What influences the productivity of methods for teaching 11th grade students mathematical concepts such as limit, continuity and differentiability in NIS PM Almaty?

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Introduction

The problem of teaching advanced mathematical concepts at the high school level has attracted attention of numerous researchers in social sciences (a comprehensive review of the topic can be found in Clements, Bishop,..., Leung (2013)). The transition from intuitive thinking to rigorous thinking plays pivotal role in the process of mathematical development of students. According to Arzarello, Michelli,..., Robutti (1988) and Frant & de Costra (2000), in the typical high school curriculum, students first encounter with rigorous thinking during the geometry course. Complete understanding of calculus requires transition to rigorous thinking. Unfortunately, this metamorphosis may sometimes be unsuccessful due to various factors, most notably: underdeveloped *concept images* in calculus (Juter, 2012), inadequate understanding of the topic by instructor (Viholainen, 2006). *Concept image* is defined as 'the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes' in Tall & Vinner (1981).

The present study is devoted to understanding whether the currently used methods in NIS PM Almaty are effective in teaching 11^{th} grade students the concepts of limit, continuity and differentiability and analyzing the reasons for their effectiveness. Student survey is used to objectively measure the effectiveness of the teaching methods; teacher interviewing is employed to find the possible causes for the observed effectiveness.

Literature review

The present study is primarily concerned with teaching of the following concepts: limit, continuity and differentiability. Since these concepts are taught almost exclusively during the calculus course, it is appropriate to consider several existing perspectives on high school calculus courses, particularly on issues of rigour and constructivism in calculus course (description of requirements from U.S. Advanced placement course perspective may be found in Bressoud (2010) though it will not be discussed here).

The first factor affecting the entire structure of the calculus course is the level of rigour employed (for a conceptual illustration refer to Diagram 1 in Appendix 1). In NIS PM Almaty, the level of rigour during 11^{th} grade is fairly low as the fundamental ideas of continuity and differentiability are explained in intuitive way, with no formal definitions or proofs. This corresponds to valid naturally conceptual/procedural link with weak concept images in terms of the classification outlined in Juter (2011). It is explained in Juter (2011) that potentially, this kind of link may lead to fragmentation and ambiguity in concepts' understanding. On the other hand, in Tall (2002) it is claimed that using formal proofs from the outset may lead to incomplete development of the mathematical intuition of students and misconception of what is mathematics at large.

The second question is that of the choice of an epistemological model for the course. Constructivism, as defined in Tall (2002), is an epistemological approach in mathematics whose basic tenet is that to proof of existence of an object with given properties invariably involves constructing such an object (it would be more appropriate to consider its pedagogical counterpart, constructivist psychology, but the subtle distinction is largely immaterial to the purposes of present study). The vast majority of mathematicians today do not believe the last statement (in particular, proof by contradiction is acceptable by standard of rigour of professional mathematics), as indicated in Chabert (2008); one of historical precedents is the proof of existence of transcendental numbers by Cantor (his proof was famously rejected by Kronecker, an eminent constructivist). In Tall (2002) opposite point of view was expressed, namely it is stated that constructivism is the default approach in basic mathematical education due to general characteristics of students' cognitive development.

Methodology

In this section the methods used to collect primary/secondary sources for the present study are examined. Regarding primary sources, two independent methods, namely interview with a mathematics teacher and student survey, were used.

The student survey is aimed at collecting data on students' understanding of calculus. The data thus collected is used to find possible discrepancy between the former and the latter.

The group under examination is that of 11^{th} grade students of NIS PM Almaty in the advanced mathematics group (10 hours of mathematics weekly). The size of the described population is equal to 50 students. The survey used standard confidence level of 95%; it is expected that the results will be skewed so the margin of error is 10% and response distribution is 30% (the results of a similar survey in Juter (2012) exhibited a comparable degree of asymmetry). Then the minimum sample size for this choice of parameters equals 30 students.

All respondents were informed that the results of the survey will be disclosed after statistical processing. Given that no personal identifying information besides age range and gender is requested in the survey, no ethical concerns arise.

