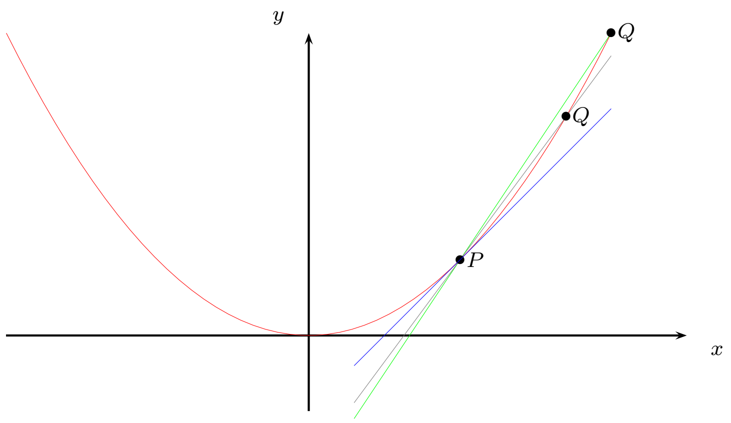
Chapter 5

Derivatives

5.1 Derivatives

Consider again the curve $y = x^2$:



In Chapter 2, we were interested in finding the slope of the line as Q approaches P. This slope can be written as

$$m := \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}.$$

When we studied the instantaneous velocity of a falling object with position function $s = f(t)$ at time $t = a$, we encounter the limit

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Such limits suggest that we should study the quantity

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

more carefully. This leads us to the next definition:

Definition 5.1.1. *The* **derivative of a function** f **at a number** a*, denoted by* $f'(a)$*, is*

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

if this limit exists.

Sometimes we will also write

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

Example 5.1.1. Find the derivative of the function $f(x) = x^2 - 8x + 9$ at the number a.

5.1.1 Tangent line

Using the definition of derivative, we define the slope of a curve $y = f(x)$ at $x = a$ as

$$m = f'(a).$$

The **tangent line** to $y = f(x)$ at $(a, f(a))$ is the line through $(a, f(a))$ with slope $f'(a)$.

Example 5.1.2. Find the equation of the tangent line to the parabola $y = x^2 - 8x + 9$ at the point $(3, -6)$.

5.1.2 Velocity

The instantaneous velocity of a particle with position function given by $s = f(t)$ at time $t = a$ can now be written as $f'(a)$. The speed of the particle is given by $|f'(a)|$.

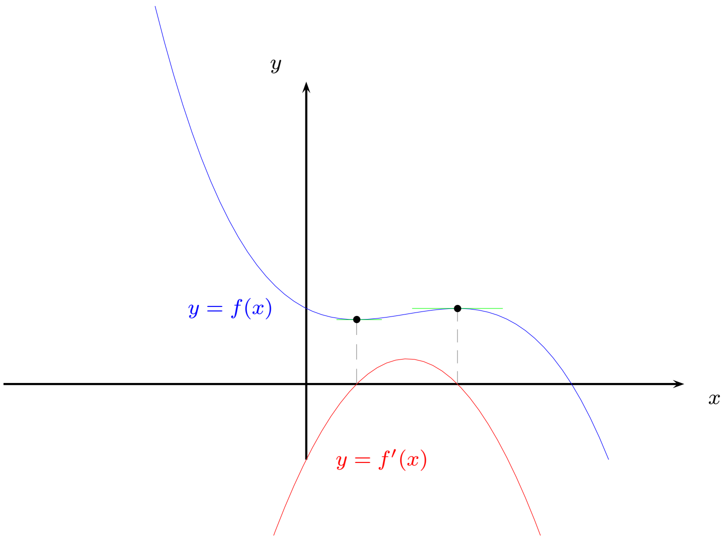
Example 5.1.3. The position of a particle is given by the equation of motion $s = 1/(1+t)$, where t is measured in seconds and s in meters. Find the velocity and the speed after 2 seconds.

