Trigonometric Coefficients for Regular N-Sided Polygons

Eli Ruminer

Table of Contents

1 Introduction

This paper derives a function that generates a coefficient that transforms sine and cosine into calculating points for any n-sided regular polygon. This is done through using inverse trigonometric functions to create a periodic function that increases and decreases linearly over each period. This allows the creation of a coefficent which when multiplied by sine and cosine serves to transform them into an equivalent function for non-circle polygons.
After deriving the equation we then prove that it represents a point on the perimeter of a unit n-gon by simplifying the point-slope formula of a constructing line of the unit n-gon and two points between the modified sine and cosine showing they are the same.
Through this, we can derive a general formula for any shape using the sign function and the newly proven coefficient function. This general function requires the use of an "input" angle, which is calculated using x and y for each point, but due to this can be rotated by simply adding or subtracting a constant from the angle calculation inside the coefficient function's input.

2 Theroems

Theorem 1.1 $\arccos(\cos(ax))$ is periodic over $\frac{2\pi}{a*n}$.
$\cos(ax)$ is periodic over $\frac{2\pi}{a*n}$ thus $\arccos(\cos(ax))$ must also be.
Theorem 1.2 For $x \in \mathbb{R}$ $\arccos(\cos(ax)) \in [0, \pi]$.
$\cos(x)$ has a domain over $\mathbb{R}$ and a range of [-1,1], $\arccos(x)$ has a domain over [-1,1] and a range of $[0, \pi]$. Due to the range of cos being the domain of arccos, all real numbers can be an input of x, and all numbers on the interval [-1,1] can be output.
Theorem 1.3 $\cos(n(a + \frac{2\pi}{\mathrm{n}})) = \cos(na)$
$\cos(n(a + \frac{2\pi}{\mathrm{n}}))$ expands to $\cos(na + 2\pi))$ which due to cosine's periodic nature evaluates to $\cos(na)$
Theorem 2.1 $\sin(\alpha) - \sin(\beta) = 2\sin(\frac{\alpha+\beta}{2})\cos(\frac{\alpha-\beta}{2})$
Theorem 2.2 $\cos(\alpha) - \cos(\beta) = -2\sin(\frac{\alpha+\beta}{2})\sin(\frac{\alpha-\beta}{2})$

3 Trigonometric Equations

Let $n = \{x : x \in \mathbb{Z}^+ \text{ and } x \geq 3\}$, and

$$\mathrm{d_n}(\theta) = \cos(\frac{\pi}{\mathrm{n}})\sec(\frac{\pi - \arccos(\cos(n\theta))}{n})$$
$$\cos_n(\theta) = \mathrm{d_n}(\theta)\cos(\theta)$$
$$\sin_n(\theta) = \mathrm{d_n}(\theta)\sin(\theta)$$

This is derived through the original functions:

$$\mathrm{h_n}(\theta) = \sin(\frac{\pi}{2} - \frac{\pi}{\mathrm{n}} + (\theta \mod \frac{2\pi}{\mathrm{n}})) - \sin(\frac{\pi}{2} - \frac{\pi}{\mathrm{n}})$$
$$\mathrm{d_n}(\theta) = \frac{\mathrm{h_n}(\theta)}{\sin(\frac{\pi}{2} - \frac{\pi}{\mathrm{n}} + (\theta \mod \frac{2\pi}{\mathrm{n}}))}$$
$$\cos_n(\theta) = (1 - \mathrm{d_n}(\theta))\cos(\theta)$$
$$\sin_n(\theta) = (1 - \mathrm{d_n}(\theta))\sin(\theta)$$

