

1. (a)  $T_1$  is a linear transformation with standard matrix  $\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ .
- (b)  $T_2$  is not a linear transformation.
- (c)  $T_3$  is a linear transformation with standard matrix  $\begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ .
- (d)  $T_4$  is not a linear transformation.
- (e)  $T_5$  is a linear transformation with standard matrix  $(y_1 \ y_2 \ \cdots \ y_n)$ .
- (f)  $T_6$  is not a linear transformation.

2. (a) There is enough information.

$$T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \frac{1}{5} \begin{pmatrix} x + 2y \\ 3x + 2y + 4z \\ -y + z \\ x + 4y + 6z \end{pmatrix} \quad \text{for } \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3.$$

The standard matrix is  $\begin{pmatrix} 1 & 2 & 0 \\ 3 & 2 & 4 \\ 0 & -1 & 1 \\ 1 & 4 & 6 \end{pmatrix}$ .

- (b) The information is not enough because the two vectors do not form a basis for  $\mathbb{R}^2$ .
- (c) There is enough information.

$$T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \frac{1}{5} \begin{pmatrix} x - y \\ x + y \end{pmatrix} \quad \text{for } \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2.$$

The standard matrix is  $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ .

- (d) There is enough information.

$$T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \frac{1}{5} \begin{pmatrix} x + 17y - 8z \\ x + 22y - 8z \end{pmatrix} \quad \text{for } \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3.$$

The standard matrix is  $\begin{pmatrix} \frac{1}{5} & \frac{17}{5} & \frac{-8}{5} \\ \frac{1}{5} & \frac{22}{5} & \frac{-8}{5} \end{pmatrix}$ .

(e) There is enough information.

$$T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} y+z \\ z \\ x+z \end{pmatrix} \quad \text{for } \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3.$$

The standard matrix is  $\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$

(f) The information is not enough because the three vectors do not form a basis for  $\mathbb{R}^3$ .

3. (a)  $(S \circ T)\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 2x \\ 2y \\ x+y \end{pmatrix}.$

$T \circ S$  is not defined.

(b)  $(S \circ T)\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} -2x - y + 3z \\ -x - y + 3z \\ -3x - 2y + 6z \end{pmatrix}.$

$$(T \circ S)\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 2x + 2y \\ 2x + y \end{pmatrix}.$$

4. ( $\Rightarrow$ ) It is a particular case of Theorem 7.1.4.2.

( $\Leftarrow$ ) Suppose

$$T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v}) \quad \text{for all } \mathbf{u}, \mathbf{v} \in \mathbb{R}^n \text{ and } c, d \in \mathbb{R}. \quad (*)$$

Let  $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$  be the standard basis for  $\mathbb{R}^n$  and let  $\mathbf{A}$  be the  $m \times n$  matrix  $(T(\mathbf{e}_1) \ T(\mathbf{e}_2) \ \dots \ T(\mathbf{e}_n))$ .

For any  $\mathbf{u} = (u_1, u_2, \dots, u_n)^T \in \mathbb{R}^n$ ,  $\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + \dots + u_n\mathbf{e}_n$ . By applying (\*) repeatedly, we have

$$\begin{aligned} T(\mathbf{u}) &= u_1T(\mathbf{e}_1) + u_2T(\mathbf{e}_2) + \dots + u_nT(\mathbf{e}_n) \\ &= (T(\mathbf{e}_1) \ T(\mathbf{e}_2) \ \dots \ T(\mathbf{e}_n)) \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} \\ &= \mathbf{A}\mathbf{u}. \end{aligned}$$

Thus  $T$  is a linear transformation.

