

Understanding the Behavior of Quasiperiodic Orbits

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May 18, 2018

1 Abstract

Quasiperiodicity is a dynamical behavior that particularly has large applications in Hamiltonian systems, one example being the Solar System, which contains quasiperiodic orbits, that is, orbits that are almost but not periodic. Not only is it used in studying planetary motion, but researchers at NASA have sought this behavior to be a possible fuel-efficient path for a spacecraft to travel further distances. Implementing an algorithm based on the work of Das, Saiki, Sander, and Yorke and of Levnajic and Mezic, we observed the dynamics of the two dimensional standard map, that is, how the map's appearance is influenced by different values of the parameter being greater than or equal to zero in the system of equations used in obtaining the map, and also by varying the number of initial conditions and trajectory points, for an understanding of the behavior of quasiperiodic orbits. In addition, we are currently extending the techniques used for obtaining and understanding the standard map to study the four dimensional Froeschle map, and looking at new parameters, which we intend to apply to other equations that have not been studied yet.

2 Introduction

In simplest terms, a quasiperiodic orbit is an orbit that is close to but is not quite periodic. This occurs frequently in Hamiltonian systems as they are conservative systems. It is conjectured that generically there are only three dynamical behaviors. Quasiperiodicity is one out of those three dynamical behaviors and falls in between periodicity and chaos in terms of complexity. It is known to be present in planetary motion. For example, the moon has quasiperiodic behavior on a three-dimensional torus:

- 27.3 days for the Moon to orbit Earth and complete one revolution relative to the stars
- 8.85 years for the moon's orbit to rotate once. The orientation of the moon's orbit is not fixed in space, but rotates over time. This is known as orbital precession. The

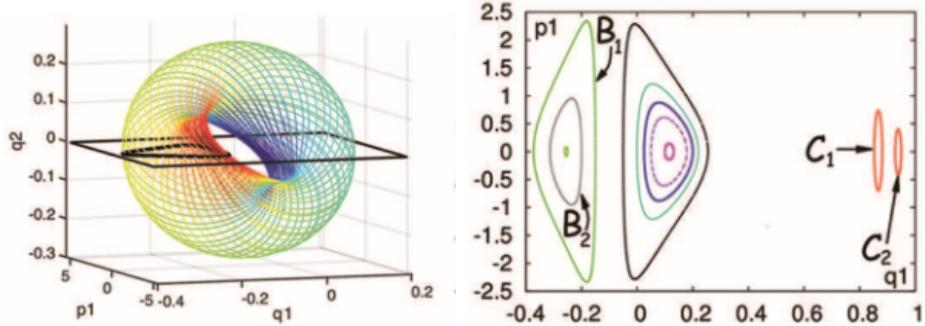


Figure 1: Images from Das et al [2]. Left: Quasiperiodic trajectory on a torus. Right: Poincare Return Map of the figure on the left.

longest diameter of the Moon’s orbit (known as the Moon’s major axis) makes one complete revolution every 8.85 Earth years, as it rotates slowly in the same direction as the Moon itself.

- 18.60 years for the lunar nodes (points at which the Moon crosses the ecliptic plane) to complete one circle

As another example, consider the three body problem consisting of the earth, the moon, and a satellite of negligible mass, where the system is considered in rotating coordinate at is restricted to a plane. The motion of the satellite is quasiperiodic, mapping onto a two dimensional torus in four dimensional space as shown by the figure on the left, where the satellite’s location and momentum are given by generalized position vector (q_1, q_2) and the generalized momentum vector (p_1, p_2) . Everything is assumed to be in a plane.

The right image in Figure 1, which was taken with permission from Das et. al [2], is a Poincare Return Map for a three body problem and a variety of quasiperiodic trajectories B_1 , B_2 , C_1 and C_2 . The black curve in the left image in Figure 1, also from Das et al. [2], is an example of quasiperiodic trajectory and also corresponds to the quasiperiodic trajectory B_1 in the left image in Figure 1.

