Trigonometric Functions on the Perimeter of any Regular Polygon

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1   Introduction

This paper derives and proves a set of functions that serve to find the point of intersection between an angle in standard position and an n-sided regular polygon through only algebra. Through these, the standard equation for any regular polygon can be derived.

2   Theroems

Theorem 1.1 $\arccos(\cos(ax))$ is periodic over $\frac{2\pi}{a*n}$.
$\cos(ax)$ is periodic over $\frac{2\pi}{a*n}$ thus $\arccos(\cos(ax))$ must also be.
Theorem 1.2 For $x \in \mathbb{R}$ $\arccos(\cos(ax)) \in [0, \pi]$.
$\cos(x)$ has a domain over $\mathbb{R}$ and a range of [-1,1], $\arccos(x)$ has a domain over [-1,1] and a range of $[0, \pi]$. Due to the range of cos being the domain of arccos, all real numbers can be an input of x, and all numbers on the interval [-1,1] can be an output.
Theorem 1.3 $\cos(n(a + \frac{2\pi}{n})) = \cos(na)$
$\cos(n(a + \frac{2\pi}{n}))$ expands to $\cos(na + 2\pi))$ which due to cosine's periodic nature evaluates to $\cos(na)$
Theorem 2.1 $\sin(\alpha) - \sin(\beta) = 2\sin(\frac{\alpha+\beta}{2})\cos(\frac{\alpha-\beta}{2})$
Theorem 2.2 $\cos(\alpha) - \cos(\beta) = -2\sin(\frac{\alpha+\beta}{2})\sin(\frac{\alpha-\beta}{2})$

3   Trigonometric Equations

Let $n = \{x : x \in \mathbb{Z}^+ \text{ and } x \geq 3\}$

$$d_n(\theta) = \cos(\frac{\pi}{n})\sec(\frac{\pi - \arccos(\cos(n\theta))}{n})$$
$$\cos_n(\theta) = d_n(\theta)\cos(\theta)$$
$$\sin_n(\theta) = d_n(\theta)\sin(\theta)$$

This is derived through the original functions:

$$h_n(\theta) = \sin(\frac{\pi}{2} - \frac{\pi}{n} + (\theta \mod \frac{2\pi}{n})) - \sin(\frac{\pi}{2} - \frac{\pi}{n})$$
$$d_n(\theta) = \frac{h_n(\theta)}{\sin(\frac{\pi}{2} - \frac{\pi}{n} - (\theta \mod \frac{2\pi}{n}))}$$
$$\cos_n(\theta) = (1 - d_n)cos(\theta)$$
$$\sin_n(\theta) = (1 - d_n)sin(\theta)$$

These functions follow the premise that any n-sided regular polygon (shape s_n) centered at (0,0) can be circumscribed within a circle where each vertex touches the circumference of the circumcircle.

Arc a_n can be constructed which is $\frac{1}{n^{th}}$ of a circle, its diameter will be the same length as any line which constructs s_n, and it's circumference will equal to the circumference of the section of the circumcircle spanning from adjacent vertices on s_n.

Thus, the height of a_n at a given point on a circle subtracted from the radius of the circle will result in the distance from the midpoint of s_n to the point on the perimeter of s_n which intersects the ray of angle θ drawn in standard position.

$h_n(\theta)$ serves to find the height of a_n at a position by splitting the circumcircle into n^{ths} (resulting in a period of $\frac{2\pi}{n}$), finding the height of a point on a_n through sin and changing from the "left" vertex to the "right" vertex over the period of a_n. The use of mod makes the function repeat over each period. It then subtracts by the height of the "right" vertex to equal 0 when $\theta = 0$

$d_n(\theta)$ divides the height ($h_n(\theta)$) by the angle between the assumed intersection point on s_n and the intersection point on the circumcircle to get the actual distance.

These functions then simplify to:

$$d_n(\theta) = \cos(\frac{\pi}{n})\sec(\frac{\pi}{n} - (\theta \mod \frac{2\pi}{n}))$$
$$\cos_n(\theta) = d_n(\theta)cos(\theta)$$
$$\sin_n(\theta) = d_n(\theta)sin(\theta)$$

Which is not algebraic due to mod (and the resultant floor function behind it), yet it can be noted that due to the nature of the original $h_n(\theta)$, $\theta \mod \frac{2\pi}{n}$ can be substituted for an equivalent function which has an equal period of $\frac{2\pi}{n}$ but increases lineraly over $[0, \frac{2\pi}{n}]$ to $[0, \frac{\pi}{n}]$ and decreases linearly at the same rate from $[\frac{\pi}{n}, \frac{2\pi}{n}]$.

