Asset allocation with a stochastic wage rate

The goal of this project is to characterize $w_i^*(t)$, the optimal proportion invested in each risky asset and $C^*(t)$, the optimal consumption of the agent seeking to maximize his intertemporal expected utility of consumption and terminal wealth under her budget constraint and with a stochastic investment opportunity set. Here, this set is introduced by the stochastic wage rate that will affect asset prices and thus wealth.

Main assumptions

1. Individuals continuously make investment and consumption choices based on a long term objective brought by their lifetime consumption and the value of the "bequest" they leave at their death.

2. Markets are "perfect".

3. The prices of risky assets follow It processes.

4. Transaction are done in continuous time and financial markets are opened without interruption.

Model setup

Let us consider the optimal allocation of an investor who can allocate her wealth in a money market account (i.e.cash), nominal bonds and a stock index. The money market account follows the dynamics:

$$\frac{dM_t}{M_t} = r_t dt$$

where r_t denotes the short-term interest rate. There is no volatility term since we consider the money market account riskless. The stock index is assumed to evolve according to the dynamics:

$$\frac{dS_t}{S_t} = (r_t + \lambda_S)dt + \sigma_S dW_t^1$$

where λ_S is the constant expected excess return from investing in stocks and σ_S is the constant stock index volatility. W^1 is a standard Brownian motion.

The short-term interest rate dynamics are described by an Ornstein-Uhlenbeck process:

$$dr_t = k(\theta - r_t)dt - \sigma_r dW_t^2$$

where W^2 is a Brownian motion. k captures the degree of mean-reversion, θ is the long-run mean of the short-term interest rate and $\sigma_r > 0$ is the interest rate volatility. The negative volatility here leads to a reflected Brownian motion which means that the Brownian motion is bounded, this makes sense since the interest rate should not in theory go below 0 and cannot go above a very high value.

The price of a default-free bond B satisfies the following dynamics:

$$\frac{dB_t}{B_t} = (r_t + \lambda_B)dt + \sigma_B dW_t^2$$

with $\lambda_B = \lambda_r D(r,t)$ the expected excess return on the bond. λ_r is the constant risk premium per unit of interest rate risk, and $\sigma_B = -\sigma_r D(r,t) > 0$. D is the duration of the bond defined by $D = -\frac{\partial B}{\partial r}\frac{1}{B}$ which is the constant elasticity of the bond price with respect to the short interest rate.

Let's assume that the individual receives a continuous flow of positive wage denoted by I_t at time t. This wage is risky income from labor until retirement date $\tilde{T} \leq T$. From the retirement date $\tilde{T}$ until the individual's final horizon date T, the individual receives a riskless stream of wage proportional to her wage at retirement with a fixed replacement rate $\bar{P}$. For any time after retirement, the wage of the individual is given by $I_t = \bar{P}I_{\tilde{T}]}, \tilde{T} \leq t \leq T$. We assume that I_t evolves following the dynamics:

$$dI_t = \begin{cases} (\xi_0(t) + \xi_1 r_t)I_t dt + \sigma_I I_t dW_t^3 & , for t \leq \tilde{T} \\ 0dt & , \tilde{T} < t \leq T \end{cases}$$

where $\xi_0(t)$ is a real-valued positive function and $\xi_1 r_t$ is the constant interest rate coefficient. σ_I is the volatility of the wage process and W^3 is a Brownian motion.

Let us suppose that all three Brownian motions are correlated with constant correlation coefficient $\rho_{ij}, i, j = 1, 2, 3$.

Now, let us call Y_t the wealth of the investor at time t. At each point in time, the investor allocates her wealth to consumption c (a dollar amount), a fraction w^S to the stock index, a fraction w^B to bonds and the rest to the money market account. The investor wants to chose a dynamic portfolio strategy in order to maximize the expected utility of her interim consumption as well as her utility from bequest at time T.

Let us consider the utility function:

$$U(c_t) = \frac{1}{1-\gamma} c_t^{1-\gamma}$$

which is an isoelastic utility function with constant relative risk aversion.

and the bequest function:

$$B(Y_T) = \frac{1}{1-\gamma} Y_T^{1-\gamma}$$

where $\gamma > 1$ is the parameter of risk aversion. The intertemporal preference parameter is denoted by ρ.

