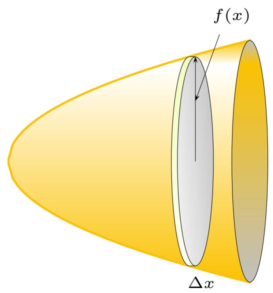
Chapter 9

Applications of Integration

9.1 Volumes

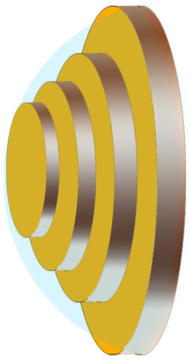
We all know that the volume of the sphere of radius r is given by $\frac{4}{3}\pi r^3$. But how do we show that this is the case?

When we want compute area, we use rectangles. If we want to estimate the volume of a solid, we use solid disks (see diagram):



$$\boxed{\Delta V = \pi f^2(x)\Delta x}$$

Suppose we want to estimate the volume of the hemisphere. The following diagram shows that we can use several disks. Note that as the number of disks increases we get better approximations of the volume.



In this case, the function $f(x) = \sqrt{r^2 - x^2}$ and so the volume near x is

$$\Delta V = \pi(r^2 - x^2)\Delta x.$$

Hence the volume of the sphere is

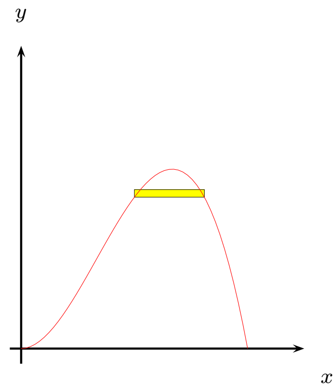
$$2\int_0^r \pi(r^2 - x^2)\, dx = 2\pi\left(r^3 - \frac{r^3}{3}\right) = \frac{4}{3}\pi r^3.$$

Example 9.1.1. Find the volume of the solid obtained by rotating about the x-axis the region under the curve $y = \sqrt{x}$ from 0 to 1.

Example 9.1.2. Find the volume of the solid obtained by rotating the region bounded by $y = x^3, y = 8$ and $x = 0$ about the y-axis.

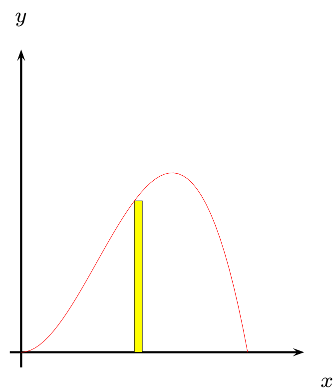
9.2 Volumes by Cylindrical Shells

Some volume problems are very difficult to handle by the methods of the preceding section. For instance, let us consider the problem of finding the volume of the solid obtained by rotating about the y-axis the region bounded by $y = 2x^2 - x^3$ and $y = 0$.



If we slice perpendicular to the y-axis we get a washer. But to compute the inner radius and the outer radius of the washer we would have to solve the cubic equation $y = 2x^2 - x^3$ for x in terms of y; that is not easy.

We now discuss a new method called the method of cylindrical shells. Instead of using the washer, we use cylinders. We consider the vertical strip and rotate it about the y-axis:



The volume of the little strip when rotated about the y-axis is

$$\Delta V = 2\pi x f(x)\Delta x.$$

Hence the volume of the solid from a to b is

$$\int_a^b 2\pi x f(x)\, dx.$$

Theorem 9.2.1. *The volume of the solid obtained by rotating about the y-axis the region under the curve $y = f(x)$ from a to b is*

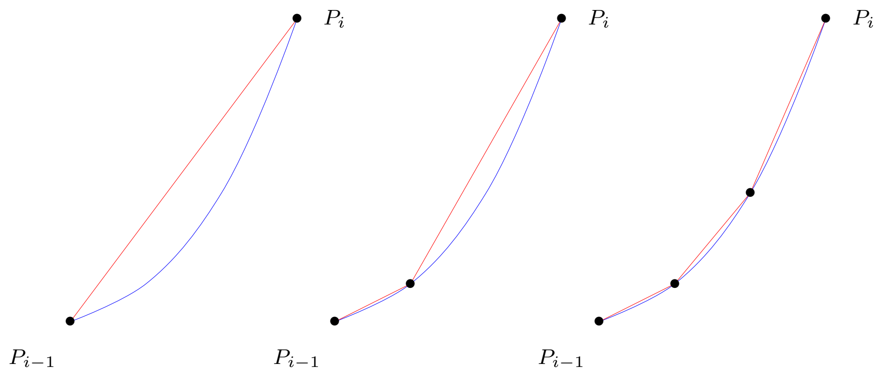
$$V = \int_a^b 2\pi x f(x)\, dx, \quad \text{where } 0 \leq a < b.$$

Example 9.2.1. Find the volume of the solid obtained by rotating about the y-axis the region between $y = x$ and $y = x^2$.

Example 9.2.2. Use Cylindrical shells to find the volume of the solid obtained by rotating about the x-axis the region under the curve $y = \sqrt{x}$ from 0 to 1.

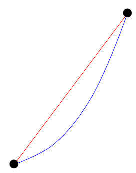
9.3 Arc Length

How do we measure the length of a curve? Consider the following diagram:



We see that we can approximate the length of a curve using line segments. Using the same idea as in the case of area and volume, we proceed as follow:

Suppose we have the following tiny segment of the curve (magnified). We estimate the length of the tiny segment using the length of the line.



If the curve is the graph of $y = f(x)$ and $P_{i-1} = (t, f(t))$ and $P_i = (t + \Delta t, f(t + \Delta t))$, then the length of the line is (by Pythagoras' Theorem and the fact that the slope of the curve is $f'(t)$):

$$\Delta L = \sqrt{(\Delta t)^2 + (f'(t))^2 (\Delta t)^2} = \sqrt{1 + (f'(t))^2}\Delta t.$$

Using the similar approach as in the area, we conclude that the length is

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

Example 9.3.1. Find the length of the arc of the parabola $y^2 = x$ from $(0, 0)$ to $(1, 1)$.

9.4 Surface Area

Suppose we want to know the surface area of a sphere. How do we compute the area? We view the sphere as rotating $y = \sqrt{1 - x^2}, \quad -1 \le x \le 1$ about the x-axis.

We now consider the general situation where we find the area of a surface of revolution. Let $y = f(x), a \le x \le b$. Consider a small segment of the curve. The arc length is

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\Delta x.$$

So the area of revolution of this segment is

$$2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\Delta x.$$

Therefore the surface area of revolution is

$$2\pi \int_a^b f(x)\sqrt{1+(f'(x))^2}\,dx.$$

Example 9.4.1. Find the area of the surface generated by rotating the curve $y = e^x$, $0 \le x \le 1$, about the x-axis.