Eric’s Math Corner

BOOLEAN ALGEBRA

Yes, Boolean

- Variables take the values True (1) or False (0)
- Variables are sometimes called propositions
  - p: “the sky is blue”
  - q: “the ocean is Barbie Pink”
- Basic operators:
  - Not (¬): Changes the truthiness of a single value
  - Or (∨): Returns true if either (or both) of two input is true
  - And (∧): Returns true if both inputs are true

Truth Tables

p	q	$\neg p$ (not p)	$p \lor q$ (p or q)	$p \land q$ (p and q)
True	True	False	True	True
True	False	False	True	False
False	True	True	True	False
False	False	True	False	False

- Brute Force logic
  - Left: all possible truth values of input variables
    - 2^{inputs} combinations
  - Right: evaluated expressions in terms of given truth values

“Algebra”

- OR can be thought of like addition, sort of
  - It’s symmetric
  - It has false as an identity
- AND can be thought of like multiplication, sort of
  - It’s symmetric
  - It has true as an identity
- “Additive” Identity
  - $\mathrm{p} \vee 0 \approx p + 0 = p$
- “Additive” symmetry
  - $1 \vee 0 \approx 1 + 0 = 0 + 1 \approx 0 \vee 1$
- “Multiplication” by 0
  - $0 \wedge 1 \approx 1 \times 0 = 0$
- “Multiplicative” Identity
  - $p \wedge 1 \approx p \times 1 = p$

Exercise: “Algebra”

- If you do your homework (*p*) AND don’t make a mess in the kitchen (*q*) then you can watch a movie (*r*)
- How would you write this with logic statements?
- When can you not watch the movie?

"Algebra" cont.

- Distribution of NOT
  - De Morgan's laws
- Distribution of AND/OR
- Order of Operations
  - Parens > NOT > AND > OR

- $\neg(p \lor q) \equiv \neg p \land \neg q$
- $\neg(p \land q) \equiv \neg p \lor \neg q$
- $p \lor q \land r \equiv (p \lor q) \land (p \lor r)$
- $p \land (q \lor r) \equiv p \land q \lor p \land r$

Exercise: Proof of DeMorgan's Laws

p	q	$\neg(p \vee q)$	$\neg p \wedge \neg q$	$\neg(p \wedge q)$	$\neg p \vee \neg q$
True	True				
True	False				
False	True				
False	False				

Proof of DeMorgan's

p	q	$\neg(p \vee q)$	$\neg p \wedge \neg q$	$\neg(p \wedge q)$	$\neg p \vee \neg q$
True	True	False	False	False	False
True	False	False	False	True	True
False	True	False	False	True	True
False	False	True	True	True	True

NOR: "NOT OR"
or "Negated OR"

NAND: "NOT AND"
or "Negated AND"

Exercise: Proof of Distributivity

p	q	r	$p \wedge (q \vee r)$	$(p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r)$	$(p \vee q) \wedge (p \vee r)$
True	True	True				
True	True	False				
True	False	True				
True	False	False				
False	True	True				
False	True	False				
False	False	True				
False	False	False				

Proof of Distributivity

p	q	r	$p \wedge (q \vee r)$	$(p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r)$	$(p \vee q) \wedge (p \vee r)$
True	True	True	True	True	True	True
True	True	False	True	True	True	True
True	False	True	True	True	True	True
True	False	False	False	False	True	True
False	True	True	False	False	True	True
False	True	False	False	False	False	False
False	False	True	False	False	False	False
False	False	False	False	False	False	False

Composite Operators

p	q	$p \oplus q$ (p xor q)	$p \Rightarrow q$ (p implies q)	$p \Leftrightarrow q$ (p iff q)
True	True	False	True	True
True	False	True	False	False
False	True	True	True	False
False	False	False	True	True

- XOR
  - Exclusive OR
  - Exactly one input is true
- Implies
  - If then
- Biconditional
  - Aka:
    - “iff”: if and only if
    - logically equivalent
    - XNOR

Exercise: Equations for XOR, Implies, Iff

- Derive equations for XOR, implies, and the biconditional in terms of AND, OR, and NOT

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{p \oplus q}$ (p xor q)	$\boldsymbol{p \Rightarrow q}$ (p implies q)	$\boldsymbol{p \Leftrightarrow q}$ (p iff q)
True	True	False	True	True
True	False	True	False	False
False	True	True	True	False
False	False	False	True	True

Equations for XOR, Implies, Iff

- Derive minimal equations for XOR, implies, and the biconditional in terms of AND, OR, and NOT
- XOR:
  - $(p \vee q) \wedge \neg(p \wedge q)$
- Implies:
  - $\neg p \vee q$
- Biconditional:
  - $(p \wedge q) \vee \neg(p \vee q)$

Note: Since iff is the negation to XOR, we can derive the equation by applying DeMorgan's to the XOR equation

Exercise: Implies

- Implies means that a truthy hypothesis (p) forces the conclusion (q) to be true
- q is allowed to be true independent of p

- If the battery is charged then the computer has power
  - If the computer has no power, what's up with the battery?
  - If the battery is dead, is the computer powered?

