Answers/Solutions of Exercise 6 (Version: November 2, 2012)

1. (a) The characteristic equation is $(\lambda+1)(\lambda-3)=0$; eigenvalues are -1 and 3; $\{(0,1)^{\mathrm{T}}\}$ is a basis for E_{-1} and $\{(1,2)^{\mathrm{T}}\}$ is a basis for E_3.

   (b) The characteristic equation is $(\lambda-2)^2=0$; the eigenvalue is 2; $\{(1,1)^{\mathrm{T}}\}$ is a basis for E_2.

   (c) The characteristic equation is $\lambda^2-4=0$; eigenvalues are -2 and 2; $\{(-2,1)^{\mathrm{T}}\}$ is a basis for E_{-2} and $\{(2,1)^{\mathrm{T}}\}$ is a basis for E_2 .

   (d) The characteristic equation is $\lambda^2=0$; the eigenvalue is 0; $\{(1,0),(0,1)^{\mathrm{T}}\}$ is a basis for E_0.

   (e) The characteristic equation is $\lambda(\lambda-2)^2=0$; eigenvalues are 0 and 2; $\{(-1,1,0)^{\mathrm{T}}\}$ is a basis for E_0 and $\{(1,1,0)^{\mathrm{T}}\}$ is a basis for E_2.

   (f) The characteristic equation is $(\lambda-2)(\lambda^2-9)=0$; eigenvalues are 2, -3 and 3; $\{(0,0,1)^{\mathrm{T}}\}$ is a basis for E_2, $\{(-1,3,0)^{\mathrm{T}}\}$ is a basis for E_{-3} and $\{(1,3,0)^{\mathrm{T}}\}$ is a basis for E_3.

   (g) The characteristic equation is $(\lambda-1)^3=0$; the eigenvalue is 1; $\{(0,0,1)^{\mathrm{T}}\}$ is a basis for E_1.

   (h) The characteristic equation is $(\lambda+1)(\lambda-1)^2=0$; eigenvalues are -1 and 1; $\{(-1,-1,1)^{\mathrm{T}}\}$ is a basis for E_{-1} and $\{(1,2,0)^{\mathrm{T}},(1,0,2)^{\mathrm{T}}\}$ is a basis for E_1.

   (i) The characteristic equation is $(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)=0$; eigenvalues are 1,2,3 and 4; $\{(0,0,0,1)^{\mathrm{T}}\}$is a bais for E_1, $\{(0,0,1,1)^{\mathrm{T}}\}$ is a basis for E_2, $\{(0,2,4,3)^{\mathrm{T}}\}$ is a basis for E_3 and $\{(3,9,12,8)^{\mathrm{T}}\}$ is a basis for E_4.

   (j) The characteristic equation is $\lambda^4-2\lambda^2+1=0$; eigenvalues are -1 and 1; $\{(-1,0,1,0)^{\mathrm{T}},(0,-1,0,1)^{\mathrm{T}}\}$ is a basis for E_{-1} and $\{(1,0,1,0)^{\mathrm{T}},(0,1,0,1)^{\mathrm{T}}\}$ is a basis for E_1.

2. (a) $\det(\lambda\boldsymbol{I}-\boldsymbol{A})=\begin{vmatrix}\lambda-a & -b\\ -c & \lambda-d\end{vmatrix}=\lambda^2+(-a-d)\lambda+(ad-bc)$

   Hence $m=-a-d=-\mathrm{tr}(\boldsymbol{A})$ and $n=\det(\boldsymbol{A})$.

   (b) Direct verification shows that $\boldsymbol{A}^2+m\boldsymbol{A}+n\boldsymbol{I}=\boldsymbol{0}$.

3. (a) Let $\boldsymbol{x}$ be an eigenvector of $\boldsymbol{A}$ associated with λ, i.e. $\boldsymbol{A}\boldsymbol{x}=\lambda\boldsymbol{x}$. We prove that $\boldsymbol{A}^n\boldsymbol{x}=\lambda^n\boldsymbol{x}$ by induction on n.

   It is given that $\boldsymbol{A}^1\boldsymbol{x}=\lambda^1\boldsymbol{x}$. Assume that $\boldsymbol{A}^k\boldsymbol{x}=\lambda^k\boldsymbol{x}$. Then

$$\boldsymbol{A}^{k+1}\boldsymbol{x}=\boldsymbol{A}(\boldsymbol{A}^k\boldsymbol{x})=\boldsymbol{A}(\lambda^k\boldsymbol{x})=\lambda^k\boldsymbol{A}\boldsymbol{x}=\lambda^k\lambda\boldsymbol{x}=\lambda^{k+1}\boldsymbol{x}.$$

By mathematical induction, $\boldsymbol{A}^n\boldsymbol{x} = \lambda^n\boldsymbol{x}$ and hence λ^n is an eigenvalue of $\boldsymbol{A}$ for all positive integer n.

