Chapter 4

Continuous Functions

4.1 Continuity of a function at a point

Definition 4.1.1. *A function f is* **continuous at a number** *a if*

$$\lim_{x \to a} f(x) = f(a).$$

Note that if we want to check whether a function f is continuous at a, we need to check three things:

1. $f(a)$ is defined (i.e., a is in the domain of f),

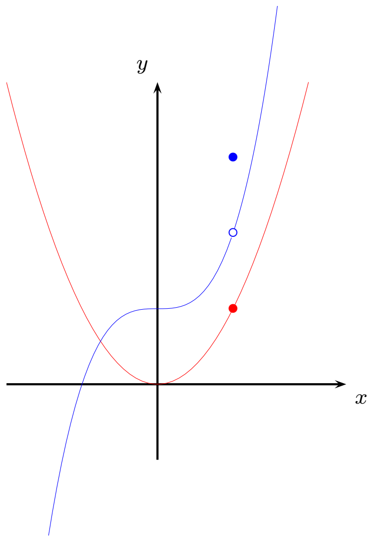
2. $\lim_{x \to a} f(x)$ exists,

3. $\lim_{x \to a} f(x) = f(a)$.

Equivalently, we can use the ϵ, δ definition: *A function f is continuous at a number a if for every $\epsilon > 0$, there exists $\delta_\epsilon > 0$ such that*

$$|f(x) - f(a)| < \epsilon \quad \textit{whenever} \quad |x - a| < \delta_\epsilon.$$

The following graph shows a function (in red) that is continuous at a and another function (in blue) that is not continuous at a.

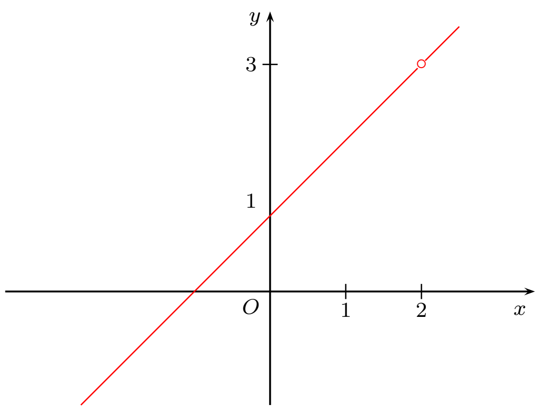


Example 4.1.1. Which of the following functions are discontinuous?

(a)

$$f(x) = \frac{x^2 - x - 2}{x - 2}.$$

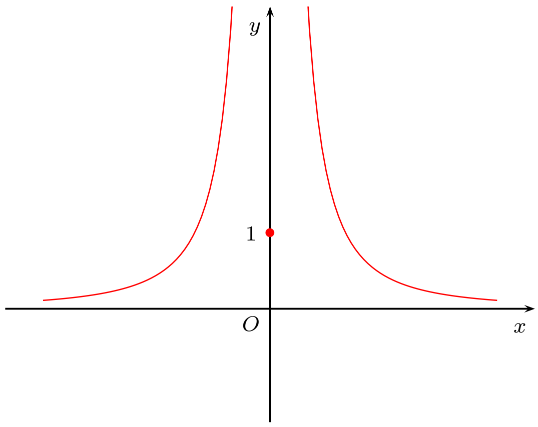
Such kind of discontinuity at 2 is called a **removable discontinuity**.



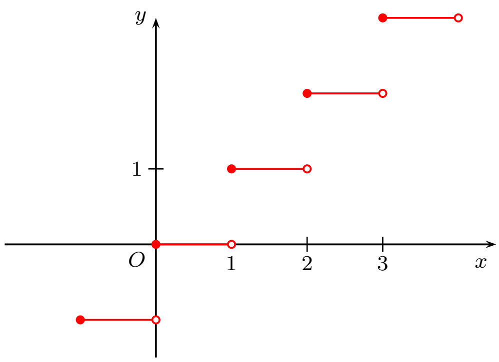
(b)

$$f(x) = \begin{cases} 1/x^2 & \text{if } x \neq 0, \\ 1 & \text{if } x = 0. \end{cases}$$

This discontinuity at 0 is called an **infinite discontinuity**.



(c) $f(x) = \lfloor x \rfloor$, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x. This is called a **jump discontinuity** at each $n \in \mathbf{Z}$.



4.2 Continuity of a function on an interval

Definition 4.2.1. *A function is* **continuous from the right at a number** a *if*

$$\lim_{x \to a^+} f(x) = f(a),$$

and f *is* **continuous from the left at** a *if*

$$\lim_{x \to a^-} f(x) = f(a).$$

At each $n \in \mathbf{Z}$, the greatest integer function $\lfloor x \rfloor$ is continuous from the right but discontinuous from the left.

Definition 4.2.2. *A function is* **continuous on an interval** *if it is continuous at every number in the interval. (If f is only defined on one side of an endpoint of the interval, we understand that continuous at the endpoint to mean continuous from the right or continuous from the left.)*

Example 4.2.1. Show that the function $f(x) = 1 - \sqrt{1-x^2}$ is continuous on the interval $[-1, 1]$.