Mental Gymnastics with Implies

Mental Gymnastic with Implies cont.

- "Vacuously True" Statements
  - $False \Rightarrow q = True$
  - Truthiness of result is irrelevant if the hypothesis is nonsense
- The whole proposition: "If my grandmother has wheels then she is a bike" is actually *True*

p: My Grandmother has wheels	q: My Grandmother is a bike	$p \Rightarrow q$
True	True	True
True	False	False
False	True	True
False	False	True

Logic Reduction Rules

Name	Reduction
Conjunction	p, q $\therefore p \wedge q$
Simplification	$p \wedge q$ $\therefore p$
Addition	p $\therefore p \vee q$
Hypothetical Syllogism	$p \Rightarrow q, q \Rightarrow r$ $\therefore p \Rightarrow r$
Modus Tollens	$p, p \Rightarrow q$ $\therefore q$
Modus Ponens	$\neg q, p \Rightarrow q$ $\therefore \neg p$
Disjunctive Resolution	$p \vee q, \neg p \vee r$ $\therefore q \vee r$

- Given “Facts” (logical expressions known to be true) apply reduction rules to derive equivalent facts
- Super formal/mechanical method of proof

Encoding More Meaning

PREDICATE LOGIC

What’s a Predicates, Precious?

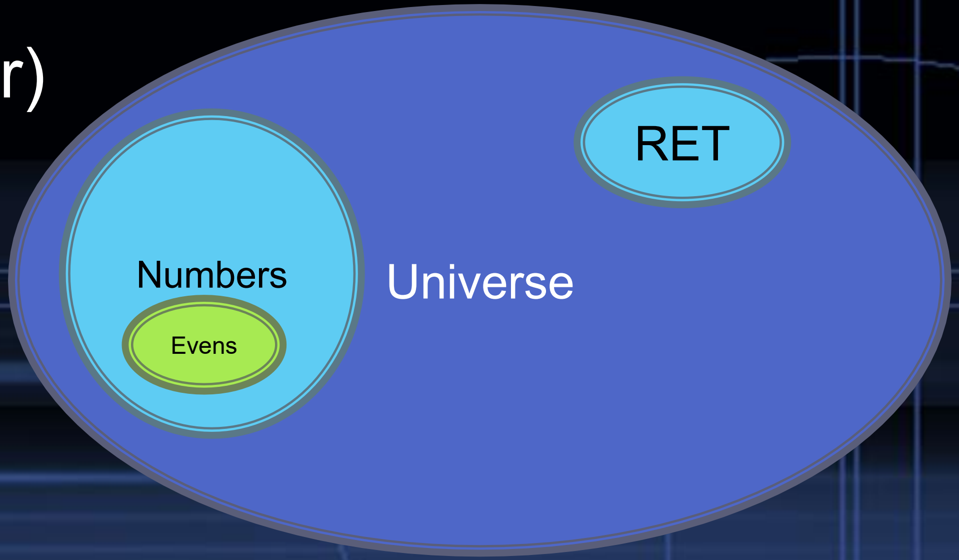
- Function accepting a(n) argument(s) which evaluates to True or False
- Inputs come from a “universe”
  - Collection of all reasonable inputs
- With a diverse enough collection of predicates, you can encode all language and meaning

Examples

- R(x): It’s raining in <place x>
  - Universe: all places
  - R(Hillsboro): False (according to forecast)
  - R(Moscow): True (according to forecast)
  - R(324152): complete nonsense (outside of universe)

Examples

- E(n): <n> is an even number
  - Universe: anything I feel like
  - E(3): False
  - E(324171902831742829107501124903123587752): True
  - E(5.5): False (not even)
  - E(Bob Bigwood): False (not a number)



Universal Quantifier

- Uses the ∀ symbol over a given universe
  - Read “for all”
- Evaluates the truthiness of the predicate for all values in the universe, then ANDs them together
  - Recall E(n): <n> is an even number
  - Let my universe {1, 2, 3, 4, 5}, then $\forall_x E(x)$ is false
  - Let my universe {2, 4, 8, 16,…}, then $\forall_x E(x)$ is true