5. (a) For any  $\mathbf{u} \in \mathbb{R}^n$ ,  $(T_1 + T_2)(\mathbf{u}) = T_1(\mathbf{u}) + T_2(\mathbf{u}) = \mathbf{A}\mathbf{u} + \mathbf{B}\mathbf{u} = (\mathbf{A} + \mathbf{B})\mathbf{u}$ . So  $T_1 + T_2$  is a linear transformation and the standard matrix for  $T_1 + T_2$  is  $\mathbf{A} + \mathbf{B}$ .
- (b) For any  $\mathbf{u} \in \mathbb{R}^n$ ,  $(\lambda T)(\mathbf{u}) = \lambda T(\mathbf{u}) = \lambda \mathbf{A}\mathbf{u} = (\lambda \mathbf{A})\mathbf{u}$ . So  $\lambda T$  is a linear transformation and the standard matrix for  $\lambda T$  is  $\lambda \mathbf{A}$ .
6. (a) (i)  $T$  is invertible and the inverse of  $T$  is  $T$  itself.  
(ii)  $T$  is not invertible. Assume there exists an inverse  $S : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ . Then  $(1, 0)^T = S \circ T((1, 0)^T) = S((1, 0)^T) = S \circ T((0, 1)^T) = (0, 1)^T$ , a contradiction.
- (b)  $\mathbf{A}^{-1}$ .
7. (a) Note that  $(\mathbf{n} \cdot \mathbf{x})\mathbf{n} = \mathbf{n}\mathbf{n}^T\mathbf{x}$  where LHS is the scalar  $\mathbf{n} \cdot \mathbf{x}$  multiplied to the vector  $\mathbf{n}$  while all operations on RHS are matrix multiplications. (To verify the equation, let  $\mathbf{n} = (a_1, \dots, a_n)^T$  and  $\mathbf{x} = (x_1, \dots, x_n)^T$  and then check that both sides give us the same vector.)
- For any  $\mathbf{x} \in \mathbb{R}^n$ ,  $P(\mathbf{x}) = \mathbf{x} - (\mathbf{n} \cdot \mathbf{x})\mathbf{n} = \mathbf{I}\mathbf{x} - \mathbf{n}\mathbf{n}^T\mathbf{x} = (\mathbf{I} - \mathbf{n}\mathbf{n}^T)\mathbf{x}$ . So  $P$  is a linear transformation and the standard matrix for  $P$  is  $\mathbf{I} - \mathbf{n}\mathbf{n}^T$ .
- (b) Since for all  $\mathbf{x} \in \mathbb{R}^n$ ,

$$\begin{aligned} (P \circ P)(\mathbf{x}) &= P(P(\mathbf{x})) = P(\mathbf{x} - (\mathbf{n} \cdot \mathbf{x})\mathbf{n}) \\ &= \mathbf{x} - (\mathbf{n} \cdot \mathbf{x})\mathbf{n} - \{\mathbf{n} \cdot [\mathbf{x} - (\mathbf{n} \cdot \mathbf{x})\mathbf{n}]\}\mathbf{n} \\ &= \mathbf{x} - (\mathbf{n} \cdot \mathbf{x})\mathbf{n} - \{(\mathbf{n} \cdot \mathbf{x}) - (\mathbf{n} \cdot \mathbf{x})(\mathbf{n} \cdot \mathbf{n})\}\mathbf{n} \\ &= \mathbf{x} - (\mathbf{n} \cdot \mathbf{x})\mathbf{n} = P(\mathbf{x}), \end{aligned}$$

$$P \circ P = P.$$

**Alternatively**, since  $\mathbf{n}$  is a unit vector,  $\mathbf{n}^T\mathbf{n} = \mathbf{n} \cdot \mathbf{n} = 1$ . Thus

$$(\mathbf{I} - \mathbf{n}\mathbf{n}^T)^2 = (\mathbf{I} - \mathbf{n}\mathbf{n}^T)(\mathbf{I} - \mathbf{n}\mathbf{n}^T) = \mathbf{I} - 2\mathbf{n}\mathbf{n}^T + \mathbf{n}\mathbf{n}^T\mathbf{n}\mathbf{n}^T = \mathbf{I} - \mathbf{n}\mathbf{n}^T.$$

By Theorem 7.1.11,  $P \circ P = P$ .

8. (a) Suppose  $T$  is not the zero transformation. So there exists  $\mathbf{x} \in \mathbb{R}^n$  such that  $T(\mathbf{x}) \neq 0$ . Define  $\mathbf{u} = T(\mathbf{x})$ . Then  $\mathbf{u}$  is a nonzero vector and

$$T(\mathbf{u}) = T(T(\mathbf{x})) = (T \circ T)(\mathbf{x}) = T(\mathbf{x}) = \mathbf{u}.$$

- (b) Suppose  $T$  is not the identity transformation. So there exists  $\mathbf{y} \in \mathbb{R}^n$  such that  $T(\mathbf{y}) \neq \mathbf{y}$ . Define  $\mathbf{v} = T(\mathbf{y}) - \mathbf{y}$ . Then  $\mathbf{v}$  is a nonzero vector and

$$T(\mathbf{v}) = T(T(\mathbf{y}) - \mathbf{y}) = (T \circ T)(\mathbf{y}) - T(\mathbf{y}) = T(\mathbf{y}) - T(\mathbf{y}) = \mathbf{0}.$$

