

CS 231

Geometric transformations

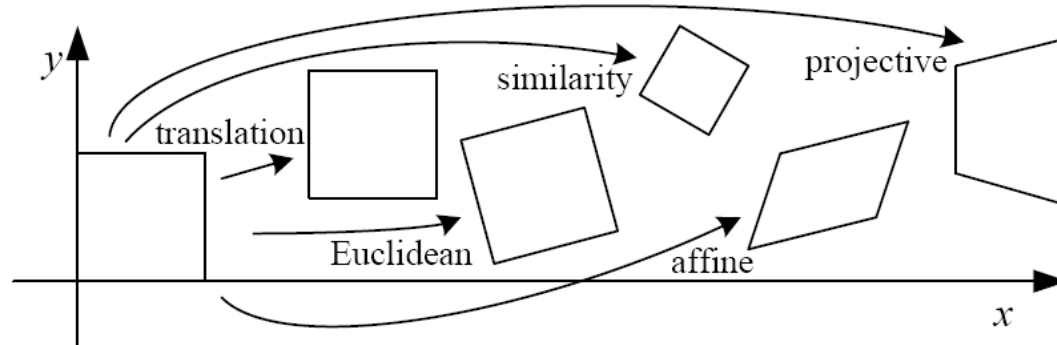
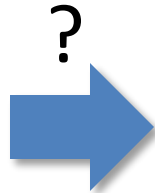


Image alignment



Why don't these image line up exactly?

What is the geometric relationship between these two images?



Answer: Similarity transformation (translation, rotation, uniform scale)

What is the geometric relationship between these two images?



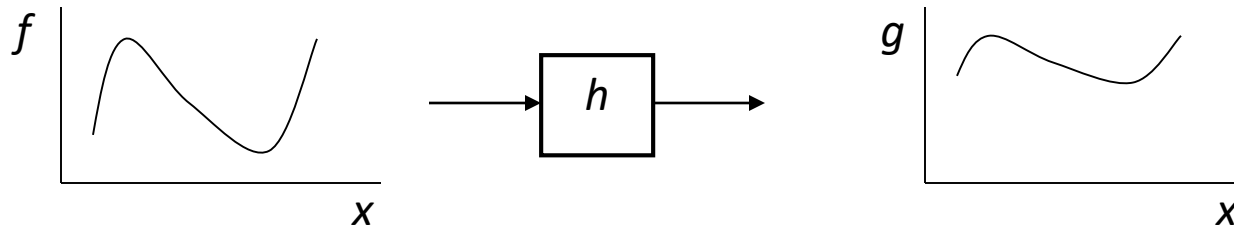
What is the geometric relationship between these two images?



Very important for creating mosaics!

Image Warping

- image filtering: change *range* of image
 - $g(x) = h(f(x))$



- image warping: change *domain* of image
 - $g(x) = f(h(x))$

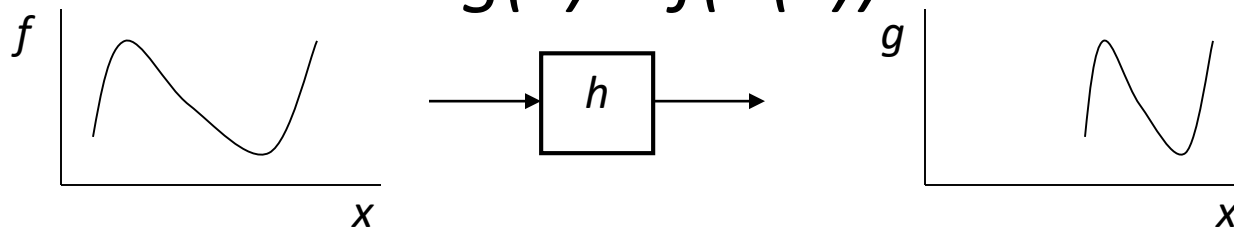
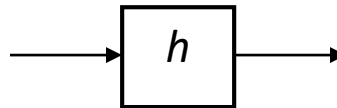


