Eric’s Math Corner

# SETS AND CARDINALITIES (FEAT. HILBERT’S HOTEL)

# A Word on Sets

- A Set is a mathematical construct for holding things
  - Things need not be of the same type
- Usually denoted with curly braces:
  - {my, set, contents, here}
- Can be any size
  - Empty: $\emptyset = \{\}$
  - Finite: $\{1, 2, 3\}$
  - Infinite: $\{1, 2, 3, \ldots\}$
- Sets cannot contain themselves

# Set Examples

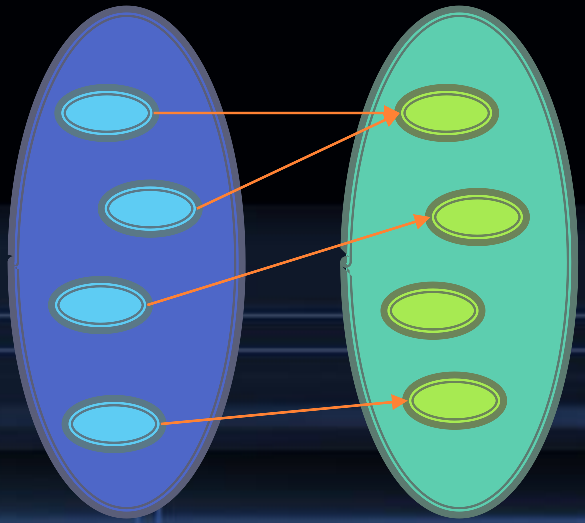
- Set of Integers: {…-3, -2, -1, 0, 1, 2, 3,…}
- Set of even numbers: {…-4, -2, 0, 2, 4, 6,…}
- Set of continuous, real-valued functions
- Set of Mtn Dew Flavors: {Original, Code Red, Voltage, Baja Blast, Live Wire,…}
- Set of Jesus' 12 Apostles: {Peter, James, John, Andrew, Phillip, Bartholemew, Matthew, Thomas, James the Less, Simon Zelotes, Judas, Judas Iscariot}
- Mixed items are ok: {1, "Hello World!", $f(x) = x^2$, D&D}

# Russel’s Paradox

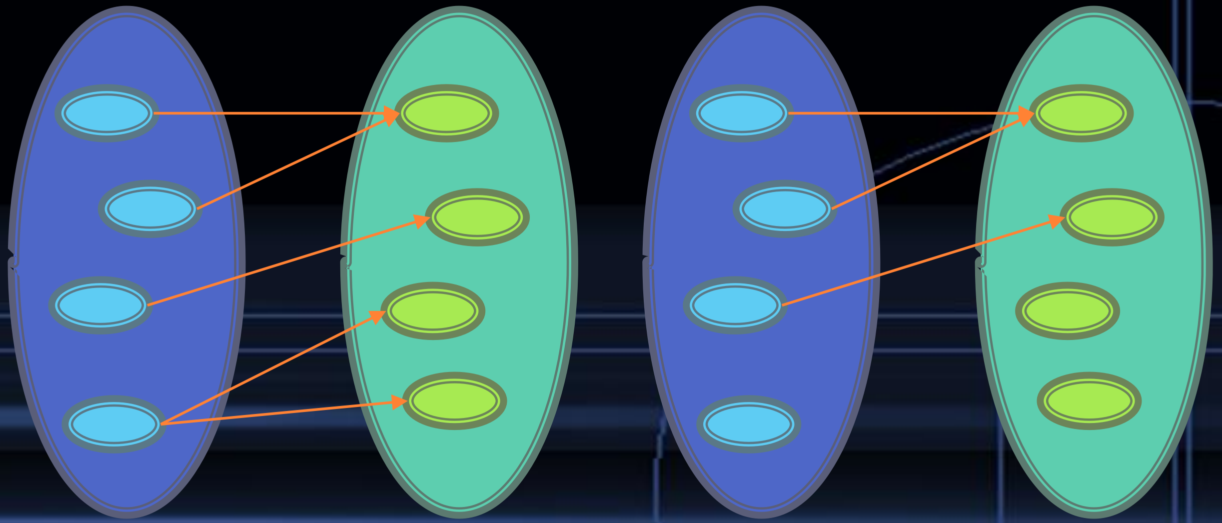
- Let $R = \{x \mid x \notin x\}$ be the set of all sets that do not contain themselves.
  - If $R$ does not contain itself, then $R \in R$, a contradiction
  - If $R$ does contains itself, then $R \notin R$, again a contradiction
- If a set will contain sets, we define it carefully
- “Set of all sets” is not allowed
  - “Class of all sets” may cameo in another Math Corner