These requirements can be met by $\frac{\arccos(\cos(n\theta))}{n}$, which due to **Theorem 1.1** is periodic over $\frac{2\pi}{n}$ and due to **Theorem 1.2** has a domain of $n\theta \in \mathbb{R}$ and a range of $[0, \frac{pi}{n}]$ thus changing the above equations to:

$$d_n(\theta) = \cos(\frac{\pi}{n})\sec(\frac{\pi - \arccos(\cos(n\theta))}{n})$$
$$\cos_n(\theta) = d_n(\theta)cos(\theta)$$
$$\sin_n(\theta) = d_n(\theta)sin(\theta)$$

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3.1 Identities

Identity 1.1

$$\cos_n(\theta + \frac{2\pi}{n}) = d_n(\theta)\cos(\theta + \frac{2\pi}{n})$$

$$\sin_n(\theta + \frac{2\pi}{n}) = d_n(\theta)\sin(\theta + \frac{2\pi}{n})$$

This identity is true due to **Theorem 1.3**

Identity 1.2 When $0 \leq \theta \leq \frac{2\pi}{n}$ then $d_n(\theta) = \cos(\frac{\pi}{n})\sec(\frac{\pi}{n} - \theta)$

When θ only spans one period, $\arccos(\cos(n\theta))$ is not required to make the function periodic.

Identity 1.3 Let $i = \mathbb{Z}$ then:

$$\cos_n(\frac{2i\pi}{n}) = (1)\cos(\frac{2i\pi}{n}) = \cos(\frac{2i\pi}{n})$$

$$\sin_n(\frac{2i\pi}{n}) = (1)\sin(\frac{2i\pi}{n}) = \sin(\frac{2i\pi}{n})$$

This is true because $\arccos(\cos(n\theta))$ evaluates to 0 whenever $\theta = \frac{2i\pi}{n}$, which means $\sec(\frac{\pi}{n} - \frac{\arccos(\cos(n\theta))}{n})$ simplifies to $\sec(\frac{\pi}{n})$ which means $d_n = \sec(\frac{\pi}{n})\cos(\frac{\pi}{n}) = 1$

4 Proof

Let $i = \mathbb{Z}$. Assume s_n is made of n vertices and n line segments with the ith vertex at the point $(\cos(\frac{2i\pi}{n}), \sin(\frac{2i\pi}{n}))$. The ith line segment, L_i, spans between v_i and v_{i+1} and is represented by the equation

$$L_i = \frac{\sin((i+1)\frac{2\pi}{n}) - \sin(\frac{2i\pi}{n})}{\cos((i+1)\frac{2\pi}{n}) - \cos(\frac{2i\pi}{n})}x + \sin(\frac{2i\pi}{n}) - \frac{\sin((i+1)\frac{2\pi}{n}) - \sin(\frac{2i\pi}{n})}{\cos((i+1)\frac{2\pi}{n}) - \cos(\frac{2i\pi}{n})}\cos(\frac{2i\pi}{n})$$

which due to **Theorem 2.1** and **Theorem 2.2** simplifies to

$$L_i = -\cot((2i+1)\frac{\pi}{n})x + \sin(\frac{2i\pi}{n}) + \cot((2i+1)\frac{\pi}{n})x$$

Let $0 \leq a < b \leq \frac{2\pi}{n}$, a "modified" line can be drawn with the equation $\sin_n(\theta)$ and $\cos_n(\theta)$ which would be written as

$$y_m = \frac{\sin_n(b + \frac{2i\pi}{n}) - \sin_n(a + \frac{2i\pi}{n})}{\cos_n(b + \frac{2i\pi}{n}) - \cos_n(a + \frac{2i\pi}{n})}x + \sin_n(a) - \frac{\sin_n(b + \frac{2i\pi}{n}) - \sin_n(a + \frac{2i\pi}{n})}{\cos_n(b + \frac{2i\pi}{n}) - \cos_n(a + \frac{2i\pi}{n})}\cos_n(a)$$

the slope of y_m can be simplified through these steps:

$$\frac{d_n(b + \frac{2i\pi}{n})\sin(b + \frac{2i\pi}{n}) - d_n(a + \frac{2i\pi}{n})\sin(a + \frac{2i\pi}{n})}{d_n(b + \frac{2i\pi}{n})\cos(b + \frac{2i\pi}{n}) - d_n(a + \frac{2i\pi}{n})\cos(a + \frac{2i\pi}{n})}$$

