Chapter 3

Precise Definition of Limits

3.1 The precise definition of limit

Our definition of limit in Chapter 2 is inadequate for some purposes because such phrases as "x is close to a" and "$f(x)$ gets closer and closer to L" are vague. In order to prove conclusively, for example, that

$$\lim_{x \to 3}(4x - 5) = 7,$$

we must make the definition of a limit precise.

Definition 3.1.1. *Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then we say that the limit of $f(x)$ as x approaches a is L and we write*

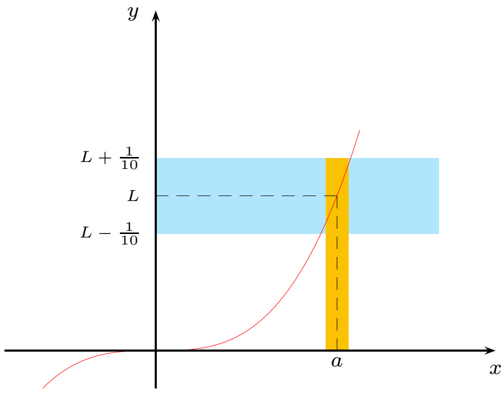
$$\lim_{x \to a} f(x) = L$$

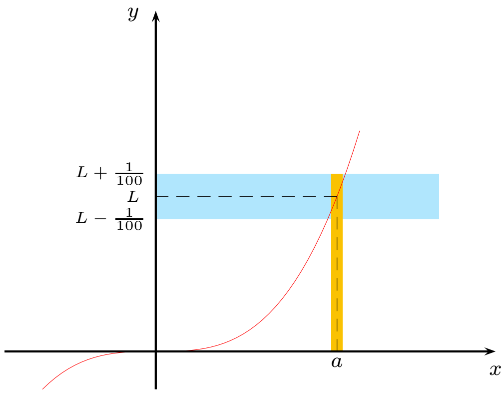
if **for every number** $\epsilon > 0$ *there is* **a number** $\delta_\epsilon > 0$ *such that*

$$|f(x) - L| < \epsilon \quad \textit{whenever} \quad 0 < |x - a| < \delta_\epsilon.$$

The number δ_ϵ depends on ϵ. It is not unique.

Let us look at a typical situation:





3.2 Some Examples

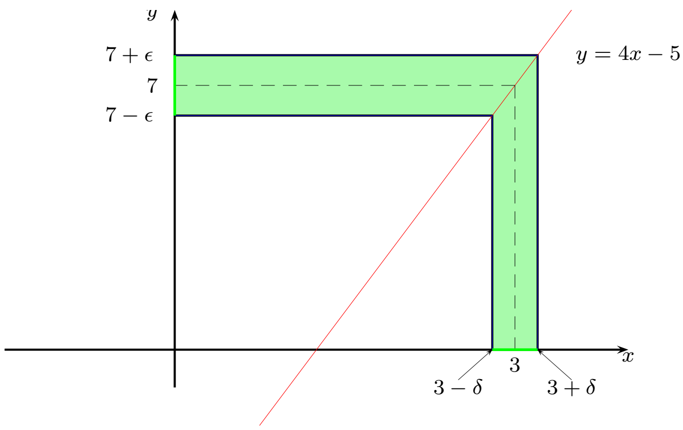
We can now show rigorously the following:

Example 3.2.1. Show that

$$\lim_{x\to 3}(4x-5)=7.$$

Solution.

A picture of the situation is given as follow:



Let $\epsilon > 0$ be given. We "work backwards" to find δ. Consider

$$|(4x-5)-7| < \epsilon.$$

This gives

$$|4x-12| < \epsilon,$$

or equivalently

$$|x-3| < \epsilon/4.$$

In other words we may choose $\delta = \epsilon/4$.

And now our proof: Let $\delta = \epsilon/4$. Suppose $0 < |x-3| < \delta$. Then

$$|(4x-5)-7| = 4|x-3| < 4\delta = \epsilon.$$

By definition of limit, we find that

$$\lim_{x\to 3}(4x-5) = 7.$$

Example 3.2.2. Show that

$$\lim_{x\to 3} x^2 = 9.$$

Solution.

We first work backwards again to find δ. Suppose we want $|x^2-9| < \epsilon$. Let $0 < |x-3| < \delta$. This implies that $-\delta < x-3 < \delta$, or equivalently

$$-\delta + 6 < x + 3 < \delta + 6,$$

Hence, if

$$\delta < 1$$

then

$$|x+3| < \delta + 6 < 7.$$

Then

$$|x^2 - 9| = |x+3||x-3| < 7\delta \le \epsilon$$

whenever

$$\delta \le \epsilon/7.$$

Hence we may choose

$$\delta = \min(1, \epsilon/7).$$

Now the proof: Let $\epsilon > 0$ be given. Let $\delta = \min(1, \epsilon/7)$. If $0 < |x-3| < \delta$, then

$$0 < |x-3| < 1 \quad \text{and} \quad 0 < |x-3| < \epsilon/7,$$

then

$$|x+3| < 7 \quad \text{and} \quad |x-3| < \epsilon/7.$$

Hence

$$|x^2 - 9| = |x+3||x-3| < 7 \cdot \epsilon/7 = \epsilon.$$

We are now able to use the definition of limit to show the **Sum Law**:

$$\lim_{x\to a} f(x) = L \quad \text{and} \quad \lim_{x\to a} g(x) = M$$

implies that

$$\lim_{x\to a}(f(x) + g(x)) = L + M.$$

Proof of the Sum Law.

By assumption we know that for every $\epsilon > 0$ there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that if $0 < |x-a| < \min(\delta_1, \delta_2) =: \delta$, then

$$|f(x) - L| < \epsilon/2 \quad \text{and} \quad |g(x) - M| < \epsilon/2.$$

Hence, for $0 < |x-a| < \min(\delta_1, \delta_2) =: \delta$,

$$|f(x) + g(x) - L - M| < |f(x) - L| + |g(x) - M| < \epsilon.$$

Therefore, $\lim_{x\to a}(f(x) + g(x)) = L + M$.

There are also precise definitions of left-hand limit, right-hand limit and infinite limits. We give these definitions here:

Definition 3.2.1 (Definition of a left-hand limit).

$$\lim_{x \to a^-} f(x) = L$$

if for every number $\epsilon > 0$ there is a number $\delta > 0$ such that

$$|f(x) - L| < \epsilon \quad \textit{whenever} \quad a - \delta < x < a.$$

Definition 3.2.2 (Definition of a right-hand limit).

$$\lim_{x \to a^+} f(x) = L$$

if for every number $\epsilon > 0$ there is a number $\delta > 0$ such that

$$|f(x) - L| < \epsilon \quad \textit{whenever} \quad a < x < a + \delta.$$

Definition 3.2.3 (Definition of an infinite limit). *Let f be a function defined on some open interval that contains a, except possibly a itself. Then*

$$\lim_{x \to a} f(x) = \infty$$

means that for every number $M > 0$ there is a number $\delta > 0$ such that

$$f(x) > M \quad \textit{whenever} \quad 0 < |x - a| < \delta.$$

Definition 3.2.4 (Definition of a negative infinite limit). *Let f be a function defined on some open interval that contains a, except possibly a itself. Then*

$$\lim_{x \to a} f(x) = -\infty$$

means that for every number $N < 0$ there is a number $\delta > 0$ such that

$$f(x) < N \quad \textit{whenever} \quad 0 < |x - a| < \delta.$$

Example 3.2.3. Show that

$$\lim_{x \to 0} \frac{1}{x^2} = \infty.$$