# Functions

- A function maps objects in one set to another set
  - Synonymous with the definition of Pure Function in CS
- Notation: $f: A \rightarrow B$: $f$ maps $A$ to $B$
- Each input gives exactly one output



Domain: $A$ Codomain: $B$



Not functions

# Functions as Sets

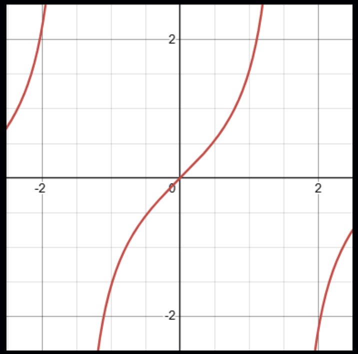
- A function $f: A \to B$ can be thought of as a subset of the Cartesian Product $A \times B$
- If $f(a) = b$, then we say $(a, b)$ is in the set defining the function
- In the case for $f(x) = x^2$, we have pairs $(0, 0)$, $(1, 1)$, $(-1, 1)$, $(2, 4)$, $(-5, 25)$, and in general every pair of the form $(x, x^2)$ where $x$ is a real number
  - Notice: the first coordinate in each pair must be unique

# Turtles All The Way Down

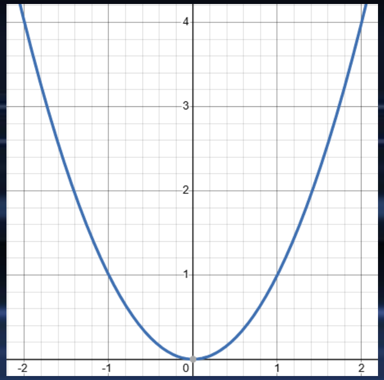
- Everything in maths is eventually a set under the hood
  - Natural Numbers: $0 = \emptyset, 1 = \{\emptyset\}, 2 = \{\emptyset, \{\emptyset\}\}$ etc.
  - Tuples: $(a, b, c) = \{\{a\}, \{a, b\}, \{a, b, c\}\}$
  - Functions: $\{(x_1, f(x_1)), (x_2, f(x_2)) \cdots\}$
  - Integers, rational numbers, real numbers, etc: derivations of Natural numbers
- In the fundamental laws of math (ZFC), sets are the only thing we assume the existence of. Everything else is a byproduct.

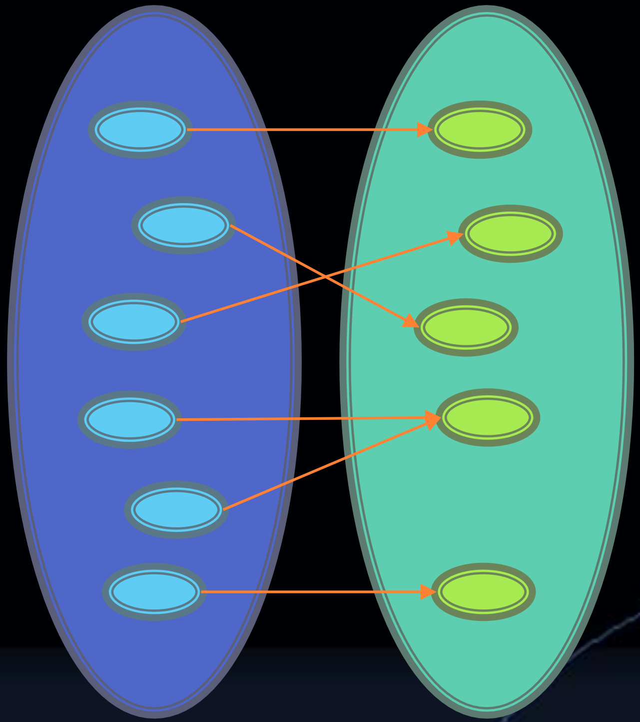
# Surjective "Onto" Functions

- Each output is mapped to by at least one input
- Everything "gets hit"
- Examples:
  - $f(x) = \tan(x)$
  - $f(x) = \ln(x)$



