

# Asymptotic Study of Ensemble Methods for Imbalanced Classification

M. Mayala, **E. Scornet**, C. Tillier, O. Wintenberger

Probability Statistics seminar, LMV, February 2026



1. Random Forests construction
2. U-statistics and link with RF
3. Asymptotic analysis of Infinite Centered RF
4. Numerical experiments

1. Random Forests construction
2. U-statistics and link with RF
3. Asymptotic analysis of Infinite Centered RF
4. Numerical experiments

Consider  $Z_i := (X_i, Y_i)$  i.i.d. copies of the pair  $(X, Y)$

- ▶ **Input variable**  $X \in \mathcal{X} = [0, 1]^d$
- ▶ **Output variable**  $Y \in \{0, 1\}$ .
- ▶ **Regression function:**  $\mu(x) = \mathbb{P}(Y = 1|X = x)$ .

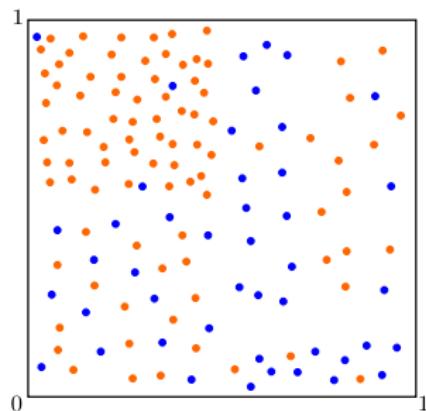
**Goal:** estimation of  $\mu$  using **random forests**.



- ▶ Non-parametric method
- ▶ Based on [bagging](#) and [random feature selections](#)
- ▶ Aggregate the predictions of  $M$  trees

# Construction of Decision trees - classification

5 / 40

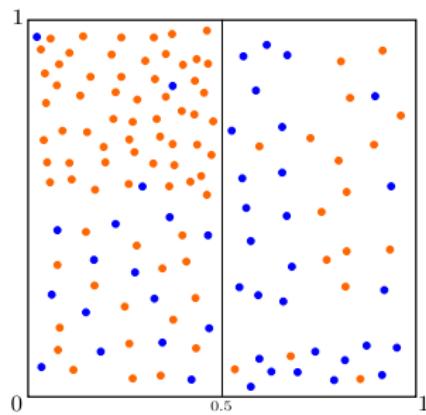


$$k = 0$$



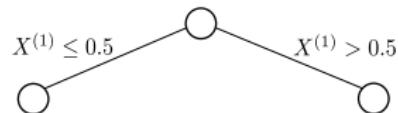
# Construction of Decision trees - classification

5 / 40



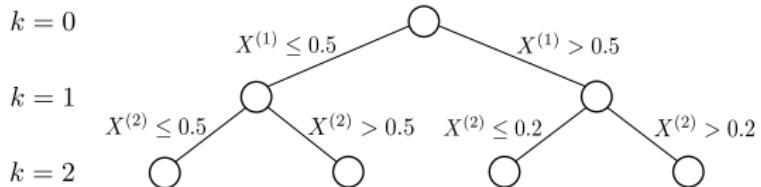
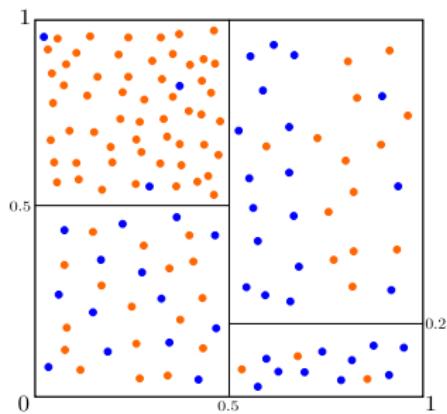
$$k = 0$$

$$k = 1$$



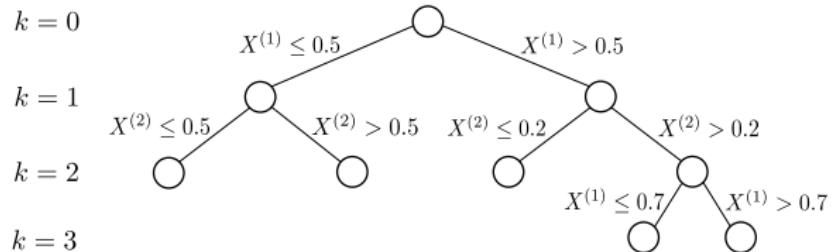
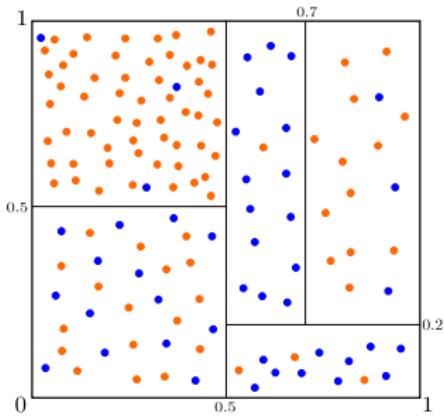
# Construction of Decision trees - classification

5 / 40



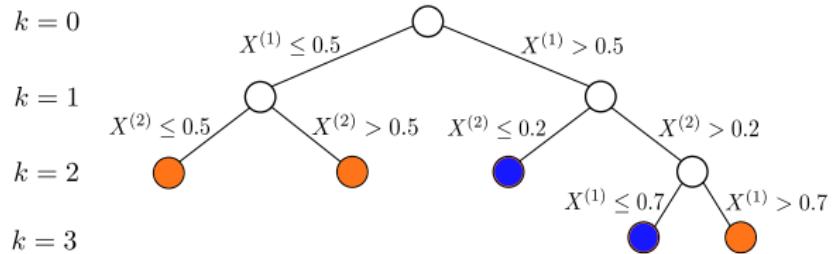
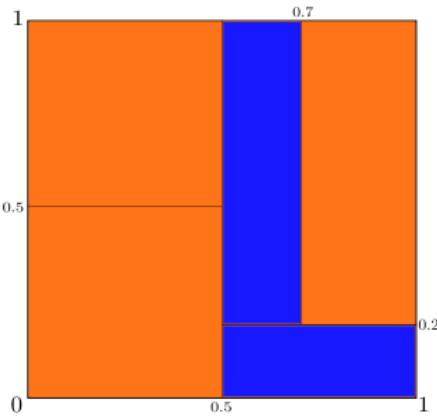
# Construction of Decision trees - classification

5 / 40



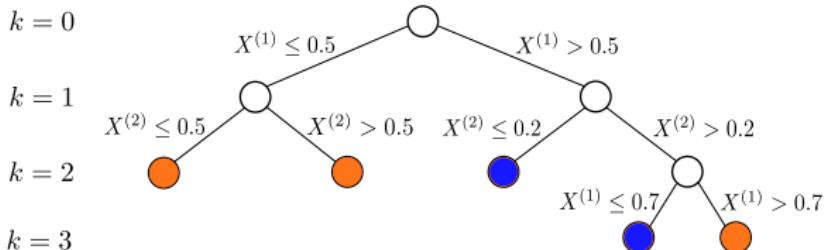
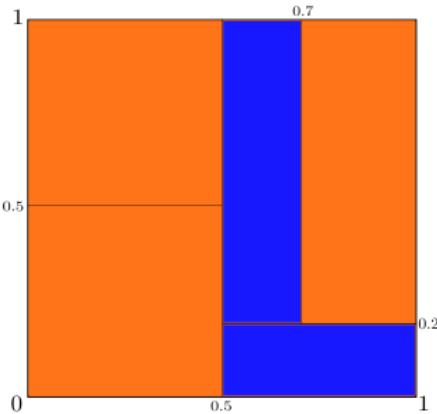
# Construction of Decision trees - classification

5 / 40



# Construction of Decision trees - classification

5 / 40



Decrease in impurity for a split  $(j, s)$ :