Wealth dynamics

We first have to determine the wealth dynamics of the investor. We have:

$dY_t = \left(w_t^S \frac{dS_t}{S_t} + w_t^B \frac{dB_t}{B_t} + (1 - w_t^S - w_t^B) \frac{dM_t}{M_t} \right) Y_t + (I_t - c_t) dt$

By replacing, the dynamics $\frac{dS_t}{S_t}$, $\frac{dB_t}{B_t}$ and $\frac{dM_t}{M_t}$, we get:

$= \left((r_t + w_t^S \lambda_S + w_t^B \lambda_B) Y_t + I_t - c_t \right) dt + w_t^S \sigma_S Y_t dW_t^1 + w_t^B \sigma_B dW_t^2$

Let us denote $r_t + w_t^S \lambda_S + w_t^B \lambda_B$ by μ_t^Y and $w_t^S \sigma_S Y_t dW_t^1 + w_t^B \sigma_B dW_t^2$ by $\sigma_t^Y Y_t dW_t^Y$

So finally:

$$dY_t = (\mu_t^Y Y_t + I_t - c_t) dt + \sigma_t^Y Y_t dW_t^Y$$

Optimization problem

The investor is supposed to choose the portfolio (w_t^S,w_t^B,$1-w_t^S-w_t^B$) and the instantaneous consumption $C(t)$ in order to maximize her expected intertemporal utility:

$$\max_{\left(w_(t)^S,w_t^B,C(t)\right)_{0\leq t\leq T}} E_0\left[\int_0^T exp(-\rho t)U[C(t)]dt + exp(-\rho T)B(Y(T),T)\right]$$

under the budget constraint:

$$dY_t = (\mu_t^Y Y_t + I_t - c_t)dt + \sigma_t^Y Y_t dW_t^Y$$

To find the optimality conditions, we must transform the maximization into a dynamic programming problem to apply Bellman's optimality principle. A necessary condition for a program to be optimal from t to T implies that whatever the decisions between t and t+h at t+h, the program must be optimal between t+h and T as well.

We then introduce the indirect utility function $J(Y_t, I_t, r_t, t)$ such that:

$$J(Y_t, I_t, r_t, t) = max_{\left(w_(t)^S,w_t^B,C(t)\right)} E_t\left[\int_t^T exp(-\rho[s-t])\frac{1}{1-\gamma}c_t^{1-\gamma}ds + exp(-\rho[T-t])\frac{1}{1-\gamma}Y_T^{1-\gamma}\right]$$

J is the indirect utility of the optimal program when the investor reaches time t. We are going to maximize all future decisions from t to T.

Derivation of the Hamilton-Jacobi-Bellman (HJB) equation

The full derivation and explanations of the HJB equation can be found in the appendix.

The HJB equation for $t < \tilde{T}$ (before retirement) is:

$\rho J = \max_{c,w^S,w^B}[\frac{1}{1-\gamma}c^{1-\gamma} + J_t + J_Y(\mu_t^Y Y + I - c) + J_r k(\theta - r) + J_I(\xi_0 + \xi_1 r)I + \frac{1}{2}J_{YY}\sigma_Y^2 Y^2 + \frac{1}{2}J_{rr}\sigma_r^2 + \frac{1}{2}J_{II}\sigma_I^2 I^2 + J_{Yr}(-w^S\rho_{12}\sigma_S\sigma_r - w^B\sigma_B\sigma_r)Y + J_{Ir}(-\rho_{23}\sigma_r\sigma_I I) + J_{YI}(w^S\rho_{13}\sigma_S\sigma_I w^S Y I + \rho_{23}\sigma_B\sigma_I w^B Y I)]$

with boundary condition:
$\lim_{t\nearrow\tilde{T}} J(Y,I,r,t) = \lim_{t\searrow\tilde{T}} J(Y,I,r,t)$ which gives us continuity of the indirect utility function at the retirement date $\tilde{T}$.

The HJB equation for $\tilde{T} < t \leq T$ (after retirement) is:

$\rho J = \max_{c,w^S,w^B}[\frac{1}{1-\gamma}c^{1-\gamma} + J_t + J_Y(\mu_t^Y Y + I - c) + J_r k(\theta - r) + \frac{1}{2}J_{YY}\sigma_Y^2 Y^2 + \frac{1}{2}J_{rr}\sigma_r^2 + J_{Yr}(-w^S\rho_{12}\sigma_S\sigma_r - w^B\sigma_B\sigma_r)Y]$

with terminal boundary condition:

$J(Y,I,r,t) = \frac{1}{1-\gamma}Y^{1-\gamma}$

Explicit resolution in the case after retirement

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For simplicity, we will focus on maximizing the HJB equation after retirement.

