CS 540 Introduction to Artificial Intelligence

Reinforcement Learning II

University of Wisconsin-Madison

Fall 2023

- Please fill out course evaluation

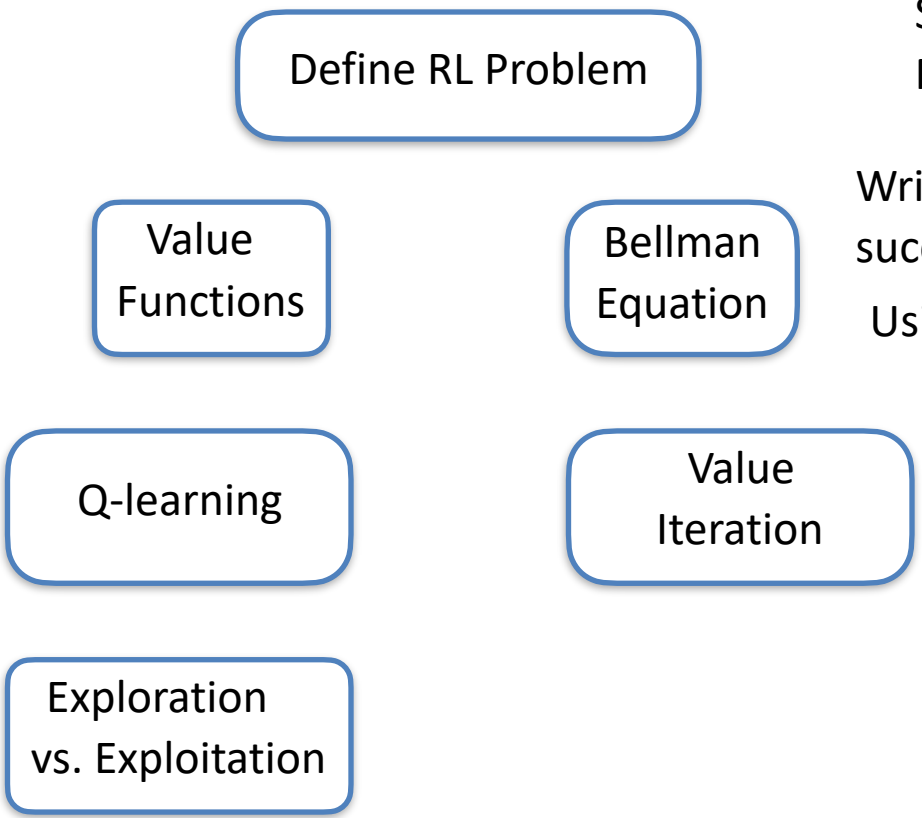
Class Roadmap

Uninformed Search
Informed Search
Games I
Games II
Reinforcement Learning I
Reinforcement Learning I

Outline

- Review of reinforcement learning setting.
  - MDPs, value functions, Q-learning
- Bellman equations and dynamic programming
- From dynamic programming to Q-learning

Key Ideas in Reinforcement Learning



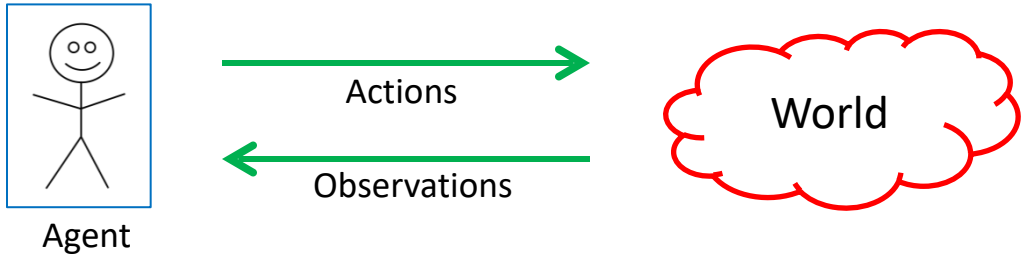
States, Actions, Transitions, Rewards, Markov property, discounting

Writing the value of one state in terms of successor states.

Using values to choose optimal actions.

