NATIONAL UNIVERSITY OF SINGAPORE

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MA1101R Linear Algebra 1 **Tutorial 9**

1. Let

$u_1 = (1, 2, 2, -1), u_2 = (1, 1, -1, 1), u_3 = (-1, 1, -1, -1) u_4 = (-2, 1, 1, 2).$

(a) Show that $S = \{u_1, u_2, u_3, u_4\}$ is an orthogonal set.

(b) Obtain an orthonormal set S' by normalizing u_1, u_2, u_3 and u_4.

(c) Is S' an orthonormal basis for $\mathbf{R}^4$?

(d) If $w = (0, 1, 2, 3)$ find $(w)_S$ and $(w)_{S'}$.

(e) Let $V = \text{span}\{u_1, u_2, u_3\}$. Find all vectors that are orthogonal to V.

(f) Find the projection of w onto V.

(Textbook, p. 171, Problem 10)

(a) $\begin{pmatrix}1\\2\\2\\-1\end{pmatrix} \cdot \begin{pmatrix}1\\1\\-1\\1\end{pmatrix} = 1+2-2-1=0$ $\quad \begin{pmatrix}1\\2\\2\\-1\end{pmatrix} \cdot \begin{pmatrix}-2\\1\\1\\2\end{pmatrix} = -2+2+2-2=0$

$\begin{pmatrix}1\\2\\2\\-1\end{pmatrix} \cdot \begin{pmatrix}-1\\1\\-1\\-1\end{pmatrix} = -1+2-2+1 = 0$ $\quad \begin{pmatrix}1\\1\\-1\\1\end{pmatrix} \cdot \begin{pmatrix}-1\\1\\-1\\-1\end{pmatrix} = -1+1+1-1=0$

$\begin{pmatrix}1\\1\\-1\\1\end{pmatrix} \cdot \begin{pmatrix}-2\\1\\1\\2\end{pmatrix} = -2+1-1+2=0$ $\quad \begin{pmatrix}-1\\1\\-1\\-1\end{pmatrix} \cdot \begin{pmatrix}-2\\1\\1\\2\end{pmatrix} = 2+1-1-2=0$

(b) $\hat{\underset{\sim}{u}}_1 = \frac{1}{\sqrt{1+2^2+2^2+1}} \begin{pmatrix}1\\2\\2\\-1\end{pmatrix} = \frac{1}{\sqrt{10}} \begin{pmatrix}1\\2\\2\\-1\end{pmatrix}$ $\quad \hat{\underline{u}}_2 = \frac{1}{2} \begin{pmatrix}1\\1\\-1\\1\end{pmatrix}$

$\hat{\underline{u}}_3 = \frac{1}{2} \begin{pmatrix}-1\\1\\-1\\-1\end{pmatrix}$ $\quad \hat{\underline{u}}_4 = \frac{1}{\sqrt{10}} \begin{pmatrix}-2\\1\\1\\2\end{pmatrix}$

(c) $\begin{pmatrix}1 & 1 & -1 & -2\\2 & 1 & 1 & 1\\2 & -1 & -1 & 1\\-1 & 1 & -1 & 2\end{pmatrix} \sim \begin{pmatrix}1 & 1 & -1 & -2\\0 & -1 & 3 & 5\\0 & -3 & 1 & 5\\0 & 2 & -2 & 0\end{pmatrix}$

$\sim \begin{pmatrix}1 & 1 & -1 & -2\\0 & -1 & 3 & 5\\0 & 0 & 8 & 10\\0 & 0 & -4 & -10\end{pmatrix} \sim \begin{pmatrix}1 & 1 & -1 & -2\\0 & -1 & 3 & 5\\0 & 0 & 8 & 10\\0 & 0 & 0 & -40\end{pmatrix}$

$\underline{u}_1, \underline{u}_2, \underline{u}_3, \underline{u}_4$ are linearly independent.

