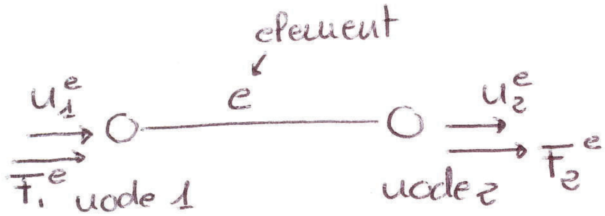
Analysis of a system of trusses (or springs)

see Fish and Belytschko ⇒ chapter ~~1~~ 2

Goal : how to describe the behavior of the entire system starting from the equations that govern individual elements → "Assembly" process

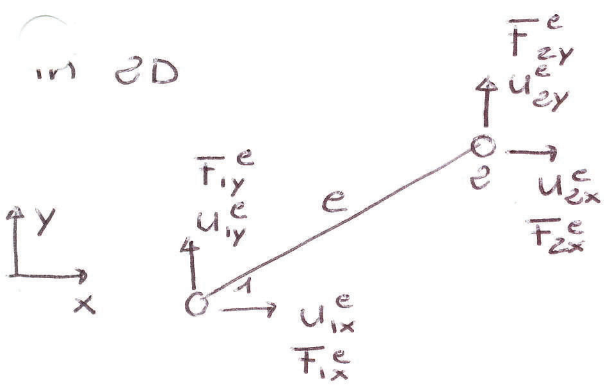
Behavior of a single bar element



nodal displ → $(u_1^e \; u_2^e)$

nodal internal forces → $(F_1^e \; F_2^e)$

in 2D



Governing equations for the bar:

1) Equilibrium $\quad F_1^e + F_2^e = 0$

2) constitutive relation : $\quad \sigma^e = E^e \varepsilon^e$ positive in tension

$\varepsilon^e = \dfrac{\delta^e}{L^e}$ ← elongation, ← initial length

$\sigma^e = \dfrac{P^e}{A^e}$ ← internal force across any section, ← cross sectional area

3) Deformation of the structure must be compatible

↓

no gaps or overlaps

For the bar

$$P^e = F_2^e = \sigma^e A^e = E^e \varepsilon^e A^e$$
$$= E^e \frac{\delta^e}{L^e} A^e$$

$$\delta^e = u_2^e - u_1^e$$

$$F_2^e = \underbrace{\frac{E^e A^e}{L^e}}_{k^e} (u_2^e - u_1^e)$$

From equie $\rightarrow F_1^e = -F_2^e = -k^e(u_2^e - u_1^e)$

In matrix form

$$\begin{vmatrix} F_1^e \\ F_2^e \end{vmatrix} = k^e \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} \begin{vmatrix} u_1^e \\ u_2^e \end{vmatrix} \Rightarrow \underline{F}^e = \underline{\underline{K}}^e \underline{d}^e$$

$\underline{F}^e$ elem force vector

$\underline{\underline{K}}^e$ elem stiffness matrix

$\underline{d}^e$ elem displ vector

Note:

1) $\underline{\underline{K}}^e$ is symmetric $\Rightarrow \underline{\underline{K}}^e = (\underline{\underline{K}}^e)^T$

2) $\underline{\underline{K}}^e$ is singular $\Rightarrow \det \underline{\underline{K}}^e = 0$

$\downarrow$

therefore $\underline{\underline{K}}^e$ is ~~never~~ not invertible $\rightarrow$ we can't invert it to get the displ $\rightarrow \cancel{\underline{d}^e = (\underline{\underline{K}}^e)^{-1} \underline{F}^e}$

The bar is not constrained in space $\Rightarrow$ it can attain multipl positions in space for the same nodal forces

$$\begin{vmatrix} F_1^e \\ F_2^e \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} \begin{vmatrix} 1 \\ 2 \end{vmatrix} = \begin{vmatrix} -1 \\ 1 \end{vmatrix}$$

$k_e = 1$

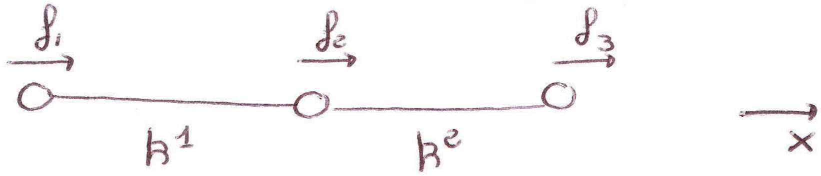
$$\begin{vmatrix} F_1^e \\ F_2^e \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} \begin{vmatrix} 3 \\ 4 \end{vmatrix} = \begin{vmatrix} -1 \\ 1 \end{vmatrix}$$