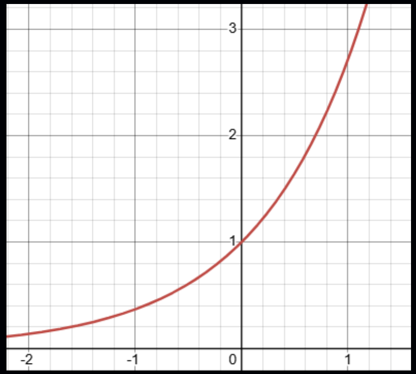

- Unexamples:
  - $f(x) = x^2$
  - $f(x) = e^x$



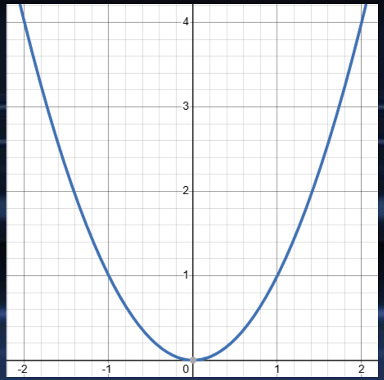



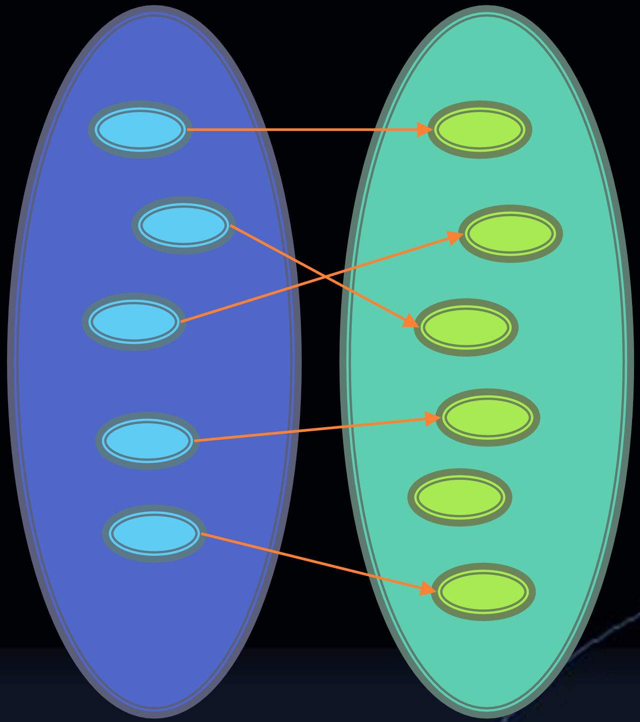
# Injective “One-to-one” Functions

- Each output is unique
- No “collisions”
- Examples:
  - $f(x) = e^x$
  - $f(x) = \arctan(x)$




- Unexamples:
  - $f(x) = x^2$
  - $f(x) = \sin(x)$



Sets and Cardinality

# HOW BIG IS A SET?

# Cardinality

- The Cardinality of a set is the number of elements it contains.
- For finite sets, literally just count the items
- Denoted by bars around the set: $|\cdot|$
  - $|\emptyset| = 0$
  - $|\{1, 2, 3\}| = 3$
- Infinite Cardinality?
  - Yes, gimme a minute

# Functions for Measuring Cardinality

- If there is a one-to-one function $f: A \to B$, then $|B| \geq |A|$
  - Think: for each element $a$ of $A$, there is a corresponding element of $B$, given by $f(a)$, so there are at least as many elements in $B$ as in $A$
- If there is an onto function $g: A \to B$, then $|B| \leq |A|$
  - Think: for each element $b$ of $B$, there is at least one element $a$ of $A$ for which $g(a) = b$, so there are at least as many elements in $A$ as in $B$
- If there is a function $h: A \to B$ which is one-to-one and onto, then $|A| = |B|$
  - Think: we have a direct correspondence of the elements
  - Or: $h$ is one-to-one so $|B| \geq |A|$ and onto so $|B| \leq |A|$, implying equality

Sets and Cardinality

# HOW BIG IS INFINITY?

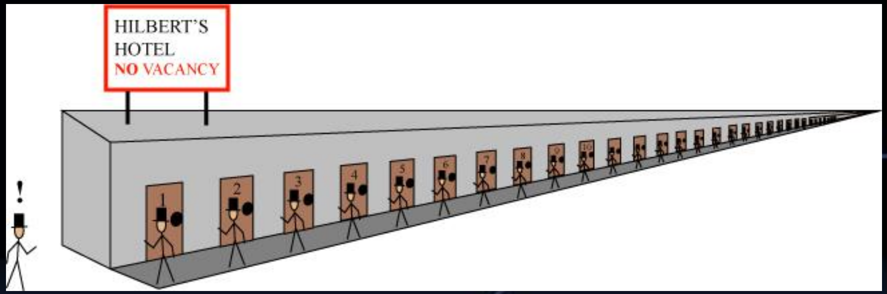
# Hilbert's Grand Hotel

Hilbert's Hotel has infinitely many rooms, numbered 1, 2, 3, 4, and so on, and every room is occupied.

