Chapter 8

Inverse functions and techniques of integration

8.1 The inverse function

A function is called one to one if it never takes on the same value twice; that is,

$$f(x_1) \neq f(x_2) \quad \text{whenever } x_1 \neq x_2.$$

Example 8.1.1. Is the function $y = x^3$ one to one?

One to one functions are important because they are precisely the functions that possess inverse functions according to the following definition:

Definition 8.1.1. *Let f be a one to one function with domain A and range B. Then its inverse function f^{-1} has domain B and range A and is defined by*

$$f^{-1}(y) = x \Longleftrightarrow f(x) = y$$

for any y in B.

Warning: Do not treat -1 in f^{-1} for an exponent. Thus, $f^{-1}(x)$ does not mean $\frac{1}{f(x)}$.

Observe that $f(x) = y$ means that $f^{-1}(y) = x$. If we substitute the first equation to the second equation, we get

$$f^{-1}(f(x)) = x$$

for every x in A. Similarly we also have

$$f(f^{-1}(y)) = y$$

for all y in B.

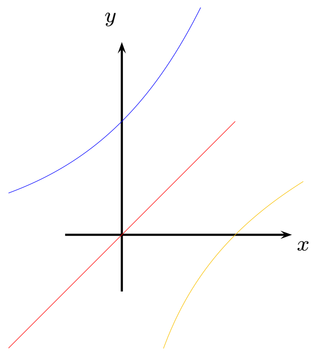
8.2 How do we find the inverse of a one to one function f?

Step 1. Write $y = f(x)$

Step 2. Solve this equation for x in terms of y (if possible)

Step 3. To express f^{-1} as a function of x, interchange x and y The resulting equation is $y = f^{-1}(x)$.

The graph of f^{-1} is obtained by reflecting the graph of f about the line $y = x$.



8.3 The Calculus of Inverse Functions

Theorem 8.3.1. *If f is a one to one continuous function defined on an interval then its inverse function f^{-1} is also continuous.*

Theorem 8.3.2. *If f is a one to one differentiable function with inverse function $g = f^{-1}$ and $f'(g(a)) \neq 0$, then the inverse function is differentiable at a and*

$$g'(a) = \frac{1}{f'(g(a))}.$$

Example 8.3.1. The function $y = x^2, x \in \mathbf{R}$ is one to one on $[0, 2]$. Find $g'(1)$ where g is the inverse of $f(x) = x^2$.

8.4 The natural Logarithm Function

The natural logarithm function is defined by

$$\ln x = \int_1^x \frac{1}{t}\, dt, x > 0.$$

Note that $\ln 1 = 0$. We define the number e to be the number for which

$$\ln e = 1.$$

By the first part of the fundamental theorem of calculus, we have

$$\frac{d \ln x}{dx} = \frac{1}{x}.$$

More generally, we have

$$\frac{d \ln ax}{dx} = \frac{1}{ax} a = \frac{1}{x}.$$

Therefore, $\ln ax$ and $\ln x$ defers by a constant, say $\ln ax = \ln x + C$. Set $x = 1$ and we get $C = \ln a$. Thus, we have

$$\ln ax = \ln a + \ln x.$$

Similarly consider

$$\frac{d \ln \frac{1}{x}}{dx} = \frac{1}{1/x} \frac{-x^{-2}}{=} - \frac{1}{x}.$$

Hence $\ln \frac{1}{x} + \ln x$ differs by a constant and the constant is 0. Hence

$$\ln \frac{1}{x} = -\ln x.$$

Finally, since

$$\frac{d \ln x^r}{dx} = \frac{1}{x^r} r x^{r-1} = \frac{r}{x},$$

we find that $\ln x^r - r \ln x$ is a constant. The constant is 0 and we have

$$\ln x^r = r \ln x.$$

What happens if we want to evaluate

$$\int_{-a}^{-b} \frac{1}{t}\, dt,$$

where a, b are positive real numbers? We compute this using the substitution: Let $u = -t$. Then

$$\int_{a}^{b} \frac{1}{-u}\, d(-u) = \ln b - \ln a.$$

This prompts us to say that the indefinite integral is

$$\int \frac{1}{t}\, dt = \ln |t| + C.$$

(Warning: One has to be careful when changing the indefinite to definite integral. The values must both be positive or both negative.)

Example 8.4.1. Evaluate

$$\int_0^{\pi/6} \tan 2x \, dx.$$

8.4.1 Logarithmic Differentiation

The derivatives of positive functions given by formulas that involve products, quotients and powers can often be found more quickly if we take the natural logarithm of both sides before differentiating. This enables us to use the laws of logarithms to simplify the formulas before differentiation. The process, called logarithm differentiation, is illustrated in the next example.

Example 8.4.2. Find $\frac{dy}{dx}$ if

$$y = \frac{(x^2+1)(x+3)^{1/2}}{x-1}, x > 1.$$

8.5 The Exponential function

Recall that the number e is such that $\ln e = 1$. We can compute e^r when r is a positive integer as $e^r = e \cdot e \cdots e$ (r times). We also have $e^{-r} = \frac{1}{e^r}$ if $r > 0$. When $r = a/b$ is rational we have $E = e^{a/b}$ as the number satisfying

$$E^b = e^a.$$

What about the value of e^x when x is any real number?