System of bars

Assembly operations → they are always used in FEM.

problem

Analyze the behavior of the system composed of two truss elements loaded by external forces



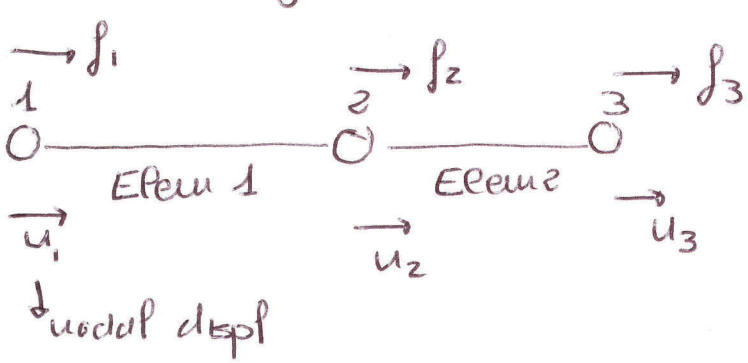
f_1, f_2, f_3 → external forces

Positive directions of all forces are along positive x-axis

solution

~~[illegible]~~ → preprocessing

We break the system into smaller parts ("elements") connected through "nodes"



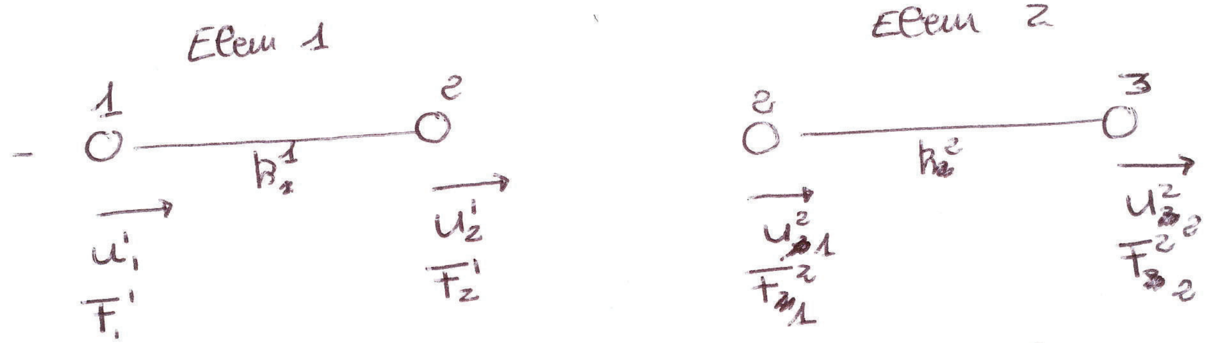
Analysis

Step 1 → development of eqns for a single element

$$\underline{F}^e = \underline{\underline{K}}^e \underline{d}^e$$

Step 2 → Assembly → from equs of a single elem to eqns of the entire system

- Split the structure into component elements



$$\begin{vmatrix} F_1^1 \\ F_2^1 \end{vmatrix} = k_1^1 \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} \begin{vmatrix} u_1^1 \\ u_2^1 \end{vmatrix} \qquad \begin{vmatrix} F_1^2 \\ F_2^2 \end{vmatrix} = k^2 \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} \begin{vmatrix} u_1^2 \\ u_2^2 \end{vmatrix}$$

Looking for a link between local and global variables

$\underset{\text{local}}{u_1^1} = \underset{\text{global}}{u_1}$ → they are arbitrarily chosen

$u_2^1 = u_2^2 = u_2$

$u_2^2 = u_3$

$$\begin{vmatrix} F_1^1 \\ F_2^1 \end{vmatrix} = k^1 \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} \begin{vmatrix} u_1 \\ u_2 \end{vmatrix} \qquad \begin{vmatrix} F_1^2 \\ F_2^2 \end{vmatrix} = k^2 \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} \begin{vmatrix} u_2 \\ u_3 \end{vmatrix}$$

Node element connectivity chart

Elem	node 1	node 2
1	1	2
2	2	3

← local node number (node 1, node 2)

← global node number (table entries)