Back to Our General Model

We have an **agent** **interacting** with the **world**



- Agent receives a reward based on state of the world
  - **Goal**: maximize reward / utility **($$$)**
  - Note: **data** consists of actions & observations
    - Compare to unsupervised learning and supervised learning

Markov Decision Process (MDP)

The formal mathematical model:

- **State set** S. Initial state s_0. **Action set** A
- **State transition model**: $P(s_{t+1}|s_t, a_t)$
  - Markov assumption: transition probability only depends on s_t and a_t, and not previous actions or states.
- **Reward function: *r*(s_t)**
- **Policy**: $\pi(s) : S \rightarrow A$, action to take at a particular state.

$$s_0 \xrightarrow{a_0} s_1 \xrightarrow{a_1} s_2 \xrightarrow{a_2} \dots$$

Defining the Optimal Policy

For policy π, **expected utility** over all possible state sequences from s_0 produced by following that policy:

$$V^{\pi}(s_0) = \sum_{\substack{\text{sequences} \\ \text{starting from } s_0}} P(\text{sequence})U(\text{sequence})$$

Called the **value function** (for π, s_0)

Discounting Rewards

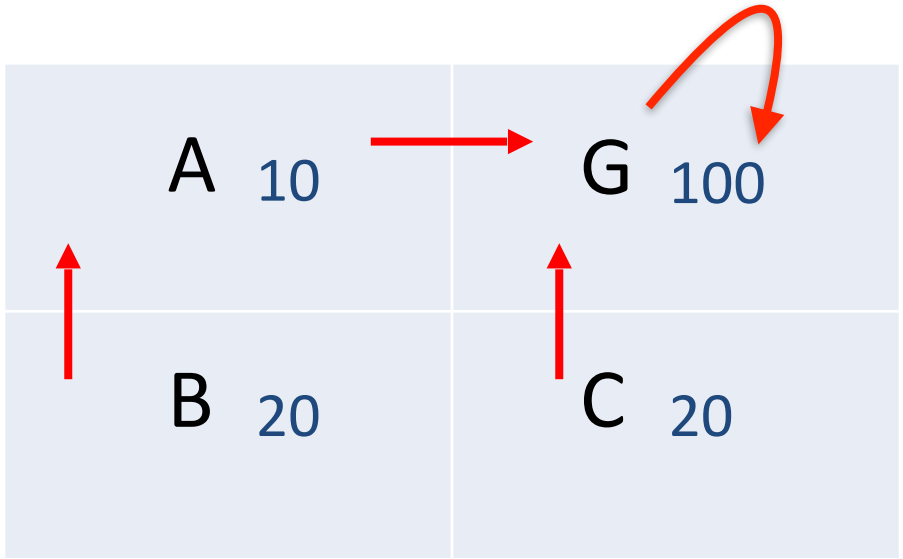
One issue: these are infinite series. **Convergence**?

- Solution

$$U(s_0, s_1 \ldots) = r(s_0) + \gamma r(s_1) + \gamma^2 r(s_2) + \ldots = \sum_{t \geq 0} \gamma^t r(s_t)$$

- Discount factor γ between 0 and 1
  - Set according to how important **present** is VS **future**
  - Note: has to be less than 1 for convergence

Example



Deterministic transitions; $\gamma = 0.8$; policy shown with red arrows.

Values and Policies

- Now that $V^{\pi}(s_0)$ is defined what a should we take?
  - First, set V*(s) to be expected utility for **optimal** policy from s
  - What's the expected utility of an action?
    - Specifically, action a in state s?

$$\sum_{s'} P(s'|s,a)V^*(s')$$

All the states we could go to

Transition probability

Expected rewards

Obtaining the Optimal Policy

Assume, we know the expected utility of an action.

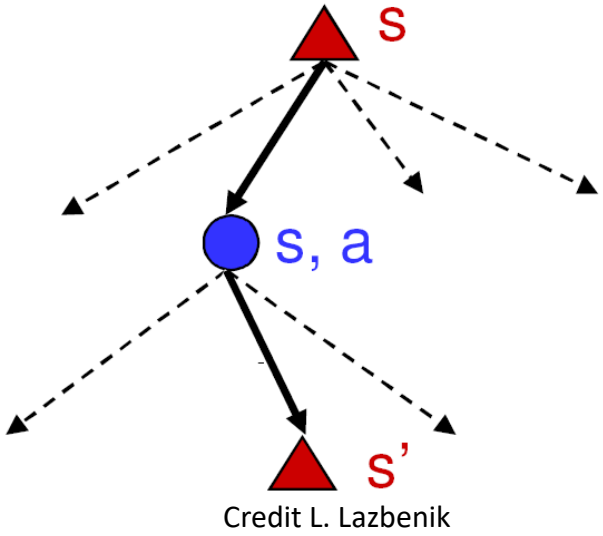
- So, to get the optimal policy, compute

