GENG5503-Exam Appendix

Continuous State-Space Equations

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t)\\ y(t) &= Cx(t) + Du(t)\end{aligned}$$

Transfer Function

$$G(s) = C(sI - A)^{-1}B + D$$

$$G(s) = \frac{b_1 s^2 + b_2 s + b_3}{s^3 + a_1 s^2 + a_2 s + a_3}$$

Controller canonical form

$$\begin{aligned}\dot{x}_c &= A_c x_c + b_c u\\ y &= c_c x_c\end{aligned}$$

$$A_c = \begin{bmatrix} -a_1 & -a_2 & -a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},\ b_c = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ \text{and } c_c = \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix}$$

Observer canonical form

$$\begin{aligned}\dot{x}_o &= A_o x_o + b_o u\\ y &= c_o x_o\end{aligned}$$

$$A_o = \begin{bmatrix} -a_1 & 1 & 0 \\ -a_2 & 0 & 1 \\ -a_3 & 0 & 0 \end{bmatrix},\ b_o = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix},\ \text{and } c_o = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

Controllability canonical form

$$
\begin{aligned}
\dot{x}_{co} &= A_{co}x_{co} + b_{co}u \\
y &= c_{co}x_{co}
\end{aligned}
$$

$$
\begin{aligned}
A_{co} &= \begin{bmatrix} 0 & 0 & -a_3 \\ 1 & 0 & -a_2 \\ 0 & 1 & -a_1 \end{bmatrix}, \; b_{co} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\
c_{co} &= \begin{bmatrix} \beta_1 & \beta_2 & \beta_3 \end{bmatrix} = \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix} \begin{bmatrix} 1 & a_1 & a_2 \\ 0 & 1 & a_1 \\ 0 & 0 & 1 \end{bmatrix}^{-1}
\end{aligned}
$$

Observability canonical form

$$
\begin{aligned}
\dot{x}_{ob} &= A_{ob}x_{ob} + b_{ob}u \\
y &= c_{ob}x_{ob}
\end{aligned}
$$

$$
\begin{aligned}
A_{ob} &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix}, \; c_{ob} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \\
b_{ob} &= \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ a_1 & 1 & 0 \\ a_2 & a_1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}
\end{aligned}
$$

Diagonal form

$$
\begin{aligned}
\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix} &= \begin{bmatrix} \lambda_1 & 0 & & 0 \\ 0 & \lambda_2 & & 0 \\ & & \ddots & \\ 0 & 0 & & \lambda_n \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} u(t) \\
y(t) &= \begin{bmatrix} c_1 & c_2 & \dots & c_n \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}
\end{aligned}
$$

Markov Parameters

$$m_i = cA^{i-1}b, \quad i = 1, 2, \ldots$$

State-Transition Matrix

$$e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!} = \mathcal{L}^{-1}\left\{(sI - A)^{-1}\right\}$$

Solution of non-homogeneous state equation

$$x(t) = e^{At}x(0) + e^{At}\int_0^t e^{-A\tau}Bu(\tau)d\tau$$

$$y(t) = Cx(t) = Ce^{At}x(0) + Ce^{At}\int_0^t e^{-A\tau}Bu(\tau)d\tau$$

Similarity Transformation

$$\begin{aligned} \dot{x} &= Ax + bu \\ y &= cx \end{aligned}$$

$$\hat{x} = Px \quad \text{or} \quad x = P^{-1}\hat{x}$$

$$\begin{aligned} \dot{\hat{x}} &= \hat{A}\hat{x} + \hat{b}u \\ y &= \hat{c}\hat{x} \end{aligned}$$

$$\begin{aligned} \hat{A} &= PAP^{-1} \\ \hat{b} &= Pb \\ \hat{c} &= cP^{-1} \end{aligned}$$

$$x = P\hat{x}$$

$$
\begin{aligned}
\dot{\hat{x}} &= \hat{A}\hat{x} + \hat{b}u \\
y &= \hat{c}\hat{x}
\end{aligned}
$$

$$
\begin{aligned}
\hat{A} &= P^{-1}AP \\
\hat{b} &= P^{-1}b \\
\hat{c} &= cP
\end{aligned}
$$