The interview with a calculus instructor is used in order to find out the main difficulties in teaching calculus in NIS PM Almaty and what curriculum reforms would be useful. The interviewee was informed that parts of interview essential to the present study would be translated and disclosed. Personal information of the interviewee remains confidential.

Secondary sources were evaluated using RAVEN evaluation methods and can be reliably used in a research project.

Results

Exactly 30 NIS students have responded to the survey (10 survey questions can be found in Appendix 1). This number of respondents ensures sufficient statistical reliability (see previous section for details).

The age of the respondents varies significantly, ranging from 13-14 age group to 17-18 age group. The majority of respondents (50%) are in 15-16 age group as indicated in Figure 1.

Regarding specifically the efficiency of teaching methods, the majority of students believe that they understand the concepts of limit and continuous& differentiable functions well (respectively 57% and 50% of the respondents). This demonstrates that teaching methods at NIS PM Almaty are successful in forming confidence of the students.

On the other hand, the answers to specific mathematics questions seem to indicate confusion in understanding of the concept of limit. For instance, the majority of respondents have given incorrect answer to question 5, which is a quite elementary question about limits. The understanding of continuous&differentiable functions and difference between them seems, on the contrary, to be quite satisfactory as the majority of respondents have given correct answers to questions 8, 9 and 10.

Complete interview answers can be found in Appendix 2. Let us note here few relevant details. Firstly, the interviewee believes that invalid conceptual links can form as a result of omission of counterexamples (e.g. everywhere non-differentiable continuous functions) from curriculum. Secondly, the interviewee believes that there introductory calculus course in 11^{th} is beneficial for students as the 12^{th} grade course does not cover the basic topics.

Discussion

This section is devoted to cross-analysis of primary and secondary sources collected during this study in order to verify our main conjecture (Table 1 in Appendix 1 summarizes the findings of present section).

Firstly, the primary sources indicate that there is significant misunderstanding of the concepts such as limit, continuity and differentiability by students who took the course. Given the vital role these concepts play in understanding of calculus, primary sources cast doubt on productivity of teaching methods in the course. This is in agreement with prediction of Juter (2012), where survey analogous to ours conducted was conducted with quite negative results. It should be noted though that there are quite substantial differences between two studies as the sample population in Juter (2012) was that of university students while in the present study the respondents were junior high school students. One could intuitively expect that understanding of high school students would be even more fragmentary compared to university students but this is not fully confirmed by our results (in Juter (2012) around half of the students believed that any continuous functions is differentiable whereas in present study only

33% think so). The existence of misunderstanding of crucial calculus concepts is confirmed therefore by both primary and secondary sources.

It is natural to consider the reasons for this misunderstanding. A plausible explanation, based on theory of concept images as outlined in Tall & Vinner (1981), is that omissions of definitions results in formation on invalid conceptual links. This explanation has been confirmed by interviewee — he thinks that '...designers of the course...did not succeed in [avoiding formation of invalid conceptual links in students' minds]'. Moreover, in survey 9 respondents out of 30 have given positive answers to both questions 8 and 9. Positive answers to these two questions are necessarily mutually inconsistent (first answer essentially asserts that differentiability is much stronger property than continuity while second asserts that there exist differentiable discontinuous functions) — this contradiction clearly shows the existence of invalid conceptual link. Thus we observe agreement of primary and secondary sources regarding the reasons of inefficiency of teaching methods in NIS PM calculus course — the lack of rigour (the perspective of Tall (2002) on the necessary level of rigour turned out to be irrelevant for NIS PM Almaty).

Finally, it is necessary to consider the proper epistemological model. The point of view of Tall (2002), asserting that constructivism is the best model in mathematics education, is confirmed by interview conducted for present study: in the answer to question 8 it is stated that 'when students do not see any counterexamples they fall into a logical trap'. Consequently, the idea that constructivism is an appropriate epistemological model in mathematics education is verified by primary and secondary sources (interestingly enough, the perspective of Tall (2002) is more relevant than that of Chabert (2008)).

