Asset allocation with a stochastic wage rate

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The general problem

How to characterize $w_i^*(t)$, the optimal proportion invested in each risky asset and $C^*(t)$, the optimal consumption of the agent seeking to maximize his intertemporal expected utility of consumption and terminal wealth under her budget constraint and with a stochastic investment opportunity set. Here, this set is introduced by the stochastic wage rate that will affect asset prices and thus wealth.

Main assumptions

- Individuals continuously make investment and consumption choices based on a long term objective brought by their lifetime consumption and the value of the "bequest" they leave at their death
- The prices of risky assets follow Ito processes
- Transaction are done in continuous time and financial markets are opened without interruption

Model Setup: Definitions

- Let us call Y_t the wealth of the investor at time t. At each point in time, the investor allocates her wealth to consumption c, a fraction w^S to the stock index, a fraction w^B to bonds and the rest to the money market account
- Let us consider the utility function:

$$U(c_t) = \frac{1}{1-\gamma} c_t^{1-\gamma}$$

  which is an isoelastic utility function with constant relative risk aversion.
- The bequest function:

$$B(Y_T) = \frac{1}{1-\gamma} Y_T^{1-\gamma}$$

  where $\gamma > 1$ is the parameter of risk aversion. The intertemporal preference parameter is denoted by ρ.

Model Setup: Price dynamics

- Money market account dynamics:

$$\frac{dM_t}{M_t} = r_t dt$$

  where r_t is the short-term interest rate.

- Stock index dynamics:

$$\frac{dS_t}{S_t} = (r_t + \lambda_S)dt + \sigma_S dW_t^1$$

  where λ_S is the constant expected excess return from investing in stocks and W^1 is a standard Brownian motion

- Short-term interest rate dynamics(Ornstein-Uhlenbeck process)

$$dr_t = k(\theta - r_t)dt - \sigma_r dW_t^2$$

  where W^2 is a Brownian motion. k captures the degree of mean-reversion, θ is the long-run mean of the short-term interest rate

Model Setup: Price dynamics cont'

- Price of default-free bond dynamics:

$$\frac{dB_t}{B_t} = (r_t + \lambda_B)dt + \sigma_B dW_t^2$$

where $\lambda_B = \lambda_r D(r,t)$ the expected excess return on the bond. λ_r is the constant risk premium per unit of interest rate risk, and $\sigma_B = -\sigma_r D(r,t) > 0$ and D is the duration of the bond

- Wage dynamics:

$$dI_t = \begin{cases} (\xi_0(t) + \xi_1 r_t)I_t dt + \sigma_I I_t dW_t^3 & , fort \leq \tilde{T} \\ 0dt & , \tilde{T} < t \leq T \end{cases}$$

where $\xi_0(t)$ is a real-valued positive function and ξ_1 is the constant interest rate coefficient.

Correlation coefficients

Let us suppose that all three Brownian motions are correlated with constant correlation coefficient $\rho_{ij}, i,j = 1,2,3$.

Model Setup: Wealth Dynamics

- The investor's wealth evolve according to the following dynamics:

$$dY_t = \left(w_t^S \frac{dS_t}{S_t} + w_t^B \frac{dB_t}{B_t} + (1 - w_t^S - w_t^B) \frac{dM_t}{M_t} \right) Y_t + (I_t - c_t)dt$$

- By replacing, the dynamics $\frac{dS_t}{S_t}$, $\frac{dB_t}{B_t}$ and $\frac{dM_t}{M_t}$, we get:

$$= \left((r_t + w_t^S \lambda_S + w_t^B \lambda_B) Y_t + I_t - c_t \right) dt + w_t^S \sigma_S Y_t dW_t^1 + w_t^B \sigma_B dW_t^2$$

- Finally, we get

$$dY_t = (\mu_t^Y Y_t + I_t - c_t)dt + \sigma_t^Y Y_t dW_t^Y$$

Optimization problem

The investor is supposed to choose the portfolio $(w_t^S, w_t^B, 1 - w_t^S - w_t^B)$ and the instantaneous consumption $C(t)$ in order to maximize her expected intertemporal utility:

$$\max_{\left(w_(t)^S, w_t^B, C(t)\right)_{0 \le t \le T}} E_0 \left[\int_0^T exp(-\rho t) U[C(t)] dt + exp(-\rho T) B(Y(T), T) \right]$$

under the budget constraint:

$$dY_t = (\mu_t^Y Y_t + I_t - c_t) dt + \sigma_t^Y Y_t dW_t^Y$$

Optimality principle

To find the optimality conditions, we must transform the maximization into a dynamic programming problem to apply Bellman's optimality principle. A necessary condition for a program to be optimal from t to T implies that whatever the decisions between t and t+h at t+h, the program must be optimal between t+h and T as well.