These functions follow the premise that any n-sided regular polygon (shape $\mathrm{s_n}$) centered at (0,0) has a circumcircle where each vertex touches the circumference of the circumcircle.
Arc a can be constructed which is $\frac{1}{\mathrm{n^{th}}}$ of the circumcircle, its diameter is the same length as any line which constructs $\mathrm{s_n}$, and it's circumference equals to the circumference of the section of the circumcircle spanning from adjacent vertices on $\mathrm{s_n}$.
Thus, the height of a at a given point on a circle subtracted from the radius of the circle will result in the distance from the midpoint of $\mathrm{s_n}$ to the point on the perimeter of $\mathrm{s_n}$ which intersects the ray of angle θ drawn in standard position.
$\mathrm{h_n}(\theta)$ serves to find the height of a at a position by splitting the circumcircle into $\mathrm{n^{ths}}$ (resulting in a period of $\frac{2\pi}{\mathrm{n}}$), finding the height of a point on a through sine and changing from the "left" vertex to the "right" vertex over the period of $\frac{2\pi}{\mathrm{n}}$ (the length of a). The use of mod makes the function periodic. It then subtracts by the height of a vertex so that $\mathrm{h_n}(\theta) = 0$ when $\theta = 0$
$\mathrm{d_n}(\theta)$ divides the height ($\mathrm{h_n}(\theta)$) by the angle between the assumed intersection point on $\mathrm{s_n}$ and the intersection point on the circumcircle to get the actual distance.
These functions then simplify to:

$$\mathrm{d_n}(\theta) = \cos(\frac{\pi}{\mathrm{n}})\sec(\frac{\pi}{\mathrm{n}} - (\theta \mod \frac{2\pi}{\mathrm{n}}))$$
$$\cos_n(\theta) = \mathrm{d_n}(\theta)cos(\theta)$$
$$\sin_n(\theta) = \mathrm{d_n}(\theta)sin(\theta)$$

Which is not algebraic due to mod (and the resultant floor function behind it), yet it can be noted that due to the nature of the original $\mathrm{h_n}(\theta)$, $\theta \mod \frac{2\pi}{\mathrm{n}}$ can be substituted for an

equivalent function which has an equal period of $\frac{2\pi}{\mathrm{n}}$, increases linearly over $[0, \frac{\pi}{\mathrm{n}}]$ to $[0, \frac{\pi}{\mathrm{n}}]$ and decreases linearly at the same rate from $[\frac{\pi}{\mathrm{n}}, \frac{2\pi}{\mathrm{n}}]$.
These requirements can be met by $\frac{\arccos(\cos(n\theta))}{n}$, which due to **Theorem 1.1** and **Theorem 1.2** is periodic over $\frac{2\pi}{\mathrm{n}}$ has a domain of $n\theta \in \mathbb{R}$, and thus $\theta \in \mathbb{R}$ and a range of $[0, \frac{\pi}{n}]$, thus the above equations can be changed to:

$$\mathrm{d_n}(\theta) = \cos(\frac{\pi}{\mathrm{n}})\sec(\frac{\pi - \arccos(\cos(n\theta))}{n})$$
$$\cos_n(\theta) = \mathrm{d_n}(\theta) cos(\theta)$$
$$\sin_n(\theta) = \mathrm{d_n}(\theta) sin(\theta)$$

3.1 Identities

Identity 1.1 Let $i = \mathbb{Z}$, due to **Theorem 1.3**:

$$\cos_n(\theta + \mathrm{i}\,\frac{2\pi}{\mathrm{n}}) = \mathrm{d_n}(\theta)\cos(\theta + \mathrm{i}\,\frac{2\pi}{\mathrm{n}})$$
$$\sin_n(\theta + \mathrm{i}\,\frac{2\pi}{\mathrm{n}}) = \mathrm{d_n}(\theta)\sin(\theta + \mathrm{i}\,\frac{2\pi}{\mathrm{n}})$$

Identity 1.2 When $0 \leq \theta \leq \frac{2\pi}{\mathrm{n}}$ then $\mathrm{d_n}(\theta) = \cos(\frac{\pi}{\mathrm{n}})\sec(\frac{\pi}{\mathrm{n}} - \theta)$
When θ only spans one period, $\frac{\arccos(\cos(n\theta))}{n}$ is not required to make the function periodic.
Identity 1.3 Let $\mathrm{i} = \mathbb{Z}$ then:

$$\cos_n(\mathrm{i}\,\frac{2\pi}{\mathrm{n}}) = (1)\cos(\mathrm{i}\,\frac{2\pi}{\mathrm{n}}) = \cos(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})$$
$$\sin_n(\mathrm{i}\,\frac{2\pi}{\mathrm{n}}) = (1)\sin(\mathrm{i}\,\frac{2\pi}{\mathrm{n}}) = \sin(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})$$