(b) Let $\boldsymbol{x}$ be an eigenvector of $\boldsymbol{A}$ associated with λ. Then

$$\boldsymbol{A}\boldsymbol{x} = \lambda\boldsymbol{x} \quad\Rightarrow\quad \boldsymbol{x} = \boldsymbol{A}^{-1}(\lambda\boldsymbol{x}) = \lambda\boldsymbol{A}^{-1}\boldsymbol{x} \quad\Rightarrow\quad \tfrac{1}{\lambda}\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{x}.$$

Thus $\frac{1}{\lambda}$ is an eigenvalue of $\boldsymbol{A}^{-1}$.

(c) λ is an eigenvalue of $\boldsymbol{A}$

$$\begin{aligned} &\Rightarrow \det(\lambda\boldsymbol{I} - \boldsymbol{A}) = 0 \\ &\Rightarrow \det\left((\lambda\boldsymbol{I} - \boldsymbol{A})^{\mathrm{T}}\right) = 0 \\ &\Rightarrow \det(\lambda\boldsymbol{I} - \boldsymbol{A}^{\mathrm{T}}) = 0 \\ &\Rightarrow \lambda \text{ is an eigenvalue of } \boldsymbol{A}^{\mathrm{T}}. \end{aligned}$$

4. (a) Let $\boldsymbol{x}$ be an eigenvector of $\boldsymbol{A}$ associated with λ, i.e. $\boldsymbol{A}\boldsymbol{x} = \lambda\boldsymbol{x}$ and $\boldsymbol{x}$ is a nonzero vector. Then

$$\boldsymbol{A}^2 = \boldsymbol{A} \quad\Rightarrow\quad \boldsymbol{A}^2\boldsymbol{x} = \boldsymbol{A}\boldsymbol{x} \quad\Rightarrow\quad \lambda^2\boldsymbol{x} = \lambda\boldsymbol{x} \quad\Rightarrow\quad \lambda(\lambda-1)\boldsymbol{x} = \boldsymbol{0}$$

Since $\boldsymbol{x}$ is nonzero, $\lambda = 0$ or 1.

(b) Since $\boldsymbol{A}$ has 2 distinct eigenvalues, it is diagonalizable. Let $\boldsymbol{P} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an invertible matrix such that $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then

$$\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} ad & -ab \\ cd & -cb \end{pmatrix} \quad \text{where } ad-bc \neq 0.$$

We can simplify the expression to $\boldsymbol{A} = \begin{pmatrix} r & s \\ t & 1-r \end{pmatrix}$ where $st = r(1-r)$.

5. (a) Let $\boldsymbol{x}$ be a nonzero eigenvector of $\boldsymbol{A}$ associated with λ, i.e. $\boldsymbol{A}\boldsymbol{x} = \lambda\boldsymbol{x}$.

$$\boldsymbol{A}^2 = \boldsymbol{0} \quad\Rightarrow\quad \boldsymbol{A}^2\boldsymbol{x} = \boldsymbol{0}\boldsymbol{x} \quad\Rightarrow\quad \boldsymbol{A}(\lambda\boldsymbol{x}) = \boldsymbol{0} \quad\Rightarrow\quad \lambda^2\boldsymbol{x} = \boldsymbol{0}$$

Since $\boldsymbol{x}$ is nonzero, $\lambda = 0$.

(b) No. Suppose $\boldsymbol{A}$ is diagonalizable. Then there exists invertible $\boldsymbol{P}$ such that $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \boldsymbol{0}$. Then $\boldsymbol{A} = \boldsymbol{P}\boldsymbol{0}\boldsymbol{P}^{-1} = \boldsymbol{0}$, a contradiction.

(c) Consider the vector equation

$$a\boldsymbol{u} + b\boldsymbol{A}\boldsymbol{u} = \boldsymbol{0}. \tag{$*$}$$

Pre-multiplying $\boldsymbol{A}$ to both side of $(*)$, we have

$$\boldsymbol{A}(a\boldsymbol{u} + \boldsymbol{A}\boldsymbol{u}) = \boldsymbol{A}\boldsymbol{0} \quad\Rightarrow\quad a\boldsymbol{A}\boldsymbol{u} = \boldsymbol{0}. \quad (\because \boldsymbol{A}^2 = \boldsymbol{0}.)$$

As $\boldsymbol{Au} \neq \mathbf{0}$, $a = 0$. Substituting $a = 0$ into $(*)$, we have $b\boldsymbol{Au} = \mathbf{0}$ and hence $b = 0$. Since $(*)$ has only the trivial solution, $\boldsymbol{u}$ and $\boldsymbol{Au}$ are linearly independent.