Image Warping

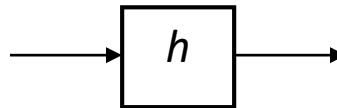
- image filtering: change *range* of image

- $g(x) = h(f(x))$



- image warping: change *domain* of image

- $g(x) = f(h(x))$



Parametric (global) warping

- Examples of parametric warps:



translation



rotation

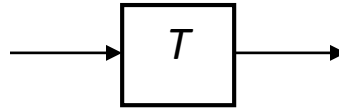


aspect

Parametric (global) warping



$\mathbf{p} = (x, y)$



$\mathbf{p}' = (x', y')$

- Transformation T is a coordinate-changing machine:
$$\mathbf{p}' = T(\mathbf{p})$$
- What does it mean that T is global?
 - Is the same for any point \mathbf{p}
 - can be described by just a few numbers (parameters)
- Let's consider *linear* xforms (can be represented by a 2D matrix):

$$\mathbf{p}' = \mathbf{T}\mathbf{p} \quad \begin{bmatrix} x' \\ y' \end{bmatrix} = \mathbf{T} \begin{bmatrix} x \\ y \end{bmatrix}$$

Common linear transformations

- Uniform scaling by s :



(0,0) ●



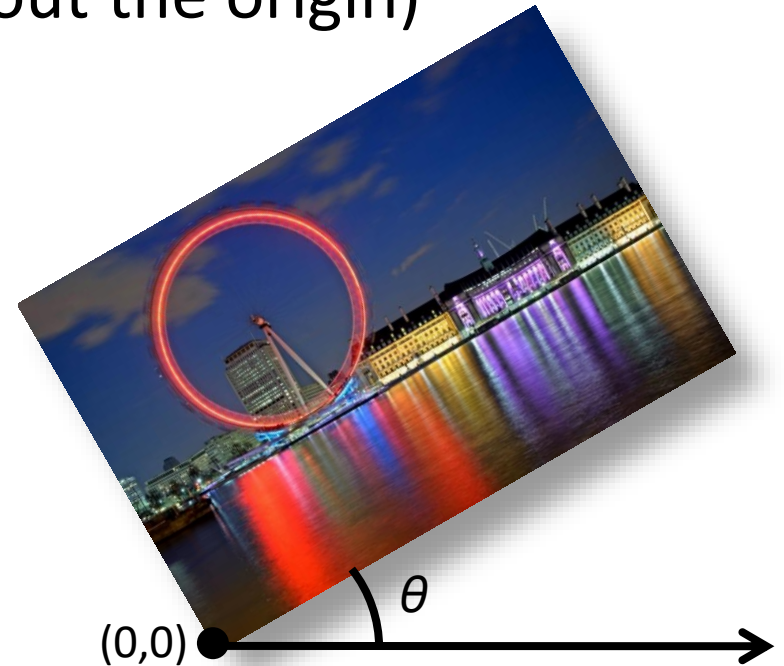
(0,0) ●

$$\mathbf{S} = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix}$$

What is the inverse?

Common linear transformations

- Rotation by angle θ (about the origin)



$$\mathbf{R} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

What is the inverse?

For rotations:

$$\mathbf{R}^{-1} = \mathbf{R}^T$$

2x2 Matrices

- What types of transformations can be represented with a 2x2 matrix?

2D mirror about Y axis?

$$\begin{aligned}x' &= -x \\ y' &= y\end{aligned}\quad \mathbf{T} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

2D mirror across line $y = x$?

$$\begin{aligned}x' &= y \\ y' &= x\end{aligned}\quad \mathbf{T} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

2x2 Matrices

- What types of transformations can be represented with a 2x2 matrix?

2D Translation?

$$x' = x + t_x \quad \text{NO!}$$

$$y' = y + t_y$$

Translation is not a linear operation on 2D coordinates

All 2D Linear Transformations

- Linear transformations are combinations of ...