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$$\frac{\cancel{\cos(\frac{\pi}{n})}}{\cancel{\cos(\frac{\pi}{n})}} \frac{\sec(\frac{\pi}{n}-b)\sin(b+\frac{2\,i\,\pi}{n})-\sec(\frac{\pi}{n}-a)\sin(a+\frac{2\,i\,\pi}{n})}{\sec(\frac{\pi}{n}-b)\cos(b+\frac{2\,i\,\pi}{n})-\sec(\frac{\pi}{n}-a)\cos(a+\frac{2\,i\,\pi}{n})} * \frac{\cos(\frac{\pi}{n}-a)\cos(\frac{\pi}{n}-b)}{\cos(\frac{\pi}{n}-a)\cos(\frac{\pi}{n}-b)}$$

$$\frac{\sin(b+\frac{2\,i\,\pi}{n})\cos(\frac{\pi}{n}-a)-\sin(a+\frac{2\,i\,\pi}{n})\cos(\frac{\pi}{n}-b)}{\cos(b+\frac{2\,i\,\pi}{n})\cos(\frac{\pi}{n}-a)-\cos(a+\frac{2\,i\,\pi}{n})\cos(\frac{\pi}{n}-b)}$$

$$\frac{\sin(b+\frac{2\,i\,\pi}{n})(\cos(\frac{\pi}{n})\cos(a)+\sin(\frac{\pi}{n})\sin(a))-\sin(a+\frac{2\,i\,\pi}{n})(\cos(\frac{\pi}{n})\cos(b)+\sin(\frac{\pi}{n})\sin(b))}{\cos(b+\frac{2\,i\,\pi}{n})(\cos(\frac{\pi}{n})\cos(a)+\sin(\frac{\pi}{n})\sin(a))-\cos(a+\frac{2\,i\,\pi}{n})(\cos(\frac{\pi}{n})\cos(b)+\sin(\frac{\pi}{n})\sin(b))} * \frac{\sec(\frac{\pi}{n})}{\sec(\frac{\pi}{n})}$$

$$\frac{\sin(b+\frac{2\,i\,\pi}{n})(\cos(a)+\tan(\frac{\pi}{n})\sin(a))-\sin(a+\frac{2\,i\,\pi}{n})(\cos(b)+\tan(\frac{\pi}{n})\sin(b))}{\cos(b+\frac{2\,i\,\pi}{n})(\cos(a)+\tan(\frac{\pi}{n})\sin(a))-\cos(a+\frac{2\,i\,\pi}{n})(\cos(b)+\tan(\frac{\pi}{n})\sin(b))} \quad \textbf{(3.1)}$$

Splitting the equation up the numerator simplifies like so:

$$\sin(b+\frac{2\,i\,\pi}{n})(\cos(a)+\tan(\frac{\pi}{n})\sin(a))-\sin(a+\frac{2\,i\,\pi}{n})(\cos(b)+\tan(\frac{\pi}{n})\sin(b))$$

$$(\sin(b)\cos(\frac{2\,i\,\pi}{n})+\cos(b)\sin(\frac{2\,i\,\pi}{n}))(\cos(a)+\tan(\frac{\pi}{n})\sin(a))-\ldots$$
$$\ldots(\sin(a)\cos(\frac{2\,i\,\pi}{n})+\cos(a)\sin(\frac{2\,i\,\pi}{n}))(\cos(b)+\tan(\frac{\pi}{n})\sin(b))$$

$$\cos(a)\sin(b)\cos(\frac{2\,i\,\pi}{n})+\cancel{\sin(a)\sin(b)\cos(\frac{2\,i\,\pi}{n})\tan(\frac{\pi}{n})}+\cancel{\cos(a)\cos(b)\sin(\frac{2\,i\,\pi}{n})}+\ldots$$
$$\ldots\sin(a)\cos(b)\sin(\frac{2\,i\,\pi}{n})\tan(\frac{\pi}{n})-\sin(a)\cos(b)\cos(\frac{2\,i\,\pi}{n})-\cancel{\sin(a)\sin(b)\cos(\frac{2\,i\,\pi}{n})\tan(\frac{\pi}{n})}-\ldots$$
$$\ldots\cancel{\cos(a)\cos(b)\sin(\frac{2\,i\,\pi}{n})}-\cos(a)\sin(b)\sin(\frac{2\,i\,\pi}{n})\tan(\frac{\pi}{n})$$

$$\cos(\frac{2\,i\,\pi}{n})(\cos(a)\sin(b)-\sin(a)\cos(b))+\sin(\frac{2\,i\,\pi}{n})\tan(\frac{\pi}{n})(\sin(a)\cos(b)-\cos(a)\sin(b))$$

