{s}}) = \cos(\mathrm{n}\frac{2\pi}{\mathrm{s}})$$
$$\sin_s(\mathrm{n}\frac{2\pi}{\mathrm{s}}) = (1) \sin(\mathrm{n}\frac{2\pi}{\mathrm{s}}) = \sin(\mathrm{n}\frac{2\pi}{\mathrm{s}})$$

$\arccos(\cos(s\theta))$ evaluates to 0 whenever $\theta = \mathrm{n}\frac{2\pi}{\mathrm{s}}$, which means $\mathrm{d_s}(\mathrm{n}\frac{2\pi}{\mathrm{s}}) = \sec(\frac{\pi}{\mathrm{s}}) \cos(\frac{\pi}{\mathrm{s}}) = 1$

[1] **J. L. Spence**, **"Perodic absolute value functions"**.

# 3 Polygonal Equations

From these formulas, the equation for any regular polygon can be derived. The equation is found through a mutation of the ellipse equation by $\mathrm{d_s}(\theta)$:

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=\mathrm{d_s}(\theta)^2 * 1$$

To derive theta, one would start with $\theta = \arctan(\frac{(y-k)a}{(x-h)b})$, which would find the angle to the point relative to the center, but due to x being squared, the polygon is mirrored over $x = h$, which will not work for odd sided polygons due to their anti-symmetry over $x = h$(this is also the case over the line $y = k$ but all regular polygons, regardless of side parity, are mirrored over $y = k$). This can be fixed by noting that the angle is calculated correctly in Quadrants I and IV (relative to $(h, k)$ as the origin), and $\pi$ radians off from the expected angle in Quadrants II and III. Thus when $x < h$, $\pi$ must be added to $\theta$. Using the sign function and modifying it to equal 1 when $x < h$ and 0 when $x \geq h$ results in the formula:

$$-\frac{1}{2}(\frac{x-h}{|x-h|}-1)$$

which when multiplied by $\pi$ and added to $\theta$ becomes:

$$\arctan\left(\frac{(y-k)a}{(x-h)b}\right)-\frac{\pi}{2}\left(\frac{x-h}{|x-h|}-1\right)$$

meaning that the standard equation is:

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=\mathrm{d_s}\left(\arctan\left(\frac{(y-k)a}{(x-h)b}\right)-\frac{\pi}{2}\left(\frac{x-h}{|x-h|}-1\right)\right)^2$$

As $\mathrm{d_s}(\theta)$ takes an angle, a constant value can be added to $\theta$ to make any shape rotate.

## 3.1 Intersections

Let the regular Polygon $P_s$ with s sides (where ($s \in \mathbb{Z}$ and $s \geq 3$)) be represented by the equation

$$P_s(x,y)=\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}-\mathrm{d_s}\left(\arctan\left(\frac{(y-k)a}{(x-h)b}\right)-\frac{\pi}{2}\left(\frac{x-h}{|x-h|}-1\right)\right)^2$$

A point $(x, y)$ is on the perimeter of $P_s$ if $P_s(x, y) = 0$, inside the area of $P_s$ if $P_s(x, y) < 0$, and outside the area of $P_s$ if $P_s(x, y) > 0$.
Knowing this 2 equations can be made:

$$P_s(x,y)+|P_s(x,y)|=\begin{cases}0 & \text{if } (x,y) \text{ is inside of } P_s\\ 2P_s(x,y) & \text{if } (x,y) \text{ is outside of } P_s\end{cases}$$

$$|P_s(x,y)-|P_s(x,y)||=\begin{cases}0 & \text{if } (x,y) \text{ is outside of } P_s\\ 2|P_s(x,y)| & \text{if } (x,y) \text{ is inside of } P_s\end{cases}$$

Using these 2 identities, operations can be performed on a set of polygons.

