

### Answers/Solutions of Exercise 3 (Version: September 21, 2012)

1.  $\mathbf{u} = (1, \sqrt{3})$ ,  $\mathbf{v} = (-\sqrt{3}, -1)$ ,  $\mathbf{u} + \mathbf{v} = (1 - \sqrt{3}, -1 + \sqrt{3})$ ,  
 $3\mathbf{u} - 2\mathbf{v} = (3 + 2\sqrt{3}, 2 + 3\sqrt{3})$ .

2. (a) Substituting  $(x, y) = (1, 2)$  and  $(2, -1)$  into the equation  $ax + by = c$ , we has a system of linear equations

$$\begin{cases} a + 2b - c = 0 \\ 2a - b - c = 0 \end{cases}$$

which implies  $a = \frac{3}{5}c$  and  $b = \frac{1}{5}c$ . In set notation, the line is

$$\{(x, y) \mid 3x + y = 5\} \text{ (implicit)} \quad \text{and} \quad \{(\frac{5-t}{3}, t) \mid t \in \mathbb{R}\} \text{ (explicit)}.$$

- (b) Substituting  $(x, y, z) = (0, 1, -1)$ ,  $(1, -1, 0)$  and  $(0, 2, 0)$  into the equation  $ax + by + cz = d$ , we has a system of linear equations

$$\begin{cases} b - c - d = 0 \\ a - b - d = 0 \\ 2b - d = 0 \end{cases}$$

which implies  $a = \frac{3}{2}d$ ,  $b = \frac{1}{2}d$  and  $c = -\frac{1}{2}d$ . In set notation, the plane is

$$\{(x, y, z) \mid 3x + y - z = 2\} \text{ (implicit)} \quad \text{and} \quad \{(\frac{2-s+t}{3}, s, t) \mid t \in \mathbb{R}\} \text{ (explicit)}.$$

- (c) In explicit form, the line is

$$\{(1, -1, 0) + t(-1, 2, -1) \mid t \in \mathbb{R}\} = \{(1 - t, -1 + 2t, -t) \mid t \in \mathbb{R}\}.$$

To find the implicit form, we need to find two non-parallel planes containing the two points  $(0, 1, -1)$  and  $(1, -1, 0)$ . The intersection of the two planes will give us the required line. Substituting  $(0, 1, -1)$  and  $(1, -1, 0)$  into  $ax + by + cz = d$  we has a system of linear equations

$$\begin{cases} b - c - d = 0 \\ a - b - d = 0 \end{cases}$$

We obtain  $a = c + 2d$  and  $b = c + d$ . There are infinitely many such planes. For example, we can write the line implicitly as

$$\{(x, y, z) \mid x + y + z = 0 \text{ and } 2x + y = 1\}.$$

3.  $A = B = C = F$  and  $A, D, E$  are all different.

4. (a)  $U$  and  $V$  contains the origin but  $W$  does not.

$$(b) \begin{cases} 2x - y + 3z = 0 \\ 3x + 2y - z = 0 \end{cases} \Leftrightarrow \begin{cases} x = -\frac{5}{7}t \\ y = \frac{11}{7}t \\ z = t \end{cases} \text{ where } t \in \mathbb{R}$$

$$\text{So } U \cap V = \{(-\frac{5}{7}t, \frac{11}{7}t, t) \mid t \in \mathbb{R}\}.$$

$$\begin{cases} 3x + 2y - z = 0 \\ x - 3y - 2z = 1 \end{cases} \Leftrightarrow \begin{cases} x = \frac{1}{11}(2 + 7t) \\ y = \frac{1}{11}(-3 - 5t) \\ z = t \end{cases} \text{ where } t \in \mathbb{R}$$

$$\text{So } V \cap W = \{(\frac{2+7t}{11}, \frac{-3-5t}{11}, t) \mid t \in \mathbb{R}\}.$$

5. (a)  $A$  is a line joining the points  $(1, 1, 1)$  and  $(2, 3, 4)$ .

(b) Let  $B = \{(x, y, z) \mid x + y - z = 1 \text{ and } x - 2y + z = 0\}$ . Since  $x + y - z = 1$  and  $x + y - z = 1$  are two non-parallel lines,  $B$  is the line of intersection of the two planes. To show that  $A = B$ , it suffices to show that the line  $A$  lies on both planes. This is true because  $(1 + t) + (1 + 2t) - (1 + 3t) = 1$  and  $(1 + t) - 2(1 + 2t) + (1 + 3t) = 0$  for all  $t \in \mathbb{R}$ .

$$(c) \text{ For example, } \mathbf{M} = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -2 & 1 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

6. Since

$$\begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ a & b & c & d \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ b & c & d \end{vmatrix} - 0 + \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ a & b & d \end{vmatrix} - 0 = a + b - d - c,$$

$$V = \{(a, b, c, d) \mid a + b - d - c = 0\} = \{(x, y, z, w) \mid x + y - z - w = 0\} = T.$$

On the other hand,  $S \neq T$  because  $(1, -1, 0, 0) \in T$  but  $(1, -1, 0, 0) \notin S$ .

