Homoclinic Orbits Near the Hamiltonian-Hopf Bifurcation in the Suspension Bridge Equation

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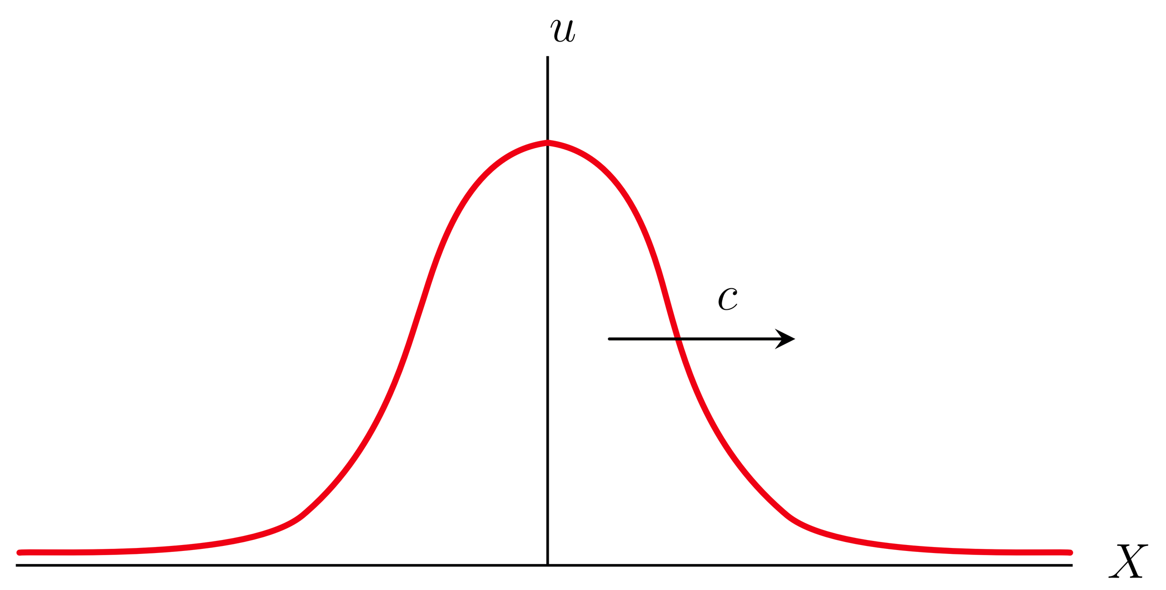
The Good

Initial Setting

We start from the PDE that models the deflection of the roadway in a suspension bridge

$$\frac{\partial^2 U}{\partial T^2} = -\frac{\partial^4 U}{\partial X^4} - e^U + 1$$

and we focus on traveling wave solutions $U(T,X) = u(X - cT)$ describing a disturbance u propagating at velocity c along the surface of the bridge.



By taking $t = X - cT$ we reach the ODE

$$u'''' + c^2 u'' + e^u - 1 = 0.$$

Due to the reversibility symmetry of the PDE we can focus on symmetric solutions for each $\beta = c^2 \in (0,2)$.

It is known that there exists a symmetric homoclinic orbit for all parameter values $\beta \in (0, 1.9]$ as seen in [2, 5]. The goal is to extend this result to the rest of the interval $(0,2)$.

Methodology

The idea is to construct a similar Computer Assisted Proof (CAP) as in [5]. We split the problem into two parts:

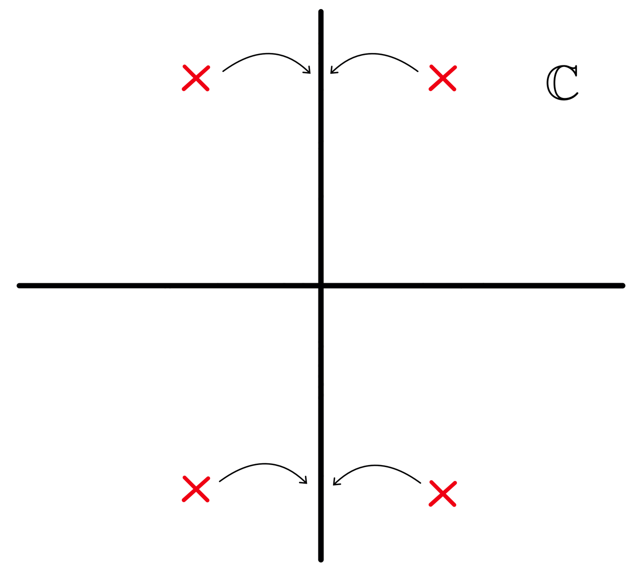
- A rigorous parameterization of the local (un)stable manifolds at the equilibrium $0 \in \mathbb{R}^4$.
- Solving a boundary value problem for the part of the orbit between the local invariant manifolds by using continuation and Chebyshev series.

For smaller values of β the boundary value problem is the most difficult part, as the orbit makes a bigger and bigger excursion away from the origin. However, for values of β close to 2 it is more difficult to obtain the local (un)stable manifold of the origin, as the real part of the eigenvalues tends to zero.

The Bad

The Hamiltonian-Hopf Bifurcation Problem

When β approaches 2, the eigenvalues of the system tend to purely imaginary, leading to a Hamiltonian-Hopf bifurcation in which the invariant manifolds and the homoclinic connection collapse to the origin. Thus, the spectral gap becomes smaller as β approaches 2.



