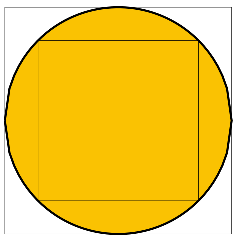
Chapter 7

Integrals

7.1 The area problem

The area of a circle of radius 1 is defined to be the number π. In an attempt to estimate this number π, Archimedes used inscribed and circumscribed polygons to approximate its value. An example of such approximations is shown as follow:

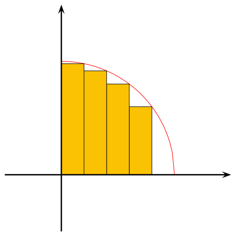


These approximations give

$$2 \leq \pi \leq 4.$$

Using polygons with more sides, Archimedes was able to approximate π to two decimal places. He is perhaps the first person to have used the idea of Calculus.

We now use Archimedes' idea in a slightly modified way. We will estimate the area of part of the circle in the first quadrant. Consider the following graph:



Note that the area of the region above is approximated by four rectangles lying below the curve $y = \sqrt{1-x^2}$. We denote R_n to be the area of the rectangles determined by n equally spaced intervals. In this case we have $R_5 \simeq 0.76$ and we have

$$0.76 < \pi/4.$$

Similarly consider the following diagram:

Note that the area of the figure above is approximated by five rectangles lying above the curve $y = \sqrt{1-x^2}$. We denote L_n to be the area of the rectangles determined by n equally spaced intervals. In this case we have $L_5 \simeq 1.92$ and we have

$$\pi/4 < 0.96.$$

We now make the observation that if we increase the number of intervals, we have the following: then

$$\lim_{n\to\infty} L_n = \lim_{n\to\infty} R_n = \pi/4.$$

Here,

$$\lim_{n\to\infty} a_n = L$$

means that for every $\epsilon > 0$ there exists a positive integer N_ϵ such that

$$|a_n - L| < \epsilon \quad \text{whenever} \quad n > N_\epsilon.$$

This is called the limit of a sequence and we will discuss it in more details in the Chapter on sequences.

We now write this explicitly in terms of symbols. Given an interval $[a, b]$, we subdivide it into n equal intervals (this is not necessary in general). Call our intervals

$$[x_0, x_1], [x_1, x_2], \cdots, [x_{n-1}, x_n].$$

Let the length of the interval be denoted by

$$\Delta x = \frac{b-a}{n}.$$

Then

$$L_n = f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x$$

while

$$R_n = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x.$$

The area A bounded by the x axis and the graph of f (assuming that this graph is above the x-axis) is therefore

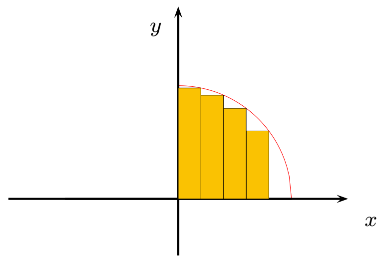
$$A = \lim_{n\to\infty} L_n = \lim_{n\to\infty} R_n.$$

Here we have used the left hand points and right hand points to approximate the area under a graph of a continuous function $y = f(x)$. In general we may replace these points by any points (called *sample points*) in the interval $[x_{i-1}, x_i]$.

Example 7.1.1. Let A be the area of the region that lies under the graph of $f(x) = \cos x$ betweein $x = 0$ and $x = b$ where $0 \le b \le \pi/2$. Using right endpoints, find an expression for A as a limit. Do not evaluate the limit.

7.2 The Distance Problem

If the velocity of a car is given by the graph $v(t) = \sqrt{1-t^2}$, where t is time. Then to compute the distance traveled given that we only know the velocity at $t = 0.2, 0.4, 0.6, 0.8$ and 1, we find that the distance is exactly the area given by the rectangles (see the diagram below):



If we know more values for the velocity, we see that the distance we calculated would be closer and closer to the area bounded by the graph of v. This shows that the area bounded by the x-axis and the graph of v can be viewed as the distance traveled by the car.

7.3 The Definite Integral

We saw in the previous section that a limit of the form

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x$$

arises when we compute an area. We shall denote such a sum as

$$\int_a^b f(x)\,dx.$$

It is called the definite integral of f from a to b.

The Definition is as follow:

Definition 7.3.1. *If f is continuous on $[a,b]$, we divide the interval $[a,b]$ into n intervals of equal width $\Delta x = (b-a)/n$. We let $x_0 = a, x_1, \cdots, x_n (= b)$ be the endpoints of these subintervals and we let $x_1^*, x_2^*, \cdots, x_n^*$ be any* **sample points** *in these subintervals, so x_i^* lies in the ith subinterval $[x_{i-1}, x_i]$. Then the definite integral of f from a to b is*

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x.$$

It can be shown (in a course in analysis) that the limit on the right hand side in the definition always exists if $f(x)$ is continuous on $[a, b]$. In other words, the limit is independent of the sample points chosen. This is why we can take right end points or left end points to define R_n and L_n and still conclude that

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} L_n.$$

Remarks. The sum $\sum_{i=1}^{n} f(x_i^*)\Delta x$ is called a Riemann sum.

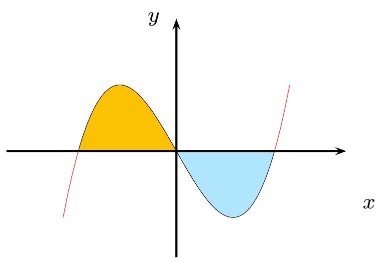
Example 7.3.1 (Section 5.1, Problem 20). Determine a region whose area is equal to the given limit

$$\lim_{n\to\infty} \sum_{i=1}^{\infty} \frac{2}{n}\left(5 + \frac{2i}{n}\right)^{10}.$$

We have seen that the area bounded by the graph of f ($f(x) \geq 0$ for all $x \in [a, b]$) and the x-axis can be computed if f is continuous. The area is given by

$$A = \lim_{n\to\infty} \sum_{j=1}^{n} f(x_j)\Delta x.$$

When $f(x)$ is not positive in $[a, b]$, we can still compute A. Suppose $f(x)$ is negative on $[a, c]$ and positive on $[c, b]$, then A is the difference of the area above the x-axis on $[c, b]$ and that below the x-axis on $[a, c]$.



