Network science / Graph mining

Metrics for analyzing a connected world

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Outline

Outline

Graphs ?

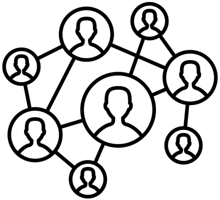


Figure: A graph: entities (nodes/vertices) and connections (edges)

An abstraction/representation for reasoning about characteristics of

- physical networks (computers, roads, circuits).
- relational data.

Focus on the structure rather than on the details of modeled objects

The omnipresence of graphs in applications

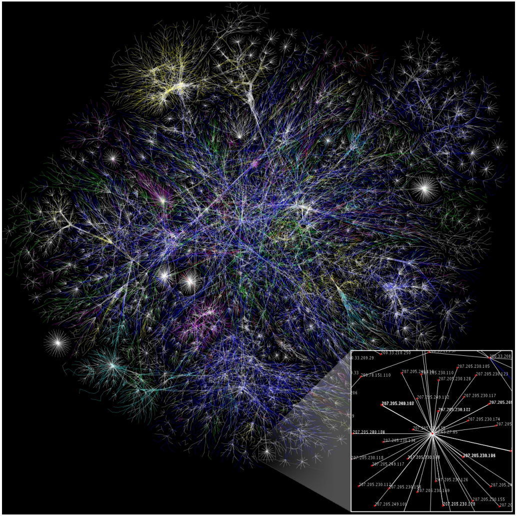


Figure: The Internet AS graph

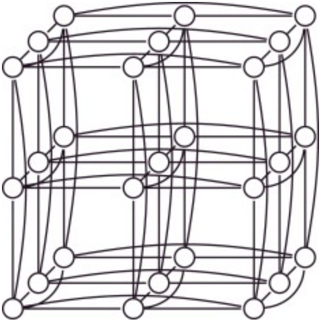


Figure: Interconnecting system-on-chips in a datacenter rack

The omnipresence of graphs in applications

- exemple use in social nets, epidemics...

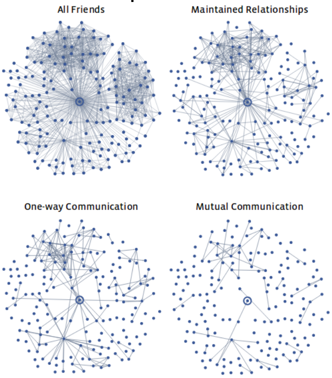


Figure 3.8: Four different views of a Facebook user's network neighborhood, showing the structure of links corresponding respectively to all declared friendships, maintained relationships, one-way communication, and reciprocal (i.e. mutual) communication. (Image from [286].)

Figure: From Networks, Crowds, and Markets: Reasoning about a Highly Connected World . By David Easley and Jon Kleinberg. Cambridge University Press, 2010.

e.g.: recommendations on YouTube

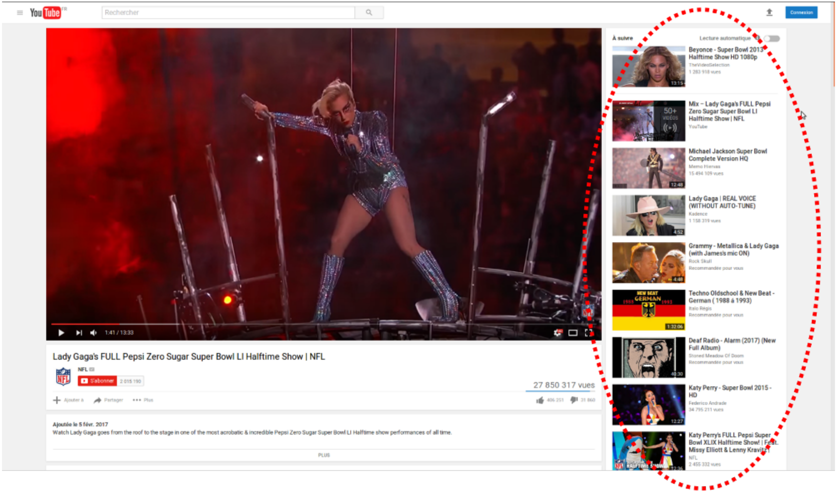


Figure: Recommandations: contextual, personalized?

