Trigonometric Coefficients for Regular S-Sided Polygons

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1 Introduction

This paper creates and proves a function that finds the distance between the center of a regular n-sided polygon and a point on its perimeter at a given angle without knowing that point's position. This when multiplied by sine or cosine allows finding those points on the perimeter of the polygon. This allows for the creation of a general equation for any regular n-sided polygon similar to the equation of a circle. This equation can then be used to find areas of intersection between polygons.
Following this, the modified sine and cosine functions are then proved to always generate a point on the perimeter of an s-sided polygon, given s and an angle.

2 Trigonometric Equations

Let $s = \{x : x \in \mathbb{Z}^+ \text{ and } x \geq 3\}$, and

$$\mathrm{d_s}(\theta) = \cos(\frac{\pi}{\mathrm{s}})\sec(\frac{\pi - \arccos(\cos(s\theta))}{s})$$
$$\cos_s(\theta) = \mathrm{d_s}(\theta)\cos(\theta)$$
$$\sin_s(\theta) = \mathrm{d_s}(\theta)\sin(\theta)$$

This is derived through the original functions:

$$\mathrm{h_s}(\theta) = \sin(\frac{\pi}{2} - \frac{\pi}{\mathrm{s}} + (\theta \mod \frac{2\pi}{\mathrm{s}})) - \sin(\frac{\pi}{2} - \frac{\pi}{\mathrm{s}})$$
$$\mathrm{d_s}(\theta) = \frac{\mathrm{h_s}(\theta)}{\sin(\frac{\pi}{2} - \frac{\pi}{\mathrm{s}} + (\theta \mod \frac{2\pi}{\mathrm{s}}))}$$
$$\cos_s(\theta) = (1 - \mathrm{d_s}(\theta))\cos(\theta)$$
$$\sin_s(\theta) = (1 - \mathrm{d_s}(\theta))\sin(\theta)$$

These functions follow the premise that any s-sided regular polygon ($\mathrm{P_s}$) centered at (0,0) has a circumcircle where each vertex touches the circumference of the circumcircle.
Arc a can be constructed which is $\frac{1}{\mathrm{s^{th}}}$ of the circumcircle, its diameter is the same length as any line which constructs $\mathrm{P_s}$, and it's circumference equals to the circumference of the section of the circumcircle spanning from adjacent vertices on $\mathrm{P_s}$.
Thus, the height of a at a given point on a circle subtracted from the radius of the circle will result in the distance from the center of $\mathrm{P_s}$ to the point on the perimeter of $\mathrm{P_s}$ which intersects the ray of angle θ drawn in standard position.
$\mathrm{h_s}(\theta)$ serves to find the height of a at a position by splitting the circumcircle into $\mathrm{s^{ths}}$ (resulting in a period of $\frac{2\pi}{\mathrm{s}}$), finding the height of a point on a through sine and changing from the "left" vertex to the "right" vertex over the period of $\frac{2\pi}{\mathrm{s}}$ (the length of a). The use of mod makes the function periodic. It then subtracts by the height of a vertex so that

$\mathrm{h_s}(\theta) = 0$ when $\theta = 0$
$\mathrm{d_s}(\theta)$ divides the height, $\mathrm{h_s}(\theta)$, by the angle between the assumed intersection point on $\mathrm{P_s}$ and the intersection point on the circumcircle to get the actual distance.
These functions then simplify to:

$$\mathrm{d_s}(\theta) = \cos(\frac{\pi}{\mathrm{s}})\sec(\frac{\pi}{\mathrm{s}} - (\theta \mod \frac{2\pi}{\mathrm{s}}))$$
$$\cos_s(\theta) = \mathrm{d_s}(\theta)\cos(\theta)$$
$$\sin_s(\theta) = \mathrm{d_s}(\theta)\sin(\theta)$$