Discrete State-Space Equations

$$
\begin{aligned}
x(k+1) &= Gx(k) + Hu(k) \\
y(k) &= Cx(k) + Du(k)
\end{aligned}
$$

Transfer Function

$$
G(z) = C(xI - G)^{-1}H + D
$$

State-transition matrix

$$
\phi(k) = G^k
$$

$$
x(k) = \phi(k)x(0) + \sum_{j=0}^{k-1} \phi(k-j-1)Hu(j)
$$

$$
y(k) = C\phi(k)x(0) + C\sum_{j=0}^{k-1} \phi(k-j-1)Hu(j) + Du(k)
$$

Computation of state-transition matrix

$$
G^k = \phi(k) = Z^{-1}\left[(zI - G)^{-1}z\right]
$$

Discretization

$$x((k+1)T) = G(T)x(kT) + H(T)u(kT)$$

where

$$G(T) = e^{AT} = I + AT + \frac{1}{2!}A^2T^2 + \dots$$

$$H(T) = \left(\int_0^T e^{At}dt\right) B$$

$$y(kT) = Cx(kT) + Du(kT)$$

Linearization Equations

- Consider a general nonlinear model with *n* state variables, *m* input variables, and *r* output variables

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, \cdots, x_n, u_1, \cdots, u_m) \\ &\vdots \\ \dot{x}_n &= f_n(x_1, \cdots, x_n, u_1, \cdots, u_m) \\ y_1 &= g_1(x_1, \cdots, x_n, u_1, \cdots, u_m) \\ &\vdots \\ y_r &= g_r(x_1, \cdots, x_n, u_1, \cdots, u_m) \end{aligned}$$

Vector notation:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$$
$$\mathbf{y} = \mathbf{g}(\mathbf{x}, \mathbf{u})$$

- Elements of the linearization matrices

$$A_{ij} = \left.\frac{\partial f_i}{\partial x_j}\right|_{\mathbf{x}_s, \mathbf{u}_s} \qquad B_{ij} = \left.\frac{\partial f_i}{\partial u_j}\right|_{\mathbf{x}_s, \mathbf{u}_s}$$

$$C_{ij} = \left.\frac{\partial g_i}{\partial x_j}\right|_{\mathbf{x}_s, \mathbf{u}_s} \qquad D_{ij} = \left.\frac{\partial g_i}{\partial u_j}\right|_{\mathbf{x}_s, \mathbf{u}_s}$$

State-space form:

$$\dot{\overline{\mathbf{x}}} = \mathbf{A}\,\overline{\mathbf{x}} + \mathbf{B}\,\overline{\mathbf{u}}$$
$$\overline{\mathbf{y}} = \mathbf{C}\,\overline{\mathbf{x}} + \mathbf{D}\,\overline{\mathbf{u}}$$

or

$$\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \mathbf{B}\,\mathbf{u}$$
$$\mathbf{y} = \mathbf{C}\,\mathbf{x} + \mathbf{D}\,\mathbf{u}$$

(The “overbar” is usually dropped)

Taylor’s Series

$$f(x) = f(x_s) + \left.\frac{\partial f}{\partial x}\right|_{x_s}(x - x_s) + \frac{1}{2}\left.\frac{\partial^2 f}{\partial x^2}\right|_{x_s}(x - x_s)^2 + \text{high order terms}$$

Controllability and Observability

$$\mathcal{C} = \left[\begin{array}{cccc} B & AB & \dots & A^{n-1}B \end{array}\right] \qquad \mathcal{O} = \left[\begin{array}{c} C \\ CA \\ \vdots \\ CA^{n-1} \end{array}\right]$$

