## Answers/Solutions of Exercise 1 (Version: May 25, 2012)

1. (a), (c), (f), (i), (l) are linear equations and (b), (d), (e), (h), (j) are not.

   For (g), although the equation does not look like a linear equation, if we apply the logarithmic function to both sides of the equation, it becomes $(x_1 + x_2 + x_3)\log(2) = \log(5)$ which is a linear equation in $x_1$, $x_2$ and $x_3$.

   (k) is a linear equation if $m = 0$ or 1.

2. (a) $x = t$, $y = -\frac{2}{5}t$ where $t$ is an arbitrary parameter.

   (b) $x = r$, $y = s$, $z = t$, $w = \frac{1}{8}(-1.5 + 2r + 5s - 6t)$ where $r$, $s$, $t$ are arbitrary parameters.

   (c) $x_2 = a$, $x_3 = b$, $x_4 = c$, $x_5 = d$, $x_1 = \frac{1}{3}(1 + 8a - 2b - c + 4d)$ where $a$, $b$, $c$, $d$ are arbitrary parameters.

3. (a) One such linear equation is $x - 2y = 1$.

   (b) If we set $x = t$, then $y = \frac{1}{2}(t - 1)$.

4. (a) $x + 4y - z = 3$.

   (b) For example, $\begin{cases} x = s \\ y = \frac{3}{4} - \frac{1}{4}s + \frac{1}{4}t \\ z = t \end{cases}$ where $s$, $t$ are arbitrary parameters;

   and $\begin{cases} x = s \\ y = t \\ z = -3 + s + 4t \end{cases}$ where $s$, $t$ are arbitrary parameters.

   (c) For example, $\begin{cases} x + 4y - z = 3 \\ 2x + 6y - 2z = 6. \end{cases}$

5. (a) It is a plane that intersects the $x$, $y$ and $z$-axes at the points $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ respectively.

   (b) (i) It is a line through the origin and containing all the points $(s, s)$ for any real number $s$.

   (ii) It is a plane containing the $z$-axis that intersects the $xy$-plane at the line $x = s$, $y = s$, $z = 0$ where $s$ is any real number.

   (c) It is the line of intersection of the two planes $x + y + z = 1$ and $x - y = 0$.

6. (a) Since

$$3\left[\frac{1}{5}(-4+3t)\right]+4\left[\frac{1}{5}(-7+4t)\right]-5t=-\frac{12}{5}+\frac{9}{5}t-\frac{28}{5}+\frac{16}{5}t-5t=-8$$

and

$$\frac{1}{5}(-4+3t)-2\left[\frac{1}{5}(-7+4t)\right]-5t=-\frac{4}{5}+\frac{3}{5}t+\frac{14}{5}-\frac{8}{5}t+t=2,$$

$x=\frac{1}{5}(-4+3t)$, $y=\frac{1}{5}(-7+4t)$, $z=t$ is a solution to the linear system.

(b) For example, $x=-\frac{4}{5}$, $y=-\frac{7}{5}$, $z=0$ and $x=-\frac{1}{5}$, $y=-\frac{3}{5}$, $z=1$.

7. (a) Either the three lines are parallel but not all three are the same or two of the lines intersect at a point but this point does not lie on the third line.

(b) All three lines are distinct and intersect at a single point, or two of the lines are identical and they intersect the third line at a point.

(c) All three lines are identical.

8. (a) Either the three planes are parallel but not all three are the same or two of the planes intersect at a line but this line is parallel to and does not lie on the third plane.

(b) Two of the planes intersect at a line and this line is not parallel to the third plane (i.e. the three planes intersect at a single point).

(c) Either two of the planes intersect at a line and this line lies on the third plane (i.e. the three planes intersect at a common line) or all three planes are identical.

9. (a) Yes. For example,

$$\begin{cases} x+y+z=0 \\ x+y+z=1. \end{cases}$$

(b) Yes. For example,

$$\begin{cases} x+\ \ y=0 \\ x-\ \ y=0 \\ 2x+4y=0. \end{cases}$$

(c) No. A linear system with more unknowns than equations will either have no solution or infinitely many solutions.