- (c) Let  $\mathbf{A}$  be the standard matrix for  $T$ . If  $T$  is not the zero transformation and the identity transformation, then by (a) and (b), 1 and 0 are the eigenvalues of  $\mathbf{A}$ . So by the result of Question 6.4,

$$\mathbf{A} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} r & s \\ t & 1-r \end{pmatrix} \quad \text{where } st = r(1-r).$$

9. (a) Similar to Question 7.7, for any  $\mathbf{x} \in \mathbb{R}^n$ ,  $F(\mathbf{x}) = \mathbf{x} - 2(\mathbf{n} \cdot \mathbf{x})\mathbf{n} = \mathbf{I}\mathbf{x} - 2\mathbf{n}\mathbf{n}^T\mathbf{x} = (\mathbf{I} - 2\mathbf{n}\mathbf{n}^T)\mathbf{x}$ . So  $F$  is a linear transformation and the standard matrix for  $F$  is  $\mathbf{I} - 2\mathbf{n}\mathbf{n}^T$ .

- (b) Since for all  $\mathbf{x} \in \mathbb{R}^n$ ,

$$\begin{aligned} (F \circ F)(\mathbf{x}) &= F(F(\mathbf{x})) = F(\mathbf{x} - 2(\mathbf{n} \cdot \mathbf{x})\mathbf{n}) \\ &= \mathbf{x} - 2(\mathbf{n} \cdot \mathbf{x})\mathbf{n} - 2\{\mathbf{n} \cdot [\mathbf{x} - 2(\mathbf{n} \cdot \mathbf{x})\mathbf{n}]\}\mathbf{n} \\ &= \mathbf{x} - 2(\mathbf{n} \cdot \mathbf{x})\mathbf{n} - 2\{(\mathbf{n} \cdot \mathbf{x}) - 2(\mathbf{n} \cdot \mathbf{x})(\mathbf{n} \cdot \mathbf{n})\}\mathbf{n} \\ &= \mathbf{x} - 2(\mathbf{n} \cdot \mathbf{x})\mathbf{n} - 2\{-(\mathbf{n} \cdot \mathbf{x})\}\mathbf{n} = \mathbf{x}, \end{aligned}$$

$F \circ F$  is the identity transformation.

**Alternatively,**

$$(\mathbf{I} - 2\mathbf{n}\mathbf{n}^T)^2 = (\mathbf{I} - 2\mathbf{n}\mathbf{n}^T)(\mathbf{I} - 2\mathbf{n}\mathbf{n}^T) = \mathbf{I} - 4\mathbf{n}\mathbf{n}^T + 4\mathbf{n}\mathbf{n}^T\mathbf{n}\mathbf{n}^T = \mathbf{I}.$$

By Theorem 7.1.11,  $F \circ F$  is the identity transformation.

- (c) Note that  $(\mathbf{I} - 2\mathbf{n}\mathbf{n}^T)^T = \mathbf{I} - 2(\mathbf{n}\mathbf{n}^T)^T = \mathbf{I} - 2\mathbf{n}\mathbf{n}^T$ . Thus

$$(\mathbf{I} - 2\mathbf{n}\mathbf{n}^T)(\mathbf{I} - 2\mathbf{n}\mathbf{n}^T)^T = (\mathbf{I} - 2\mathbf{n}\mathbf{n}^T)^2 = \mathbf{I}$$

by (b). The standard matrix is an orthogonal matrix.

10. (a) By Theorem 7.1.4.2,

$$\begin{aligned} T(\mathbf{u} + \mathbf{v}) \cdot T(\mathbf{u} + \mathbf{v}) &= (T(\mathbf{u}) + T(\mathbf{v})) \cdot (T(\mathbf{u}) + T(\mathbf{v})) \\ &= T(\mathbf{u}) \cdot T(\mathbf{u}) + 2(T(\mathbf{u}) \cdot T(\mathbf{v})) + T(\mathbf{v}) \cdot T(\mathbf{v}) \\ &= \|T(\mathbf{u})\|^2 + \|T(\mathbf{v})\|^2 + 2(T(\mathbf{u}) \cdot T(\mathbf{v})) \\ &= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2(T(\mathbf{u}) \cdot T(\mathbf{v})). \end{aligned} \tag{1}$$

On the other hand,

$$\begin{aligned} T(\mathbf{u} + \mathbf{v}) \cdot T(\mathbf{u} + \mathbf{v}) &= \|T(\mathbf{u} + \mathbf{v})\|^2 \\ &= \|\mathbf{u} + \mathbf{v}\|^2 \\ &= (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) \\ &= \mathbf{u} \cdot \mathbf{u} + 2(\mathbf{u} \cdot \mathbf{v}) + \mathbf{v} \cdot \mathbf{v} \\ &= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2(\mathbf{u} \cdot \mathbf{v}). \end{aligned} \tag{2}$$

Thus (1) and (2) imply  $T(\mathbf{u}) \cdot T(\mathbf{v}) = \mathbf{u} \cdot \mathbf{v}$ .