Hankel matrix

$$M[1, n-1] = \left[\begin{array}{cccc} h_1 & h_2 & \dots & h_n \\ h_2 & h_2 & \dots & h_{n+1} \\ \vdots & & & \vdots \\ h_n & h_{n+1} & \dots & h_{2n-1} \end{array}\right] = \mathcal{O}\mathcal{C}$$

Non-controllable decomposition

$$\begin{aligned} \bar{A} &= T^{-1}AT = \left[\begin{array}{cc} \bar{A}_c & \bar{A}_{c\bar{c}} \\ 0 & \bar{A}_{\bar{c}} \end{array}\right] \\ \bar{b} &= T^{-1}b = \left[\begin{array}{c} \bar{b}_c \\ 0 \end{array}\right] \\ \bar{c} &= cT = \left[\begin{array}{cc} \bar{c}_c & \bar{c}_{\bar{c}} \end{array}\right] \end{aligned}$$

Non-observable decomposition

$$\begin{aligned} \bar{A} &= TAT^{-1} = \left[\begin{array}{cc} \bar{A}_o & 0 \\ \bar{A}_{\bar{o}o} & \bar{A}_{\bar{o}} \end{array}\right] \\ \bar{b} &= Tb = \left[\begin{array}{c} \bar{b}_o \\ \bar{b}_{\bar{o}} \end{array}\right] \\ \bar{c} &= cT^{-1} = \left[\begin{array}{cc} \bar{c}_o & 0 \end{array}\right] \end{aligned}$$

Lyapunov Stability

Sylvester's Criterion

A quadratic form $v(x) = x^T P x$ is positive definite if all the successive principal minors of P are positive; that is

$$p_{11} > 0, \qquad \begin{vmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{vmatrix} > 0, \ \ldots, \begin{vmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{12} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & & \vdots \\ p_{1n} & p_{2n} & \cdots & p_{nn} \end{vmatrix} > 0$$

$$A^T P + PA = -Q$$

Solution unique if and only if

$$\lambda_i + \lambda_j \neq 0, \forall i, j$$

$$G^T P G - P = -Q$$

Solution unique if and only if

$$\lambda_i \left[G \right] . \lambda_j \left[G^T \right] \neq 1 \quad \forall i, j$$

Routh Table

Write the polynomial in s in the following form:

$$a_0 s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n = 0$$

where the coefficients are real numbers.

$$\begin{array}{lllllll}
s^n & a_0 & a_2 & a_4 & a_6 & . & . \\
s^{n-1} & a_1 & a_3 & a_5 & a_7 & . & . \\
s^{n-2} & b_1 & b_2 & b_3 & b_4 & . & . \\
s^{n-3} & c_1 & c_2 & c_3 & c_4 & . & . \\
\vdots & \vdots & \vdots & & & & \\
s^2 & d_1 & d_2 & & & & \\
s^1 & e_1 & & & & & \\
s^0 & f_1 & & & & &
\end{array}$$

where the coefficients b_1, b_2, and so on are computed as follows:

$$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1}$$

$$
\begin{aligned}
b_2 &= \frac{a_1a_4 - a_0a_5}{a_1} \\
\vdots &\quad \vdots
\end{aligned}
$$

The coefficients c_1, c_2, and so on are computed as follows:

$$
\begin{aligned}
c_1 &= \frac{b_1a_3 - a_1b_2}{b_1} \\
c_2 &= \frac{b_1a_5 - a_1b_3}{b_1} \\
\vdots &\quad \vdots
\end{aligned}
$$