Conclusion

The purpose of present study was to find out factors possibly affecting the productivity of teaching methods in NIS PM Almaty. The study largely focused on comprehension of the concepts of limit, continuity and differentiability.

The main conjecture of this study, stating that students do not fully understand the difference between continuous and differentiable functions and subtleties of the rigorous definition of limit, was confirmed by both primary and secondary sources. Furthermore, the reason for this misunderstanding was identified as formation of an invalid conceptual link stemming from omission of counterexamples in curriculum of the course.

It would be worthwhile to examine limitations of the present study. Given the uniformity of curricula used in NIS network schools, it is expected that the results of the study may be applicable in other schools of the network. Possible limitation of the study is that it only applies to PM schools (i.e. schools with physics-mathematics focus); the situation in CB (chemistry-biology) schools may be very different and has not been researched yet. Besides that, the present study is primarily concerned with rigorous definitions in calculus course; important intuitive aspects of calculus understanding, such as mental maps for derivative and integral, were not explored in any detail.

References

[1] Edited by Clements, M.A.K., Bishop, A., Keitel-Kreidt, C., Kilpatrick, J., Leung, F.K.-S. (2013). Third International Handbook of Mathematics Education, New York, U.S.: Springer-Verlag.

[2] Arzarello, F., Micheletti, C., Olivero, F., Paola, D., & Robutti, O. (1998). A model for analysing the transition to formal proofs in geometry. In A. Olivier & K. Newstead (Eds.), Proceedings of the 22nd Conference of the International Group for the Psychology of Mathematics Education(Vol. 2, pp. 24–31). Stellenbosh, South Africa: International Group for the Psychology of Mathematics Education.

[3] Frant, J. B., & de Costra, R. M. (2000, July). Proofs in geometry: Different concepts build upon very different cognitive mechanisms. Paper presented at the ICME 9, TSG12: Proof and Proving in Mathematics Education, Tokyo, Japan.

[4] Tall, D. & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. Educational Studies in Mathematics, 12, 151-169.

[5] Juter, K. (2012). The Validity of Students' Conceptions of Differentiability and Continuity, Kristianstad University preprint.

[6] Viholainen, A. (2006). In Novotná, J., Moraová, H., Krátká, M. & Stehlíková, N. (Eds.). Proceedings 30^{th} Conference of the international Group for the Psychology of Mathematics Education, Vol. 5, pp. 329-336. Prague: PME.

[7] Bressoud, D. M. (2010). The changing face of calculus: First-semester calculus as a high school course. The calculus collection: A resource for AP and beyond, 54.

[8] Chabert, J. (2008). Algorithms, in T. Gowers (Ed.), the Princeton companion to mathematics. Princeton University Press, Princeton, NJ.

[9] Juter, K. (2011). University students linking limits, derivatives, integrals and continuity. In Proceedings of the seventh congress of the European Society for Research in Mathematics Education (CERME 7) (pp. 2043-2052).

[10] Tall, D. (2002). The psychology of advanced mathematical thinking. In Advanced mathematical thinking (pp. 3-21). Springer Netherlands.

Appendix 1

What is your age?

30 ответов

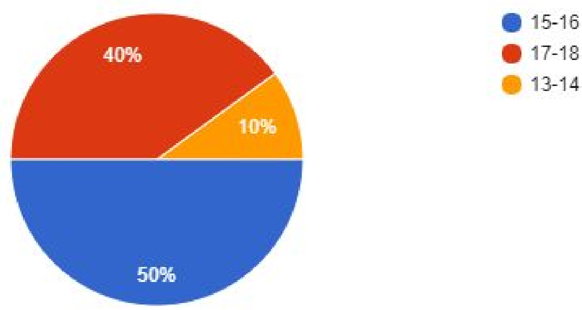

Figure 1

What is your gender?