Let us define:

$\Phi(Y,c,w^S,w^B,t) = [\frac{1}{1-\gamma}c^{1-\gamma} + J_t + J_Y(\mu_t^Y Y + I - c) + J_r k(\theta - r) + \frac{1}{2}J_{YY}\sigma_Y^2 Y^2 + \frac{1}{2}J_{rr}\sigma_r^2 + J_{Yr}(-w^S\rho_{12}\sigma_S\sigma_r - w^B\sigma_B\sigma_r)Y - \rho J]$
Then, the problem can be rewritten as follows:

$\max_{c,w^S,w^B} \Phi(Y,c,w^S,w^B,t) = 0$

The system to be solved is then:

$$\begin{cases} \Phi(Y,c^*,w_S^*,w_B^*,t) = 0 \\ \Phi_{w_S}(Y,c^*,w_S^*,w_B^*,t) = 0 \\ \Phi_{w_B}(Y,c^*,w_S^*,w_B^*,t) = 0 \\ \Phi_c(Y,c^*,w_S^*,w_B^*,t) = 0 \end{cases}$$

The first order conditions are:

$\Phi_c(Y,c,w^S,w^B,t) = 0$
$\Leftrightarrow U_c(C^*) = J_Y$

(envelope condition) $\Leftrightarrow c = J_Y^{-\frac{1}{\gamma}}$

$\Phi_{w_B}(Y, c, w^S, w^B, t) = 0$

Here we have to remember that $(\sigma_t^Y)^2 Y_t^2 h = \left((w_t^S)^2\sigma_S^2 + (w_t^B)^2\sigma_B^2 + 2\rho_{12}\sigma_S\sigma_B w_t^S w_t^B\right) Y_t^2 h$

$\Leftrightarrow \frac{1}{2} J_{YY} Y^2 (2w^B\sigma_B^2 + 2\rho_{12}\sigma_S\sigma_B w^S) - \sigma_B\sigma_r Y J_{Yr}$

By solving for w^B, we obtain:

$$w^B = \frac{\sigma_r J_{Yr} - \rho_{12}\sigma_S w^S Y J_{YY}}{\sigma_B Y J_{YY}}$$

$\Phi_{w_S}(Y, c, w^S, w^B, t) = 0$
$\Leftrightarrow \frac{1}{2} J_{YY} Y^2 (2w^S\sigma_S^2 + 2\rho_{12}\sigma_S\sigma_B w^B) - \rho_{12}\sigma_S\sigma_r Y J_{Yr} = 0$

Solving for w^S gives us:

$$w^S = \frac{\rho_{12}\sigma_S\sigma_r Y J_{Yr} - \rho_{12}\sigma_S\sigma_B w^B Y^2 J_{YY}}{\sigma_S^2 Y^2 J_{YY}}$$

Replacing w^S in w^B gives:

$$w^B = \frac{\sigma_r J_{Yr} - \rho_{12}^2\sigma_r J_{Yr}}{(1+\rho_{12}^2)\sigma_B Y J_{YY}}$$

and now by replacing w^B in w^S with the previous line, we obtain:

$$w^S = \frac{\rho_{12}\sigma_r J_{Yr}}{\sigma_S Y J_{YY}} - \frac{\rho_{12}\sigma_r J_{Yr} - \rho_{12}^3\sigma_r J_{Yr}}{Y^3 J_{YY}^2 (1+\rho_{12}^2)}$$

To properly solve the problem, it is necessary to guess an appropriate solution. Merton (1969) chose an isoelastic bequest function:

$$B(W, T) = \frac{\epsilon^{1-\gamma}[W(T)]^\gamma}{\gamma}$$

and solves the problem by guessing an appropriate solution:

$$\hat{J}(W,t) = \frac{b(t)}{\gamma}[W(t)]^{\gamma}$$

However, in this project, I was not able to determine J_{Y_r} which is why this is as far as I could go.

Economic implications of the model

All the comments below are based on results obtained in the paper "Dynamic asset allocation with stochastic income and interest rates" by Munk and Sorensen (2010).

a) Income

The model can be solved numerically and different levels of education come into the model by specifying different parameters for the wage dynamics. We consider 'No high school' and 'College' as levels of education. The calibration yields $\xi_1 = -0.24$ for the individual with no high school education and $\xi_1 = 0.49$ for the individual with a college degree. The parameters are calibrated using US-income data.

