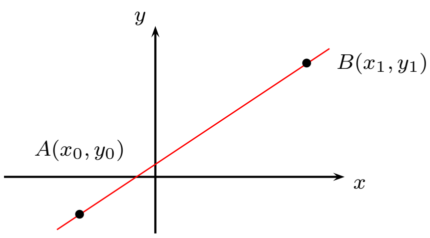
Chapter 2

A First Encounter with Limits

2.1 The straight line

In the previous lesson, we saw how we can represent a function using a graph. We now ask whether we could determine the function if we are given a straight line on an xy-plane. We all know that a straight line is determined by two distinct points. Consider the following line passing through A and B:



Let (x, y) be an arbitrary point on the line. Then by using similar triangles, we see that

$$\frac{y - y_0}{x - x_0} = \frac{y_1 - y_0}{x_1 - x_0}.$$

Hence the equation of the line is

$$y = m(x - x_0) + y_0,$$

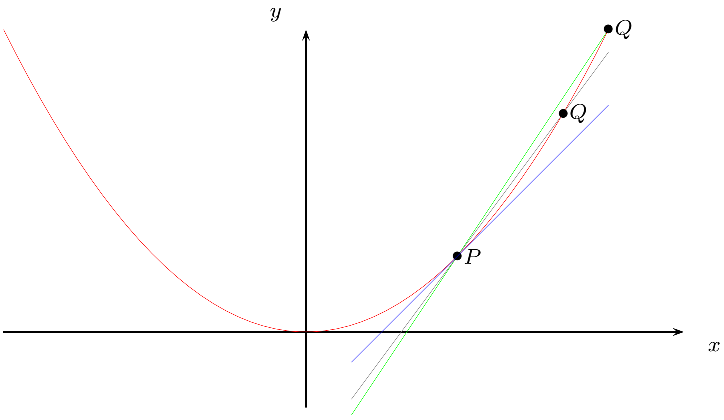
where

$$m = \frac{y_1 - y_0}{x_1 - x_0}.$$

Here the number m is called the **slope** (or **gradient**) of the line.

2.2 The tangent

Consider the curve $y = x^2$:



Fix a point P and let Q be any point in the graph of f. When $Q \neq P$, we can calculate the slope m_{PQ} of the secant line joining points P and Q. Note that as Q approaches P, the slope varies. When Q is exactly P, we see that the resulting line touches the curve at only one point. The slope of the resulting line is the "limit" of the slopes of the secant lines and we express symbolically by writing

$$\lim_{Q \to P} m_{PQ} = m$$

Since Q can be expressed as (x, x^2), the slope m_{PQ} can be expressed as

$$m_{PQ} = \frac{x^2 - 1}{x - 1},$$

and we may also write the slope m symbolically as

$$m = \lim_{x \to 1} \frac{x^2 - 1}{x - 1}.$$

This is perhaps our first encounter of "limit' in this course.

The line that passes through P with slope m is called the **tangent line** of the curve $y = x^2$ at point P. We will be able to give a precise definition of the tangent line after we discuss the derivative of a function.

Example 2.2.1.

Suppose a ball is dropped from a tower 450m above the ground. Find the velocity of the ball after 5 seconds.

It is known that if $s(t)$ denotes the distance fallen after t seconds then (just an approximation)

$$s(t) = 4.9t^2.$$

The average velocity within the time interval $[5, 5+h]$ is then

$$V_h = \frac{4.9(5+h)^2 - 4.9(5^2)}{h}.$$

The instantaneous velocity is then defined to be

$$\lim_{h \to 0} V_h.$$

If we plot the curve $s = 4.9t^2$, we see that finding the instantaneous velocity at $t = 5$ is the same as finding the slope of the curve at the point $(5, s(5))$.

2.3 Limit of a function

Having seen how limits arise in the previous section, let us investigate the various method of computing limits.

Consider the function

$$f(x) = x^2 - x + 2.$$

By substituting various values close to $x = 2$, we observe that as x approaches 2 (from both the left and right side of $x = 2$) $f(x)$ approaches 4. We write

$$\lim_{x \to 2}(x^2 - x + 2) = 4.$$

In general we have the following notation:

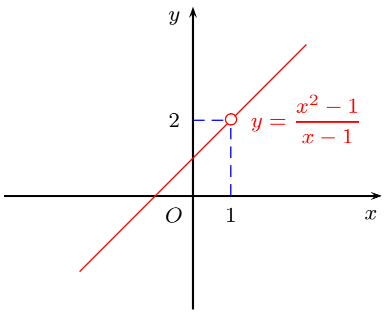
Definition 2.3.1 (Intuitive definition of limit). *We write*

$$\lim_{x \to a} f(x) = L$$

and say **"the limit of $f(x)$, as x approaches a, equals L"** *if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a.*

Example 2.3.1.