$$\sin(\frac{2\,i\,\pi}{n})\tan(\frac{\pi}{n})(\sin(a)\cos(b)-\cos(a)\sin(b))-\cos(\frac{2\,i\,\pi}{n})(\sin(a)\cos(b)-\cos(a)\sin(b))$$

$$(\sin(\frac{2\,i\,\pi}{n})\tan(\frac{\pi}{n})-\cos(\frac{2\,i\,\pi}{n}))(\sin(a)\cos(b)-\cos(a)\sin(b))$$

$$(\sin(\frac{2\,i\,\pi}{n})\tan(\frac{\pi}{n})-\cos(\frac{2\,i\,\pi}{n}))\sin(a-b)$$

and the denominator simplifies to

$$\cos(b+\frac{2\,\mathrm{i}\,\pi}{n})(\cos(a)+\tan(\frac{\pi}{n})\sin(a))-\cos(a+\frac{2\,\mathrm{i}\,\pi}{n})(\cos(b)+\tan(\frac{\pi}{n})\sin(b))$$

$$(\cos(b)\cos(\frac{2\,\mathrm{i}\,\pi}{n})-\sin(b)\sin(\frac{2\,\mathrm{i}\,\pi}{n}))(\cos(a)+\tan(\frac{\pi}{n})\sin(a))-\ldots$$
$$\ldots(\cos(a)\cos(\frac{2\,\mathrm{i}\,\pi}{n})-\sin(a)\sin(\frac{2\,\mathrm{i}\,\pi}{n}))(\cos(b)+\tan(\frac{\pi}{n})\sin(b))$$

$$\cancel{\cos(a)\cos(b)\cos(\frac{2\,\mathrm{i}\,\pi}{n})}+\sin(a)\cos(b)\cos(\frac{2\,\mathrm{i}\,\pi}{n})\tan(\frac{\pi}{n})-\cos(a)\sin(b)\sin(\frac{2\,\mathrm{i}\,\pi}{n})-\ldots$$
$$\ldots\cancel{\sin(a)\sin(b)\sin(\frac{2\,\mathrm{i}\,\pi}{n})\tan(\frac{\pi}{n})}-\cancel{\cos(a)\cos(b)\cos(\frac{2\,\mathrm{i}\,\pi}{n})}-\cos(a)\sin(b)\cos(\frac{2\,\mathrm{i}\,\pi}{n})\tan(\frac{\pi}{n})+\ldots$$
$$\ldots\sin(a)\cos(b)\sin(\frac{2\,\mathrm{i}\,\pi}{n})+\cancel{\sin(a)\sin(b)\sin(\frac{2\,\mathrm{i}\,\pi}{n})\tan(\frac{\pi}{n})}$$

$$(\cos(\frac{2\,\mathrm{i}\,\pi}{n})\tan(\frac{\pi}{n}))(\sin(a)\cos(b)-\cos(a)\sin(b))+\sin(\frac{2\,\mathrm{i}\,\pi}{n})(\sin(a)\cos(b)-\cos(a)\sin(b))$$
$$(\cos(\frac{2\,\mathrm{i}\,\pi}{n})\tan(\frac{\pi}{n})+\sin(\frac{2\,\mathrm{i}\,\pi}{n}))(\sin(a)\cos(b)-\cos(a)\sin(b))$$
$$(\cos(\frac{2\,\mathrm{i}\,\pi}{n})\tan(\frac{\pi}{n})+\sin(\frac{2\,\mathrm{i}\,\pi}{n}))\sin(a-b)$$

meaning that we can now simplify the original fraction (denoted **3.1**) to become:

$$\frac{(\sin(\frac{2\,\mathrm{i}\,\pi}{n})\tan(\frac{\pi}{n})-\cos(\frac{2\,\mathrm{i}\,\pi}{n}))\sin(a-b)}{(\cos(\frac{2\,\mathrm{i}\,\pi}{n})\tan(\frac{\pi}{n})+\sin(\frac{2\,\mathrm{i}\,\pi}{n}))\sin(a-b)}$$

$$\frac{\sin(\frac{2\,\mathrm{i}\,\pi}{n})\tan(\frac{\pi}{n})-\cos(\frac{2\,\mathrm{i}\,\pi}{n})}{\cos(\frac{2\,\mathrm{i}\,\pi}{n})\tan(\frac{\pi}{n})+\sin(\frac{2\,\mathrm{i}\,\pi}{n})}*\frac{\cos(\frac{\pi}{n})}{\cos(\frac{\pi}{n})}$$