- Scale,
- Rotation,
- Shear, and
- Mirror

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

- Properties of linear transformations:

- Origin maps to origin
- Lines map to lines
- Parallel lines remain parallel
- Ratios are preserved
- Closed under composition

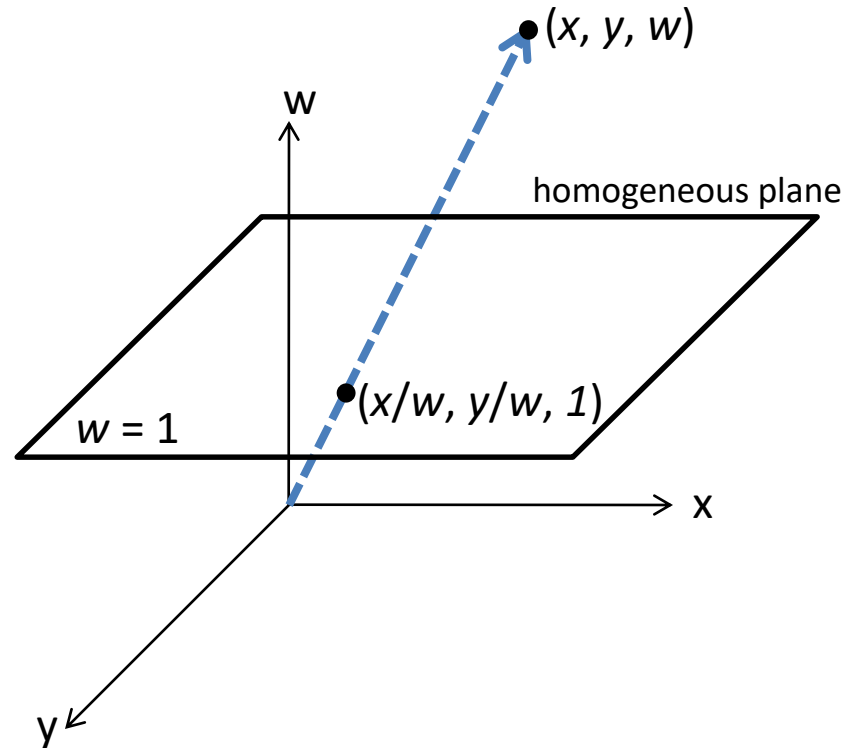
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} \begin{bmatrix} i & j \\ k & l \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Homogeneous coordinates

Trick: add one more coordinate:

$$(x, y) \Rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

homogeneous image
coordinates



Converting *from* homogeneous coordinates

$$\begin{bmatrix} x \\ y \\ w \end{bmatrix} \Rightarrow (x/w, y/w)$$

Translation

- Solution: homogeneous coordinates to the rescue

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x + t_x \\ y + t_y \\ 1 \end{bmatrix}$$

Affine transformations

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$$



any transformation with
last row $[0 \ 0 \ 1]$ we call an
affine transformation

$$\begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix}$$

Basic affine transformations

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Translate

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Scale

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

2D *in-plane* rotation

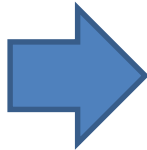
$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & sh_x & 0 \\ sh_y & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Shear

Affine Transformations

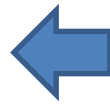
- Affine transformations are combinations of ...
 - Linear transformations, and
 - Translations
$$\begin{bmatrix} x' \\ y' \\ w \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$
- Properties of affine transformations:
 - Origin does not necessarily map to origin
 - Lines map to lines
 - Parallel lines remain parallel
 - Ratios are preserved
 - Closed under composition

Is this an affine transformation?



