CS 540 Introduction to Artificial Intelligence

Logic

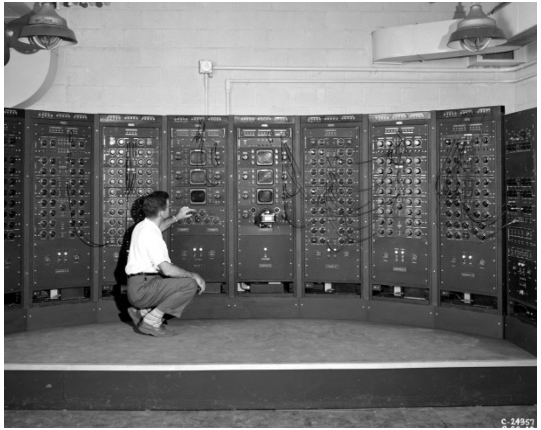
University of Wisconsin-Madison

Fall 2023

Logic & AI

Why are we studying logic?

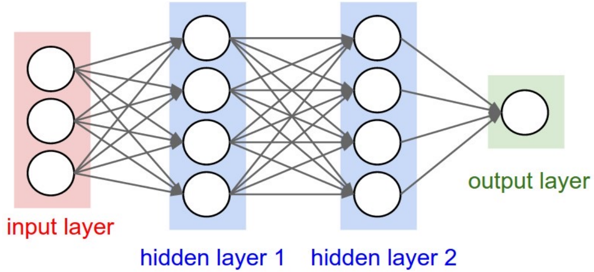
- **Traditional** approach to AI ('50s-'80s)
  - "Symbolic AI"
  - The Logic Theorist – 1956
    - Proved a bunch of theorems!
- Logic also the language of:
  - Knowledge rep., databases, etc.



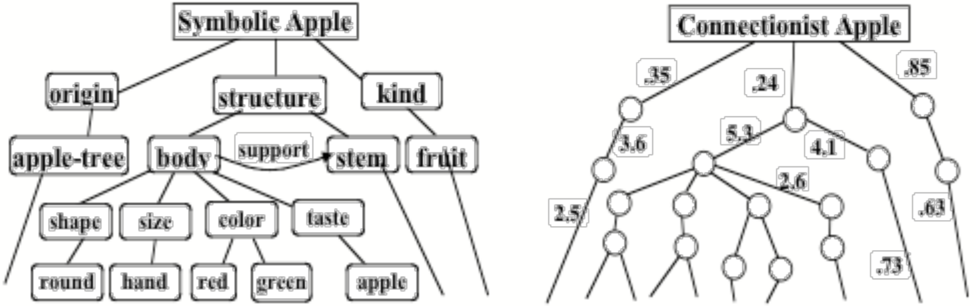
Symbolic vs Connectionist

Rival approach: **connectionist**

- Probabilistic models
- Neural networks
- **Extremely popular** last 20 years



Stanford CS231n

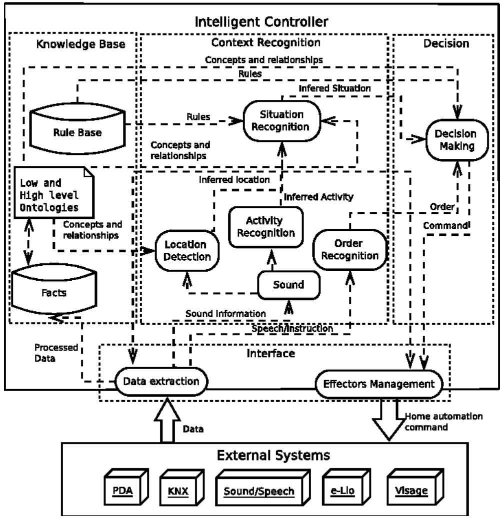


M. Minsky

Symbolic vs Connectionist

Which is better?

