Chapter 1

Functions

1.1 What is a set?

A **set** is a collection of objects. A set is usually denoted by capital letters $A, B, C, \ldots$ and it is written in the form

$$A = \{a, b, c, \ldots\}.$$

The objects $a, b, c, \ldots$ between "{" and "}" are called the **elements** of the set A. We write $a \in A$ if a is an element of A, and write $a \notin A$ if a is *not* an element of A. If every element of A is again an element of B, we say A is a **subset** of B, and write $A \subseteq B$.

We can define some operations on sets:

- Union: $A \cup B$ is the set consisting of the elements which are in A or B.
- Intersection: $A \cap B$ is the set of consisting of the elements which are in both A and B.
- Difference: $A \backslash B$ is the set consisting of the elements which are in A but not in B.
- Product: $A \times B$ is the set consisting of the elements of the form (a, b) for which $a \in A$ and $b \in B$.

There are some basic notations we use for this course:

1. $\mathbf{N} = \{0, 1, 2, 3, \ldots\}$, the set of natural numbers.
2. $\mathbf{Z} = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$: the set of integers.

3. $\mathbf{Z}^+ = \{1, 2, 3, \ldots\}$: the set of positive integers.
4. $\mathbf{Q} = \{m/n \mid m \in \mathbf{Z}, n \in \mathbf{Z}^+\}$: the set of rational numbers.
5. $\mathbf{R}$: the set of real numbers.

1.2 What is a function?

Let A and B be two sets. If there is a rule that assigns each element of A to a unique element in B, we call the rule a **function**. A function is usually denoted by f. The unique element that an element $a \in A$ is assigned by f is denoted by $f(a)$.

Example 1.2.1

Let A be the set of students participating in a competition. Consider the rule

$$f(a) = \begin{cases} \text{gold medal if } a \text{ scores more than or equal to 80 marks,} \\ \text{silver medal if } a \text{ scores more than or equal to 70 but less than 80,} \\ \text{bronze medal if } a \text{ scores more than or equal to 60 but less than 70,} \\ \text{certificate of participation, otherwise.} \end{cases}$$

This rule is clearly a function.

1.3 The Domain and the Range of a function

If f is a function from A to B, written as $f : A \to B$, then A is called the **domain** of f and B is called the **codomain** of f. The **range** of f is the set of all possible values of $f(x)$ as x varies throughout the domain.

The symbol we use for element of A (in this case, a) is called an **independent variable**. The symbol (sometimes written as $b = f(a)$) we use for element of the range is called a **dependent variable**.

In this course, our sets A and B are often subsets of the real numbers $\mathbf{R}$.[1]

[1] Just think of these numbers as numbers representing the distance from the origin O to a point R in a number line. When R is to the left of O, we denote the distance as a negative real number and when R is to the right, the distance corresponds to a positive real number.

Example 1.3.1

Find the domain of the function

$$g(x) = \frac{x}{3x-1}.$$

The function is not defined at $x = \frac{1}{3}$. Hence the domain is $\mathbf{R} \backslash \{\frac{1}{3}\}$.

1.4 The graph of a function

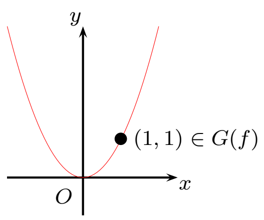
Let A and B be subsets of $\mathbf{R}$. We define the **graph** of the function $f : A \to B$ as the set

$$G(f) := \{(x, f(x)) \mid x \in A\}.$$

Since A and B are subsets of $\mathbf{R}$, we can represent each element of $G(f)$ as a point on the cartesian plane. This allows us to have a pictorial view of f.

Example 1.4.1

The following is the graph of $y = x^2$:



From the above graph, we see that if the domain of A is $\mathbf{R}$ then the range of f is $\mathbf{R}^+ \cup \{0\}$, the set of non-negative real numbers. The graph of a function allows us to "see" not only the range of f but also other behaviors of f.

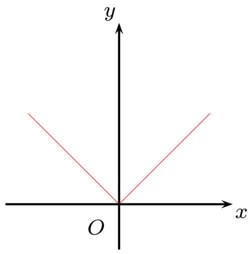
1.5 More examples of functions

Example 1.5.1

The absolute value function $f(x) = |x|$ is defined by

$$|x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

The graph is :

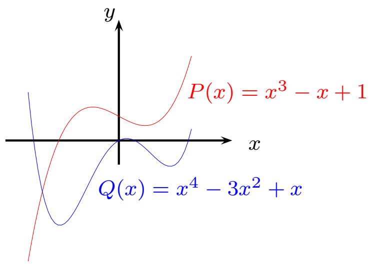


Example 1.5.2

A function P is called a polynomial (over $\mathbf{R}$) if

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0,$$

and $a_i \in \mathbf{R}$. If $a_n \neq 0$, then we say that n is the degree of $P(x)$. A polynomial of degree 1 is called a linear function. A polynomial of degree 2 is a quadratic function. Here are some of the graphs of polynomials:



Example 1.5.3

A rational function $R(x)$ is a function of the form

$$R(x) = \frac{P(x)}{Q(x)},$$

where $P(x)$ and $Q(x)$ are polynomials, $(Q(x) \neq 0)$.