A Patron comes to the reception desk and asks for a room.

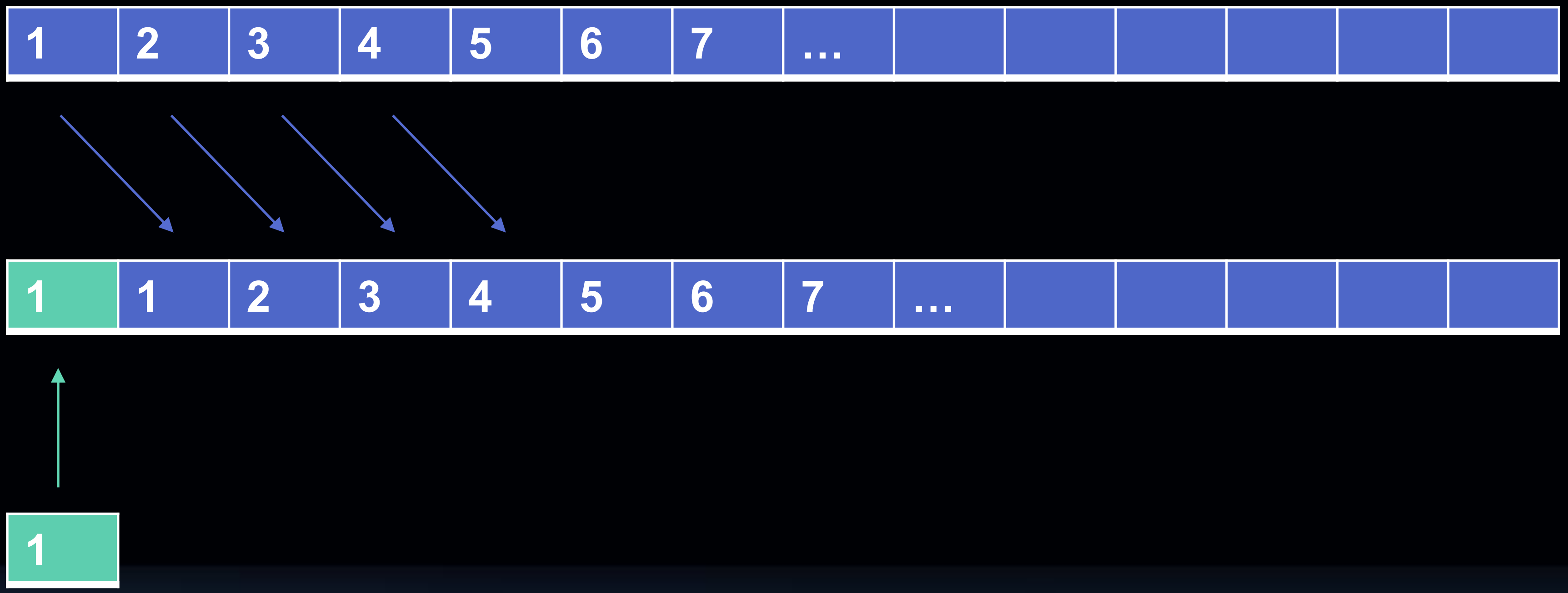
"I can fit you in" Hilbert says.

How can Hilbert fit the guest in?