$$\frac{\sin(\frac{2\,\mathrm{i}\,\pi}{n})\sin(\frac{\pi}{n})-\cos(\frac{2\,\mathrm{i}\,\pi}{n})\cos(\frac{\pi}{n})}{\cos(\frac{2\,\mathrm{i}\,\pi}{n})\sin(\frac{\pi}{n})+\sin(\frac{2\,\mathrm{i}\,\pi}{n})\cos(\frac{\pi}{n})}$$

$$\frac{-(\cos(\frac{2\,\mathrm{i}\,\pi}{n})\cos(\frac{\pi}{n})-\sin(\frac{2\,\mathrm{i}\,\pi}{n})\sin(\frac{\pi}{n}))}{\sin(\frac{\pi}{n})\cos(\frac{2\,\mathrm{i}\,\pi}{n})+\cos(\frac{\pi}{n})\sin(\frac{2\,\mathrm{i}\,\pi}{n})}$$

$$\frac{-\cos(\frac{2\,\mathrm{i}\,\pi}{n}+\frac{\pi}{n})}{\sin(\frac{2\,\mathrm{i}\,\pi}{n}+\frac{\pi}{n})}$$

$$-\cot(\frac{2\,\mathrm{i}\,\pi}{n}+\frac{\pi}{n})$$

$$-\cot((2\,\mathrm{i}+1)\frac{\pi}{n})$$

meaning that the equation of the "modified" line becomes:

$$y_m = -\cot((2\,i+1)\,\frac{\pi}{n})x + \sin_n(a) + \cot((2\,i+1)\,\frac{\pi}{n})\cos_n(a)$$

This leads to the conclusion that when $0 \le a < b \le \frac{2\pi}{n}$ the slopes of y_m and y are equal. This leads to the fact that if $a = 0$, due to **Identity 1.3**, then for any applicable value of b y_m becomes:

$$y_m = -\cot((2\,i+1)\,\frac{\pi}{n})x + \sin(a) + \cot((2\,i+1)\,\frac{\pi}{n})\cos(a)$$

Meaning that $y_m = L_i$ when $a = 0$ and due to **Identity 1.3** $\sin_n(a) = \sin(a)$ and $\cos_n(a) = \cos(a)$ thus the point at point p_a (located at $(\cos_n(a), \sin_n(a))$) is equal to point v_i. Because $y_m = L_i$, point p_b (located at $(\cos_n(b), \sin_n(b))$) is always on line segment L_i, thus when $a \ne 0$ p_a must also always be on line segment L_i.
This means that for any value x where $0 \le x \le \frac{2\pi}{n}$, the point p_t (found at $(\cos_n(x + \frac{2\,i\,\pi}{n}), \sin_n(x + \frac{2\,i\,\pi}{n}))$) falls on line L_i of shape s_n, and since $i \in \mathbb{Z}$, any point p_θ (found at $(\cos_n(\theta), \sin_n(\theta))$) will always be located on the perimeter of shape s_n.

5 Shape Equations

From these formulas, the equation for any regular polygon can be derived. The equation is found through a mutation of the ellipse equation by $d_n(\theta)$:

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = d_n^2(\theta) * 1$$

To derive theta, one would start with the assumption that $\theta = \arctan(\frac{(y-k)a}{(x-h)b})$, which would find the angle at the correct point offset from the center, but, due to the left side of the equation involving x^2, the shape is mirrored over $x = h$, which works perfectly for even sided polygons as they are symmetric over $x = h$. Yet, odd-sided polygons do not work due to their anti-symmetry over $x = h$. This can be fixed by firstly noting that the angle is calculated correctly in Quadrants I and IV, and π radians off from the expected angle in Quadrants II and III. Thus when $x < h$, π must be added to θ. Using the sign function, which is defined as

$$\mathrm{sgn}(x) = \frac{x}{|x|} = \frac{|x|}{x}$$

and modifying sgn to equal 1 when $x < h$ and 0 when $x \ge h$ results in the formula:

$$-\frac{1}{2}(\frac{x-h}{|x-h|} - 1)$$

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which when multiplied by π and added to θ, θ becomes equal to:

$$\arctan(\frac{(y-k)a}{(x-h)b}) - \frac{\pi}{2}(\frac{x-h}{|x-h|} - 1)$$

meaning that the standard equation, with the above equation substitued for θ becomes:

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = d_n^2(\arctan(\frac{(y-k)a}{(x-h)b}) - \frac{\pi}{2}(\frac{x-h}{|x-h|} - 1))$$

Interestingly, due to the existence of θ inside this equation, a constant value can be added to it to make any shape rotate, allowing for quick and efficient rotation calculations.