30 ответов

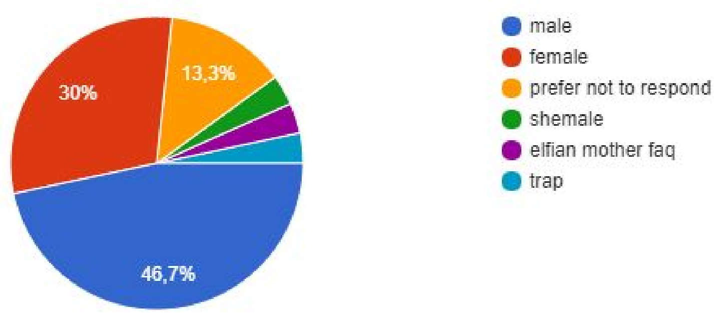

Figure 2

Do you think you understand well what is the limit of a sequence?

30 ответов

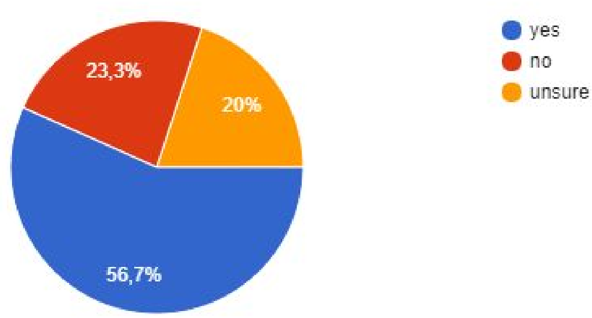

Figure 3

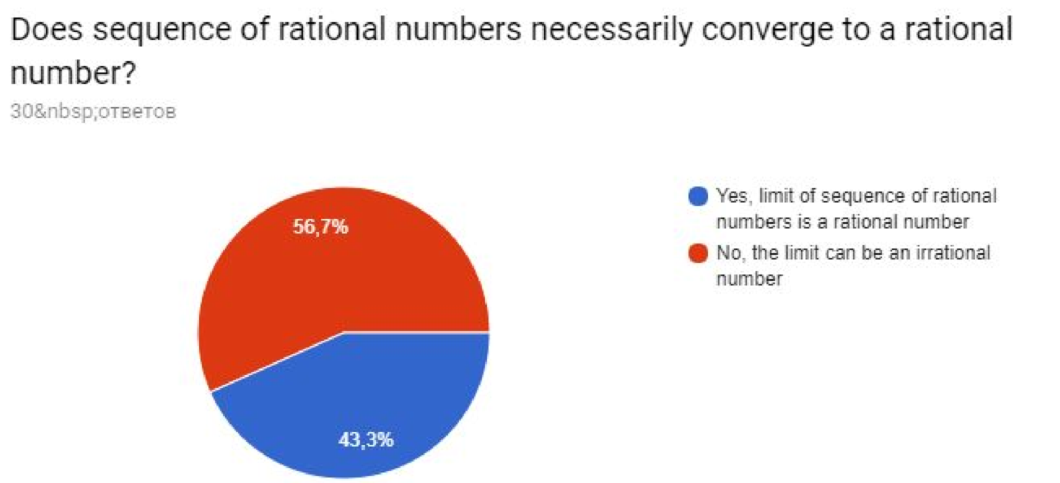

Figure 4

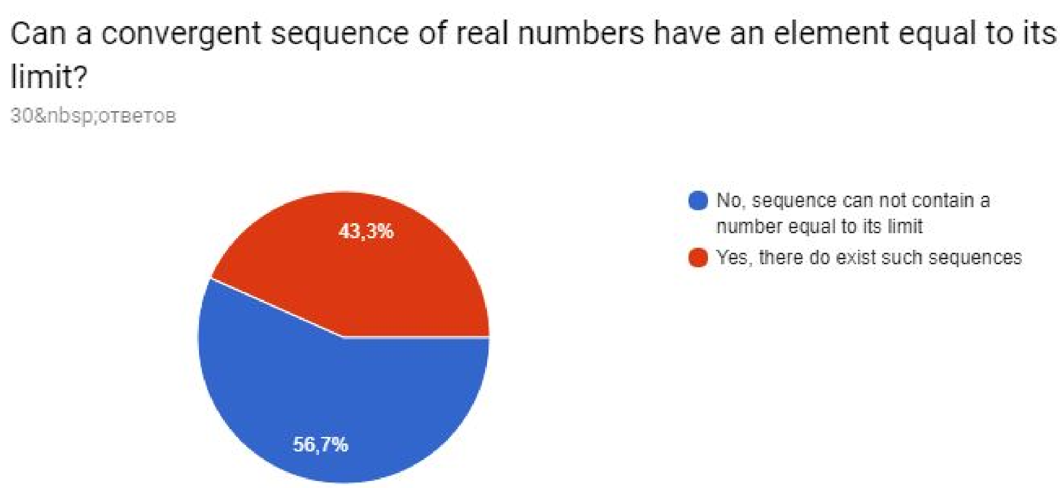

Figure 5

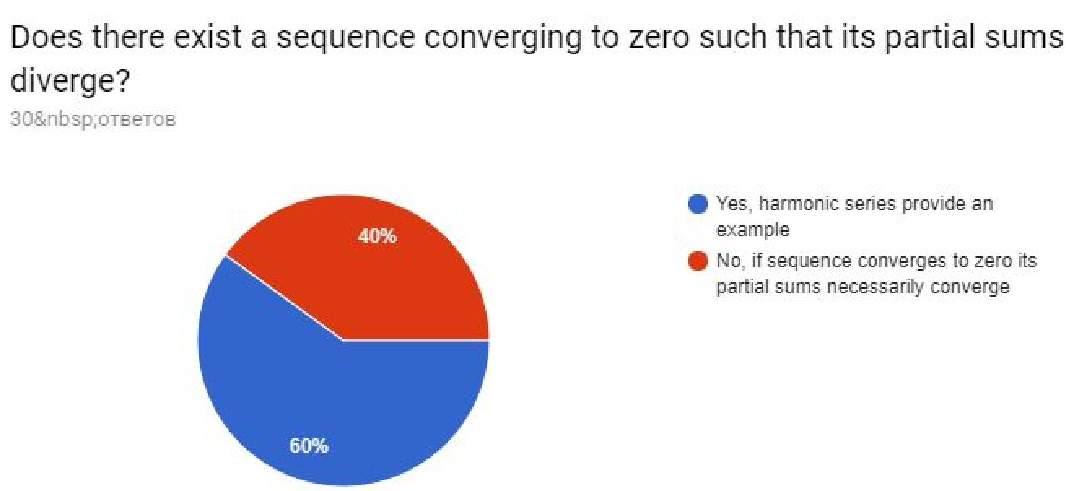

Figure 6