# Slide down

Move by $f(room) = (room) + 1$




New guest goes to room 1

# Vacancy Issues Intensify

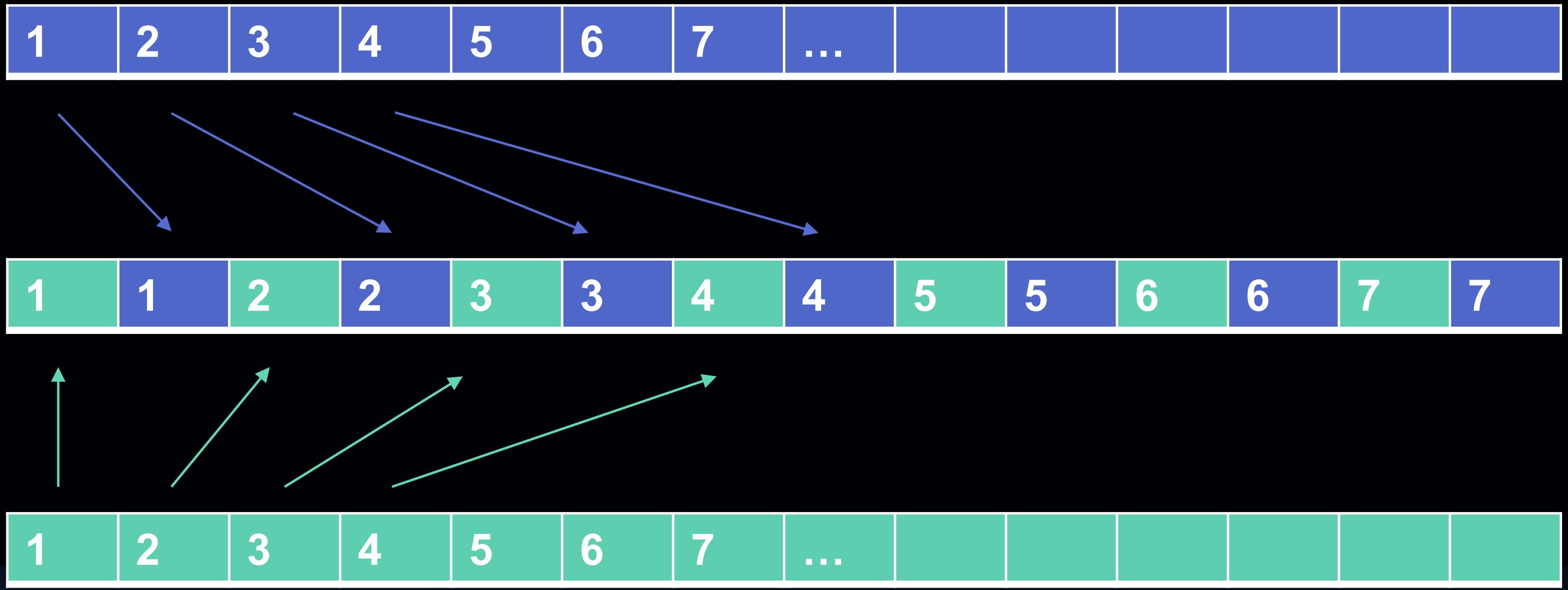
A tour bus with infinitely many seats (numbered 1, 2, 3,…) arrives at Hilbert's (still completely full) hotel. The Tour operator explains the situation to Hilbert, and asks if he can accommodate the guests.

"I can fit them in, no problem," Hilbert replies again.

How will he do it?

