Universidad del Norte
Facultad de Ingeniería de Sistemas y Computación
Estructuras Discretas

Funciones Generadoras Ordinarias - OGF

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Profesor: Alfonso M. Mancilla Herrera,
Monitor: Enrique A. Niebles Saco.

"Don't limit yourself. Many people limit themselves to what they think they can do. You can go as far as your mind lets you. What you believe, remember, you can achieve".
– Mary Kay Ash

1. Funciones Generadoras Ordinarias

1.1. Ejercicio 3

Halle la OGF de la siguiente sucesión:

$$\left\{ \sum_{0<k<n} 3^k \cdot (n-k) \right\}_{n\geq 2}$$

Definimos nuestra OGF

$$F(z) = \sum_{n\geq 2} \left(\sum_{0<k<n} 3^k \cdot (n-k) \right) z^n$$

Reescribimos los límites de la sumatoria interna

$$F(z) = \sum_{n\geq 2} \left(\sum_{1\leq k\leq n-1} 3^k \cdot (n-k) \right) z^n$$

$$F(z) = \sum_{n\geq 2} \left(\sum_{1\leq k+1\leq n-1} 3^{k+1} \cdot (n-(k+1)) \right) z^n$$

$$F(z) = \sum_{n\geq 2} \left(\sum_{0\leq k\leq n-2} 3^{k+1} \cdot (n-k-1) \right) z^n$$

Desplazamos hacia la izquierda 2 unidades la sumatoria externa, esto es reemplazar n por $n+2$.

$$F(z) = \sum_{n+2\geq 2} \left(\sum_{0\leq k\leq n+2-2} 3^{k+1} \cdot (n+2-k-1) \right) z^{n+2}$$

$$F(z) = 3z^2 \cdot \sum_{n\geq 0} \left(\sum_{0\leq k\leq n} 3^k \cdot (n-k+1) \right) z^n$$

De la sumatoria interna, podemos deducir que $h_n = 3^n$ y $g_{n-k} = n-k+1$. De esta última, se tiene que $g_n = n+1$. De esta forma,

$$F(z) = 3z^2 \cdot H(z) \cdot G(z)$$

A continuación, se halla $H(z)$

$$H(z) = \sum_{n\geq 0} 3^n z^n = \frac{1}{1-3z}$$

De forma similar, $G(z)$

$$G(z) = \sum_{n\geq 0} (n+1)z^n = \sum_{n\geq 0} nz^n + \sum_{n\geq 0} z^n$$

$$G(z) = \frac{z}{(1-z)^2} + \frac{1}{1-z} = \frac{1}{(1-z)^2}$$

Finalmente,

$$F(z) = 3z^2 \cdot \underbrace{\frac{1}{1-3z}}_{H(z)} \cdot \underbrace{\frac{1}{(1-z)^2}}_{G(z)}$$

$$\boxed{F(z) = \frac{3z^2}{(1-3z)(1-z)^2}}$$

1.2. Ejercicio 4

Hallar la OGF de la siguiente secuencia

$$\left\{ \sum_{2\leq k\leq n} \binom{k}{2} \cdot 2^{n-k} \right\}_{n\geq 3}$$

Reescribimos la sucesión. Desplazamos 2 unidades hacia la izquierda la sucesión

$$\left\{ \sum_{2\leq k+2\leq n} \binom{k+2}{2} \cdot 2^{n-(k+2)} \right\}_{n\geq 3}$$

$$\left\{ \sum_{0\leq k\leq n-2} \binom{k+2}{2} \cdot 2^{n-k-2} \right\}_{n\geq 3}$$

Planteamos la OGF $F(z)$

$$F(z) = \sum_{n\geq 3} \underbrace{\left[\sum_{0\leq k\leq n-2} \binom{k+2}{2} \cdot 2^{n-k-2}\right]}_{f_n} z^n$$

Ahora desplazamos 2 unidades hacia la izquierda la sumatoria externa

$$F(z) = \sum_{n+2\geq 3} \left[\sum_{0\leq k\leq n+2-2} \binom{k+2}{2} \cdot 2^{n+2-k-2}\right] z^{n+2}$$

$$F(z) = z^2 \sum_{n\geq 1} \left[\sum_{0\leq k\leq n} \binom{k+2}{2} \cdot 2^{n-k}\right] z^n$$

Para resolver la sumatoria externa, partiremos de la definición de series de potencia

$$F(z) = z^2 \left\{ \sum_{n\geq 0} \left[\sum_{0\leq k\leq n} \underbrace{\binom{k+2}{2}}_{h_k} \cdot \underbrace{2^{n-k}}_{g_{n-k}}\right] z^n - f_0 \right\}$$

A partir de lo anterior, calculamos las OGF de las sumatorias internas

$$\sum_{n\geq 0} \left[\sum_{0\leq k\leq n} \binom{k+2}{2} \cdot 2^{n-k}\right] z^n = H(z) \cdot G(z)$$

Calculando $H(z)$

$$\sum_{n\geq 0} \binom{n+2}{2} z^n$$

Partimos de

$$\sum_{n\geq M} \binom{n}{M} z^n = \frac{z^M}{(1-z)^{M+1}}$$

Para $M = 2$

$$\sum_{n\geq 2} \binom{n}{2} z^n = \frac{z^2}{(1-z)^3}$$

Desplazo 2 unidades hacia la izquierda

$$\sum_{n+2\geq 2} \binom{n+2}{2} z^{n+2} = \frac{z^2}{(1-z)^3}$$

$$H(z) = \sum_{n\geq 0} \binom{n+2}{2} z^{n} = \frac{1}{(1-z)^3}$$

Para hallar $G(z)$, partiremos de la OGF más básica.

$$\sum_{n\geq 0} z^n = \frac{1}{1-z}$$

Al escalar en un factor de 2, se obtiene que

$$\sum_{n\geq 0} (2z)^n = \sum_{n\geq 0} 2^n z^n = \frac{1}{1-2z}$$

$$G(z) = \sum_{n\geq 0} 2^n z^n = \frac{1}{1-2z}$$

$$F(z) = z^2 \left(\frac{1}{(1-z)^3} \cdot \frac{1}{1-2z} - f_0 \right)$$

Se procede a calcular f_0, esto es evaluar f_n con $n = 0$.

$$f_0 = \sum_{0\leq k\leq 0} \binom{k+2}{2} \cdot 2^{0-k}$$

$$f_0 = \binom{0+2}{2} \cdot 2^{0-0} = \binom{2}{2} \cdot 2^0 = 1$$

Finalmente,

$$\boxed{F(z) = \frac{z^2}{(1-z)^3(1-2z)} - 1}$$