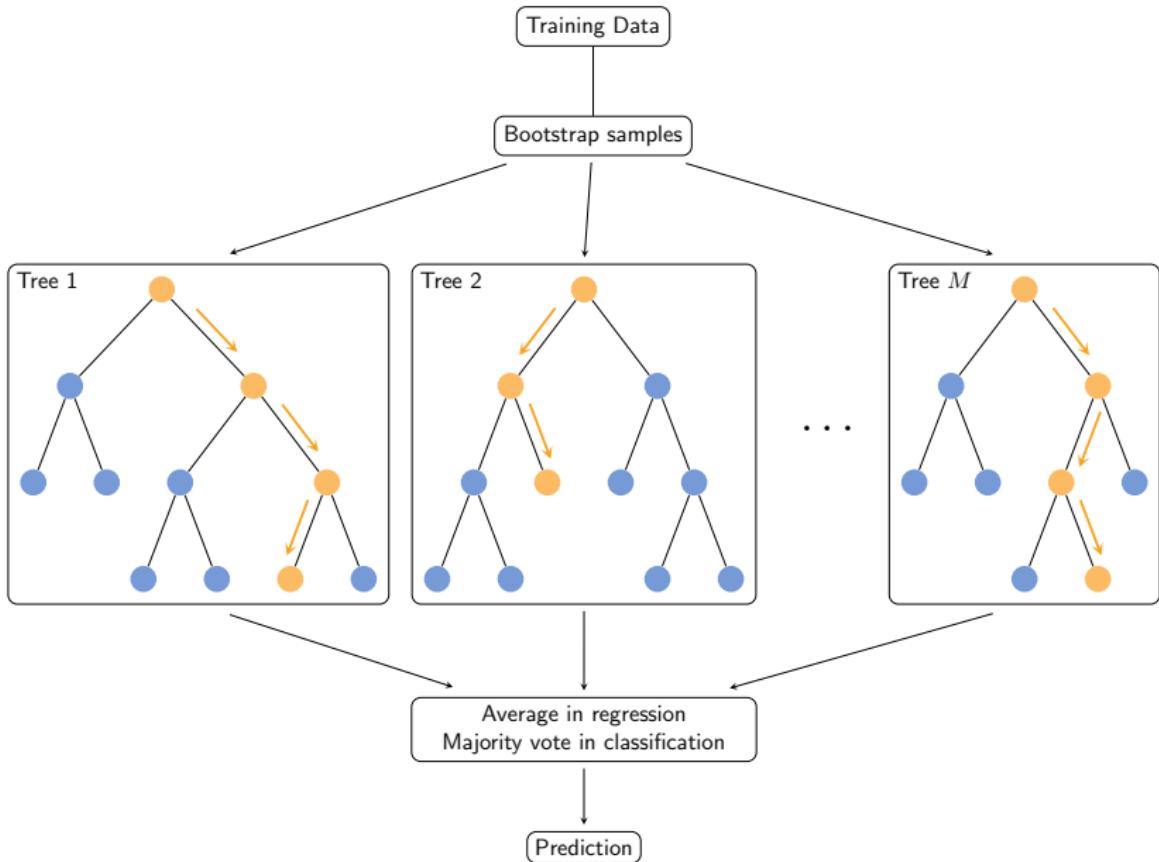
$$\Delta Imp(j, s; A) = Imp(A) - p_L Imp(A_L) - p_R Imp(A_R),$$

where  $p_L$  (resp.  $p_R$ ) is the fraction of observations in  $A$  that fall into  $A_L$  (resp.  $A_R$ ). The best split  $(j^*, s^*)$  is then chosen as

$$(j^*, s^*) \in \operatorname{argmax}_{j,s} \Delta Imp(j, s; A).$$

# A classical random forest

6 / 40



Let  $\mathbf{Z}_S := (Z_{i_1}, \dots, Z_{i_s})$  be a training subsample of size  $s$ .

- ▶ The tree prediction at a point  $x \in \mathcal{X}$  is

$$T^s(x; \Theta; \mathbf{Z}_S) := \frac{\sum_{i \in S} Y_i \mathbb{1}\{X_i \in L_\Theta(x)\}}{N_{L_\Theta(x)}(\mathbf{Z}_S)},$$

- ▶  $L_\Theta(x)$  is the leaf of the tree built with randomness  $\Theta$
- ▶  $N_{L_\Theta(x)}(\mathbf{Z}_S) := \sum_{i \in S} \mathbb{1}\{X_i \in L_\Theta(x)\}$ , number of observations
- ▶ Finite forest with  $M$  trees

$$\hat{\mu}_{M,s}(x; \mathbf{Z}_n) := \frac{1}{M} \sum_{m=1}^M T^s(x; \Theta_m; \mathbf{Z}_{S_m}).$$

- ▶ Infinite forest

$$\hat{\mu}_{M,s}(x; \mathbf{Z}_n) \xrightarrow{M \rightarrow \infty} \underbrace{\mathbb{E}[T^s(x; \Theta; \mathbf{Z}_S) | \mathbf{Z}_n]}_{\hat{\mu}^s(x)}$$

- ▶ **Simplified RF versions**, whose construction is **independent of the dataset**.

(Biau et al.; Biau, 2012; Genuer, 2012; Arlot and Genuer, 2014; Scornet, 2016; Mourtada et al., 2020; Klusowski, 2021)

- ▶ **Simplified RF versions**, whose construction is **independent of the dataset**.  
(Biau et al.; Biau, 2012; Genuer, 2012; Arlot and Genuer, 2014; Scornet, 2016; Mourtada et al., 2020; Klusowski, 2021)
- ▶ Analysis of more data-dependent forests:
  - ▶ **Asymptotic normality** of random forests (Mentch and Hooker, 2016; Wager and Athey, 2018),
  - ▶ **Variable importance** (Louppe et al., 2013; Li et al., 2019; Scornet, 2023),
  - ▶ **(Rate of) consistency** (Scornet et al., 2015; Wager and Walther, 2015; Klusowski and Tian, 2024).

- ▶ **Simplified RF versions**, whose construction is **independent of the dataset**.  
(Biau et al.; Biau, 2012; Genuer, 2012; Arlot and Genuer, 2014; Scornet, 2016; Mourtada et al., 2020; Klusowski, 2021)
- ▶ Analysis of more data-dependent forests:
  - ▶ **Asymptotic normality** of random forests (Mentch and Hooker, 2016; Wager and Athey, 2018),
  - ▶ **Variable importance** (Louppe et al., 2013; Li et al., 2019; Scornet, 2023),
  - ▶ **(Rate of) consistency** (Scornet et al., 2015; Wager and Walther, 2015; Klusowski and Tian, 2024).
- ▶ Literature review on random forests:
  - ▶ **Methodological review** (Criminisi et al., 2011; Boulesteix et al., 2012),
  - ▶ **Theoretical review** (Biau and Scornet, 2016; Scornet and Hooker, 2025)

1. Random Forests construction
2. U-statistics and link with RF
3. Asymptotic analysis of Infinite Centered RF
4. Numerical experiments

## Subsampling IRF

$$\hat{\mu}^s(x) := \binom{n}{s}^{-1} \sum_{S \subset \{1, \dots, n\}, |S|=s} \mathbb{E}[T^s(x; \Theta; Z_S) \mid Z_S].$$

## Subsampling IRF

$$\hat{\mu}^s(x) := \binom{n}{s}^{-1} \sum_{S \subset \{1, \dots, n\}, |S|=s} \mathbb{E}[T^s(x; \Theta; Z_S) | Z_S].$$

Connection to U-statistics<sup>1</sup>

Assume we are given i.i.d.  $Z_1, \dots, Z_n$  and we want to estimate  $\mu = \mathbb{E}[h(Z_1, \dots, Z_s)]$ . Then the unbiased estimator with minimal variance is the U-statistics defined as

$$U_n = \frac{1}{\binom{n}{s}} \sum_i h(Z_{i_1}, \dots, Z_{i_s}).$$

---

<sup>1</sup>(Lee, 2019)

$$\widehat{\mu}^s(x) = \underbrace{\mathbb{E}[T^s(x; \Theta; Z_S)] + \frac{s}{n} \sum_{i=1}^n T_1^s(x; Z_i)}_{\text{H\u{a}jek projection}} + \sum_{r=2}^s \binom{s}{r} \widehat{\mu}_{n,r}^s(x).$$

- ▶ First two terms: **H\u{a}jek projection**<sup>2</sup>  $\stackrel{\circ}{\widehat{\mu}}(x)$
- ▶  $T_1^s = \mathbb{E}[T^s(x; \Theta; Z_S) | Z_1]$
- ▶  $\widehat{\mu}_{n,r}^s(x)$  **uncorrelated** terms

---

<sup>2</sup>der Vaart (2000)

<sup>3</sup><https://www.stat.berkeley.edu/~pitman/s205f02/lecture10.pdf>

$$\widehat{\mu}^s(x) = \underbrace{\mathbb{E}[T^s(x; \Theta; Z_S)] + \frac{s}{n} \sum_{i=1}^n T_1^s(x; Z_i)}_{\text{H\u00e1jek projection}} + \sum_{r=2}^s \binom{s}{r} \widehat{\mu}_{n,r}^s(x).$$

- ▶ First two terms: **H\u00e1jek projection**<sup>2</sup>  $\stackrel{\circ}{\widehat{\mu}}(x)$
- ▶  $T_1^s = \mathbb{E}[T^s(x; \Theta; Z_S) | Z_1]$
- ▶  $\widehat{\mu}_{n,r}^s(x)$  **uncorrelated** terms

## Sketch of proof

- ▶ Prove that  $\text{Var}[\widehat{\mu}^s(x)] \sim \text{Var}[\stackrel{\circ}{\widehat{\mu}}(x)]$ .
- ▶ CLT for  $\stackrel{\circ}{\widehat{\mu}}(x) \Rightarrow$  CLT for  $\widehat{\mu}^s(x)$
- ▶ Use Lindeberg condition for triangular arrays to obtain a CLT on  $\stackrel{\circ}{\widehat{\mu}}(x)$ <sup>3</sup>

---

<sup>2</sup>der Vaart (2000)

