

# Trigonometric Coefficients for Regular N-Sided Polygons

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# 1 Introduction

This paper derives a function that generates a coefficient that transforms sine and cosine into calculating points for any n-sided regular polygon. This is done through using inverse trigonometric functions to create a periodic function that increases and decreases linearly over each period, allowing the creation of a coefficient which when multiplied by sine and cosine serves to transform them into an equivalent function for non-circle polygons.

After deriving the equation we then prove that it represents a point on the perimeter of a unit n-gon by simplifying the slope-intercept form of a constructing line of the unit n-gon and two points between the modified sine and cosine showing they are the same.

Through this, we can derive a general formula for any shape using the sign function and the newly proven coefficient function. This general function requires the use of an "input" angle, which is calculated using x and y for each point, and due to this can be rotated by simply adding or subtracting a constant from the angle calculation inside the coefficient function's input.

# 2 Theroems

**Theorem 1.1**  $\arccos(\cos(ax))$  is periodic over  $\frac{2\pi}{a}$ .

$\cos(ax)$  is periodic over  $\frac{2\pi}{a}$  thus  $\arccos(\cos(ax))$  must also be.

**Theorem 1.2** For  $x \in \mathbb{R}$ ,  $\arccos(\cos(ax)) \in [0, \pi]$ .

$\cos(x)$  has a domain over  $\mathbb{R}$  and a range of  $[-1, 1]$ ,  $\arccos(x)$  has a domain over  $[-1, 1]$  and a range of  $[0, \pi]$ . Due to the range of  $\cos$  being the domain of  $\arccos$ , all real numbers can be an input of  $x$ , and all numbers on the interval  $[-1, 1]$  can be output.

**Theorem 1.3**  $\cos(n(a + \frac{2\pi}{n})) = \cos(na)$

$\cos(n(a + \frac{2\pi}{n})) = \cos(na + 2\pi)$  which due to cosine's periodic nature evaluates to  $\cos(na)$

**Theorem 2.1**  $\sin(a) - \sin(b) = 2 \sin(\frac{a-b}{2}) \cos(\frac{a+b}{2})$

**Theorem 2.2**  $\cos(a) - \cos(b) = -2 \sin(\frac{a+b}{2}) \sin(\frac{a-b}{2})$

**Theorem 2.3**  $\frac{\sin(a) - \sin(b)}{\cos(a) - \cos(b)} = -\cot(\frac{a+b}{2})$

$$\begin{aligned} \frac{\sin(a) - \sin(b)}{\cos(a) - \cos(b)} &= \frac{2 \sin(\frac{a-b}{2}) \cos(\frac{a+b}{2})}{-2 \sin(\frac{a+b}{2}) \sin(\frac{a-b}{2})} \\ &= \frac{2 \cos(\frac{a+b}{2})}{-2 \sin(\frac{a+b}{2})} \\ &= -\cot(\frac{a+b}{2}) \end{aligned}$$

### 3 Trigonometric Equations

Let  $n = \{x : x \in \mathbb{Z}^+ \text{ and } x \geq 3\}$ , and

$$\begin{aligned} d_n(\theta) &= \cos\left(\frac{\pi}{n}\right) \sec\left(\frac{\pi - \arccos(\cos(n\theta))}{n}\right) \\ \cos_n(\theta) &= d_n(\theta) \cos(\theta) \\ \sin_n(\theta) &= d_n(\theta) \sin(\theta) \end{aligned}$$

This is derived through the original functions:

$$\begin{aligned} h_n(\theta) &= \sin\left(\frac{\pi}{2} - \frac{\pi}{n} + (\theta \bmod \frac{2\pi}{n})\right) - \sin\left(\frac{\pi}{2} - \frac{\pi}{n}\right) \\ d_n(\theta) &= \frac{h_n(\theta)}{\sin\left(\frac{\pi}{2} - \frac{\pi}{n} + (\theta \bmod \frac{2\pi}{n})\right)} \\ \cos_n(\theta) &= (1 - d_n(\theta)) \cos(\theta) \\ \sin_n(\theta) &= (1 - d_n(\theta)) \sin(\theta) \end{aligned}$$

These functions follow the premise that any  $n$ -sided regular polygon (shape  $s_n$ ) centered at  $(0,0)$  has a circumcircle where each vertex touches the circumference of the circumcircle.

