NATIONAL UNIVERSITY OF SINGAPORE

$$[T]_S = \begin{pmatrix} \frac{1}{5} & \frac{17}{5} & -\frac{8}{5} \\ \frac{1}{5} & \frac{27}{5} & -\frac{8}{5} \end{pmatrix}.$$

SEMESTER 1, 2014/2015

MA1101R Linear Algebra 1 **Tutorial 11**

1. For each of the following linear transformation, (i) determine whether there is enough information for us to find the formula of T, and (ii) find the formula and the standard matrix for T if possible.

   (a) $T : \mathbf{R}^3 \to \mathbf{R}^2$ such that

$$T\left(\begin{pmatrix}1\\1\\1\end{pmatrix}\right) = \begin{pmatrix}2\\3\end{pmatrix}, T\left(\begin{pmatrix}2\\1\\3\end{pmatrix}\right) = \begin{pmatrix}-1\\0\end{pmatrix}, \quad \text{and} \quad T\left(\begin{pmatrix}-1\\1\\2\end{pmatrix}\right) = \begin{pmatrix}0\\1\end{pmatrix}.$$

   (b) $T : \mathbf{R}^3 \to \mathbf{R}^3$ such that

$$T\left(\begin{pmatrix}1\\0\\1\end{pmatrix}\right) = \begin{pmatrix}1\\1\\2\end{pmatrix}, T\left(\begin{pmatrix}1\\1\\0\end{pmatrix}\right) = \begin{pmatrix}1\\0\\1\end{pmatrix}, \quad \text{and} \quad T\left(\begin{pmatrix}1\\0\\0\end{pmatrix}\right) = \begin{pmatrix}0\\0\\1\end{pmatrix}.$$

(Textbook, p. 229, Problem 2)

(b) $\left(\begin{array}{ccc|c} 1 & 1 & 1 & x \\ 0 & 1 & 0 & y \\ 1 & 0 & 0 & z \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & x \\ 0 & 1 & 0 & y \\ 0 & 0 & -1 & z-x+y \end{array}\right)$

$c = x-y-z, \quad b = y, \quad a = z.$

$$\therefore T\Big(\underbrace{\begin{pmatrix}1\\0\\1\end{pmatrix}a + \begin{pmatrix}1\\1\\0\end{pmatrix}b + \begin{pmatrix}1\\0\\0\end{pmatrix}c}_{\begin{pmatrix}x\\y\\z\end{pmatrix}}\Big) = \begin{pmatrix}1\\1\\2\end{pmatrix}z + \begin{pmatrix}1\\0\\1\end{pmatrix}y + \begin{pmatrix}0\\0\\1\end{pmatrix}(x-y-z)$$

$$= \begin{pmatrix} z+y \\ z \\ x+z \end{pmatrix}$$

$$[T]_S = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

2^{nd} method:

$$T\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 2 & 1 & 1 \end{pmatrix}$$

$$T\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 2 & 1 & 1 \end{pmatrix}$$

$$\therefore \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}[T]_S\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 2 & 1 & 1 \end{pmatrix}$$

where $S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$.

$$\therefore [T]_S = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 2 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}^{-1}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & -1 & -1 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & 1 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} -1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & 1 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & -1 \end{array}\right)$$

$$\therefore [T]_S = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 2 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & -1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

2. Let $T : \mathbf{R}^n \to \mathbf{R}^n$ be a linear operator. if there exits a linear operator $S : \mathbf{R}^n \to \mathbf{R}^n$ such that $S \circ T$ is the identity transformation, i.e.,

$$(S \circ T)(u) = u$$

for all $u \in \mathbf{R}^n$ then T is said to be invertible and S is called the inverse of T.

(a) For each of the following, determine whether T is invertible and find the inverse of T if possible.

(i) $T : \mathbf{R}^2 \to \mathbf{R}^2$ such that

$$T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} y \\ x \end{pmatrix}$$

for $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbf{R}^2$.

(ii) $T : \mathbf{R}^2 \to \mathbf{R}^2$ such that

$$T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x + y \\ 0 \end{pmatrix}$$

for $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbf{R}^2$.

(b) Suppose T is invertible and A is the standard matrix for T. Find the standard matrix for the inverse of T.

(Textbook, p. 231, Problem 6)

(a)(i) Let S be the standard basis.

$$[T]_S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad [T^{-1}]_S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

(ii) $[T]_S = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$. T is not invertible.