(d) Let $\boldsymbol{P} = \begin{pmatrix} \boldsymbol{u} & \boldsymbol{Au} \end{pmatrix}$. By (c), $\boldsymbol{P}$ is invertible. Since

$$\boldsymbol{AP} = \begin{pmatrix} \boldsymbol{Au} & \boldsymbol{A}^2\boldsymbol{u} \end{pmatrix} = \begin{pmatrix} \boldsymbol{Au} & \mathbf{0} \end{pmatrix}$$

and

$$\boldsymbol{P}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0\boldsymbol{u} + \boldsymbol{Au} & 0\boldsymbol{u} + 0\boldsymbol{Au} \end{pmatrix} = \begin{pmatrix} \boldsymbol{Au} & \mathbf{0} \end{pmatrix},$$

$\boldsymbol{AP} = \boldsymbol{P}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ which implies $\boldsymbol{P}^{-1}\boldsymbol{AP} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

6. (a) Since $\det(-\boldsymbol{I} - \boldsymbol{A}) = 0$, -1 is an eigenvalue of $\boldsymbol{A}$.

   (b) $\{(1,1,0)^{\mathrm{T}}, (0,0,1)^{\mathrm{T}}\}$ is a basis for E_{-1} and hence $\dim(E_{-1}) = 2$.

   (c) For example, $\boldsymbol{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

7. (a) Since $\det(2\boldsymbol{I} - \boldsymbol{A}) = 0$, 2 is an eigenvalue of $\boldsymbol{A}$.

   (b) $\{(1,2,0)^{\mathrm{T}}, (-3,0,1)^{\mathrm{T}}\}$ is a basis for the eigenspace associated with 2.

   (c) Let E_2 be the eigenspace of $\boldsymbol{A}$ associated with 2 and let E'_λ be the eigenspace of $\boldsymbol{B}$ associated with λ.

   Since E_2 and E'_λ are subspaces of $\mathbb{R}^3$ and have dimension 2, they are two planes in $\mathbb{R}^3$ that contain the origin. So $E_2 \cap E'_\lambda$ is either a line through the origin or a plane containing the origin. In both cases, we can find a nonzero vector $\boldsymbol{u} \in E_2 \cap E'_\lambda$, i.e. $\boldsymbol{Au} = 2\boldsymbol{u}$ and $\boldsymbol{Bu} = \lambda\boldsymbol{u}$, such that

$$(\boldsymbol{A} + \boldsymbol{B})\boldsymbol{u} = \boldsymbol{Au} + \boldsymbol{Bu} = 2\boldsymbol{u} + \lambda\boldsymbol{u} = (2 + \lambda)\boldsymbol{u}.$$

   So $2 + \lambda$ is an eigenvalue of $\boldsymbol{A} + \boldsymbol{B}$.

8. Note that for $i = 1, 2, \ldots, n$, $\boldsymbol{A}^n\boldsymbol{u_i} = \boldsymbol{A}^{n-1}\boldsymbol{u_{i+1}} = \cdots = \boldsymbol{A}^i\boldsymbol{u_n} = \mathbf{0}$.

   Let $\boldsymbol{v} \in \mathbb{R}^n$ be an eigenvector of $\boldsymbol{A}$ associated with eigenvalue λ, i.e. $\boldsymbol{Av} = \lambda\boldsymbol{v}$. Since $\{\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_n}\}$ is a basis for $\mathbb{R}^n$,

$$\boldsymbol{v} = c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + \cdots + c_n\boldsymbol{u_n}$$

   for some $c_1, c_2, \ldots, c_n \in \mathbb{R}$. Then

$$\boldsymbol{A}^n\boldsymbol{v} = c_1\boldsymbol{A}^n\boldsymbol{u_1} + c_2\boldsymbol{A}^n\boldsymbol{u_2} + \cdots + c_n\boldsymbol{A}^n\boldsymbol{u_n} = \mathbf{0}.$$

From the proof of Question 6.3(a), $\boldsymbol{A}^n\boldsymbol{v} = \lambda^n\boldsymbol{v}$. Since $\boldsymbol{v} \neq \boldsymbol{0}$, $\lambda = 0$. Hence we have shown that $\boldsymbol{A}$ has only one eigenvalue 0.

As $\lambda = 0$, we get $\boldsymbol{Av} = \boldsymbol{0}$. Then

$$\boldsymbol{0} = \boldsymbol{Av} = c_1\boldsymbol{Au_1} + c_2\boldsymbol{Au_2} + \cdots + c_n\boldsymbol{Au_n} = c_1\boldsymbol{u_2} + c_2\boldsymbol{u_3} + \cdots + c_{n-1}\boldsymbol{u_n}.$$

Since $\boldsymbol{u_2}, \boldsymbol{u_3}, \ldots, \boldsymbol{u_n}$ are linearly independent, $c_1 = 0$, $c_2 = 0$, ..., $c_{n-1} = 0$, i.e. $\boldsymbol{v} = c_n\boldsymbol{u_n}$. Hence all eigenvectors of $\boldsymbol{A}$ are scalar multiples of $\boldsymbol{u_n}$.

9. (a) Diagonalizable. Let $\boldsymbol{P} = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$. Then $\boldsymbol{P}^{-1}\boldsymbol{AP} = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}$.