5.2 Derivative as a function

We have seen in the previous section that the derivative of $f(x)$ at $x = a$ is defined by

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}.$$

If we replace a by x, we can view $f'(x)$ as a function of x. The following graph illustrate the behavior of $f'(x)$ with a given function $f(x)$.



Note that the graph $y = f'(x)$ crosses the line $y = 0$ exactly when the curve $y = f(x)$ "turns". The x-coordinates of the two intersecting points of $y = f'(x)$ and $y = 0$ give the values of x when $(x, f(x))$ is a turning point. These information allow us to plot the curve $y = f(x)$. We will discuss the relation between $f'(x)$ and curve plotting in subsequent chapters.

Example 5.2.1. If $f(x) = x^3 - x$, find a formula for $f'(x)$.

Example 5.2.2. If $f(x) = \sqrt{x-1}$, find the derivative of f. State the domain of f'.

Example 5.2.3. Find f' if $f(x) = \dfrac{1-x}{2+x}$.

There are other notations for $f'(x)$. These are

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = D_x f(x).$$

We also write

$$f'(a) = \left.\frac{dy}{dx}\right|_{x=a}.$$

5.3 Differentiable functions

Definition 5.3.1. *A function f is* **differentiable at** a *if $f'(a)$ exists. It is* **differentiable on an open interval** (a, b) *if it is differentiable at every number in the interval.*

Example 5.3.1. Where is the function $f(x) = |x|$ differentiable?

In the next result we show that a function that is differentiable at a is always continuous at a.

Theorem 5.3.1. *If f is differentiable at a then f is continuous at a.*

Proof. Consider

$$f(x) - f(a) = \frac{f(x) - f(a)}{x - a}(x - a).$$

Taking limits on both sides, we find that

$$\lim_{x \to a}(f(x) - f(a)) = f'(a)(a - a) = 0.$$

Hence

$$\lim_{x \to a} f(x) = \lim_{x \to a}(f(x) - f(a)) + \lim_{x \to a} f(a) = f(a).$$

5.4 Differentiation formulas

In this section, we derive the derivatives of basic functions and other properties of derivatives of functions. We have

Theorem 5.4.1. *1.* $\frac{d}{dx}c = 0$ *if c is a constant.*

2. For positive integer n,

$$\frac{d}{dx}x^n = nx^{n-1}.$$

3. For a constant c and a differentiable function f,

$$\frac{d}{dx}(cf) = c\frac{df}{dx}.$$

4. If f and g are both differentiable, then

$$\frac{d(f \pm g)}{dx} = \frac{df}{dx} \pm \frac{dg}{dx}.$$

5.

$$\frac{d(fg)}{dx} = f\frac{dg}{dx} + g\frac{df}{dx}.$$

6.

$$\frac{d(f/g)}{dx} = \frac{g\frac{df}{dx} - f\frac{dg}{dx}}{g^2}.$$

Using the above, we can now differentiate any rational functions.

Example 5.4.1. Differentiate

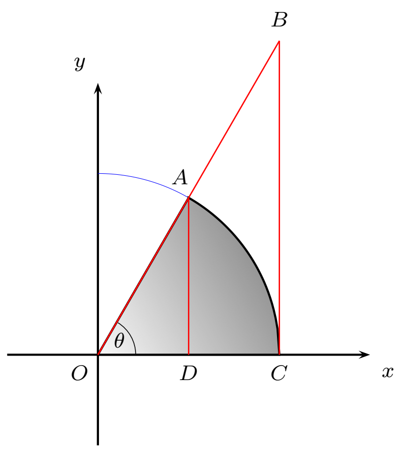
$$y = \frac{u^6 - 2u^3 + 5}{u^2}$$

with respect to u.