To obtain a function $f(x)$ that is defined for any real number and coincides with e^r when r is rational, we first observe that

$$\ln e^r = r \ln e = r.$$

So for rational r, we have

$$e^r = (\ln)^{-1} r.$$

We are therefore led to the following definition:

Definition 8.5.1. *For every real number x,*

$$e^x = \ln^{-1} x = \exp x.$$

This gives us a way to define a^x for any positive real number a and real number x.

Definition 8.5.2. *For every real number x and $a > 0$,*

$$a^x = e^{x \ln a}.$$

Some properties of e^x:

1. $e^u e^v = e^{u+v}$
2. $e^{-u} = \frac{1}{e^u}$
3. $(e^u)^v = e^{uv}$
4. $\dfrac{de^x}{dx} = e^x$.

Theorem 8.5.1. *The number e can be calculated as the limit*

$$e = \lim_{x \to 0} (1 + x)^{1/x}.$$

We can also establish the formula

$$\frac{dx^r}{dx} = rx^{r-1}.$$

8.6 Inverse Trigonometric functions

From Theorem 8.3.2, we know that if $y = \sin x$ and u is the inverse of y, then

$$\frac{du}{dx} = \frac{1}{\sqrt{1 - x^2}}, -1 \leq x \leq 1.$$

(Note that it is in the range $-1 \leq x \leq 1$ that u is one to one.)

In a similar way we have

$$\begin{aligned} \frac{d}{dx}(\cos^{-1} x) &= -\frac{1}{\sqrt{1 - x^2}} & \qquad \frac{d}{dx}(\csc^{-1} x) &= -\frac{1}{x\sqrt{x^2 - 1}} \\ \frac{d}{dx}(\sec^{-1} x) &= \frac{1}{x\sqrt{x^2 - 1}} & \qquad \frac{d}{dx}(\tan^{-1} x) &= \frac{1}{1 + x^2} \\ \frac{d}{dx}(\cot^{-1} x) &= -\frac{1}{1 + x^2}. \end{aligned}$$

Example 8.6.1. Define $\sinh x = \frac{e^x - e^{-x}}{2}$ and $\cosh x = \frac{e^x + e^{-x}}{2}$. Show that

$$\frac{d \sinh x}{dx} = \cosh x$$

and

$$\cosh^2 x - \sinh^2 x = 1.$$

Hence deduce that

$$\frac{\sinh^{-1} x}{dx} = \frac{1}{\sqrt{1 + x^2}}.$$

8.7 Techniques of Integration

In this section we will learn new techniques of deriving indefinite integrals.

8.7.1 Integration by parts

Consider the formula

$$\frac{d}{dx}(f(x)g(x)) = f(x)g'(x) + g(x)f'(x).$$

This means that

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx.$$

Example 8.7.1. Find $\int x \sin x\,dx$.

Example 8.7.2. Find $\int \ln x\,dx$.

Example 8.7.3. Find $\int e^x \sin x\,dx$.

Example 8.7.4. Prove the reduction formula

$$\int \sin^n x\,dx = -\frac{1}{n}\cos x \sin^{n-1} x + \frac{n-1}{n}\int \sin^{n-2} x\,dx$$

where $n \geq 2$ is an integer.

8.8 Integration of rational functions by partial fractions

In this section, we will learn how to integrate function of the form

$$\frac{A(x)}{B(x)}$$

where $A(x)$ and $B(x)$ are polynomials. If the degree of $A(x)$ is greater than that of $B(x)$ we do long division to obtain

$$A(x) = B(x)Q(x) + A_1(x).$$

We can then integrate

$$Q(x) + \frac{A_1(x)}{B(x)}.$$

From now on, we will assume that the degree of $A(x)$ is smaller than that of $B(x)$ in the expression $A(x)/B(x)$.

Case 1 Suppose $B(x) = (a_1x+b_1)(a_2x+b_2)\cdots(a_kx+b_k)$ where each of the linear factors are distinct. In this case (can you show this?)

$$\frac{A(x)}{B(x)} = \frac{A_1}{a_1x+b_1} + \cdots + \frac{A_k}{a_kx+b_k}.$$

One then solve for $A_1, \cdots, A_k$.

Example 8.8.1. Evaluate

$$\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx.$$

Case 2 Case 2 is an extension of case 1. Suppose $B(x) = (a_1x+b_1)^{r_1}(a_2x+b_2)^{r_2}\cdots(a_kx+b_k)^{r_k}$ where each of the linear factors are distinct. In this case

$$\frac{A(x)}{B(x)} = \frac{A_{1,1}}{a_1x+b_1} + \cdots + \frac{A_{1,r_1}}{(a_1x+b_1)^{r_1}} + + \cdots + \frac{A_{k,1}}{a_kx+b_k} + \cdots + \frac{A_{k,r_k}}{(a_kx+b_k)^{r_k}}.$$

Example 8.8.2. Evaluate

$$\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}\,dx.$$

Case 3 (Quadratic factors)

Suppose $B(x) = (a_1x^2+b_1x+c_1)(a_2x^2+b_2x+c_2)\cdots(a_kx^2+b_kx+c_k)$. In this case

$$\frac{A(x)}{B(x)} = \frac{A_1x+B_1}{a_1x^2+b_1x+c_1} + \cdots + \frac{A_kx+B_k}{a_kx^2+b_kx+c_k}.$$

Example 8.8.3. Evaluate

$$\int \frac{2x^2-x+4}{x^3+4x}\,dx.$$