   (b) Not diagonalizable.

   (c) Diagonalizable. Let $\boldsymbol{P} = \begin{pmatrix} -2 & 2 \\ 1 & 1 \end{pmatrix}$. Then $\boldsymbol{P}^{-1}\boldsymbol{AP} = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}$.

   (d) Diagonalizable. Let $\boldsymbol{P} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Then $\boldsymbol{P}^{-1}\boldsymbol{AP} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

   (e) Not diagonalizable.

   (f) Diagonalizable. Let $\boldsymbol{P} = \begin{pmatrix} 0 & -1 & 1 \\ 0 & 3 & 3 \\ 1 & 0 & 0 \end{pmatrix}$. Then $\boldsymbol{P}^{-1}\boldsymbol{AP} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

   (g) Not diagonalizable.

   (h) Diagonalizable. Let $\boldsymbol{P} = \begin{pmatrix} -1 & 1 & 1 \\ -1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}$. Then $\boldsymbol{P}^{-1}\boldsymbol{AP} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

   (i) Diagonalizable. Let $\boldsymbol{P} = \begin{pmatrix} 0 & 0 & 0 & 3 \\ 0 & 0 & 2 & 9 \\ 0 & 1 & 4 & 12 \\ 1 & 1 & 3 & 8 \end{pmatrix}$. Then $\boldsymbol{P}^{-1}\boldsymbol{AP} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$.

   (j) Diagonalizable. Let $\boldsymbol{P} = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$. Then $\boldsymbol{P}^{-1}\boldsymbol{AP} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

10. (a) Eigenvalues are $-i$ and i.

    Let $\boldsymbol{P} = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}$. Then $\boldsymbol{P}^{-1}\boldsymbol{AP} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$.

    (b) Eigenvalues are $2 - i$ and $2 + i$.

    Let $\boldsymbol{P} = \begin{pmatrix} 1+i & 1-i \\ 2 & 2 \end{pmatrix}$. Then $\boldsymbol{P}^{-1}\boldsymbol{AP} = \begin{pmatrix} 2-i & 0 \\ 0 & 2+i \end{pmatrix}$.

(c) Eigenvalues are 0, $2-i$ and $2+i$.

Let $\boldsymbol{P} = \begin{pmatrix} 1 & 1+3i & 1-3i \\ 0 & 5i & -5i \\ 0 & 5 & 5 \end{pmatrix}$. Then $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2-i & 0 \\ 0 & 0 & 2+i \end{pmatrix}$.

11. (a) Let $\boldsymbol{P} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$. Then $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$.

(b) $\boldsymbol{A}^{10} = \begin{pmatrix} 1 & 0 & 4^{10}-1 \\ 0 & 4^{10} & 0 \\ 0 & 0 & 4^{10} \end{pmatrix}$

(c) For example, let $\boldsymbol{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ and $\boldsymbol{B} = \boldsymbol{P}\boldsymbol{C}\boldsymbol{P}^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$. Then $\boldsymbol{B}^2 = \boldsymbol{A}$.

12. Let $\boldsymbol{P} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$ and $\boldsymbol{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$. Then the matrix $\boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^{-1} = \begin{pmatrix} -1 & -\frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ has the required eigenvalues and eigenvectors.

13. The matrix is diagonalizable if and only if $a \neq b$.

14. (a) The eigenvalues are 2, 0, 1 and -1.

(b) $\boldsymbol{u_1}$ is an eigenvector associated with 2.
$\boldsymbol{u_2}$ is an eigenvector associated with 0.
$\boldsymbol{u_3} + \boldsymbol{u_4}$ is an eigenvector associated with 1.
$\boldsymbol{u_3} - \boldsymbol{u_4}$ is an eigenvector associated with -1.

(c) Note that $\boldsymbol{u_1}$, $\boldsymbol{u_2}$, $\boldsymbol{u_3}$, $\boldsymbol{u_3}+\boldsymbol{u_4}$, $\boldsymbol{u_3}+\boldsymbol{u_4}$ are linearly independent eigenvectors. By Theorem 6.2.3, $\boldsymbol{B}$ is diagonalizable.

Alternatively Solution: Since $\boldsymbol{B}$ has 4 distinct eigenvalues, by Theorem 6.2.7, $\boldsymbol{B}$ is diagonalizable.

15. (a) (i) $\boldsymbol{B} = \boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} \Rightarrow \boldsymbol{B}^n = \underbrace{(\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P})(\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P})\cdots(\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P})}_{n \text{ times}} = \boldsymbol{P}^{-1}\boldsymbol{A}^n\boldsymbol{P}$

So $\boldsymbol{A}^n$ is similar to $\boldsymbol{B}^n$.

(ii) $\boldsymbol{B} = \boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} \Rightarrow \boldsymbol{B}^{-1} = (\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P})^{-1} = \boldsymbol{P}^{-1}\boldsymbol{A}^{-1}\boldsymbol{P}$
So $\boldsymbol{A}^{-1}$ is similar to $\boldsymbol{B}^{-1}$.