<sup>3</sup><https://www.stat.berkeley.edu/~pitman/s205f02/lecture10.pdf>

**(H1)** The individual trees  $T^s(x; \Theta; Z_S)$  are bounded a.s. and satisfy  $nV_1^s \rightarrow \infty$ , as  $n \rightarrow \infty$ , where

$$V_1^s = \text{Var}[\mathbb{E}[T^s(x; \Theta; Z_S) | Z_1]].$$

Theorem 1 (Mayala, Wintenberger, Tillier and Dombry '24)

Let  $\hat{\mu}^s(x)$  be the subsampling IRF at point  $x \in \mathcal{X}$ . Under **(H1)**,

$$\sqrt{\frac{n}{s^2 V_1^s}} (\hat{\mu}^s(x) - \mathbb{E}[\hat{\mu}^s(x)]) \xrightarrow[n \rightarrow \infty]{d} \mathcal{N}(0, 1).$$

- (H1)** The individual trees  $T^s(x; \Theta; Z_S)$  are bounded a.s. and satisfy  $nV_1^s \rightarrow \infty$ , as  $n \rightarrow \infty$ , where

$$V_1^s = \text{Var}[\mathbb{E}[T^s(x; \Theta; Z_S) | Z_1]].$$

- (H2)**

$$\sqrt{\frac{n}{s^2 V_1^s}} |\mathbb{E}[\hat{\mu}^s(x)] - \mu(x)| \rightarrow 0.$$

Theorem 1 (Mayala, Wintenberger, Tillier and Dombry '24)

Let  $\hat{\mu}^s(x)$  be the subsampling IRF at point  $x \in \mathcal{X}$ . Under **(H1)** and **(H2)**,

$$\sqrt{\frac{n}{s^2 V_1^s}} (\hat{\mu}^s(x) - \mu(x)) \xrightarrow[n \rightarrow \infty]{d} \mathcal{N}(0, 1).$$

First CLT on random forests by [Mentch and Hooker \(2016\)](#) with

- ▶ Different asymptotics in the number of trees
- ▶ Valid for any subsample size  $s = o(\sqrt{n})$
- ▶ But require  $\lim_{n,s \rightarrow \infty} V_1^s \neq 0$ , where

$$V_1^s = \text{Var}[\mathbb{E}[T^s(x; \Theta; Z_S) | Z_1]].$$

- ▶ Not true for fully-grown trees used in RF

Extension to subsampling with replacement (and variance estimation) by [Zhou et al. \(2021\)](#).

[Wager and Athey \(2018\)](#) establish a CLT for trees such that :

- ▶ (honest tree) dataset is split in two parts (building the tree/estimating the mean in each leaf)
- ▶ ( $\alpha$ -regular) leaves at least a fraction  $\alpha$  of samples in each child
- ▶ (split randomization) positive probability of splitting each variable

For such trees, [Wager and Athey \(2018\)](#) establish a CLT

- ▶ Valid for any subsample size  $s \simeq n^\beta$ , for all  $\beta \in (\eta, 1)$  where  $\eta$  depends on the tree structure
- ▶ Centered at the true value of the regression function  $\mu(x)$ .
- ▶ But with non-explicit convergence rates and variance

[Peng et al. \(2022\)](#) established CLT for various random forests, whose trees are built independently of the labels of the data.

All results are built on Hájek projections.

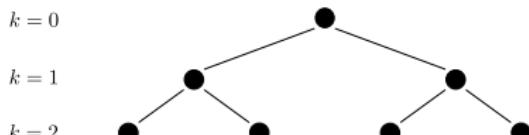
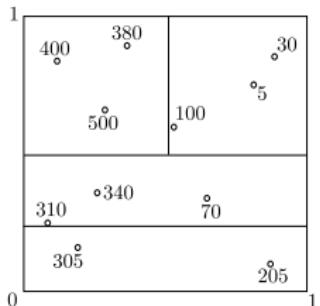
1. Random Forests construction
2. U-statistics and link with RF
3. Asymptotic analysis of Infinite Centered RF
4. Numerical experiments

# Centered Random Forests (CRF)

15 / 40

Construction of a centered tree (Breiman, 2004; Biau, 2012):

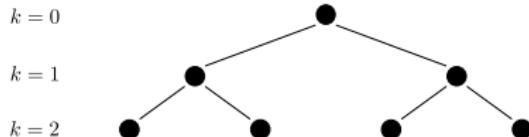
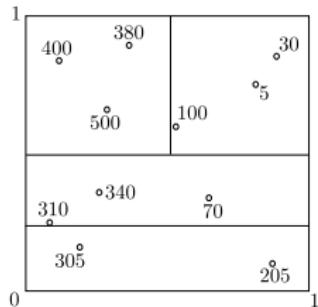
- ▶ Select  $s$  among  $n$  observations without replacement,
- ▶ Start at the root  $[0, 1]^d$ , and at each node,
  1. a feature is uniformly chosen among all possible  $d$  features
  2. Along this feature, split is made at the center of the cell
- ▶ Stop when each cell has been split exactly  $k$  times.



If the new point  $x$  falls into an empty cell, the tree arbitrarily predicts 0.

# Infinite Centered Random Forests (ICRF)

16 / 40



Let  $\mathbf{Z}_S := (Z_{i_1}, \dots, Z_{i_s})$  be a training subsample of size  $s$ . The centered tree prediction at  $x \in [0, 1]^d$  is

$$T^s(x; \Theta; \mathbf{Z}_S) := \frac{\sum_{i \in S} Y_i \mathbb{1}\{X_i \in L_\Theta(x)\}}{N_{L_\Theta(x)}(\mathbf{Z}_S)}.$$

## Infinite CRF

$$\hat{\mu}_s^{\text{ICRF}}(x) := \binom{n}{s}^{-1} \sum_{S \subset \{1, \dots, n\}, |S|=s} \mathbb{E}[T^s(x; \Theta; \mathbf{Z}_S) \mid \mathbf{Z}_S].$$

## Infinite CRF

$$\hat{\mu}_s^{\text{ICRF}}(x) := \binom{n}{s}^{-1} \sum_{S \subset \{1, \dots, n\}, |S|=s} \mathbb{E}[T^s(x; \Theta; Z_S) \mid Z_S].$$

- (H0) **Covariate Condition:**  $X$  is uniformly distributed on  $\mathcal{X}$ .
- (H1) **Smoothness Condition:** The regression function  $\mu$  is a  $L$ -lipschitz function with respect to the max norm.
- (G1) **Tree complexity Condition:** The subsample size  $s$  and the tree depth  $k$  tend to infinity and satisfy  $s/(k2^k) \rightarrow \infty$ , as  $n \rightarrow \infty$ .

### Theorem (Mayala, Scornet, Tillier and Wintenberger 2025)

Let  $d \geq 2$  and  $\hat{\mu}_s^{\text{ICRF}}(x)$  be the ICRF estimator at point  $x$ . Assume **(H0)**, **(H1)** and **(G1)** hold and

$$\frac{n2^k}{s^2 k^{(d-1)/2}} \rightarrow \infty, \quad (1)$$

as  $n, s, k \rightarrow \infty$ . Then, for all  $x \in [0, 1]^d$ , we have

$$\sqrt{\frac{nk^{(d-1)/2}}{2^k}} \left( \hat{\mu}_s^{\text{ICRF}}(x) - \mathbb{E}[\hat{\mu}_s^{\text{ICRF}}(x)] \right) \xrightarrow[n \rightarrow \infty]{d} \mathcal{N}(0, C(d)\mu(x)(1 - \mu(x))).$$

## Theorem (Mayala, Scornet, Tillier and Wintenberger 2025)

Let  $d \geq 2$  and  $\hat{\mu}_s^{\text{ICRF}}(x)$  be the ICRF estimator at point  $x$ . Assume **(H0)**, **(H1)** and **(G1)** hold and

$$\frac{n2^k}{s^2 k^{(d-1)/2}} \rightarrow \infty,$$

as  $n, s, k \rightarrow \infty$ . Then, for all  $x \in [0, 1]^d$ , we have

$$\sqrt{\frac{nk^{(d-1)/2}}{2^k}} \left( \hat{\mu}_s^{\text{ICRF}}(x) - \mathbb{E}[\hat{\mu}_s^{\text{ICRF}}(x)] \right) \xrightarrow[n \rightarrow \infty]{d} \mathcal{N}(0, C(d)\mu(x)(1 - \mu(x))).$$

- ▶ First CLT for random forests, with an explicit convergence rate, asymptotic variance, and condition on tree complexity.