Where do we go from here?

$$\begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix}$$



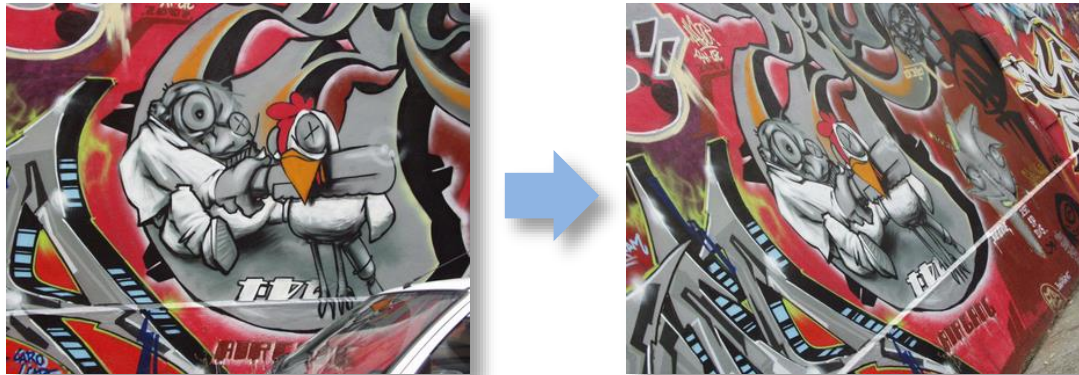
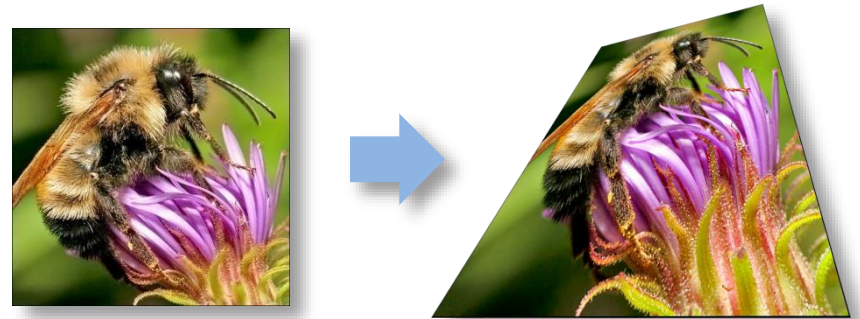
what happens when we
mess with this row?

affine transformation

Projective Transformations aka Homographies aka Planar Perspective Maps

$$\mathbf{H} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & 1 \end{bmatrix}$$

Called a *homography*
(or *planar perspective map*)



Homographies

- Example on board

Image warping with homographies

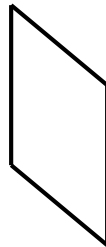
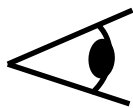
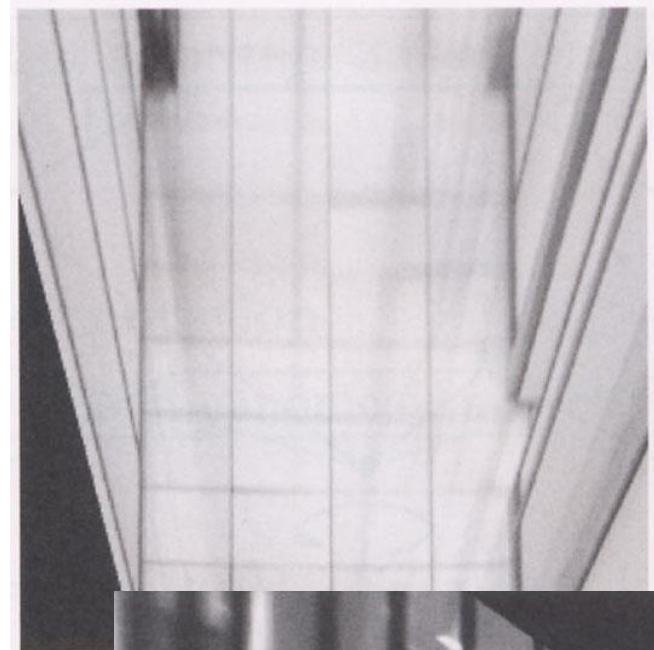
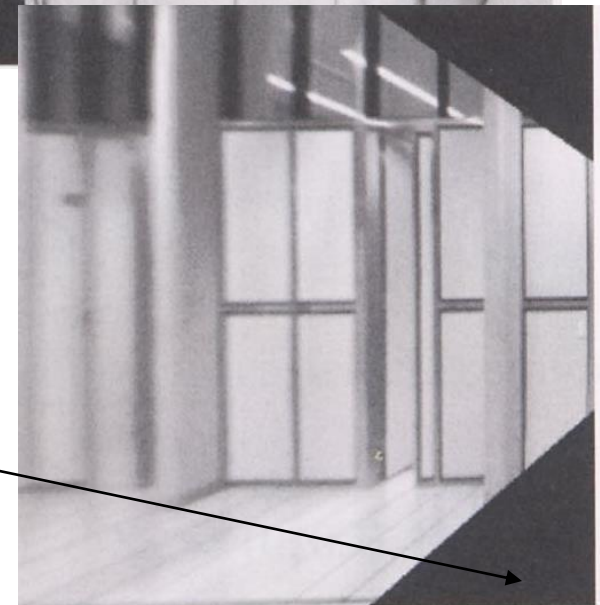