(d) Yes. For example,

$$\begin{cases} x+\ \ y=1 \\ 2x+2y=2 \\ 3x+3y=3. \end{cases}$$

10. (i) $\boldsymbol{B}$ can be obtained from $\boldsymbol{A}$ by adding the first row to the second row.

(ii) $\boldsymbol{C}$ can be obtained from $\boldsymbol{B}$ by multiplying the first row by 5.

(iii) $\boldsymbol{D}$ can be obtained from $\boldsymbol{C}$ by first adding $-5$ times of the second row to the first row and then multiplying the second row by 2.

Thus the four matrices are row equivalent.

11. $\boldsymbol{B}$ can be obtained from $\boldsymbol{A}$ by the following series of elementary row operations:

(i) Add $-2$ times of the second row to the first row.

(ii) Add $-1$ times of the third row to the second row.

(iii) Multiply the first row by 3.

(iv) Multiply the second row by 6.

(v) Multiply the third row by 9.

12. (a) The matrix is neither in row-echelon nor reduced row-echelon form. A system of linear equations corresponding to the augmented matrix is

$$\begin{cases} x_1 \phantom{+x_2+x_3} = 5 \\ \phantom{x_1+x_2+} x_3 = 3 \\ x_1 + x_2 \phantom{+x_3} = 4. \end{cases}$$

The solution to the system is $x_1 = 5$, $x_2 = -1$, $x_3 = 3$.

(b) The matrix is in row-echelon form but not reduced row-echelon form. A system of linear equations corresponding to the augmented matrix is

$$\begin{cases} x_1 \phantom{-x_2} + 3x_3 = 0 \\ \phantom{x_1} -x_2 + 2x_3 = 0 \\ \phantom{x_1-x_2+2} x_3 = 0. \end{cases}$$

The solution to the system is $x_1 = x_2 = x_3 = 0$.

(c) The matrix is in both row-echelon and reduced row-echelon form. A system of linear equations corresponding to the augmented matrix is

$$\begin{cases} x_1 \phantom{+x_2-x_3} = 0 \\ \phantom{x_1+} x_2 - x_3 = 0 \\ \phantom{x_1+x_2-x_3} 0 = 1\ . \end{cases}$$

The system is inconsistent.

(d) The matrix is in row-echelon form but not reduced row echelon-form. A system of linear equations corresponding to the augmented matrix is

$$\begin{cases} -2x_1 \phantom{+3x_2} - x_3 - 7x_4 = 8 \\ \phantom{-2x_1} 3x_2 \phantom{- x_3} + 3x_4 = 2 \\ \phantom{-2x_1 3x_2 - x_3 +3} x_4 = -1. \end{cases}$$

A general solution to the system is $x_1 = \frac{1}{2}(-1-t), x_2 = \frac{5}{3}, x_3 = t, x_4 = -1$ where $t$ is an arbitrary parameter.

(e) The matrix is neither in row-echelon nor reduced row-echelon form. A system of linear equations corresponding to the augmented matrix is

$$\begin{cases} x_1 + 2x_3 - 2x_4 + 3x_5 = -2 \\ x_3 + x_4 + 3x_5 = 2 \\ x_4 + 5x_5 = 5. \end{cases}$$

(We omit the equation $0x_1+0x_2+0x_3+0x_4+0x_5 = 0$.) A general solution to the system is $x_1 = 14 - 17t$, $x_2 = s$, $x_3 = -3 + 2t$, $x_4 = 5 - 5t$, $x_5 = t$ where $s$, $t$ are arbitrary parameters.

(f) The matrix is in both row and reduced row-echelon form. A system of linear equations corresponding to the augmented matrix is

$$\begin{cases} x_1 - 2x_3 + 2x_5 = -2 \\ x_2 + 2x_5 = 4 \\ x_4 - x_5 = 1 \\ x_6 = 1. \end{cases}$$

A general solution to the system is $x_1 = -2+2s-2t$, $x_2 = 4-2t$, $x_3 = s$, $x_4 = 1+t$, $x_5 = t$, $x_6 = 1$ where $s$, $t$ are arbitrary parameters.

