Chapter 6

Applications of differentiation

6.1 Maximum and Minimum values

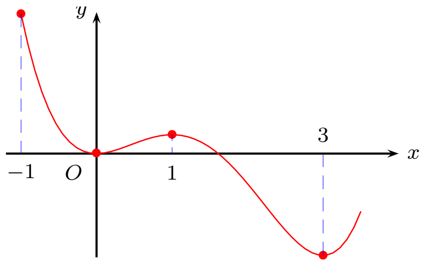
Some of the most important applications of differential calculus are optimization problems, in which we are required to find the optimal way of doing something. Here are some examples:

- What is the shape of a can that minimizes manufacturing costs?
- What is the maximum acceleration of a space shuttle?
- At what angle should blood vessels branch so as to minimize the energy expended by the heart pumping blood?

These problems can be reduced to finding the maximum and minimum values of a function.

Definition 6.1.1. *Let f be a function and let D be its domain. f has an* **absolute maximum** *(or global maximum) at c if $f(c) \geq f(x)$ for all $x \in D$. The number $f(c)$ is called the* **maximum value** *of f on D. Similarly if $f(c) \leq f(x)$ for all $x \in D$, then we say that f has an* **absolute minimum** *(or global minimum) at c. The maximum and minimum values of f are called the* **extreme values** *of f.*

Example 6.1.1. Consider the graph of $f(x) = 3x^4 - 16x^3 + 18x^2$ for $-1 \leq x \leq 3.5$.



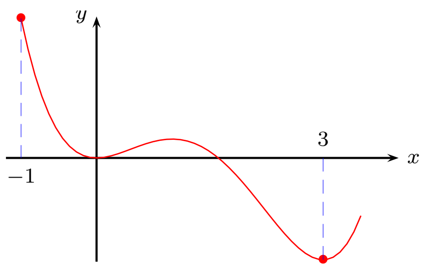
Note that $(-1, 37)$ is the maximum point while $(3, -27)$ is the minimum point. There are other points that are of interest to us. The point $(1, 5)$ is "highest" compared to its neighbours, and $(0, 0)$ is "lowest" compared to its neighbours. We have the following definition for such points:

Definition 6.1.2. *A function f has a* **local maximum** *at c if $f(c) \geq f(x)$ for all x in some open interval containing c. Similarly, f has a* **local minimum** *at c if $f(c) \leq f(x)$ for all x in some open interval containing c.*

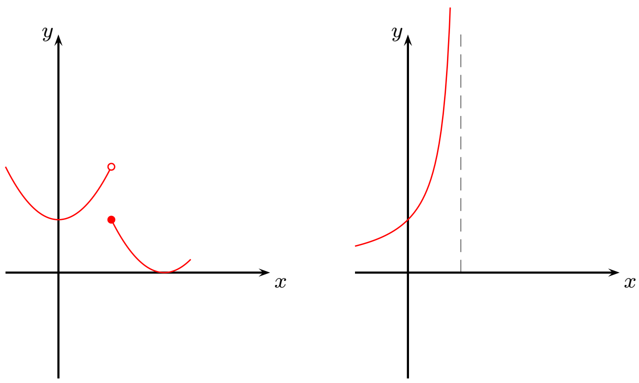
Theorem 6.1.1 (The Extreme Value Theorem). *If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.*

Although the Extreme Value Theorem is intuitively clear, its proof is difficult and will be omitted.

Consider the following curve as shown above, the maximum and minimum exists: In the closed interval $[-1, 3.5]$, it attains the absolute maximum value 37 at $x = -1$, and attains the absolute minimum value -27 at $x = 3$.



Example 6.1.2. In the following, we see the graph of functions which do not have maximum and minimum: For the first graph, the conclusion fails because f is not continuous. As for the second graph, the domain is not a closed interval.



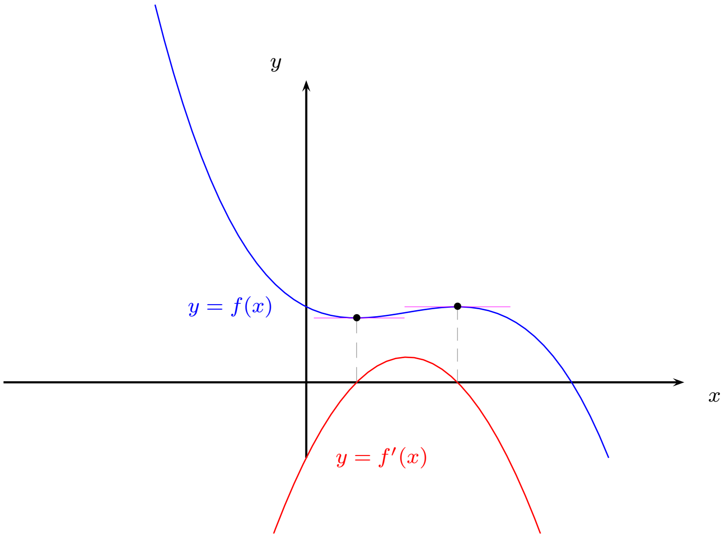
The Extreme value theorem says that a continuous function on a closed interval has a maximum value and a minimum value, but it does not tell us how to find these points. The procedures that we will follow in order to find these values are as follow:

(a) Find local maximums and local minimums.