- (b) ( $\Leftarrow$ ) Suppose  $\mathbf{A}$  is an orthogonal matrix of order  $n$ . Then by Question 5.32, for all  $\mathbf{u} \in \mathbb{R}^n$ ,

$$\|T(\mathbf{u})\| = \|\mathbf{A}\mathbf{u}\| = \|\mathbf{u}\|.$$

So  $T$  is an isometry.

- ( $\Rightarrow$ ) Suppose  $T$  is an isometry on  $\mathbb{R}^n$ . Let  $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$  be the standard basis for  $\mathbb{R}^n$ . Then

$$\begin{aligned} (\mathbf{A}\mathbf{e}_i) \cdot (\mathbf{A}\mathbf{e}_j) &= (\mathbf{A}\mathbf{e}_i)^T \mathbf{A}\mathbf{e}_j = \mathbf{e}_i^T \mathbf{A}^T \mathbf{A}\mathbf{e}_j \\ &= \text{the } (i, j)\text{-entry of } \mathbf{A}^T \mathbf{A}. \end{aligned} \quad (3)$$

On the other hand, by (a),

$$(\mathbf{A}\mathbf{e}_i) \cdot (\mathbf{A}\mathbf{e}_j) = T(\mathbf{e}_i) \cdot T(\mathbf{e}_j) = \mathbf{e}_i \cdot \mathbf{e}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases} \quad (4)$$

By (3) and (4),  $\mathbf{A}^T \mathbf{A} = \mathbf{I}$ . By Remark 5.4.4,  $\mathbf{A}$  is an orthogonal matrix.

- (c) All isometries on  $\mathbb{R}^2$  are of the form

$$T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x \cos(\theta) + \delta y \sin(\theta) \\ x \sin(\theta) - \delta y \cos(\theta) \end{pmatrix} \quad \text{for } \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$$

where  $\delta = \pm 1$  and  $0 \leq \theta < 2\pi$ .

11. The standard matrix of  $T$  is  $\begin{pmatrix} 2 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$ .

$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix} \xrightarrow[\text{Elimination}]{\text{Gauss-Jordan}} \begin{pmatrix} 1 & 0 & \frac{1}{3} \\ 0 & 1 & -\frac{2}{3} \end{pmatrix}$$

- (a)  $\{(2, 1)^T, (1, -1)^T\}$  is a basis for  $\text{R}(T)$ . (For this example, any two linearly independent vectors in  $\mathbb{R}^2$  is a basis for  $\text{R}(T)$ . Why?)  
 (b)  $\{(-\frac{1}{3}, \frac{2}{3}, 1)^T\}$  is a basis for  $\text{Ker}(T)$ .  
 (c)  $\text{rank}(T) + \text{nullity}(T) = \dim(\text{R}(T)) + \dim(\text{Ker}(T)) = 2 + 1 = 3 = \dim(\mathbb{R}^3)$ .  
 (d) For example,  $\{(-\frac{1}{3}, \frac{2}{3}, 1)^T, (0, 1, 0)^T, (0, 0, 1)^T\}$  is a basis for  $\mathbb{R}^3$ .

$$12. \begin{pmatrix} 3 & -1 & 2 & 7 \\ 1 & 2 & 3 & 0 \\ 0 & 1 & 1 & -1 \end{pmatrix} \xrightarrow[\text{Elimination}]{\text{Gauss-Jordan}} \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

- (a)  $\{(3, 1, 0)^T, (-1, 2, 1)^T\}$  is a basis for  $R(T)$ .
- (b)  $\{(-1, -1, 1, 0)^T, (-2, 1, 0, 1)^T\}$  is a basis for  $\text{Ker}(T)$ .
- (c)  $\text{rank}(T) + \text{nullity}(T) = \dim(R(T)) + \dim(\text{Ker}(T)) = 2 + 2 = 4 = \dim(\mathbb{R}^4)$ .

13. (a) 2. (b) 2. (c) 2.

14. (a)  $\{\mathbf{0}\}$ . (b)  $\mathbb{R}^n$ .