This is true because $\arccos(\cos(n\theta))$ evaluates to 0 whenever $\theta = \mathrm{i}\,\frac{2\pi}{\mathrm{n}}$, which means $\sec(\frac{\pi}{\mathrm{n}} - \frac{\arccos(\cos(n\theta))}{n})$ simplifies to $\sec(\frac{\pi}{\mathrm{n}})$ so $\mathrm{d_n}(\mathrm{i}\,\frac{2\pi}{\mathrm{n}}) = \sec(\frac{\pi}{\mathrm{n}})\cos(\frac{\pi}{\mathrm{n}}) = 1$

4 Proof

Let $\mathrm{i}, n = \mathbb{Z}$. Shape $\mathrm{s_n}$ is made of n vertices and n line segments with the $\mathrm{i^{th}}$ vertex ($\mathrm{v_i}$) at the point $(\cos(\mathrm{i}\,\frac{2\pi}{\mathrm{n}}), \sin(\mathrm{i}\,\frac{2\pi}{\mathrm{n}}))$. The $\mathrm{i^{th}}$ line segment, $\mathrm{L_i}$, spans between $\mathrm{v_i}$ and $\mathrm{v_{i+1}}$ and is represented by the equation

$$\mathrm{L_i} = \frac{\sin((\mathrm{i}+1)\,\frac{2\pi}{\mathrm{n}}) - \sin(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})}{\cos((\mathrm{i}+1)\,\frac{2\pi}{\mathrm{n}}) - \cos(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})}x + \sin(\mathrm{i}\,\frac{2\pi}{\mathrm{n}}) - \frac{\sin((\mathrm{i}+1)\,\frac{2\pi}{\mathrm{n}}) - \sin(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})}{\cos((\mathrm{i}+1)\,\frac{2\pi}{\mathrm{n}}) - \cos(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})}\cos(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})$$

which due to **Theorem 2.1** and **Theorem 2.2** simplifies to

$$\mathrm{L_i} = -\cot((2\,\mathrm{i}+1)\,\frac{\pi}{\mathrm{n}})x + \sin(\mathrm{i}\,\frac{2\pi}{\mathrm{n}}) + \cot((2\,\mathrm{i}+1)\,\frac{\pi}{\mathrm{n}})\cos(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})$$

Let $0 \leq a < b \leq \frac{2\pi}{\mathrm{n}}$, $\alpha = a + \mathrm{i}\,\frac{2\pi}{\mathrm{n}}$, and $\beta = b + \mathrm{i}\,\frac{2\pi}{\mathrm{n}}$: a "modified" line can be drawn with the equation $\sin_n(\theta)$ and $\cos_n(\theta)$ which would be written as

$$y_m = \frac{\sin_n(\beta) - \sin_n(\alpha)}{\cos_n(\beta) - \cos_n(\alpha)}x + \sin_n(a) - \frac{\sin_n(\beta) - \sin_n(\alpha)}{\cos_n(\beta) - \cos_n(\alpha)}\cos_n(a)$$

the slope of y_m can be simplified through these steps:

$$\frac{\mathrm{d_n}(\beta)\sin(\beta) - \mathrm{d_n}(\alpha)\sin(\alpha)}{\mathrm{d_n}(\beta)\cos(\beta) - \mathrm{d_n}(\alpha)\cos(\alpha)}$$

$$\cancel{\frac{\cos(\frac{\pi}{\mathrm{n}})}{\cos(\frac{\pi}{\mathrm{n}})}}\,\frac{\sec(\frac{\pi}{\mathrm{n}}-b)\sin(\beta) - \sec(\frac{\pi}{\mathrm{n}}-a)\sin(\alpha)}{\sec(\frac{\pi}{\mathrm{n}}-b)\cos(\beta) - \sec(\frac{\pi}{\mathrm{n}}-a)\cos(\alpha)} * \frac{\cos(\frac{\pi}{\mathrm{n}}-a)\cos(\frac{\pi}{\mathrm{n}}-b)}{\cos(\frac{\pi}{\mathrm{n}}-a)\cos(\frac{\pi}{\mathrm{n}}-b)}$$