Indirect Utility function

$$J(Y_t, I_t, r_t, t) = max_{(w_(t)^S, w_t^B, C(t))} E_t[\int_t^T exp(-\rho[s-t])\frac{1}{1-\gamma}c_t^{1-\gamma}ds +$$

$$exp(-\rho[T-t])\frac{1}{1-\gamma}Y_T^{1-\gamma}]$$

J is the indirect utility of the optimal program when the investor reaches time t. We are going to maximize all future decisions from t to T.

Derivation of the Hamilton-Jacobi-Bellman equation (1)

- Separation of the expected utility

$$J(Y_t, I_t, r_t, t) = max_{(w_(t)^S, w_t^B, C(t))} E_t[\int_t^{t+h} exp(-\rho[s-t])\frac{1}{1-\gamma}c_t^{1-\gamma}ds +$$

$$\int_{t+h}^{t} exp(-\rho[s-t-h+h])\frac{1}{1-\gamma}c_t^{1-\gamma}ds + exp(-\rho[T-t-h+h])\frac{1}{1-\gamma}Y_T^{1-\gamma}$$

$$= max_{(w_(t)^S, w_t^B, C(t))}\left[E_t\left[\int_t^{t+h} exp(-\rho[s-t])\frac{1}{1-\gamma}c_t^{1-\gamma}ds\right]\right] +$$

$$exp(-\rho h) max_{(w_(t)^S, w_t^B, C(t))}[E_t[\int_{t+h}^{t} exp(-\rho[s-t-h])\frac{1}{1-\gamma}c_t^{1-\gamma}ds$$

$$+exp(-\rho[T-t-h])\frac{1}{1-\gamma}Y_T^{1-\gamma}]]$$

(1)

Derivation of the Hamilton-Jacobi-Bellman equation (2)

Now, for the second part of equation (1):

$$= max_{(w_(t)^S, w_t^B, C(t))} E_t[E_t + h[\int_{t+h}^{t} exp(-\rho[s-t-h])\frac{1}{1-\gamma}c_t^{1-\gamma}ds +$$

$$exp(-\rho[T-t-h])\frac{1}{1-\gamma}Y_T^{1-\gamma}]]$$

$$= max_t E_t[max_{t+h} E_t + h[\int_{t+h}^{t} exp(-\rho[s-t-h])\frac{1}{1-\gamma}c_t^{1-\gamma}ds +$$

$$exp(-\rho[T-t-h])\frac{1}{1-\gamma}Y_T^{1-\gamma}]]$$

Finally, we have:

$$= max_{(w_(t)^S, w_t^B, C(t))} E_t\left[J(Y, I, r, t+h)\right]$$

Derivation of the Hamilton-Jacobi-Bellman equation (3)

- Mean-Value theorem
  Now for the first part of equation (1), we have:

$$= \left[E_t\left[\int_t^{t+h} exp(-\rho[s-t])\frac{1}{1-\gamma}c_t^{1-\gamma}ds\right]\right]$$

Using the mean value theorem, we have that:

$$\int_t^{t+h} exp(-\rho[s-t])\frac{1}{1-\gamma}c_t^{1-\gamma}ds = \frac{1}{1-\gamma}c_{\bar{t}}^{1-\gamma}exp(-\rho[\bar{t}-t])h$$

for some $\bar{t}$ st $t \leq \bar{t} \leq t+h$.
Finally for J, we find:

$$J(Y, I, r, t) = \max_{w_(t)^S, w_t^B, C(t)} E_t[exp(-\rho[\bar{t}-t])\frac{1}{1-\gamma}c_{\bar{t}}^{1-\gamma}h + exp(-\rho h)J(Y, I, r, t+h)]$$

Derivation of the Hamilton-Jacobi-Bellman equation (4)