$\{\hat{\underline{u}}_1, \hat{\underline{u}}_2, \hat{\underline{u}}_3, \hat{\underline{u}}_4\}$ is an orthonormal basis for $\mathbb{R}^4$

(c) $$\begin{pmatrix}0\\1\\2\\3\end{pmatrix} = \alpha\begin{pmatrix}1\\2\\2\\-1\end{pmatrix} + \beta\begin{pmatrix}1\\1\\-1\\1\end{pmatrix} + \gamma\begin{pmatrix}-1\\1\\-1\\-1\end{pmatrix} + \delta\begin{pmatrix}-2\\1\\1\\2\end{pmatrix}$$

$$\left(\begin{array}{cccc|c}1 & 1 & -1 & -2 & 0\\ 2 & 1 & 1 & 1 & 1\\ 2 & -1 & -1 & 1 & 2\\ -1 & 1 & -1 & 2 & 3\end{array}\right) \sim \left(\begin{array}{cccc|c}1 & 1 & -1 & -2 & 0\\ 0 & -1 & 3 & 5 & 1\\ 0 & -3 & 1 & 5 & 2\\ 0 & 2 & -2 & 0 & 3\end{array}\right)$$

$$\sim \left(\begin{array}{cccc|c}1 & 1 & -1 & -2 & 0\\ 0 & -1 & 3 & 5 & 1\\ 0 & 0 & 8 & 10 & 1\\ 0 & 0 & -4 & -10 & -5\end{array}\right)$$

$$\sim \left(\begin{array}{cccc|c}1 & 1 & -1 & -2 & 0\\ 0 & -1 & 3 & 5 & 1\\ 0 & 0 & 8 & 10 & 1\\ 0 & 0 & 0 & -40 & -36\end{array}\right)$$

$\delta = \frac{9}{10}$, $8\gamma + 9 = 1$, $\gamma = -1$, $-\beta - 3 + \frac{45}{10} = 1 \Rightarrow \beta = -4 + \frac{9}{2} = \frac{1}{2}$

$\alpha + \frac{1}{2} + 1 - \frac{18}{10} = 0$, $\alpha = \frac{9}{5} - \frac{3}{2} = \frac{18-15}{10} = \frac{3}{10}$.

$\therefore (w)_S = \left(\frac{3}{10}, \frac{1}{2}, -1, \frac{9}{10}\right)$

$(w)_{S'} = \left(\frac{3}{10}\sqrt{10}, \frac{1}{2}\cdot 2, (-1)\cdot 2, \frac{9}{10}\sqrt{10}\right) = \left(\frac{3}{\sqrt{10}}, 1, -2, \frac{9}{\sqrt{10}}\right)$.

(c) $\operatorname{Span}\left\{\begin{pmatrix}-2\\1\\1\\2\end{pmatrix}\right\}$ is orthogonal to V.

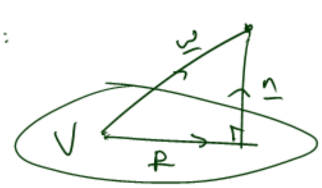
(d) Projection is given by

$$\left(\frac{\begin{pmatrix}0\\1\\2\\3\end{pmatrix}\cdot\begin{pmatrix}1\\2\\2\\-1\end{pmatrix}}{10}\right)\underline{u}_1 + \begin{pmatrix}0\\1\\2\\3\end{pmatrix}\cdot\begin{pmatrix}1\\1\\-1\\1\end{pmatrix}\frac{\underline{u}_2}{4} + \begin{pmatrix}0\\1\\2\\3\end{pmatrix}\cdot\begin{pmatrix}-1\\1\\-1\\-1\end{pmatrix}\frac{\underline{u}_3}{4}$$

$$= \frac{(2+4-3)}{10}\begin{pmatrix}1\\2\\2\\-1\end{pmatrix} + \frac{(1-2+3)}{4}\begin{pmatrix}1\\1\\-1\\1\end{pmatrix} + \frac{(1-2-3)}{4}\begin{pmatrix}-1\\1\\-1\\-1\end{pmatrix}$$

$$= \frac{3}{10}\begin{pmatrix}1\\2\\2\\-1\end{pmatrix} + \frac{2}{4}\begin{pmatrix}1\\1\\-1\\1\end{pmatrix} + (-1)\begin{pmatrix}-1\\1\\-1\\-1\end{pmatrix} = \begin{pmatrix}\frac{9}{5}\\ \frac{1}{10}\\ \frac{11}{10}\\ \frac{6}{5}\end{pmatrix}$$

Note:



If $\underline{p} \in V$, we can let $\underline{p} = a_1\underline{v}_1 + \dots + a_r\underline{v}_r$. $\{\underline{v}_1, \dots, \underline{v}_r\}$ is an orthogonal basis.

$\therefore \underline{w} = a_1\underline{v}_1 + \dots + a_r\underline{v}_r + \underline{n}$.