### Theorem (Mayala, Scornet, Tillier and Wintenberger 2025)

Let  $d \geq 2$  and  $\hat{\mu}_s^{\text{ICRF}}(x)$  be the ICRF estimator at point  $x$ . Assume **(H0)**, **(H1)** and **(G1)** hold and

$$\frac{n2^k}{s^2 k^{(d-1)/2}} \rightarrow \infty,$$

as  $n, s, k \rightarrow \infty$ . Then, for all  $x \in [0, 1]^d$ , we have

$$\sqrt{\frac{nk^{(d-1)/2}}{2^k}} \left( \hat{\mu}_s^{\text{ICRF}}(x) - \mathbb{E}[\hat{\mu}_s^{\text{ICRF}}(x)] \right) \xrightarrow[n \rightarrow \infty]{d} \mathcal{N}(0, C(d)\mu(x)(1 - \mu(x))).$$

- ▶ (1) imposes that  $s = o(n)$ , since  $2^k \leq s$  to avoid empty cells.  
Usual condition in the literature (see e.g. [Wager and Athey, 2018](#); [Peng et al., 2022](#))
- ▶ This result holds for trees close to bootstrapped ( $s = n$ ) fully grown trees ( $k = \log_2(s)$ ) but rate of convergence is very slow for such trees

## Theorem (Mayala, Scornet, Tillier and Wintenberger 2025)

Let  $d \geq 2$  and  $\hat{\mu}_s^{\text{ICRF}}(x)$  be the ICRF estimator at point  $x$ . Assume **(H0)**, **(H1)** and **(G1)** hold and

$$\frac{n2^k}{s^2 k^{(d-1)/2}} \rightarrow \infty,$$

as  $n, s, k \rightarrow \infty$ . Then, for all  $x \in [0, 1]^d$ , we have

$$\sqrt{\frac{nk^{(d-1)/2}}{2^k}} \left( \hat{\mu}_s^{\text{ICRF}}(x) - \mathbb{E}[\hat{\mu}_s^{\text{ICRF}}(x)] \right) \xrightarrow[n \rightarrow \infty]{d} \mathcal{N}(0, C(d)\mu(x)(1 - \mu(x))).$$

- ▶ Beware! The CLT is not centered at the true value of the regression function  $\mu(x)$

### Theorem (Mayala, Scornet, Tillier and Wintenberger 2025)

Let  $d \geq 2$  and  $\hat{\mu}_s^{\text{ICRF}}(x)$  be the ICRF estimator at point  $x$ . Assume **(H0)**, **(H1)** and **(G1)** hold and

$$\frac{n2^k}{s^2 k^{(d-1)/2}} \rightarrow \infty, \quad \text{and} \quad 2^k k^{-\frac{d-1}{2}} n^{-\frac{d \log 2}{1+d \log 2}} \rightarrow \infty,$$

as  $n, s, k \rightarrow \infty$ . Then, for all  $x \in [0, 1]^d$ , we have

$$\sqrt{\frac{nk^{(d-1)/2}}{2^k}} \left( \hat{\mu}_s^{\text{ICRF}}(x) - \mu(x) \right) \xrightarrow[n \rightarrow \infty]{d} \mathcal{N}(0, C(d)\mu(x)(1 - \mu(x))).$$

### Theorem (Mayala, Scornet, Tillier and Wintenberger 2025)

Let  $d \geq 2$  and  $\hat{\mu}_s^{\text{ICRF}}(x)$  be the ICRF estimator at point  $x$ . Assume **(H0)**, **(H1)** and **(G1)** hold and

$$\frac{n2^k}{s^2 k^{(d-1)/2}} \rightarrow \infty, \quad \text{and} \quad 2^k k^{-\frac{d-1}{2}} n^{-\frac{d \log 2}{1+d \log 2}} \rightarrow \infty,$$

as  $n, s, k \rightarrow \infty$ . Then, for all  $x \in [0, 1]^d$ , we have

$$\sqrt{\frac{nk^{(d-1)/2}}{2^k}} \left( \hat{\mu}_s^{\text{ICRF}}(x) - \mu(x) \right) \xrightarrow[n \rightarrow \infty]{d} \mathcal{N}(0, C(d)\mu(x)(1 - \mu(x))).$$

- ▶ First CLT for random forests with an asymptotically unbiased estimator and explicit constants and conditions on tree structure.
- ▶ Comes at the price of an additional assumption

### Theorem (Mayala, Scornet, Tillier and Wintenberger 2025)

Let  $d \geq 2$  and  $\hat{\mu}_s^{\text{ICRF}}(x)$  be the ICRF estimator at point  $x$ . Assume **(H0)**, **(H1)** and **(G1)** hold and

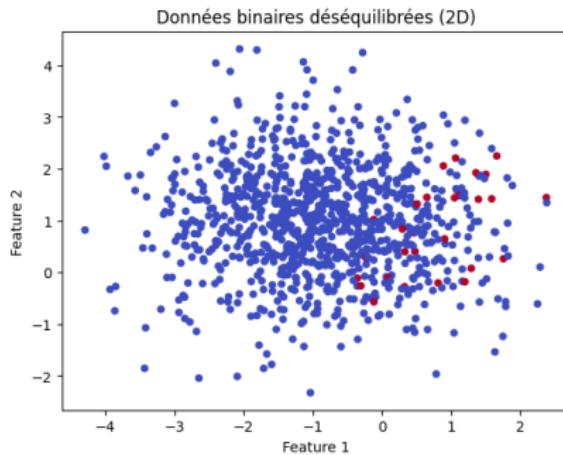
$$\frac{n2^k}{s^2 k^{(d-1)/2}} \rightarrow \infty, \quad \text{and} \quad 2^k k^{-\frac{d-1}{2}} n^{-\frac{d \log 2}{1+d \log 2}} \rightarrow \infty,$$

as  $n, s, k \rightarrow \infty$ . Then, for all  $x \in [0, 1]^d$ , we have

$$\sqrt{\frac{nk^{(d-1)/2}}{2^k}} \left( \hat{\mu}_s^{\text{ICRF}}(x) - \mu(x) \right) \xrightarrow[n \rightarrow \infty]{d} \mathcal{N}(0, C(d)\mu(x)(1 - \mu(x))).$$

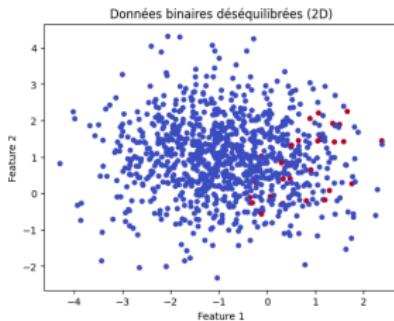
- ▶ The first (resp. second) condition imposes that  $s$  is not too large (resp. too small). Let  $s = n^\alpha$  and  $2^k = n^\beta$ . These two conditions can be rewritten as

$$\frac{d \log 2}{1 + d \log 2} < \alpha < 1 \quad \text{and} \quad \max \left( \frac{d \log 2}{1 + d \log 2}, 2\alpha - 1 \right) < \beta < \alpha.$$



Imbalance setting:  $\mathbb{P}[Y = 1] \ll \mathbb{P}[Y = 0]$

- ▶ Learning algorithms may struggle in such settings
- ▶ Tendency to only predict the majority class
- ▶ Recall of the minority class may be very low
- ▶ Need for designing specific methods

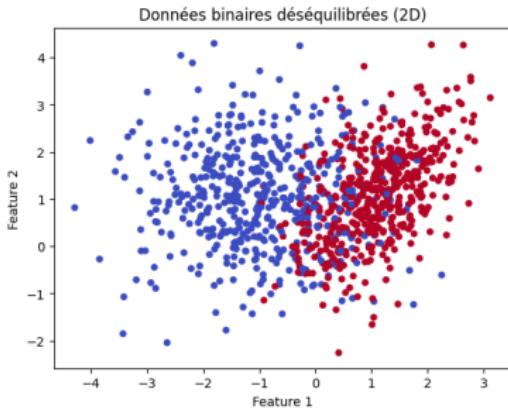
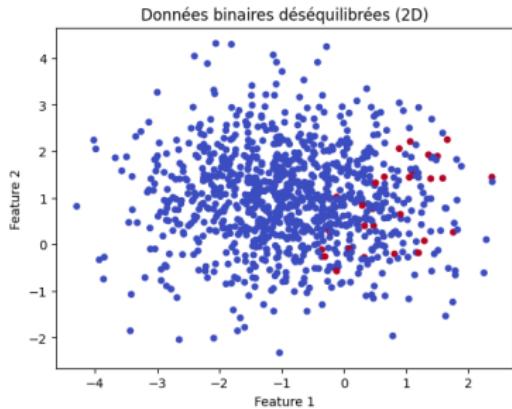


Rebalancing strategies ([Ramyachitra and Manikandan, 2014](#); [Krawczyk, 2016](#)):

- ▶ Undersample the majority class / oversample the minority class
- ▶ Assign weights to samples ([King and Zeng, 2001](#))
- ▶ Change the loss (focal loss [Lin et al., 2017](#))
- ▶ Generate synthetic data
  - ▶ SMOTE-like strategies ([Chawla et al., 2002](#))
  - ▶ GAN ([Xu et al., 2019](#)), Diffusion ([Jolicoeur-Martineau et al.](#))

# Rebalancing framework

20 / 40



Now, assume that we are given a i.i.d. rebalanced training set  $Z'_n := (Z'_1, \dots, Z'_n)$ , with  $Z'_i = (X'_i, Y'_i)$  such that

$$\begin{cases} \mu'(x) := \mathbb{P}(Y' = 1 | X' = x), x \in \mathcal{X}, \\ \mathbb{P}(Y' = 1) = p', \\ \mathbb{P}(X' \in \cdot | Y' = j) = \mathbb{P}(X \in \cdot | Y = j), j = \{0, 1\}. \end{cases}$$