Which is not algebraic due to modulo, yet it can be noted that due to the nature of the original $\mathrm{h_s}(\theta)$, $\theta \mod \frac{2\pi}{\mathrm{s}}$ can be substituted for an equivalent function which has an equal period of $\frac{2\pi}{\mathrm{s}}$, increases linearly from 0 to $\frac{\pi}{\mathrm{s}}$ over $[0, \frac{\pi}{\mathrm{s}}]$ and decreases linearly at the same rate over $[\frac{\pi}{\mathrm{s}}, \frac{2\pi}{\mathrm{s}}]$.
These requirements can be met by $\frac{\arccos(\cos(n\theta))}{n}$[1], which is periodic over $\frac{2\pi}{\mathrm{s}}$, has a domain of $n\theta \in \mathbb{R}$, and thus $\theta \in \mathbb{R}$, and a range of $[0, \frac{\pi}{s}]$, thus the above equations can be changed to:

$$\mathrm{d_s}(\theta) = \cos(\frac{\pi}{\mathrm{s}})\sec(\frac{\pi - \arccos(\cos(s\theta))}{s})$$
$$\cos_s(\theta) = \mathrm{d_s}(\theta)\cos(\theta)$$
$$\sin_s(\theta) = \mathrm{d_s}(\theta)\sin(\theta)$$

2.1 Identities

Identity 1.1 Let $\mathrm{n} = \mathbb{Z}$:

$$\cos_s(\theta + \mathrm{n}\frac{2\pi}{\mathrm{s}}) = \mathrm{d_s}(\theta)\cos(\theta + \mathrm{n}\frac{2\pi}{\mathrm{s}})$$
$$\sin_s(\theta + \mathrm{n}\frac{2\pi}{\mathrm{s}}) = \mathrm{d_s}(\theta)\sin(\theta + \mathrm{n}\frac{2\pi}{\mathrm{s}})$$

Identity 1.2 When $0 \leq \theta \leq \frac{2\pi}{\mathrm{s}}$ then $\mathrm{d_s}(\theta) = \cos(\frac{\pi}{\mathrm{s}})\sec(\frac{\pi}{\mathrm{s}} - \theta)$
When θ only spans one period, $\frac{\arccos(\cos(s\theta))}{s}$ is not required to make the function periodic.
Identity 1.3 Let $\mathrm{n} = \mathbb{Z}$ then:

$$\cos_s(\mathrm{n}\frac{2\pi}{\mathrm{s}}) = (1)\cos(\mathrm{n}\frac{2\pi}{\mathrm{s}}) = \cos(\mathrm{n}\frac{2\pi}{\mathrm{s}})$$
$$\sin_s(\mathrm{n}\frac{2\pi}{\mathrm{s}}) = (1)\sin(\mathrm{n}\frac{2\pi}{\mathrm{s}}) = \sin(\mathrm{n}\frac{2\pi}{\mathrm{s}})$$

This is true because $\arccos(\cos(s\theta))$ evaluates to 0 whenever $\theta = \mathrm{n}\frac{2\pi}{\mathrm{s}}$, which means $\sec(\frac{\pi - \arccos(\cos(s\theta))}{s})$ simplifies to $\sec(\frac{\pi}{\mathrm{s}})$ so $\mathrm{d_s}(\mathrm{n}\frac{2\pi}{\mathrm{s}}) = \sec(\frac{\pi}{\mathrm{s}})\cos(\frac{\pi}{\mathrm{s}}) = 1$

[1] **J. L. Spence**, **"Perodic absolute value functions"**.

3 Polygonal Equations

From these formulas, the equation for any regular polygon can be derived. The equation is found through a mutation of the ellipse equation by $\mathrm{d_s}(\theta)$:

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = \mathrm{d_s}(\theta)^2 * 1$$

To derive theta, one would start with $\theta = \arctan(\frac{(y-k)a}{(x-h)b})$, which would find the angle at the correct point offset from the center, but due to x being squared, the shape is mirrored over $x = h$, which will work for even sided polygons as they are symmetric over $x = h$, but not for odd sided polygons due to their anti-symmetry (this is also the case over the line $y = k$ but all shapes, regardless of side parity, are mirrored over $y = k$). This can be fixed by noting that the angle is calculated correctly in Quadrants I and IV (relative to (h, k) as the origin), and π radians off from the expected angle in Quadrants II and III. Thus when $x < h$, π must be added to θ. Using the sign function and modifying it to equal 1 when $x < h$ and 0 when $x \geq h$ results in the formula:

$$-\frac{1}{2}(\frac{x-h}{|x-h|} - 1)$$

which when multiplied by π and added to θ becomes:

$$\arctan(\frac{(y-k)a}{(x-h)b}) - \frac{\pi}{2}(\frac{x-h}{|x-h|} - 1)$$

meaning that the standard equation is:

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = \mathrm{d_s}(\arctan(\frac{(y-k)a}{(x-h)b}) - \frac{\pi}{2}(\frac{x-h}{|x-h|} - 1))^2$$

As $\mathrm{d_s}$ takes an angle, a constant value can be added to θ to make any shape rotate, allowing for easy rotation calculations.

