

Trigonometric Equations for Regular N-Sided Polygons

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1 Introduction

This paper derives and proves a set of functions that serve to find the point of intersection between an angle in standard position and an n-sided regular polygon through only algebra. Through these, the standard equation for any regular polygon can be derived.

2 Theroems

Theorem 1.1 $\arccos(\cos(ax))$ is periodic over $\frac{2\pi}{a*n}$.

$\cos(ax)$ is periodic over $\frac{2\pi}{a*n}$ thus $\arccos(\cos(ax))$ must also be.

Theorem 1.2 For $x \in \mathbb{R}$ $\arccos(\cos(ax)) \in [0, \pi]$.

$\cos(x)$ has a domain over \mathbb{R} and a range of $[-1, 1]$, $\arccos(x)$ has a domain over $[-1, 1]$ and a range of $[0, \pi]$. Due to the range of \cos being the domain of \arccos , all real numbers can be an input of \cos , and all numbers on the interval $[-1, 1]$ can be output.

Theorem 1.3 $\cos(n(a + \frac{2\pi}{n})) = \cos(na)$

$\cos(n(a + \frac{2\pi}{n}))$ expands to $\cos(na + 2\pi)$ which due to cosine's periodic nature evaluates to $\cos(na)$

Theorem 2.1 $\sin(\alpha) - \sin(\beta) = 2 \sin(\frac{\alpha+\beta}{2}) \cos(\frac{\alpha-\beta}{2})$

Theorem 2.2 $\cos(\alpha) - \cos(\beta) = -2 \sin(\frac{\alpha+\beta}{2}) \sin(\frac{\alpha-\beta}{2})$

3 Trigonometric Equations

Let $n = \{x : x \in \mathbb{Z}^+ \text{ and } x \geq 3\}$, and

$$\begin{aligned} d_n(\theta) &= \cos\left(\frac{\pi}{n}\right) \sec\left(\frac{\pi - \arccos(\cos(n\theta))}{n}\right) \\ \cos_n(\theta) &= d_n(\theta) \cos(\theta) \\ \sin_n(\theta) &= d_n(\theta) \sin(\theta) \end{aligned}$$

This is derived through the original functions:

$$\begin{aligned} h_n(\theta) &= \sin\left(\frac{\pi}{2} - \frac{\pi}{n} + (\theta \bmod \frac{2\pi}{n})\right) - \sin\left(\frac{\pi}{2} - \frac{\pi}{n}\right) \\ d_n(\theta) &= \frac{h_n(\theta)}{\sin\left(\frac{\pi}{2} - \frac{\pi}{n} + (\theta \bmod \frac{2\pi}{n})\right)} \\ \cos_n(\theta) &= (1 - d_n(\theta)) \cos(\theta) \\ \sin_n(\theta) &= (1 - d_n(\theta)) \sin(\theta) \end{aligned}$$

These functions follow the premise that any n-sided regular polygon (shape s_n) centered at (0,0) has a circumcircle where each vertex touches the circumference of the circumcircle. Arc a can be constructed which is $\frac{1}{n^{\text{th}}}$ of the circumcircle, its diameter is the same length

as any line which constructs s_n , and it's circumference equals to the circumference of the section of the circumcircle spanning from adjacent vertices on s_n .

Thus, the height of a at a given point on a circle subtracted from the radius of the circle will result in the distance from the midpoint of s_n to the point on the perimeter of s_n which intersects the ray of angle θ drawn in standard position.

$h_n(\theta)$ serves to find the height of a at a position by splitting the circumcircle into n^{th} (resulting in a period of $\frac{2\pi}{n}$), finding the height of a point on a through sine and changing from the "left" vertex to the "right" vertex over the period of $\frac{2\pi}{n}$ (the length of a). The use of mod makes the function periodic. It then subtracts by the height of a vertex so that $h_n(\theta) = 0$ when $\theta = 0$

$d_n(\theta)$ divides the height ($h_n(\theta)$) by the angle between the assumed intersection point on s_n and the intersection point on the circumcircle to get the actual distance.