### 3.1.1 Addition

Given the polygons $P_{s_1}$ and $P_{s_2}$ the addition of the two polygons creates a new polygon $P_a$ where (x,y) is inside $P_a$ if (x,y) is inside $P_{s_1}$ and/or inside $P_{s_2}$:

$$P_a(x,y) = |(C_{s_1}(x,y) + |C_{s_1}(x,y)|)(C_{s_2}(x,y) + |C_{s_2}(x,y)|)|$$

Where (x,y) is inside $P_a$ if $P_a(x,y) = 0$.
More generally for any set of Polygons $C$, the addition of all polygons in the set $P_C$ is represented by

$$P_C = \left| \prod_{i=1}^{|C|} (C_i(x,y) + |C_i(x,y)|) \right|$$

where (x,y) is inside $P_C$ if $P_C(x,y) = 0$

### 3.1.2 Subtraction

A point (x,y) is inside the difference between two polygons $P_d = P_{s_1} - P_{s_2}$ if $P_{s_1}(x,y) < 0$ and $P_{s_2}(x,y) > 0$. So $P_d$ is represented by:

$$P_d = |(C_{s_1}(x,y) - |C_{s_2}(x,y)|)(C_{s_2}(x,y) + |C_{s_2}(x,y)|)|$$

Where (x,y) is inside $P_d$ if $P_d(x,y) > 0$.

### 3.1.3 Exclusive Area

The point (x,y) is in the exclusive area of a set of polygons $C$ when it is inside one polygon of the set but no other. This is the same as the difference of the addition of all polygons in the set and their intersection ($E_C = P_C - I_C$). This can be represented as:

$$I_C(x,y) = \frac{-1}{2} \sum_{i=1}^{|C|} \left( \frac{C_i(x,y)}{|C_i(x,y)|} - 1 \right)$$

$$E_C(x,y) = \prod_{j=1}^{|C|} (C_j(x,y) + |C_j(x,y)|) + (I_C(x,y) - 1 + |I_C(x,y) - 1|)$$

Where (x,y) is inside $E_C$ if $E_C(x,y) = 0$.

### 3.1.4 Union

The point (x,y) is within the union of the set of polygons $C$ if (x,y) is within every polygon within $C$. The union of $C$ is expressed as:

$$U_C = \left| \prod_{i=1}^{|C|} (C_i(x,y) - |C_i(x,y)|) \right|$$

Where (x,y) is within $U_C$ if $U_C(x,y) > 0$.

# 4 Proof

Let $\mathrm{n}, s = \mathbb{Z}, s \geq 3$. Shape $P_s$ is made of s vertices and s line segments with the $\mathrm{n}^{\mathrm{th}}$ vertex $v_n(\cos(\mathrm{n}\,\frac{2\pi}{\mathrm{s}}), \sin(\mathrm{n}\,\frac{2\pi}{\mathrm{s}}))$. The $\mathrm{n}^{\mathrm{th}}$ line segment, $L_n$, spans between $v_n$ and $v_{n+1}$ and is represented by the equation

$$L_n = \frac{\sin((\mathrm{n}+1)\,\frac{2\pi}{\mathrm{s}}) - \sin(\mathrm{n}\,\frac{2\pi}{\mathrm{s}})}{\cos((\mathrm{n}+1)\,\frac{2\pi}{\mathrm{s}}) - \cos(\mathrm{n}\,\frac{2\pi}{\mathrm{s}})} x + \sin(\mathrm{n}\,\frac{2\pi}{\mathrm{s}}) - \frac{\sin((\mathrm{n}+1)\,\frac{2\pi}{\mathrm{s}}) - \sin(\mathrm{n}\,\frac{2\pi}{\mathrm{s}})}{\cos((\mathrm{n}+1)\,\frac{2\pi}{\mathrm{s}}) - \cos(\mathrm{n}\,\frac{2\pi}{\mathrm{s}})} \cos(\mathrm{n}\,\frac{2\pi}{\mathrm{s}})$$

which simplifies to

$$L_n = -\cot((2\,\mathrm{n}+1)\,\frac{\pi}{\mathrm{s}})x + \sin(\mathrm{n}\,\frac{2\pi}{\mathrm{s}}) + \cot((2\,\mathrm{n}+1)\,\frac{\pi}{\mathrm{s}}) \cos(\mathrm{n}\,\frac{2\pi}{\mathrm{s}})$$