e.g.: recommendations on YouTube

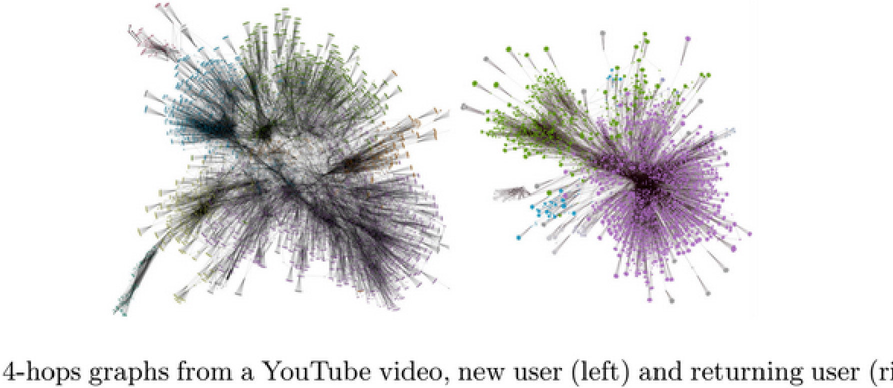
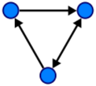
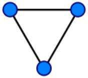


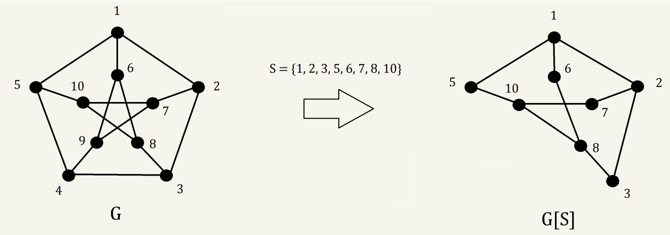
Figure: Blank profile vs. my recommandations

- *Directed* and *undirected* graphs:
  - in directed graphs, edges have orientation (arrow end)

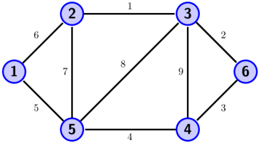




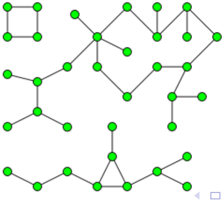
- A *subgraph* of G: formed by a subset of vertices and edges from G.



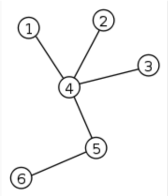
- Edge *weight*: value assigned as a label to an edge.
  - e.g., distance in km of a road from city 1 to 2.



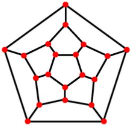
- Graph *connectivity:*
  - A graph is *connected* if there is a *path* btw any pair of vertices.
  - Otherwise, *connected components* are the subgraphs in which paths exist.



- A *cycle*: a path in which a vertex is reachable from itself.
  - Example of an *acyclic* connected graph: a *tree*



- A *planar* graph: can be draw without any edges crossing each other.



Special *topologies*

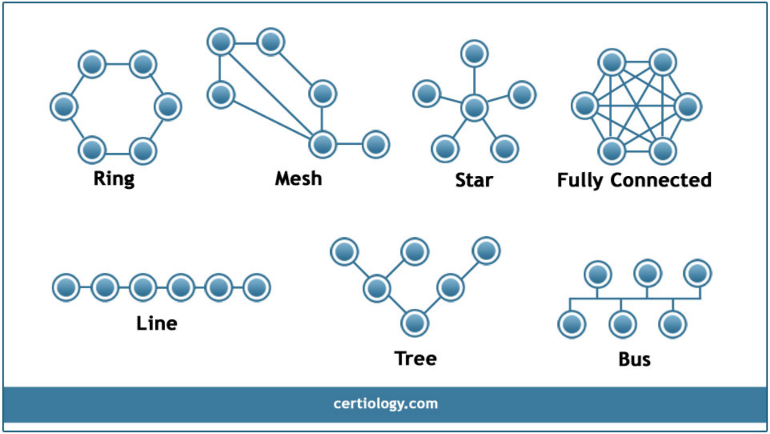
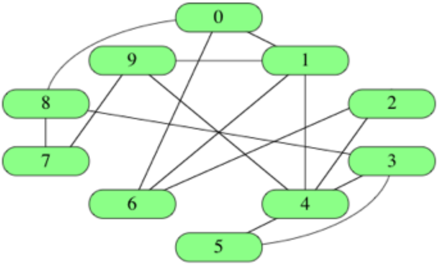


Figure: Graphs to remember, often used as illustrations

Adjacency list or edge list representations



Edge list:

[[0,1], [0,6], [0,8], [1,4], [1,6], [1,9], [2,4], [2,6], [3,4], [3,5], [3,8],[4,5], [4,9], [7,8], [7,9]]

$O(|V|)$ access time to find an edge, but $O(|E|)$ space in memory.