$$\pi^*(s) = \text{argmax}_a \sum_{s'} P(s'|s,a)V^*(s')$$

All the states we could go to

Transition probability

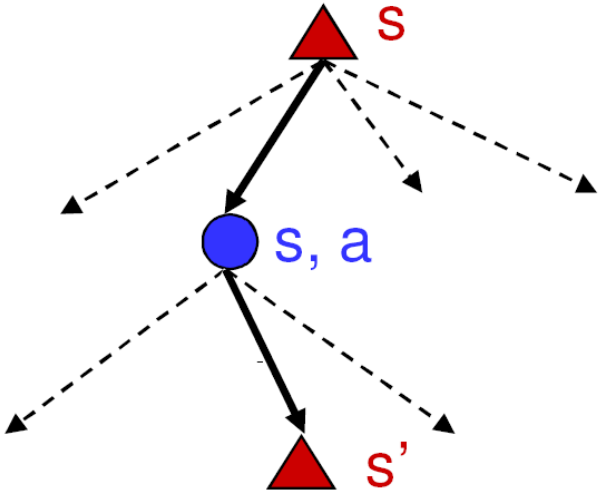
Expected rewards



Credit L. Lazbenik

Bellman Equations

Let's walk over one step for the value function:



Credit L. Lazbenik

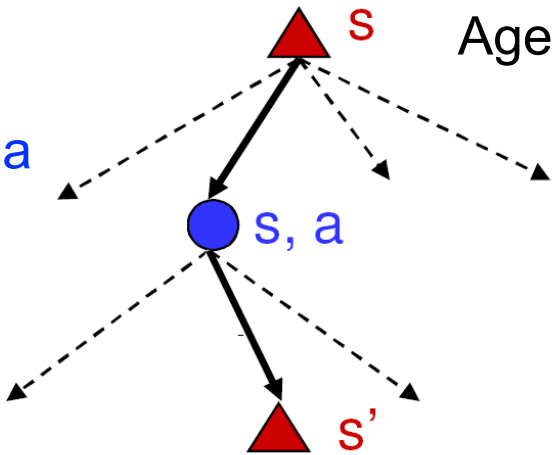
$$V^*(s) = r(s) + \gamma \max_a \sum_{s'} P(s'|s,a)V^*(s')$$

Current state reward

Discounted expected future **rewards**

Richard Bellman: Inventor of dynamic programming.

The Bellman equation



Agent receives reward r(s)

Agent chooses action a

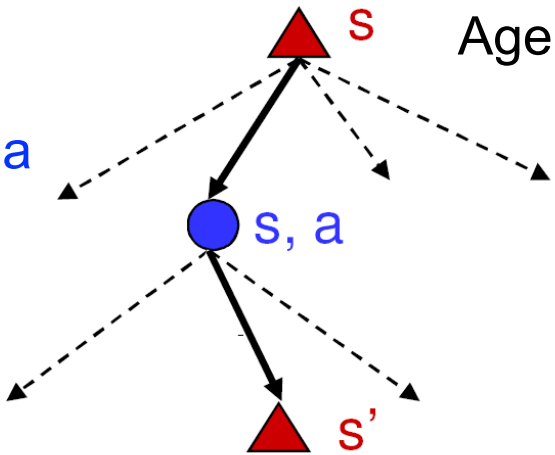
Environment returns

$$s' \sim P(\cdot \mid s, a)$$

Define state value $V^*(s)$ as the expected sum of discounted rewards if the agent follows an *optimal* policy starting in state s.

Image source: L. Lazbenik

The Bellman equation



Agent receives reward r(s)

Agent chooses action a

Environment returns

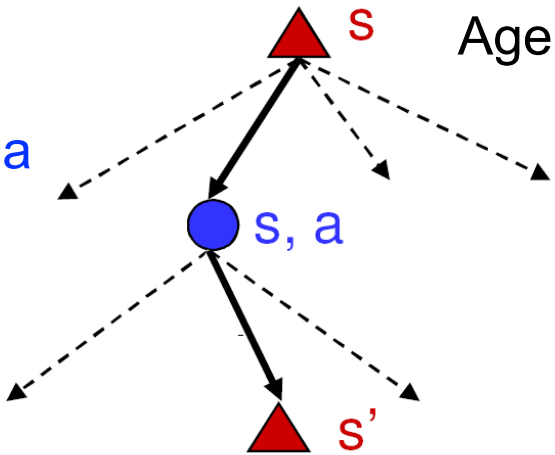
$$s' \sim P(\cdot \mid s, a)$$

- What is the expected utility of taking action **a** in state **s**?