These functions then simplify to:

$$\begin{aligned} d_n(\theta) &= \cos\left(\frac{\pi}{n}\right) \sec\left(\frac{\pi}{n} - (\theta \bmod \frac{2\pi}{n})\right) \\ \cos_n(\theta) &= d_n(\theta) \cos(\theta) \\ \sin_n(\theta) &= d_n(\theta) \sin(\theta) \end{aligned}$$

Which is not algebraic due to mod (and the resultant floor function behind it), yet it can be noted that due to the nature of the original $h_n(\theta)$, $\theta \bmod \frac{2\pi}{n}$ can be substituted for an equivalent function which has an equal period of $\frac{2\pi}{n}$, increases linearly over $[0, \frac{\pi}{n}]$ to $[0, \frac{\pi}{n}]$ and decreases linearly at the same rate from $[\frac{\pi}{n}, \frac{2\pi}{n}]$.

These requirements can be met by $\frac{\arccos(\cos(n\theta))}{n}$, which due to **Theorem 1.1** and **Theorem 1.2** is periodic over $\frac{2\pi}{n}$ has a domain of $n\theta \in \mathbb{R}$ and a range of $[0, \frac{\pi}{n}]$, thus the above equations can be changed to:

$$\begin{aligned} d_n(\theta) &= \cos\left(\frac{\pi}{n}\right) \sec\left(\frac{\pi - \arccos(\cos(n\theta))}{n}\right) \\ \cos_n(\theta) &= d_n(\theta) \cos(\theta) \\ \sin_n(\theta) &= d_n(\theta) \sin(\theta) \end{aligned}$$

3.1 Identities

Identity 1.1 Let $i = \mathbb{Z}$, due to **Theorem 1.3**:

$$\begin{aligned}\cos_n(\theta + i \frac{2\pi}{n}) &= d_n(\theta) \cos(\theta + i \frac{2\pi}{n}) \\ \sin_n(\theta + i \frac{2\pi}{n}) &= d_n(\theta) \sin(\theta + i \frac{2\pi}{n})\end{aligned}$$

Identity 1.2 When $0 \leq \theta \leq \frac{2\pi}{n}$ then $d_n(\theta) = \cos(\frac{\pi}{n}) \sec(\frac{\pi}{n} - \theta)$

When θ only spans one period, $\frac{\arccos(\cos(n\theta))}{n}$ is not required to make the function periodic.

Identity 1.3 Let $i = \mathbb{Z}$ then:

$$\begin{aligned}\cos_n(i \frac{2\pi}{n}) &= (1) \cos(i \frac{2\pi}{n}) = \cos(i \frac{2\pi}{n}) \\ \sin_n(i \frac{2\pi}{n}) &= (1) \sin(i \frac{2\pi}{n}) = \sin(i \frac{2\pi}{n})\end{aligned}$$

This is true because $\arccos(\cos(n\theta))$ evaluates to 0 whenever $\theta = i \frac{2\pi}{n}$, which means $\sec(\frac{\pi}{n} - \frac{\arccos(\cos(n\theta))}{n})$ simplifies to $\sec(\frac{\pi}{n})$ so $d_n = \sec(\frac{\pi}{n}) \cos(\frac{\pi}{n}) = 1$

4 Proof

Let $i = \mathbb{Z}$. Shape s_n is made of n vertices and n line segments with the i^{th} vertex at the point $(\cos(i \frac{2\pi}{n}), \sin(i \frac{2\pi}{n}))$. The i^{th} line segment, L_i , spans between v_i and v_{i+1} and is represented by the equation

$$L_i = \frac{\sin((i+1) \frac{2\pi}{n}) - \sin(i \frac{2\pi}{n})}{\cos((i+1) \frac{2\pi}{n}) - \cos(i \frac{2\pi}{n})} x + \sin(i \frac{2\pi}{n}) - \frac{\sin((i+1) \frac{2\pi}{n}) - \sin(i \frac{2\pi}{n})}{\cos((i+1) \frac{2\pi}{n}) - \cos(i \frac{2\pi}{n})} \cos(i \frac{2\pi}{n})$$

which due to **Theorem 2.1** and **Theorem 2.2** simplifies to

$$L_i = -\cot((2i+1) \frac{\pi}{n}) x + \sin(i \frac{2\pi}{n}) + \cot((2i+1) \frac{\pi}{n}) \cos(i \frac{2\pi}{n})$$