$\underline{w}\cdot\underline{v}_j = a_j(\underline{v}_j\cdot\underline{v}_j)$ since $\underline{v}_j\cdot\underline{v}_k = 0$ $(k\neq j)$ & $\underline{v}\cdot\underline{n} = 0$ $\because \underline{v}_j \perp \underline{v}_k$ & $\underline{v}_i \perp \underline{n}$

$\therefore a_j = \frac{(\underline{w}\cdot\underline{v}_j)}{(\underline{v}_j\cdot\underline{v}_j)}$. $\therefore \underline{p} = \left(\frac{\underline{w}\cdot\underline{v}_1}{\underline{v}_1\cdot\underline{v}_1}\right)\underline{v}_1 + \dots + \left(\frac{\underline{w}\cdot\underline{v}_r}{\underline{v}_r\cdot\underline{v}_r}\right)\underline{v}_r$. $\square$

2. (a) Find an orthonormal basis for the solution space of the equation $x + y - z = 0$.
   (b) Find the projection of $(1, 0, -1)$ onto the plane $x + y - z = 0$.
   (c) Extend the set obtained in Part (a) to an orthonormal basis for $\mathbf{R}^3$.

(Textbook, p. 171, Problem 14)

(a) $\begin{pmatrix}x\\y\\z\end{pmatrix} = s\begin{pmatrix}-1\\1\\0\end{pmatrix} + t\begin{pmatrix}1\\0\\1\end{pmatrix}, \quad s, t \in \mathbb{R}$

By Gran Schmidt process,

$$\begin{pmatrix}1\\0\\1\end{pmatrix} - \frac{\begin{pmatrix}-1\\1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\0\\1\end{pmatrix}}{\begin{pmatrix}-1\\1\\0\end{pmatrix}\cdot\begin{pmatrix}-1\\1\\0\end{pmatrix}}\begin{pmatrix}-1\\1\\0\end{pmatrix} = \begin{pmatrix}1\\0\\1\end{pmatrix} + \begin{pmatrix}-1\\1\\0\end{pmatrix}\frac{1}{2} = \begin{pmatrix}1/2\\1/2\\1\end{pmatrix}$$

Orthonormal basis is $\left\{\begin{pmatrix}-1\\1\\0\end{pmatrix}\frac{1}{\sqrt2}, \begin{pmatrix}1\\1\\2\end{pmatrix}\frac{1}{\sqrt6}\right\}$

(b) Projection of $\begin{pmatrix}1\\0\\-1\end{pmatrix}$ onto the plane is

$$\frac{1}{\sqrt2}\left(\begin{pmatrix}1\\0\\-1\end{pmatrix}\cdot\begin{pmatrix}-1\\1\\0\end{pmatrix}\right)\frac{1}{\sqrt2}\begin{pmatrix}-1\\1\\0\end{pmatrix} + \frac{1}{\sqrt6}\frac{1}{\sqrt6}\left(\begin{pmatrix}1\\0\\-1\end{pmatrix}\cdot\begin{pmatrix}1\\1\\2\end{pmatrix}\right)\begin{pmatrix}1\\1\\2\end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix}1\\-1\\0\end{pmatrix} + \frac{1}{6}\begin{pmatrix}-1\\-1\\-2\end{pmatrix} = \begin{pmatrix}1/3\\-2/3\\-1/3\end{pmatrix}$$

$\begin{pmatrix}1/3\\-2/3\\-1/3\end{pmatrix}$

(c) $\begin{pmatrix}x\\y\\z\end{pmatrix}\cdot\begin{pmatrix}1\\1\\-1\end{pmatrix} = 0$. $\begin{pmatrix}1\\1\\-1\end{pmatrix}$ is orthorgonal to the plane.

$\therefore \left\{\frac{1}{\sqrt3}\begin{pmatrix}1\\1\\-1\end{pmatrix}, \begin{pmatrix}1\\1\\2\end{pmatrix}\frac{1}{\sqrt6}, \begin{pmatrix}-1\\1\\0\end{pmatrix}\frac{1}{\sqrt2}\right\}$ is an orthonormal basis for $\mathbb{R}^3$.