(b) $T(\underline{e_1} \dots \underline{e_n}) = (\underline{e_1} \dots \underline{e_n})[T]_S$

$T^{-1}T(\underline{e_1} \dots \underline{e_n}) = T^{-1}(\underline{e_1} \dots \underline{e_n})[T]_S$

$(\underline{e_1}, \dots \underline{e_n}) = (\underline{e_1} \dots \underline{e_n})[T^{-1}]_S[T]_S \Rightarrow I_n = [T^{-1}]_S[T]_S$

$\therefore [T^{-1}]_S = [T]_S^{-1}$

3. Let $\cancel{n}\ \underset{\sim}{u}$ be a unit vector in $\mathbf{R}^n$. Define $P : \mathbf{R}^n \to \mathbf{R}^n$ such that

$$P(x) = x - (\cancel{n}\,\underset{\sim}{u} \cdot x)\cancel{n}\,\underset{\sim}{u}$$

for all $x \in \mathbf{R}^n$.

(a) Show that P is a linear transformation and find the standard matrix for P.

(b) Prove that $P \circ P = P$.

(Textbook, p. 231, Problem 7)

3(a)

$$P(\underset{\sim}{x}) = \underset{\sim}{x} - (\underset{\sim}{u} \cdot \underset{\sim}{x})\underset{\sim}{u}$$

$$= \underset{\sim}{x} - \underset{\sim}{u}(\underset{\sim}{u} \cdot \underset{\sim}{x})$$

$$= \underset{\sim}{x} - \underset{\sim}{u}\,\underset{\sim}{u}^T \underset{\sim}{x} = (I - \underset{\sim}{u}\underset{\sim}{u}^T)\underset{\sim}{x}$$

$\therefore P$ is a linear transformation.

Let $\underset{\sim}{u} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$.

$$\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} (\alpha_1 \cdots \alpha_n) = \begin{pmatrix} \alpha_1^2 & \alpha_1\alpha_2 \cdots & \alpha_1\alpha_n \\ \alpha_2\alpha_1 & \alpha_2^2 \cdots & \alpha_2\alpha_n \\ \vdots & \ddots & \\ \alpha_n\alpha_1 & \cdots & \alpha_n^2 \end{pmatrix}$$

$$P(\underset{\sim}{e_j}) = \begin{array}{l} \\ \\ \\ j\text{th row} \rightarrow \\ \\ \end{array} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} - \begin{pmatrix} \alpha_1\alpha_j \\ \alpha_2\alpha_j \\ \vdots \\ \alpha_j^2 \\ \vdots \\ \alpha_n\alpha_j \end{pmatrix} = -\alpha_1\alpha_j \underset{\sim}{e_1} - \alpha_2\alpha_j \underset{\sim}{e_2} - \cdots + (1-\alpha_j^2)\underset{\sim}{e_j} + \cdots - \alpha_n\alpha_j \underset{\sim}{e_n}$$

$$[P]_S = \begin{pmatrix} 1-\alpha_1^2 & -\alpha_1\alpha_2 & \cdots & -\alpha_1\alpha_j & \cdots & -\alpha_1\alpha_n \\ -\alpha_1\alpha_2 & 1-\alpha_2^2 & \cdots & & & \vdots \\ \vdots & \vdots & & 1-\alpha_j^2 & & \vdots \\ -\alpha_1\alpha_n & -\alpha_2\alpha_n & & -\alpha_j\alpha_n & \cdots & 1-\alpha_n^2 \end{pmatrix}$$

$$P \circ P(\underline{x}) = P(\underline{x} - (\underline{u}\cdot\underline{x})\underline{u})$$

$$= P(\underline{x}) - (\underline{u}\cdot\underline{x})\,P(\underline{u})$$

$$= P(\underline{x}) - (\underline{u}\cdot\underline{x})\underbrace{(\underline{u} - (\underline{u}\cdot\underline{u})\underline{u})}_{=0}$$

$\underline{u}\cdot\underline{u} = 1$

$$= P(\underline{x})$$

$$\therefore P \circ P = P$$

4. Let V be a subspace of $\mathbf{R}^n$. Define a mapping $P : \mathbf{R}^n \to \mathbf{R}^n$ such that for all $u \in \mathbf{R}^n$, $P(u)$ is the projection of u onto V.

   (a) Show that P is a linear transformation.

   (b) Suppose $n = 3$ and V is the plane $ax + by + cz = 0$ where a, b, c are not all zeroes. Find $\ker(P)$ and $\mathrm{R}(P)$.

