Probability Density Function

$$P(a \leq X \leq b) = \int_a^b f_X(x)\,\mathrm{d}x$$

$$\frac{P(X \approx y)}{P(X \approx z)} \approx \frac{\epsilon f_X(y)}{\epsilon f_X(z)} = \frac{f_X(y)}{f_X(z)}$$

Non-negativity: $f_X(x) \geq 0$ for all $x \in \mathbb{R}$

Normalization: $\int_{-\infty}^{+\infty} f_X(x)\,\mathrm{d}x = 1$

$$\mathbb{E}(X) = \lim_{n\to\infty} \sum_{i=1}^{n} x_i \cdot \Delta x \cdot f(x_i) = \int_a^b x f(x) dx$$

$$\mathrm{Var}(X) = \lim_{n\to\infty} \sum_{i=1}^{n} (x_i - \mu)^2 \cdot \Delta x \cdot f(x_i)$$

$$\boxed{= \int_a^b (x-\mu)^2 f(x) dx} = \int_a^b x^2 f(x) dx - \mu^2$$

$$\boxed{Var(cX + d) = c^2 \mathrm{Var}(X)} \qquad = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2$$

Normal Distribution

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$X \sim N(\mu, \sigma^2)$$

Linear function

- Let $X \sim N(\mu, \sigma^2)$, if $W = a + bX$, then:

$$W \sim N(\underbrace{a + b\mu}_{\text{Mean}}, \underbrace{b^2\sigma^2}_{\text{Variance}})$$

$$Z = \frac{X - \mu}{\sigma}$$

$$X \sim N(0, 1)$$

Uniform Distribution

$$\mathbb{E}(X) = \int_a^b x f(x) dx$$

$$= \int_a^b \frac{x}{b-a} dx$$

$$= \frac{b^2 - a^2}{2(b-a)} = \frac{a+b}{2}$$

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2$$

$$= \int_a^b \frac{x^2}{b-a} dx - \frac{(a+b)^2}{4}$$

$$= \frac{b^3 - a^3}{3(b-a)} - \frac{(a+b)^2}{4}$$

$$= \frac{(b-a)^2}{12}$$

Exponential Distribution

$$f(x) = \lambda e^{-\lambda x}$$

$$P(X \leq x) = F(x)$$

$$= \int_0^x f(x) dx = \int_0^x \lambda e^{-\lambda x} dx$$

$$= -e^{-\lambda x}\Big|_0^x = 1 - e^{-\lambda x}$$

$$\mathbb{E}(X) = \int_a^b x f(x) dx$$

$$= \int_0^\infty x\lambda e^{-\lambda x} dx = -xe^{-\lambda x}\Big|_0^\infty - \int_0^\infty -e^{-\lambda x} dx$$

$$= 0 + \frac{1}{\lambda}\int_0^\infty \lambda e^{-\lambda x} dx = \frac{1}{\lambda}$$

$$\text{Var}(X) = \mathbb{E}[X^2] - [\mathbb{E}(X)]^2$$

$$= \frac{2}{\lambda}\mathbb{E}(X) - [\mathbb{E}(X)]^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

- Binomial:
  $X \sim Bin(n, p) = Bin(12, 0.5)$
- $\mathbb{E}(X) = np = 12 \times 0.5 = 6$
- $\text{Var}(X) = npq = 3$
- Normal:
  $X \sim N(\mu, \sigma^2) = N(6, 3)$

- Use continuity correction factor: $\frac{1}{2}$
- Want to find the binomial prob $P(a \le X \le b)$,
  we can use the normal prob $P\left(a - \frac{1}{2} \le X \le b + \frac{1}{2}\right)$

Central Limit Theorem

By CLT

CLT: $Y_n = \frac{S_n - n\mu}{\sqrt{n\sigma^2}}, P(Y_n \le y) \to \mathcal{N}(0,1)$

$\mathbb{E}[X_i] = p = 0.95$
$\text{Var}(X_i) = p(1-p) = 0.0475$

$Y_{1000} = \frac{\sum_{i=1}^{1000} X_i - 1000 \cdot 0.95}{\sqrt{1000 \cdot 0.0475}}$ is approximately $\mathcal{N}(0,1)$

$$P(X \le 940)$$

$$= P\left(\frac{X - 1000 \cdot 0.95}{\sqrt{1000 \cdot 0.0475}} \le \frac{940 - 1000 \cdot 0.95}{\sqrt{1000 \cdot 0.0475}}\right)$$

$$= P(Y_{1000} \le \frac{940 - 1000 \cdot 0.95}{\sqrt{1000 \cdot 0.0475}})$$

$$P(X = 950) \approx$$
$$P(949.5 \le Z \le 950.5)$$

$$P(X \ge 5)$$

$$= P\left(\frac{X - 10/1.8}{\sqrt{10/1.8^2}} \ge \frac{5 - 10/1.8}{\sqrt{10/1.8^2}}\right)$$

$$\approx P\left(Y \ge \frac{5 - 10/1.8}{\sqrt{10/1.8^2}}\right)$$ By CLT