$$\sum_{s'} P(s'|s,a)V^*(s')$$

Image source: L. Lazbenik

The Bellman equation



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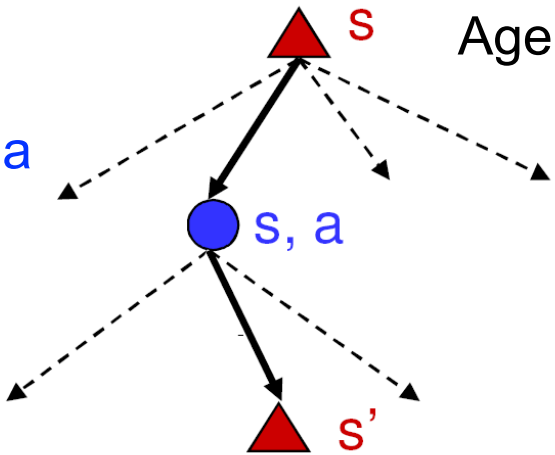
- What is the recursive expression for $V^*(s)$ in terms of $V^*(s')$ - the utilities of its successors?

$$V^*(s) = r(s) + \gamma \sum_{s'} P(s'|s, \pi^*(s))V^*(s')$$

Image source: L. Lazbenik

The Bellman equation



Agent receives reward r(s)

Agent chooses action a

Environment returns

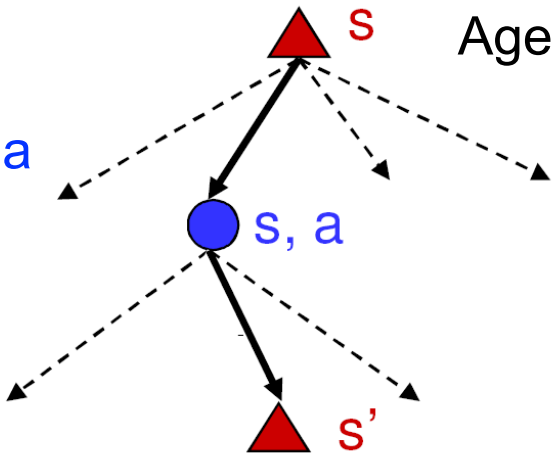
$$s' \sim P(\cdot \mid s, a)$$

- How do we choose the action?

$$\pi^*(s) = \arg\max_a \sum_{s'} P(s'|s,a)V^*(s')$$

Image source: L. Lazbenik

The Bellman equation



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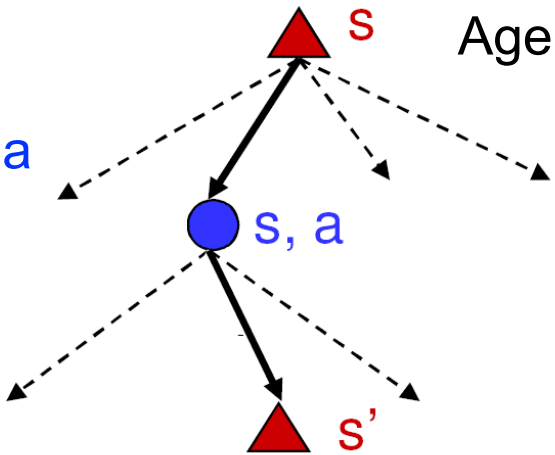


Agent receives reward r(s)

