THE UNIVERSITY OF WESTERN AUSTRALIA

Department of Mathematics and Statistics

MATH1011 MULTIVARIABLE CALCULUS

FORMULA SHEET

The second derivative test for functions of two variables

For a real-valued function $f(x, y)$ of two variables defined on a subset D of $\mathbb{R}^2$ and for $\mathbf{c} = (a, b) \in D$, we define the Hessian matrix

$$\begin{bmatrix} f_{xx}(\mathbf{c}) & f_{xy}(\mathbf{c}) \\ f_{yx}(\mathbf{c}) & f_{yy}(\mathbf{c}) \end{bmatrix},$$

with determinant given by

$$D_{\mathbf{c}} = f_{xx}(\mathbf{c}) f_{yy}(\mathbf{c}) - [f_{xy}(\mathbf{c})]^2 .$$

If $\nabla f(\mathbf{c}) = (f_x(\mathbf{c}), f_y(\mathbf{c})) = \mathbf{0}$ (that is, $\mathbf{c}$ is a critical point of f):

1. If $D_{\mathbf{c}} > 0$ and $f_{xx}(\mathbf{c}) > 0$, then f has a local minimum at $\mathbf{c}$.

2. If $D_{\mathbf{c}} > 0$ and $f_{xx}(\mathbf{c}) < 0$, then f has a local maximum at $\mathbf{c}$.

3. If $D_{\mathbf{c}} < 0$, then $\mathbf{c}$ is a saddle point of f.

4. When $D_{\mathbf{c}} = 0$, the Second Derivatives Test gives no information.

Taylor polynomials for functions of one variable

Let $f(x)$ be a real-valued function of one variable defined on some interval I and having continuous derivatives $f'(x), f''(x), \ldots, f^{(n)}(x)$ on I for some integer $n \geq 1$. Let a be an interior point of I. The n^{th} degree Taylor polynomial of f about a is defined by

$$T_{n,a}(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n,$$

where $n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$ is the factorial of n.

Taylor polynomials for functions of two variables

Let D be an open disc in $\mathbb{R}^2$, let $f : D \longrightarrow \mathbb{R}$, and let $\mathbf{c} = (a, b) \in D$. If f has continuous and bounded partial derivatives up to second order in D, then the second-order Taylor polynomial of f about (a, b) is given by

$$\begin{aligned} f(x,y) =& f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) \\ &+ \frac{1}{2!}\left[f_{xx}(a,b)(x-a)^2 + 2f_{xy}(a,b)(x-a)(y-b) + f_{yy}(a,b)(y-b)^2\right]. \end{aligned}$$

Trigonometric properties

Fundamental property: $\sin^2 x + \cos^2 x = 1,$

$$\tan^2 x + 1 = \sec^2 x,$$

$$1 + \cot^2 x = \operatorname{cosec}^2 x.$$

Odd/even property: $\sin(-x) = -\sin x$, $\cos(-x) = \cos x$.

Addition formula: $\sin(x+y) = \sin x \cos y + \cos x \sin y,$

$$\cos(x+y) = \cos x \cos y - \sin x \sin y,$$

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}.$$

Half-angle formula: $\sin(2x) = 2\sin x \cos x,$

$$\cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x,$$

$$\tan(2x) = \frac{2\tan x}{1 - \tan^2 x}.$$

Product formula: $\sin x \cos y = \frac{1}{2}[\sin(x+y) + \sin(x-y)],$

$$\sin x \sin y = \tfrac{1}{2}[\cos(x-y) - \cos(x+y)],$$

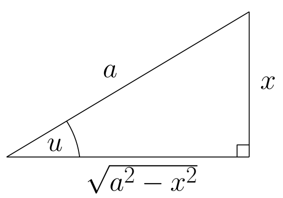
$$\cos x \cos y = \tfrac{1}{2}[\cos(x+y) + \cos(x-y)].$$

Integration

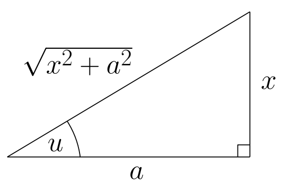
<u>Integration by inverse trigonometric substitution:</u>

Integral involves	Then substitute	Restriction on u	Use the identity
$\sqrt{a^2 - x^2}$	$x = a\sin u$	$-\frac{\pi}{2} \leq u \leq \frac{\pi}{2}$	$1 - \sin^2 u = \cos^2 u$
$\sqrt{a^2 + x^2}$	$x = a\tan u$	$-\frac{\pi}{2} < u < \frac{\pi}{2}$	$1 + \tan^2 u = \sec^2 u$
$\sqrt{x^2 - a^2}$	$x = a\sec u$	$0 \leq u < \frac{\pi}{2}$	$\sec^2 u - 1 = \tan^2 u$

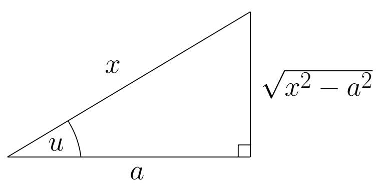
To return to the original variable x use the reference triangles illustrated below.



