Computational Physics, Project 5 - **Shock waves**

Ena Salihovic
Jacobs University Bremen

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Background

Burgers' equation is used as a simple model of shock waves and turbulence. It is of the following form:

$$\frac{\partial u}{\partial t} + \varepsilon u \frac{\partial u}{\partial x} = 0 \tag{1}$$

where εu is equal to the speed of the wave, c. This equation is an extension of the advection equation in which the wave speed is proportional to the amplitude of the wave [1]. Because of this, the shape of the wave is not preserved in time: parts of the wave with large amplitudes propagate faster than those with small amplitudes. This gives rise to the shock waves.

The leapfrog integration is a method for numerically integrating differential equations of the form. [2]

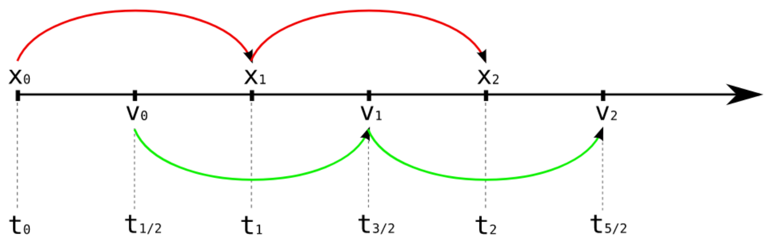

Figure 1: Visualization of Leapfrog algorithm [2]

The Lax–Wendroff method belongs to the class of conservative schemes (78) and can be derived in various ways. The 1D Burgers equation is a simple (if not the simplest) non-linear hyperbolic equation commonly used as a model equation to illustrate various numerical schemes for nonlinear hyperbolic differential equations [3].

Objective

The goal of this simulation was to write a program to solve Burgers' equation via the Lax-Wendroff method. The following arrays were defined in the code: $u0$ for initial wave data and u for the numerical solution. The initial wave was taken to be sinusoidal: $u0[i] = 3sin(3.2x)$, and speed was taken to be $c = 1$. Boundary conditions were incorporated as follows $u[0] = 0$ and $u[99] = 0$, which can be seen in the code. The CFL number $\beta = \varepsilon/(\Delta x/\Delta t)$ was varied and it was found that the stability condition

$\beta < 1$ is correct for this nonlinear problem.

The goal was also to plot the initial wave and the solution for several time values on the same graph in order to see the formation of a shock wave. The code was ran for several increasingly large CFL numbers, and the results can be seen in the next following pages.

Results

CFL = 0.2

Figures (2), (3), and (4) show the initial wave as well as the numerical solution for Courant number $\beta = 0.2$ for different time steps.

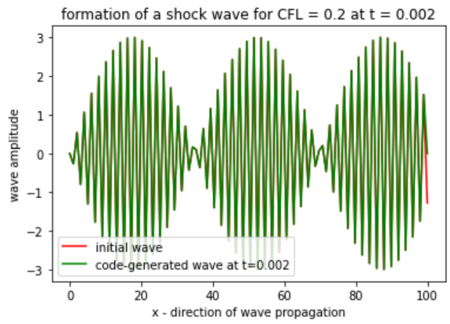

Figure 2: Initial wave and numerical solution for CFL = 0.2

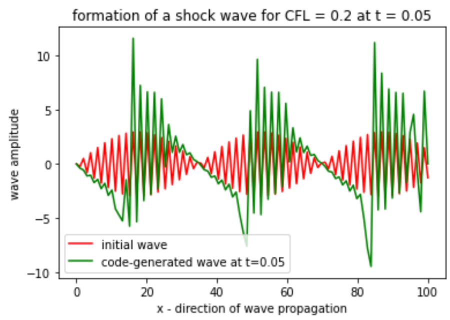

Figure 3: Initial wave and numerical solution for CFL = 0.2

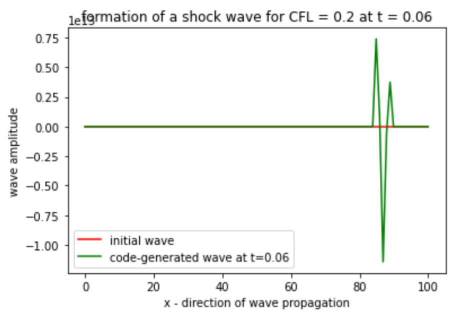

Figure 4: Initial wave and numerical solution for CFL = 0.2

CFL = 0.4

Figures (5), (6), and (7) show the initial wave as well as the numerical solution for Courant number $\beta = 0.4$ for different time steps.