Alternatively: Choose a vector not in $\mathrm{Span}\left\{\begin{pmatrix}1\\1\\2\end{pmatrix}, \begin{pmatrix}-1\\1\\0\end{pmatrix}\right\}$

For example, pick $\begin{pmatrix}0\\0\\1\end{pmatrix}$

Check: $\begin{pmatrix}1 & -1 & a\\1 & 1 & b\\2 & 0 & c\end{pmatrix} \sim \begin{pmatrix}1 & -1 & a\\0 & 2 & b-a\\0 & 2 & c-2a\end{pmatrix}$

The vector $\begin{pmatrix}0\\0\\1\end{pmatrix} - \frac{\begin{pmatrix}0\\0\\1\end{pmatrix}\cdot\begin{pmatrix}1\\1\\2\end{pmatrix}}{6}\begin{pmatrix}1\\1\\2\end{pmatrix} - \frac{\begin{pmatrix}0\\0\\1\end{pmatrix}\cdot\begin{pmatrix}-1\\1\\0\end{pmatrix}}{2}\begin{pmatrix}-1\\1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix} - \frac{2}{6}\begin{pmatrix}1\\1\\2\end{pmatrix} = \begin{pmatrix}-1/3\\-1/3\\1/3\end{pmatrix}$ Choose $a=0, c=1, b=0$

is orthogonal to V.

$\therefore \left\{\begin{pmatrix}1\\1\\-1\end{pmatrix}\frac{1}{\sqrt3}, \begin{pmatrix}1\\1\\2\end{pmatrix}\frac{1}{\sqrt6}, \begin{pmatrix}-1\\1\\0\end{pmatrix}\frac{1}{\sqrt2}\right\}$ forms an orthonormal basis of $\mathbb{R}^3$.

3. (a) Let

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix}.$$

(i) Solve the linear system $A\mathbf{x} = \mathbf{b}$.

(ii) Find the least square solution to $A\mathbf{x} = \mathbf{b}$.

(b) Suppose a linear system $A\mathbf{x} = \mathbf{b}$ is consistent. Show that the solution set of $A\mathbf{x} = \mathbf{b}$ is equal to the solution set of

$$A^T A\mathbf{x} = A^T\mathbf{b}.$$

(Textbook, p. 174, Problem 27)

(a) $\left(\begin{array}{cc|c} 1 & 1 & 3 \\ 1 & 2 & 4 \\ 1 & 0 & 2 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{array}\right)$ $\quad \underset{\sim}{x} = \begin{pmatrix} a \\ b \end{pmatrix}$, $b = 1$, $a = 2$

The sol^n in both cases is $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

(b) From a problem in Tut. 8, we know that

$B\underset{\sim}{x} = \underset{\sim}{0} \iff B^T B\underset{\sim}{x} = \underset{\sim}{0}$.

Let $\underset{\sim}{u}$ be a sol^n of $A\underset{\sim}{y} = \underset{\sim}{b}$. ($u$ exists because the system is consistent.

Let $A\underset{\sim}{x} = \underset{\sim}{b}$. then $A(\underset{\sim}{x} - \underset{\sim}{u}) = \underset{\sim}{0}$

But $A(x - \underset{\sim}{u}) = \underset{\sim}{0} \iff A^T A(\underset{\sim}{x} - \underset{\sim}{u}) = \underset{\sim}{0}$

$\therefore A\underset{\sim}{x} = \underset{\sim}{b} \iff A^T A\underset{\sim}{x} = A^T \underset{\sim}{u}$.

Note that we can also use Th^m in the book to find least square sol^n. $\therefore$ it is the sol^n to $A^T A\underset{\sim}{x} = A^T\underset{\sim}{b}$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 3 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 9 \\ 7 \end{pmatrix} \Rightarrow \begin{pmatrix} 3 & 3 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 9 \\ 2 \end{pmatrix}$$

$y = 1$, $3x + 3y = 9 \Rightarrow x = 2$.

4. Let
$$A = \begin{pmatrix} 0 & -1 & 0 \\ 2 & -3 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

(a) Show that -1 is an eigenvalue of A.

(b) Show that $\dim(E_{-1}) = 2$.

(c) Find a 3×3 matrix B such that -3 is an eigenvalue of BA.

(Textbook, p. 202, Problem 6)

(a) $$\det \begin{pmatrix} -x & -1 & 0 \\ 2 & -3-x & 0 \\ 0 & 0 & -1-x \end{pmatrix} = (-1-x)(-x(-3-x)+2) = 0$$

$\therefore$ $x = -1$ is an eigenvalue.