7. (a) For example,  $P = \{(1 + s - t, s, t) \mid s, t \in \mathbb{R}\}$ .

(b)  $A$  lies in  $P$  because  $a - a + 1 = 1$ . Since both  $B$  and  $C$  pass through  $(0, 0, 0)$  and  $(0, 0, 0) \notin P$ ,  $B$  and  $C$  does not lies in  $P$ .

(c)  $B$  intersects  $P$  at one point,  $(1, 0, 0)$ .

(d) The plane  $x - y + z = 0$  contains  $C$  but not  $A$  and  $B$ .

(e) No. By Discussion 1.4.11, the solution set of a consistent nonzero linear system in three variables represents a point, a line or a plane in  $\mathbb{R}^3$ . Suppose we have a nonzero linear system whose solution set contains both  $B$  and  $C$ . Then the solution set must be a plane. However, the plane containing both  $B$  and  $C$  is the  $xz$ -plane which does not contain  $A$ . So the solution set cannot contain  $A$ .

8.  $(2, 3, -7, 3)$ ,  $(0, 0, 0, 0)$  and  $(-4, 6, -13, 4)$  are vectors in  $\text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$  while  $(1, 1, 1, 1)$  is not.

9.  $S_4$  and  $S_6$  span  $\mathbb{R}^3$  while  $S_1, S_2, S_3$  and  $S_5$  do not span  $\mathbb{R}^3$ .

10. (a) Since  $(1, 1, 0)$  and  $(5, 2, 3)$  satisfy the equation  $x - y - z = 0$ ,  $(1, 1, 0), (5, 2, 3) \in V$  and hence  $\text{span}(S) \subseteq V$ .

Note that a general solution of  $x - y - z = 0$  is  $x = s + t$ ,  $y = s$ ,  $z = t$  where  $s, t \in \mathbb{R}$ . Let  $(s + t, s, t)$  be any vector in  $V$ . Consider the following equation:

$$a(1, 1, 0) + b(5, 2, 3) = (s + t, s, t) \Leftrightarrow \begin{cases} a + 5b = s + t \\ a + 2b = s \\ 3b = t. \end{cases}$$

Since

$$\left( \begin{array}{cc|c} 1 & 5 & s+t \\ 1 & 2 & s \\ 0 & 3 & t \end{array} \right) \xrightarrow[\text{Elimination}]{\text{Gaussian}} \left( \begin{array}{cc|c} 1 & 5 & s+t \\ 0 & 3 & t \\ 0 & 0 & 0 \end{array} \right),$$

the system is consistent for all  $s, t \in \mathbb{R}$ . So  $V \subseteq \text{span}(S)$ .

We have shown that  $\text{span}\{(1, 1, 0), (5, 2, 3)\} = V$ .

(b) Since

$$\left( \begin{array}{ccc} 1 & 5 & 0 \\ 1 & 2 & 0 \\ 0 & 3 & 1 \end{array} \right) \xrightarrow[\text{Elimination}]{\text{Gaussian}} \left( \begin{array}{ccc} 1 & 5 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{array} \right),$$

by Discussion 3.2.5,  $\text{span}\{(1, 1, 0), (5, 2, 3), (0, 0, 1)\} = \mathbb{R}^3$ .

$$11. \quad (a) \quad \left( \begin{array}{cc|c} 1 & 0 & 2 \\ -1 & 1 & -2 \\ -5 & 1 & 0 \end{array} \right) \xrightarrow[\text{Elimination}]{\text{Gaussian}} \left( \begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 10 \end{array} \right)$$

Since  $\mathbf{u}_2 \notin \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ ,  $\text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} \neq \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ .

$$(b) \left( \begin{array}{ccc|c|c} 1 & 2 & -1 & 1 & 0 \\ 6 & 4 & 2 & -2 & 8 \\ 4 & -1 & 5 & -5 & 9 \end{array} \right) \xrightarrow[\text{Elimination}]{\text{Gaussian}} \left( \begin{array}{ccc|c|c} 1 & 2 & -1 & 1 & 0 \\ 0 & -8 & 8 & -8 & 8 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

The systems are consistent and thus  $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\} \subseteq \text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ .

$$\left( \begin{array}{ccc|c|c} 1 & 0 & 1 & 2 & -1 \\ -2 & 8 & 6 & 4 & 2 \\ -5 & 9 & 4 & -1 & 5 \end{array} \right) \xrightarrow[\text{Elimination}]{\text{Gaussian}} \left( \begin{array}{ccc|c|c} 1 & 0 & 1 & 2 & -1 \\ 0 & 8 & 8 & 8 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

The systems are consistent and thus  $\text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} \subseteq \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ .

So  $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\} = \text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ .

$$12. (a) \left( \begin{array}{cccc|c} -1 & 3 & 0 & -4 & 1 \\ 2 & 1 & 1 & 3 & 0 \\ 1 & 4 & 1 & -1 & 2 \\ 0 & 0 & 3 & 6 & 5 \end{array} \right) \xrightarrow[\text{Elimination}]{\text{Gaussian}} \left( \begin{array}{cccc|c} -1 & 3 & 0 & -4 & 1 \\ 0 & 7 & 1 & -5 & 2 \\ 0 & 0 & 3 & 6 & 5 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right)$$

Since  $\mathbf{u}_2 \notin \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ ,  $\text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\} \not\subseteq \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ .

$$(b) \left( \begin{array}{cccc|c|c|c|c} 2 & 1 & 0 & 1 & -1 & 3 & 0 & -4 \\ 0 & 0 & 3 & 1 & 2 & 1 & 1 & 3 \\ 2 & 2 & 6 & 2 & 1 & 4 & 1 & -1 \\ 0 & 5 & 9 & -1 & 0 & 0 & 3 & 6 \end{array} \right) \xrightarrow[\text{Elimination}]{\text{Gaussian}} \left( \begin{array}{cccc|c|c|c|c} 2 & 1 & 0 & 1 & -1 & 3 & 0 & -4 \\ 0 & 1 & 6 & 1 & 2 & 1 & 1 & 3 \\ 0 & 0 & 3 & 1 & 2 & 1 & 1 & 3 \\ 0 & 0 & 0 & 1 & 4 & 2 & 5 & 12 \end{array} \right)$$

The systems are consistent and thus  $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\} \subseteq \text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ .

(c)  $\text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\} = \mathbb{R}^4$ .

(d)  $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\} \neq \mathbb{R}^4$ .

13. By Example 3.2.8.2,  $S_1$  does not span  $\mathbb{R}^3$ .

Since  $\mathbf{w} - \mathbf{u} = -(\mathbf{u} - \mathbf{v}) - (\mathbf{v} - \mathbf{w})$ ,  $\text{span}(S_2) = \text{span}\{\mathbf{u} - \mathbf{v}, \mathbf{v} - \mathbf{w}\}$  and by Example 3.2.8.2,  $S_2$  does not span  $\mathbb{R}^3$ .

$S_3$  spans  $\mathbb{R}^3$ : Since  $\text{span}(S_3) \subseteq \mathbb{R}^3$ , we only need to show  $\text{span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\} \subseteq \text{span}(S_3)$ . Note that

$$\begin{aligned} \mathbf{u} &= \frac{1}{2}[(\mathbf{u} - \mathbf{v}) + (\mathbf{v} - \mathbf{w}) + (\mathbf{u} + \mathbf{w})], \\ \mathbf{v} &= \frac{1}{2}[-(\mathbf{u} - \mathbf{v}) + (\mathbf{v} - \mathbf{w}) + (\mathbf{u} + \mathbf{w})], \\ \mathbf{w} &= \frac{1}{2}[-(\mathbf{u} - \mathbf{v}) - (\mathbf{v} - \mathbf{w}) + (\mathbf{u} + \mathbf{w})]. \end{aligned}$$

By Theorem 3.2.10,  $\text{span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\} \subseteq \text{span}(S_3)$ . Hence  $\text{span}(S_3)$  spans  $\mathbb{R}^3$ .

Using the same argument as for  $S_3$ , we can show that both  $S_4$  and  $S_5$  also span  $\mathbb{R}^3$ .

14. (a) True. Let  $\mathbf{u} = (u)$  for  $u \neq 0$ . Then for any  $(c) \in \mathbb{R}^1$ ,  $(c) = \frac{c}{u} \mathbf{u}$ .
- (b) False. For example, let  $\mathbf{u} = (1, 1)$ ,  $\mathbf{v} = (2, 2)$ .
- (c) False. For example, let  $S_1 = \{(1, 0), (0, 1)\}$ ,  $S_2 = \{(1, 0), (0, 2)\}$ .
- (d) False. For example, let  $S_1 = \{(1, 0)\}$ ,  $S_2 = \{(0, 1)\}$ .