Let
$$f(x) = \frac{x^2-1}{x-1}.$$
This is a function we encountered in the previous section. Note that this function is not defined at $x = 1$. However, we can guess the limit
$$\lim_{x\to 1} \frac{x^2-1}{x-1}$$
to be 2 by letting x as close to 1 as possible.



We will defer the precise definition of a limit to a later section. In the meantime, let us learn some ways of computing limits.

2.4 Calculating limits using the limit laws

Suppose c is a real number. Then
$$\lim_{x\to a} c = c.$$
This is because c is independent of x and the behavior of $f(x) = c$ is not affected by the behavior of x.

Next, if $f(x) = x$, then
$$\lim_{x\to a} x = a.$$
We can show this using the precise definition of limit. For the time being, we observe that as x is close to a, $f(x) = x$ is close to a.

We now state the Limit Laws. By using the Limit Laws and the above simple observations, we will be able to compute many simple limits.

Theorem 2.4.1 (Limit Laws). *Suppose that c is a constant and that the limits*

$$\lim_{x\to a} f(x) \quad \textit{and} \quad \lim_{x\to a} g(x)$$

exist. Then

1. $\lim\limits_{x\to a}(f(x) \pm g(x)) = \lim\limits_{x\to a} f(x) \pm \lim\limits_{x\to a} g(x)$,

2. $\lim\limits_{x\to a} cf(x) = c\lim\limits_{x\to a} f(x)$,

3. $\lim\limits_{x\to a} f(x)g(x) = \lim\limits_{x\to a} f(x)\lim\limits_{x\to a} g(x)$,

4. $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}$, *if* $\lim_{x\to a} g(x) \neq 0$.

One can prove all the above statements using the precise definition of limit. This will be done in a later section.

The product rule shows that

$$\lim_{x\to a} x^n = (\lim_{x\to a} x)^n = a^n.$$

One can also show that the following rule holds:

$$\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}.$$

Together with the other rules we are able to calculate

$$\lim_{x\to 5}(2x^2 - 3x + 4)$$

as follow:

$$\begin{aligned}
\lim_{x\to 5}(2x^2 - 3x + 4) &= \lim_{x\to 5} 2x^2 - \lim_{x\to 5} 3x + 4 \\
&= 2(\lim_{x\to 5} x)^2 - 3\lim_{x\to 5} x + 4 \\
&= 2\cdot 5^2 - 3\cdot 5 + 4 = 39.
\end{aligned}$$

We see that if we substitute $x = 5$ directly into $2x^2 - 3x + 4$, we get $f(5) = 39$. In fact we have the following:

Theorem 2.4.2 (Direct substitution property). *If f is a polynomial or a rational function and a is in the domain of f, then*

$$\lim_{x \to a} f(x) = f(a).$$

Functions with the direct substitution property are called *continuous* at a and will be studied in a later section.

The requirement that "a is in the domain of f" in Theorem 2.4.2 is important as the following example shows:

Example 2.4.1.

Find

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1}.$$

Note that if

$$f(x) = \frac{x^2 - 1}{x - 1},$$

we cannot compute $f(1)$ because 1 is not in the domain of f. However we may first simplify $f(x)$ for $x \neq 1$ as

$$g(x) = \frac{x^2 - 1}{x - 1} = x + 1.$$

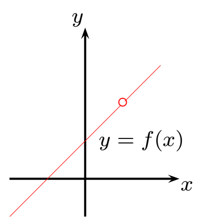
Note that $f(x)$ agrees with $g(x)$ except at $x = 1$. Also,

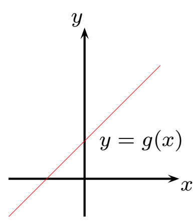
$$\lim_{x \to 1} g(x) = 2.$$

Since $f(x)$ agrees with $g(x)$, by the definition of limit we see that $f(x)$ must be close to 2 when x is close to 1. Hence,

$$\lim_{x \to 1} f(x) = \lim_{x \to 1} g(x) = 2.$$

This is therefore the slope of the tangent line to the curve $y = x^2$ at $(1, 1)$ in Section 2.





In general, if $f(x) = g(x)$ for all x in the domain of f except at $x = a$, then

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x).$$

Example 2.4.2.