Given  $\mathbf{Z}'_S := (Z'_{i_1}, \dots, Z'_{i_s})$ , with  $S = \{i_1, \dots, i_s\} \subset I$  and  $|S| = s$ , centered tree evaluated at point  $x$  trained on the  $s$ -subsample  $Z'_S$  that takes the form

$$T^s(x; \Theta; \mathbf{Z}'_S) := \frac{\sum_{i \in S} Y'_i \mathbb{1}\{X'_i \in L_\Theta(x)\}}{N_{L_\Theta(x)}(\mathbf{X}'_S)},$$

## Rebalanced ICRF

$$\hat{\mu}_{\text{RB},s}^{\text{ICRF}}(x) := \binom{n}{s}^{-1} \sum_{S \subset \{1, \dots, n\}, |S|=s} \mathbb{E}[T^s(x; \Theta; \mathbf{Z}'_S) \mid \mathbf{Z}'_S].$$

## Theorem 2 (Mayala, Scornet, Tillier and Wintenberger, 2025)

Let  $d \geq 2$  and  $\hat{\mu}_{\text{RB},s}^{\text{ICRF}}(x)$  be the Rebalanced ICRF estimator at point  $x$ . Assume **(H0)**, **(H1)** and **(G1)** hold and

$$\frac{n2^k}{s^2 k^{(d-1)/2}} \rightarrow \infty, \quad \text{and} \quad 2^k k^{-\frac{d-1}{2}} n^{-\frac{d \log 2}{1+d \log 2}} \rightarrow \infty,$$

as  $n, s, k \rightarrow \infty$ . Then, for all  $x \in [0, 1]^d$  we have

$$\sqrt{\frac{n}{s^2 V'_{1,s}}} (\hat{\mu}_{\text{RB},s}^{\text{ICRF}}(x) - \mu'(x)) \xrightarrow[n \rightarrow \infty]{d} \mathcal{N}(0, 1),$$

where

$$\frac{c'_1(x)2^k}{s^2 k^{(d-1)/2}} \leq \frac{V'_{1,s}}{\mu'(x)(1 - \mu'(x))} \leq \frac{c'_2(x)2^k}{s^2 k^{(d-1)/2}}.$$

## Theorem 2 (Mayala, Scornet, Tillier and Wintenberger, 2025)

Let  $d \geq 2$  and  $\hat{\mu}_{\text{RB},s}^{\text{ICRF}}(x)$  be the Rebalanced ICRF estimator at point  $x$ . Assume **(H0)**, **(H1)** and **(G1)** hold and

$$\frac{n2^k}{s^2 k^{(d-1)/2}} \rightarrow \infty, \quad \text{and} \quad 2^k k^{-\frac{d-1}{2}} n^{-\frac{d \log 2}{1+d \log 2}} \rightarrow \infty,$$

as  $n, s, k \rightarrow \infty$ . Then, for all  $x \in [0, 1]^d$  we have

$$\sqrt{\frac{n}{s^2 V'_{1,s}}} (\hat{\mu}_{\text{RB},s}^{\text{ICRF}}(x) - \mu'(x)) \xrightarrow[n \rightarrow \infty]{d} \mathcal{N}(0, 1),$$

where

$$\frac{c'_1(x)2^k}{s^2 k^{(d-1)/2}} \leq \frac{V'_{1,s}}{\mu'(x)(1 - \mu'(x))} \leq \frac{c'_2(x)2^k}{s^2 k^{(d-1)/2}}.$$

- ▶ Due to the rebalancing step,  $X'$  is not uniform on  $[0, 1]^d$  anymore: exact constant/rates cannot be derived anymore but the asymptotic remains the same.

## Theorem 2 (Mayala, Scornet, Tillier and Wintenberger, 2025)

Let  $d \geq 2$  and  $\hat{\mu}_{\text{RB},s}^{\text{ICRF}}(x)$  be the Rebalanced ICRF estimator at point  $x$ . Assume **(H0)**, **(H1)** and **(G1)** hold and

$$\frac{n2^k}{s^2 k^{(d-1)/2}} \rightarrow \infty, \quad \text{and} \quad 2^k k^{-\frac{d-1}{2}} n^{-\frac{d \log 2}{1+d \log 2}} \rightarrow \infty,$$

as  $n, s, k \rightarrow \infty$ . Then, for all  $x \in [0, 1]^d$  we have

$$\sqrt{\frac{n}{s^2 V'_{1,s}}} (\hat{\mu}_{\text{RB},s}^{\text{ICRF}}(x) - \mu'(x)) \xrightarrow[n \rightarrow \infty]{d} \mathcal{N}(0, 1),$$

where

$$\frac{c'_1(x)2^k}{s^2 k^{(d-1)/2}} \leq \frac{V'_{1,s}}{\mu'(x)(1 - \mu'(x))} \leq \frac{c'_2(x)2^k}{s^2 k^{(d-1)/2}}.$$

- ▶ There is a bias: due to rebalancing, the CLT is not centered at  $\mu(x)$  but at  $\mu'(x)$ !

Due to the rebalancing approach, we have

$$\begin{cases} \mu'(x) := \mathbb{P}(Y' = 1 | X' = x), x \in \mathcal{X}, \\ \mathbb{P}(Y' = 1) = p', \\ \mathbb{P}(X' \in \cdot | Y' = j) = \mathbb{P}(X \in \cdot | Y = j), j = \{0, 1\}. \end{cases}$$

Using standard calculation, we obtain

$$\mu(x) = \frac{p(1 - p')\mu'(x)}{p'(1 - p)(1 - \mu'(x)) + (1 - p')p\mu'(x)}.$$

Thanks to the function

$$g(u) = \frac{p(1 - p')u}{p'(1 - p)(1 - u) + (1 - p')pu},$$

one can debias the RB-ICRF estimate using  $\mu(x) = g(\mu'(x))$ .

## Importance sampling ICRF

$$\widehat{\mu}_{IS,s}^{ICRF}(x) := \frac{n_1(1 - p')\widehat{\mu}_{RB,s}^{ICRF}(x)}{p'n_0(1 - \widehat{\mu}_{RB,s}^{ICRF}(x)) + n_1(1 - p')\widehat{\mu}_{RB,s}^{ICRF}(x)}.$$

$$\hat{\mu}_{IS,s}^{ICRF}(x) := \frac{n_1(1-p')\hat{\mu}_{RB,s}^{ICRF}(x)}{p'n_0(1-\hat{\mu}_{RB,s}^{ICRF}(x)) + n_1(1-p')\hat{\mu}_{RB,s}^{ICRF}(x)}.$$

### Corollary (Mayala, Scornet, Tillier and Wintenberger, 2025)

Let  $d \geq 2$  and  $\hat{\mu}_{IS,s}^{ICRF}(x)$  be the importance sampling ICRF estimator. Let  $p \neq 0, p' \neq 1$  and assume **(H0)**, **(H1)** and **(G1)** hold and

$$\frac{n2^k}{s^2 k^{(d-1)/2}} \rightarrow \infty, \quad \text{and} \quad 2^k k^{-\frac{d-1}{2}} n^{-\frac{d \log 2}{1+d \log 2}} \rightarrow \infty,$$

as  $n, s, k \rightarrow \infty$ . Then, for all  $x \in [0, 1]^d$ , we have

$$\frac{1}{g'(\mu'(x))} \sqrt{\frac{n}{s^2 V'_{1,s}}} (\hat{\mu}_{IS,s}^{ICRF}(x) - \mu(x)) \xrightarrow[n \rightarrow \infty]{d} \mathcal{N}(0, 1).$$

Aim: comparing asymptotic variances of ICRF and IS-ICRF in high imbalanced settings ( $p \rightarrow 0$ ).

Aim: comparing asymptotic variances of ICRF and IS-ICRF in high imbalanced settings ( $p \rightarrow 0$ ).

### Problem - Assumptions

- ▶  $p \rightarrow 0$
- ▶  $f_X(x) = pf_{X|Y=1}(x) + (1-p)f_{X|Y=0}(x)$  is the uniform density  
⇒ Conditional distributions  $f_{X|Y=0,1}$  must change when  $p \rightarrow 0$ .

Aim: comparing asymptotic variances of ICRF and IS-ICRF in high imbalanced settings ( $p \rightarrow 0$ ).

### New framework (H3)

We fix  $f_{X|Y=0}$  and  $f_{X|Y=1}$  such that

- ▶ Both are  $L$ -Lipschitz
- ▶  $0 < b_1 \leq f_{X|Y=0}(\cdot), f_{X|Y=1}(\cdot) \leq b_2 < \infty$
- ▶ There exists  $p''$  such that

$$p''f_{X|Y=1} + (1 - p'')f_{X|Y=0}$$

is the uniform density on  $[0, 1]^d$ .

Aim: comparing asymptotic variances of ICRF and IS-ICRF in high imbalanced settings ( $p \rightarrow 0$ ).