Agent chooses action a

Environment returns

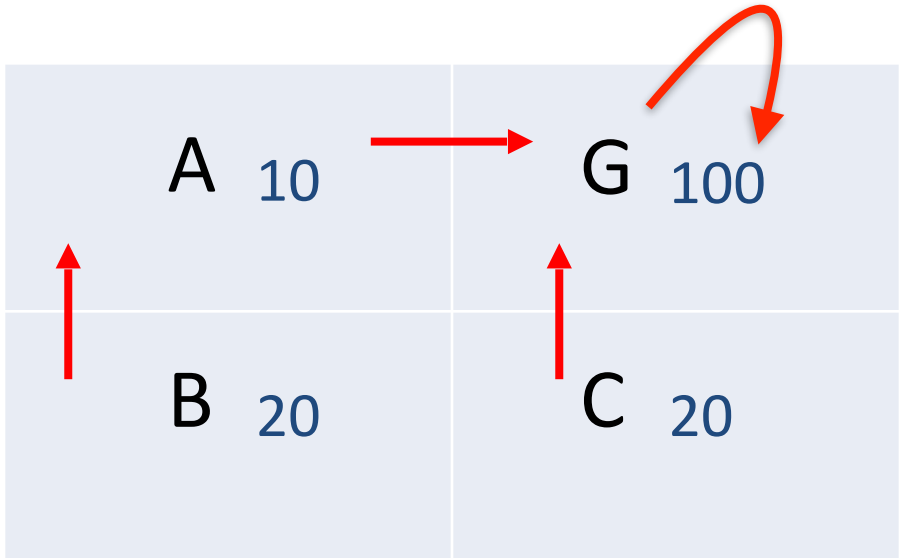
$$s' \sim P(\cdot \mid s, a)$$

- The same reasoning gives the Bellman equation for a general policy:

$$V^{\pi}(s) = r(s) + \gamma \sum_{s'} P(s'|s, \pi(s))V^{\pi}(s')$$

Image source: L. Lazbenik

Example



Deterministic transitions; $\gamma = 0.8$; policy shown with red arrows.

Value Iteration

Q: how do we find $V^*(s)$?

- Why do we want it? Can use it to get the best policy
- Know: reward $r(s)$, transition probability $P(s'|s,a)$
- Also know $V^*(s)$ satisfies Bellman equation:

$$V^*(s) = r(s) + \gamma \max_a \sum_{s'} P(s'|s,a) V^*(s')$$

A: Use the property. Start with $V_0(s)=0$. Then, update

$$V_{i+1}(s) = r(s) + \gamma \max_a \sum_{s'} P(s'|s,a) V_i(s')$$

Value Iteration Algorithm

Input: Transition function P, reward function r, precision $\delta > 0$

1. For all states s, set V(s) = 0.
2. $\Delta \leftarrow \infty$
3. While $\Delta > \delta$:
4. Loop for each state s:
5. $$V(s) \leftarrow r(s) + \max_a \gamma \sum_{s'} P(s' \mid s, a) V(s')$$
6. $\Delta \leftarrow$ maximum change in $V(s)$ for any state s
7. End Loop
8. End While

Here, P and r are known so no need for exploration or interaction with real world.

Value Iteration Demo

https://cs.stanford.edu/people/karpathy/reinforcejs/gridworld_dp.html

Break & Quiz

Q 2.1 Consider an MDP with 2 states {*A, B*} and 2 actions: **"stay"** at current state and "**move**" to other state. Let **r** be the reward function such that **r**(*A*) = 1, **r**(*B*) = 0. Let γ be the discounting factor. Let π: π(*A*) = π(*B*) = **move** (i.e., an "always move" policy). What is the value function $V^{\pi}(A)$?

- A. 0
- B. 1 / (1 -γ)
- C. 1 / (1 -γ^2)
- D. 1

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- B. $1/(1-\gamma)$
- **C. $1/(1-\gamma^2)$**
- D. 1

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- A. 0
- B. $1/(1-\gamma)$
- **C. $1/(1-\gamma^2)$** (States: A,B,A,B,… rewards 1,0, γ^2,0, γ^4,0, …)
- D. 1

Q-Learning

- Reinforcement learning without knowledge of r or P
- Learn from data of the form: $\{(s_t, a_t, r_t, s_{t+1})\}$.
- Learns an action-value function Q*(s,a) that tells us the expected value of taking a in state s.
  - Note: $V^*(s) = \max_a Q^*(s, a)$.
- Optimal policy is formed as $\pi^*(s) = \arg\max_a Q^*(s, a)$

Q-Learning

Estimate Q*(*s*,*a*) from data $\{(s_t, a_t, r_t, s_{t+1})\}$:

1. Initialize Q(.,.) arbitrarily (eg all zeros)
    1. Except terminal states $Q(s_{terminal},.)=0$
2. Iterate over data until Q(.,.) converges:

$$Q(s_t, a_t) \leftarrow (1-\alpha)Q(s_t, a_t) + \alpha(r_t + \gamma \max_b Q(s_{t+1}, b))$$

Learning rate

Equivalent update: $Q(s_t, a_t) \leftarrow Q(s_t, a_t) + \alpha(r(s_t) + \gamma \max_{a'} Q(s_{t+1}, a') - Q(s_t, a_t))$