Bass Gura's formula for Controller Design

$$
k = \begin{bmatrix} (\alpha_1 - a_1) & (\alpha_2 - a_2) & \ldots & (\alpha_n - a_n) \end{bmatrix} W^{-1}M^{-1}
$$

$$
\begin{aligned}
M &= \begin{bmatrix} b \; Ab \; \ldots \; A^{n-1}b \end{bmatrix} \\
W &= \begin{bmatrix} 1 & a_1 & a_2 & \ldots & a_{n-1} \\ 0 & 1 & a_1 & \ldots & a_{n-2} \\ \vdots & . & . & . & \vdots \\ 0 & 0 & 0 & & 1 \end{bmatrix}
\end{aligned}
$$

where a_i's are the coefficient of the characteristic polynomial

$$
|sI - A| = s^n + a_1s^{n-1} + \ldots + a_{n-1}s + a_n
$$

If the desired pole locations are μ_1, μ_2, $\ldots$, μ_n. Then the desired characteristic equation becomes

$$
\prod_{i=1}^{n}(s - \mu_i) = s^n + \alpha_1s^{n-1} + \ldots + \alpha_{n-1}s + \alpha_n = \alpha(s)
$$

Ackermann's formula for Controller Design

$$
k = \begin{bmatrix} 0 & 0 & \ldots & 1 \end{bmatrix} \begin{bmatrix} b & Ab & \ldots & A^{n-1}b \end{bmatrix}^{-1} \alpha(A)
$$

Direct Method for Controller Design

$$|sI - A + bk| = \prod_{i=1}^{n}(s - \mu_i) = \alpha(s)$$

Bass Gura's Formula for Observer Design

$$l = (W\mathcal{O})^{-1} \begin{bmatrix} \alpha_1 - a_1 \\ \alpha_2 - a_2 \\ \vdots \\ \alpha_n - a_n \end{bmatrix}$$

where

$$\mathcal{O} = \begin{bmatrix} c \\ cA \\ \vdots \\ cA^{n-1} \end{bmatrix} \quad and \quad W = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ a_1 & 1 & 0 & 0 & 0 \\ \vdots & . & . & . & \vdots \\ a_{n-2} & a_{n-3} & \dots & 1 & 0 \\ a_{n-1} & a_{n-2} & \dots & a_1 & 1 \end{bmatrix}$$

$$|sI - A| = s^n + a_1 s^{n-1} + \ldots + a_{n-1}s + a_n$$

If the desired pole locations are $\mu_1,\ \mu_2,\ \ldots,\ \mu_n$. Then the desired characteristic equation becomes

$$(s-\mu_1)(s-\mu_2)\ldots(s-\mu_n) = s^n + \alpha_1 s^{n-1} \ldots \alpha_{n-1}s + \alpha_n = \alpha(s)$$

Ackermann's Formula for Observer Design

$$l = \alpha(A) \begin{bmatrix} c \\ cA \\ \vdots \\ cA^{n-1} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

Direct Method for Observer Design

$$|sI - A + lc| = \prod_{i=1}^{n}(s-\mu_i) = \alpha(s)$$

Reduced Order Observer Equations

$$\begin{bmatrix} \dot{x}_m \\ \dot{x}_u \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_m \\ x_u \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u$$

$$y = \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} x_m \\ x_u \end{bmatrix}$$

$$\dot{z} = Dz + Fy + Gu$$

$$\hat{x}_u = z + Ly$$

where

$$D = A_{22} - LA_{12}$$

$$F = DL + A_{21} - LA_{11}$$

$$G = B_2 - LB_1$$

ZOH Transfer Function

$$G_h(s) = \frac{1-e^{-Ts}}{s}$$

$$X(z) = \mathcal{Z}\left\{X(s)\right\} = (1-z^{-1})\mathcal{Z}\left\{\frac{G(s)}{s}\right\}$$

Impulse Function

$$\int_{-\epsilon}^{\epsilon} \delta(t)dt = 1 \text{ where } \epsilon \text{ is a small positive number}$$

$$\int_{-\infty}^{\infty} \phi(t)\delta(t)dt = \phi(0) \text{ where } \phi(t) \text{ is a regular continuous function}$$

Z-Transform Shifting Theorems

$$Z_U\{x(k-k_0)\} = x(-k_0) + x(-k_0+1)z^{-1} + \ldots + x(-1)z^{-k_0+1} + z^{-k_0}X_U(z)$$

$$Z_U\{x(k+k_0)\} = z^{k_0}X_U(z) - x(0)z^{k_0} - x(1)z^{k_0-1} - \ldots - x(k_0-1)z$$