(b) Compute the value of $y = f(x)$ at the end points of the interval.

(c) Compare the values computed in (a) and (b) to seek out the global maximum and minimum.

In order to carry out step (a), we first observe from the following graph that local maximum and minimum appear occur from points satisfying $f'(x) = 0$. In fact this is always the case.

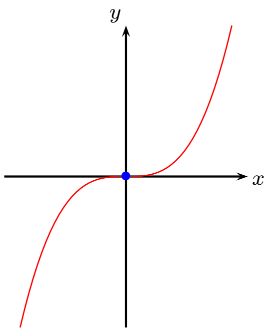


Theorem 6.1.2 (Fermat's Theorem). *If f has a local maximum or minimum at c and if $f'(c)$ exists then $f'(c) = 0$.*

Proof to be given in Class. It can also be found in Textbook page 226.

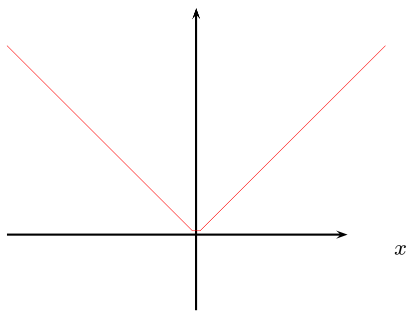
Warning: The Theorem only says that if c is a local maximum or minimum then $f'(c) = 0$ when it exists. It does not say that if $f'(c) = 0$ then c has to be a maximum or a minimum.

For example, if $f(x) = x^3$ then $f'(x) = 3x^2$. Solving the equation $f'(x) = 0$, we have $x = 0$. But $x = 0$ is neither a maximum nor a minimum of f as shown in the graph below:



The next example shows that even if $f'(c)$ does not exist, $(c, f(c))$ may still be a local maximum or a local minimum.

Example 6.1.3. Consider the following example:



Note that $f'(0)$ does not exist but $(0,0)$ is a local minimum.

From the above examples, we see that if we want to find local maximum and local minimum, we need to look for those c such that either $f'(c) = 0$ or $f'(c)$ does not exist. We call such c a critical number for f.

Definition 6.1.3. *A* **critical number** *of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.*

Using the above definition, we may restate Theorem 6.1.2 as

Theorem 6.1.3 (Fermat's Theorem). *If f has a local maximum or minimum at c, then c is a critical number of f.*

Example 6.1.4. Find the critical numbers of $f(x) = x^{3/5}(4 - x)$.

Once we determine the critical numbers, we are able to determine the global maximum and minimum of a continuous function in a closed interval.
The procedure of finding extremal values of f on the closed interval $[a, b]$ now becomes:

(a) Find the values of f at critical numbers of f in (a, b).

(b) Find the values of f at the endpoints of the interval.

(c) The largest (resp. smallest) of the values from (a) and (b) is the absolute maximum (resp. minimum) value.

We illustrate the procedures using the following examples.

Example 6.1.5. Find the absolute maximum and minimum values of the function

$$f(x) = x^3 - 3x^2 + 1, \quad -\frac{1}{2} \le x \le 4.$$

Example 6.1.6. The Hubble Space Telescope was deployed on April 24, 1990, by the space shuttle *Discovery.* A model for the velocity of the shuttle during this mission, from liftoff at $t = 0$ until the solid rocket boosters were jettisoned at $t = 126$ s, is given by

$$v(t) = 0.001302t^3 - 0.09029t^2 + 23.61t - 3.083$$

(in feet per second). Using this model, estimate the absolute maximum and minimum values of the acceleration of the shuttle between liftoff and the jettisoning of the boosters.

6.2 The Mean Value Theorem

Theorem 6.2.1 (Rolle's Theorem). *Let f be a function that satisfies the following three hypotheses:*

1. *f is continuous on the closed interval $[a, b]$,*

2. *f is differentiable on the open interval (a, b),*

3. *$f(a) = f(b)$.*

Then there is a number c in (a, b) such that $f'(c) = 0$.