Q-learning Algorithm

Input: step size α, exploration probability ϵ

1. set Q(s,a) = 0 for all s, a.
2. For each episode:
3. Get initial state s.
4. While (s not a terminal state):
5. Perform a = ϵ-greedy(Q, s), receive r, s'
6. $$Q(s,a) = (1-\alpha)Q(s,a) + \alpha(r + \gamma \max_{a'} Q(s',a'))$$
7. $s \leftarrow s'$
8. End While
9. End For

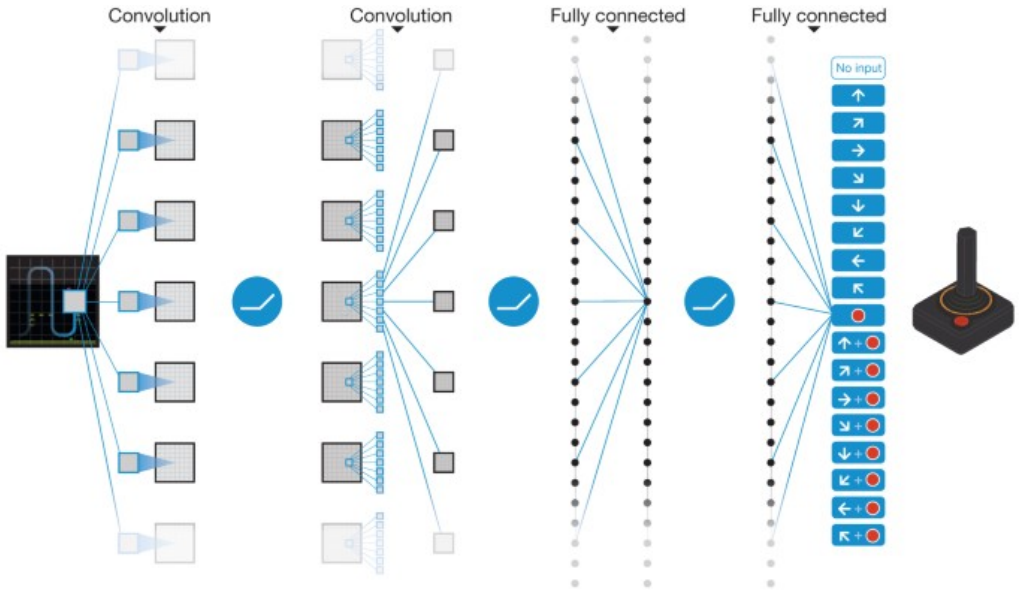
Explore: take action to see what happens.

Update action-value based on result.

Converges to Q*(s,a) in limit if all states and actions visited infinitely often.

Deep Q-Learning

How do we get Q(*s*,*a*) with a large number of states?



Mnih et al, "Human-level control through deep reinforcement learning"

Deep Q-Learning

How do we get Q(*s*,*a*) with a large number of states?

- Deep Q-learning uses a neural network to approximate Q(s,a)
  - Let $Q_\theta : S \times A \to \mathbb{R}$ be the neural network with weights and biases denoted θ.
- Training is similar to supervised regression:
  - (s, a) as input and $y = r(s) + \gamma \max_{a'} Q_\theta(s', a')$ as label.
  - Note that output of the neural network is used in the label.
  - Loss function: $\mathscr{L}(\theta) = (y - Q_\theta(s, a))^2$

Break & Quiz

Q 2.1 For Q learning to converge to the true Q function, we must

- A. Visit every state and try every action infinitely often.
- B. Perform at least 20,000 iterations.
- C. Re-start with different random initial table values.
- D. Prioritize exploitation over exploration.

Break & Quiz

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Break & Quiz

Q 2.1 For Q learning to converge to the true Q function, we must

- **A. Visit every state and try every action infinitely often.**
- B. Perform at least 20,000 iterations. (No: this is dependent on the particular problem, not a general constant).
- C. Re-start with different random initial table values. (No: this is not necessary in general).
- D. Prioritize exploitation over exploration. (No: insufficient exploration means potentially unupdated state action pairs).