3 Properties of Quasiperiodicity

Definition. For the orbit of a map T to be quasiperiodic, it

1. must lie on a closed curve X (or torus in higher dimensions), such that
2. by a smooth change of coordinates (i.e, at least twice differentiable), the dynamics of T becomes a pure rotation on the circle (respectively, torus) by a fixed **irrational**

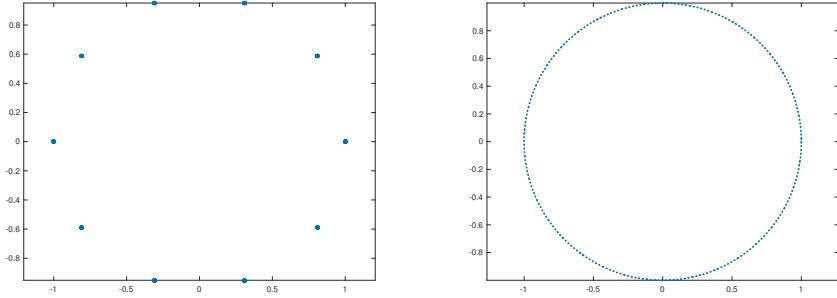


Figure 2: Orbits for rational and irrational ρ . Left: A periodic orbit for when $\rho = \frac{3}{10}$. Right: a dense orbit starting at any x for $f(x)$ when $\rho = \sqrt{10}$

rotation number(s) ρ , i.e.,

$$\theta_i \rightarrow \theta_i + \rho_i \pmod{1}.$$

The rotation number is the rate of rotation as the reader will also find in Alligood et. al [1]. Below is a more formal definition of the rotation number as stated by de la Llave and Luque [5]:

Definition. Let f be the lift of an orientation-preserving homeomorphism of a circle. Then, the rotation number of f is defined as

$$\rho(f) = \pi \left(\lim_{|n| \rightarrow \infty} \frac{f^n(x_0) - x_0}{n} \right)$$

3.1 Quasiperiodic vs Periodic

Rational ρ Implies Periodicity:

Suppose $x \in [0, 1]$ and that $f(x) = x + \rho(\text{mod}1)$. If $\rho = \frac{p}{q}$ for $p, q \in \mathbb{Z}$, that is, ρ is a rational number, in simplified form, where $q \neq 0$, then every point has period q , as indicated by the figure 3, left, where $\rho = \frac{3}{10}$, therefore producing an orbit of 10 points.

Irrational ρ Means Quasiperiodicity:

If we have $x \rightarrow x + \rho(\text{mod}1)$ and that ρ is an irrational number (up to machine precision). $\forall n \in \mathbb{Z}$, $n\rho$ is also irrational for all n . Therefore,

$$(f \circ \dots \circ f)(x) = x + n\rho(\text{mod}1) \neq x$$

So, $f(x) = x + \rho(\text{mod}1)$ can never be periodic. In fact $f(x) = x + \rho(\text{mod}1)$ starting at any x has a dense orbit, as evident from the figure 3, right, where $\rho = \sqrt{10}$.

When we say ρ must be an irrational number in order for $f(x)$ to be a dense orbit starting at any x , we also want to mention that the numbers chosen for ρ must be rationally independent to each other. More formally if c_1, c_2, \dots, c_n are in the set of rational numbers and $\rho_1, \rho_2, \dots, \rho_n$ are in the set of real numbers, then

$$c_1\rho_1 + c_2\rho_2 + \dots + c_n\rho_n = 0$$

is the trivial solution in which every c_k is zero [6].