$$\frac{\sin(\beta)\cos(\frac{\pi}{\mathrm{n}}-a) - \sin(\alpha)\cos(\frac{\pi}{\mathrm{n}}-b)}{\cos(\beta)\cos(\frac{\pi}{\mathrm{n}}-a) - \cos(\alpha)\cos(\frac{\pi}{\mathrm{n}}-b)}$$

$$\frac{\sin(\beta)(\cos(\frac{\pi}{\mathrm{n}})\cos(a) + \sin(\frac{\pi}{\mathrm{n}})\sin(a)) - \sin(\alpha)(\cos(\frac{\pi}{\mathrm{n}})\cos(b) + \sin(\frac{\pi}{\mathrm{n}})\sin(b))}{\cos(\beta)(\cos(\frac{\pi}{\mathrm{n}})\cos(a) + \sin(\frac{\pi}{\mathrm{n}})\sin(a)) - \cos(\alpha)(\cos(\frac{\pi}{\mathrm{n}})\cos(b) + \sin(\frac{\pi}{\mathrm{n}})\sin(b))} * \frac{\sec(\frac{\pi}{\mathrm{n}})}{\sec(\frac{\pi}{\mathrm{n}})}$$

$$\frac{\sin(\beta)(\cos(a) + \tan(\frac{\pi}{\mathrm{n}})\sin(a)) - \sin(\alpha)(\cos(b) + \tan(\frac{\pi}{\mathrm{n}})\sin(b))}{\cos(\beta)(\cos(a) + \tan(\frac{\pi}{\mathrm{n}})\sin(a)) - \cos(\alpha)(\cos(b) + \tan(\frac{\pi}{\mathrm{n}})\sin(b))} \qquad (\mathbf{3.1})$$

Splitting the equation up the numerator simplifies like so:

$$\sin(\beta)(\cos(a) + \tan(\frac{\pi}{\mathrm{n}})\sin(a)) - \sin(\alpha)(\cos(b) + \tan(\frac{\pi}{\mathrm{n}})\sin(b))$$

$$(\sin(b)\cos(\mathrm{i}\,\frac{2\pi}{\mathrm{n}}) + \cos(b)\sin(\mathrm{i}\,\frac{2\pi}{\mathrm{n}}))(\cos(a) + \tan(\frac{\pi}{\mathrm{n}})\sin(a)) - \ldots$$
$$\ldots(\sin(a)\cos(\mathrm{i}\,\frac{2\pi}{\mathrm{n}}) + \cos(a)\sin(\mathrm{i}\,\frac{2\pi}{\mathrm{n}}))(\cos(b) + \tan(\frac{\pi}{\mathrm{n}})\sin(b))$$

$$\cos(a)\sin(b)\cos(\mathrm{i}\,\frac{2\pi}{\mathrm{n}}) + \cancel{\sin(a)\sin(b)\cos(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})\tan(\frac{\pi}{\mathrm{n}})} + \cancel{\cos(a)\cos(b)\sin(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})} + \ldots$$
$$\ldots\sin(a)\cos(b)\sin(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})\tan(\frac{\pi}{\mathrm{n}}) - \sin(a)\cos(b)\cos(\mathrm{i}\,\frac{2\pi}{\mathrm{n}}) - \cancel{\sin(a)\sin(b)\cos(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})\tan(\frac{\pi}{\mathrm{n}})} - \ldots$$
$$\ldots\cancel{\cos(a)\cos(b)\sin(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})} - \cos(a)\sin(b)\sin(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})\tan(\frac{\pi}{\mathrm{n}})$$

$$\cos(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})(\cos(a)\sin(b)-\sin(a)\cos(b))+\sin(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})\tan(\frac{\pi}{\mathrm{n}})(\sin(a)\cos(b)-\cos(a)\sin(b))$$

$$\sin(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})\tan(\frac{\pi}{\mathrm{n}})(\sin(a)\cos(b)-\cos(a)\sin(b))-\cos(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})(\sin(a)\cos(b)-\cos(a)\sin(b))$$

$$(\sin(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})\tan(\frac{\pi}{\mathrm{n}})-\cos(\mathrm{i}\,\frac{2\pi}{\mathrm{n}}))(\sin(a)\cos(b)-\cos(a)\sin(b))$$

$$(\sin(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})\tan(\frac{\pi}{\mathrm{n}})-\cos(\mathrm{i}\,\frac{2\pi}{\mathrm{n}}))\sin(a-b)$$