# $f(x) = 2x$ for the win

Move by $f(room) = 2 \times (room)$

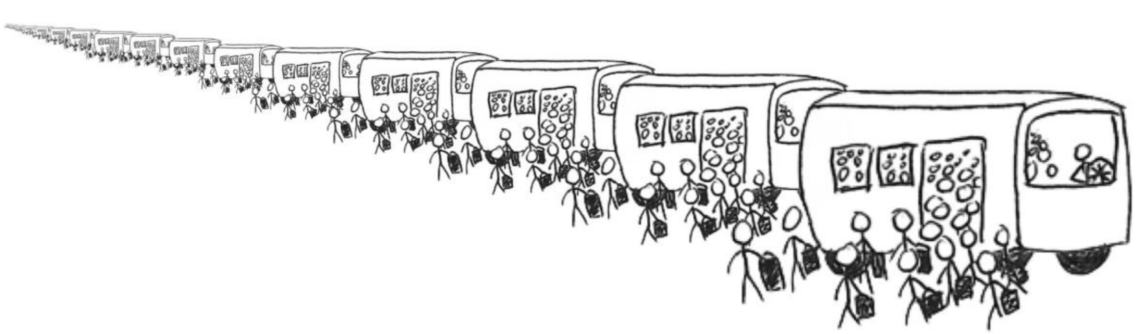


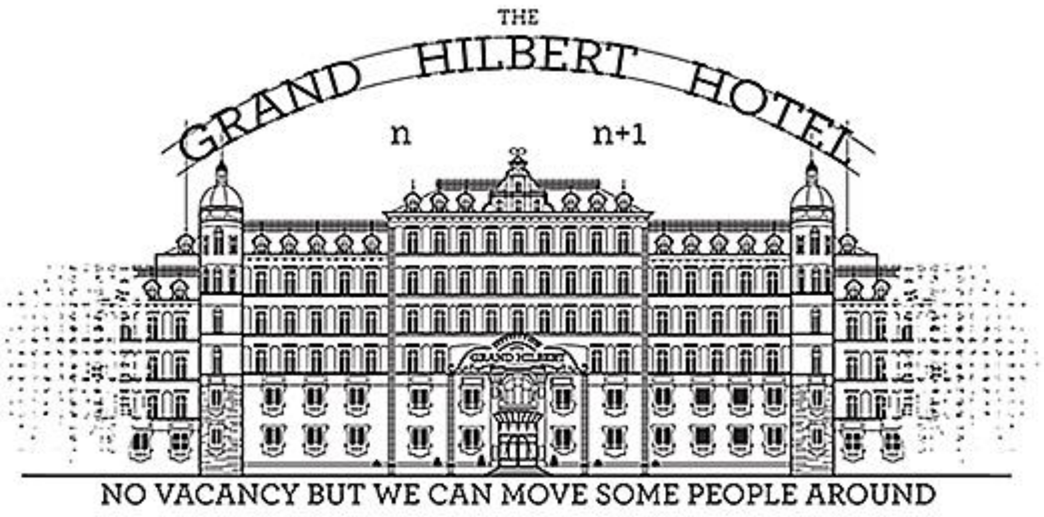

Move by $f(seat) = 2 \times (seat) - 1$

# plz stop

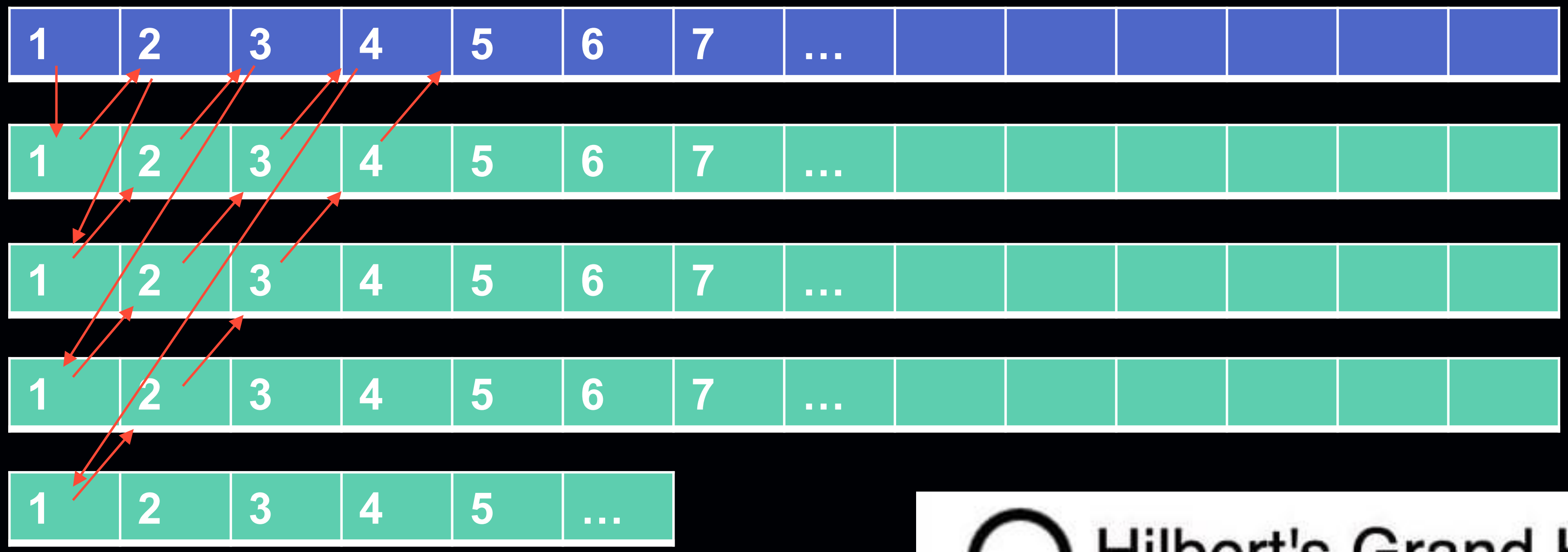
An infinite number of tour busses (numbered 1, 2, 3,…) arrive each with infinitely many seats (numbered). Again, Hilbert agrees to make accommodations.