4 Computing Rotation Numbers for the Standard Map

4.1 The Standard Taylor Chirokov Map

The Standard Map is an area preserving map on the two dimensional torus in which both chaos and quasiperiodicity occur for a large set of parameter values, r and is represented by the following system of two equations:

$$S_1 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + y \\ y + r \sin(x + y) \end{pmatrix} \pmod{2\pi}.$$

The graphs shown in Figure 3 were generated in MATLAB by implementing an algorithm based on the work of Das et. al [2]; a specific value for r was chosen along with 800 initial conditions for x and y and the number of trajectory points, $N = 8000$. The link to the MATLAB code used in generating the Standard Map for different values of r can be found in the Appendix. Let us take a look at the Standard Map for when $r = 1.4$ in Figure 5. The orbits in the figure are the quasiperiodic regions and everything else around them are the regions of chaos. What was understood is that the appearance of the Standard Map varies in terms of the number of quasiperiodic and chaotic regions. When $r = 0$, one would expect to obtain a plot of only straight lines. As the values of r are increased, the number of quasiperiodic regions increase and the regions of chaos decrease; this is noticed after r values of 1.2. For example, refer to the Standard Map for when $r = 1$. You will notice how there are more regions of chaos for when $r = 1$ than for when $r = 1.4$.

Another feature the reader will be able to visualize for the Standard Maps in Figure 3 is how a torus can be formed. In other words, suppose we have a square. Either the top and lower boundaries or left and right boundaries can be aligned and glued together to form a cylinder. Now imagine glueing the right and left boundaries of the obtained cylinder: then we have a torus. As Alligood et. al [1] mentions, this visualization can be applied for the Standard Maps shown in Figure 3 as well; the reader will then notice that some of these orbits will wrap around the torus.

4.2 Results

The next step after studying the behavior of quasiperiodic orbits within the Standard Map was to compute the rotation number. The rotation number will be equal to the average