Let $0 \leq a < b \leq \frac{2\pi}{\mathrm{s}}$, $\alpha = a + \mathrm{n}\,\frac{2\pi}{\mathrm{s}}$, and $\beta = b + \mathrm{n}\,\frac{2\pi}{\mathrm{s}}$: a line can be drawn with $\sin_s(\theta)$ and $\cos_s(\theta)$ with the equation:

$$y_n = \frac{\sin_s(\beta) - \sin_s(\alpha)}{\cos_s(\beta) - \cos_s(\alpha)} x + \sin_s(\alpha) - \frac{\sin_s(\beta) - \sin_s(\alpha)}{\cos_s(\beta) - \cos_s(\alpha)} \cos_s(\alpha)$$

the slope of $y_n$ can be expanded to

$$\frac{\sin(\beta)(\cos(a) + \tan(\frac{\pi}{\mathrm{s}})\sin(a)) - \sin(\alpha)(\cos(b) + \tan(\frac{\pi}{\mathrm{s}})\sin(b))}{\cos(\beta)(\cos(a) + \tan(\frac{\pi}{\mathrm{s}})\sin(a)) - \cos(\alpha)(\cos(b) + \tan(\frac{\pi}{\mathrm{s}})\sin(b))}$$

Which simplifies to

$$\frac{\sin(a-b)(\sin(\mathrm{n}\,\frac{2\pi}{\mathrm{s}})\tan(\frac{\pi}{\mathrm{s}}) - \cos(\mathrm{n}\,\frac{2\pi}{\mathrm{s}}))}{\sin(a-b)(\cos(\mathrm{n}\,\frac{2\pi}{\mathrm{s}})\tan(\frac{\pi}{\mathrm{s}}) + \sin(\mathrm{n}\,\frac{2\pi}{\mathrm{s}}))} = -\cot((2\,\mathrm{n}+1)\,\frac{\pi}{\mathrm{s}})$$

thus the equation of the line becomes:

$$y_n = -\cot((2\,\mathrm{n}+1)\,\frac{\pi}{\mathrm{s}})x + \sin_s(\alpha) + \cot((2\,\mathrm{n}+1)\,\frac{\pi}{\mathrm{s}}) \cos_s(\alpha)$$

This leads to the conclusion that when $\mathrm{n}\frac{2\pi}{\mathrm{s}} \leq a < b \leq (\mathrm{n}+1)\frac{2\pi}{n}$ the slopes of $y_n$ and $L_n$ are equal.
If $a = 0$, due to **Identity 1.3**, then for any applicable value of $b$, $y_n$ becomes:

$$y_n = -\cot((2\,\mathrm{n}+1)\,\frac{\pi}{\mathrm{s}})x + \sin(\mathrm{n}\,\frac{2\pi}{\mathrm{s}}) + \cot((2\,\mathrm{n}+1)\,\frac{\pi}{\mathrm{s}})\cos(\mathrm{n}\,\frac{2\pi}{\mathrm{s}})$$

Meaning that $y_n = L_n$ when $a = 0$ (denoted $y_{n,a=0}$) and due to **Identity 1.3**: the point $A(\cos_s(a + \mathrm{n}\,\frac{2\pi}{\mathrm{s}}), \sin_s(a + \mathrm{n}\,\frac{2\pi}{\mathrm{s}})) = v_n$.
Because $y_{n,a=0} = L_n$, the point $B(\cos_s(b + \mathrm{n}\,\frac{2\pi}{\mathrm{s}}), \sin_s(b + \mathrm{n}\,\frac{2\pi}{\mathrm{s}})))$ is always on line segment $L_n$, thus when $a \neq 0$, $A$ must also always be on line segment $L_n$.
This means that for any value $t$ where $0 \leq t \leq \frac{2\pi}{\mathrm{s}}$, the point $T(\cos_s(t+\mathrm{n}\,\frac{2\pi}{\mathrm{s}}), \sin_s(t+\mathrm{n}\,\frac{2\pi}{\mathrm{s}}))$, falls on line $L_n$ of $P_s$, and since $\mathrm{n} \in \mathbb{Z}$, any point $X(\cos_s(\theta), \sin_s(\theta))$, will always be on the perimeter of $P_s$.

## References

[1] J. L. Spence, "Periodic absolute value functions," *School Science and Mathematics*, vol. 61, no. 9, pp. 664–666, 1961. DOI: https://doi.org/10.1111/j.1949-8594.1961.tb08220.x. eprint: https://onlinelibrary.wiley.com/doi/pdf/10.1111/j.1949-8594.1961.tb08220.x. [Online]. Available: https://onlinelibrary.wiley.com/doi/abs/10.1111/j.1949-8594.1961.tb08220.x.