(iii) Suppose there exists an invertible matrix $\boldsymbol{Q}$ such that $\boldsymbol{Q}^{-1}\boldsymbol{A}\boldsymbol{Q}$ is a diagonal matrix. Let $\boldsymbol{R} = \boldsymbol{P}^{-1}\boldsymbol{Q}$. Then $\boldsymbol{R}$ is invertible and $\boldsymbol{R}^{-1}\boldsymbol{B}\boldsymbol{R} = \boldsymbol{Q}^{-1}\boldsymbol{P}\boldsymbol{B}\boldsymbol{P}^{-1}\boldsymbol{Q} = \boldsymbol{Q}^{-1}\boldsymbol{A}\boldsymbol{Q}$ is a diagonal matrix.

(b) Since $\boldsymbol{A}$ is a triangular matrix, its eigenvalues are 0, 1 and -1. Also it is easy to find from the characteristic equation of $\boldsymbol{B}$ that the eigenvalues of $\boldsymbol{B}$ are 0, 1 and -1. By Theorem 6.2.7, both $\boldsymbol{A}$ and $\boldsymbol{B}$ are diagonalizable. So there exist invertible matrices $\boldsymbol{R}$ and $\boldsymbol{Q}$ such that

$$\boldsymbol{R}^{-1}\boldsymbol{A}\boldsymbol{R} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \boldsymbol{Q}^{-1}\boldsymbol{B}\boldsymbol{Q}.$$

Let $\boldsymbol{P} = \boldsymbol{R}\boldsymbol{Q}^{-1}$. Then $\boldsymbol{P}$ is invertible matrix and $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \boldsymbol{Q}\boldsymbol{R}^{-1}\boldsymbol{A}\boldsymbol{R}\boldsymbol{Q}^{-1} = \boldsymbol{B}$.

16. (a) Let $\boldsymbol{A} = (a_{ij})_{n\times n}$. Then $a_{1i} + a_{2i} + \cdots + a_{ni} = 1$ for $i = 1, 2, \ldots, n$.

(i) $$\boldsymbol{A}^{\mathrm{T}} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} a_{11} + a_{21} + \cdots + a_{n1} \\ a_{12} + a_{22} + \cdots + a_{n2} \\ \vdots \\ a_{1n} + a_{2n} + \cdots + a_{nn} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}.$$

Thus 1 is an eigenvalue of $\boldsymbol{A}^{\mathrm{T}}$. By 3c, 1 is an eigenvalue of $\boldsymbol{A}$.

(ii) By 3c, λ is an eigenvalue of $\boldsymbol{A}^{\mathrm{T}}$.

Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$ be a eigenvector of $\boldsymbol{A}^{\mathrm{T}}$ associated with the eigenvalue λ, i.e. $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{x} = \lambda\boldsymbol{x}$. Choose $k \in \{1, 2, \ldots, n\}$ such that $|x_k| = \max\{\, |x_i| \mid i = 1, 2, \ldots, n\,\}$, i.e. $|x_k| \geq |x_i|$ for $i = 1, 2, \ldots, n$. Since $\boldsymbol{x}$ is a nonzero vector, $|x_k| > 0$.

By comparing the kth coordinate of both sides of $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{x} = \lambda\boldsymbol{x}$, we have

$$\begin{aligned}
& a_{1k}x_1 + a_{2k}x_2 + \cdots + a_{nk}x_n = \lambda x_k \\
\Rightarrow\ & |\lambda|\,|x_k| = |a_{1k}x_1 + a_{2k}x_2 + \cdots + a_{nk}x_n| \\
& \qquad \leq |a_{1k}x_1| + |a_{2k}x_2| + \cdots + |a_{nk}x_n| \\
& \qquad \leq a_{1k}|x_1| + a_{2k}|x_2| + \cdots + a_{nk}|x_n| \quad (\because a_{ij} \geq 0 \text{ for all } i, j) \\
& \qquad \leq (a_{1k} + a_{2k} + \cdots + a_{nk})|x_k| \\
& \qquad = |x_k| \\
\Rightarrow\ & |\lambda| \leq 1.
\end{aligned}$$

(b) (i) Yes.

(ii) Let $\boldsymbol{P} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix}$. Then $\boldsymbol{P}^{-1}\boldsymbol{B}\boldsymbol{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.95 & 0 \\ 0 & 0 & 0.9 \end{pmatrix}$.