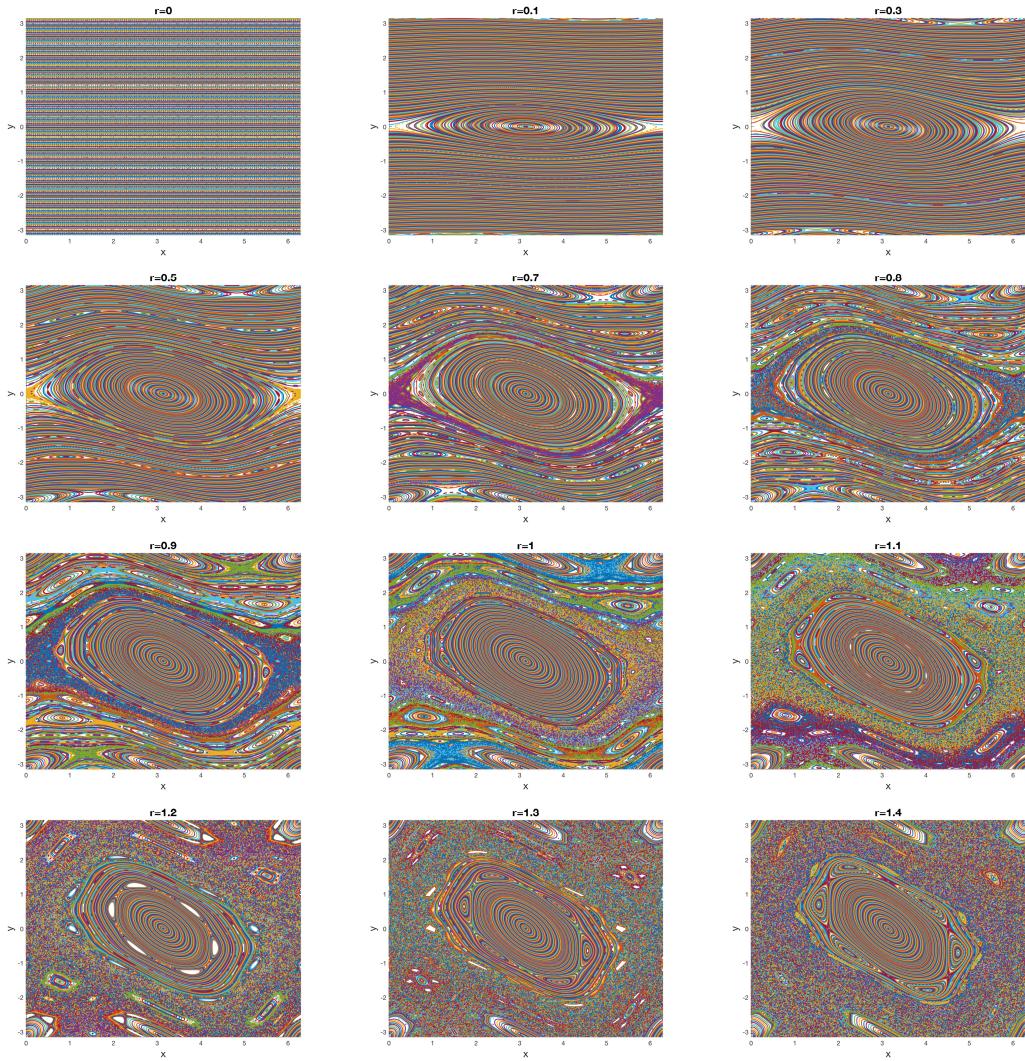


Figure 3: The Standard Map for $0 \leq r \leq 1.4$. The following graphs display both chaos and quasiperiodicity. They were obtained for $N = 8000$ trajectory points, and 800 initial conditions of x and y for the listed values of the parameter r provided in the plots.

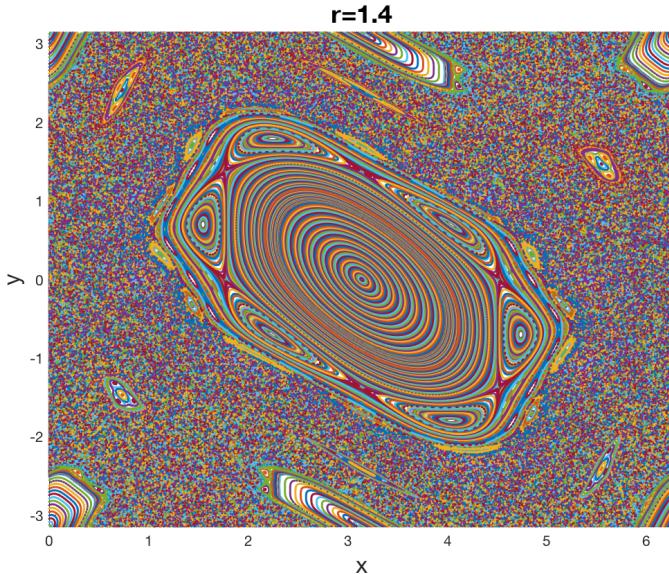


Figure 4: Standard Map for $r = 1.4$, 800 initial conditions of x and y and $N = 8000$ trajectory points.

of the change in the values of ρ over a chosen orbit within the Standard Map. So, finding the rotation number is a matter of selecting a specific orbit within our Standard Map for a particular value of r , and by choosing a specific x and y coordinate. We show in Figure 5 the specific orbit that was selected from the Standard Map corresponding to $r = 1.4$ and how we observe the angles; let us name them θ_i and θ_{i+1} . We refer to the difference in between θ_i and θ_{i+1} , or in between preceding angles, as Δ_i , which in turn is ρ_n . The rotation number would be the average of $\Delta\rho_n$ over the orbit. So, we compute the average by integrating the trajectory of the orbit and dividing by the total number of trajectory points N . Additionally, our process for computing the rotation number included obtaining a scatterplot for θ_i and Δ_i , where $\Delta_i = \theta_{i+1} - \theta_i = \rho_n$; the result is shown in Figure 5 along with the entire process that was used in computing the rotation number. The reader will notice that we have a blue and a red curve in our scatterplot. The red curve is simply the blue curve translated down by 1, as the reader will find the the MATLAB Code to computing the rotation number, as provided in Appendix A. We plotted the two curves in order to compare and check the accuracy of our computed rotation number. Below is the algorithm, also based on the work of Das et. al [2] for computing the rotation number and the result we obtained for it:

1. Set our parameter r to be fixed to $r = 1.4$

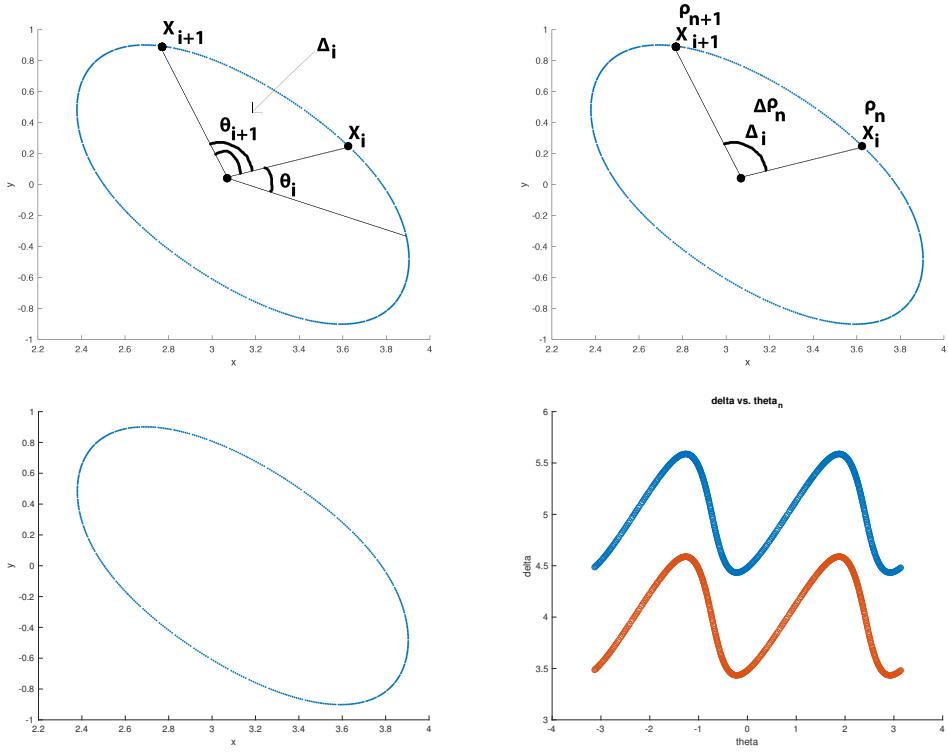


Figure 5: Process for computing the rotation number. Top left: How we chose θ_i and θ_{i+1} . The image of the orbit for $r = 1.4$. Top right: How Δ_i was determined, and, therefore ρ_n . Lower Left: A specific orbit for when $r = 1.4$, $x = 3.5$ and $y = 0.4$, and $N = 500$. Lower Right: A scatterplot of Δ vs. θ . The red curve and blue curve are the same; the main difference is that the red curve is the blue curve translated downward by 1. Both curves represent $\Delta_i = \theta_{i+1} - \theta_i$.