5.5 Derivatives of trigonometric functions

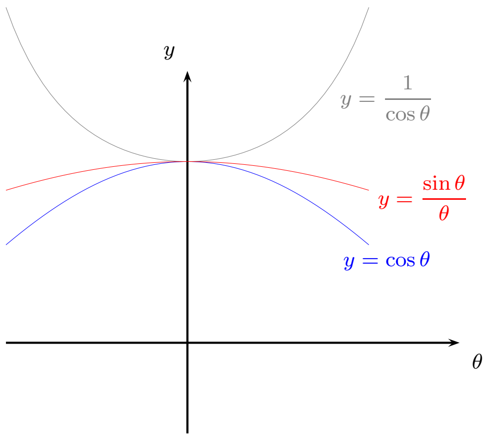
When we mention continuous functions, we say that the trigonometric functions $\sin x$ and $\cos x$ are continuous. However, we have not shown this fact. In this section, we will show that these functions are differentiable. We will first show that

$$\lim_{h \to 0} \frac{\sin h}{h} = 1.$$



The area of the triangle QAD is less than the area of segment OAC, and the area segment OAC is less than the triangle OBC. If the length OC is 1, we will get, for $0 < \theta < \pi/2$, the inequality

$$\cos\theta < \frac{\sin\theta}{\theta} < \frac{1}{\cos\theta}.$$



By Squeeze Theorem, we conclude that

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1.$$

We are now ready to show the following:

Theorem 5.5.1.

1. $$\frac{d}{dx}\sin x = \cos x.$$

2. $$\frac{d}{dx}\cos x = -\sin x.$$

5.6 The Chain rule

The rules we learn so far do not allow us to differentiate

$$F(x) = \sqrt{x^2+1}.$$

Observe that $F(x)$ is a composite function. We can write

$$F(x) = f(g(x)),$$

where

$$f(x) = \sqrt{x} \quad \text{and} \quad g(x) = x^2 + 1.$$

The chain rule allows us to differentiate composite functions.

Theorem 5.6.1 (Chain Rule). *If f and g are both differentiable and $F = f \circ g$ is the composite function defined by*

$$F(x) = f(g(x)),$$

then F is differentiable and

$$F'(x) = f'(g(x))g'(x).$$

Example 5.6.1. Use Chain rule to evaluate $f'(x)$ for the following functions:

1. $\dfrac{1}{\sqrt[3]{x^2+x+1}}$,

2. $\left(x^3-1\right)^{100}$,

3. $\left(\dfrac{x-2}{2x+1}\right)^9$.

5.7 Implicit differentiation

So far, we have learnt to differentiate function of the type $y = f(x)$. Suppose we are given

$$x^3 + y^3 = 6xy,$$

and we would like to compute dy/dx, what can we do? We use Chain rule to help us.

By Chain rule we have

$$3x^2 + 3y^2\frac{dy}{dx} = 6y + 6x\frac{dy}{dx},$$

this gives

$$\frac{dy}{dx} = \frac{3x^2 - 6y}{6x - 3y^2}.$$

We have assumed that y can be implicitly expressed as a differentiable function of x and this method of obtaining y' (when y is not expressed as a function of x explicitly) is called the method of implicit differentiation.

Insert proof of the Chain Rule

5.8 Higher derivatives

If f is differentiable then f' is also a function and so we may continue to differentiate f' to obtain $(f')'$. The function $(f')'$ written as f'' is called the **second derivative** of f. It is also written as

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}.$$

When $s(t)$ is the position function of an object that moves in a straight line, we know that its derivative represents the velocity $v(t)$. The second derivative of $s(t)$ is called the acceleration of the object and it represents the instantaneous rate of change of velocity.

Example 5.8.1.

The position of a particle is given by the equation

$$s = t^3 - 6t^2 + 9t$$

where t is measured in seconds and s in meters.

(a) Find the acceleration at time t. What is the acceleration after 4 s?

(b) Graph the position, velocity and acceleration functions for $0 \le t \le 5$.

(c) When is the particle speeding up? When is it slowing down?

There is nothing to stop us from defining higher derivatives. We define

$$f''' = (f'')'.$$

We also write

$$f^{(3)} = f'''.$$

In general we define $f^{(0)} := f$, and for integer $n \ge 1$,

$$f^{(n)} = (f^{(n-1)})'.$$

Example 5.8.2.

If $f(x) = 1/x$, find $f^{(n)}$.

more examples