Adjacency list:

[[1, 6, 8], [0, 4, 6, 9], [4, 6], [4, 5, 8], [1, 2, 3, 5, 9], [3, 4], [0, 1, 2], [8, 9], [0, 3, 7], [1, 4, 7]]

$O(1)$ access time to vertex , but $O(|V|)$ to access a given edge.[1]

[1] https://www.khanacademy.org/computing/computer-science/algorithms/graph-representation/a/representing-graphs

Matrix representation

	0	1	2	3	4	5	6	7	8	9
0	0	1	0	0	0	0	1	0	1	0
1	1	0	0	0	1	0	1	0	0	1
2	0	0	0	0	1	0	1	0	0	0
3	0	0	0	0	1	1	0	0	1	0
4	0	1	1	1	0	1	0	0	0	1
5	0	0	0	1	1	0	0	0	0	0
6	1	1	1	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0	1	1
8	1	0	0	1	0	0	0	1	0	0
9	0	1	0	0	1	0	0	1	0	0

Figure: Matrix representation of previous graph

Find edge presence in $O(1)$ time, but $\Theta(V^2)$ space in memory.
1's to be replaced by edge weigths for weighted graphs.

Example tool families for manipulating graphs

GraphX

Figure: For massive graphs (cannot fit into on server's memory)

X − Stream

Figure: Big graph processing on a single machine

neo4j

Figure: For a database-like handling of graphs

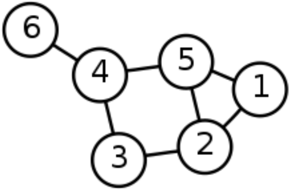
NetworkX

Figure: Prototyping in Python, lots of contributions

Stanford Large Network Dataset Collection

Outline

Basic notations



- $G(V,E)$: graph G with node set V, connected by edge set E.
  - $V = \{1,2,3,4,5,6\}$;
    $E = [[1,5],[1,2],[2,3],[2,5],[3,4],[4,5],[4,6]]$
- Number of nodes is $n = |V|$, edges is $m = |E|$.
- Neighbors of node i are set $\Gamma(i)$.
  - $\Gamma(1) = \{2,5\}$

Degree of a node

- The degree d_v of node v is equal to $|\Gamma(v)|$ (its number of neighbors).
- Degree span: $0 \leq d_v \leq n-1$ (if no self loops).
- *Degree distribution* P(d) is the probability distribution of each degree in the current graph:

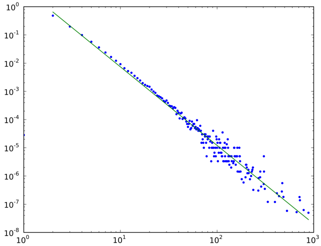


Figure: Degree distribution: x-axis is degree, y-axis is probability

- In(out)-degree of v: counts incoming(outgoing) edges only.

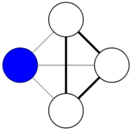
Clustering coefficient

- Every two nodes in a *clique* are neighbors.
- *Local clustering coefficient of a node i measures "how close are* $\Gamma(i)$ *from being a clique":*

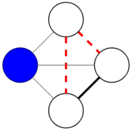
$$C_i = \frac{2|e_{jk} : v_j, v_k \in \Gamma(v_i), e_{jk} \in E|}{d_i(d_i - 1)}$$

- Average clustering coefficient:

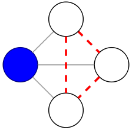
$$\bar{C} = \frac{1}{n}\sum_{i=1}^{n} C_i$$



c = 1



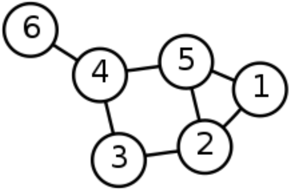
c = 1/3



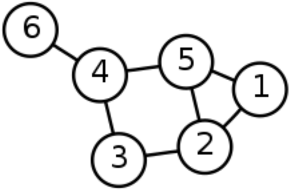
c = 0

Path lengths

- *Path*: sequence of adjacent nodes connecting two nodes (if exists).
  - e.g., two paths btw 6 and 1: (4,5,1) and (4,3,2,5,1).
  - One *hop*: one transition from a node to another.
- *Shortest path*: path of minimal cardinality.
  - Distance $dist(6,1) = |(4,5,1)| = 3$
- *Single-source shortest path* (SSSP): shortest paths from node i to all other nodes ($V \setminus i$).
- *All-pairs shortest paths (APSP)*: SSSP from $\forall i \in V$.