(b) $$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$\alpha - \beta = 0$

$\underline{\alpha = \beta}.$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} s + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} t$$

$\therefore$ Basis for eigenspace is $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

(c)

$BA\underline{v} = (-3)\underline{v}$

$$\begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} A \underline{v} = -\begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} \underline{v} = -3\underline{v}$$

5. Let $\{u_1, u_2, \cdots, u_n\}$ be a basis for $\mathbf{R}^n$ and let A be an $n \times n$ matrix such that

$$Au_i = u_{i+1}$$

for $i = 1, 2, \cdots, n-1$ and $Au_n = 0$. Show that the only eigenvalue of A is 0 and find all the eigenvectors of A.

(Textbook, p. 203, Problem 8)

(5) $A\underline{u}_1 = \underline{u}_2$

$A\underline{u}_2 = \underline{u}_3$

$\vdots \quad \vdots \quad \vdots$

$A\underline{u}_{n-1} = \underline{u}_n \qquad A\underline{u}_n = \underline{0}$

Let $\underline{v} = \alpha_1 \underline{u}_1 + \cdots + \alpha_n \underline{u}_n \neq 0$ be an eigenvector.

$A(\alpha_1 \underline{u}_1 + \cdots + \alpha_n \underline{u}_n) = \lambda\alpha_1 \underline{u}_1 + \cdots + \lambda \alpha_n \underline{u}_n$

But $A(\alpha_1 \underline{u}_1 + \cdots + \alpha_n \underline{u}_n) = \alpha_1 \underline{u}_2 + \cdots + \alpha_{n-1} \underline{u}_n = \lambda \alpha_1 \underline{u}_1 + \cdots + \underline{\underline{\lambda \alpha_n \underline{u}_n}}$

Since $\underline{u}_1, \ldots, \underline{u}_n$ is l.i, we have

$\lambda \alpha_n = \alpha_{n-1}, \quad \lambda\alpha_{n-1} = \alpha_{n-2}, \ldots, \quad \lambda\alpha_2 = \alpha_1, \ \lambda\alpha_1 = 0.$

If $\lambda \neq 0$, then $\alpha_1 = \alpha_2 = \alpha_3 = \cdots = \alpha_n = 0. \Rightarrow \underline{v} = \underline{0}$ $\times$.

$\therefore \lambda = 0$. So $\lambda = 0$ is the only eigenvalue.

Now $\lambda = 0 \Rightarrow \alpha_1 = \alpha_2 = \cdots = \alpha_{n-1} = 0$

α_n is arbitrary because the eq^n $\lambda\alpha_n = 0$ holds since $\lambda = 0$.

$\therefore \underset{\sim}{v} = t\underline{u}_n$ for any $t \in \mathbb{R}$.

Second Solutions:

Note first that when we write $A = (a_{ij})_{n\times n}$,

we mean that A sends $\underset{\sim}{e_1}$ to $\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix}$, $\underset{\sim}{e_2}$ to $\begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \\ \vdots \\ a_{n2} \end{pmatrix}$ ... etc.

$$\therefore \quad A(\underset{\sim}{e_1}\ \underset{\sim}{e_2} \dots \underset{\sim}{e_n}) = (\underset{\sim}{e_1}\ \underset{\sim}{e_2} \dots \underset{\sim}{e_n})[A]_S$$

where $[A]_S$ is the matrix with respect to the standard basis.

Note since $(\underset{\sim}{e_1} \dots \underset{\sim}{e_n})$ is the identity matrix, A can be identified as $[A]_S$.

Suppose we have another basis $\{\underset{\sim}{u_1}, \dots, \underset{\sim}{u_n}\}$ for $\mathbb{R}^n$.

Note $\underset{\sim}{e_1}, \dots, \underset{\sim}{e_n}$ can then be expressed in terms of $\underset{\sim}{u_1}, \dots, \underset{\sim}{u_n}$ and vice versa. For example

$$(\underset{\sim}{e_1} \dots \underset{\sim}{e_n}) = (\underset{\sim}{u_1} \dots \underset{\sim}{u_n})T$$

where T is the transition matrix from S to $\mathcal{U}$.

Note that $(\underset{\sim}{e_1} \dots \underset{\sim}{e_n}) = I_n$. $\therefore \quad T = (\underset{\sim}{u_1} \dots \underset{\sim}{u_n})^{-1}$.