We have $(\xi_0(t) + \xi_1 r_t)I_t$ in the wealth of individuals. Consequently, if ξ_1 is negative (-0.24), then the individual with no high school education will constantly be in a worse off in terms of wealth than the individual with a college degree who has a positive ξ_1 (0.49). The interest rate r_t is a proxy for the economy: when there is economic growth, the interest rate should be high and when there is a sluggish economy, the interest rate should be low to drive investment and consumption. If we assume that the interest rate is always positive ($r_t \geq 0$), then the individual will always lose wealth no matter what the economy is doing (except in the case where $r_t = 0$), she will lose even more in a strong economy. On the other hand, the individual with a college degree will always see her wealth increase by half the current interest rate in the economy.

In a strong economy, because there are generally not enough skilled workers (college degree), the companies which get more capital will compete to hire skilled workers at a higher salary. For the unskilled workers (no high school education), the overall unemployment will decrease but they will accept any job they can find without much bargaining power and they compete now with more individuals. However, it would normally make sense that they get more bargaining power in a strong economy.

In a sluggish economy, when r_t is low and close to zero, the difference of wealth between skilled workers and unskilled workers becomes smaller. The skilled workers will have more difficulty finding their preferred jobs and the unskilled workers will always be able to find jobs that do not necessitate a college degree, jobs that the skilled workers would not be interested in.

b) Wealth and wage over the life-cycle (assuming relative risk aversion $\gamma = 4$)

The wage is higher for college graduates, and increasing more steeply. The consumption and the wealth of the college graduate is generally higher than the individual with no high school education.

The exception is for < 35 years, the individual without high school education has higher wealth but the college graduate still has higher consumption in the early years because he foresees a higher future wage.

c) Optimal portfolios for different levels of education

During the retirement phase, the portfolio weights are similar for both individuals. There is near zero weight in bonds after 70-75 years to insure against interest rate risk and more an more weight is allocated into cash.

Before retirement, the optimal portfolios of the two different individuals are very different: the college graduate invests more in stocks and the rest in bonds while for the individual with no high school education, there is a large allocation in stocks, some in bonds and a significant allocation in cash. The reason for the difference here is the big difference in ξ_1 between the two individuals:

The individual with no high school education has a negative ξ_1, she needs less of the bond and more of cash to insure against interest rate risk. At the same time, the college graduate has a positive ξ_1, the evolution of her wealth is positively correlated with short-term interest rate so she substitutes cash by bonds.

d) Shortcomings of the model

-The hypothetical setup of the bond
-The deterministic time of death (can be modeled as a stochastic process)
-The bond does not contain any default risk
-The wealth can be negative (should have non-negativity restriction)
-Estimation risk (a lot of parameters need to be estimated)

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Appendix

Derivation of the HJB equation:

a) Optimality principle and law of Iterated expectation

We will proceed in a recursive way:

$J(Y_t, I_t, r_t, t) = max_{\left(w_(t)^S, w_t^B, C(t)\right)} E_t[\int_t^{t+h} exp(-\rho[s-t])\frac{1}{1-\gamma}c_t^{1-\gamma}ds +$
$\int_{t+h}^t exp(-\rho[s-t-h+h])\frac{1}{1-\gamma}c_t^{1-\gamma}ds + exp(-\rho[T-t-h+h])\frac{1}{1-\gamma}Y_T^{1-\gamma}]$

$= max_{\left(w_(t)^S, w_t^B, C(t)\right)} \left[E_t\left[\int_t^{t+h} exp(-\rho[s-t])\frac{1}{1-\gamma}c_t^{1-\gamma}ds\right]\right] +$
$exp(-\rho h) max_{\left(w_(t)^S, w_t^B, C(t)\right)} \left[E_t\left[\int_{t+h}^T exp(-\rho[s-t-h])\frac{1}{1-\gamma}c_t^{1-\gamma}ds + exp(-\rho[T-t-h])\frac{1}{1-\gamma}Y_T^{1-\gamma}\right]\right]$
(1)

For the second part of equation (1):