and the denominator simplifies to

$$\cos(\beta)(\cos(a)+\tan(\frac{\pi}{\mathrm{n}})\sin(a))-\cos(\alpha)(\cos(b)+\tan(\frac{\pi}{\mathrm{n}})\sin(b))$$

$$(\cos(b)\cos(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})-\sin(b)\sin(\mathrm{i}\,\frac{2\pi}{\mathrm{n}}))(\cos(a)+\tan(\frac{\pi}{\mathrm{n}})\sin(a))-\dots$$
$$\dots(\cos(a)\cos(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})-\sin(a)\sin(\mathrm{i}\,\frac{2\pi}{\mathrm{n}}))(\cos(b)+\tan(\frac{\pi}{\mathrm{n}})\sin(b))$$

$$\cancel{\cos(a)\cos(b)\cos(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})}+\sin(a)\cos(b)\cos(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})\tan(\frac{\pi}{\mathrm{n}})-\cos(a)\sin(b)\sin(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})-\dots$$
$$\dots\cancel{\sin(a)\sin(b)\sin(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})\tan(\frac{\pi}{\mathrm{n}})}-\cancel{\cos(a)\cos(b)\cos(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})}-\cos(a)\sin(b)\cos(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})\tan(\frac{\pi}{\mathrm{n}})+\dots$$
$$\dots\sin(a)\cos(b)\sin(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})+\cancel{\sin(a)\sin(b)\sin(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})\tan(\frac{\pi}{\mathrm{n}})}$$

$$(\cos(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})\tan(\frac{\pi}{\mathrm{n}}))(\sin(a)\cos(b)-\cos(a)\sin(b))+\sin(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})(\sin(a)\cos(b)-\cos(a)\sin(b))$$
$$(\cos(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})\tan(\frac{\pi}{\mathrm{n}})+\sin(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})(\sin(a)\cos(b)-\cos(a)\sin(b))$$
$$(\cos(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})\tan(\frac{\pi}{\mathrm{n}})+\sin(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})\sin(a-b)$$

meaning that we can now simplify the original fraction (denoted **3.1**) to become:

$$\frac{(\sin(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})\tan(\frac{\pi}{\mathrm{n}})-\cos(\mathrm{i}\,\frac{2\pi}{\mathrm{n}}))\sin(a-b)}{(\cos(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})\tan(\frac{\pi}{\mathrm{n}})+\sin(\mathrm{i}\,\frac{2\pi}{\mathrm{n}}))\sin(a-b)}$$
$$\frac{\sin(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})\tan(\frac{\pi}{\mathrm{n}})-\cos(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})}{\cos(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})\tan(\frac{\pi}{\mathrm{n}})+\sin(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})}*\frac{\cos(\frac{\pi}{\mathrm{n}})}{\cos(\frac{\pi}{\mathrm{n}})}$$
$$\frac{\sin(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})\sin(\frac{\pi}{\mathrm{n}})-\cos(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})\cos(\frac{\pi}{\mathrm{n}})}{\cos(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})\sin(\frac{\pi}{\mathrm{n}})+\sin(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})\cos(\frac{\pi}{\mathrm{n}})}$$
$$\frac{-(\cos(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})\cos(\frac{\pi}{\mathrm{n}})-\sin(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})\sin(\frac{\pi}{\mathrm{n}}))}{\sin(\frac{\pi}{\mathrm{n}})\cos(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})+\cos(\frac{\pi}{\mathrm{n}})\sin(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})}$$
$$\frac{-\cos(\mathrm{i}\,\frac{2\pi}{\mathrm{n}}+\frac{\pi}{\mathrm{n}})}{\sin(\mathrm{i}\,\frac{2\pi}{\mathrm{n}}+\frac{\pi}{\mathrm{n}})}$$
$$-\cot(\mathrm{i}\,\frac{2\pi}{\mathrm{n}}+\frac{\pi}{\mathrm{n}})$$
$$-\cot((2\,\mathrm{i}+1)\,\frac{\pi}{\mathrm{n}})$$

so the equation of the "modified" line becomes:

$$y_m=-\cot((2\,\mathrm{i}+1)\,\frac{\pi}{\mathrm{n}})x+\sin_n(a+\mathrm{i}\,\frac{2\pi}{\mathrm{n}})+\cot((2\,\mathrm{i}+1)\,\frac{\pi}{\mathrm{n}})\cos_n(a+\mathrm{i}\,\frac{2\pi}{\mathrm{n}})$$