- Future: combination; best-of-both-worlds.
  - "Neurosymbolic AI"
  - **Example**: Markov Logic Networks



Propositional Logic Basics

Logic Vocabulary:

- Sentences, symbols, connectives, parentheses
  - Symbols: P, Q, R, … (**atomic** sentences)
  - Connectives:

$\wedge$	and	[conjunction]
$\vee$	or	[disjunction]
$\Rightarrow$	implies	[implication]
$\Leftrightarrow$	is equivalent	[biconditional]
$\neg$	not	[negation]

  - Literal: P or negation $\neg$P

Propositional Logic Basics

Examples:

- $(P \vee Q) \Rightarrow S$
  - "If it is cold or it is raining, then I need a jacket"
- $Q \Rightarrow P$
  - "If it is raining, then it is cold"
- $\neg R$
  - "It is not hot"

Propositional Logic Basics

Several rules in place

- Precedence: $\neg$, $\wedge$, $\vee$, $\Rightarrow$, $\Leftrightarrow$
- Use parentheses when needed
- Sentences: **well-formed** or not well-formed:
  - $P \Rightarrow Q \Rightarrow S$ **not well-formed (not associative!)**

Sentences & Semantics

- Sentences: built up from symbols with connectives

  - **Interpretation**: assigning True / False to symbols (a row in truth table)
  - **Semantics**: interpretations for which sentence evaluates to True
  - **Model**: (of a set of sentences) interpretation for which all sentences are True

Evaluating a Sentence

- Example:

P	Q	$\neg P$	$P \wedge Q$	$P \vee Q$	$P \Rightarrow Q$	$P \Leftrightarrow Q$
$false$	$false$	$true$	$false$	$false$	$true$	$true$
$false$	$true$	$true$	$false$	$true$	$true$	$false$
$true$	$false$	$false$	$false$	$true$	$false$	$false$
$true$	$true$	$false$	$true$	$true$	$true$	$true$

- Note:
  - If P is false, P$\Rightarrow$Q is true regardless of Q ("5 is even implies 6 is odd" is True!)
  - Causality not needed: ("5 is odd implies the Sun is a star" is True!)

Evaluating a Sentence: Truth Table

- **Ex**:

P	Q	R	$\neg P$	$Q \wedge R$	$\neg P \vee Q \wedge R$	$\neg P \vee Q \wedge R \Rightarrow Q$
0	0	0	1	0	1	0
0	0	1	1	0	1	0
0	1	0	1	0	1	1
0	1	1	1	1	1	1
1	0	0	0	0	0	1
1	0	1	0	0	0	1
1	1	0	0	0	0	1
1	1	1	0	1	1	1

- **Satisfiable**
  - There exists some interpretation where the sentence is true.

Break & Quiz

Q 1.1: Suppose P is false, Q is true, and R is true. Does this assignment satisfy

(i) $\neg(\neg p \rightarrow \neg q) \wedge r$

(ii) $(\neg p \vee \neg q) \rightarrow (p \vee \neg r)$

- A. Both
- B. Neither
- C. Just (i)
- D. Just (ii)

Break & Quiz

Q 1.1: Suppose P is false, Q is true, and R is true. Does this assignment satisfy

(i) $\neg(\neg p \rightarrow \neg q) \land r$

(ii) $(\neg p \lor \neg q) \rightarrow (p \lor \neg r)$

- A. Both
- B. Neither
- **C. Just (i)**
- D. Just (ii)

Break & Quiz

Q 1.1: Suppose P is false, Q is true, and R is true. Does this assignment satisfy

(i) $\neg(\neg p \rightarrow \neg q) \wedge r$

(ii) $(\neg p \vee \neg q) \rightarrow (p \vee \neg r)$

- A. Both
- B. Neither
- **C. Just (i)**
- D. Just (ii)

Plug interpretation into each sentence.

For (i): $(\neg p \rightarrow \neg q)$ will be false so $\neg(\neg p \rightarrow \neg q)$ will be true and r is true by assignment.

For (ii): $(\neg p \vee \neg q)$ is true and $(p \vee \neg r)$ is false which makes the implication false.

Break & Quiz

Q 1.2: Let A =“Aldo is Italian” and B =“Bob is English”. Formalize “Aldo is Italian or if Aldo isn’t Italian then Bob is English”.

- a. $A \vee (\neg A \rightarrow B)$
- b. $A \vee B$
- c. $A \vee (A \rightarrow B)$
- d. $A \rightarrow B$

Break & Quiz

Q 1.2: Let A =“Aldo is Italian” and B =“Bob is English”. Formalize “Aldo is Italian or if Aldo isn’t Italian then Bob is English”.

- **a. $A \vee (\neg A \rightarrow B)$**
- **b. $A \vee B$ (equivalent!)**
- c. $A \vee (A \rightarrow B)$
- d. $A \rightarrow B$

Break & Quiz

Q 1.2: Let A ="Aldo is Italian" and B ="Bob is English". Formalize "Aldo is Italian or if Aldo isn't Italian then Bob is English".