### New framework (H3)

We fix  $f_{X|Y=0}$  and  $f_{X|Y=1}$  such that

- ▶ Both are  $L$ -Lipschitz
- ▶  $0 < b_1 \leq f_{X|Y=0}(\cdot), f_{X|Y=1}(\cdot) \leq b_2 < \infty$
- ▶ There exists  $p''$  such that

$$p''f_{X|Y=1} + (1 - p'')f_{X|Y=0}$$

is the uniform density on  $[0, 1]^d$ .

**(G1) Tree complexity Condition:** The subsample size  $s$  and the tree depth  $k$  tend to infinity and satisfy  $s/(k2^k) \rightarrow \infty$ , as  $n \rightarrow \infty$ .

## Corollary (Mayala, Scornet, Tillier and Wintenberger, 2025)

Let  $d \geq 2$ ,  $p \neq 0$  and  $p' \neq 1$ . Grant **(H3)** and **(G1)**. Assume that

$$\frac{n2^k}{s^2 k^{(d-1)/2}} \rightarrow \infty, \quad \text{and} \quad 2^k k^{-\frac{d-1}{2}} n^{-\frac{d \log 2}{1+d \log 2}} \rightarrow \infty,$$

as  $n, s, k \rightarrow \infty$ . Then, for all  $x \in [0, 1]^d$ , we have

$$\sqrt{\frac{n}{s^2 V_{1,s}}} (\hat{\mu}_s^{\text{ICRF}}(x) - \mu(x)) \xrightarrow[n \rightarrow \infty]{d} \mathcal{N}(0, 1),$$

and

$$\frac{1}{g'(\mu'(x))} \sqrt{\frac{n}{s^2 V'_{1,s}}} (\hat{\mu}_{|S,s}^{\text{ICRF}}(x) - \mu(x)) \xrightarrow[n \rightarrow \infty]{d} \mathcal{N}(0, 1).$$

Thus, for all  $k$  large enough,

$$\frac{V'_{1,s}}{V_{1,s}} g'(\mu'(x))^2 = O(p).$$

1. Random Forests construction
2. U-statistics and link with RF
3. Asymptotic analysis of Infinite Centered RF
4. Numerical experiments

(Sim. setting)  $n$  i.i.d. pairs  $(X_i, Y_i)$  distributed as  $(X, Y)$ , with  $X \sim U([0, 1]^2)$  and

$$\mathbb{P}(Y = 1 | X = x) = \mu(x) = \frac{1}{1 + \exp(-(\beta_0 + 3x_1 + 2x_2))},$$

where  $\beta_0$  is such that  $\mathbb{P}(Y = 1) = 0.1$ .

For each  $(n, \alpha, \beta)$ , we repeat  $B = 1000$  times:

1. Generate a dataset with  $n$  observations (Sim. setting).
2. A RF<sup>4</sup> is trained with default parameters and  $s = n^\alpha$ ,  
 $\text{max.depth} = \beta \log_2 n$ .
3. The forest prediction  $\hat{\mu}_s^{\text{ICRF}}(x)$  is evaluated at  $x = (0.7, 0.7)$ .

We use these predictions to estimate

$$\mathbb{E}[\hat{\mu}_s^{\text{ICRF}}(x)] - \mu(x) \quad \text{and} \quad \log \left( \mathbb{E} \left[ \left( \hat{\mu}_s^{\text{ICRF}}(x) - \mathbb{E}[\hat{\mu}_s^{\text{ICRF}}(x)] \right)^2 \right] \right).$$

---

<sup>4</sup>R package ranger (see Wright and Ziegler, 2017)

According to our theoretical analysis, if CLT were to hold in  $L^2$ , we would obtain

$$\begin{aligned} & \log(\mathbb{E}[(\hat{\mu}_s^{\text{ICRF}}(x) - \mathbb{E}[\hat{\mu}_s^{\text{ICRF}}(x)])^2]) \\ & \sim -(1 - \beta) \log n - \frac{d - 1}{2} \log \log n + C_{1,d,\beta}(x) \end{aligned}$$

with

$$C_{1,d,\beta}(x) = \log(C(d)\mu(x)(1 - \mu(x))) - \frac{d - 1}{2} \log \beta - \frac{d - 1}{2} \log \log 2.$$

According to our theoretical analysis, if CLT were to hold in  $L^2$ , we would obtain

$$\begin{aligned} & \log(\mathbb{E}[(\hat{\mu}_s^{\text{ICRF}}(x) - \mathbb{E}[\hat{\mu}_s^{\text{ICRF}}(x)])^2]) \\ & \sim -(1 - \beta) \log n - \frac{d - 1}{2} \log \log n + C_{1,d,\beta}(x) \end{aligned}$$

with

$$C_{1,d,\beta}(x) = \log(C(d)\mu(x)(1 - \mu(x))) - \frac{d - 1}{2} \log \beta - \frac{d - 1}{2} \log \log 2.$$

Thus,  $\log(\mathbb{E}[(\hat{\mu}_s^{\text{ICRF}}(x) - \mathbb{E}[\hat{\mu}_s^{\text{ICRF}}(x)])^2])$

- ▶ is approximately linear in  $\log n$
- ▶ with lower slopes for larger values of  $\beta$
- ▶ depends on  $\beta$  but not on the subsample size.

# Rates of convergence - variance

30 / 40

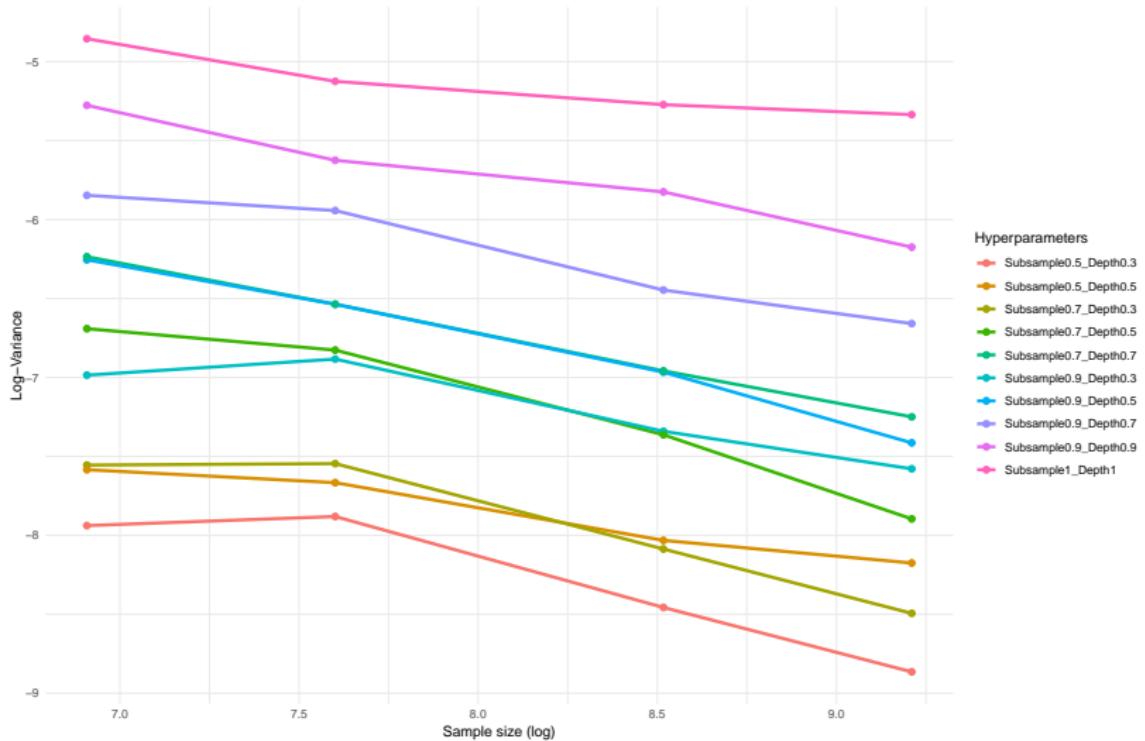


Figure: Log-variance of the classic RF

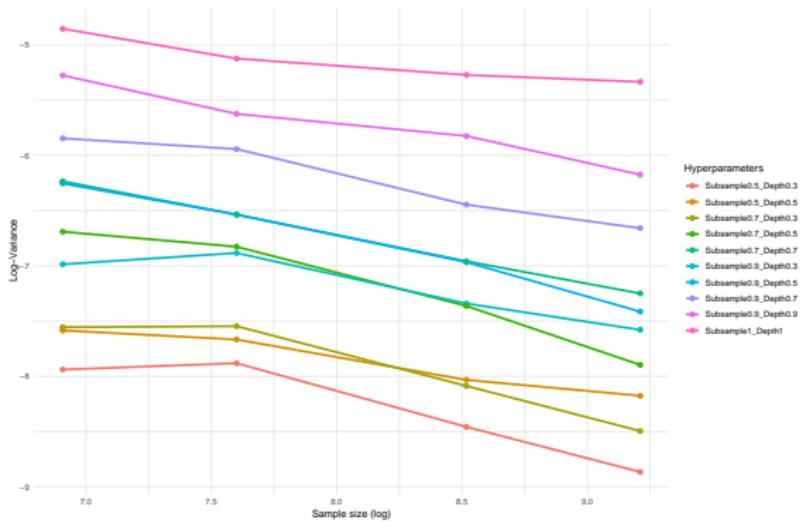


Figure: Log-variance of the classic RF

Theory tells us that the log-variance

- ▶ is approximately linear in  $\log n$  ✓
- ▶ with lower slopes for larger values of  $\beta$  ✓
- ▶ depends on  $\beta$  but not on the subsample size  $\simeq$