Reference triangle for
$x = a\sin u$



Reference triangle for
$x = a\tan u$

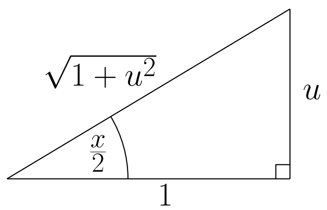


Reference triangle for
$x = a\sec u$

Integration by half-angle substitution:

The substitution $u = \tan\left(\frac{x}{2}\right) \Rightarrow x = 2\tan^{-1} u$ with reference triangle shown to the right turns an integral with a quotient involving $\sin x$ and/or $\cos x$ into an integral of a rational function of u, where

$$\sin x = \frac{2u}{1+u^2} \quad \text{and} \quad \cos x = \frac{1-u^2}{1+u^2}.$$



Integration by partial fractions:

A rational function $f(x) = \dfrac{P(x)}{Q(x)}$ with $\deg(P(x)) < \deg(Q(x))$ can be decomposed into partial fractions as follows:

Case 1: Denominator has distinct linear factors

$$f(x) = \frac{P(x)}{(x-a_1)\cdots(x-a_k)} = \frac{A_1}{x-a_1} + \cdots + \frac{A_k}{x-a_k},$$

where $a_1, \ldots, a_k$ are pairwise distinct.

Case 2: Denominator has repeated linear factors

$$f(x) = \frac{P(x)}{(x-a)^c} = \frac{B_1}{x-a} + \frac{B_2}{(x-a)^2} + \cdots + \frac{B_{c-1}}{(x-a)^{c-1}} + \frac{B_c}{(x-a)^c}.$$

Case 3: Denominator has an irreducible factor of degree 2

$$f(x) = \frac{P(x)}{(x-a)(x^2+bx+c)} = \frac{A_1}{x-a} + \frac{C_1 x + C_2}{x^2+bx+c}.$$

Integration by parts:

$$\int u\,dv = uv - \int v\,du.$$

Use the following table as a *guide*:

u	dv
Polynomial	Exponential Trigonometric
Logarithmic Inverse trigonometric	Polynomial

Centre of mass

In $\mathbb{R}^3$, given a mass density function $\rho(x, y, z)$ the total mass of a body is

$$M = \iiint_R \rho(x, y, z)\, dV.$$

The first moments and the centre of mass are

$$M_{yz} = \iiint_R x\rho(x, y, z)\, dV \quad , \quad M_{xz} = \iiint_R y\rho(x, y, z)\, dV \quad , \quad M_{xy} = \iiint_R z\rho(x, y, z)\, dV,$$

$$(C_x, C_y, C_z) = \left(\frac{M_{yz}}{M}, \frac{M_{xz}}{M}, \frac{M_{xy}}{M}\right).$$

In $\mathbb{R}^2$, given a mass density function $\rho(x, y, z)$ the total mass of a body is

$$M = \iint_R \rho(x, y)\, dA,$$

and the first moments and the centre of mass are:

$$M_y = \iint_R x\rho(x, y)\, dA \quad , \quad M_x = \iint_R y\rho(x, y)\, dA,$$

$$(C_x, C_y) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right).$$

Change of coordinates in double integrals

Given a change of coordinates $(x, y) = \mathbf{g}(u, v) = (\phi(u, v), \psi(u, v))$ we have

$$\iint_R f(x, y)\, dx\, dy = \iint_S f(\mathbf{g}(u, v)) \left|\det\left(\frac{\partial \mathbf{g}(u, v)}{\partial(u, v)}\right)\right| du\, dv,$$

where $\left(\dfrac{\partial \mathbf{g}(u, v)}{\partial(u, v)}\right)$ is the Jacobian matrix of the transformation $\mathbf{g}(u, v)$ at (u, v):

$$\left(\frac{\partial \mathbf{g}(u, v)}{\partial(u, v)}\right) = \begin{bmatrix} \dfrac{\partial \phi}{\partial u}(u, v) & \dfrac{\partial \phi}{\partial v}(u, v) \\ \dfrac{\partial \psi}{\partial u}(u, v) & \dfrac{\partial \psi}{\partial v}(u, v) \end{bmatrix}.$$

In polar coordinates (r, θ) where

$$x = r\cos\theta \quad , \quad y = r\sin\theta,$$

we have

$$\iint_R f(x, y)\, dx\, dy = \iint_S f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta.$$

Change of coordinates in triple integrals

In cylindrical coordinates (r, θ, ξ) where

$$x = r\cos\theta \quad , \quad y = r\sin\theta \quad , \quad z = \xi,$$

we have

$$\iiint\limits_R f(x,y,z)\,dx\,dy\,dz = \iiint\limits_S f(r\cos\theta, r\sin\theta, \xi)\,r\,dr\,d\theta\,d\xi.$$