Now
$$A(\underset{\sim}{e_1} \dots \underset{\sim}{e_n}) = (\underset{\sim}{e_1} \dots \underset{\sim}{e_n})[A]_S$$

$$A(u_1, \dots, u_n)\,T = (\underset{\sim}{u_1} \dots \underset{\sim}{u_n})\,T\,[A]_S$$

(A is a function and $[A]_S$ is the matrix representing A with respect to the basis S)

$$\therefore \quad A(\underset{\sim}{u_1} \dots u_n) = (\underset{\sim}{u_1} \dots u_n)\,T[A]_S T^{-1}$$

But $A(\underset{\sim}{u_1}, \dots, \underset{\sim}{u_n}) = (\underset{\sim}{u_1}, \dots \underset{\sim}{u_n})[A]_{\mathcal{U}}$

where $[A]_{\mathcal{U}}$ is the matrix we obtained if we use the basis $\mathcal{U}$.

$$\therefore \quad [A]_{\mathcal{U}} = T[A]_S T^{-1}, \quad \text{where} \quad (\underset{\sim}{e_1} \dots \underset{\sim}{e_n}) = (\underset{\sim}{u_1} \dots \underset{\sim}{u_n})T$$

or $T = (\underset{\sim}{u_1} \dots \underset{\sim}{u_n})^{-1}$

Next, suppose $C = TMT^{-1}$.

then $\det(TMT^{-1} - xI) = \det(TMT^{-1} - TxIT^{-1})$

$= \det(T(M - Ix)T^{-1})$

$= \det(T)\det(M - Ix)\det(T^{-1})$

$= \det(M - Ix)$.

$\therefore$ eigenvalues of M is the same as eigenvalues of TMT^{-1} ★

Now, $A\underline{u}_1 = \underline{u}_2$, $A\underline{u}_2 = \underline{u}_3$, ... $A\underline{u}_{n-1} = \underline{u}_n$, $A\underline{u}_n = \underline{0}$.

This implies that $[A]_{\underline{u}} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \ddots & \vdots & \vdots \\ 0 & 1 & \ddots & & \vdots \\ \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & 0 \cdots & 1 & 0 \end{pmatrix}$

$\det([A]_u - xI) = \det \begin{pmatrix} -x & & 0 \\ 1 & -x & \\ & \ddots & \ddots \\ 0 & & 1 \; -x \end{pmatrix} = (-x)^n$.

$\therefore$ The only eigenvalue of $[A]_u$ is

0.

Next, if $\underline{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = (\underline{e}_1 \; \ldots \; \underline{e}_n) \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$,

then $\underline{v} = (\underline{u}_1 \; \ldots \; \underline{u}_n)\, T \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$

$\therefore (\underline{v})_u = T\underline{v}$, where T is the transition matrice from $\{\underline{e}_1, \ldots, \underline{e}_n\}$ to $\{\underline{u}_1, \ldots, \underline{u}_n\}$

$$\lambda(\underset{\sim}{v})_U = (A\,\underline{v})_U = T(A\,\underline{v}) \qquad \left(A(e_1 \dots e_n) = (e_1 \dots e_n)[A]_S\right)$$

$$= T[A]_S T^{-1}(T(\underset{\sim}{v})) = T[A]_S T^{-1}(\underset{\sim}{v})_U$$

$$= [A]_U(\underset{\sim}{v})_U.$$

$$[A]_U(\underset{\sim}{v})_U = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$ if $(\underline{v})_U$ is an eigenvector associated with 0.

If we write $(\underset{\sim}{v})_U = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = \alpha_1 \underline{u}_1 + \dots + \alpha_n \underline{u}_n$,

then $[A]_U(\underline{v})_U = 0\underline{u}_1 + \alpha_1 \underline{u}_2 + \dots + \alpha_{n-1}\underline{u}_n$

$$= \begin{pmatrix} 0 \\ \alpha_1 \\ \vdots \\ \alpha_{n-1} \end{pmatrix}$$

But $\underline{v}$ is an eigenvector associated with $0 \Rightarrow$

$$[A]_U(\underset{\sim}{v})_U = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha_1 \\ \vdots \\ \alpha_{n-1} \end{pmatrix}$$

$\therefore \quad \alpha_1 = \alpha_2 = \dots = \alpha_{n-1} = 0.$

$\therefore$ The eigenvector $(\underset{\sim}{v})_U$ is of the form $\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ t \end{pmatrix}$,

or of the form $t\underline{u}_n$.