Therefore

We expand the matrices and vectors to obtain

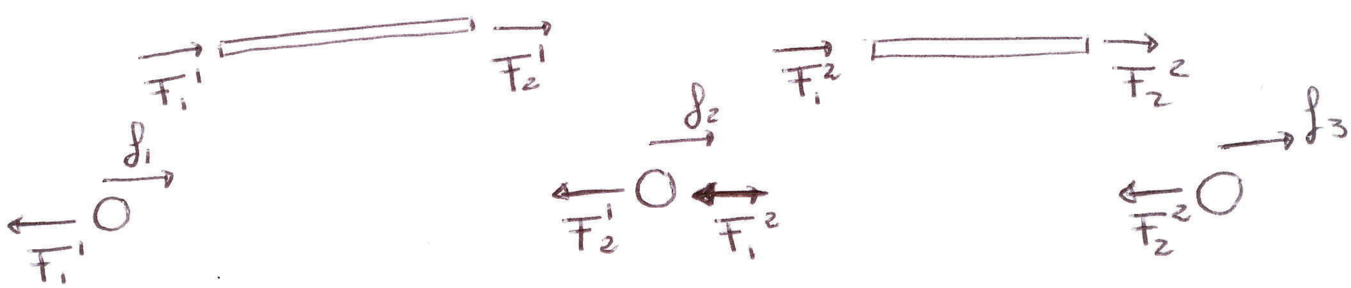
$$\begin{vmatrix} F_1^1 \\ F_2^1 \\ 0 \end{vmatrix} = \begin{vmatrix} k^1 & -k^1 & 0 \\ -k^1 & k^1 & 0 \\ 0 & 0 & 0 \end{vmatrix} \begin{vmatrix} u_1 \\ u_2 \\ u_3 \end{vmatrix}$$

$$\begin{vmatrix} 0 \\ F_1^2 \\ F_2^2 \end{vmatrix} = k^2 \begin{vmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{vmatrix} \begin{vmatrix} u_1 \\ u_2 \\ u_3 \end{vmatrix}$$

expanded elem force vector — Expanded elem stiffness matrix — global displ vect

How do we relate element local nodal forces to global forces?

We draw free body diagrams of nodes and elem



Equil conditions for the nodes

$$\begin{cases} f_1 - F_1^1 = 0 & \leftarrow \text{node } 1 \\ f_2 - F_2^1 - F_1^2 = 0 & \leftarrow \text{node } 2 \\ f_3 - F_2^2 = 0 & \leftarrow \text{node } 3 \end{cases}$$

in vector form $\Longrightarrow$ $\underline{F} = \begin{vmatrix} f_1 \\ f_2 \\ f_3 \end{vmatrix} = \begin{vmatrix} F_1^1 \\ F_2^1 + F_1^2 \\ F_2^2 \end{vmatrix}$

global nodal force vector

$\underline{F} = \underline{F}^1_{exp} + \underline{F}^2_{exp}$ ✓ expanded elem force vectors

$$\underline{F}^1_{exp} = \begin{vmatrix} F^1_1 \\ F^1_2 \\ 0 \end{vmatrix} \qquad \underline{F}^2_{exp} = \begin{vmatrix} 0 \\ F^2_1 \\ F^2_2 \end{vmatrix}$$

but $\underline{F}^1_{exp} = \underline{\underline{K}}^1_{exp}\,\underline{d}$ $\qquad \underline{F}^2_{exp} = \underline{\underline{K}}^2_{exp}\,\underline{d}$ $\qquad \underline{d} = \begin{vmatrix} u_1 \\ u_2 \\ u_3 \end{vmatrix}$

$$\Downarrow$$

$$\underbrace{\underline{F}}_{\text{global nodal force vector}} = \underbrace{\left(\underline{\underline{K}}^1_{exp} + \underline{\underline{K}}^2_{exp}\right)}_{\text{global stiffness matrix } \underline{\underline{K}}} \underbrace{\underline{d}}_{\text{global nodal displ vector}}$$

$$\underline{\underline{K}} = \sum_e \underline{\underline{K}}^e = \begin{vmatrix} k^1 & -k^1_* & 0 \\ -k^1_* & k^1_* + k^2 & -k^2 \\ 0 & -k^2 & k^2 \end{vmatrix}$$

Note: - $\underline{\underline{K}}$ is symmetric

- $\underline{\underline{K}}$ is singular $\Rightarrow$ ~~bna.~~ boundary conditions must be prescribed

$\underline{F} = \underline{\underline{K}}\,\underline{d} \Rightarrow$ these are the 3 equil equs at the 3 nodes