15. (a) Let  $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$  be an orthonormal basis for  $V$ . By Theorem 5.2.15,

$$\begin{aligned} P(\mathbf{u}) &= (\mathbf{u} \cdot \mathbf{v}_1)\mathbf{v}_1 + (\mathbf{u} \cdot \mathbf{v}_2)\mathbf{v}_2 + \cdots + (\mathbf{u} \cdot \mathbf{v}_k)\mathbf{v}_k \\ &= \mathbf{v}_1\mathbf{v}_1^T\mathbf{u} + \mathbf{v}_2\mathbf{v}_2^T\mathbf{u} + \cdots + \mathbf{v}_k\mathbf{v}_k^T\mathbf{u} \\ &= (\mathbf{v}_1\mathbf{v}_1^T + \mathbf{v}_2\mathbf{v}_2^T + \cdots + \mathbf{v}_k\mathbf{v}_k^T)\mathbf{u} \end{aligned}$$

Note that  $\mathbf{v}_1\mathbf{v}_1^T + \mathbf{v}_2\mathbf{v}_2^T + \cdots + \mathbf{v}_k\mathbf{v}_k^T$  is an  $n \times n$  matrix. So  $P$  is a linear transformation.

(b)  $\text{Ker}(P) = \text{span}\{(a, b, c)\}$  and  $R(P) = V$ .

16. ( $\Rightarrow$ ) Let  $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$  such that  $T(\mathbf{u}) = T(\mathbf{v})$ . Then  $T(\mathbf{u} - \mathbf{v}) = T(\mathbf{u}) - T(\mathbf{v}) = \mathbf{0}$  and hence  $\mathbf{u} - \mathbf{v} \in \text{Ker}(T)$ . Since  $\text{Ker}(T) = \{\mathbf{0}\}$ ,  $\mathbf{u} - \mathbf{v} = \mathbf{0}$ , i.e.  $\mathbf{u} = \mathbf{v}$ . Thus  $T$  is one-to-one.

( $\Leftarrow$ ) By Theorem 7.1.4.1,  $T(\mathbf{0}) = \mathbf{0}$ . Since  $T$  is one-to-one, for all  $\mathbf{v} \in \mathbb{R}^n$ , if  $\mathbf{v} \neq \mathbf{0}$ ,  $T(\mathbf{v}) \neq T(\mathbf{0}) = \mathbf{0}$ . Thus  $\text{Ker}(T) = \{\mathbf{0}\}$ .

17. (a) Let  $\mathbf{u} \in \text{Ker}(S)$ , i.e.  $S(\mathbf{u}) = \mathbf{0}$ . Then  $T \circ S(\mathbf{u}) = T(S(\mathbf{u})) = T(\mathbf{0}) = \mathbf{0}$  and hence  $\mathbf{u} \in \text{Ker}(T \circ S)$ . Thus  $\text{Ker}(S) \subseteq \text{Ker}(T \circ S)$ .

(b) Let  $\mathbf{v} \in R(T \circ S)$ , i.e. there exists  $\mathbf{u} \in \mathbb{R}^n$  such that  $\mathbf{v} = T \circ S(\mathbf{u})$ . Put  $\mathbf{w} = S(\mathbf{u}) \in \mathbb{R}^m$ . Then  $\mathbf{v} = T(S(\mathbf{u})) = T(\mathbf{w})$ . This means that  $\mathbf{v} \in R(T)$ . Thus  $R(T \circ S) \subseteq R(T)$ .

18. For this question, it is helpful if you first compute  $T((1, 0)^T)$  and  $T((0, 1)^T)$  and then sketch the vectors on the  $xy$ -plane.

(a)  $T$  is the dilation by a factor of 3.

(b)  $T$  is the contraction by a factor of 0.5.

$$(c) \begin{pmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} \cos(150^\circ) & \sin(150^\circ) \\ \sin(150^\circ) & -\cos(150^\circ) \end{pmatrix}$$

$T$  is the reflection about the line  $y = x \tan(75^\circ)$ .

$$(d) \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} \cos(30^\circ) & -\sin(30^\circ) \\ \sin(30^\circ) & \cos(30^\circ) \end{pmatrix}$$

$T$  is the anti-clockwise rotation about the origin through an angle  $30^\circ$ .

$$(e) \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} \cos(-30^\circ) & -\sin(-30^\circ) \\ \sin(-30^\circ) & \cos(-30^\circ) \end{pmatrix}$$

$T$  is the clockwise rotation about the origin through an angle  $30^\circ$ .

$$(f) \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} \cos(30^\circ) & \sin(30^\circ) \\ \sin(30^\circ) & -\cos(30^\circ) \end{pmatrix}$$

$T$  is the reflection about the line  $y = x \tan(15^\circ)$ .