Laplace Transforms

$f(t)$	$F(s)$
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
$e^{-at}u(t)$	$\frac{1}{s+a}$
$te^{-at}u(t)$	$\frac{1}{(s+a)^2}$
$t^n e^{-at}u(t)$	$\frac{n!}{(s+a)^{n+1}}$

Eigenvalues of a $n \times n$ matrix A are the roots of the characteristic equation:

$$|\lambda I - A| = 0$$

Eigenvectors of a $n \times n$ matrix: Any non zero vector x_i such that

$$Ax_i = \lambda x_i$$

is said to be an eigenvector associated with an eigenvalue λ_i of A where A is a $n \times n$ matrix.

The eigenvectors are obtained by solving homogeneous algebraic equations of the form:

$$[A - \lambda_i I]\, x_i = 0$$

Determinants

$$det(A) = |A| = \sum_{i=1}^{n} a_{ij}C_{ij}$$

(cofactor expansion along the j-th column)

$$det(A) = |A| = \sum_{j=1}^{n} a_{ij}C_{ij}$$

(cofactor expansion along the i-th row)

Minors and Cofactors: If A is a square matrix, then the **minor of entry** a_{ij} is denoted by M_{ij} and is defined to be determinant of the submatrix that remains after the i-th row and j-th column are deleted from A. The number $(-1)^{i+j}M_{ij}$ is denoted by C_{ij} and is called the **cofactor of entry** a_{ij}.

Matrix of Cofactors and Adjoint

If A is any $n \times n$ matrix and C_{ij} is the cofactor of a_{ij}, then the matrix

$$\begin{bmatrix} C_{11} & C_{12} & \dots & C_{1n} \\ C_{21} & C_{22} & \dots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \dots & C_{nn} \end{bmatrix}$$

is called the matrix of cofactors of A. The transpose of this matrix is called the adjoint of A and is denoted by adj(A).

Inverse of a Matrix

If A is an $n \times n$ matrix, then its inverse is given by the following formula:

$$A^{-1} = \frac{1}{det(A)} adj(A)$$

Cayley-Hamilton Theorem states that every square matrix satisfies its own characteristic equation. If

$$|\lambda I - A| = \lambda^n + a_1\lambda^{n-1} + \ldots + a_{n-1}\lambda + a_n = 0$$

is the characteristic equation then from Cayley Hamilton theorem we have

$$A^n + a_1A^{n-1} + \ldots + a_{n-1}A + a_nI = 0$$

Rank of a matrix

A matrix A is called of rank m if the maximum number of linearly independent rows (or columns) is m. Hence, if there exists a $m \times m$ submatrix M of A such that $|M| \neq 0$, and the determinant of every $r \times r$ submatrix (where $r \geq m + 1$) of A is zero, then the rank of A is m.