3.1 Intersections

Let the regular Polygon P_s with s sides (where ($s \in \mathbb{Z}$ and $s \geq 3$) be represented by the equation

$$P_s(x, y) = \frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} - \mathrm{d_s}(\arctan(\frac{(y-k)a}{(x-h)b}) - \frac{\pi}{2}(\frac{x-h}{|x-h|} - 1))^2$$

A point (x, y) is on the perimeter of P_s if $P_s(x, y) = 0$, inside the area of P_s if $P_s(x, y) < 0$, and outside the area of P_s if $P_s(x, y) > 0$.

Thus (x,y) is inside the intersection of two polygons P_{s_1} and P_{s_2} if $P_{s_1}(x,y) < 0$ and $P_{s_2}(x,y) < 0$. This can be made into one inequality of:

$$|(P_{s_1}(x,y) - |P_{s_1}(x,y)|)(P_{s_2}(x,y) - |P_{s_2}(x,y)|)| > 0$$

More generally for any set of Polygons C, a point (x,y) is inside the intersection of all polygons in the set (the "addition" of the polygons) if

$$\left|\prod_{i=1}^{|C|}(C_i(x,y) - |C_i(x,y)|)\right| > 0$$

4 Proof

Let $\mathrm{n}, s = \mathbb{Z}, s \geq 3$. Shape $\mathrm{P_s}$ is made of s vertices and s line segments with the $\mathrm{n^{th}}$ vertex ($\mathrm{n_i}$) at the point $(\cos(\mathrm{n}\,\frac{2\pi}{\mathrm{s}}), \sin(\mathrm{n}\,\frac{2\pi}{\mathrm{s}}))$. The $\mathrm{n^{th}}$ line segment, $\mathrm{L_n}$, spans between $\mathrm{v_n}$ and $\mathrm{v_{n+1}}$ and is represented by the equation

$$\mathrm{L_n} = \frac{\sin((\mathrm{n}+1)\,\frac{2\pi}{\mathrm{s}}) - \sin(\mathrm{n}\,\frac{2\pi}{\mathrm{s}})}{\cos((\mathrm{n}+1)\,\frac{2\pi}{\mathrm{s}}) - \cos(\mathrm{n}\,\frac{2\pi}{\mathrm{s}})} x + \sin(\mathrm{n}\,\frac{2\pi}{\mathrm{s}}) - \frac{\sin((\mathrm{n}+1)\,\frac{2\pi}{\mathrm{s}}) - \sin(\mathrm{n}\,\frac{2\pi}{\mathrm{s}})}{\cos((\mathrm{n}+1)\,\frac{2\pi}{\mathrm{s}}) - \cos(\mathrm{n}\,\frac{2\pi}{\mathrm{s}})} \cos(\mathrm{n}\,\frac{2\pi}{\mathrm{s}})$$

which simplifies to

$$\mathrm{L_n} = -\cot((2\,\mathrm{n}+1)\,\frac{\pi}{\mathrm{s}})x + \sin(\mathrm{n}\,\frac{2\pi}{\mathrm{s}}) + \cot((2\,\mathrm{n}+1)\,\frac{\pi}{\mathrm{s}})\cos(\mathrm{n}\,\frac{2\pi}{\mathrm{s}})$$

Let $0 \leq a < b \leq \frac{2\pi}{\mathrm{s}}$, $\alpha = a + \mathrm{n}\,\frac{2\pi}{\mathrm{s}}$, and $\beta = b + \mathrm{n}\,\frac{2\pi}{\mathrm{s}}$: a "modified" line can be drawn with the equation $\sin_s(\theta)$ and $\cos_s(\theta)$ which would be written as

$$y_m = \frac{\sin_s(\beta) - \sin_s(\alpha)}{\cos_s(\beta) - \cos_s(\alpha)} x + \sin_s(\alpha) - \frac{\sin_s(\beta) - \sin_s(\alpha)}{\cos_s(\beta) - \cos_s(\alpha)} \cos_s(\alpha)$$

the slope of y_m can be expanded to

$$\frac{\sin(\beta)(\cos(a) + \tan(\frac{\pi}{\mathrm{s}})\sin(a)) - \sin(\alpha)(\cos(b) + \tan(\frac{\pi}{\mathrm{s}})\sin(b))}{\cos(\beta)(\cos(a) + \tan(\frac{\pi}{\mathrm{s}})\sin(a)) - \cos(\alpha)(\cos(b) + \tan(\frac{\pi}{\mathrm{s}})\sin(b))}$$