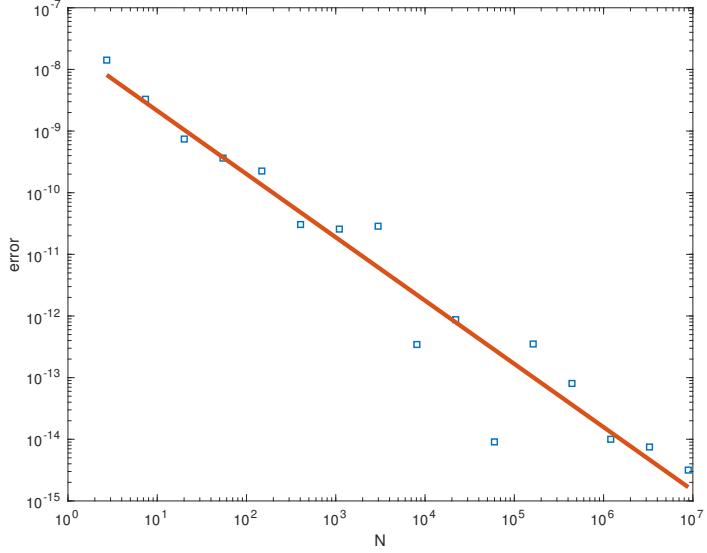


Figure 6: Loglog plot of Error vs N for $N = e^{17}$ with its line of best fit, which displays a slope of -1 and an order of convergence of 1 .

2. Set initial conditions $x_0 = 3.5$ and $y_0 = 0.4$ and chose $N = 500$ iterates, where N is the number of trajectory points. As the reader will find later in this subsection along with the reason why the following is done, the value for N is not kept constant; it is varied to e^{17} . By fixing the initial conditions and varying N , we make better progress in getting an approximation of the rotation number.
3. Find the change in between two values of angles, i.e., $\Delta_{i+1} = \theta_{i+1} - \theta_i$ as shown in the upper two drawings in Figure 5.
4. Plot a scatterplot of Δ vs. θ as shown in the lower right image in Figure 5.
5. Numerically compute the **rotation number** when $r = 1.4$ which is done by taking the average of $\Delta\rho_n$ over the orbit, that is $\rho = \frac{1}{N} \sum_{k=1}^N \Delta_i$, where Δ_i corresponds to $\Delta\rho_n$ as shown in the drawings in Figure 5. For $r = 1.4$, $\rho = 1.9290$. A value of $\rho = 2$ or approximately near was expected, so our result is accurate.

We then proceeded to finding the difference in between preceding values of ρ and then divided by N ; this has been referred to as the error. The goal then was to obtain a loglog plot of Error vs. N , which enabled us in proceeding to find the line of best linear fit as