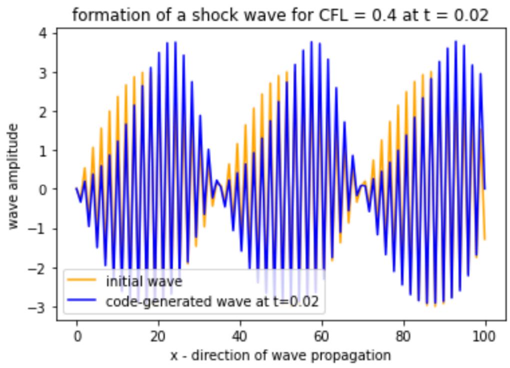

Figure 5: Initial wave and numerical solution for CFL = 0.4

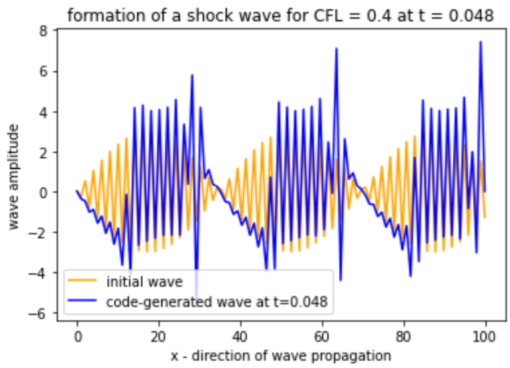

Figure 6: Initial wave and numerical solution for CFL = 0.4

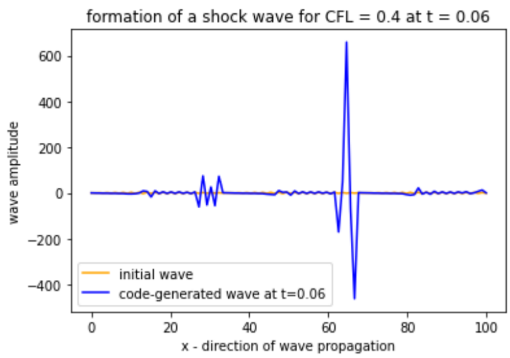

Figure 7: Initial wave and numerical solution for CFL = 0.4

CFL = 0.6

Figures (8), (9), and (10) show the initial wave as well as the numerical solution for Courant number $\beta = 0.6$ for different time steps.