Let $0 \leq a < b \leq \frac{2\pi}{n}$, $\alpha = a + i \frac{2\pi}{n}$, and $\beta = b + i \frac{2\pi}{n}$: a "modified" line can be drawn with the equation $\sin_n(\theta)$ and $\cos_n(\theta)$ which would be written as

$$y_m = \frac{\sin_n(\beta) - \sin_n(\alpha)}{\cos_n(\beta) - \cos_n(\alpha)} x + \sin_n(a) - \frac{\sin_n(\beta) - \sin_n(\alpha)}{\cos_n(\beta) - \cos_n(\alpha)} \cos_n(a)$$

the slope of y_m can be simplified through these steps:

$$\frac{d_n(\beta) \sin(\beta) - d_n(\alpha) \sin(\alpha)}{d_n(\beta) \cos(\beta) - d_n(\alpha) \cos(\alpha)}$$

$$\begin{aligned}
& \frac{\cancel{\cos(\frac{\pi}{n})} \sec(\frac{\pi}{n}-b) \sin(\beta) - \sec(\frac{\pi}{n}-a) \sin(\alpha)}{\cancel{\cos(\frac{\pi}{n})} \sec(\frac{\pi}{n}-b) \cos(\beta) - \sec(\frac{\pi}{n}-a) \cos(\alpha)} * \frac{\cos(\frac{\pi}{n}-a) \cos(\frac{\pi}{n}-b)}{\cos(\frac{\pi}{n}-a) \cos(\frac{\pi}{n}-b)} \\
& \frac{\sin(\beta) \cos(\frac{\pi}{n}-a) - \sin(\alpha) \cos(\frac{\pi}{n}-b)}{\cos(\beta) \cos(\frac{\pi}{n}-a) - \cos(\alpha) \cos(\frac{\pi}{n}-b)} \\
& \frac{\sin(\beta)(\cos(\frac{\pi}{n}) \cos(a) + \sin(\frac{\pi}{n}) \sin(a)) - \sin(\alpha)(\cos(\frac{\pi}{n}) \cos(b) + \sin(\frac{\pi}{n}) \sin(b))}{\cos(\beta)(\cos(\frac{\pi}{n}) \cos(a) + \sin(\frac{\pi}{n}) \sin(a)) - \cos(\alpha)(\cos(\frac{\pi}{n}) \cos(b) + \sin(\frac{\pi}{n}) \sin(b))} * \frac{\sec(\frac{\pi}{n})}{\sec(\frac{\pi}{n})} \\
& \frac{\sin(\beta)(\cos(a) + \tan(\frac{\pi}{n}) \sin(a)) - \sin(\alpha)(\cos(b) + \tan(\frac{\pi}{n}) \sin(b))}{\cos(\beta)(\cos(a) + \tan(\frac{\pi}{n}) \sin(a)) - \cos(\alpha)(\cos(b) + \tan(\frac{\pi}{n}) \sin(b))} \quad (3.1)
\end{aligned}$$

Splitting the equation up the numerator simplifies like so:

$$\begin{aligned}
& \sin(\beta)(\cos(a) + \tan(\frac{\pi}{n}) \sin(a)) - \sin(\alpha)(\cos(b) + \tan(\frac{\pi}{n}) \sin(b)) \\
& (\sin(b) \cos(i \frac{2\pi}{n}) + \cos(b) \sin(i \frac{2\pi}{n}))(\cos(a) + \tan(\frac{\pi}{n}) \sin(a)) - \dots \\
& \dots (\sin(a) \cos(i \frac{2\pi}{n}) + \cos(a) \sin(i \frac{2\pi}{n}))(\cos(b) + \tan(\frac{\pi}{n}) \sin(b)) \\
& \cos(a) \sin(b) \cos(i \frac{2\pi}{n}) + \cancel{\sin(a) \sin(b) \cos(i \frac{2\pi}{n}) \tan(\frac{\pi}{n})} + \cancel{\cos(a) \cos(b) \sin(i \frac{2\pi}{n})} + \dots \\
& \dots \sin(a) \cos(b) \sin(i \frac{2\pi}{n}) \tan(\frac{\pi}{n}) - \cancel{\sin(a) \cos(b) \cos(i \frac{2\pi}{n})} - \cancel{\sin(a) \sin(b) \cos(i \frac{2\pi}{n}) \tan(\frac{\pi}{n})} - \dots \\
& \dots \cancel{\cos(a) \cos(b) \sin(i \frac{2\pi}{n})} - \cos(a) \sin(b) \sin(i \frac{2\pi}{n}) \tan(\frac{\pi}{n}) \\
& \cos(i \frac{2\pi}{n})(\cos(a) \sin(b) - \sin(a) \cos(b)) + \sin(i \frac{2\pi}{n}) \tan(\frac{\pi}{n})(\sin(a) \cos(b) - \cos(a) \sin(b)) \\
& \sin(i \frac{2\pi}{n}) \tan(\frac{\pi}{n})(\sin(a) \cos(b) - \cos(a) \sin(b)) - \cos(i \frac{2\pi}{n})(\sin(a) \cos(b) - \cos(a) \sin(b)) \\
& (\sin(i \frac{2\pi}{n}) \tan(\frac{\pi}{n}) - \cos(i \frac{2\pi}{n}))(\sin(a) \cos(b) - \cos(a) \sin(b)) \\
& (\sin(i \frac{2\pi}{n}) \tan(\frac{\pi}{n}) - \cos(i \frac{2\pi}{n})) \sin(a - b)
\end{aligned}$$