How?





# I don't know the formula for this




Etc.


Hilbert's Grand Hotel

-1/12 days ago NEW

G. Cantor's review

-

Came here after seeing great reviews. However the hotel was full.
It was nice that they made place for me. But as soon as more guests came in, they ask me to change rooms over and over.
This was incredibly tiresome when a countable number of guests arrived.

# Countably Infinite

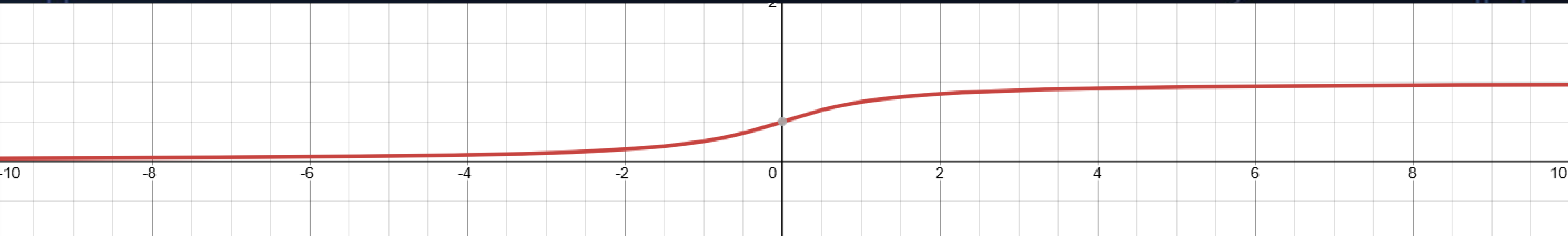
- "Countable" because they can be lined up 1, 2, 3,…
- Sometimes denoted $\aleph_0$: "Aleph Null"
- Examples:
  - Positive Even numbers: $\{2, 4, 6, 8, \cdots\}$
  - The Natural numbers: $\mathrm{N} = \{1, 2, 3, \cdots\}$
  - The Integers: $\mathrm{Z} = \{\cdots, -2, -1, 0, 1, 2, 3, \cdots\}$
  - The Primes: $\{2, 3, 5, 7, 11, 13, 17, \cdots\}$
  - The Rational numbers: $\mathrm{Q} = \{\frac{a}{b} : b \neq 0; a, b \in Z\}$
  - The number of rational numbers between any other two rational numbers 🤯

# How Big are the Real Numbers?

- The Real numbers include the Rational numbers, as well as all irrational and transcendental numbers, like $\pi, e, \sqrt{2}$, and $\sqrt[\phi]{4.8e^{1+\pi}} \approx 34.09337145943007763975951916 3$
- Could you line them up in order?
- Are there actually more of them?

# The Trick(s)

- Because of Arctan—A one-to-one function, which is onto the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$—we can get away with only counting the reals in the interval $(0, 1)$ because they have the same size
- $f(x) = \dfrac{\arctan(x)}{\pi} + \dfrac{1}{2}$



# Cantor's Diagonalization Argument

- Pretend we have a listing of the reals in (0, 1) as decimals
- Let $d_n(x)$ mean the $n$-th decimal digit of the decimal expansion of $x$
  - if $x$ terminates before $n$, return 0
  - $x$ is not allowed to end in repeating 9's (it's the same as terminating)
- We will construct a missing number

$f(x): N \to R$

| | |
|---|---|
| 1 | 0.14159265358979… |
| 2 | 0.50000000000000… |
| 3 | 0.71828182845904… |
| 4 | 0.61803398874989… |
| 5 | 0.4142135623731… |
| 6 | 0.0909090909090… |
| 7 | 0.1428571428571… |
| 8 | 0.0312500000000… |
| 9 | 0.0123456790123… |

# Cantor's Diagonalization Argument cont.

- We will pick a missing number, $y$
- Make a picker function:

$$p(x) = \begin{cases} 2 \; if \; x = 1 \\ 1 \; otherwise \end{cases}$$

- Set $d_1(y) = p(d_1(f(1)))$
  - $y$ = 0.2…
  - $y \neq f(1)$ because they differ in the first digit