Table of Laplace and Z-transforms

	$X(s)$	$x(t)$	$x(kT)$ or $x(k)$	$X(z)$
1.	–	–	Kronecker delta $\delta_0(k)$ 1 $k = 0$ 0 $k \neq 0$	1
2.	–	–	$\delta_0(n-k)$ 1 $n = k$ 0 $n \neq k$	z^{-k}
3.	$\frac{1}{s}$	$1(t)$	$1(k)$	$\frac{1}{1-z^{-1}}$
4.	$\frac{1}{s+a}$	e^{-at}	e^{-akT}	$\frac{1}{1-e^{-aT}z^{-1}}$
5.	$\frac{1}{s^2}$	t	kT	$\frac{Tz^{-1}}{(1-z^{-1})^2}$
6.	$\frac{2}{s^3}$	t^2	$(kT)^2$	$\frac{T^2z^{-1}(1+z^{-1})}{(1-z^{-1})^3}$
7.	$\frac{6}{s^4}$	t^3	$(kT)^3$	$\frac{T^3z^{-1}(1+4z^{-1}+z^{-2})}{(1-z^{-1})^4}$
8.	$\frac{a}{s(s+a)}$	$1-e^{-at}$	$1-e^{-akT}$	$\frac{(1-e^{-aT})z^{-1}}{(1-z^{-1})(1-e^{-aT}z^{-1})}$
9.	$\frac{b-a}{(s+a)(s+b)}$	$e^{-at}-e^{-bt}$	$e^{-akT}-e^{-bkT}$	$\frac{(e^{-aT}-e^{-bT})z^{-1}}{(1-e^{-aT}z^{-1})(1-e^{-bT}z^{-1})}$
10.	$\frac{1}{(s+a)^2}$	te^{-at}	kTe^{-akT}	$\frac{Te^{-aT}z^{-1}}{(1-e^{-aT}z^{-1})^2}$
11.	$\frac{s}{(s+a)^2}$	$(1-at)e^{-at}$	$(1-akT)e^{-akT}$	$\frac{1-(1+aT)e^{-aT}z^{-1}}{(1-e^{-aT}z^{-1})^2}$
12.	$\frac{2}{(s+a)^3}$	t^2e^{-at}	$(kT)^2e^{-akT}$	$\frac{T^2e^{-aT}(1+e^{-aT}z^{-1})z^{-1}}{(1-e^{-aT}z^{-1})^3}$
13.	$\frac{a^2}{s^2(s+a)}$	$at-1+e^{-at}$	$akT-1+e^{-akT}$	$\frac{[(aT-1+e^{-aT})+(1-e^{-aT}-aTe^{-aT})z^{-1}]z^{-1}}{(1-z^{-1})^2(1-e^{-aT}z^{-1})}$
14.	$\frac{\omega}{s^2+\omega^2}$	$\sin \omega t$	$\sin \omega kT$	$\frac{z^{-1}\sin \omega T}{1-2z^{-1}\cos \omega T+z^{-2}}$
15.	$\frac{s}{s^2+\omega^2}$	$\cos \omega t$	$\cos \omega kT$	$\frac{1-z^{-1}\cos \omega T}{1-2z^{-1}\cos \omega T+z^{-2}}$
16.	$\frac{\omega}{(s+a)^2+\omega^2}$	$e^{-at}\sin \omega t$	$e^{-akT}\sin \omega kT$	$\frac{e^{-aT}z^{-1}\sin \omega T}{1-2e^{-aT}z^{-1}\cos \omega T+e^{-2aT}z^{-2}}$
17.	$\frac{s+a}{(s+a)^2+\omega^2}$	$e^{-at}\cos \omega t$	$e^{-akT}\cos \omega kT$	$\frac{1-e^{-aT}z^{-1}\cos \omega T}{1-2e^{-aT}z^{-1}\cos \omega T+e^{-2aT}z^{-2}}$
18.	–	–	a^k	$\frac{1}{1-az^{-1}}$
19.	–	–	a^{k-1} $k = 1, 2, 3, \ldots$	$\frac{z^{-1}}{1-az^{-1}}$
20.	–	–	ka^{k-1}	$\frac{z^{-1}}{(1-az^{-1})^2}$
21.	–	–	k^2a^{k-1}	$\frac{z^{-1}(1+az^{-1})}{(1-az^{-1})^3}$
22.	–	–	k^3a^{k-1}	$\frac{z^{-1}(1+4az^{-1}+a^2z^{-2})}{(1-az^{-1})^4}$
23.	–	–	k^4a^{k-1}	$\frac{z^{-1}(1+11az^{-1}+11a^2z^{-2}+a^3z^{-3})}{(1-az^{-1})^5}$
24.	–	–	$a^k \cos k\pi$	$\frac{1}{1+az^{-1}}$

$x(t) = 0$ for $t < 0$
$x(kT) = x(k) = 0$ for $k < 0$
Unless otherwise noted, $k = 0, 1, 2, 3, \ldots$