Arc  $a$  can be constructed which is  $\frac{1}{n^{\text{th}}}$  of the circumcircle, its diameter is the same length as any line which constructs  $s_n$ , and it's circumference equals to the circumference of the section of the circumcircle spanning from adjacent vertices on  $s_n$ .

Thus, the height of  $a$  at a given point on a circle subtracted from the radius of the circle will result in the distance from the center of  $s_n$  to the point on the perimeter of  $s_n$  which intersects the ray of angle  $\theta$  drawn in standard position.

$h_n(\theta)$  serves to find the height of  $a$  at a position by splitting the circumcircle into  $n^{\text{th}}$  (resulting in a period of  $\frac{2\pi}{n}$ ), finding the height of a point on  $a$  through sine and changing from the "left" vertex to the "right" vertex over the period of  $\frac{2\pi}{n}$  (the length of  $a$ ). The use of mod makes the function periodic. It then subtracts by the height of a vertex so that  $h_n(\theta) = 0$  when  $\theta = 0$

$d_n(\theta)$  divides the height,  $h_n(\theta)$ , by the angle between the assumed intersection point on  $s_n$  and the intersection point on the circumcircle to get the actual distance.

These functions then simplify to:

$$\begin{aligned} d_n(\theta) &= \cos\left(\frac{\pi}{n}\right) \sec\left(\frac{\pi}{n} - (\theta \bmod \frac{2\pi}{n})\right) \\ \cos_n(\theta) &= d_n(\theta) \cos(\theta) \\ \sin_n(\theta) &= d_n(\theta) \sin(\theta) \end{aligned}$$

Which is not algebraic due to modulo (mod), yet it can be noted that due to the nature of the original  $h_n(\theta)$ ,  $\theta \bmod \frac{2\pi}{n}$  can be substituted for an equivalent function which has an

equal period of  $\frac{2\pi}{n}$ , increases linearly from 0 to  $\frac{\pi}{n}$  over  $[0, \frac{\pi}{n}]$  and decreases linearly at the same rate over  $[\frac{\pi}{n}, \frac{2\pi}{n}]$ .

These requirements can be met by  $\frac{\arccos(\cos(n\theta))}{n}$ , which due to **Theorem 1.1** and **Theorem 1.2** is periodic over  $\frac{2\pi}{n}$  has a domain of  $n\theta \in \mathbb{R}$ , and thus  $\theta \in \mathbb{R}$ , and a range of  $[0, \frac{\pi}{n}]$ , thus the above equations can be changed to:

$$\begin{aligned} d_n(\theta) &= \cos\left(\frac{\pi}{n}\right) \sec\left(\frac{\pi - \arccos(\cos(n\theta))}{n}\right) \\ \cos_n(\theta) &= d_n(\theta) \cos(\theta) \\ \sin_n(\theta) &= d_n(\theta) \sin(\theta) \end{aligned}$$

### 3.1 Identities

**Identity 1.1** Let  $i = \mathbb{Z}$ , due to **Theorem 1.3**:

$$\begin{aligned} \cos_n(\theta + i \frac{2\pi}{n}) &= d_n(\theta) \cos(\theta + i \frac{2\pi}{n}) \\ \sin_n(\theta + i \frac{2\pi}{n}) &= d_n(\theta) \sin(\theta + i \frac{2\pi}{n}) \end{aligned}$$

**Identity 1.2** When  $0 \leq \theta \leq \frac{2\pi}{n}$  then  $d_n(\theta) = \cos(\frac{\pi}{n}) \sec(\frac{\pi}{n} - \theta)$

When  $\theta$  only spans one period,  $\frac{\arccos(\cos(n\theta))}{n}$  is not required to make the function periodic.