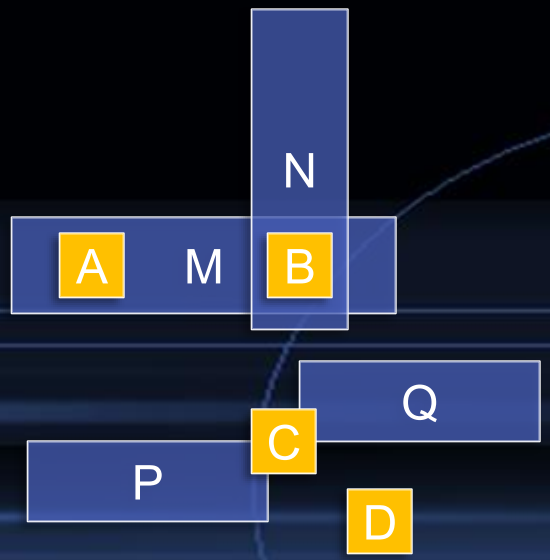
Existential Quantifier

- Uses the ∃ symbol over a given universe
  - Read “there exists”
- Evaluates the truthiness of the predicate for all values in the universe, then ORs them together
  - Let my universe {1, 2, 3, 4, 5}, then $\exists_x E(x)$ is true
  - Let my universe {3, 9, 27, 81…}, then $\exists_x E(x)$ is false

Example: Fruit

```
Rule R1
  m M0
  v V0 | m.contains(v)

Rule R2
  m M0
  v V0 | NOT m.contains(v)

Rule R3
  m M0
  NOT v V0 | m.contains(v)
```

- What predicates are present?
- What quantifiers are used?
- Which Vias will be flagged by each rule?



Example: Fruit

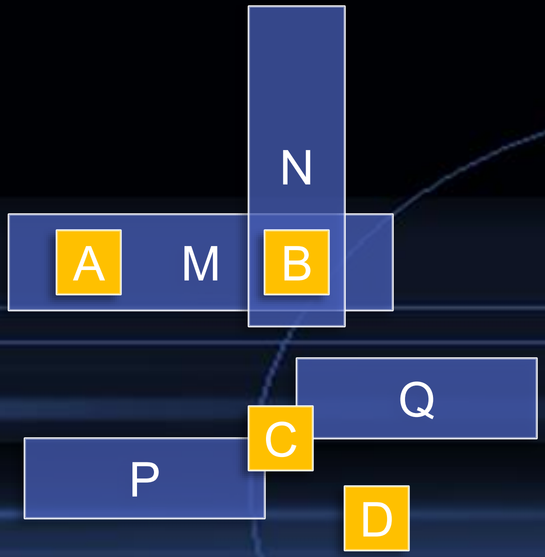
```
Rule R1
   m M0
   v V0 | m.contains(v)

Rule R2
   m M0
   v V0 | NOT m.contains(v)

Rule R3
   m M0
   NOT v V0 | m.contains(v)
```

R1: $\exists_{m \in M_0} \exists_{v \in V_0}\ m.contains(v)$

R2: $\exists_{m \in M_0} \exists_{v \in V_0} \neg m.contains(v)$

R3: $\exists_{m \in M_0} \neg \exists_{v \in V_0}\ m.contains(v)$

$= \exists_{m \in M_0} \forall_{v\_ \in V_0}\ \neg m.contains(v)$



Exercise: Negating Quantifiers

- Consider
  - $\forall_x E(x) = E(x_1) \wedge E(x_2) \wedge E(x_3) \wedge \cdots$
  - $\exists_x E(x) = E(x_1) \vee E(x_2) \vee E(x_3) \vee \cdots$

- $\neg\forall_x E(x) = ?$
- $\neg\exists_x E(x) = ?$

Hint: DeMorgan's

Negating Quantifiers

- Given
  - $\forall_x E(x) = E(x_1) \wedge E(x_2) \wedge E(x_3) \wedge \cdots$
  - $\exists_x E(x) = E(x_1) \vee E(x_2) \vee E(x_3) \vee \cdots$
- Apply DeMorgan's
  - $\neg\forall_x E(x) = \neg E(x_1) \vee \neg E(x_2) \vee \neg E(x_3) \vee \cdots = \exists_x \neg E(x)$
  - $\neg\exists_x E(x) = \neg E(x_1) \wedge \neg E(x_2) \wedge \neg E(x_3) \wedge \cdots = \forall_x \neg E(x)$
- The negation "passes through" the quantifier, "flipping its type" on the way through

Bonus: Induction

- Suppose a predicate $P(k)$ is true for an integer *k*, and we want to prove that the predicate is true for all integers greater than *k*, we can prove
$$\forall_{n \geq k}\, P(n) \Rightarrow P(n+1)$$
instead of
$$\forall_{n \geq k}\, P(n)$$

Why?