(Textbook, p. 233, Problem 15)

(a) let $\underline{v}_1, \ldots, \underline{v}_m$ be a orthonormal basis of V

$$\underline{v}_1 = \begin{pmatrix} a_{11} \\ \vdots \\ a_{1n} \end{pmatrix}$$

$$P(\underline{u}) = (\underline{v}_1, \underline{u})\, \underline{v}_1 + \ldots + (\underline{v}_n, \underline{u})\, \underline{v}_n$$

$$= \underline{v}_1 \underline{v}_1^T \underline{u} + \ldots + \underline{v}_n \underline{v}_n^T \underline{u}.$$

$$= (\underline{v}_1 \underline{v}_1^T + \ldots + \underline{v}_n \underline{v}_n^T)\, \underline{u}.$$

OR $P(a\underline{x} + \underline{y}) = (\underline{v}_1 \cdot (a\underline{x} + \underline{y}))\, \underline{v}_1 + \ldots + (\underline{v}_n \cdot (a\underline{x} + \underline{y}))\, \underline{v}_n$

$$= a(\underline{v}_1 \cdot \underline{x})\, \underline{v}_1 + (\underline{v}_1 \cdot \underline{y})\, \underline{v}_1 + \ldots + a(\underline{v}_n \cdot \underline{x})\, \underline{v}_n + (\underline{v}_n \cdot \underline{y})\, \underline{v}_n$$

$$= a\big((\underline{v}_1 \cdot \underline{x})\, \underline{v}_1 + \ldots + (\underline{v}_n \cdot \underline{x})\, \underline{v}_n\big) + (\underline{v}_1 \cdot \underline{y})\, \underline{v}_1 + \ldots + (\underline{v}_n \cdot \underline{y})\, \underline{v}_n$$

$$= aP(\underline{x}) + P(\underline{y}).$$

$\therefore$ P is linear

(b) $\underline{n = 3}$:

Let $\underline{v}_1$ & $\underline{v}_2$ be orthonormal vectors that span V.

$$P(\underline{u}) = (\underline{u} \cdot \underline{v}_1)\, \underline{v}_1 + (\underline{u} \cdot \underline{v}_2)\, \underline{v}_2.$$

Suppose $\underline{w} \in \ker(P)$. Then

$$P(\underline{w}) = \underline{0}$$

$$(\underline{w} \cdot \underline{v}_1)\, \underline{v}_1 + (\underline{w} \cdot \underline{v}_2)\, \underline{v}_2 = \underline{0}$$

But $\underline{v}_1$ & $\underline{v}_2$ are l.i.

$\Rightarrow$ $\underline{w} \cdot \underline{v}_1 = 0$ & $\underline{w} \cdot \underline{v}_2 = 0$

$\Rightarrow$ $\underline{w}$ is orthogonal to $\underline{v}_1$ & $\underline{v}_2$.

Note $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$.

$\therefore$ $\underline{w} \in \text{Span}\left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right\}$.

Now, $\ker(P)$ is a subspace of $\mathbb{R}^3$ &

$\ker(P) \cap V = \phi$. $\Rightarrow \dim \ker(P) = 1$ $\therefore$

$$\ker(P) = \text{Span}\left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right\}.$$

Let $\underline{v} \in V$. $\underline{v} = a\underline{v}_1 + b\underline{v}_2$.

$P(\underline{v}) = aP(\underline{v}_1) + bP(\underline{v}_2)$ $\quad$ $\textcircled{}$ $\because$ $\underline{v}_1$ $\perp$ $\underline{v}_2$.

But $P(\underline{v}_1) = (\underline{v}_1 \cdot \underline{v}_1)\, \underline{v}_1 + (\underline{v}_1 \cdot \underline{v}_2)\, \underline{v}_2$

$= \underline{v}_1$. Similarly, $P(\underline{v}_2) = \underline{v}_2$

$\therefore$ $P(\underline{v}) = a\underline{v}_1 + b\underline{v}_2 = \underline{v}$. The above shows

Now: $P: \mathbb{R}^3 \to V$. $\Rightarrow$ $R(P) \subset V$.

$V \subset R(P)$ $\Rightarrow$ $V = R(P)$ $\square$

5. Let $S : \mathbf{R}^n \to \mathbf{R}^n$ and $T : \mathbf{R}^n \to \mathbf{R}^k$ be linear transformations.
   (a) Show that $\mathrm{Ker}(S) \subset \mathrm{Ker}(T \circ S)$.
   (b) Show that $\mathrm{R}(T \circ S) \subset \mathrm{R}(T)$.

(Textbook, p. 233, Problem 17)

5(a) $\underset{\sim}{x} \in \mathrm{Ker}(S)$. $S\underset{\sim}{x} = \underline{0}$

$TS\underset{\sim}{x} = T\underline{0} = \underline{0}$

$\therefore$ $\mathrm{Ker}\, S \subset \mathrm{Ker}(T \circ S)$

(b) Let $\underset{\sim}{x} \in R(T \circ S)$. $\underset{\sim}{x} = T \circ S(\underset{\sim}{y})$

$\underset{\sim}{x} = T(S(\underset{\sim}{y}))$

$\therefore$ $\underset{\sim}{x} \in R(T)$.