(g)  $T$  is the scaling along axes in the directions of  $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$  and  $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$  by factors of 3 and 2 respectively.

19. The standard matrices for  $F_1$  and  $F_2$  are

$$\begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \cos(2\phi) & \sin(2\phi) \\ \sin(2\phi) & -\cos(2\phi) \end{pmatrix}$$

respectively. Thus the standard matrices for  $F_2 \circ F_1$  is

$$\begin{aligned} & \begin{pmatrix} \cos(2\phi) & \sin(2\phi) \\ \sin(2\phi) & -\cos(2\phi) \end{pmatrix} \begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix} \\ &= \begin{pmatrix} \cos(2\phi)\cos(2\theta) + \sin(2\phi)\sin(2\theta) & \cos(2\phi)\sin(2\theta) - \sin(2\phi)\cos(2\theta) \\ \sin(2\phi)\cos(2\theta) - \cos(2\phi)\sin(2\theta) & \sin(2\phi)\sin(2\theta) + \cos(2\phi)\cos(2\theta) \end{pmatrix} \\ &= \begin{pmatrix} \cos(2(\phi - \theta)) & -\sin(2(\phi - \theta)) \\ \sin(2(\phi - \theta)) & \cos(2(\phi - \theta)) \end{pmatrix} \end{aligned}$$

which is the standard matrix for the anti-clockwise rotation about the origin through an angle  $2(\phi - \theta)$ .

20. (a) The standard matrix for  $T_1 \circ T_2$  is  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$  and the

standard matrix for  $T_2 \circ T_1$  is  $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$ .

Hence  $T_1 \circ T_2 = T_2 \circ T_1$ .

(b) The standard matrix for  $T_1 \circ T_2$  is  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$  and the standard matrix for  $T_2 \circ T_1$  is  $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$ .

Hence  $T_1 \circ T_2 \neq T_2 \circ T_1$ .

(c) The standard matrix for  $T_1 \circ T_2$  is  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$  and the standard matrix for  $T_2 \circ T_1$  is  $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ .

Hence  $T_1 \circ T_2 \neq T_2 \circ T_1$ .

21. (a) True. Let the standard matrices for  $R_1$  and  $R_2$  be  $\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$  and  $\begin{pmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{pmatrix}$  respectively. Then the standard matrix for  $R_2 \circ R_1$  is

$$\begin{pmatrix} \cos(\theta)\cos(\phi) - \sin(\phi)\sin(\theta) & -\cos(\theta)\sin(\phi) - \sin(\theta)\cos(\phi) \\ \sin(\theta)\cos(\phi) + \cos(\theta)\sin(\phi) & -\sin(\theta)\sin(\phi) + \cos(\theta)\cos(\phi) \end{pmatrix} \\ = \begin{pmatrix} \cos(\theta + \phi) & -\sin(\theta + \phi) \\ \sin(\theta + \phi) & \cos(\theta + \phi) \end{pmatrix}.$$

Thus  $R_2 \circ R_1$  is a rotation in  $\mathbb{R}^2$ .

(b) True. The standard matrix for  $R_1 \circ R_2$  is  $\begin{pmatrix} \cos(\phi + \theta) & -\sin(\phi + \theta) \\ \sin(\phi + \theta) & \cos(\phi + \theta) \end{pmatrix}$  which is also the standard matrix for  $R_2 \circ R_1$ .

(c) True. Let the standard matrices for  $F$  and  $R$  be  $\begin{pmatrix} \cos(2\phi) & \sin(2\phi) \\ \sin(2\phi) & -\cos(2\phi) \end{pmatrix}$  and  $\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$  respectively. Then the standard matrix for  $F \circ R$  is

$$\begin{pmatrix} \cos(2\phi - \theta) & \sin(2\phi - \theta) \\ \sin(2\phi - \theta) & -\cos(2\phi - \theta) \end{pmatrix}.$$

Thus  $F \circ R$  is a reflection in  $\mathbb{R}^2$ .

(d) False. For example, see Question 7.20(c).

(e) False. For example, let  $F_1$  and  $F_2$  be reflections about the  $x$ -axis and the line  $y = x$  respectively. Then  $F_2 \circ F_1$  is the anti-clockwise rotation about the origin through an angle  $90^\circ$ .