$max_{\left(w_(t)^S, w_t^B, C(t)\right)} \left[E_t\left[\int_{t+h}^T exp(-\rho[s-t-h])\frac{1}{1-\gamma}c_t^{1-\gamma}ds + exp(-\rho[T-t-h])\frac{1}{1-\gamma}Y_T^{1-\gamma}\right]\right]$

$= max_{\left(w_(t)^S, w_t^B, C(t)\right)} E_t\left[E_{t+h}\left[\int_{t+h}^t exp(-\rho[s-t-h])\frac{1}{1-\gamma}c_t^{1-\gamma}ds + exp(-\rho[T-t-h])\frac{1}{1-\gamma}Y_T^{1-\gamma}\right]\right]$
by the law of iterated expectations

$= max_t E_t\left[max_{t+h} E_{t+h}\left[\int_{t+h}^t exp(-\rho[s-t-h])\frac{1}{1-\gamma}c_t^{1-\gamma}ds + exp(-\rho[T-t-h])\frac{1}{1-\gamma}Y_T^{1-\gamma}\right]\right]$
by the optimality principle

The intuition behind the optimality principle is that whatever the result obtained following $(w_(t)^S, w_t^B, C(t))$, from $s = t$ to $s = t + h$, if the program is optimal from t to T, then the choice at $s = t + h$ must also be an optimum on $[t + h, T]$. Formally, it means that $\max_t(\max_{t+h}) = \max_t$.

Finally, we have:

$= max_{\left(w_(t)^S, w_t^B, C(t)\right)} E_t\left[J(Y, I, r, t+h)\right]$

Now for the first part of equation (1), we have:

$= \left[E_t\left[\int_t^{t+h} exp(-\rho[s-t])\frac{1}{1-\gamma}c_t^{1-\gamma}ds\right]\right]$

The mean value theorem tells us that if f is a continuous function defined on $[a;b]$, $m \leq f(x) \leq M \forall x \in [a,b]$, then there exists a real number $a \leq \tilde{x} \leq b$ such that:

$$\int_a^b f(x)dx = f(\tilde{x})(b-a)$$

Using this theorem, we then have that:

$\int_t^{t+h} exp(-\rho[s-t])\frac{1}{1-\gamma}c_t^{1-\gamma}ds = \frac{1}{1-\gamma}c_{\bar{t}}^{1-\gamma}exp(-\rho[\bar{t}-t])h$ for some $\bar{t}$ st $t \leq \bar{t} \leq t+h$.

Finally for J, we find:

$J(Y,I,r,t) = \max_{\left(w(t)^S, w_t^B, C(t)\right)} E_t\left[exp(-\rho[\bar{t}-t])\frac{1}{1-\gamma}c_{\bar{t}}^{1-\gamma}h + exp(-\rho h)J(Y,I,r,t+h)\right]$

b) Taylor expansion

We then use a Taylor expansion to expand J(Y,I,r,t+h) around (Y,I,r,t):

$[J(Y,I,r,t+h) = J(Y,I,r,t)+J_t(Y,I,r,t)h+J_Y(Y,I,r,t)[Y(t+h)-Y(t)]+\frac{1}{2}J_{YY}(Y,I,r,t)[Y(t+h)-Y(t)]^2+J_I(Y,I,r,t)[I(t+h)-I(t)]+\frac{1}{2}J_{II}(Y,I,r,t)[I(t+h)-I(t)]^2+J_r(Y,I,r,t)[r(t+h)-r(t)]+\frac{1}{2}J_{rr}(Y,I,r,t)[r(t+h)-r(t)]^2+J_{YI}(Y,I,r,t)[Y(t+h)-Y(t)][I(t+h)-I(t)]+J_{Yr}(Y,I,r,t)[Y(t+h)-Y(t)][r(t+h)-r(t)]+J_{Ir}(Y,I,r,t)[I(t+h)-I(t)][r(t+h)-r(t)]$

where $J_{jk} = \frac{\partial^2 J}{\partial j \partial k}, \quad j,k = I,r,Y$
and $J_k = \frac{\partial J}{\partial k}, \quad k = I,r,Y$

At time t, $Y(t)$ is not random:

$E_t[J(Y,I,r,t+h)] = J(Y,I,r,t) + J_t h + J_Y E_t[\triangle\, Y(t)] + \frac{1}{2}J_{YY}E_t[\triangle\, Y(t)]^2 + J_I E_t[(\triangle\, I(t)] + \frac{1}{2}J_{II}E_t[\triangle\, I(t)]^2 + J_r E_t[\triangle\, r(t)] + \frac{1}{2}J_{rr}E_t[\triangle\, r(t)]^2 + J_{YI}E_t[\triangle\, Y(t) \triangle\, I(t)] + J_{Yr}E_t[\triangle\, Y(t) \triangle\, r(t)] + J_{Ir}E_t[\triangle\, I(t) \triangle\, r(t)]$

We now have two cases, one case after retirement and one case before retirement. Indeed, if we refer back to the dynamics of the income I_t, we can see that the process is different depending on the retirement of the individual.

c) HJB equation before retirement:

From the dynamics in the model setup, we get the expectations:

$E_t[\triangle\, Y(t)] = h(\mu_t^Y Y_t + I_t - c_t)$

$E_t[\triangle\, Y(t)]^2 = \left((w_t^S)^2\sigma_S^2 + (w_t^B)^2\sigma_B^2 + 2\rho_{12}\sigma_S\sigma_B w_t^S w_t^B\right) Y_t^2 h$

For simplicity, let us define the previous equality as $(\sigma_t^Y)^2 Y_t^2 h$

$E_t[(\triangle\, I(t)] = (\xi_0(t) + \xi_1 r_t) I_t h$

$E_t[\triangle\, I(t)]^2 = \sigma_I^2 I_t^2 h$

$E_t[\triangle\, r(t)] = k(\theta - r_t)h$

$E_t[\triangle\, r(t)]^2 = \sigma_r^2 h$

$E_t[\triangle\, Y(t) \triangle\, r(t)] = (-w^S\rho_{12}\sigma_S\sigma_r - w^B\sigma_B\sigma_r)Y_t h$

$E_t[\triangle\, Y(t) \triangle\, I(t)] = (\rho_{13}\sigma_S\sigma_I w^S Y I + \rho_{23}\sigma_B\sigma_I w^B Y I)h$

$E_t[\triangle\, I(t) \triangle\, r(t)] = (-\rho_{23}\sigma_r\sigma_I I)h$

where $h = dt$

So we have:

$\max_t E_t[exp(-\rho h) J(Y, I, r, t+h)] = \max_t E_t[exp(-\rho h)(J(Y, I, r, t) + J_t h + J_Y(\mu_t^Y Y + I - c)h + J_r k(\theta - r)h + J_I(\xi_0 + \xi_1 r)Ih + \frac{1}{2}J_{YY}\sigma_Y^2 Y^2 h + \frac{1}{2}J_{rr}\sigma_r^2 h + \frac{1}{2}J_{II}\sigma_I^2 I^2 h + J_{Yr}(-w^S\rho_{12}\sigma_S\sigma_r - w^B\sigma_B\sigma_r)Yh + J_{Ir}(-\rho_{23}\sigma_r\sigma_I I)h + J_{YI}(w^S\rho_{13}\sigma_S\sigma_I w^S Y I + \rho_{23}\sigma_B\sigma_I w^B Y I)h)]$

Now replacing J(Y,I,r,t+h) in the indirect utility function J(Y,I,r,t), we find for some $\bar{t} \in [t, t+h]$:

$J(Y, I, r, t) = \max_{c, w^S, w^B}[exp(-\rho(\bar{t} - t))\frac{1}{1-\gamma}c^{1-\gamma}h + exp(-\rho h)(J(Y, I, r, t) + J_t h + J_Y(\mu_t^Y Y + I - c)h + J_r k(\theta - r)h + J_I(\xi_0 + \xi_1 r)Ih + \frac{1}{2}J_{YY}\sigma_Y^2 Y^2 h + \frac{1}{2}J_{rr}\sigma_r^2 h + \frac{1}{2}J_{II}\sigma_I^2 I^2 h + J_{Yr}(-w^S\rho_{12}\sigma_S\sigma_r -$

$w^B\sigma_B\sigma_r)Yh + J_{Ir}(-\rho_{23}\sigma_r\sigma_I I)h + J_{YI}(w^S\rho_{13}\sigma_S\sigma_I w^S YI + \rho_{23}\sigma_B\sigma_I w^B YI)h)]$

We first subtract each side of the equation by $exp(-\rho h)J$ giving us:

$[1-exp(-\rho h)]J = \max_{c,w^S,w^B}[exp(-\rho(\bar{t}-t))\frac{1}{1-\gamma}c^{1-\gamma}h + exp(-\rho h)(J(Y,I,r,t) + J_t h + J_Y(\mu_t^Y Y + I - c)h + J_r k(\theta - r)h + J_I(\xi_0+\xi_1 r)Ih + \frac{1}{2}J_{YY}\sigma_Y^2 Y^2 h + \frac{1}{2}J_{rr}\sigma_r^2 h + \frac{1}{2}J_{II}\sigma_I^2 I^2 h + J_{Yr}(-w^S\rho_{12}\sigma_S\sigma_r - w^B\sigma_B\sigma_r)Yh + J_{Ir}(-\rho_{23}\sigma_r\sigma_I I)h + J_{YI}(w^S\rho_{13}\sigma_S\sigma_I w^S YI + \rho_{23}\sigma_B\sigma_I w^B YI)h) - exp(-\rho h)J]$

We have on the right side of the equation $exp(-\rho h)J(Y,I,r,t)$ and $-exp(-\rho h)J$ which cancel out giving:

$[1-exp(-\rho h)]J = \max_{c,w^S,w^B}[exp(-\rho(\bar{t}-t))\frac{1}{1-\gamma}c^{1-\gamma}h + exp(-\rho h)(J_t h + J_Y(\mu_t^Y Y + I - c)h + J_r k(\theta - r)h + J_I(\xi_0+\xi_1 r)Ih + \frac{1}{2}J_{YY}\sigma_Y^2 Y^2 h + \frac{1}{2}J_{rr}\sigma_r^2 h + \frac{1}{2}J_{II}\sigma_I^2 I^2 h + J_{Yr}(-w^S\rho_{12}\sigma_S\sigma_r - w^B\sigma_B\sigma_r)Yh + J_{Ir}(-\rho_{23}\sigma_r\sigma_I I)h + J_{YI}(w^S\rho_{13}\sigma_S\sigma_I w^S YI + \rho_{23}\sigma_B\sigma_I w^B YI)h)]$

Now, we divide each side by h and take the limit as $h \to 0$. We notice that $\lim_{h\to 0}\frac{1}{h}[1-exp(-\rho h)] = \rho$.

Finally, the HJB equation for $t < \tilde{T}$ (before retirement) is:

$\rho J = \max_{c,w^S,w^B}[\frac{1}{1-\gamma}c^{1-\gamma} + J_t + J_Y(\mu_t^Y Y + I - c) + J_r k(\theta - r) + J_I(\xi_0 + \xi_1 r)I + \frac{1}{2}J_{YY}\sigma_Y^2 Y^2 + \frac{1}{2}J_{rr}\sigma_r^2 + \frac{1}{2}J_{II}\sigma_I^2 I^2 + J_{Yr}(-w^S\rho_{12}\sigma_S\sigma_r - w^B\sigma_B\sigma_r)Y + J_{Ir}(-\rho_{23}\sigma_r\sigma_I I) + J_{YI}(w^S\rho_{13}\sigma_S\sigma_I w^S YI + \rho_{23}\sigma_B\sigma_I w^B YI)]$

with boundary condition:
$\lim_{t\nearrow\tilde{T}} J(Y,I,r,t) = \lim_{t\searrow\tilde{T}} J(Y,I,r,t)$ which gives us continuity of the indirect utility function at the retirement date $\tilde{T}$.

d) HJB equation after retirement:

After, retirement,i.e. for $\tilde{T} < t \leq T$, the dynamics of I_t is $0dt$, so the expectations $E_t[(\triangle\, I(t)]$,$E_t[(\triangle\, I(t)]^2$, $E_t[\triangle\, Y(t)\, \triangle\, I(t)]$ and $E_t[\triangle\, I(t)\, \triangle\, r(t)]$ are all equal to zero.

The HJB equation for $\tilde{T} < t \leq T$ is:

$\rho J = \max_{c,w^S,w^B}[\frac{1}{1-\gamma}c^{1-\gamma} + J_t + J_Y(\mu_t^Y Y + I - c) + J_r k(\theta - r) + \frac{1}{2}J_{YY}\sigma_Y^2 Y^2 + \frac{1}{2}J_{rr}\sigma_r^2 + J_{Yr}(-w^S\rho_{12}\sigma_S\sigma_r - w^B\sigma_B\sigma_r)Y]$

with terminal boundary condition:

$J(Y, I, r, t) = \frac{1}{1-\gamma} Y^{1-\gamma}$