17. Let a_n (respectively, b_n) be the number of customers who pay late (respectively, early) in month n. Then for $n = 1, 2, \ldots,$

$$\begin{cases} a_n = \frac{1}{2}a_{n-1} + \frac{2}{10}b_{n-1} \\ b_n = \frac{1}{2}a_{n-1} + \frac{8}{10}b_{n-1}. \end{cases}$$

Let $\boldsymbol{x}_n = \begin{pmatrix} a_n \\ b_n \end{pmatrix}$ and $\boldsymbol{A} = \begin{pmatrix} \frac{1}{2} & \frac{1}{5} \\ \frac{1}{2} & \frac{4}{5} \end{pmatrix}$. Then $\boldsymbol{x_n} = \boldsymbol{A}\boldsymbol{x}_{n-1} = \cdots = \boldsymbol{A}^{n-1}\boldsymbol{x_1}$ where $\boldsymbol{x_1} = \begin{pmatrix} 0 \\ 10000 \end{pmatrix}$.

By Algorithm 6.2.4, we find a matrix $\boldsymbol{P} = \begin{pmatrix} 2 & 1 \\ 5 & -1 \end{pmatrix}$ such that $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} 1 & 0 \\ 0 & 0.3 \end{pmatrix}$. Then

$$\boldsymbol{x_n} = \boldsymbol{P}\begin{pmatrix} 1 & 0 \\ 0 & 0.3^{n-1} \end{pmatrix}\boldsymbol{P}^{-1}\boldsymbol{x_1} = \frac{10000}{7}\begin{pmatrix} 2 - 2(0.3)^{n-1} \\ 5 + 2(0.3)^{n-1} \end{pmatrix}.$$

So the number of customers that will pay on time in April is $b_4 = \frac{10000}{7}[5 + 2(0.3)^3] = 7220$.

The number of customers that will pay on time will stabilize in the long run and $\lim_{n\to\infty} b_n = \frac{50000}{7} \approx 7143$.

18. Let a_n, b_n and c_n be the percentage of customers choosing brand A, B and C, respectively, after n months. Then for $n = 1, 2, ...,$

$$\begin{cases} a_n = 0.97a_{n-1} + 0.01b_{n-1} + 0.02c_{n-1} \\ b_n = 0.01a_{n-1} + 0.97b_{n-1} + 0.02c_{n-1} \\ c_n = 0.02a_{n-1} + 0.02b_{n-1} + 0.96c_{n-1}. \end{cases}$$

Let $\boldsymbol{x_n} = \begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix}$ and $\boldsymbol{A} = \begin{pmatrix} 0.97 & 0.01 & 0.02 \\ 0.01 & 0.97 & 0.02 \\ 0.02 & 0.02 & 0.96 \end{pmatrix}$.

Then $\boldsymbol{x_n} = \boldsymbol{A}\boldsymbol{x}_{n-1} = \cdots = \boldsymbol{A}^n\boldsymbol{x_0}$ where $\boldsymbol{x_0} = \begin{pmatrix} 100 \\ 0 \\ 0 \end{pmatrix}$.

By Algorithm 6.2.4, we find $\boldsymbol{P} = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & -1 \\ 1 & 0 & 2 \end{pmatrix}$ such that $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.96 & 0 \\ 0 & 0 & 0.94 \end{pmatrix}$.

Then

$$\boldsymbol{x_n} = \boldsymbol{P} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.96^n & 0 \\ 0 & 0 & 0.94^n \end{pmatrix} \boldsymbol{P}^{-1}\boldsymbol{x_0} = \frac{50}{3} \begin{pmatrix} 2 + 3 \cdot 0.96^n + 0.94^n \\ 2 - 3 \cdot 0.96^n + 0.94^n \\ 2 - 2 \cdot 0.94^n \end{pmatrix}.$$

The present market shares are $\frac{50}{3}[2 + 3 \cdot 0.96^4 + 0.94^4]\% \approx 88.8\%$, $\frac{50}{3}[2 - 3 \cdot 0.96^4 + 0.94^4]\% \approx 3.9\%$ and $\frac{50}{3}[2 - 2 \cdot 0.94^4]\% \approx 7.3\%$ for brand A, B and C, respectively.

The market shares will stabilize after a long run and $\lim_{n\to\infty} \boldsymbol{x_n} = \begin{pmatrix} \frac{100}{3} \\ \frac{100}{3} \\ \frac{100}{3} \end{pmatrix}$.

19. Note that $e^x = 1 + \dfrac{1}{1!}x + \dfrac{1}{2!}x^2 + \cdots = \displaystyle\sum_{n=0}^{\infty} \frac{1}{n!}x^n$ for $x \in \mathbb{R}$.