$f(x): N \rightarrow R$

| | |
|---|---|
| 1 | 0.14159265358979… |
| 2 | 0.50000000000000… |
| 3 | 0.71828182845904… |
| 4 | 0.61803398874989… |
| 5 | 0.4142135623731… |
| 6 | 0.0909090909090… |
| 7 | 0.1428571428571… |
| 8 | 0.0312500000000… |
| 9 | 0.0123456790123… |

# Cantor's Diagonalization Argument cont.

- Set $d_2(y) = p(d_2(f(2)))$
  - $y$ = 0.21…
  - $y \neq f(2)$ bc they differ in the 2nd digit
- Set $d_3(y) = p(d_3(f(3)))$
  - $y$ = 0.211…
  - $y \neq f(3)$ bc they differ in the 3rd digit

$f(x): N \to R$

| | |
|---|---|
| 1 | 0.14159265358979… |
| 2 | 0.50000000000000… |
| 3 | 0.71828182845904… |
| 4 | 0.61803398874989… |
| 5 | 0.4142135623731… |
| 6 | 0.0909090909090… |
| 7 | 0.1428571428571… |
| 8 | 0.0312500000000… |
| 9 | 0.0123456790123… |

# Cantor's Diagonalization Argument cont.

- In general, set $d_n(y) = p(d_n(f(n)))$ so that $y \neq f(n)$ for any $n$
- $y$ = 0.211121211…
- Guaranteed to be a real number which is not in the image of $f$, so $f$ is not onto, and thus $N$ and $R$ cannot be the same size

$f(x): N \to R$

| | |
|---|---|
| 1 | 0.14159265358979… |
| 2 | 0.50000000000000… |
| 3 | 0.71828182845904… |
| 4 | 0.61803398874989… |
| 5 | 0.4142135623731… |
| 6 | 0.0909090909090… |
| 7 | 0.1428571428571… |
| 8 | 0.0312500000000… |
| 9 | 0.0123456790123… |

# "Uncountably Infinite"

- Having a cardinality strictly larger than $\aleph_0$
  - There is no way to line the objects up and number them, there are simply too many
- Examples
  - The Real Numbers
  - The collection of continuous functions $R \rightarrow R$
  - The set of Irrational numbers $(R - Q)$

# Continuum Hypothesis and Beth Numbers

- The Cardinality of the reals is not known in terms of Aleph numbers.
- We define $|R| = \beth_0$ to be the "continuum cardinality"
- The Continuum hypothesis (CH) states that $\beth_0 = \aleph_1$
  - Equivalent to saying there are no cardinalities between $\aleph_0$ and the continuum cardinality
  - OR every subset of the reals is either finite, countable, or uncountable
- The CH is unprovable, and independent of the remaining set theory

# Making Bigger Sets

- Power set of $A$, denoted $P(A)$ or $2^A$ is the set of all subsets of $A$
- $P(\{1,2,3\}) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$
- $|P(A)| = 2^{|A|}$ if $A$ is a finite set
- If $A$ is infinite, then $|P(A)|$ is a strictly larger cardinality than $|A|$
  - $|P(N)| = \aleph_1 > \aleph_0 = |N|$
  - $|P(N)| < |P(P(N))| < |P(P(P(N)))|$ etc.

# Exercise: Cantor Diagonalization on $P(N)$

- Let $f(N) \to P(N)$ be any listing of the elements of $P(N)$. Use a similar argument as before to construct a subset of $N$ which demonstrated that $f$ is not onto.

# Cantor Diagonalization on $P(N)$

- $M = \{n \in N : n \notin f(n)\}$
- $M = \{3, 4, 5, 7, 9, \dots\}$
- $M \notin f(N)$

$f(x): N \rightarrow P(N)$

| | |
|---|---|
| 1 | {1, 3, 5, 7, 9,…} |
| 2 | {2, 3, 5, 7, 11, 13, 17,…} |
| 3 | {1, 2} |
| 4 | {3, 6, 9, 12, 15,…} |
| 5 | {4, 8, 12, 16, 20, 24,…} |
| 6 | {2, 4, 6, 8, 10, 12,…} |
| 7 | {1, 4, 9, 16, 25, 36,…} |
| 8 | {8} |
| 9 | {1, 2, 27, 256, 3125,…} |