Diameter



- *Average path length*: average of all-pair shortest distances in the graph.
- *Diameter*: longest path of the APSP, i.e., greatest distance between any pair of vertices.
  - $diam(G) = |(4,5,1)| = 3$, starting at node 6.

Spectral analysis

The Laplacian matrix $L_G = D - A$*:*

- D *is the degree matrix a diagonal-matrix with* $D(i,i)$ *is the degree of the* i*th node in* G
- A *is the adjacency matrix, with* $A(i,j) = 1$ *if and only if* $(i,j) \in E$

$$L_G(i,j) = \begin{cases} deg(i) & \text{if } i = j \\ -1 & \text{if } (i,j) \in E \equiv 1 \\ 0 & \text{otherwise} \end{cases}$$

Labelled graph	Degree matrix	Adjacency matrix	Laplacian matrix
	$\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 2 & -1 & 0 & 0 & -1 & 0 \\ -1 & 3 & -1 & 0 & -1 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 3 & -1 & -1 \\ -1 & -1 & 0 & -1 & 3 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{pmatrix}$

Spectral analysis (2)

*For an (undirected) graph G and its **Laplacian matrix** $L = D - A$ with eigenvalues $\lambda_0 \leq \lambda_1 \leq ... \leq \lambda_{n-1}$:*

- $\lambda_0 = 0$, *as* $v_0 = (1, 1, ..., 1)$ *satisfies* $Lv_0 = 0$ *(row sum and column sum of* L *are* 0*)*
- *# of connected components in G is the algebraic multiplicity of the 0 eigenvalue (* $\Longrightarrow \lambda_2 = 0$ *iff G is disconnected)*
- *the smallest non-zero eigenvalue of L is called the **spectral gap***
- *the second smallest eigenvalue of L (could be zero) is the **algebraic connectivity** of G*
- ...

An example of a result: the diameter of a non complete graph G satisfies:

$$diam(G) \leq \lceil \frac{\log(vol(G)/\delta)}{\log \frac{\lambda_{n-1}+\lambda_1}{\lambda_{n-1}-\lambda_1}} \rceil,$$

with δ the minimum degree of G.

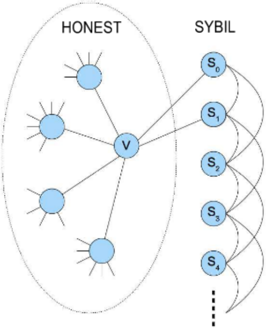
...and multiple results from graph theory, in general or for specific graphs

Conductance

- The conductance $\Phi(C)$ of a set C of vertices in a given graph G is the ratio between the number of edges going out from C and the number of edges inside C:

$$\Phi(C) = \frac{|cut(C)|}{vol(C)},$$

where $vol(C)$ is the sum of the degrees of the vertices in C.

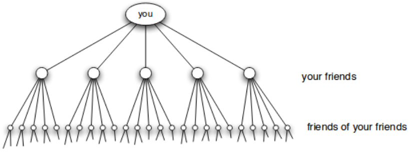


Expansion

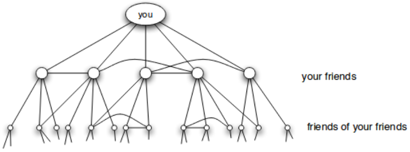
- *Expansion* of G: mean number of nodes that are reached in h hops from all nodes:

$$e_G(h) = \frac{1}{n^2} \sum_{v \in V} |C_v(h)|,$$

with $C_v(h)$ the set of reachable nodes from v in h hops.



(a) *Pure exponential growth produces a small world*



(b) *Triadic closure reduces the growth rate*

Resilience

- Measures the robustness of a graph:

$$r_G(h) = \frac{1}{|E|} \sum_{v \in V} I(v, |C_v(h)|),$$

with $I(v, |C_v(h)|)$ the number of edges that need to be removed to split $C_v(h)$ into 2 sets (of roughly the same size). h: distance (hops).