Bonus: $\varepsilon - \delta$ Continuity

- Let f be a real valued function, then we say f is **continuous** if:

$$\forall_{\varepsilon>0}\forall_{x}\exists_{\delta>0}\ |x-y|<\delta \Rightarrow |f(x)-f(y)|<\varepsilon$$

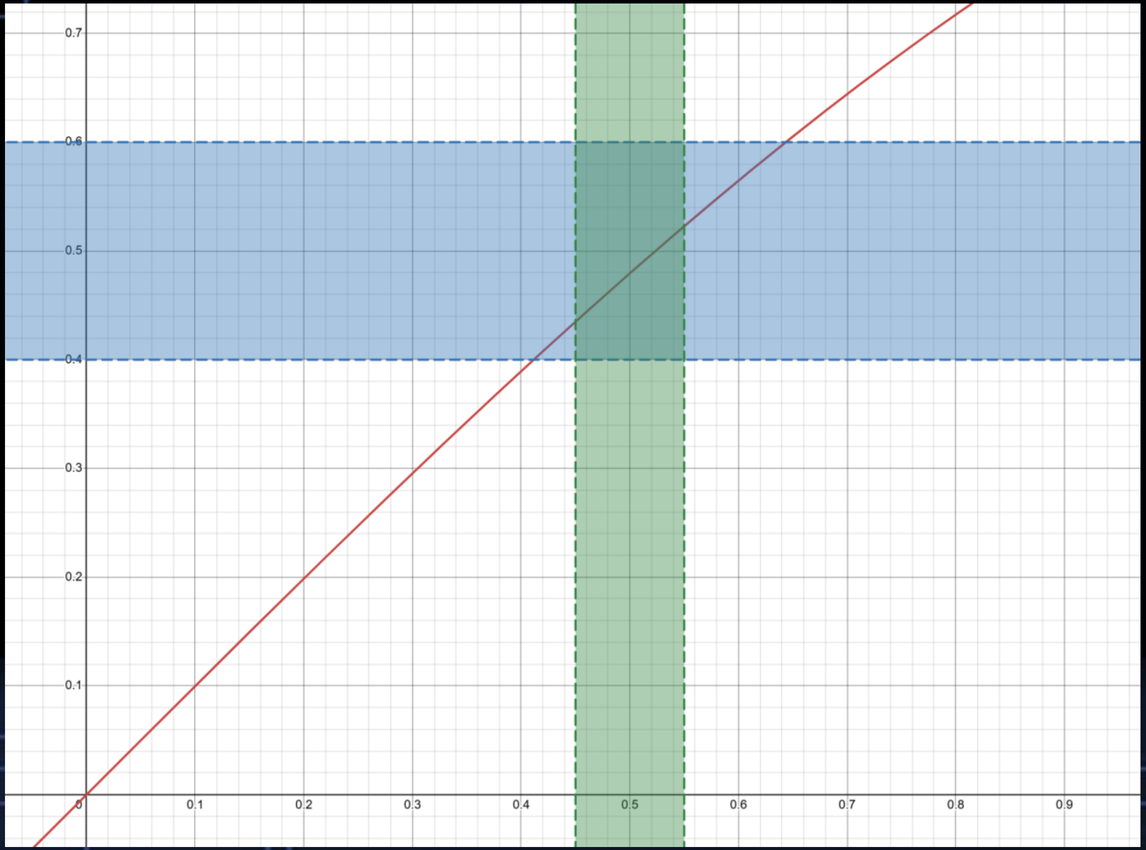
- We say f is **uniformly continuous** if:

$$\forall_{\varepsilon>0}\exists_{\delta>0}\forall_{x}\ |x-y|<\delta \Rightarrow |f(x)-f(y)|<\varepsilon$$

How does reordering the quantifiers change the meaning?

How would you form the definition of discontinuity at x?

Picture of $\varepsilon - \delta$ continuity



- Given any x, and blue neighborhood of width 2ε around $f(x)$, I can define a green neighborhood of width 2δ around x that maps into the blue one