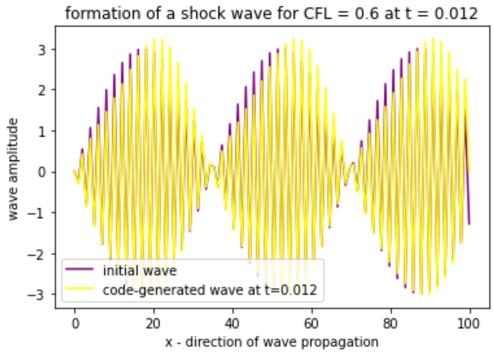

Figure 8: Initial wave and numerical solution for CFL = 0.6

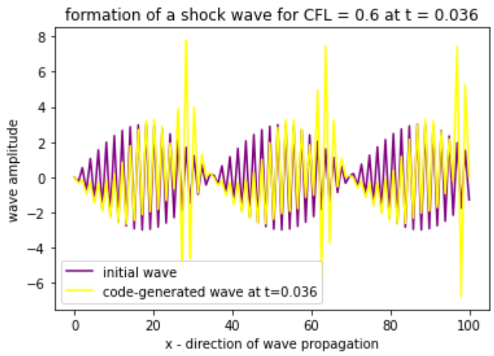

Figure 9: Initial wave and numerical solution for CFL = 0.6

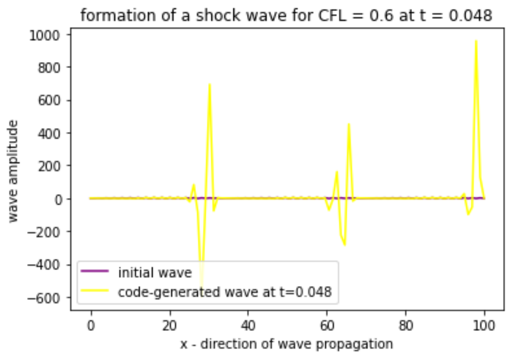

Figure 10: Initial wave and numerical solution for CFL = 0.6

CFL = 0.8

Figures (11), (12), and (13) show the initial wave as well as the numerical solution for Courant number $\beta = 0.8$ for different time steps.

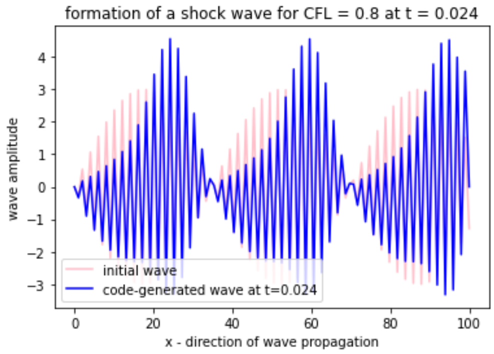

Figure 11: Initial wave and numerical solution for CFL = 0.8

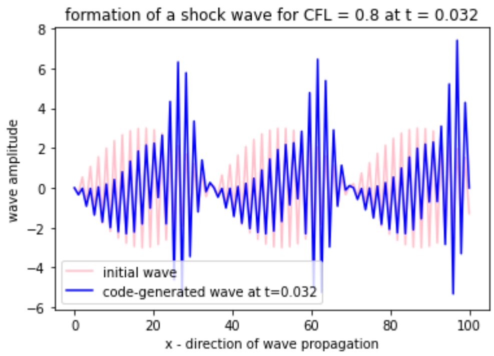

Figure 12: Initial wave and numerical solution for CFL = 0.8

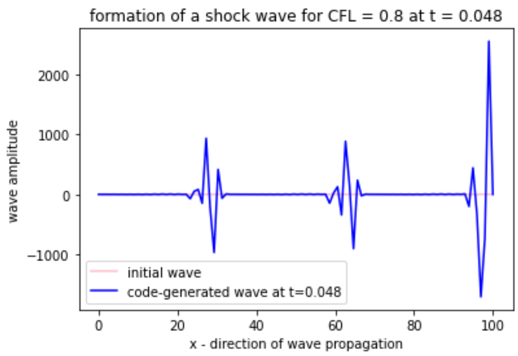

Figure 13: Initial wave and numerical solution for CFL = 0.8

CFL = 1.0

Figures (14), (15), and (16) show the initial wave as well as the numerical solution for Courant number $\beta = 1.0$ for different time steps.