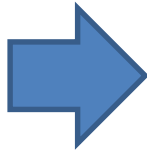
image plane in front



black area
where no pixel
maps to



Homographies



Projective Transformations

- Projective transformations ...

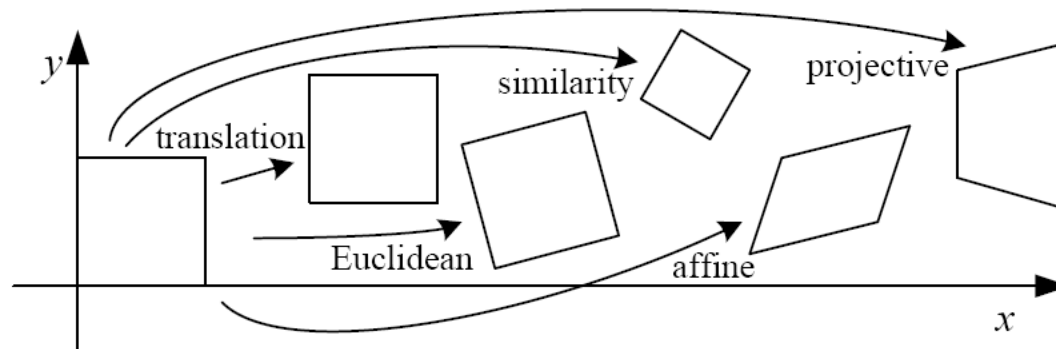
- Affine transformations, and
- Projective warps






$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

- Properties of projective transformations:

- Origin does not necessarily map to origin
- Lines map to lines
- Parallel lines do not necessarily remain parallel
- Ratios are not preserved
- Closed under composition

2D image transformations



Name	Matrix	# D.O.F.	Preserves:	Icon
translation	$\begin{bmatrix} \mathbf{I} & \mathbf{t} \end{bmatrix}_{2 \times 3}$	2	orientation + ...	
rigid (Euclidean)	$\begin{bmatrix} \mathbf{R} & \mathbf{t} \end{bmatrix}_{2 \times 3}$	3	lengths + ...	
similarity	$\begin{bmatrix} s\mathbf{R} & \mathbf{t} \end{bmatrix}_{2 \times 3}$	4	angles + ...	
affine	$\begin{bmatrix} \mathbf{A} \end{bmatrix}_{2 \times 3}$	6	parallelism + ...	
projective	$\begin{bmatrix} \tilde{\mathbf{H}} \end{bmatrix}_{3 \times 3}$	8	straight lines	

These transformations are a nested set of groups

- Closed under composition and inverse is a member