**Identity 1.3** Let  $i = \mathbb{Z}$  then:

$$\begin{aligned} \cos_n(i \frac{2\pi}{n}) &= (1) \cos(i \frac{2\pi}{n}) = \cos(i \frac{2\pi}{n}) \\ \sin_n(i \frac{2\pi}{n}) &= (1) \sin(i \frac{2\pi}{n}) = \sin(i \frac{2\pi}{n}) \end{aligned}$$

This is true because  $\arccos(\cos(n\theta))$  evaluates to 0 whenever  $\theta = i \frac{2\pi}{n}$ , which means  $\sec(\frac{\pi - \arccos(\cos(n\theta))}{n})$  simplifies to  $\sec(\frac{\pi}{n})$  so  $d_n(i \frac{2\pi}{n}) = \sec(\frac{\pi}{n}) \cos(\frac{\pi}{n}) = 1$

## 4 Proof

Let  $i, n = \mathbb{Z}, n \geq 3$ . Shape  $s_n$  is made of  $n$  vertices and  $n$  line segments with the  $i^{\text{th}}$  vertex ( $v_i$ ) at the point  $(\cos(i \frac{2\pi}{n}), \sin(i \frac{2\pi}{n}))$ . The  $i^{\text{th}}$  line segment,  $L_i$ , spans between  $v_i$  and  $v_{i+1}$  and is represented by the equation

$$L_i = \frac{\sin((i+1) \frac{2\pi}{n}) - \sin(i \frac{2\pi}{n})}{\cos((i+1) \frac{2\pi}{n}) - \cos(i \frac{2\pi}{n})} x + \sin(i \frac{2\pi}{n}) - \frac{\sin((i+1) \frac{2\pi}{n}) - \sin(i \frac{2\pi}{n})}{\cos((i+1) \frac{2\pi}{n}) - \cos(i \frac{2\pi}{n})} \cos(i \frac{2\pi}{n})$$

which due to **Theorem 2.1** and **Theorem 2.2** simplifies to

$$L_i = -\cot((2i+1)\frac{\pi}{n})x + \sin(i\frac{2\pi}{n}) + \cot((2i+1)\frac{\pi}{n})\cos(i\frac{2\pi}{n})$$

Let  $0 \leq a < b \leq \frac{2\pi}{n}$ ,  $\alpha = a + i\frac{2\pi}{n}$ , and  $\beta = b + i\frac{2\pi}{n}$ : a "modified" line can be drawn with the equation  $\sin_n(\theta)$  and  $\cos_n(\theta)$  which would be written as

$$y_m = \frac{\sin_n(\beta) - \sin_n(\alpha)}{\cos_n(\beta) - \cos_n(\alpha)}x + \sin_n(\alpha) - \frac{\sin_n(\beta) - \sin_n(\alpha)}{\cos_n(\beta) - \cos_n(\alpha)}\cos_n(\alpha)$$

the slope of  $y_m$  can be simplified through these steps:

$$\begin{aligned} & \frac{d_n(\beta)\sin(\beta) - d_n(\alpha)\sin(\alpha)}{d_n(\beta)\cos(\beta) - d_n(\alpha)\cos(\alpha)} \\ & \frac{\cancel{\cos(\frac{\pi}{n})} \sec(\frac{\pi}{n}-b)\sin(\beta) - \sec(\frac{\pi}{n}-a)\sin(\alpha)}{\cancel{\cos(\frac{\pi}{n})} \sec(\frac{\pi}{n}-b)\cos(\beta) - \sec(\frac{\pi}{n}-a)\cos(\alpha)} * \frac{\cos(\frac{\pi}{n}-a)\cos(\frac{\pi}{n}-b)}{\cos(\frac{\pi}{n}-a)\cos(\frac{\pi}{n}-b)} \\ & \frac{\sin(\beta)\cos(\frac{\pi}{n}-a) - \sin(\alpha)\cos(\frac{\pi}{n}-b)}{\cos(\beta)\cos(\frac{\pi}{n}-a) - \cos(\alpha)\cos(\frac{\pi}{n}-b)} \\ & \frac{\sin(\beta)(\cos(\frac{\pi}{n})\cos(a) + \sin(\frac{\pi}{n})\sin(a)) - \sin(\alpha)(\cos(\frac{\pi}{n})\cos(b) + \sin(\frac{\pi}{n})\sin(b))}{\cos(\beta)(\cos(\frac{\pi}{n})\cos(a) + \sin(\frac{\pi}{n})\sin(a)) - \cos(\alpha)(\cos(\frac{\pi}{n})\cos(b) + \sin(\frac{\pi}{n})\sin(b))} * \frac{\sec(\frac{\pi}{n})}{\sec(\frac{\pi}{n})} \\ & \frac{\sin(\beta)(\cos(a) + \tan(\frac{\pi}{n})\sin(a)) - \sin(\alpha)(\cos(b) + \tan(\frac{\pi}{n})\sin(b))}{\cos(\beta)(\cos(a) + \tan(\frac{\pi}{n})\sin(a)) - \cos(\alpha)(\cos(b) + \tan(\frac{\pi}{n})\sin(b))} \quad (3.1) \end{aligned}$$

Splitting the equation up the numerator simplifies like so:

$$\begin{aligned} & \sin(\beta)(\cos(a) + \tan(\frac{\pi}{n})\sin(a)) - \sin(\alpha)(\cos(b) + \tan(\frac{\pi}{n})\sin(b)) \\ & (\sin(b)\cos(i\frac{2\pi}{n}) + \cos(b)\sin(i\frac{2\pi}{n}))(\cos(a) + \tan(\frac{\pi}{n})\sin(a)) - \dots \\ & \dots (\sin(a)\cos(i\frac{2\pi}{n}) + \cos(a)\sin(i\frac{2\pi}{n}))(\cos(b) + \tan(\frac{\pi}{n})\sin(b)) \\ & \cos(a)\sin(b)\cos(i\frac{2\pi}{n}) + \cancel{\sin(a)\sin(b)\cos(i\frac{2\pi}{n})\tan(\frac{\pi}{n})} + \cancel{\cos(a)\cos(b)\sin(i\frac{2\pi}{n})} + \dots \\ & \dots \sin(a)\cos(b)\sin(i\frac{2\pi}{n})\tan(\frac{\pi}{n}) - \sin(a)\cos(b)\cos(i\frac{2\pi}{n}) - \cancel{\sin(a)\sin(b)\cos(i\frac{2\pi}{n})\tan(\frac{\pi}{n})} - \dots \\ & \dots \cancel{\cos(a)\cos(b)\sin(i\frac{2\pi}{n})} - \cos(a)\sin(b)\sin(i\frac{2\pi}{n})\tan(\frac{\pi}{n}) \end{aligned}$$

$$\cos(i \frac{2\pi}{n})(\cos(a) \sin(b) - \sin(a) \cos(b)) + \sin(i \frac{2\pi}{n}) \tan(\frac{\pi}{n})(\sin(a) \cos(b) - \cos(a) \sin(b))$$

$$\sin(i \frac{2\pi}{n}) \tan(\frac{\pi}{n})(\sin(a) \cos(b) - \cos(a) \sin(b)) - \cos(i \frac{2\pi}{n})(\sin(a) \cos(b) - \cos(a) \sin(b))$$

$$(\sin(i \frac{2\pi}{n}) \tan(\frac{\pi}{n}) - \cos(i \frac{2\pi}{n}))(\sin(a) \cos(b) - \cos(a) \sin(b))$$

$$(\sin(i \frac{2\pi}{n}) \tan(\frac{\pi}{n}) - \cos(i \frac{2\pi}{n})) \sin(a - b)$$

and the denominator simplifies to

$$\cos(\beta)(\cos(a) + \tan(\frac{\pi}{n}) \sin(a)) - \cos(\alpha)(\cos(b) + \tan(\frac{\pi}{n}) \sin(b))$$