- Taylor expansion

  We then use a Taylor expansion to expand $J(Y, I, r, t+h)$ around (Y, I, r, t) and take its expectation at time t:

$$E_t[J(Y, I, r, t+h)] = J(Y, I, r, t) + J_t h + J_Y E_t[\triangle\ Y(t)] + \frac{1}{2} J_{YY} E_t[\triangle\ Y(t)]^2 + J_I E_t[(\triangle\ I(t)] + \frac{1}{2} J_{II} E_t[\triangle\ I(t)]^2 + J_r E_t[\triangle\ r(t)] + \frac{1}{2} J_{rr} E_t[\triangle\ r(t)]^2 + J_{YI} E_t[\triangle\ Y(t) \triangle\ I(t)] + J_{Yr} E_t[\triangle\ Y(t) \triangle\ r(t)] + J_{Ir} E_t[\triangle\ I(t) \triangle\ r(t)]$$

  We now have two cases, one case after retirement and one case before retirement. Indeed, if we refer back to the dynamics of the income I_t, we can see that the process is different depending on the retirement of the individual.

- HJB equation before retirement:

  From the dynamics in the model setup, we get the expectations:

  $E_t[\triangle\ Y(t)] = h(\mu_t^Y Y_t + I_t - c_t)$
  $E_t[\triangle\ Y(t)]^2 = \left((w_t^S)^2\sigma_S^2 + (w_t^B)^2\sigma_B^2 + 2\rho_{12}\sigma_S\sigma_B w_t^S w_t^B\right) Y_t^2 h$
  defined as $(\sigma_t^Y)^2 Y_t^2 h$

  $E_t[(\triangle\ I(t)] = (\xi_0(t) + \xi_1 r_t) I_t h$
  $E_t[\triangle\ I(t)]^2 = \sigma_I^2 I_t^2 h$
  $E_t[\triangle\ r(t)] = k(\theta - r_t)h$
  $E_t[\triangle\ r(t)]^2 = \sigma_r^2 h$
  $E_t[\triangle\ Y(t)\ \triangle\ r(t)] = (-w^S\rho_{12}\sigma_S\sigma_r - w^B\sigma_B\sigma_r)Y_t h$
  $E_t[\triangle\ Y(t)\ \triangle\ I(t)] = (\rho_{13}\sigma_S\sigma_I w^S YI + \rho_{23}\sigma_B\sigma_I w^B YI)h$
  $E_t[\triangle\ I(t)\ \triangle\ r(t)] = (-\rho_{23}\sigma_r\sigma_I I)h$

  where $h = dt$

Derivation of the Hamilton-Jacobi-Bellman equation (6)

Now replacing J(Y,I,r,t+h) in the indirect utility function J(Y,I,r,t), we find for some $t \in [t, t+h]$:

$$J(Y,I,r,t) = \max_{c,w^S,w^B}[exp(-\rho(\bar{t}-t))\frac{1}{1-\gamma}c^{1-\gamma}h + exp(-\rho h)(J(Y,I,r,t)+$$

$$J_t h + J_Y(\mu_t^Y Y + I - c)h + J_r k(\theta - r)h + J_I(\xi_0 + \xi_1 r)Ih + \frac{1}{2}J_{YY}\sigma_Y^2 Y^2 h +$$

$$\frac{1}{2}J_{rr}\sigma_r^2 h + \frac{1}{2}J_{II}\sigma_I^2 I^2 h + J_{Yr}(-w^S\rho_{12}\sigma_S\sigma_r - w^B\sigma_B\sigma_r)Yh +$$

$$J_{Ir}(-\rho_{23}\sigma_r\sigma_I I)h + J_{YI}(w^S\rho_{13}\sigma_S\sigma_I w^S YI + \rho_{23}\sigma_B\sigma_I w^B YI)h)]$$

Derivation of the Hamilton-Jacobi-Bellman equation (7)

- HJB equation before retirement:

By subtracting each side of the equation by $exp(-\rho h)J$ and dividing each side by by h and taking the limit as $h \to 0$. We notice that $\lim_{h\to 0} \frac{1}{h}[1 - exp(-\rho h)] = \rho$ which gives us:

$$\rho J = \max_{c,w^S,w^B}[\frac{1}{1-\gamma}c^{1-\gamma} + J_t + J_Y(\mu_t^Y Y + I - c) + J_r k(\theta - r) + J_I(\xi_0 + \xi_1 r)I +$$

$$\frac{1}{2}J_{YY}\sigma_Y^2 Y^2 + \frac{1}{2}J_{rr}\sigma_r^2 + \frac{1}{2}J_{II}\sigma_I^2 I^2 + J_{Yr}(-w^S\rho_{12}\sigma_S\sigma_r - w^B\sigma_B\sigma_r)Y +$$

$$J_{Ir}(-\rho_{23}\sigma_r\sigma_I I) + J_{YI}(w^S\rho_{13}\sigma_S\sigma_I w^S YI + \rho_{23}\sigma_B\sigma_I w^B YI)]$$

with boundary condition:
$\lim_{t\nearrow\tilde{T}} J(Y,I,r,t) = \lim_{t\searrow\tilde{T}} J(Y,I,r,t)$ which gives us continuity of the indirect utility function at the retirement date $\tilde{T}$.

