NATIONAL UNIVERSITY OF SINGAPORE

SEMESTER 1, 2014/2015

MA1101R Linear Algebra 1 **Tutorial 8**

1. Determine the possible rank and nullity of each of the following matrices

   (a) $A = \begin{pmatrix} 1 & 1 & a \\ 1 & a & 1 \\ a & 1 & 1 \end{pmatrix}$,

   (b) $B = \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & c \\ d & e & f \end{pmatrix}$, where a, b, c, d, e, f are real numbers.

   (Textbook, p. 144, Problem 15)

2. Let A and B be two matrices of the same size. Show that

   $$\text{rank}(A + B) \leq \text{rank}(A) + \text{rank}(B).$$

   (Textbook, p. 146, Problem 23)

3. Let A be an $m \times n$ matrix.

   (a) Show that the nullspace of A is equal to the nullspace of $A^T A$.

   (b) Show that $\text{nullity}(A) = \text{nullity}(A^T A)$ and $\text{rank}(A) = \text{rank}(A^T A)$.

   (c) Is it true that $\text{nullity}(A) = \text{nullity}(AA^T)$? Justify your answer.

   (d) Is it true that $\text{rank}(A) = \text{rank}(AA^T)$? Justify your answer.

   (Textbook, p. 146, Problem 25)

4. Let $\{u_1, \cdots, u_n\}$ be an orthogonal set of vectors in a vector space. Show that
$$\|u_1 + \cdots + u_n\|^2 = \|u_1\|^2 + \cdots + \|u_n\|^2.$$
For $n = 2$, interpret the result geometrically in $\mathbf{R}^2$.

(Textbook, p. 171, Problem 9)

5. Determine which of the following statements are true. Justify your answer.
   (a) If u, v, w are vectors in $\mathbf{R}^n$ such that $\|u\| = \|v\|$, then $\|u + w\| = \|v + w\|$.
   (b) If u, v, w are vectors in $\mathbf{R}^n$ such that $\|u\| = \|w\|$ and w is orthogonal to both u and v then $\|u + w\| = \|v + w\|$.
   (c) If u, v, w are vectors in $\mathbf{R}^n$ such that u is orthogonal to both v and w then u and $v + w$ are orthogonal.
   (d) If u, v, w are vectors in $\mathbf{R}^n$ such that u, v are orthogonal and v, w are orthogonal then u and w are orthogonal.

(Textbook, p. 172, Problem 20)