and the denominator simplifies to

$$\cos(\beta)(\cos(a) + \tan(\frac{\pi}{n}) \sin(a)) - \cos(\alpha)(\cos(b) + \tan(\frac{\pi}{n}) \sin(b))$$

$$\begin{aligned}
& (\cos(b) \cos(i \frac{2\pi}{n}) - \sin(b) \sin(i \frac{2\pi}{n})) (\cos(a) + \tan(\frac{\pi}{n}) \sin(a)) - \dots \\
& \dots (\cos(a) \cos(i \frac{2\pi}{n}) - \sin(a) \sin(i \frac{2\pi}{n})) (\cos(b) + \tan(\frac{\pi}{n}) \sin(b))
\end{aligned}$$

$$\begin{aligned}
& \cancel{\cos(a) \cos(b) \cos(i \frac{2\pi}{n})} + \sin(a) \cos(b) \cos(i \frac{2\pi}{n}) \tan(\frac{\pi}{n}) - \cos(a) \sin(b) \sin(i \frac{2\pi}{n}) - \dots \\
& \dots \cancel{\sin(a) \sin(b) \sin(i \frac{2\pi}{n}) \tan(\frac{\pi}{n})} - \cancel{\cos(a) \cos(b) \cos(i \frac{2\pi}{n})} - \cos(a) \sin(b) \cos(i \frac{2\pi}{n}) \tan(\frac{\pi}{n}) + \dots \\
& \dots \sin(a) \cos(b) \sin(i \frac{2\pi}{n}) + \cancel{\sin(a) \sin(b) \sin(i \frac{2\pi}{n}) \tan(\frac{\pi}{n})}
\end{aligned}$$

$$\begin{aligned}
& (\cos(i \frac{2\pi}{n}) \tan(\frac{\pi}{n})) (\sin(a) \cos(b) - \cos(a) \sin(b)) + \sin(i \frac{2\pi}{n}) (\sin(a) \cos(b) - \cos(a) \sin(b)) \\
& (\cos(i \frac{2\pi}{n}) \tan(\frac{\pi}{n}) + \sin(i \frac{2\pi}{n})) (\sin(a) \cos(b) - \cos(a) \sin(b)) \\
& (\cos(i \frac{2\pi}{n}) \tan(\frac{\pi}{n}) + \sin(i \frac{2\pi}{n})) \sin(a - b)
\end{aligned}$$

meaning that we can now simplify the original fraction (denoted **3.1**) to become:

$$\begin{aligned}
& \frac{(\sin(i \frac{2\pi}{n}) \tan(\frac{\pi}{n}) - \cos(i \frac{2\pi}{n})) \sin(a - b)}{(\cos(i \frac{2\pi}{n}) \tan(\frac{\pi}{n}) + \sin(i \frac{2\pi}{n})) \sin(a - b)} \\
& \frac{\sin(i \frac{2\pi}{n}) \tan(\frac{\pi}{n}) - \cos(i \frac{2\pi}{n})}{\cos(i \frac{2\pi}{n}) \tan(\frac{\pi}{n}) + \sin(i \frac{2\pi}{n})} * \frac{\cos(\frac{\pi}{n})}{\cos(\frac{\pi}{n})} \\
& \frac{\sin(i \frac{2\pi}{n}) \sin(\frac{\pi}{n}) - \cos(i \frac{2\pi}{n}) \cos(\frac{\pi}{n})}{\cos(i \frac{2\pi}{n}) \sin(\frac{\pi}{n}) + \sin(i \frac{2\pi}{n}) \cos(\frac{\pi}{n})} \\
& \frac{-(\cos(i \frac{2\pi}{n}) \cos(\frac{\pi}{n}) - \sin(i \frac{2\pi}{n}) \sin(\frac{\pi}{n}))}{\sin(\frac{\pi}{n}) \cos(i \frac{2\pi}{n}) + \cos(\frac{\pi}{n}) \sin(i \frac{2\pi}{n})} \\
& \frac{-\cos(i \frac{2\pi}{n} + \frac{\pi}{n})}{\sin(i \frac{2\pi}{n} + \frac{\pi}{n})} \\
& -\cot(i \frac{2\pi}{n} + \frac{\pi}{n}) \\
& -\cot((2i+1) \frac{\pi}{n})
\end{aligned}$$