*P*roof. There are three cases:

Case 1. $f(x) = k$, a constant. If this is the case, $f'(x) = 0$ and any number in (a, b) can be taken as the number c.

Case 2. $f(x) > f(a)$ for some $x \in (a, b)$. By the extreme value theorem, f has a maximum value in $[a, b]$. Since $f(a) = f(b)$, f must attain its maximum at a number c in the open interval (a, b). Then f has a local maximum at c, and by hypothesis 2 f is differentiable at c. Therefore $f'(c) = 0$ by Theorem 6.1.2.

Case 3. $f(x) < f(a)$ for some $x \in (a, b)$. This is similar to case 2.

Example 6.2.1. Prove that the equation $x^3 + x - 1 = 0$ has exactly one real root.

We are now ready to prove the mean value theorem.

Theorem 6.2.2 (Mean Value Theorem). *Let f be a function that satisfies the following hypotheses:*

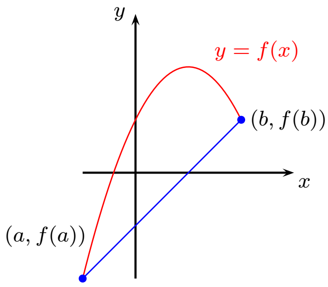
1. *f is continuous on the closed interval $[a, b]$,*

2. *f is differentiable on the open interval (a, b).*

Then there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

To prove the mean value theorem, we only need to define a function $h(x)$ satisfying the hypotheses in Rolle's Theorem. Consider the following graph:



We let $h(x)$ to be the distance function between the blue line and the red line. More precisely, let

$$h(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - b) + f(b).$$

The proof will then follow from Rolle's Theorem.

Example 6.2.2. To illustrate the Mean Value Theorem with a specific function, consider $f(x) = x^3 - x$, $a = 0$, $b = 2$. Since f is a polynomial, it is continuous and differentiable for all x, so it is certainly continuous on $[0, 2]$ and differentiable on $(0, 2)$. Therefore by the Mean Value Theorem, there is a number c in $(0, 2)$ such that

$$f(2) = f(0) = f'(c)(2 - 0).$$

Now $f(2) = 6$ and $f(0) = 0$ and $f'(x) = 3x^2 - 1$, so this equation becomes

$$6 = (3c^2 - 1)2 = 6c^2 - 2,$$

giving $c = \frac{2}{\sqrt{3}}$ or $-\frac{2}{\sqrt{3}}$.

Example 6.2.3. Suppose $f(0) = -3$ and $f'(x) \leq 5$ for all values of x. How large can $f(2)$ possibly be?

One of the most interesting application of the mean value theorem is that it leads us to the following theorem:

Theorem 6.2.3. *If $f'(x) = 0$ for all x in an interval (a, b) then f is constant on (a, b).*

Corollary 6.2.4. *If $f'(x) = g'(x)$ for all x in an interval (a, b) then $f - g$ is constant on (a, b): That is, $f(x) = g(x) + c$ where c is a constant.*

6.3 How derivatives affect the shape of a graph

The first derivative of a function f tells us the possible positions of local minimum and maximum in an interval. In this section, we discuss briefly what the second derivative of f tells us about the behavior of the graph of f.

We first make the following observations:

Theorem 6.3.1 (Increasing and Decreasing Test).

(a) *If $f'(x) > 0$ on an interval, then f is increasing on that interval.*

(b) *If $f'(x) < 0$ on an interval, then f is decreasing on that interval.*

Proof of (a). Let x_1, x_2 be two numbers in the interval such that $x_1 < x_2$. Then we need to show that $f(x_1) < f(x_2)$.

Since $f'(x) > 0$, we find, by the mean value theorem, that

$$f(x_2) - f(x_1) = (x_2 - x_1)f'(c),$$

for some $c \in (x_1, x_2)$. Hence $f(x_2) > f(x_1)$. The proof of (b) is similar.

Example 6.3.1. Find where the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is increasing and where it is decreasing.

Theorem 6.3.2 (The First Derivative Test). *Suppose c is a critical number of a continuous function f.*

(a) *If f' changes from positive to negative at c, then f has a local maximum at c.*

(b) *If f' changes from negative to positive at c, then f has a local minimum at c.*

(c) *If f' does not change sign at c (for example, if f' is positive on both sides of c or negative on both sides), then f has no local maximum or minimum at c.*

6.3.1 What does f'' say about f?

Definition 6.3.1. If the graph of f lies above all of its tangents on an interval I, then it is called concave upward on I. If the graph of f lies below all of its tangents on I, it is called concave downward on I.

We have the following:

Theorem 6.3.3 (Concavity Test).

(a) *If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I.*

(b) *If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.*

(Proof in Appendix F page A43 of the textbook.)