Summary of RL

- Reinforcement learning setup
- Mathematical formulation: MDP
- Value functions & the Bellman equation
- Value iteration
- Q-learning

Search and RL Review

- Search
  - Uninformed vs Informed
  - Optimization
- Games
  - Minimax search
- Reinforcement Learning
  - MDPs, value iteration, Q-learning

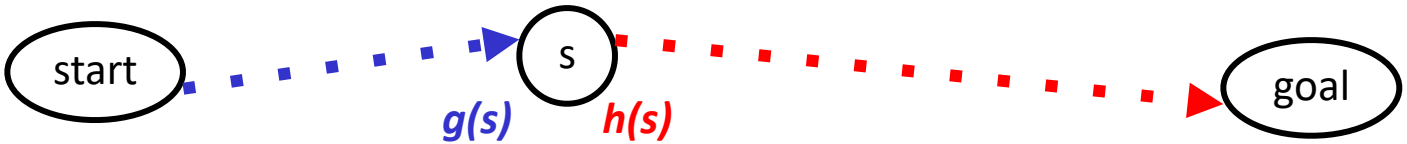
Uninformed vs Informed Search

Uninformed search (all of what we saw). Know:

- Path cost ***g***(*s*) from start to node *s*
- Successors.

Informed search. Know:

- All uninformed search properties, plus
- Heuristic **h(s)** from s to goal (recall game heuristic)

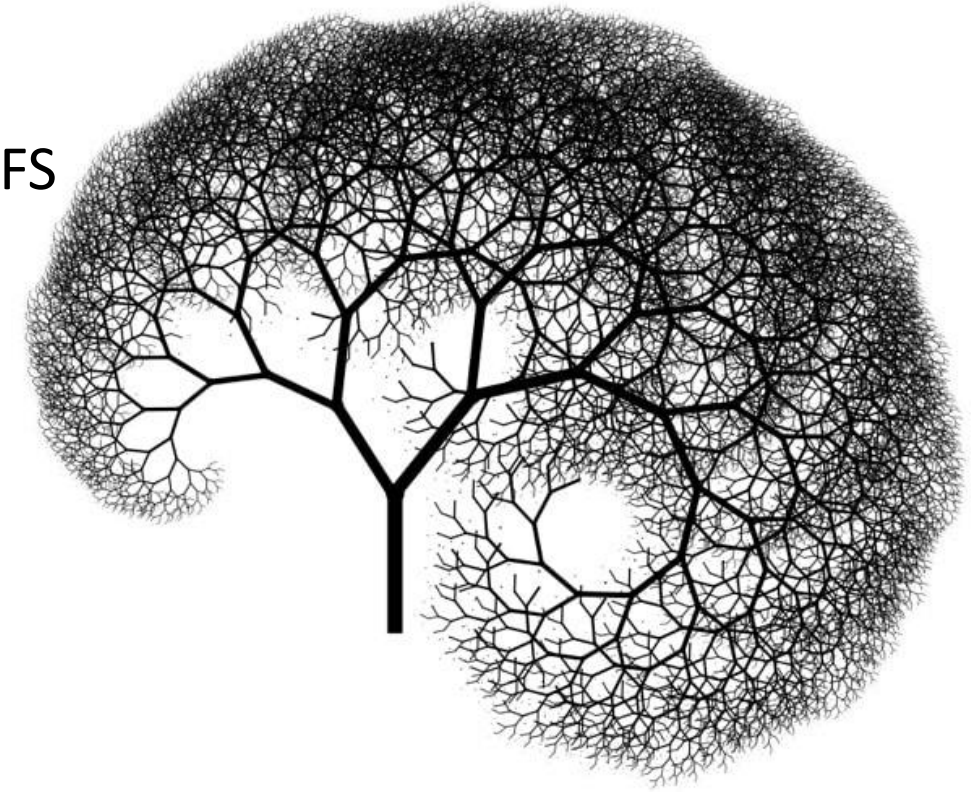


Uninformed Search: Iterative Deepening DFS

Repeated limited DFS

- Search like BFS, fringe like DFS
- **Properties**:
  - Complete
  - Optimal (if edge cost 1)
  - Time O(b^d)
  - Space O(bd)

A good option!