13. (i) $\boldsymbol{A}$ is row equivalent to $\left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

The corresponding linear system represents the plane $x + 2y + 3z = 0$.

(ii) $\boldsymbol{B}$ is row equivalent to $\left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right)$

The corresponding linear system represents the line of intersection of the two planes $x + 2y + 3z = 0$ and $x + y + z = 0$.

(iii) The linear system corresponding to $\boldsymbol{C}$ has a unique solution $x = 0$, $y = 0$, $z = 0$.

By Theorem 1.2.7, since the linear systems corresponding to the three matrices have different solution sets, the three matrices are not row equivalent to each other.

14. For the solution set to be a plane, there must be one leading entry in the reduced-row echelon form and two arbitrary parameters. Thus, we must have $a = 1$, $b = c = e = f = 0$. Since the plane does not contain the origin, $d \neq 0$. (Since it is given that the matrix is in reduced row-echelon form, $a$ cannot be 0.)

15. (a) There are 11 possible forms:

$$\text{(i)} \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right), \quad \text{(ii)} \left(\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 0 & 0 & 0 \end{array}\right), \quad \text{(iii)} \left(\begin{array}{ccc|c} 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \end{array}\right),$$

$$\text{(iv)} \left(\begin{array}{ccc|c} 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{array}\right), \quad \text{(v)} \left(\begin{array}{ccc|c} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right), \quad \text{(vi)} \left(\begin{array}{ccc|c} 1 & 0 & * & * \\ 0 & 1 & * & * \end{array}\right),$$

$$\text{(vii)} \left(\begin{array}{ccc|c} 1 & * & 0 & * \\ 0 & 0 & 1 & * \end{array}\right), \quad \text{(viii)} \left(\begin{array}{ccc|c} 1 & * & * & 0 \\ 0 & 0 & 0 & 1 \end{array}\right), \quad \text{(ix)} \left(\begin{array}{ccc|c} 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{array}\right),$$

$$\text{(x)} \left(\begin{array}{ccc|c} 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 \end{array}\right), \quad \text{(xi)} \left(\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right).$$

(b) In Part (a), forms (v), (viii), (x) and (xi) represent inconsistent systems.

(c) Each of the forms in Part (a) is not row equivalent to any others.

16. (a) $x_1 = -31$, $x_2 = -5$, $x_3 = 20$.

(b) $w = 12 + 4t$, $x = 13 + 5t$, $y = 9 + 3t$, $z = t$ where $t$ is an arbitrary parameter.

(c) $x_1 = -2 - 3s - t$, $x_2 = s$, $x_3 = 1 - \frac{t}{3}$, $x_4 = t$ where $s$, $t$ are arbitrary parameters.

(d) The system is inconsistent.

(e) $x_1 = 4$, $x_2 = 0$, $x_3 = 1$.

(f) $w = \frac{1}{12}(5 - s - 27t)$, $x = \frac{1}{12}(-1 + 5s - 9t)$, $y = s$, $z = t$ where $s$, $t$ are arbitrary parameters.

(g) $x_1 = -2 + \frac{1}{2}s - \frac{1}{2}t$, $x_2 = 3 - s + t$, $x_3 = -2 - t$, $x_4 = s$, $x_5 = t$ where $s$, $t$ are arbitrary parameters.

17. From Question 1.16(e), we see that $x^2 = 4$, $y^2 = 0$, $z^2 = 1$. So the solutions are $(x, y, z) = (2, 0, 1)$, $(2, 0, -1)$, $(-2, 0, 1)$, $(-2, 0, -1)$.