$$(\cos(b) \cos(i \frac{2\pi}{n}) - \sin(b) \sin(i \frac{2\pi}{n}))(\cos(a) + \tan(\frac{\pi}{n}) \sin(a)) - \dots$$

$$\dots (\cos(a) \cos(i \frac{2\pi}{n}) - \sin(a) \sin(i \frac{2\pi}{n}))(\cos(b) + \tan(\frac{\pi}{n}) \sin(b))$$

$$\begin{aligned} & \cancel{\cos(a) \cos(b) \cos(i \frac{2\pi}{n})} + \sin(a) \cos(b) \cos(i \frac{2\pi}{n}) \tan(\frac{\pi}{n}) - \cos(a) \sin(b) \sin(i \frac{2\pi}{n}) - \dots \\ & \dots \cancel{\sin(a) \sin(b) \sin(i \frac{2\pi}{n}) \tan(\frac{\pi}{n})} - \cancel{\cos(a) \cos(b) \cos(i \frac{2\pi}{n})} - \cos(a) \sin(b) \cos(i \frac{2\pi}{n}) \tan(\frac{\pi}{n}) + \dots \\ & \dots \sin(a) \cos(b) \sin(i \frac{2\pi}{n}) + \cancel{\sin(a) \sin(b) \sin(i \frac{2\pi}{n}) \tan(\frac{\pi}{n})} \end{aligned}$$

$$(\cos(i \frac{2\pi}{n}) \tan(\frac{\pi}{n}))(\sin(a) \cos(b) - \cos(a) \sin(b)) + \sin(i \frac{2\pi}{n})(\sin(a) \cos(b) - \cos(a) \sin(b))$$

$$(\cos(i \frac{2\pi}{n}) \tan(\frac{\pi}{n}) + \sin(i \frac{2\pi}{n}))(\sin(a) \cos(b) - \cos(a) \sin(b))$$

$$(\cos(i \frac{2\pi}{n}) \tan(\frac{\pi}{n}) + \sin(i \frac{2\pi}{n})) \sin(a - b)$$

meaning that we can now simplify the original fraction (denoted **3.1**) to become:

$$\begin{aligned}
& \frac{(\sin(i \frac{2\pi}{n}) \tan(\frac{\pi}{n}) - \cos(i \frac{2\pi}{n})) \sin(a - b)}{(\cos(i \frac{2\pi}{n}) \tan(\frac{\pi}{n}) + \sin(i \frac{2\pi}{n})) \sin(a - b)} \\
& \frac{\sin(i \frac{2\pi}{n}) \tan(\frac{\pi}{n}) - \cos(i \frac{2\pi}{n})}{\cos(i \frac{2\pi}{n}) \tan(\frac{\pi}{n}) + \sin(i \frac{2\pi}{n})} * \frac{\cos(\frac{\pi}{n})}{\cos(\frac{\pi}{n})} \\
& \frac{\sin(i \frac{2\pi}{n}) \sin(\frac{\pi}{n}) - \cos(i \frac{2\pi}{n}) \cos(\frac{\pi}{n})}{\cos(i \frac{2\pi}{n}) \sin(\frac{\pi}{n}) + \sin(i \frac{2\pi}{n}) \cos(\frac{\pi}{n})} \\
& \frac{-(\cos(i \frac{2\pi}{n}) \cos(\frac{\pi}{n}) - \sin(i \frac{2\pi}{n}) \sin(\frac{\pi}{n}))}{\sin(\frac{\pi}{n}) \cos(i \frac{2\pi}{n}) + \cos(\frac{\pi}{n}) \sin(i \frac{2\pi}{n})} \\
& \frac{-\cos(i \frac{2\pi}{n} + \frac{\pi}{n})}{\sin(i \frac{2\pi}{n} + \frac{\pi}{n})} \\
& -\cot(i \frac{2\pi}{n} + \frac{\pi}{n}) \\
& -\cot((2i+1) \frac{\pi}{n})
\end{aligned}$$

so the equation of the "modified" line becomes:

$$y_m = -\cot((2i+1) \frac{\pi}{n})x + \sin_n(\alpha) + \cot((2i+1) \frac{\pi}{n}) \cos_n(\alpha)$$

This leads to the conclusion that when  $i \frac{2\pi}{n} \leq a < b \leq (i+1) \frac{2\pi}{n}$  the slopes of  $y_m$  and  $L_i$  are equal.