(f) False. Using the example in (e),  $F_1 \circ F_2$  is the clockwise rotation about the origin through an angle  $90^\circ$ .

22. For this question, it is helpful if you first compute  $T((1, 0, 0)^T)$ ,  $T(0, 1, 0)^T$  and  $T((0, 0, 1)^T)$  and then figure out the positions of the vectors in  $\mathbb{R}^3$ .

(a)  $T$  is the dilation by a factor of 2.

(b)  $T$  is the contraction by a factor of  $\frac{1}{3}$ .

(c)  $T$  is the anti-clockwise rotation about the  $x$ -axis through an angle  $\theta = \cos^{-1}(\frac{4}{5})$ .

(d)  $T$  is the reflection about the plane spanned by  $(1, 0, 0)^T$  and  $(0, \cos(\phi), \sin(\phi))^T$  where  $\phi = \frac{1}{2} \cos^{-1}(\frac{3}{5})$ .

(e)  $T$  is the scaling along axes in the directions of  $(1, 0, 0)^T$ ,  $(0, \frac{4}{5}, \frac{3}{5})^T$  and  $(0, -\frac{3}{5}, \frac{4}{5})^T$  by factors of 2, 1 and 0.5 respectively.

23. (a) The standard matrix for  $T_1 \circ T_2$  is

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.5 \end{pmatrix} = \begin{pmatrix} 0 & -0.5 & 0 \\ 0.5 & 0 & 0 \\ 0 & 0 & 0.5 \end{pmatrix}$$

and the standard matrix for  $T_2 \circ T_1$  is

$$\begin{pmatrix} 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.5 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -0.5 & 0 \\ 0.5 & 0 & 0 \\ 0 & 0 & 0.5 \end{pmatrix}.$$

Hence  $T_1 \circ T_2 = T_2 \circ T_1$ .

(b) The standard matrix for  $T_1 \circ T_2$  is

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

and the standard matrix for  $T_2 \circ T_1$  is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Hence  $T_1 \circ T_2 = T_2 \circ T_1$ .

(c) The standard matrix for  $T_1 \circ T_2$  is

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and the standard matrix for  $T_2 \circ T_1$  is

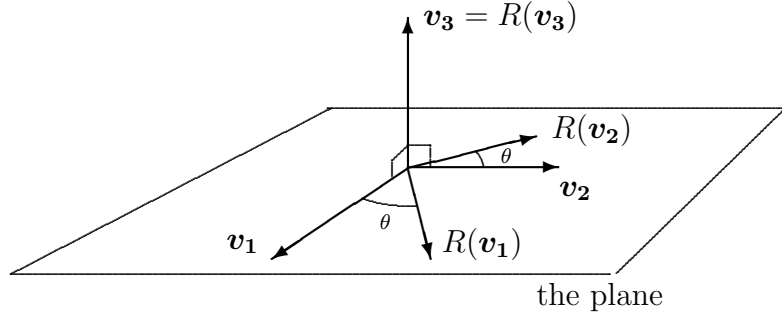
$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Hence  $T_1 \circ T_2 \neq T_2 \circ T_1$ .

24. By sketching the effect of the rotation on the standard basis vectors, we find that  $R(\mathbf{e}_1) = -\mathbf{e}_1$ ,  $R(\mathbf{e}_2) = \mathbf{e}_3$  and  $R(\mathbf{e}_3) = \mathbf{e}_2$ . So the standard matrix for  $R$  is

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \text{ and } R\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -x \\ z \\ y \end{pmatrix} \text{ for } \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3.$$

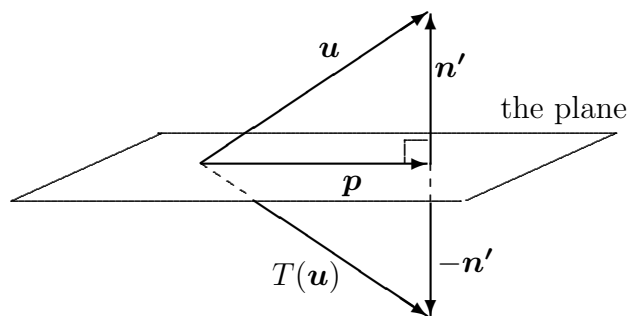
25. Let  $\mathbf{v}_1 = \frac{1}{\sqrt{2}}(-1, 1, 0)^T$ ,  $\mathbf{v}_2 = \frac{1}{\sqrt{6}}(1, 1, 2)^T$  and  $\mathbf{v}_3 = \frac{1}{\sqrt{3}}(1, 1, -1)^T$  where  $\mathbf{v}_1, \mathbf{v}_2$  are vectors on the plane  $x + y - z = 0$  while  $\mathbf{v}_3$  lies on the axis of rotation.