Do you understand the difference between continuous and differentiable functions well?

30 ответов

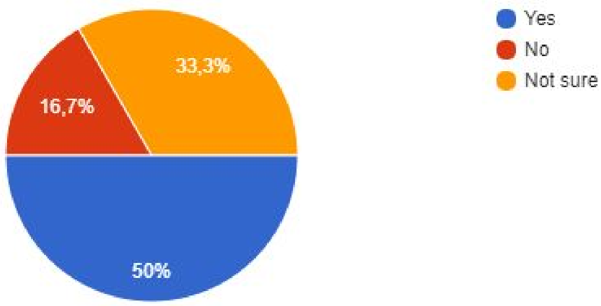

Figure 7

Do there exist continuous functions which are not differentiable at infintely many points?

30 ответов

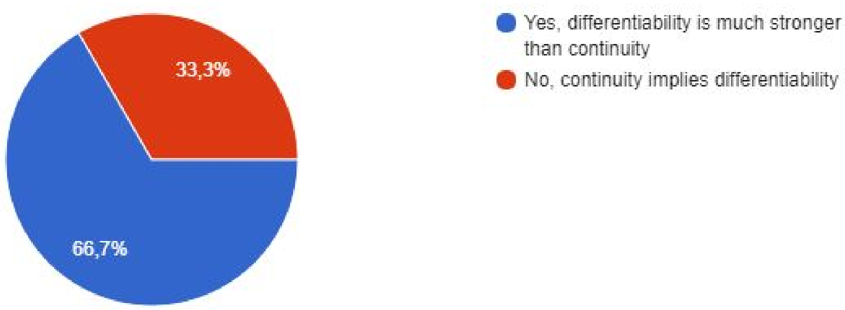

Figure 8

Do there exist everywhere differentiable functions which are not continuous?