(a) Since $\boldsymbol{A}^n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & 3^n \end{pmatrix}$ for $n = 1, 2, \ldots,$

$$e^{\boldsymbol{A}} = \begin{pmatrix} 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots & 0 & 0 \\ 0 & 1 + \frac{1}{1!}2 + \frac{1}{2!}2^2 + \cdots & 0 \\ 0 & 0 & 1 + \frac{1}{1!}3 + \frac{1}{2!}3^2 + \cdots \end{pmatrix} = \begin{pmatrix} e & 0 & 0 \\ 0 & e^2 & 0 \\ 0 & 0 & e^3 \end{pmatrix}.$$

(b) Let $\boldsymbol{P} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$. Then $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$. Since $\boldsymbol{A}^n = \boldsymbol{P} \begin{pmatrix} 2^n & 0 \\ 0 & 4^n \end{pmatrix} \boldsymbol{P}^{-1}$ for $n = 1, 2, \ldots,$

$$e^{\boldsymbol{A}} = \boldsymbol{P} \begin{pmatrix} 1 + \frac{1}{1!}2 + \frac{1}{2!}2^2 + \cdots & 0 \\ 0 & 1 + \frac{1}{1!}4 + \frac{1}{2!}4^2 + \cdots \end{pmatrix} \boldsymbol{P}^{-1} = \frac{1}{2} \begin{pmatrix} e^4 + e^2 & e^4 - e^2 \\ e^4 - e^2 & e^4 + e^2 \end{pmatrix}.$$

(c) Let $\boldsymbol{P} = \begin{pmatrix} -1 & 1 & 1 \\ -1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}$. Then $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Since $\boldsymbol{A}^n =$

$$\boldsymbol{P}\begin{pmatrix}(-1)^n & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\boldsymbol{P}^{-1} \text{ for } n = 1, 2, \ldots,$$

$$\begin{aligned} e^{\boldsymbol{A}} &= \boldsymbol{P}\begin{pmatrix}1 - \frac{1}{1!} + \frac{1}{2!} - \cdots & 0 & 0\\ 0 & 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots & 0\\ 0 & 0 & 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots\end{pmatrix}\boldsymbol{P}^{-1}\\ &= \begin{pmatrix}e^{-1} & \frac{1}{2}(e - e^{-1}) & \frac{1}{2}(e - e^{-1})\\ -e + e^{-1} & \frac{1}{2}(3e - e^{-1} & \frac{1}{2}(e - e^{-1})\\ e - e^{-1} & \frac{1}{2}(-e + e^{-1}) & \frac{1}{2}(e + e^{-1})\end{pmatrix}.\end{aligned}$$

20. In the following, we use the procedure discussed in Example 6.2.11.2.

(a) Let $\boldsymbol{x_n} = \begin{pmatrix}a_n\\ a_{n+1}\end{pmatrix}$ and $\boldsymbol{A} = \begin{pmatrix}0 & 1\\ -2 & 3\end{pmatrix}$. Then $\boldsymbol{x_n} = \boldsymbol{A}\boldsymbol{x_{n-1}} = \cdots = \boldsymbol{A}^n\boldsymbol{x_0}$. Let $\boldsymbol{P} = \begin{pmatrix}1 & 1\\ 2 & 1\end{pmatrix}$. Then $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix}2 & 0\\ 0 & 1\end{pmatrix}$. Thus

$$\begin{aligned}\begin{pmatrix}a_n\\ a_{n+1}\end{pmatrix} = \boldsymbol{x_n} &= \boldsymbol{P}\begin{pmatrix}2^n & 0\\ 0 & 1\end{pmatrix}\boldsymbol{P}^{-1}\boldsymbol{x_0}\\ &= \begin{pmatrix}1 & 1\\ 2 & 1\end{pmatrix}\begin{pmatrix}2^n & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ 2 & 1\end{pmatrix}^{-1}\begin{pmatrix}0\\ 1\end{pmatrix} = \begin{pmatrix}2^n - 1\\ 2^{n+1} - 1\end{pmatrix}.\end{aligned}$$

Thus $a_n = 2^n - 1$.

(b) Let $\boldsymbol{x_n} = \begin{pmatrix}a_n\\ a_{n+1}\end{pmatrix}$ and $\boldsymbol{A} = \begin{pmatrix}0 & 1\\ 2 & 1\end{pmatrix}$. Then $\boldsymbol{x_n} = \boldsymbol{A}\boldsymbol{x_{n-1}} = \cdots = \boldsymbol{A}^n\boldsymbol{x_0}$. Let $\boldsymbol{P} = \begin{pmatrix}1 & -1\\ 2 & 1\end{pmatrix}$. Then $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix}2 & 0\\ 0 & -1\end{pmatrix}$. Thus

$$\begin{aligned}\begin{pmatrix}a_n\\ a_{n+1}\end{pmatrix} = \boldsymbol{x_n} &= \boldsymbol{P}\begin{pmatrix}2^n & 0\\ 0 & (-1)^n\end{pmatrix}\boldsymbol{P}^{-1}\boldsymbol{x_0}\\ &= \begin{pmatrix}1 & -1\\ 2 & 1\end{pmatrix}\begin{pmatrix}2^n & 0\\ 0 & (-1)^n\end{pmatrix}\begin{pmatrix}1 & -1\\ 2 & 1\end{pmatrix}^{-1}\begin{pmatrix}1\\ 0\end{pmatrix} = \begin{pmatrix}\frac{1}{3}[2^n + 2(-1)^n]\\ \frac{1}{3}[2^{n+1} - 2(-1)^n]\end{pmatrix}.\end{aligned}$$

Thus $a_n = \frac{1}{3}[2^n + 2(-1)^n]$.