$\underline{F} = \underline{\underline{K}}\,\underline{d}$

$$\begin{cases} f_1 = \overbrace{k^1 u_1 - k^1 u_2}^{F_1^1} \\ f_2 = \underbrace{-k^1 u_1 + (k^1}_{F_2^1} + \underbrace{k^2)\, u_2 - k^2 u_3}_{F_1^2} \\ f_3 = \underbrace{-k^2 u_2 + k^2 u_3}_{F_2^2} \end{cases}$$

equil equs at 3 nodes

Notice that

$$f_1 + f_2 + f_3 = F_1^1 + (\underset{\uparrow \atop -F_1^1}{F_2^1} + F_1^2) + \underset{\uparrow \atop -F_1^2}{F_2^2} = 0$$

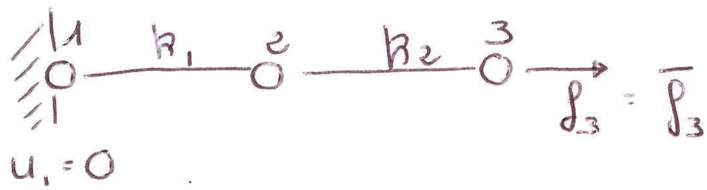
$\downarrow$

the structure is in static equil

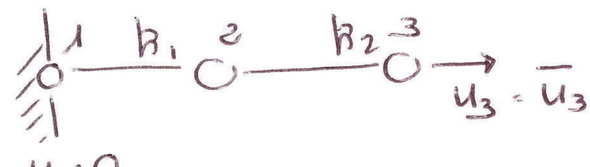
<u>Step 3</u> : solution of the eqns

$\nearrow$ partition approach

sw $\underline{\underline{K}}$ is singular $\Rightarrow$ boundary cond must be specified

case A 

$$\begin{cases} \underline{d}_E = |u_1| \quad \underline{d}_F = \begin{vmatrix} u_2 \\ u_3 \end{vmatrix} \\ \underline{F}_E = |f_1| \quad \underline{F}_F = \begin{vmatrix} f_2 \\ f_3 \end{vmatrix} \end{cases}$$

case b 

$$\begin{cases} \underline{d}_F = |u_2| \quad \underline{d}_E = \begin{vmatrix} u_1 \\ u_3 \end{vmatrix} \\ \underline{F}_F = |f_2| \quad \underline{F}_E = \begin{vmatrix} f_1 \\ f_3 \end{vmatrix} \end{cases}$$

We partition the nodes into $\rightarrow$ E-nodes (essential nodes)

their displ is known

$\downarrow$

$\underline{d} = \begin{vmatrix} \underline{d}_E \\ \underline{d}_F \end{vmatrix}$

$\rightarrow$ F-nodes (free nodes)

their displ are unknown

$$\underline{F} = \underline{\underline{K}}\,\underline{d}$$

$$\Downarrow$$

unknown

$$\begin{vmatrix} \underline{F}_E \\ \underline{F}_F \end{vmatrix} = \begin{vmatrix} \underline{\underline{K}}_{EE} & \underline{\underline{K}}_{EF} \\ \underline{\underline{K}}_{FE} & \underline{\underline{K}}_{FF} \end{vmatrix} \begin{vmatrix} \underline{d}_E \\ \underline{d}_F \end{vmatrix} \quad \text{unknown}$$

$\cancel{\underline{\underline{K}}_{FE} = \underline{\underline{K}}_{EF}^T}$ ~~since~~ $\underline{\underline{K}} = \underline{\underline{K}}^T$

$$\underline{F}_F = \underline{\underline{K}}_{FE}\,\underline{d}_E + \underline{\underline{K}}_{FF}\,\underline{d}_F$$

$\underline{F}_F$ known, $\underline{d}_E$ known

$$\Downarrow$$

$$\underline{d}_F = \underline{\underline{K}}_{FF}^{-1}\left(\underline{F}_F - \underline{\underline{K}}_{FE}\,\underline{d}_E\right)$$

once we know $\underline{d}_F$

$$\hookrightarrow \underline{F}_E = \underline{\underline{K}}_{EE}\,\underline{d}_E + \underline{\underline{K}}_{EF}\,\underline{d}_F$$

<u>Case A</u> : $\underline{\underline{K}}_{EE} = k^1 \qquad \underline{\underline{K}}_{FF} = \begin{vmatrix} k^1 + k^2 & -k^2 \\ -k^2 & k^2 \end{vmatrix}$

$$\underline{\underline{K}}_{EF} = \underline{\underline{K}}_{FE} = \begin{bmatrix} -k^1 & 0 \end{bmatrix}$$

$$\underline{\underline{K}} = \begin{array}{c|cc} K_{EE} & K_{EF} & \\ \hline k^1 & -k^1 & 0 \\ -k^1 & k^1+k^2 & -k^2 \\ 0 & -k^2 & k^2 \\ K_{FE} & K_{FF} & \end{array}$$