30 ответов

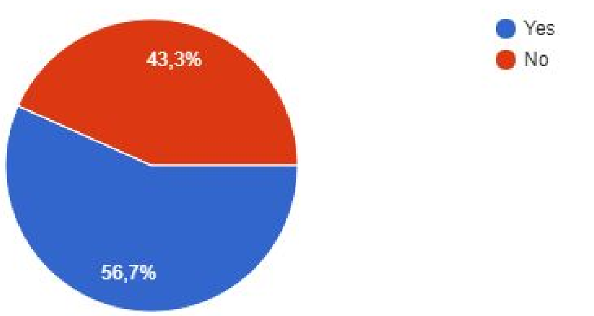

Figure 9

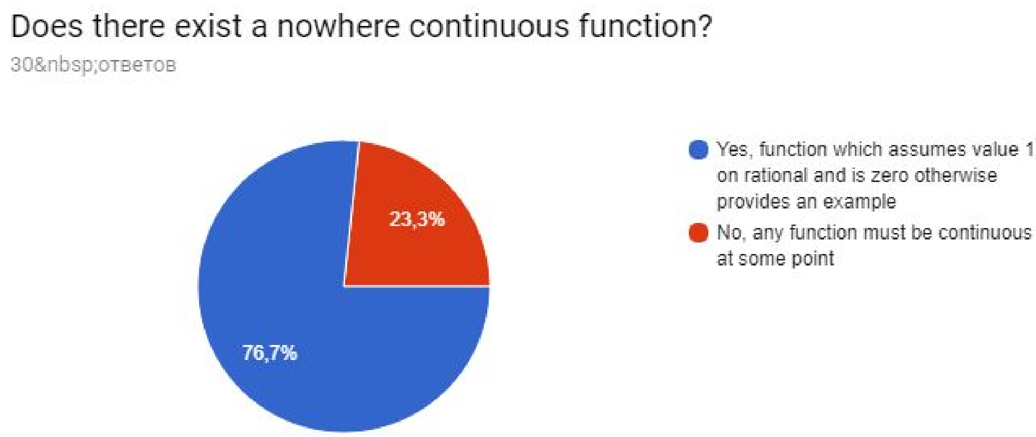

Figure 10

Survey questions

Disclaimer: This is a survey aimed at analyzing the efficiency of the mathematics teaching methods in NIS PM Almaty. The results will be disclosed after statistical processing. Your personal data, such as your gender, age, etc., will remain strictly confidential. By responding to the survey, you agree to these terms.

What is your age?
1. 15-16
2. 17-18
3. Another option

What is your gender?
1. male
2. female
3. prefer not to respond
4. Another option

Do you think you understand well what is the limit of a sequence?
1. yes
2. no
3. unsure

Does sequence of rational numbers necessarily converge to a rational number?
1. Yes, the limit of sequence of rational numbers is a rational number
2. No, the limit can be an irrational number

Can a convergent sequence of real numbers have an element equal to its limit?
1. No, sequence can not contain a number equal to its limit
2. Yes, there do exist such sequences

Does there exist a sequence converging to zero such that its partial sums diverge?

1. Yes, harmonic series provide an example
2. No, if sequence converges to zero its partial sums necessarily converge

Do you understand the difference between continuous and differentiable functions well?

1. yes
2. no
3. unsure

Do there exist continuous functions which are not differentiable at infinitely many points?

1. Yes, differentiability is much stronger than continuity
2. No, continuity implies differentiability

Do there exist everywhere differentiable functions which are not continuous?

1. Yes
2. No

Does there exist a nowhere continuous function?