21. Use cofactor expansion along the first row:

$$d_n = \begin{vmatrix} 3 & 1 & & & & 0 \\ 1 & 3 & 1 & & & \\ & 1 & 3 & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & 3 & 1 \\ & 0 & & & 1 & 3 \end{vmatrix}_{n\times n}$$

$$= 3\begin{vmatrix} 3 & 1 & & 0 & \\ 1 & 3 & \ddots & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & 3 & 1 \\ & 0 & & 1 & 3 \end{vmatrix}_{(n-1)\times(n-1)} - \begin{vmatrix} 1 & 1 & & 0 & \\ 0 & 3 & \ddots & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & 3 & 1 \\ & 0 & & 1 & 3 \end{vmatrix}_{(n-1)\times(n-1)}.$$

The first determinant above is d_{n-1}. By using cofactor expansion along the first column, we find that the second determinant is d_{n-2}. So

$$d_n = 3d_{n-1} - d_{n-2}.$$

Note that $d_1 = 3$ and $d_2 = \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} = 8$.

By the procedure discussed in Example 6.2.11.2, we obtain

$$d_n = \left(\frac{5+3\sqrt{5}}{10}\right)\left(\frac{3+\sqrt{5}}{2}\right)^n + \left(\frac{5-3\sqrt{5}}{10}\right)\left(\frac{3-\sqrt{5}}{2}\right)^n.$$

22. Consider the vector equation

$$a_1\boldsymbol{u_1} + a_2\boldsymbol{u_2} + \cdots + a_m\boldsymbol{u_m} + b_1\boldsymbol{v_1} + b_2\boldsymbol{v_2} + \cdots + b_p\boldsymbol{v_p} = \mathbf{0}. \tag{1}$$

Pre-multiplying $\boldsymbol{A}$ to both side of (1), we have

$$a_1\lambda_1\boldsymbol{u_1} + a_2\lambda_2\boldsymbol{u_2} + \cdots + a_m\lambda_m\boldsymbol{u_m} + b_1\mu\boldsymbol{v_1} + b_2\mu\boldsymbol{v_2} + \cdots + b_p\mu\boldsymbol{v_p} = \mathbf{0}. \tag{2}$$

Subtracting (2) by μ times of (1), we obtain

$$a_1(\lambda_1-\mu)\boldsymbol{u_1} + a_2(\lambda_2-\mu)\boldsymbol{u_2} + \cdots + a_m(\lambda_m-\mu)\boldsymbol{u_m} = \mathbf{0}.$$

Since $\boldsymbol{u_1}$, $\boldsymbol{u_2}$, $\ldots$, $\boldsymbol{u_m}$ are linearly independent, $a_1(\lambda_1-\mu)=0$, $a_2(\lambda_2-\mu)=0$, $\ldots$, $a_m(\lambda_m-\mu)=0$. As $\lambda_i \neq \mu$ for $i=1,2,\ldots,m$, we have $a_1=0$, $a_2=0$, $\ldots$, $a_m=0$.

Substituting $a_1 = 0$, $a_2 = 0$, $\ldots$, $a_m = 0$ into (2), we have

$$b_1\boldsymbol{v_1} + b_2\boldsymbol{v_2} + \cdots + b_p\boldsymbol{v_p} = \mathbf{0}.$$

Since $\boldsymbol{v_1}$, $\boldsymbol{v_2}$, $\ldots$, $\boldsymbol{v_p}$ are linearly independent, $b_1 = 0$, $b_2 = 0$, $\ldots$, $b_p = 0$.

We have shown that the vector equation (1) has only the trivial solution. Thus $\{\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_m}, \boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_p}\}$ is linearly independent.

23. (a) True. Let $\boldsymbol{P}$ be an invertible matrix that diagonalizes $\boldsymbol{A}$, i.e. $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \boldsymbol{D}$ where $\boldsymbol{D}$ is a diagonalizable matrix. Then

$$\boldsymbol{D} = \boldsymbol{D}^{\mathrm{T}} = (\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P})^{\mathrm{T}} = \boldsymbol{P}^{T}\boldsymbol{A}^{\mathrm{T}}(\boldsymbol{P}^{-1})^{\mathrm{T}} = \boldsymbol{P}^{T}\boldsymbol{A}^{\mathrm{T}}(\boldsymbol{P}^{\mathrm{T}})^{-1}.$$

Thus the matrix $(\boldsymbol{P}^{\mathrm{T}})^{-1}$ diagonalizes $\boldsymbol{A}^{\mathrm{T}}$.

(b) False. For example, $\boldsymbol{A} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix}$ are both diagonalizable but $\boldsymbol{A} + \boldsymbol{B} = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}$ is not diagonalizable.

(c) False. For example, $\boldsymbol{A} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$ are both diagonalizable but $\boldsymbol{A}\boldsymbol{B} = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}$ is not diagonalizable.