so the equation of the "modified" line becomes:

$$y_m = -\cot((2i+1) \frac{\pi}{n})x + \sin_n(a + i \frac{2\pi}{n}) + \cot((2i+1) \frac{\pi}{n}) \cos_n(a + i \frac{2\pi}{n})$$

This leads to the conclusion that when $0 \leq a < b \leq \frac{2\pi}{n}$ the slopes of y_m and y are equal. If $a = 0$, due to **Identity 1.3**, then for any applicable value of b , y_m becomes:

$$y_m = -\cot((2i+1)\frac{\pi}{n})x + \sin(i\frac{2\pi}{n}) + \cot((2i+1)\frac{\pi}{n})\cos(i\frac{2\pi}{n})$$

Meaning that $y_m = L_i$ when $a = 0$ and due to **Identity 1.3**: $\sin_n(a+i\frac{2\pi}{n}) = \sin(i\frac{2\pi}{n})$ and $\cos_n(a+i\frac{2\pi}{n}) = \cos(i\frac{2\pi}{n})$ thus the point at point p_a (located at $(\cos_n(a+i\frac{2\pi}{n}), \sin_n(a+i\frac{2\pi}{n}))$) is equal to v_i .

Because $y_m = L_i$, point p_b (located at $(\cos_n(b+i\frac{2\pi}{n}), \sin_n(b+i\frac{2\pi}{n}))$) is always on line segment L_i , thus when $a \neq 0$, p_a must also always be on line segment L_i .

This means that for any value x where $0 \leq x \leq \frac{2\pi}{n}$, the point p_x (found at $(\cos_n(x+i\frac{2\pi}{n}), \sin_n(x+i\frac{2\pi}{n}))$) falls on line L_i of shape s_n , and since $i \in \mathbb{Z}$, any point p_θ (found at $(\cos_n(\theta), \sin_n(\theta))$) will always be located on the perimeter of shape s_n .

5 Shape Equations

From these formulas, the equation for any regular polygon can be derived. The equation is found through a mutation of the ellipse equation by $d_n(\theta)$:

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = d_n(\theta)^2 * 1$$

To derive theta, one would start with $\theta = \arctan(\frac{(y-k)a}{(x-h)b})$, which would find the angle at the correct point offset from the center, but due to x being squared, the shape is mirrored over $x = h$, which will work for even sided polygons as they are symmetric over $x = h$, but not for odd sided polygons due to their anti-symmetry (this is also the case over the line $y = k$ but all shapes, regardless of side parity, are mirrored over $y = k$). This can be fixed by noting that the angle is calculated correctly in Quadrants I and IV (relative to (h, k) as the origin), and π radians off from the expected angle in Quadrants II and III. Thus when $x < h$, π must be added to θ . Using the sign function and modifying it to equal 1 when $x < h$ and 0 when $x \geq h$ results in the formula:

$$-\frac{1}{2}\left(\frac{x-h}{|x-h|} - 1\right)$$

which when multiplied by π and added to θ becomes:

$$\arctan\left(\frac{(y-k)a}{(x-h)b}\right) - \frac{\pi}{2}\left(\frac{x-h}{|x-h|} - 1\right)$$

meaning that the standard equation is:

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = d_n\left(\arctan\left(\frac{(y-k)a}{(x-h)b}\right) - \frac{\pi}{2}\left(\frac{x-h}{|x-h|} - 1\right)\right)^2$$

As d_n takes an angle, a constant value can be added to θ to make any shape rotate, allowing for easy rotation calculations.