Which simplifies to

$$\frac{\sin(a-b)(\sin(\mathrm{n}\,\frac{2\pi}{\mathrm{s}})\tan(\frac{\pi}{\mathrm{s}}) - \cos(\mathrm{n}\,\frac{2\pi}{\mathrm{s}}))}{\sin(a-b)(\cos(\mathrm{n}\,\frac{2\pi}{\mathrm{s}})\tan(\frac{\pi}{\mathrm{s}}) + \sin(\mathrm{n}\,\frac{2\pi}{\mathrm{s}}))} = -\cot((2\,\mathrm{n}+1)\,\frac{\pi}{\mathrm{s}})$$

thus the equation of the "modified" line becomes:

$$y_m = -\cot((2\,\mathrm{n}+1)\,\frac{\pi}{\mathrm{s}})x + \sin_s(\alpha) + \cot((2\,\mathrm{n}+1)\,\frac{\pi}{\mathrm{s}})\cos_s(\alpha)$$

This leads to the conclusion that when $\mathrm{n}\,\frac{2\pi}{\mathrm{s}} \leq a < b \leq (\mathrm{n}+1)\frac{2\pi}{n}$ the slopes of y_m and $\mathrm{L_n}$ are equal.
If $a = 0$, due to **Identity 1.3**, then for any applicable value of b, y_m becomes:

$$y_m = -\cot((2\,\mathrm{n}+1)\,\frac{\pi}{\mathrm{s}})x + \sin(\mathrm{n}\,\frac{2\pi}{\mathrm{s}}) + \cot((2\,\mathrm{n}+1)\,\frac{\pi}{\mathrm{s}})\cos(\mathrm{n}\,\frac{2\pi}{\mathrm{s}})$$

Meaning that $y_m = \mathrm{L_n}$ when $a = 0$ (denoted $y_{m,a=0}$) and due to **Identity 1.3**: $\sin_s(a + \mathrm{n}\,\frac{2\pi}{\mathrm{s}}) = \sin(\mathrm{n}\,\frac{2\pi}{\mathrm{s}})$ and $\cos_s(a + \mathrm{n}\,\frac{2\pi}{\mathrm{s}}) = \cos(\mathrm{n}\,\frac{2\pi}{\mathrm{s}})$ thus the point $A(\cos_s(a + \mathrm{n}\,\frac{2\pi}{\mathrm{s}}), \sin_s(a + \mathrm{n}\,\frac{2\pi}{\mathrm{s}}))$, is equal to $\mathrm{v_n}$.
Because $y_{m,a=0} = L_n$, the point $B(\cos_s(b + \mathrm{n}\,\frac{2\pi}{\mathrm{s}}), \sin_s(b + \mathrm{n}\,\frac{2\pi}{\mathrm{s}})))$ is always on line segment $\mathrm{L_n}$, thus when $a \neq 0$, A must also always be on line segment $\mathrm{L_n}$.
This means that for any value x where $0 \leq x \leq \frac{2\pi}{\mathrm{s}}$, the point $X(\cos_s(x + \mathrm{n}\,\frac{2\pi}{\mathrm{s}}), \sin_s(x + \mathrm{n}\,\frac{2\pi}{\mathrm{s}}))$, falls on line $\mathrm{L_n}$ of $\mathrm{P_s}$, and since $\mathrm{n} \in \mathbb{Z}$, any point $P(\cos_s(\theta), \sin_s(\theta))$, will always be on the perimeter of $\mathrm{P_s}$.

References

[1] J. L. Spence, "Periodic absolute value functions," *School Science and Mathematics*, vol. 61, no. 9, pp. 664–666, 1961. DOI: https://doi.org/10.1111/j.1949-8594.1961.tb08220.x. eprint: https://onlinelibrary.wiley.com/doi/pdf/10.1111/j.1949-8594.1961.tb08220.x. [Online]. Available: https://onlinelibrary.wiley.com/doi/abs/10.1111/j.1949-8594.1961.tb08220.x.