In spherical coordinates (ρ, θ, ϕ) where

$$x = \rho\cos\theta\sin\phi \quad , \quad y = \rho\sin\theta\sin\phi \quad , \quad z = \rho\cos\phi,$$

we have

$$\iiint\limits_R f(x,y,z)\,dx\,dy\,dz = \iiint\limits_S f(\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi.$$

Path integrals

The length of a curve $C = \{(x, f(x)) : a \leq x \leq b\}$ in $\mathbb{R}^2$ is given by

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\;dx.$$

If C is given in parametric form by $\{\mathbf{r}(t) | a \leq t \leq b\}$, then the length is given by

$$L = \int_a^b \left|\frac{d\mathbf{r}}{dt}\right|\;dt.$$

The path integral of a function f over a path $C = \{\mathbf{r}(t) | a \leq t \leq b\}$ is given by

$$\int\limits_C f\,ds = \int\limits_C f(\mathbf{r}(t))\,|\dot{\mathbf{r}}(t)|\,dt.$$

In three dimensions along a curve C parameterised by $\mathbf{r}(t) = (x(t), y(t))$ for $t \in [a,b]$ we have the path integral w.r.t. arc length s:

$$\int\limits_C f(x,y)\,ds = \int_a^b f(x(t),y(t)) \cdot \sqrt{(x'(t))^2 + (y'(t))^2}\,dt,$$

and the path interals w.r.t. the $x-$axis and $y-$axis:

$$\int\limits_C f(x,y)\,dx = \int_a^b f(x(t),y(t)) \cdot x'(t)\,dt \quad , \quad \int\limits_C f(x,y)\,dy = \int_a^b f(x(t),y(t)) \cdot y'(t)\,dt.$$

Surface areas

The surface area of a surface S be the surface given by a continuously differentiable parametrisation $S = \{\mathbf{S}(u,v) | \ (u,v) \in D\}$ for some region D in the $(u,v)-$plane is given by

$$\iint_D \left| \frac{\partial \mathbf{S}}{\partial u} \times \frac{\partial \mathbf{S}}{\partial v} \right| du\, dv = \iint_D |\mathbf{N}(u,v)|\, du\, dv.$$

Differential equations

First-order linear differential equation

1. Write the linear first-order differential equation in standard form $\dfrac{dy}{dx} + f(x)y = g(x)$.

2. Find the integrating factor $I(x) = \exp\left(\int f(x)\, dx\right)$, omitting the integration constant.

3. Find $\int I(x)g(x)\, dx$, omitting the integration constant.

4. The general solution is then $y(x) = \dfrac{1}{I(x)} \int I(x)g(x)\, dx + \dfrac{C}{I(x)}$.

Second-order linear homogeneous differential equation with constant coefficients

General form $\dfrac{d^2y}{dx^2} + p\dfrac{dy}{dx} + qy = 0$ with characteristic equation $m^2 + pm + q = 0$.

1. If the roots m_1 and m_2 are real and unequal, then the general solution is

$$y(x) = C_1 e^{m_1 x} + C_2 e^{m_2 x}.$$

2. If the roots are complex conjugates $a \pm ib$, then the general solution is

$$y(x) = C_1 e^{ax} \cos(bx) + C_2 e^{ax} \sin(bx).$$

3. If there is a single (or repeated) root m, then the general solution is

$$y(x) = C_1 e^{mx} + C_2 x e^{mx}.$$

Differentiation and integration formulas

$\dfrac{dy}{dx}$	y	$\displaystyle\int y\,dx$
0	a (constant)	$ax + C$
nx^{n-1}	x^n $(n \neq -1)$	$\dfrac{x^{n+1}}{n+1} + C$
$-\dfrac{1}{x^2}$ or $-x^{-2}$	$\dfrac{1}{x}$ or x^{-1}	$\ln x + C$
e^x	e^x	$e^x + C$
$\dfrac{1}{x}$	$\ln x$	$x \ln x - x + C$
$\cos x$	$\sin x$	$-\cos x + C$
$-\sin x$	$\cos x$	$\sin x + C$
$\sec^2 x$	$\tan x$	$\ln(\sec x) + C$
$-\cot x \operatorname{cosec} x$	$\operatorname{cosec} x$	$\ln(\operatorname{cosec} x - \cot x) + C$
$\tan x \sec x$	$\sec x$	$\ln(\sec x + \tan x) + C$
$-\operatorname{cosec}^2 x$	$\cot x$	$\ln(\sin x) + C$
$\dfrac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x$	$x \sin^{-1} x + \sqrt{1-x^2} + C$
$-\dfrac{1}{\sqrt{1-x^2}}$	$\cos^{-1} x$	$x \cos^{-1} x - \sqrt{1-x^2} + C$
$\dfrac{1}{1+x^2}$	$\tan^{-1} x$	$x \tan^{-1} x - \dfrac{1}{2} \ln\left(1 + x^2\right) + C$