Derivation of the Hamilton-Jacobi-Bellman equation (8)

- HJB equation after retirement:

After, retirement,i.e. for $\tilde{T} < t \leq T$, the dynamics of I_t is $0dt$, so the expectations $E_t[(\triangle\ I(t)]$,$E_t[(\triangle\ I(t)]^2$, $E_t[\triangle\ Y(t)\ \triangle\ I(t)]$ and $E_t[\triangle\ I(t)\ \triangle\ r(t)]$ are all equal to zero.

The HJB equation for $\tilde{T} < t \leq T$ is:

$$\rho J = \max_{c,w^S,w^B}[\frac{1}{1-\gamma}c^{1-\gamma} + J_t + J_Y(\mu_t^Y Y + I - c) + J_r k(\theta - r) +$$
$$\frac{1}{2}J_{YY}\sigma_Y^2 Y^2 + \frac{1}{2}J_{rr}\sigma_r^2 + J_{Yr}(-w^S\rho_{12}\sigma_S\sigma_r - w^B\sigma_B\sigma_r)Y]$$

with terminal boundary condition:

$$J(Y, I, r, t) = \frac{1}{1-\gamma}Y^{1-\gamma}$$

Let us define:

$$\Phi(Y,c,w^S,w^B,t) = [\frac{1}{1-\gamma}c^{1-\gamma}+J_t+J_Y(\mu_t^Y Y+I-c)+J_r k(\theta-r)+\frac{1}{2}J_{YY}\sigma_Y^2$$

$$\frac{1}{2}J_{rr}\sigma_r^2 + J_{Yr}(-w^S\rho_{12}\sigma_S\sigma_r - w^B\sigma_B\sigma_r)Y - \rho J]$$

Then, the problem can be rewritten as follows:

$\max_{c,w^S,w^B} \Phi(Y,c,w^S,w^B,t) = 0$

The system to be solved is then:

$$\begin{cases} \Phi(Y,c^*,w_S^*,w_B^*,t) = 0 \\ \Phi_{w_S}(Y,c^*,w_S^*,w_B^*,t) = 0 \\ \Phi_{w_B}(Y,c^*,w_S^*,w_B^*,t) = 0 \\ \Phi_c(Y,c^*,w_S^*,w_B^*,t) = 0 \end{cases}$$

Explicit resolution in the case after retirement (2)

The first order conditions are:

$\Phi_c(Y, c, w^S, w^B, t) = 0$
$\Leftrightarrow U_c(C^*) = J_Y$

(envelope condition) $\Leftrightarrow c = J_Y^{-\frac{1}{\gamma}}$

$\Phi_{w_B}(Y, c, w^S, w^B, t) = 0$
$\Phi_{w_S}(Y, c, w^S, w^B, t) = 0$

Solving for w^B and w^S , we find:

$$w^B = \frac{\sigma_r J_{Yr} - \rho_{12}^2 \sigma_r J_{Yr}}{(1 + \rho_{12}^2)\sigma_B Y J_{YY}}$$

$$w^S = \frac{\rho_{12}\sigma_S\sigma_r Y J_{Yr} - \rho_{12}\sigma_S\sigma_B w^B Y^2 J_{YY}}{\sigma_S^2 Y^2 J_{YY}}$$

Explicit resolution in the case after retirement (3)

To properly solve the problem, it is necessary to guess an appropriate solution. Merton (1969) chose an isoelastic bequest function:

$$B(W,T) = \frac{\epsilon^{1-\gamma}[W(T)]^{\gamma}}{\gamma}$$

and solves the problem by guessing an appropriate solution:

$$\hat{J}(W,t) = \frac{b(t)}{\gamma}[W(t)]^{\gamma}$$

However, in this project, I was not able to determine J_{Y_r} which is why this is as far as I could go.

Limits of the model

- The hypothetical setup of the bond
- The deterministic time of death (can be modeled as a stochastic process)
- The bond does not contain any default risk
- The wealth can be negative (should have non-negativity restriction)
- Estimation risk (a lot of parameters need to be estimated)