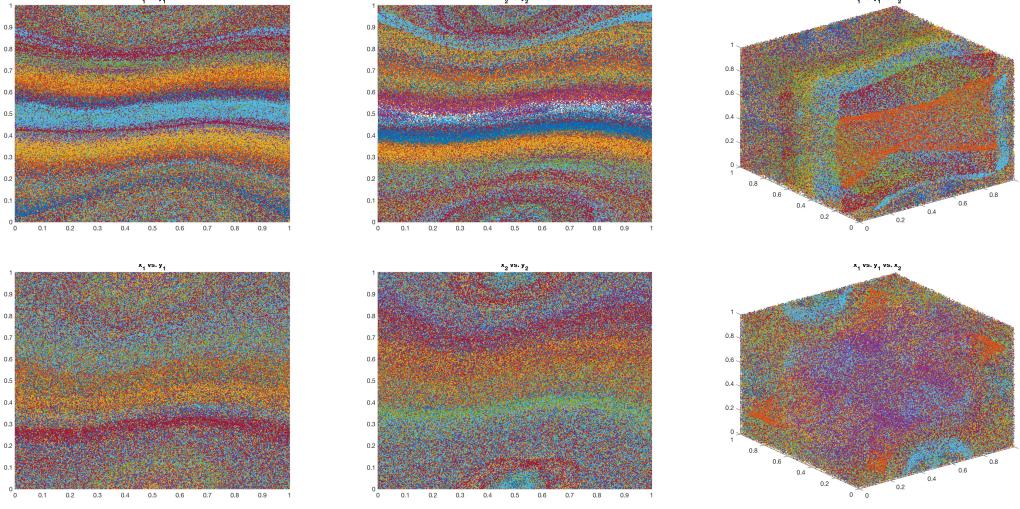


Figure 7: The Froeschle Map for two different values of ϵ and η , where $\epsilon = 2\eta$ and $N = 8000$ as based on the work of Mezic Levnajic [4]. First row: The Froeschle Map for when $\epsilon = 0.05$, $\eta = 0.025$. The top left image is a two-dimensional slice of the Froeschle Map for x_1 vs. y_1 , and the top middle is for x_2 vs. y_2 . The top right image is a three-dimensional plot of x_1 vs x_2 vs y_1 . Second Row: The Froeschle Map for when $\epsilon = 0.08$, $\eta = 0.04$. The images in this row are again the two dimensional slices and the three-dimensional plot as in the first row of images.

shown in Figure 6. Before obtaining our plot, we set $N = e^{17}$ as it was providing the most accurate result we needed for our Error Vs. N Plot. Another reason we aimed to find a line of linear fit is because we were expecting an order of convergence of 1 as it fulfills the notion of the Birkhoff average, which is known to have an error of at least the order N^{-1} for a length N trajectory in the case of quasiperiodic orbits; the reader may refer to Das et. al [2] for more knowledge on the Birkhoff average.

5 Future Work

5.1 Froeschle Map

Now that we have studied the behavior of quasiperiodic orbits within the Standard Map, our next goal is to move on to a higher-dimensional example, the Froeschle Map. This map is a four dimensional area-preserving map. Below is the system for Froeschle Map, which

the reader will also find in Levnajic and Mezic [4]:

$$\begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 + \epsilon_1 \sin(2\pi x_1) + \eta \sin(2\pi x_1 + 2\pi x_2) \\ y_1 + \epsilon_1 \sin(2\pi x_1) + \eta \sin(2\pi x_1 + 2\pi x_2) \\ x_2 + y_2 + \epsilon_2 \sin(2\pi x_2) + \eta \sin(2\pi x_1 + 2\pi x_2) \\ y_2 + \epsilon_2 \sin(2\pi x_2) + \eta \sin(2\pi x_1 + 2\pi x_2) \end{pmatrix} \pmod{1}.$$

The images in Figure 7 were generated in MATLAB again by implementing the algorithm based on the work of Das et. al [2] and also Levnajic and Mezic [4] and our current progress in our study of quasiperiodic orbits within the Froeschle Map. It was similar to what was used for the Standard Map, except this time, there were two more equations involved in the system and two parameters. In fact, if we were to set x_2 and y_2 to 0, then we would again obtain a system of two equations similar to that of the Froeschle Map. Therefore, the slice for x_1 vs. y_1 for example looks similar to the Standard Map. The same can be said for the x_2 vs y_2 slices. Here is what was noticed in the three dimensional plots. In top right 3D plot in Figure 7, we notice that there are rectangular slices, whereas in the lower left 3D plot, the slices are circular. The goal with the Froeschle Map is to take the three dimensional x_1 vs x_2 vs y_1 plots and find a specific 2 dimensional tori in them. Once that is achieved the next step would be to find the angles within that tori, θ and Δ , as was done in the case of the Standard Map and obtain a scatterplot of the change in angle versus the prior angle in order to numerically compute the rotation number for the quasiperiodic orbits within the Standard Map.

Additionally, one of our goals is to look at new parameters and apply them to equations that have not been studied yet. Another one of future works that is currently in progress includes taking the data obtained for the quasiperiodic orbits for various r values within the Standard Map into the mesh software ParaView and create a 3D visualization of how every orbit corresponds to a different rotation number, which will be evident in the varying heights of the orbits.

I would like to thank my advisor Dr. Evelyn Sander, the faculty and colleague of the George Mason University EXTREEMS-QED Program, the National Science Foundation Grant and the Department of Mathematical Sciences at George Mason University.

6 Appendix: List of MATLAB Files

1. Standard Map: stmapfinal.m
2. Froeschle Map: froeshlemapfinal.m
3. Computing Rotation Numbers: rotationnumbercomputation.m

References

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