According to our theoretical results, the bias satisfies

$$\left(\frac{\beta}{\log 2}\right)^{(d-1)/2} n^{1-\beta} (\log n)^{(d-1)/2} \left( \mathbb{E}[\hat{\mu}_s^{\text{ICRF}}(x)] - \mu(x) \right) \rightarrow 0.$$

Thus,

$$\left( \mathbb{E}[\hat{\mu}_s^{\text{ICRF}}(x)] - \mu(x) \right) = o \left( n^{\beta-1} (\log n)^{-(d-1)/2} \right)$$

# Rates of convergence - bias

32 / 40

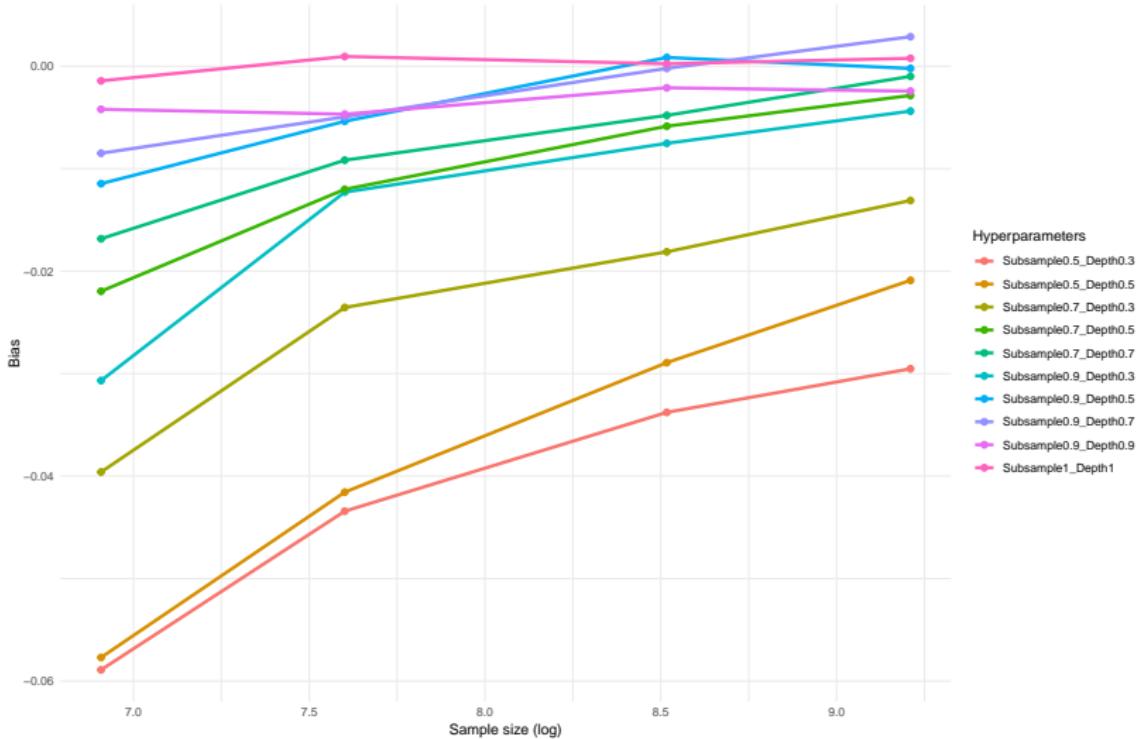


Figure: Bias of the classic RF

# Rates of convergence - bias

32 / 40

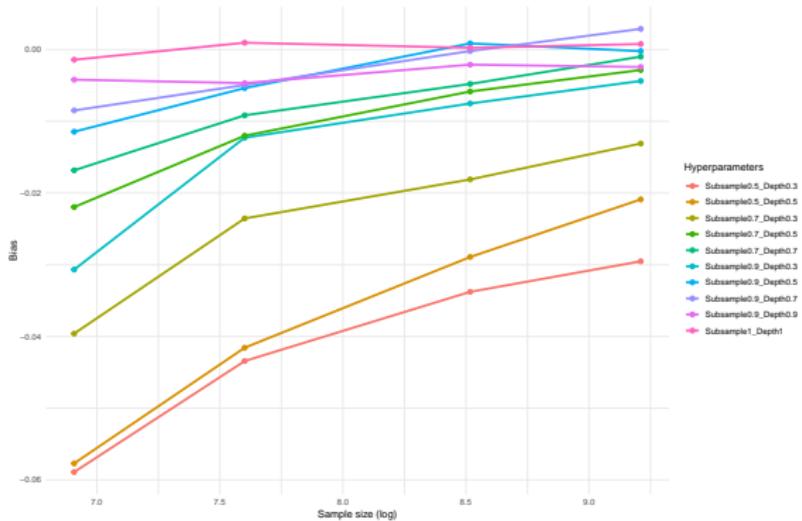


Figure: Bias of the classic RF

- ▶ Negative bias: majority of 0, RF pred. shifted toward 0.
- ▶ All biases tends to zero - Expected since tree depth increases ( $k = \beta \log_2 n$ ).

# Numerical Illustrations

33 / 40

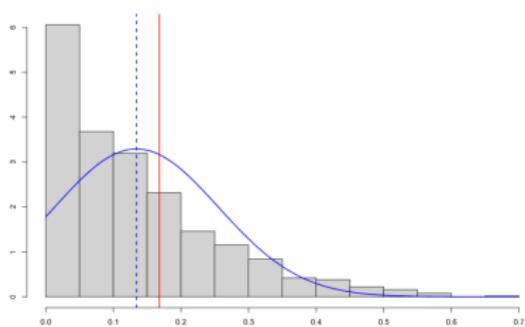


Figure: RF

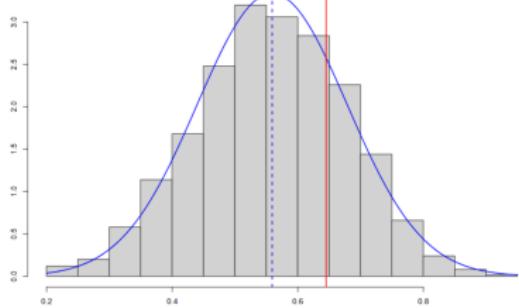


Figure: RB-RF

Histograms of predictions for each estimator with  $p' = 0.5$ ,  $n = 100$  and  $B = 1000$  replicates. The empirical variances are: 0.121 (RF), 0.119 (RB-RF).

# Numerical Illustrations

33 / 40

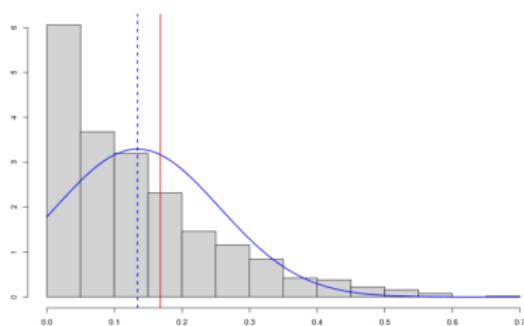


Figure: RF

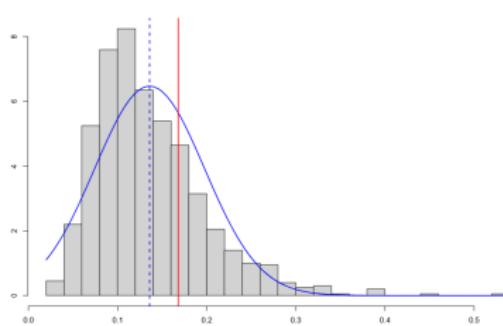


Figure: IS-RF

Histograms of predictions for each estimator with  $p' = 0.5$ ,  $n = 100$  and  $B = 1000$  replicates. The empirical variances are: 0.121 (RF), 0.061 (IS-RF).

