Algebraic geometry

Simplified

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Why do we care?

- The study of algebraic curves, and their higher-dimensional analogues- varieties finds applications in cryptography (that's computer security), computer vision and error-correcting codes (In case you like fancy names, string theory also has something to do with it)

What's a plane?

- The real plane $A^2_{\mathbb{R}}$ is the set of ordered pairs of real numbers (one of them for horizontal coordinate, another for vertical).

What's an algebraic planar curve?

- An algebraic planar curve is the zero-set of some polynomial in two variables.
- Line is given by $ax + by + c = 0$ with $a, b, c \in \mathbb{R}$ being some real constants.
- Circle is given by $x^2 + y^2 - 1 = 0$.
- Parabola is given by $y - x^2 = 0$.

What do we study?

- Algebraic curves have several aspects (sometimes interrelated):
- topological (that is study of varieties up to continuous deformation- doughnut and cup are topologically equivalent)
- algebraic (basically ring-theoretic properties of the algebra of functions on varieties)
- arithmetic (existence of rational or integer points on varieties)
- categorical (associating categorical structures of different origin to variety).

Topological aspects

- Algebraicity hypothesis is quite strong one: in particular, all complex algebraic varieties are orientable as real manifolds. We study branched covers, bundles, homotopy groups. Remarkable result is:
- Etalé fundamental group unites the theory of branched covers and Galois groups

Algebraic aspects

- We use a lot of different tools (mainly of cohomological character) to study geometric properties of varieties. Remarkable result:
- Zariski's main theorem: if x is a normal point of a variety then it is analytically normal; in other words the completion of the local ring at x is a normal integral domain.

Arithmetic aspects

- Geometric properties of variety sometimes reflect its arithmetic structure. Remarkable result:
- Faltings' theorem:any smooth curve of genus $g > 1$ over $\mathbb{Q}$ has only finitely many rational points.

Categorical aspects

- Grothendieck school used category theory rather heavily to get abstract theory of algebraic geometry. For example, a scheme can be recovered from the category of coherent sheaves on it. This way, algebraic geometry could be extended to more general objects (look up topoi theory).

For Further Reading

R. Hartshorne.
Algebraic geometry.
New York: Springer-Verlag, 1977.