Algebraic Definitions of Number Theory Functions

Eli Ruminer

Table of Contents

1 Introduction

This paper seeks to show a method of defining a floor function in algebraic terms and thus defining other number theory functions algebraicly from knowing floor.

2 Floor

We start by defining the fractional operator, which gets the decimal part of x, and removes any whole part to be

$$\{x\}$$

Where

$$0 \leq \{x\} \leq 1$$

From this we can define the floor function, $\lfloor x \rfloor$ as

$$\lfloor x \rfloor = x - \{x\}$$

To define this algebraically we first must find a function which finds $\{x\}$ to be able to subtract it. This can be found by using the function

$$\mathrm{a}(x) = \tan^{-1}(\tan(x))$$

This creates a periodic function increasing linearly from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ over $[n\pi - \frac{\pi}{2}, n\pi + \frac{\pi}{2}]$ where $n \in \mathbb{Z}$. We must first decrease the period from π to 1, and range from π to 1, giving us

$$\mathrm{b}(x) = \frac{a(x\pi)}{\pi} = \frac{\tan^{-1}(\tan(x\pi))}{\pi}$$

The function then must be phase shifted 0.5 to "start" each period at an integer, and shifted up so that it is always positive giving us

$$\begin{aligned}\{x\} = \mathrm{b}(x - \frac{1}{2}) + \frac{1}{2} &= \frac{\tan^{-1}(\tan(x\pi - \frac{\pi}{2}))}{\pi} + \frac{1}{2} \\ &= \frac{\cot^{-1}(\cot(x\pi))}{\pi}\end{aligned}$$

We then plug the new definition of $\{x\}$ into the original floor equation to find that

$$\lfloor x \rfloor = x - \frac{\cot^{-1}(\cot(x\pi))}{\pi}$$

Due to using tangent, there are undefined asymptotes at the whole numbers, thus floor cannot be defined truly algebraically, but must still have a piecewise definition of

$$\lfloor x \rfloor = \begin{cases} x & \text{if } x \in \mathbb{Z} \\ x - \frac{\cot^{-1}(\cot(x\pi))}{\pi} & \text{if } x \in \mathbb{R} \setminus \mathbb{Z} \end{cases}$$

Unless explicitly stated the non-piecewise definition will be used for the rest of the paper.

3 Ceiling and Round

From floor, we can define the other 2 types of rounding functions: Ceiling and Round.

3.1 Ceiling

Ceiling (ceil) always rounds up and can be defined as

$$\lceil x \rceil = \begin{cases} x & \text{if } x \in \mathbb{Z} \\ x - \{x\} + 1 & \text{if } x \in \mathbb{R} \setminus \mathbb{Z} \end{cases}$$

which one will notices shares $x - \{x\}$ with $\lfloor x \rfloor$ and thus

$$\begin{aligned} \lceil x \rceil &= \begin{cases} x & \text{if } x \in \mathbb{Z} \\ \lfloor x \rfloor + 1 & \text{if } x \in \mathbb{R} \setminus \mathbb{Z} \end{cases} \\ &= \begin{cases} x & \text{if } x \in \mathbb{Z} \\ x - \frac{\cot^{-1}(\cot(x\pi))}{\pi} & \text{if } x \in \mathbb{R} \setminus \mathbb{Z} \end{cases} \end{aligned}$$

3.2 Round

Round is defined as

$$\begin{aligned} \lfloor x \rceil &= \begin{cases} x - \{x\} & \text{if } \{x\} < 0.5 \\ x - \{x\} + 1 & \text{if } \{x\} \geq 0.5 \end{cases} \\ &= \begin{cases} \lfloor x \rfloor & \text{if } \{x\} < 0.5 \\ \lceil x \rceil & \text{if } \{x\} \geq 0.5 \end{cases} \end{aligned}$$

One notices that $\lfloor x \rceil = \lfloor x + 0.5 \rfloor$ thus

$$\lfloor x \rceil = x - \frac{\tan^{-1}(\tan(x\pi))}{\pi} = \begin{cases} x & \text{if } x \in \mathbb{Z} \\ x - \frac{\tan^{-1}(\tan(x\pi))}{\pi} & \text{if } x \in \mathbb{R} \setminus \mathbb{Z} \end{cases}$$

4 Mod

We start by defining the function $a \bmod b$ as

$$\begin{aligned} a \bmod b &= n \text{ where } a \equiv n (\bmod b) \\ &= a - b \left\lfloor \frac{a}{b} \right\rfloor \end{aligned}$$

which substituting the previously defined definition of floor results in

$$
\begin{aligned}
a \bmod b &= a - b\left(\frac{a}{b} - \frac{\cot^{-1}(\cot(\frac{a\pi}{b}))}{\pi}\right) \\
&= \frac{b}{\pi}\cot^{-1}(\cot(\frac{a\pi}{b}))
\end{aligned}
$$

5 Proof

$$
\frac{d}{dx}\lfloor x \rfloor = \frac{d}{dx}\left(x - \frac{\cot^{-1}(\cot(x\pi))}{\pi}\right) = 1 - 1 = 0 \tag{1}
$$

By the definition of $\lfloor x \rfloor$, when $x \in \mathbb{Z}$, $\lfloor x \rfloor = x$, and that $\frac{d}{dx}\lfloor x \rfloor = 0$, $\lfloor x \rfloor$ is constant between integers, and jumps to each integer at that integer ■