Fractalsaco

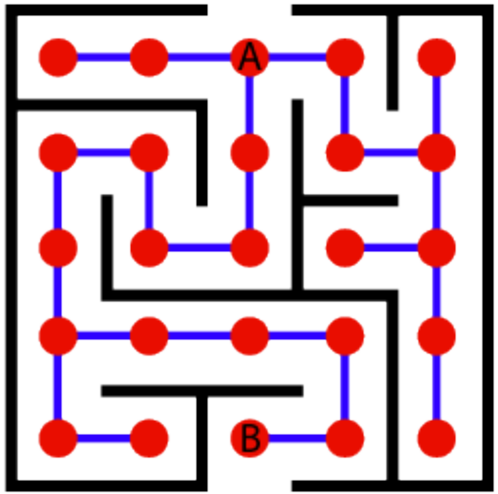
Informed Search: **A* Search**

A*: Expand best ***g(s)*** + ***h(s)***, with one requirement

- Demand that ***h(s)*** ≤ ***h*(s)***

- If heuristic has this property, “admissible”
  - Optimistic! Never over-estimates

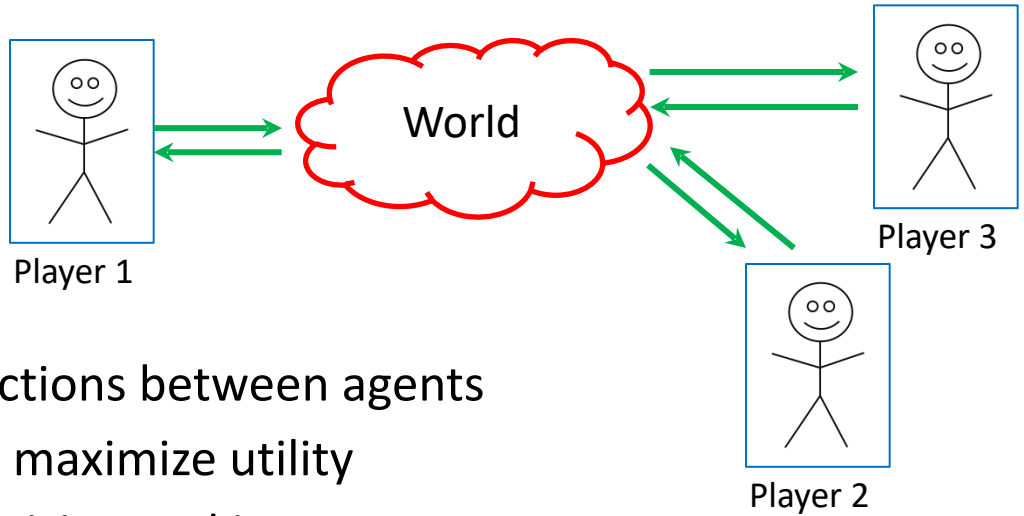
- Still need ***h(s)*** ≥ 0
  - Negative heuristics can lead to strange behavior



V. Batoćanin

Games Setup

Games setup: **multiple** agents



- Now: interactions between agents
- Still want to maximize utility
- **Strategic** decision making.

Minimax Search

Note that long games are yield huge computation

- To deal with this: limit ***d*** for the search depth
- **Q**: What to do at depth *d*, but no termination yet?
  - **A**: Use a heuristic evaluation function *e(x)*

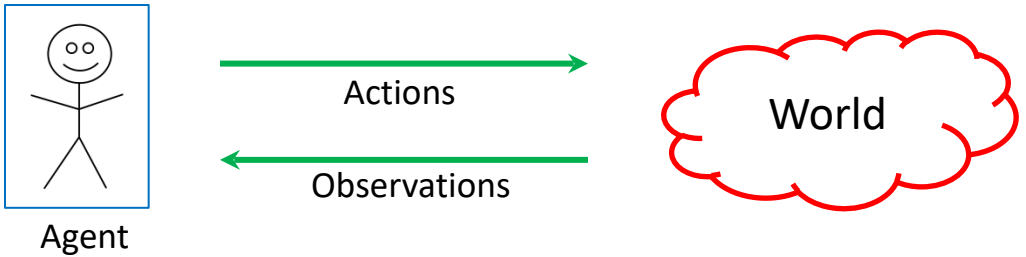
```
function MINIMAX(x, d) returns an estimate of x's utility value
  inputs: x, current state in game
          d, an upper bound on the search depth
  if x is a terminal state then return Max's payoff at x
  else if d = 0 then return e(x)
  else if it is Max's move at x then
      return max{MINIMAX(y, d−1) : y is a child of x}
  else return min{MINIMAX(y, d−1) : y is a child of x}
```

Credit: Dana Nau

Building The Theoretical Model

Basic setup:

- Set of states, S
- Set of actions A



- Information: at time t, observe state $s_t \in S$. Get reward r_t
- Agent makes choice $a_t \in A$. State changes to $s_{t+1,}$ continue

Goal: find a map from **states to actions** maximize rewards.

A "policy"