This leads to the conclusion that when $0\le a<b\le\frac{2\pi}{\mathrm{n}}$ the slopes of y_m and y are equal.
If $a=0$, due to **Identity 1.3**, then for any applicable value of b, y_m becomes:

$$y_m=-\cot((2\,\mathrm{i}+1)\,\frac{\pi}{\mathrm{n}})x+\sin(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})+\cot((2\,\mathrm{i}+1)\,\frac{\pi}{\mathrm{n}})\cos(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})$$

Meaning that $y_m=\mathrm{L_i}$ when $a=0$ and due to **Identity 1.3**: $\sin_n(a+\mathrm{i}\,\frac{2\pi}{\mathrm{n}})=\sin(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})$ and $\cos_n(a+\mathrm{i}\,\frac{2\pi}{\mathrm{n}})=\cos(\mathrm{i}\,\frac{2\pi}{\mathrm{n}})$ thus the point $\mathrm{p_a}$, located at $(\cos_n(a+\mathrm{i}\,\frac{2\pi}{\mathrm{n}}),\sin_n(a+\mathrm{i}\,\frac{2\pi}{\mathrm{n}}))$, is equal to $\mathrm{v_i}$. Because $y_{m,a=0}=L_i$, point $\mathrm{p_b}$ (located at $(\cos_n(b+\mathrm{i}\,\frac{2\pi}{\mathrm{n}}),\sin_n(b+\mathrm{i}\,\frac{2\pi}{\mathrm{n}}))$) is always on line segment $\mathrm{L_i}$, thus when $a\ne 0$, $\mathrm{p_a}$ must also always be on line segment $\mathrm{L_i}$.
This means that for any value x where $0\le x\le\frac{2\pi}{\mathrm{n}}$, the point $\mathrm{p_x}$, located at $(\cos_n(x+\mathrm{i}\,\frac{2\pi}{\mathrm{n}}),\sin_n(x+\mathrm{i}\,\frac{2\pi}{\mathrm{n}}))$, falls on line $\mathrm{L_i}$ of shape $\mathrm{s_n}$, and since $\mathrm{i}\in\mathbb{Z}$, any point p_θ, located at $(\cos_n(\theta),\sin_n(\theta))$, will always be on the perimeter of shape $\mathrm{s_n}$.

5 Shape Equations

From these formulas, the equation for any regular polygon can be derived. The equation is found through a mutation of the ellipse equation by $\mathrm{d_n}(\theta)$:

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=\mathrm{d_n}(\theta)^2*1$$

To derive theta, one would start with $\theta = \arctan(\frac{(y-k)a}{(x-h)b})$, which would find the angle at the correct point offset from the center, but due to x being squared, the shape is mirrored over $x = h$, which will work for even sided polygons as they are symmetric over $x = h$, but not for odd sided polygons due to their anti-symmetry (this is also the case over the line $y = k$ but all shapes, regardless of side parity, are mirrored over $y = k$). This can be fixed by noting that the angle is calculated correctly in Quadrants I and IV (relative to (h, k) as the origin), and π radians off from the expected angle in Quadrants II and III. Thus when $x < h$, π must be added to θ. Using the sign function and modifying it to equal 1 when $x < h$ and 0 when $x \geq h$ results in the formula:

$$-\frac{1}{2}(\frac{x-h}{|x-h|} - 1)$$

which when multiplied by π and added to θ becomes:

$$\arctan(\frac{(y-k)a}{(x-h)b}) - \frac{\pi}{2}(\frac{x-h}{|x-h|} - 1)$$

meaning that the standard equation is:

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = \mathrm{d_n}(\arctan(\frac{(y-k)a}{(x-h)b}) - \frac{\pi}{2}(\frac{x-h}{|x-h|} - 1))^2$$

As $\mathrm{d_n}$ takes an angle, a constant value can be added to θ to make any shape rotate, allowing for easy rotation calculations.