NATIONAL UNIVERSITY OF SINGAPORE

SEMESTER 1, 2014/2015

MA1101R Linear Algebra 1 **Tutorial 2**

1. Let A and B be $m \times n$ and $n \times p$ matrices respectively.

   (a) Suppose the homogeneous linear system $B\mathbf{x} = \mathbf{0}$ has infinitely many solutions, how many solutions does the system $AB\mathbf{x} = \mathbf{0}$ have?

   (b) Suppose $B\mathbf{x} = \mathbf{0}$ has only the trivial solution. Can we tell how many solutions are there for $AB\mathbf{x} = \mathbf{0}$?

(Textbook, p. 71, Problem 10)

2. Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Consider the matrix

$$AX = XA \tag{1}$$

where X is a 2×2 matrix.

   (a) Determine which of the following matrices satisfy (1):

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

   (b) Prove that if P and Q satisfy (1) then $P+Q$ and PQ also satisfy (1).

   (c) Find conditions on p, q, r, s which determine precisely when $\begin{pmatrix} p & q \\ r & s \end{pmatrix}$ satisfy (1).

(Textbook, p. 72, Problem 14)

3. Given that A is a 3×3 matrix such that

$$A\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad A\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Find a matrix X such that

$$AX = \begin{pmatrix} 1 & 0 & 4 \\ 1 & 0 & 4 \\ 1 & 0 & 7 \end{pmatrix}.$$

(Textbook, p. 74, Problem 21)

4. Determine which of the following statements are true. Justify your answer.
   (a) If A an B are symmetric matrices of the same size, then $A - B$ is symmetric.
   (b) If A and B are symmetric matrices of the same size, then AB is symmetric.
   (c) If A is a square matrix such that $A^2 = \mathbf{O}$ then $A = \mathbf{O}$.
   (d) If A is a matrix such that $AA^T = \mathbf{O}$ then $A = \mathbf{O}$.

(Textbook, p. 75, Problem 24)

5. Let A be a square matrix.
   (a) Show that if $A^2 = \mathbf{O}$ then $I - A$ is invertible and $(I - A)^{-1} = I + A$.
   (b) Show that if $A^3 = \mathbf{O}$ then $I - A$ is invertible and $(I - A)^{-1} = I + A + A^2$.
   (c) If $A^n = \mathbf{O}$ for $n \geq 4$, is $I - A$ invertible?

(Textbook, p. 75, Problem 26)