If  $a = 0$ , due to **Identity 1.3**, then for any applicable value of  $b$ ,  $y_m$  becomes:

$$y_m = -\cot((2i+1) \frac{\pi}{n})x + \sin(i \frac{2\pi}{n}) + \cot((2i+1) \frac{\pi}{n}) \cos(i \frac{2\pi}{n})$$

Meaning that  $y_m = L_i$  when  $a = 0$  (denoted  $y_{m,a=0}$ ) and due to **Identity 1.3**:  $\sin_n(a + i \frac{2\pi}{n}) = \sin(i \frac{2\pi}{n})$  and  $\cos_n(a + i \frac{2\pi}{n}) = \cos(i \frac{2\pi}{n})$  thus the point  $p_a$ , located at  $(\cos_n(a + i \frac{2\pi}{n}), \sin_n(a + i \frac{2\pi}{n}))$ , is equal to  $v_i$ .

Because  $y_{m,a=0} = L_i$ , point  $p_b$  (located at  $(\cos_n(b + i \frac{2\pi}{n}), \sin_n(b + i \frac{2\pi}{n}))$ ) is always on line segment  $L_i$ , thus when  $a \neq 0$ ,  $p_a$  must also always be on line segment  $L_i$ .

This means that for any value  $x$  where  $0 \leq x \leq \frac{2\pi}{n}$ , the point  $p_x$ , located at  $(\cos_n(x + i \frac{2\pi}{n}), \sin_n(x + i \frac{2\pi}{n}))$ , falls on line  $L_i$  of shape  $s_n$ , and since  $i \in \mathbb{Z}$ , any point  $p_\theta$ , located at  $(\cos_n(\theta), \sin_n(\theta))$ , will always be on the perimeter of shape  $s_n$ .

## 5 Shape Equations

From these formulas, the equation for any regular polygon can be derived. The equation is found through a mutation of the ellipse equation by  $d_n(\theta)$ :

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = d_n(\theta)^2 * 1$$



To derive theta, one would start with  $\theta = \arctan(\frac{(y-k)a}{(x-h)b})$ , which would find the angle at the correct point offset from the center, but due to x being squared, the shape is mirrored over  $x = h$ , which will work for even sided polygons as they are symmetric over  $x = h$ , but not for odd sided polygons due to their anti-symmetry (this is also the case over the line  $y = k$  but all shapes, regardless of side parity, are mirrored over  $y = k$ ). This can be fixed by noting that the angle is calculated correctly in Quadrants I and IV (relative to  $(h, k)$  as the origin), and  $\pi$  radians off from the expected angle in Quadrants II and III. Thus when  $x < h$ ,  $\pi$  must be added to  $\theta$ . Using the sign function and modifying it to equal 1 when  $x < h$  and 0 when  $x \geq h$  results in the formula:

$$-\frac{1}{2}(\frac{x-h}{|x-h|} - 1)$$

which when multiplied by  $\pi$  and added to  $\theta$  becomes:

$$\arctan(\frac{(y-k)a}{(x-h)b}) - \frac{\pi}{2}(\frac{x-h}{|x-h|} - 1)$$

meaning that the standard equation is:

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = d_n(\arctan(\frac{(y-k)a}{(x-h)b}) - \frac{\pi}{2}(\frac{x-h}{|x-h|} - 1))^2$$

As  $d_n$  takes an angle, a constant value can be added to  $\theta$  to make any shape rotate, allowing for easy rotation calculations.