The parameterization of the manifolds is in terms of a series expansion. The decay rate of the terms in the series is proportional to the spectral gap. When $\beta \to 2$ this makes the estimates blow up.

The Ugly

Rescaling Approach

To circumvent the bifurcation problem, we can search for a time rescaling in which the small manifolds are magnified to a standard size. To continue the CAP from there, we should find explicit approximations for such manifolds.

Normal Form Transformation

The approach used in [1] uses the normal form of the corresponding Hamiltonian of the RTBP problem to find a first approximation of the manifolds and prove their connection. For our case we use the Hamiltonian of the suspension bridge equation with two degrees of freedom

$$H = p_2^2 + p_1 p_2 - \frac{1}{2}\left(q_2 - \frac{1}{2}(\beta+2)q_1\right)^2 q_2 + e^{q_1} - q_1.$$

The linearization A is diagonalizable for $\beta < 2$ but not for $\beta = 2$.

$$\begin{pmatrix} -\frac{\sqrt{-\beta-\sqrt{\beta^2-4}}}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & \frac{\sqrt{-\beta-\sqrt{\beta^2-4}}}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & -\frac{\sqrt{-\beta+\sqrt{\beta^2-4}}}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & \frac{\sqrt{-\beta+\sqrt{\beta^2-4}}}{\sqrt{2}} \end{pmatrix} \underset{\beta\to 2}{\not\longrightarrow} \begin{pmatrix} -i & 1 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 1 \\ 0 & 0 & 0 & i \end{pmatrix}.$$

The Versal Normal Form

As seen in [3] and introduced by Arnold, the versal normal form allows for a smooth transition in β that agrees with the usual normal form at $\beta = 2$.

$$\Lambda = \begin{pmatrix} -\frac{i}{2}\sqrt{\beta+2} & 0 & 0 & -\frac{\beta-2}{4} \\ 0 & \frac{i}{2}\sqrt{\beta+2} & -\frac{\beta-2}{4} & 0 \\ 0 & 1 & \frac{i}{2}\sqrt{\beta+2} & 0 \\ 1 & 0 & 0 & -\frac{i}{2}\sqrt{\beta+2} \end{pmatrix} \xrightarrow[\beta\to 2]{} \begin{pmatrix} -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 1 & i & 0 \\ 1 & 0 & 0 & -i \end{pmatrix}.$$

For that we need to find the transformation matrix R in terms of β such that $R^{-1}AR = \Lambda$. We also need the change to be symplectic, so $R^T J R = J$.

Polar Change of Coordinates and Rescaling

Following [4] we can focus on the few first terms of the normal form and apply a polar change of coordinates to understand better the dynamics.

$$\begin{cases} \dot{r} = R, & \dot{R} = \frac{\Theta^2}{r^3} - \frac{(\beta-2)r}{4} + \eta r^3, \\ \dot{\theta} = 1 + \frac{\Theta}{r^2}, & \dot{\Theta} = 0. \end{cases}$$

With this information we can also rescale the system by a factor related to $2-\beta$ so that as $\beta \to 2$ the size of the manifolds is fixed. This yields an approximate expression for the manifolds: $R^2 = \frac{1}{2}\eta(r^2 - r^4)$ where η is the coefficient of the first nonlinear term in the normal form.

The Manifolds and the Orbit

It is shown in [1] that the invariant manifolds intersect for every β near 2. With this information we can craft an initial approximation for our CAP.

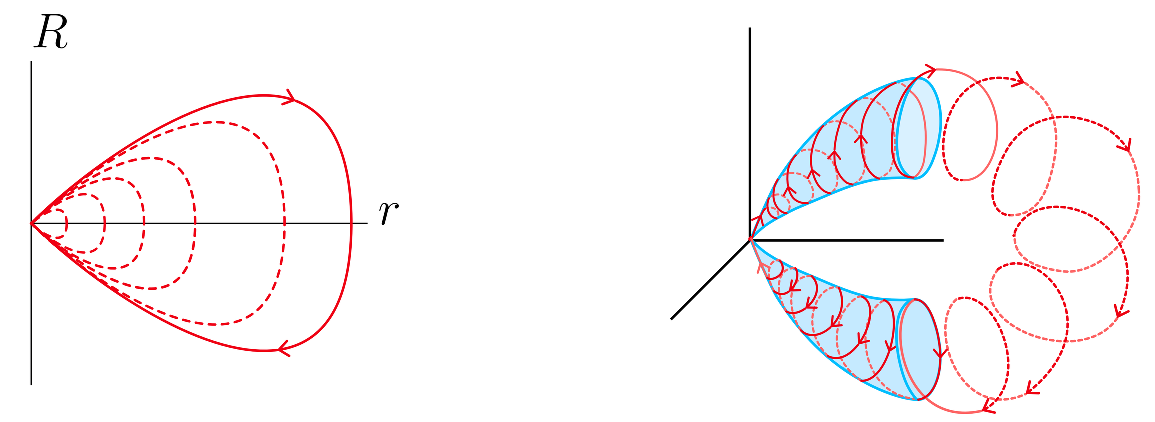


Figure 1. Left: Manifolds in terms of the radius and its momentum in polar coordinates. They shrink to 0 when $\beta \to 2$. Right: The local manifolds (blue). The homoclinic orbit (red).

References

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