18. Let $x_1 = \cos(\theta)$, $x_2 = \sin(\phi)$, $x_3 = \tan(\phi)$. Solve the linear system

$$\begin{cases} x_1 - x_2 - \ \ x_3 = 0 \\ 3x_1 - x_2 - 2x_3 = 0, \end{cases}$$

A general solution is $x_1 = \frac{1}{2}t$, $x_2 = -\frac{1}{2}t$, $x_3 = t$ where $t$ is arbitrary. There are two situations:

(a) If $t = 0$, then $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$ and $\phi = 0, \pi$.

(b) For $t \neq 0$, we have $\sin(\phi) = -\frac{1}{2}\tan(\phi)$ which implies $\cos(\phi) = -\frac{1}{2}$. Hence either $\phi = \frac{2\pi}{3}$ and $\theta = \frac{5\pi}{6}, \frac{7\pi}{6}$ or $\phi = \frac{4\pi}{3}$ and $\theta = \frac{\pi}{6}, \frac{11\pi}{6}$.

19. $$\begin{cases} D_1 = S_1 \\ D_2 = S_2 \\ D_3 = S_3 \end{cases} \Leftrightarrow \begin{cases} P_1 + P_2 + \ \ P_3 = \ \ 5 \\ P_1 - P_2 + \ \ P_3 = \ \ 1 \\ P_1 - P_2 - 2P_3 = -5 \end{cases}$$

The solution of the system is $P_1 = 1$, $P_2 = 2$, $P_3 = 2$ and hence $D_1 = S_1 = 6$, $D_2 = S_2 = 6$, $D_3 = S_3 = 10$.

20. (a) $$\begin{cases} x_1 + 310 = x_2 + 640 \\ 610 + 450 = x_1 + x_4 \\ x_2 + 600 = x_3 + 330 \\ x_3 + x_4 = 480 + 520 \end{cases} \Leftrightarrow \begin{cases} x_1 - x_2 \qquad\qquad = 330 \\ x_1 \qquad\qquad + x_4 = 1060 \\ \qquad x_2 - x_3 \qquad = -270 \\ \qquad\qquad x_3 + x_4 = 1000 \end{cases}$$

$$\left(\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 330 \\ 1 & 0 & 0 & 1 & 1060 \\ 0 & 1 & -1 & 0 & -270 \\ 0 & 0 & 1 & 1 & 1000 \end{array}\right) \xrightarrow[\text{Elimination}]{\text{Gaussian}} \left(\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 330 \\ 0 & 1 & 0 & 1 & 730 \\ 0 & 0 & -1 & -1 & -1000 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

The system has infinitely many solutions. We cannot determine the values of $x_1$, $x_2$, $x_3$, $x_4$ uniquely.

(b) $x_1 = 560$, $x_2 = 230$, $x_3 = 500$.

21. (a) (i) When $\delta = 0.0001$, $(x, y) = (-10000, 10000)$.
When $\delta = 0.0002$, $(x, y) = (-5000, 5000)$.

(ii) When $\delta = 0.0001$, $(x, y) \approx (0.499975, -0.499975)$.
When $\delta = 0.0002$, $(x, y) \approx (0.49995, -0.49995)$.

(b) The solutions of the system (i) are very different but the solutions of the system (ii) are approximately the same.

(c) The equations of the system (i) gives us two lines that are almost parallel to each other. A small change of the slope of one line will move the intersection point dramatically.

The equations of the system (ii) gives us two lines that are almost perpendicular to each other. A small change of the slope of one line will only move the intersection point slightly.

22. (a) The system always has solution. It has only one solution if $a \neq 1, -2$. It has infinitely many solutions if $a = 1$ or $-2$.
    (b) The system has no solution if $a = -2$. It has only one solution if $a \neq 2, -2$. It has infinitely many solutions if $a = 2$.
    (c) The system has no solution if $a = 0$. It has only one solution if $a > 0$. It has infinitely many solutions if $a < 0$.

23. The system has no solution if $a = 0$ and $b \neq 2$. It has only one solution if $a \neq 0$ and $b \neq 2$. It has infinitely many solutions and a general solution has one arbitrary parameter if $a \neq 0$ and $b = 2$. It has infinitely many solutions and a general solution has two arbitrary parameters if $a = 0$ and $b = 2$.