[illegible] : $\underline{F}_F = \begin{vmatrix} 0 \\ \bar{f}_3 \end{vmatrix} \qquad \underline{d}_E = 0$

$$\underline{d}_F = \begin{vmatrix} \frac{1}{k_1} & \frac{1}{k_1} \\ \frac{1}{k_1} & \frac{1}{k_1} + \frac{1}{k_2} \end{vmatrix} \begin{vmatrix} 0 \\ \bar{f}_3 \end{vmatrix} = \begin{vmatrix} \frac{\bar{f}_3}{k_1} \\ \bar{f}_3\left(\frac{1}{k_1} + \frac{1}{k_2}\right) \end{vmatrix} = \begin{vmatrix} u_2 \\ u_3 \end{vmatrix}$$

$$\underline{F}_E = -k^1 \cdot 0 + \begin{bmatrix} -k^1 & 0 \end{bmatrix} \begin{bmatrix} \bar{f}_3/k_1 \\ \bar{f}_3\left(\frac{1}{k_1} + \frac{1}{k_2}\right) \end{bmatrix} = -\bar{f}_3$$

<u>Case B</u>

$\underline{F}_F = 0 \qquad \underline{d}_E = \begin{vmatrix} 0 \\ \bar{u}_3 \end{vmatrix}$

$$\begin{array}{r} \underline{F}_E \left\{ \begin{matrix} \\ \\ \end{matrix} \right. \\ \underline{F}_F \to \end{array} \begin{vmatrix} f_1 \\ f_3 \\ f_2 \end{vmatrix} = \begin{array}{c} \begin{array}{cc} K_{EE} & K_{EF} \end{array} \\ \left|\begin{array}{cc|c} k^1 & 0 & -k_1 \\ 0 & k^2 & -k^2 \\ \hline -k^1 & -k^2 & k^1+k^2 \end{array}\right| \\ \begin{array}{cc} K_{FE} & K_{FF} \end{array} \end{array} \left|\begin{array}{c} u_1 \\ u_3 \\ \hline u_2 \end{array}\right| \begin{array}{l} \left.\begin{matrix} \\ \\ \end{matrix}\right\} \underline{d}_E \\ \underline{d}_F \end{array}$$

$\underline{\underline{K}}_{EE} = \begin{vmatrix} k_1 & 0 \\ 0 & k_2 \end{vmatrix} \qquad \underline{\underline{K}}_{FF} = k^1 + k^2$

$$\underline{d}_F = \frac{1}{k^1+k^2}\left(0 - [-k_1 \quad -k_2]\begin{vmatrix} 0 \\ \bar{u}_3 \end{vmatrix}\right)$$

$$= \frac{+ k^2 \bar{u}_3}{k^1+k^2} = u_2$$

$$\underline{F}_E = \begin{vmatrix} k_1 & 0 \\ 0 & k_2 \end{vmatrix}\begin{vmatrix} 0 \\ \bar{u}_3 \end{vmatrix} + \begin{vmatrix} -k_1 \\ -k_2 \end{vmatrix} + \frac{k^2 \bar{u}_3}{k_1+k_2}$$

$$= \begin{vmatrix} 0 \\ k^2 \bar{u}_3 \end{vmatrix} - \begin{vmatrix} \frac{k_1 k_2 \bar{u}_3}{k_1+k_2} \\ +\frac{k_2^2 \bar{u}^3}{k_1+k_2} \end{vmatrix} = \begin{vmatrix} -\frac{k_1 k_2 \bar{u}_3}{k_1+k_2} \\ \frac{k_1 k_2 \bar{u}_3 + \cancel{k_2^2 \bar{u}_3} - \cancel{k_2^2 \bar{u}_3}}{k_1+k_2} \end{vmatrix} = \begin{vmatrix} F_1 \\ F_3 \end{vmatrix}$$

Physical significance of $\underline{\underline{K}}$

$$\begin{vmatrix} f_1 \\ f_2 \\ f_3 \end{vmatrix} = \begin{vmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{vmatrix} \begin{vmatrix} u_1 \\ u_2 \\ u_3 \end{vmatrix}$$

$k_{ij} \rightarrow$ Force at node "i" due to unit displ at node "j" keeping all the other nodes fixed