Example 1.5.4

A function f is called an algebraic function if it can be constructed using algebraic operations (such as addition, subtraction, multiplication, division and taking roots) starting with polynomials.

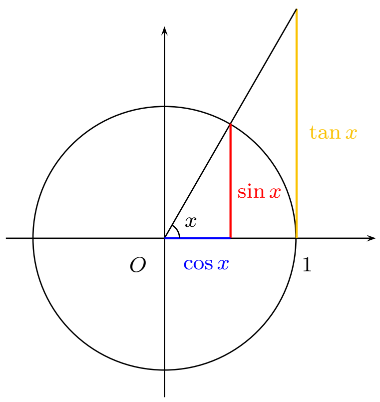
For example,

$$f(x) = \sqrt{x^2+1}, \quad g(x) = \frac{x^3+1}{x+2} + (x-2)\sqrt[5]{x^3-1}.$$

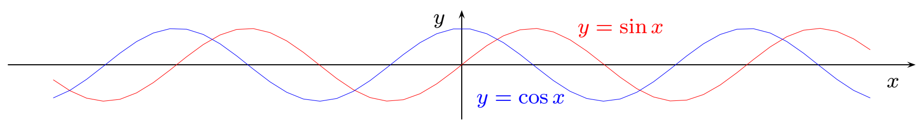
Here $\sqrt{t}$ refers to the non-negative squareroot of t and it is defined only when $t \geq 0$.

Example 1.5.5

The trigonometric functions $\sin x$, $\cos x$ and $\tan x$ are defined as ratios of the sides of a right angled triangle. If x is one of the angles (not the right angle) then $\sin x$ (where x is measured in radians) is defined to be the ratio of the side opposite the angle x to that of the hypotenuse.



The graph of the sine and cosine function are shown as follow:



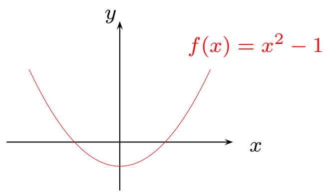
There are other functions such as the logarithm and exponential functions. We will discuss these functions after we learn about integrals.

1.6 Special classes of functions

A function is **even** if it satisfies

$$f(x) = f(-x).$$

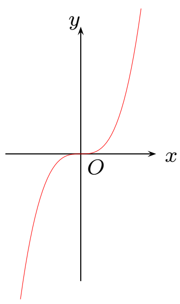
If we view the graph $G(f)$ then we see that the curve representing $G(f)$ is symmetric about the $y-$axis. The graph below is an example of a graph of an even function:



A function is **odd** if it satisfies

$$f(-x) = -f(x).$$

The graph $G(f)$ is then symmetric about the origin. In other words if the curve representing $G(f)$ is rotated 180 degrees about the origin then the resulting curve is the same as the initial curve. Here is an example of an odd function ($f(x) = x^3$):



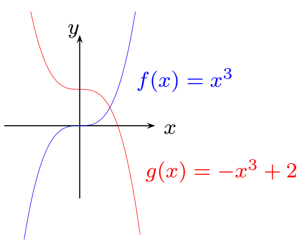
A function is said to be **increasing** on an interval I if

$$f(x_1) < f(x_2) \quad \text{whenever } x_1 < x_2 \text{ in } I.$$

It is called **decreasing** on I if

$$f(x_1) > f(x_2) \quad \text{whenever } x_1 < x_2 \text{ in } I.$$

In the following diagram, the blue curve is increasing while the red curve is decreasing:



1.7 Algebra of functions

Let f and g be functions with domains A and B, respectively. We define

$$\begin{aligned}(f+g)(x) &:= f(x)+g(x) \quad \text{domain} = A\cap B,\\ (f-g)(x) &:= f(x)-g(x) \quad \text{domain} = A\cap B,\\ (fg)(x) &:= f(x)g(x) \quad \text{domain} = A\cap B,\\ (f/g)(x) &:= f(x)/g(x) \quad \text{domain} = \{x\in A\cap B \mid g(x)\neq 0\}\ .\end{aligned}$$

Given f and g we define the **composite** of these two functions

$$(f\circ g)(x) := f(g(x)).$$

Note that $f\circ g$ is NOT the same as fg. The domain of $f\circ g$ is the set of elements x in the domain of g such that $g(x)$ is in the domain of f.

Example 1.7.1

If $f(x)=\sqrt{x}$ and $g(x)=\sqrt{2-x}$, find each function and its domain.
(a) $f\circ g$ (b) $g\circ f$ (c) $f\circ f$ (d) $g\circ g$.