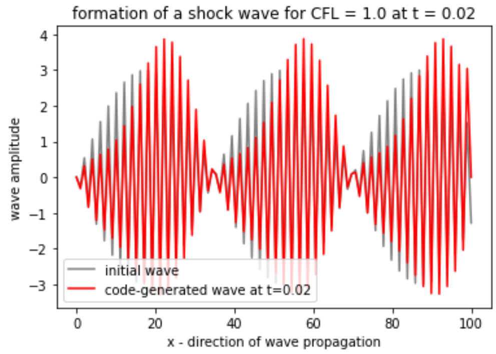

Figure 14: Initial wave and numerical solution for CFL = 1.0

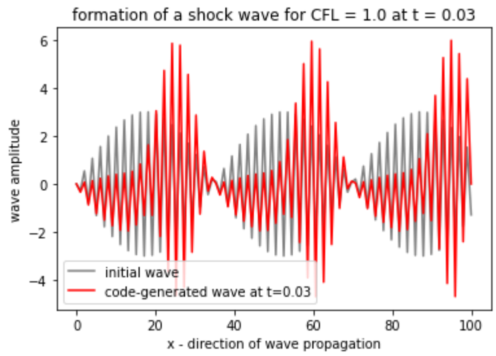

Figure 15: Initial wave and numerical solution for CFL = 1.0

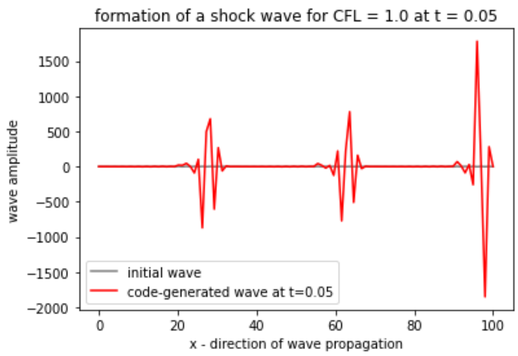

Figure 16: Initial wave and numerical solution for CFL = 1.0

CFL = 2.0

Figures (17), (18), and (19) show the initial wave as well as the numerical solution for Courant number $\beta = 2.0$ for different time steps.

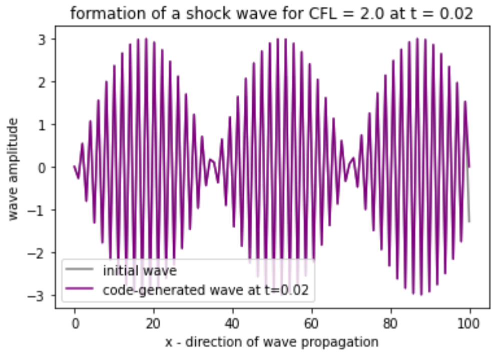

Figure 17: Initial wave and numerical solution for CFL = 2.0

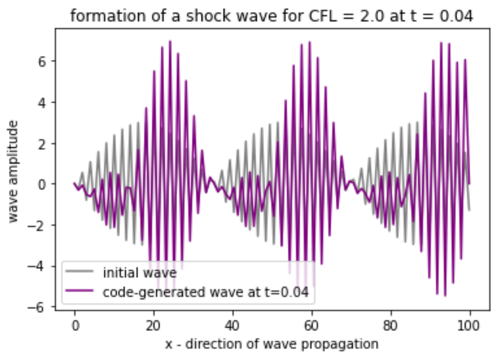

Figure 18: Initial wave and numerical solution for CFL = 2.0

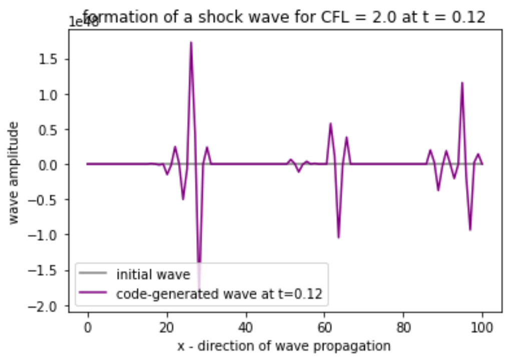

Figure 19: Initial wave and numerical solution for CFL = 2.0

Conclusion

Generally, it can be observed that as the CFL number increases, it takes less time for the formation of the shock wave to take place.

References

[1] Prof. Ulrich Kleinekathöfer (Fall term 2021). Lecture Slides 9. Retrieved November 17, 2021, from https://elearning.jacobs-university.de/mod/folder/view.php?id=14089

[2] Holm, P. D. C. (n.d.). Chapter 2. Molecular Dynamics integrators - Simulation methods in physics 1 . uni-stuttgart.de. Retrieved November 17, 2021, from https://www2.icp.uni-stuttgart.de/~icp/mediawiki/images/5/54/Skript_sim_methods_I.pdf.

[3] Chapter 6: Convection Problems and hyperbolic pdes. Numerical Methods for Engineers. (n.d.). Retrieved November 15, 2021, from http://lrhgit.github.io/tkt4140/allfiles/digital_compendium/._main021.html.