Universidad del Norte
Facultad de Ingeniería de Sistemas y Computación
Estructuras Discretas

Funciones Generadoras Ordinarias

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"Don't limit yourself. Many people limit themselves to what they think they can do. You can go as far as your mind lets you. What you believe, remember, you can achieve".
– Mary Kay Ash

1. Funciones Generadoras Ordinarias

1.1. Ejercicio 1

Dado la siguiente secuencia

$$\{n^2 \cdot 3^{n+5}\}_{n\geq 2}$$

Halle su Función Generadora Ordinaria

$$F(z) = \sum_{n\geq 2} (n^2 \cdot 3^{n+5}) z^n$$

Partamos de una OGF ya conocida:

$$\sum_{n\geq M} \binom{n}{M} z^n = \frac{z^M}{(1-z)^{M+1}}$$

$$\sum_{n\geq 2} \binom{n}{2} z^n = \frac{z^2}{(1-z)^3}$$

$$\sum_{n\geq 2} \frac{n!}{(n-2)! \cdot 2!} z^n = \frac{z^2}{(1-z)^3}$$

$$\sum_{n\geq 2} \frac{n(n-1)(n-2)!}{(n-2)! \cdot 2!} z^n = \frac{z^2}{(1-z)^3}$$

$$\sum_{n\geq 2} \frac{n(n-1)}{2} z^n = \frac{z^2}{(1-z)^3}$$

$$\sum_{n\geq 2} (n^2 - n) z^n = \frac{2z^2}{(1-z)^3}$$

$$\sum_{n\geq 2} n^2 z^n = \frac{2z^2}{(1-z)^3} + \underbrace{\sum_{n\geq 2} n z^n}_{G(z)}$$

Para continuar el ejercicio, necesitamos saber quién es $G(z)$

$$G(z) = \sum_{n\geq 2} nz^n$$

Partimos de otra OGF ya conocida,

$$\sum_{n\geq 1} \binom{n}{1} z^n = \frac{z}{(1-z)^2}$$

$$\sum_{n\geq 1} nz^n = \frac{z}{(1-z)^2}$$

Desplazamos una unidad hacia la derecha la sumatoria, esto es reemplazar n por $n-1$

$$\sum_{n-1\geq 1} (n-1)z^{n-1} = \frac{z}{(1-z)^2}$$

$$\sum_{n\geq 2} (n-1)z^n = \frac{z^2}{(1-z)^2}$$

$$\sum_{n\geq 2} nz^n = \frac{z^2}{(1-z)^2} + \sum_{n\geq 2} z^n$$

$$\sum_{n\geq 2} nz^n = \frac{z^2}{(1-z)^2} + \frac{z^2}{1-z}$$

$$\sum_{n\geq 2} nz^n = \frac{z^2}{(1-z)^2} + \frac{z^2}{1-z} \cdot \frac{1-z}{1-z}$$

$$\sum_{n\geq 2} nz^n = \frac{z^2 + z^2(1-z)}{(1-z)^2}$$

$$\sum_{n\geq 2} nz^n = \frac{2z^2 - z^3}{(1-z)^2}$$

Reemplazamos en el ejercicio principal a $G(z)$

$$\sum_{n\geq 2} n^2 z^n = \frac{2z^2}{(1-z)^3} + \frac{2z^2 - z^3}{(1-z)^2}$$

$$\sum_{n\geq 2} n^2 z^n = \frac{2z^2 + 2z^2 - z^3 - 2z^3 + z^4}{(1-z)^3}$$

$$\sum_{n\geq 2} n^2 z^n = \frac{z^4 - 3z^3 + 4z^2}{(1-z)^3}$$

Escalamos la función anterior en un factor de 3, esto es

$$H(3z) = \sum_{n\geq 2} n^2 (3z)^n = \frac{(3z)^4 - 3(3z)^3 + (3z)z^2}{(1-3z)^3}$$

$$\sum_{n\geq 2} n^2 3^n z^n = \frac{(3z)^4 - 3(3z)^3 + (3z)z^2}{(1-3z)^3}$$

Ahora multiplicamos ambas partes por 3^5

$$\sum_{n\geq 2} n^2 3^5 \cdot 3^n z^n = 3^5 \cdot \frac{(3z)^4 - 3(3z)^3 + (3z)z^2}{(1-3z)^3}$$

$$\sum_{n\geq 2} n^2 3^{n+5} z^n = 3^5 \cdot \frac{(3z)^4 - 3(3z)^3 + (3z)^2}{(1-3z)^3}$$

Finalmente,

$$\boxed{F(z) = 3^5 \cdot \frac{(3z)^4 - 3(3z)^3 + (3z)^2}{(1-3z)^3}}$$