Definition 6.3.2. A point P on a curve $y = f(x)$ is called an inflection point if f is continuous there and the curve changes from concave upward to concave downward or from concave downward to concave upward.

The Concavity Test leads us to the following theorem:

Theorem 6.3.4 (The Second Derivative Test). *Suppose f'' is continuous near c.*

(a) *If $f'(c) = 0$ and $f''(c) > 0$ then f is a local minimum at c.*

(b) *If $f'(c) = 0$ and $f''(c) < 0$ then f is a local maximum at c.*

6.4 Optimization Problems

The methods we have learnt in this chapter for finding extreme values have practical applications in many areas of life. A businessperson wants to minimize costs and maximize profits. A traveler wants to minimize transportation time. In this section, we solve such problems such as maximizing areas, volumes and profits and minimizing distances, costs and times.

Here are the steps to follow in solving optimization problems:

1. Understand the problem.

2. Draw a diagram.

3. Introduce notation: Use a symbol for the quantity you want to maximize or minimize. Call it Q.

4. Express Q in terms of appropriate quantity (call it x) such as time, distance etc.. Write down the domain of this function.

5. Use the method illustrated previously to find maximum and minimum of the functions.

Example 6.4.1. A farmer has 2400 ft of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area?

Example 6.4.2. A cylindrical can is to be made to hold 1 litre of oil. Find the dimensions that will minimize the cost of the metal to manufacture the can.

Example 6.4.3. Find the point on the parabola $y^2 = 2x$ that is closest to the point $(1, 4)$.

Example 6.4.4. A man launches his boat from point A on a bank of a straight river, 3 km wide, and wants to reach point B, 8 km downstream on the opposite bank, as quickly as possible. He could row his boat directly a across the river to point C and B and then run to B. If he can row 6 km/h and run 8 km/h, where should he land to reach B as soon as possible? (We assume that the speed of the water is negligible compared with the speed at which the man rows.)

Example 6.4.5. Find the area of the largest rectangle that can be inscribed in a semicircle of radius r.

6.5 L'Hôpital's Rule

Theorem 6.5.1 (L'Hôpital's Rule). *Suppose that $f(a) = g(a) = 0$, that f and g are differentiable on an open interval I containing a, and that $g'(x) \neq 0$ on I if $x \neq a$. Then*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)},$$

assuming that the limit on the right side exists.

Example 6.5.1. Find the limits of the following:

(a) $\displaystyle\lim_{x\to 0} \frac{\sqrt{1-x} - 1 - \frac{x}{2}}{x^2}$

(b) $\displaystyle\lim_{x\to 0} \frac{x - \sin x}{x^3}$

Proof of the L'Hôpital's Rule depends on Cauchy's Mean Value Theorem.

Theorem 6.5.2 (Cauchy's mean value theorem). *Suppose functions f and g are continuous on $[a, b]$ and differentiable throughout (a, b), and also suppose $g'(x) \neq 0$ throughout (a, b). Then there exists a number c in (a, b) at which*

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Proof. We first show that $g(b) \neq g(a)$.

If $g(b) = g(a)$, then by mean value theorem there exists a number c in (a, b) such that

$$g'(c) = \frac{g(b) - g(a)}{b - a} = 0.$$

But $g'(x) \neq 0$ for all x. Therefore, $g(b) \neq g(a)$.

Next, let

$$F(x) = f(x) - f(a) - \frac{f(b) - f(a)}{g(b) - g(a)} \cdot (g(x) - g(a)).$$

This function is continuous and differentiable wherever f and g are, and $F(a) = F(b) = 0$. Hence, by mean value theorem there exists $c \in (a, b)$ such that $F'(c) = 0$.

Therefore,

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

□

Proof of L'Hôpital rule. Suppose x is to the right of a, $g'(x) \neq 0$ and let

$$L = \lim_{x \to c} \frac{f'(x)}{g'(x)}.$$

Apply Cauchy's mean value theorem to $[a, x]$. Then there exists $c \in (a, x)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(x) - f(a)}{g(x) - g(a)}.$$

But $f(a) = g(a) = 0$, so

$$\frac{f'(c)}{g'(c)} = \frac{f(x)}{g(x)}.$$

Therefore,

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = \lim_{c \to a^+} \frac{f'(c)}{g'(c)} = L$$

(as $x \to a^+$, $c \to a^+$, because $c \in [a, x]$). Similarly if x is to the left of a, then

$$\lim_{x \to a^-} \frac{f(x)}{g(x)} = \lim_{c \to a^-} \frac{f'(c)}{g'(c)} = L.$$

Therefore,

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{c \to a} \frac{f'(c)}{g'(c)}.$$

$\square$