Theorem 4.2.1 (Some basic properties of continuous function). *Given two functions f and g which are continuous at a, the following functions are also continuous at a:*

1. $f + g$,
2. $f - g$,
3. fg,
4. f/g, *if* $g(a) \neq 0$,
5. cf, *where c is a constan*

4.3 Examples of continuous functions

The product rule of limits says that if $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$ then

$$\lim_{x\to a}[f(x)g(x)] = LM.$$

Since $\lim_{x\to a} x = a$, we see that

$$\lim_{x\to a} x^n = a^n \quad \text{for } n = 0, 1, 2, \ldots.$$

Therefore x^n is continuous at any point a. Together with the sum law, we find that the polynomial

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$$

is continuous everywhere.

If $Q(x)$ is a polynomial, then $P(x)/Q(x)$ is continuous whenever $Q(x) \neq 0$ (see Theorem 4.2.1). In other words, if we define

$$D = \{x \mid Q(x) \neq 0\},$$

then the rational function $P(x)/Q(x)$ is continuous on D.

We summarize our findings in the following:

Theorem 4.3.1.

1. *Any polynomial is continuous everywhere;*

2. *Any rational function is continuous wherever it is defined.*

Example 4.3.1. Find

$$\lim_{x\to 4}\frac{x+1}{2x^2-1}.$$

Besides the polynomials and rational functions, the root function $x^{1/n}, n \in \mathbf{Z}^+$, is also continuous in the domain where the n^{th} root of x is defined. More precisely when n is an even positive integer, $x^{1/n}$ is continuous for all non-negative real numbers x, and when n is an odd positive integer, $x^{1/n}$ is continuous for all real numbers x. At this point, we will just show that $\sqrt{x}$ is continuous when $x > 0$.

Let $f(x) = \sqrt{x}$ and $a > 0$. Note that

$$|\sqrt{x}-\sqrt{a}| = \frac{|x-a|}{\sqrt{x}+\sqrt{a}}.$$

First, if $|x-a| < a/2$ then $x > a/2 > 0$. This implies that given $\epsilon > 0$, if we choose $\delta = \min(a/2, \epsilon\sqrt{a})$ then

$$|\sqrt{x}-\sqrt{a}| = \frac{|x-a|}{\sqrt{x}+\sqrt{a}} < \frac{\delta}{\sqrt{a}} \le \epsilon \quad \text{whenever} \quad |x-a| < \delta.$$

There is another common class of continuous functions. These are the trigonometric functions. The sine and cosine functions are both functions, which are continuous everywhere. The tangent function is continuous everywhere except at the points when $\cos x = 0$, namely, $x = 0, \pm\pi/2, \pm 3\pi/2, \ldots$. To prove that $\sin x$ is continuous at $x = a$, we only need to use the addition formula for sine and use the fact that $\lim_{h\to 0}\sin h = 0$, $\lim_{h\to 0}\cos h = 1$.

We have thus seen four classes of continuous functions, namely, the polynomials, the rational functions, the root functions and the trigonometric functions. To construct more continuous functions from the known ones, we use the following Theorem:

Theorem 4.3.2. *If f is continuous at b and $\lim_{x\to a} g(x) = b$ then*

$$\lim_{x\to a} f(g(x)) = f(b).$$

In other words,

$$\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x)).$$

Corollary 4.3.3. *If g is continuous at a and f is continuous at $g(a)$, then the composite function $f \circ g$ given by $(f \circ g)(x) = f(g(x))$ is continuous at a.*

Proof of Corollary. The number a is in the domain of $f \circ g$ since a is in the domain of g and $g(a)$ is in the domain of f. Then

$$\lim_{x \to a}(f \circ g)(x) = \lim_{x \to a} f(g(x)) = f(\lim_{x \to a} g(x)) = f(g(a)) = (f \circ g)(a),$$

and so $f \circ g$ is continuous at a.

With the above Corollary, we can show that every algebraic function is continuous on its domain, for example, that

$$\sqrt[3]{x^6 + x - 1}$$

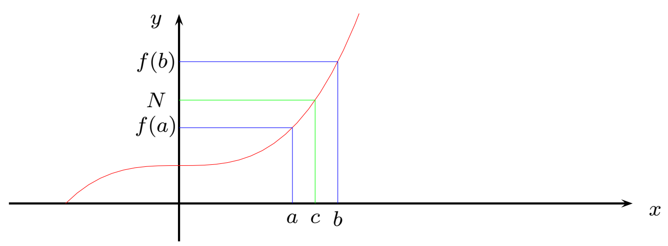
is continuous for all real x.

4.4 The Intermediate Value Theorem

Theorem 4.4.1 (Intermediate Value Theorem). *Suppose that f is* **continuous on the closed interval** $[a, b]$, *and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c such that $f(c) = N$.*

The proof of this result is beyond the scope of this course. However, intuitively this is clear. If we think of a continuous function as a function whose graph has no holes or break, then we see that any line $y = N$ drawn between $y = f(a)$ and $y = f(b)$ as in the following figure must cut the graph. This means that there exists a c such that

$$f(c) = N.$$



Example 4.4.1. Show that there is a root of the equation

$$4x^3 - 6x^2 + 3x - 2 = 0.$$