Note that in all definition of R_n, we use $\Delta x = \frac{b-a}{n}$ and we have assumed that $b > a$. If we extend this definition to allow $a > b$, then we see that

$$\int_a^b f(x)\,dx = -\int_a^b f(x)\,dx.$$

When $a = b$, we see that

$$\int_a^a f(x)\,dx = 0.$$

Theorem 7.3.1 (Properties of definite integrals).

1. $\displaystyle\int_a^b c\,dx = c(b-a)$, *where c is any constant,*

2. $\displaystyle\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx,$

3. $\displaystyle\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$, *where c is any constant,*

4. $\displaystyle\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx,$

5. *If $f(x) \geq 0$ for $a \leq x \leq b$ then $\int_a^b f(x)\,dx \geq 0$,*

6. *If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$,*

7. *If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then*

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a).$$

Example 7.3.2. Prove Properties 3 and 6.

Example 7.3.3. Use Riemann sums to evaluate $\int_0^1 x^2\,dx$. Use the properties of integrals to evaluate $\int_0^1 (4+3x^2)\,dx$.

7.4 The Fundamental Theorem of Calculus

The fundamental of Calculus is appropriately named because it establishes a connection between the two branches of calculus: Differential calculus and Integral calculus. Differential calculus arose from the tangent problem, whereas integral calculus arose from a seemingly unrelated problem, the area problem. Newton's teacher at Cambridge, Isaac Barrow (1630-1677), discovered that these two problems are actually closely related. In fact, he realized that differentiation and integration are inverse processes. The Fundamental Theorem of Calculus gives the precise inverse relationship between the derivative and the integral. It was Newton and Leibniz who exploited this relationship and used it to develop calculus into a systematic mathematical method. In particular, they saw that the Fundamental Theorem enabled them to compute areas and integrals very easily without having to compute them as limits of sums.

Theorem 7.4.1 (The Fundamental Theorem of Calculus, Part 1). *If f is continuous on $[a, b]$, then the function g defined by*

$$g(x) = \int_a^x f(t)\,dt, \quad a \leq x \leq b$$

is continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) = f(x)$.

Proof will be discussed in class

Example 7.4.1. Find the $\dfrac{d}{dx} \int_1^{x^4} \sec t\, dt$.

Theorem 7.4.2 (The Fundamental Theorem of Calculus, Part 2). *If f is continuous on $[a, b]$, then*

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any antiderivative of f, that is, a function such that $F' = f$.

Proof will be discussed in class.

Example 7.4.2. Evaluate the integral $\displaystyle\int_{-2}^{1} x^3\,dx$.

Example 7.4.3. What is wrong with the following calculation?

$$\int_{-1}^{3} \frac{1}{x^2}\,dx = \frac{x^{-1}}{-1}\bigg]_{-1}^{3} = -\frac{1}{3} - 1 = -\frac{4}{3}.$$

7.5 Indefinite integrals and the net change theorem

We know that an antiderivative is a function F such that $F'(x) = f(x)$. We will use the symbol

$$\int f(x)\,dx$$

to denote an $F(x)$. This is called an indefinite integral. Note that we use the word "an antiderivative" because if $F' = f$ and $G' = f$ then $(F - G)' = 0$. This implies that $F = G + c$ where c is a constant. This means that both F and $F + c$ are antiderivatives of f. In other word, when we write $\int f(x)\,dx$, we are referring to an entire family of functions (one antiderivative for each value of the constant C).

Warning: You should distinguish carefully between definite and indefinite integrals. An definite integral $\int_a^b f(x)\,dx$ is a number, whereas an indefinite integral $\int f(x)\,dx$ is a function (or family of functions).

The connection between definite integral and indefinite integral is given by Part 2 of the Fundamental Theorem. If f is continuous on $[a, b]$ then

$$\int_a^b f(x)\,dx = \int f(x)\,dx\bigg]_a^b.$$

Here is a list of some indefinite integrals:

$$\int cf(x)\,dx = c\int f(x)\,dx \qquad \int (f(x)+g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx$$

$$\int k\,dx = kx + c \qquad \int x^n\,dx = \frac{x^{n+1}}{n+1} + c, (n \neq -1)$$

$$\int \sin x\,dx = -\cos x + c \qquad \int \cos x\,dx = \sin x + c$$

$$\int \sec^2 x\,dx = \tan x + c \qquad \int \csc^2 x = -\cot x + c$$

$$\int \sec x \tan x\,dx = \sec x + c \qquad \int \csc x\,dx = -\csc x + c.$$

When we write $\int \frac{1}{x^2}\,dx$, it is understood that it is only valid on certain interval. For example, the above integral is only valid for $x \neq 0$.

Example 7.5.1. Find the general indefinite integral

$$\int (10x^4 - 2\sec^2 x)\,dx.$$

7.6 The substitution rule (The first technique of integration)

Consider the integral

$$\int 2x\sqrt{1-x^2}\,dx.$$

Our standard formulas do not tell us how to solve the above. Let us use the substitution $u = x^2$.

The Substitution Rule. If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

The Substitution Rule for Definite Integrals. If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

Example 7.6.1. Evaluate

$$\int_1^2 \frac{dx}{(3-5x)^2}.$$