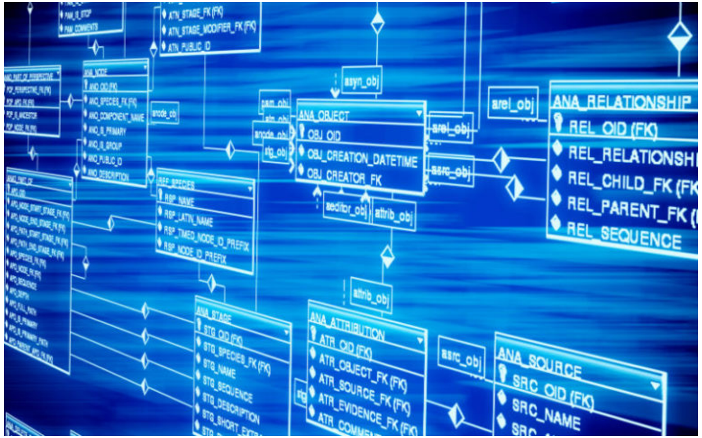
- **a. $A \lor (\neg A \rightarrow B)$**
- **b. $A \lor B$ (equivalent!)**
- c. $A \lor (A \rightarrow B)$
- d. $A \rightarrow B$

Answer a. is the exact translation of the English sentence into a logic sentence. You can see that answer b. is also correct by writing out the truth table for all answers and seeing that a and b have the same truth tables.

Or you can use the fact that $\neg A \rightarrow B = A \lor B$ and that $A \lor A \lor B = A \lor B$ to prove equivalence.

Knowledge Bases

- **Knowledge Base (KB)**: A set of sentences $\{A_1, A_2, \ldots A_n\}$.
  - Like a long sentence, connect with conjunction:
  - KB is $A_1 \wedge A_2 \wedge \cdots \wedge A_n$.



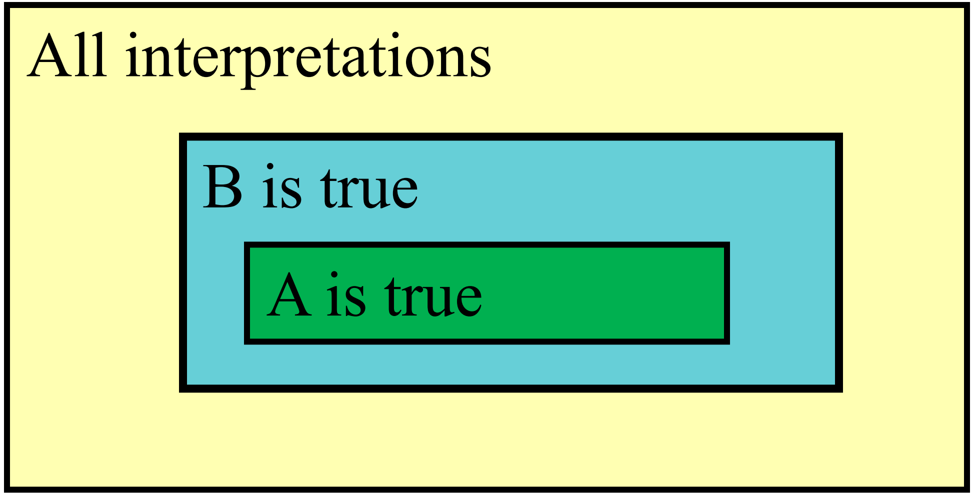
Model of a KB: interpretations where all sentences are True

Goal: inference to discover new sentences

Entailment

Entailment: a sentence B logically follows from A

- Write A ⊨ B
- A ⊨ B iff in every interpretation where A is true, B is also true



Methods of Inference: **1. Enumeration**

- Enumerate all interpretations; look at the truth table
  - "Model checking"
- Downside: 2^n interpretations for n symbols

Methods of Inference: **2. Using Rules**

- *Modus Ponens*: $(A \Rightarrow B, A) \vDash B$
- And-elimination
- Other rules on the next page
  - Commutativity, associativity, de Morgan's laws, distribution for conjunction/disjunction