24. The system has infinitely many solutions if $a = b = c = 0$. The system has a unique solution if $a, b, c$ all nonzero. For all other values of $a, b, c$, the system has no solution.

25. (a) The system has only the trivial solution.
    (b) The system has only the trivial solution.
    (c) The system has nontrivial solutions since it has more unknowns than equations.
    (d) The system has nontrivial solutions since the third equation is the sum of the first and second.

26. (a) $$\begin{cases} 3w \quad\quad - y \quad\quad = 0 \\ 8w \quad\quad\quad\quad - 2z = 0 \\ \quad\quad 2x - 2y - \ z = 0. \end{cases}$$
    (b) $w = \frac{1}{4}t$, $x = \frac{5}{4}t$, $y = \frac{3}{4}t$, $z = t$ where $t$ is an arbitrary parameter.
    (c) $w = 1$, $x = 5$, $y = 3$ and $z = 4$.

27. (a) Since $ax_0 + by_0 + cz_0 = 0$ and $dx_0 + ey_0 + fz_0 = 0$, we have $akx_0 + bky_0 + ckz_0 = 0$ and $dkx_0 + eky_0 + fkz_0 = 0$. Thus $x = kx_0$, $y = ky_0$, $z = kz_0$ is also a solution of the system.
    (b) Since $ax_0 + by_0 + cz_0 = 0$, $dx_0 + ey_0 + fz_0 = 0$, $ax_1 + by_1 + cz_1 = 0$ and $dx_1 + ey_1 + fz_1 = 0$, we have $a(x_0 + x_1) + b(y_0 + y_1) + c(z_0 + z_1) = 0$ and $d(x_0 + x_1) + e(y_0 + y_1) + f(z_0 + z_1) = 0$. So $x = x_0 + x_1$, $y = y_0 + y_1$, $z = z_0 + z_1$ is also a solution of the system.

28. We consider the following four cases.

**Case 1.** The reduced-row echelon form is the zero matrix:

$$\left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

The solutions obtained give us the whole 3D space $\mathbb{R}^3$.

**Case 2.** The reduced-row echelon form has one leading entry:

$$\left(\begin{array}{ccc|c} 1 & * & * & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right), \quad \left(\begin{array}{ccc|c} 0 & 1 & * & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \text{or} \quad \left(\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

The solutions obtained represent a plane that passes through the origin.

**Case 3.** The reduced-row echelon form has two leading entries:

$$\left(\begin{array}{ccc|c} 1 & 0 & * & 0 \\ 0 & 1 & * & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \left(\begin{array}{ccc|c} 1 & * & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \text{or} \quad \left(\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

The solutions obtained represent a line that passes through the origin.

**Case 4.** The reduced-row echelon form has three leading entries:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right).$$

There is only one solution, the origin.

29. As the line passes through the origin and the point $(1, 1, 1)$, we have $d = g = k = 0$ and $a+b+c = 0$, $e+f = 0$. Since the general solution needs two arbitrary parameters (so that the solutions form a line in the $xyz$-space), $e \neq 0$. Finally, the augmented matrix is in reduced row-echelon form, we get $a = e = 1$, $b = 0$ and hence $f = c = -1$.

30. (a) True. For example,

$$\begin{cases} x \qquad - z = 0 \\ \qquad y - z = 0. \end{cases}$$

(b) False. Given a system of linear equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \qquad \vdots \qquad\qquad\qquad\qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m, \end{cases}$$

if $x_1 = 0$, $x_2 = 0$, $\ldots$, $x_n = 0$ is a solution, then $b_1 = 0$, $b_2 = 0$, $\ldots$, $b_m = 0$ and hence the system is homogeneous.

(c) False. Use the example in (a).

(d) False. Every homogeneous system has the trivial solution.

(e) True. It is a consequence of Remark 1.5.4.1.

(f) False. Use the example in (a).

(g) True. It is a consequence of Remark 1.5.4.1.