Thus  $R(\mathbf{v}_1) = [\cos(\theta)]\mathbf{v}_1 + [\sin(\theta)]\mathbf{v}_2$ ,  $R(\mathbf{v}_2) = [-\sin(\theta)]\mathbf{v}_1 + [\cos(\theta)]\mathbf{v}_2$  and  $R(\mathbf{v}_3) = \mathbf{v}_3$ .

(The answer for this question depends on the choices of  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ . A different choice may have a slightly different answer. For example, if  $\mathbf{v}'_1 = \frac{1}{\sqrt{2}}(1, -1, 0)^T$ ,  $\mathbf{v}'_2 = \frac{1}{\sqrt{6}}(1, 1, 2)^T$  and  $\mathbf{v}'_3 = \frac{1}{\sqrt{3}}(1, 1, -1)^T$ , then  $R(\mathbf{v}'_1) = [\cos(\theta)]\mathbf{v}'_1 - [\sin(\theta)]\mathbf{v}'_2$ ,  $R(\mathbf{v}'_2) = [\sin(\theta)]\mathbf{v}'_1 + [\cos(\theta)]\mathbf{v}'_2$  and  $R(\mathbf{v}'_3) = \mathbf{v}'_3$ . Note that  $\mathbf{v}'_1 = -\mathbf{v}_1$ ,  $\mathbf{v}'_2 = \mathbf{v}_2$  and  $\mathbf{v}'_3 = \mathbf{v}_3$  in the diagram above.)

26. For any vector  $\mathbf{u}$  in  $\mathbb{R}^3$ , we can write  $\mathbf{u} = \mathbf{n}' + \mathbf{p}$  where  $\mathbf{n}' = \left(\frac{\mathbf{u} \cdot \mathbf{n}}{\mathbf{n} \cdot \mathbf{n}}\right) \mathbf{n}$  and  $\mathbf{p} = \mathbf{u} - \left(\frac{\mathbf{u} \cdot \mathbf{n}}{\mathbf{n} \cdot \mathbf{n}}\right) \mathbf{n}$ . Since  $\mathbf{n}$  is orthogonal to the plane,  $\mathbf{n}'$  is also orthogonal to the plane. Also, since  $\mathbf{p} \cdot \mathbf{n} = \mathbf{u} \cdot \mathbf{n} - \left(\frac{\mathbf{u} \cdot \mathbf{n}}{\mathbf{n} \cdot \mathbf{n}}\right) \mathbf{n} \cdot \mathbf{n} = 0$ ,  $\mathbf{p}$  is a vector in plane.



So  $T(\mathbf{u}) = \mathbf{p} - \mathbf{n}'$  is the reflection about the plane. The standard matrix for  $T$  is

$$\frac{1}{a^2 + b^2 + c^2} \begin{pmatrix} -a^2 + b^2 + c^2 & -2ab & -2ac \\ -2ab & a^2 - b^2 + c^2 & -2bc \\ -2ac & -2bc & a^2 + b^2 - c^2 \end{pmatrix}.$$

27. (a) The figure represented by  $\mathbf{B}$  is “N”.
- (b) (i) The figure represented by  $\mathbf{PB}$  is “N”. The transformation is a contraction by a factor of 0.5.
- (ii) The figure represented by  $\mathbf{PB}$  is the inverted “Z”. The transformation is a reflection about the line  $y = x$ .
- (iii) The figure represented by  $\mathbf{PB}$  is “Z”. The transformation is a clockwise rotation about the origin through an angle  $90^\circ$ .
- (iv) The figure represented by  $\mathbf{PB}$  is “N”. The transformation is a translation by distance of 1 in both the  $x$  and  $y$  direction.

28. (a)  $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$

(b)  $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$

The transformation is a translation that moves  $(x, y)^T$  to  $(x + 1, y - 2)^T$ .

(c)  $\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$

- (d) Note that the point  $(-1, 2)^T$  should be invariant under this transformation. So we first translate  $(-1, 2)^T$  to  $(0, 0)^T$  then do a rotation about the origin before performing the inverse translation.

The standard matrix is

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{3}{\sqrt{2}} - 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} + 2 \\ 0 & 0 & 1 \end{pmatrix}.$$