Logical equivalences

$$
\begin{aligned}
(\alpha \wedge \beta) &\equiv (\beta \wedge \alpha) & \text{commutativity of } \wedge \\
(\alpha \vee \beta) &\equiv (\beta \vee \alpha) & \text{commutativity of } \vee \\
((\alpha \wedge \beta) \wedge \gamma) &\equiv (\alpha \wedge (\beta \wedge \gamma)) & \text{associativity of } \wedge \\
((\alpha \vee \beta) \vee \gamma) &\equiv (\alpha \vee (\beta \vee \gamma)) & \text{associativity of } \vee \\
\neg(\neg\alpha) &\equiv \alpha & \text{double-negation elimination} \\
(\alpha \Rightarrow \beta) &\equiv (\neg\beta \Rightarrow \neg\alpha) & \text{contraposition} \\
(\alpha \Rightarrow \beta) &\equiv (\neg\alpha \vee \beta) & \text{implication elimination} \\
(\alpha \Leftrightarrow \beta) &\equiv ((\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha)) & \text{biconditional elimination} \\
\neg(\alpha \wedge \beta) &\equiv (\neg\alpha \vee \neg\beta) & \text{de Morgan} \\
\neg(\alpha \vee \beta) &\equiv (\neg\alpha \wedge \neg\beta) & \text{de Morgan} \\
(\alpha \wedge (\beta \vee \gamma)) &\equiv ((\alpha \wedge \beta) \vee (\alpha \wedge \gamma)) & \text{distributivity of } \wedge \text{ over } \vee \\
(\alpha \vee (\beta \wedge \gamma)) &\equiv ((\alpha \vee \beta) \wedge (\alpha \vee \gamma)) & \text{distributivity of } \vee \text{ over } \wedge
\end{aligned}
$$

You can use these equivalences to modify sentences.

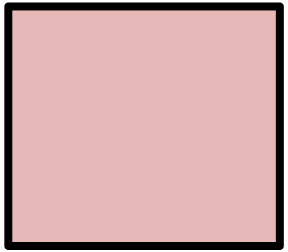
First Order Logic (FOL)

Propositional logic has some limitations

- Ex: how to say “all squares have four sides”
- No context, hard to generalize; express facts

FOL is a more expressive logic; works over

- Facts, Objects, Relations, Functions



First Order Logic Syntax

- **Term**: an object in the world
  - **Constant**: Alice, 2, Madison, Green, …
  - **Variables**: x, y, a, b, c, …
  - **Function**(term_1, …, term_n)
    - Sqrt(9), Distance(Madison, Chicago)
    - Maps one or more objects to another object
    - Can refer to an unnamed object: LeftLeg(John)
    - Represents a user defined functional relation

- A **ground term** is a term without variables.
  - Constants or functions of constants.

FOL Syntax

- **Atom**: smallest T/F expression
  - **Predicate**(term$_1$, …, term$_n$)
    - Teacher(Jerry, you), Bigger(sqrt(2), x)
    - Convention: read “Jerry (is)Teacher(of) you”
    - Maps one or more objects to a truth value
    - Represents a user defined relation
  - **term$_1$ = term$_2$**
    - Radius(Earth)=6400km, 1=2
    - Represents the equality relation when two terms refer to the same object.

FOL Syntax

- **Sentence**: T/F expression
  - Atom
  - Complex sentence using connectives: ∧ ∨ ¬ ⇒ ⇔
    - Less(x,22) ∧ Less(y,33)
  - Complex sentence using quantifiers ∀, ∃
- Sentences are evaluated under an interpretation
  - Which objects are referred to by constant symbols
  - Which objects are referred to by function symbols
  - What subsets defines the predicates

FOL Quantifiers

- Universal quantifier: ∀
- Sentence is true **for all** values of x in the domain of variable x.

- Main connective typically is ⇒
  - Forms if-then rules
  - "all humans are mammals"

$$\forall x\ \mathtt{human}(x) \Rightarrow \mathtt{mammal}(x)$$

  - Means if x is a human, then x is a mammal

FOL Quantifiers

- Existential quantifier: ∃
- Sentence is true **for some** value of x in the domain of variable x.

- Main connective typically is ∧
  - "some humans are male"

$$\exists x\ \mathbf{human(x)}\ \wedge\ \mathbf{male(x)}$$

  - Means there is an x who is a human and is a male

Break & Quiz

Q 2.1: How many entries does a truth table have for a FOL sentence with k variables where each variable can take on n values?

- A. Truth tables are not applicable to FOL.
- B. 2^k
- C. n^k
- D. It depends

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Must have one entry for every possible assignment of values to variables. That number is (C).