1. Yes, function which assumes value 1 on rational and is zero otherwise provides an example
2. No, any function must be continuous at some point

Cross-analysis of perspectives on organization of calculus course suggested by primary and secondary sources

	Confirmed by primary sources	Refuted by primary sources
Constructivist epistemological model	+	
Acceptance of proof by contradiction		+
Use of rigorous definition	+	
Intuitionist approach		+

Table 1

Illustration of hierarchy of understanding layers in calculus

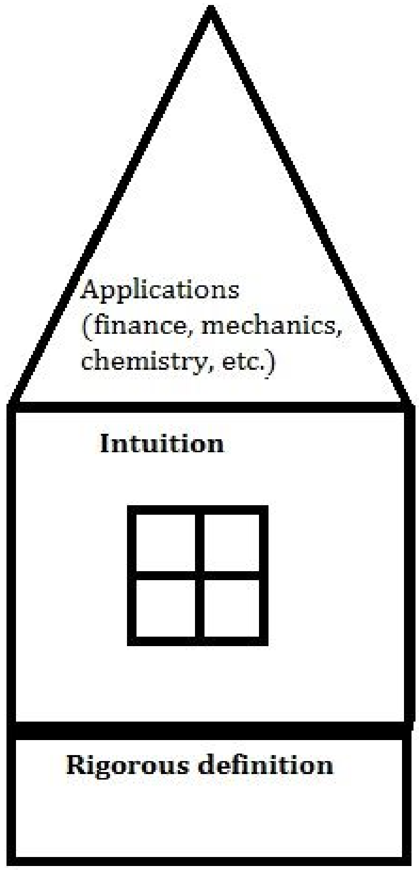

Diagram 1

Appendix 2

Interview answers:

1. Do you think that the introductory calculus course in 11^{th} is useful for NIS PM students given that they will have more comprehensive course in 12^{th} grade?

   I think that it is indeed very useful because the immense amount of material to be covered in 12^{th} grade makes it necessary to present the basic material earlier, in 11^{th} grade.

2. The 12^{th} grade course's prerequisites include topics from 11^{th} grade course. Is it possible that some essential material will be forgotten by students?

   No, I strongly doubt it.

3. Why the curriculum of 11^{th} grade course does not include rigorous definition of limit and series convergence?

   It is because of the lack of applications of this rigorous definitions to modern physics or engineering. It would be useless to include those.

4. Can invalid conceptual links form because of this omission?

   The designers of our curriculum were well aware of these problems and, therefore, tried their best to avoid formation of invalid conceptual links. Nevertheless, in view of overwhelming practical evidence I gathered when teaching the course I do not think that they succeeded in doing so.

5. Should topics of absolute and conditional convergence be covered in the course?

   I think that they are irrelevant to the course matter.

6. Some students do not understand that partial sums of sequence converging to zero may diverge. Do you see any reason for this incorrect understanding? Would inclusion of harmonic series into curriculum improve the situation?

   As I said before, this distinction is largely irrelevant. The incorrect understanding is readily explained by the omission of the distinction. I do not think that inclusion of harmonic series would somehow improve the situation.

7. Why the 'epsilon-delta' definition of continuity is not covered in the course? Do you think that students' understanding would improve if it were covered?

   It is not covered because it will not help students with their further careers.

8. From your experience you may have noticed that students rarely understand the difference between continuity and differentiability (besides the

most trivial cases such as piecewise linear functions). Do you see any pedagogical explanation for this fact?

This is a fair point. I think we should pay more attention to this subtle distinction. Some of the bright students intuitively understand that not all continuous functions are differentiable but (almost) none of them are able to construct a counterexample. Obviously, when students do not see any counterexamples they fall into a logical trap.

9. Should 'monster' functions such as Weierstrass or Riemann functions be included into curriculum, in order to illustrate the difference between continuous and differentiable functions?

   I believe that his functions are beyond the level of vast majority of students, so no.

10. The basic concept of the whole theory — that of function — is never explained rigorously during the course. One could speculate that this is the reason students frequently do not understand that the vast majority of functions are everywhere discontinuous. Do you find this plausible? Do you think that explicit examples of everywhere discontinuous functions should be introduced during the course?

    I do find suggested explanation possible though we should understand that high-school calculus course is not a place for idealists so we are in condition of shortage of time. I do not think that counterexamples should be included.