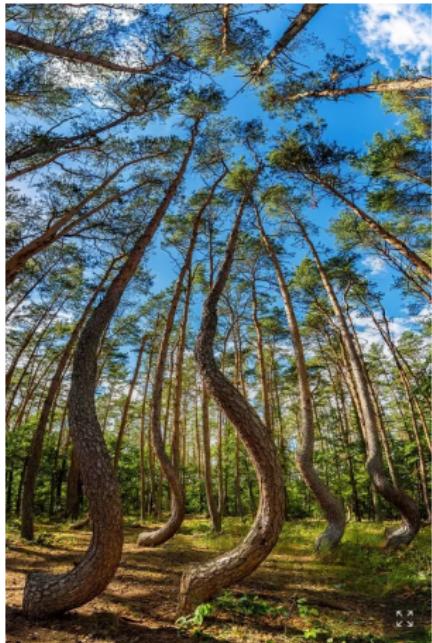
- ▶ We establish a CLT for centered random forest under the assumption that covariates are uniformly distributed on  $[0, 1]^d$ .
  - ▶ Convergence rate and asymptotic variance are made explicit
  - ▶ Assumptions on tree structure (subsampling rate, tree depth)
  - ▶ First CLT on random forests with explicit rate of convergence and assumptions on tree structure

- ▶ We establish a CLT for centered random forest under the assumption that covariates are uniformly distributed on  $[0, 1]^d$ .
  - ▶ Convergence rate and asymptotic variance are made explicit
  - ▶ Assumptions on tree structure (subsampling rate, tree depth)
  - ▶ First CLT on random forests with explicit rate of convergence and assumptions on tree structure
- ▶ We analyze imbalanced learning problems
  - ▶ CLT for rebalanced forest, with non explicit constant as the new covariate distribution is not uniform
  - ▶ CLT is not centered at the correct value - bias of rebalancing strategies
  - ▶ We correct this bias and establish a CLT for the IS estimate
  - ▶ In a high imbalanced framework,  $V_{IS} \ll V_{RF}$

- ▶ We establish a CLT for centered random forest under the assumption that covariates are uniformly distributed on  $[0, 1]^d$ .
  - ▶ Convergence rate and asymptotic variance are made explicit
  - ▶ Assumptions on tree structure (subsampling rate, tree depth)
  - ▶ First CLT on random forests with explicit rate of convergence and assumptions on tree structure
- ▶ We analyze imbalanced learning problems
  - ▶ CLT for rebalanced forest, with non explicit constant as the new covariate distribution is not uniform
  - ▶ CLT is not centered at the correct value - bias of rebalancing strategies
  - ▶ We correct this bias and establish a CLT for the IS estimate
  - ▶ In a high imbalanced framework,  $V_{IS} \ll V_{RF}$
- ▶ Numerical experiments show that our findings on centered forest can be partially extended to Breiman's random forests.

Thank you for your attention!

35 / 40



Asymptotic Normality of Infinite  
Centered Random Forests - Ap-  
plication to Imbalanced Classifi-  
cation, M. Mayala, E. Scornet, C.  
Tillier and O. Wintenberger

- S. Arlot and R. Genuer. Analysis of purely random forests bias. *arXiv preprint arXiv:1407.3939*, 2014.
- G. Biau. Analysis of a random forests model. *JMLR*, 2012.
- G. Biau and E. Scornet. A random forest guided tour. *Test*, 25:197–227, 2016.
- G. Biau, L. Devroye, and volume=9 number=9 year=2008 G. Lugosi, journal=JMLR. Consistency of random forests and other averaging classifiers.
- A.-L. Boulesteix, S. Janitza, J. Kruppa, and I.R. König. Overview of random forest methodology and practical guidance with emphasis on computational biology and bioinformatics. *Wiley Interdisciplinary Reviews: Data Mining and Knowledge Discovery*, 2:493–507, 2012.
- L. Breiman. Random forests. *Machine learning*, 45(1):5–32, 2001.
- L. Breiman. Consistency for a simple model of random forests. 2004.
- N.V. Chawla, K.W. Bowyer, L.O. Hall, and W.P. Kegelmeyer. Smote: synthetic minority over-sampling technique. *Journal of artificial intelligence research*, 2002.

- A. Criminisi, J. Shotton, and E. Konukoglu. Decision forests: A unified framework for classification, regression, density estimation, manifold learning and semi-supervised learning. *Foundations and Trends in Computer Graphics and Vision*, 7:81–227, 2011.
- A.W. Van der Vaart. *Asymptotic statistics*, volume 3. Cambridge university press, 2000.
- R. Genuer. Variance reduction in purely random forests. *Journal of Nonparametric Statistics*, 24(3):543–562, 2012.
- A. Jolicoeur-Martineau, K. Fatras, and T. Kachman. Generating and imputing tabular data via diffusion and flow-based gradient-boosted trees. In *AISTAT*.
- G. King and L. Zeng. Logistic regression in rare events data. *Political analysis*, 2001.
- J. Klusowski. Sharp analysis of a simple model for random forests. In *AISTAT*, 2021.
- J. Klusowski and P. Tian. Large scale prediction with decision trees. *Journal of the American Statistical Association*, 119(545):525–537, 2024.
- B. Krawczyk. Learning from imbalanced data: open challenges and future directions. *Progress in artificial intelligence*, 2016.

- A J Lee. *U-statistics: Theory and Practice*. Routledge, 2019.
- X. Li, Y. Wang, S. Basu, K. Kumbier, and B. Yu. A debiased mdi feature importance measure for random forests. *NeurIPS*, 2019.
- T.-Y. Lin, P. Goyal, R. Girshick, K. He, and P. Dollár. Focal loss for dense object detection. In *IEEE international conference on computer vision*, 2017.
- G. Louppe, L. Wehenkel, A. Sutera, and P. Geurts. Understanding variable importances in forests of randomized trees. In C.J.C. Burges, L. Bottou, M. Welling, Z. Ghahramani, and K.Q. Weinberger, editors, *NeurIPS*, pages 431–439, 2013.
- L. Mentch and G. Hooker. Quantifying uncertainty in random forests via confidence intervals and hypothesis tests. *JMLR*, 2016.
- J. Mourtada, S. Gaiffas, and E. Scornet. Minimax optimal rates for mondrian trees and forests. *The Annals of Statistics*, 2020.
- W. Peng, T. Coleman, and L. Mentch. Rates of convergence for random forests via generalized u-statistics. *Electronic Journal of Statistics*, 2022.
- D. Ramyachitra and P. Manikandan. Imbalanced dataset classification and solutions: a review. *International Journal of Computing and Business Research (IJCBR)*, 2014.

- E. Scornet. Random forests and kernel methods. *IEEE Transactions on Information Theory*, 62(3):1485–1500, 2016.
- E. Scornet. Trees, forests, and impurity-based variable importance in regression. In *Annales de l'Institut Henri Poincaré (B) Probabilités et statistiques*. Institut Henri Poincaré, 2023.
- E. Scornet and G. Hooker. Theory of random forests: A review. 2025.
- E. Scornet, G. Biau, and J.-P. Vert. Consistency of random forests. *The Annals of Statistics*, 2015.
- S. Wager and S. Athey. Estimation and inference of heterogeneous treatment effects using random forests. *Journal of the American Statistical Association*, 2018.
- S. Wager and G. Walther. Adaptive concentration of regression trees, with application to random forests. *arXiv preprint arXiv:1503.06388*, 2015.
- S. Wager, T. Hastie, and B. Efron. Confidence intervals for random forests: The jackknife and the infinitesimal jackknife. *The Journal of Machine Learning Research*, 2014.
- M.N. Wright and A. Ziegler. ranger: A fast implementation of random forests for high dimensional data in c++ and r. *Journal of statistical software*, 2017.

- L. Xu, M. Skoularidou, A. Cuesta-Infante, and K. Veeramachaneni. Modeling tabular data using conditional gan. *NeurIPS*, 2019.
- Z. Zhou, L. Mentch, and G. Hooker. V-statistics and variance estimation. *JMLR*, 2021.

Considering the centered forest with uniform covariates, we have

$$p_{k,\Theta}(x) = \mathbb{P}(X \in L_\Theta(x) | \Theta) = 2^{-k}.$$

However, when the distribution is not uniform, we prove that

$$\begin{aligned} p'_{k,\Theta}(x) &= \mathbb{P}(X' \in L_\Theta(x) | \Theta) \\ &= \frac{c'(x)}{2^k} (1 + \alpha'(x) \varepsilon'_\Theta(x) \text{Diam}(L_\Theta(x))) \quad \text{a.s.} \end{aligned}$$

- ▶ Random variable and not a deterministic quantity
- ▶ Depends on  $x$

# Asymptotic constant

In our theoretical result for uniform covariates

$$\sqrt{\frac{nk^{(d-1)/2}}{2^k}} (\hat{\mu}_s^{\text{ICRF}}(\mathbf{x}) - \mu(\mathbf{x})) \xrightarrow[n \rightarrow \infty]{d} \mathcal{N}(0, C(d)\mu(\mathbf{x})(1 - \mu(\mathbf{x}))).$$

with

$$C(d) = \frac{2\Gamma(d-1)}{(\log 2)^{d-1}\Gamma((d-1)/2)} \mathbb{E} \left[ \left( \frac{\|(\mathbf{N} - \bar{\mathbf{N}}\mathbf{1})\|_2}{\|(\mathbf{N} - \bar{\mathbf{N}}\mathbf{1})\|_1} \right)^{d-1} \right],$$

where  $\mathbf{N} = (N_1, \dots, N_d)$  with  $N_1, \dots, N_d$  independent  $\mathcal{N}(0, 1)$  and  $\bar{\mathbf{N}} = (1/d) \sum_{j=1}^d N_j$ .

This comes from a new control of the quantity: when  $k \rightarrow \infty$ ,

$$\mathbb{E} [\mathbb{P}(X_1 \in L_\Theta(\mathbf{x}) | X_1)^2] \sim \frac{C(d)}{2^k k^{(d-1)/2}}.$$