The University of Western Australia
Department of Mathematics and Statistics
MATH3023 Advanced Mathematics Applications

Formula Sheet

The quadratic formula

$$ax^2 + bx + c = 0 \qquad \Rightarrow \qquad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Cylindrical polar coordinates

$$\mathbf{r} = (x, y, z) = (r\cos\theta, r\sin\theta, z) \qquad\qquad dxdydz = rdrd\theta dz$$

$$\frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial z} = (r\cos\theta, r\sin\theta, 0)$$

Spherical polar coordinates

$$\mathbf{r} = (x, y, z) = (\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi) \qquad\qquad dxdydz = \rho^2 \sin\phi d\rho d\phi d\theta$$

$$\frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial \phi} = \rho\mathbf{r}\sin\phi$$

2D and 3D vector fields

$$\mathbf{F} = M\mathbf{i} + N\mathbf{j} = (M, N) \qquad\qquad \mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k} = (M, N, P)$$

Flow and flux

$$\int \mathbf{F} \cdot \mathbf{T} dS = \int \mathbf{F} \cdot d\mathbf{r} = \int M dx + N dy \qquad\qquad \int \mathbf{F} \cdot \mathbf{n} dS = \int M dy - N dx$$

Gradient, curl and divergence

$$\mathbf{F} = \nabla f = \mathbf{i}\frac{\partial f}{\partial x} + \mathbf{j}\frac{\partial f}{\partial y} + \mathbf{k}\frac{\partial f}{\partial z} = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\mathbf{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k}$$

$$\text{div } \mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$$

Line integrals of scalar and vector functions

$$\int_C f(x, y, z) ds = \int_a^b f(x(t), y(t), z(t))\sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} dt$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C M dx + N dy + P dz = \int_a^b (M, N, P) \cdot (x', y', z') dt$$

Fundamental theorem for line integrals

$$\int_A^B \mathbf{F} \cdot d\mathbf{r} = f(B) - f(A)$$

Surface integrals of scalar and vector functions

$$\int\int_S f dS = \int\int_D f(\mathbf{r}(u,v)) \left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right| dudv$$

$$\int\int_S \mathbf{F} \cdot \mathbf{n} dS = \int\int_D \mathbf{F}(\mathbf{r}(u,v)) \cdot \left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right) dudv$$

Green's theorem

$$\oint_C M dx + N dy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$$

Stokes theorem

$$\oint_C \mathbf{F} \cdot \mathbf{dr} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS$$

Divergence theorem

$$\iint_S \mathbf{F} \cdot \mathbf{n} dS = \iiint_E \nabla \cdot \mathbf{F} dV$$

De Moivre's theorem

$$z = re^{i\theta} = r\cos\theta + r\sin\theta \qquad \Rightarrow \qquad z^n = r^n e^{in\theta} = r^n \cos(n\theta) + ir^n \sin(n\theta)$$

Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Cauchy's Integral Formula

$$\oint_C \frac{f(z)}{z - z_0} dz = 2\pi i f(z_0)$$

Cauchy's Integral Formula for derivatives

$$\oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz = \frac{2\pi i}{n!} f^{(n)}(z_0)$$

A useful result

$$\oint_C \frac{dz}{(z - z_0)^k} = \begin{cases} 2\pi i, & k = 1 \\ 0, & k \neq 1 \end{cases}$$

Cauchy's Residue Theorem

$$\oint_C f(z) dz = 2\pi i \sum_{m=1}^{M} \mathrm{Res}[f, z_m]$$

Evaluation of residues: If $f(z)$ has a pole of order k at $z = z_0$ then

$$\mathrm{Res}[f, z_0] = \frac{1}{(k-1)!} \lim_{z \to z_0} \frac{d^{k-1}}{dz^{k-1}} \left[(z - z_0)^k f(z) \right]$$

Integration by parts

$$\int_a^b f(t)g'(t)dt = \Big[f(t)g(t)\Big]_a^b - \int_a^b f'(t)g(t)dt$$

Integrating factor method

$$\text{To solve} \quad \frac{df}{dt} + p(t)f = q(t) \quad \text{let} \quad \mu(t) = e^{\int p(t)dt} \qquad \Rightarrow \qquad \frac{d}{dt}(\mu f) = \mu(t)q(t)$$

Second-order homogeneous linear differential equations

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

$$y = Ae^{m_1x} + Be^{m_2x} \quad \text{or} \quad y = e^{\alpha x}\left(A\cos\beta x + B\sin\beta x\right) \quad \text{or} \quad y = Ae^{mx} + Bxe^{mx}$$

$$ax^2\frac{d^2y}{dx^2} + bx\frac{dy}{dx} + cy = 0$$

$$y = Ax^{m_1} + Bx^{m_2} \quad \text{or} \quad y = x^{\alpha}\left[A\cos(\beta\ln x) + B\sin(\beta\ln x)\right] \quad \text{or} \quad y = Ax^m + Bx^m\ln x$$

Method of undetermined coefficients

$$f(x) = Ae^{cx} \qquad \Rightarrow \qquad y_p(x) = Ce^{cx}$$

$$f(x) = A\cos(cx) \quad \text{or} \quad f(x) = A\sin(cx) \qquad \Rightarrow \qquad y_p(x) = C_1\cos(cx) + C_2\sin(cx)$$

$$f(x) = \text{polynomial of degree } n \qquad \Rightarrow \qquad y_p(x) = \text{polynomial of degree } n$$

Similarity variables

$$x \to k^c x \qquad\qquad t \to k^a t \qquad\qquad u \to k^b u$$

$$\eta = \frac{x^a}{t^c}, \quad \phi = \frac{u^a}{t^b}, \quad \psi = \frac{u^c}{x^b} \qquad \text{or} \qquad \eta = \frac{x}{t^{c/a}}, \quad \phi = \frac{u}{t^{b/a}}, \quad \psi = \frac{u}{x^{b/c}}$$

D'Alembert solution of the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial t^x} \qquad\qquad u(x,0) = U(x) \qquad\qquad \frac{\partial u}{\partial t}(x,0) = V(x)$$

$$u(x,t) = \frac{U(x+ct) + U(x-ct)}{2} + \frac{W(x+ct) - W(x-ct)}{2} \qquad\qquad W(x) = \frac{1}{c}\int V(x)dx$$

Inner product

$$\langle f, g\rangle = \int_a^b f(t)g(t)w(t)dt$$

Eigenfunction expansion

$$f(x) = \sum_{n=1}^{\infty} c_n y_n(x) \qquad \Rightarrow \qquad c_m = \frac{\langle f(x), y_m(x)\rangle}{\langle y_m(x), y_m(x)\rangle}$$