Find

$$\lim_{x\to 0} \frac{\sqrt{x^2+9}-3}{x^2}.$$

We first find a function that agrees with

$$f(x) = \frac{\sqrt{x^2+9}-3}{x^2}$$

everywhere except at $x = 0$. We rationalize $f(x)$ to obtain

$$\begin{aligned}\frac{\sqrt{x^2+9}-3}{x^2} &= \frac{(\sqrt{x^2+9}-3)(\sqrt{x^2+9}+3)}{x^2(\sqrt{x^2+9}+3)} \\ &= \frac{1}{\sqrt{x^2+9}+3}.\end{aligned}$$

Let

$$g(x) = \frac{1}{\sqrt{x^2+9}+3}.$$

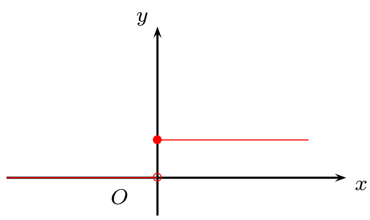
Since $g(x)$ agrees with $f(x)$ except at $x = 0$, we conclude that

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = g(0) = \frac{1}{6}.$$

2.5 One-sided limits

Consider the Heaviside function H defined by

$$H(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } x \geq 0. \end{cases}$$



From the graph, we see that as x approaches 0 from the right, $H(x)$ approaches 1. We write

$$\lim_{x \to 0^+} H(x) = 1.$$

This is called the **right-hand limit** of $H(x)$ as x approaches 0. In general, we write

$$\lim_{x \to a^+} f(x) = L$$

and say that the right hand limit of $f(x)$ as x approaches a is equal to L if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a and x is greater than a.

As x approaches 0 from the left, $H(x)$ approaches 0 and we write

$$\lim_{x \to 0^-} H(x) = 0.$$

This is called the **left-hand limit** of $H(x)$ as x approaches 0. In general, we write

$$\lim_{x \to a^-} f(x) = L$$

and say that the left hand limit of $f(x)$ as x approaches a is equal to L if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a and x is less than a.

We have

Theorem 2.5.1.

$$\lim_{x \to a} f(x) = L \quad \textit{if and only if} \quad \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L.$$

Example 2.5.1.

1. Show that
$$\lim_{x \to 0} |x| = 0.$$

2. Prove that
$$\lim_{x \to 0} \frac{|x|}{x}$$
does not exist.

3. If
$$f(x) = \begin{cases} \sqrt{x-4}, & \text{if } x > 4, \\ 8 - 2x, & \text{if } x < 4 \end{cases}$$
determine whether $\lim_{x \to 4} f(x)$ exists.

2.6 Infinite Limits

The function $f(x) = \frac{1}{x^2}$ appears to be very large when x approaches 0. When this happens we write

$$\lim_{x \to 0} \frac{1}{x^2} = \infty.$$

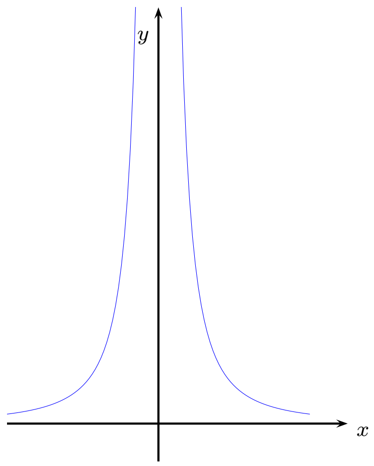
In general, suppose f is defined on both sides of a except possibly at a. Then

$$\lim_{x \to a} f(x) = \infty$$

means that the value of $f(x)$ can be made arbitrarily large by taking x sufficiently close to a but not equal to a.

If we plot the graph of f, we will see that when x approaches a, the curve "shoots" upwards to infinity as it approaches the line $x = a$. The line $x = a$ is called the vertical asymptote of the curve $y = f(x)$.

In the case of $f(x) = \frac{1}{x^2}$, we see that $x = 0$ is the asymptote of $f(x)$.



2.7 The Squeeze Theorem

We end this Chapter with an additional way of computing limits.

Theorem 2.7.1 (The Squeeze Theorem). *If $f(x) \leq g(x) \leq h(x)$ when x is near a (except possibly at a) and*

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L,$$

then

$$\lim_{x \to a} g(x) = L.$$

Example 2.7.1.

Show that

$$\lim_{x \to 0} x^2 \sin \frac{1}{x} = 0.$$

(Hint: Note that $f(x) = x^2 \sin \frac{1}{x}$ is "squeezed" between $y = x^2$ and $y = -x^2$ (see diagram).)