On the long-time behavior of discontinuous Galerkin method for Vlasov-Poisson-Fokker-Planck model

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Outline

1. Continuous framework: long-time behavior & hypocoercivity

2. Discrete framework: spectral decompn. in velocity & DG approxn. in space

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1. Continuous framework: long-time behavior & hypocoercivity

2. Discrete framework: spectral decompn. in velocity & DG approxn. in space

Problem

Vlasov-Poisson-Fokker-Planck model.

$$\begin{cases} \underbrace{\partial_t f + v\partial_x f}_{\text{free transport}} + \underbrace{E\partial_v f}_{\text{field interaction}} = \frac{1}{\tau}\underbrace{\partial_v(vf + T_0\partial_v f)}_{\text{collision}}, \\ E = -\partial_x \Phi, \quad -\partial_{xx}\Phi = \rho - \rho_0, \quad \rho = \int_{\mathbb{R}} f \mathrm{d}v. \end{cases} \tag{1}$$

- electron distribution $f(x, v, t)$; electron density $\rho(x, t)$;
- electric potential $\Phi(x, t)$; electric field $E(x, t)$;
- position $x \in \mathbb{T}$; velocity $v \in \mathbb{R}$; time $t \geq 0$;
- ion density $\rho_0(x)$; ion temperature $T_0 > 0$;
- mean free time $\tau > 0$;

Asymptotic limits

Long time regime.

$$f(x,v,t) \xrightarrow{t\to+\infty} \rho_\infty(x)\mathcal{M}(v).$$

- Maxwellian $\mathcal{M}(v)$.

$$\mathcal{M}(v) = (2\pi T_0)^{-\frac{1}{2}} \exp(-\frac{v^2}{2T_0}),$$

- Maxwellian-Boltzmann distribution $\rho_\infty(x)$.

$$\rho_\infty = c_\infty \exp(-\frac{\Phi_\infty}{T_0}), \quad -\partial_{xx}\Phi_\infty = \rho_\infty - \rho_0.$$

- Normalization coefficient c_∞ is determined by

$$\int_{\mathbb{T}} \rho_\infty(x)\mathrm{d}x = \int_{\mathbb{T}} \rho_0(x)\mathrm{d}x.$$

Quantitative estimate

Free energy estimate. Let $f_\infty(x,v) = \rho_\infty(x)\mathcal{M}(v)$,

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(f,f_\infty) = -\dfrac{1}{\tau}\mathcal{D}(f,f_\infty), \\ \mathcal{H}(f,f_\infty) := \displaystyle\int_{\mathbb{T}\times\mathbb{R}} \log\Big(\frac{f}{f_\infty}\Big) f \,\mathrm{d}x\mathrm{d}v + \frac{1}{2T_0}\int_{\mathbb{T}} |E - E_\infty|^2 \,\mathrm{d}x, \\ \mathcal{D}(f,f_\infty) := 4T_0 \displaystyle\int_{\mathbb{T}\times\mathbb{R}} \left|\partial_v \sqrt{\frac{f}{f_\infty}}\right|^2 f_\infty \mathrm{d}x\mathrm{d}v. \end{cases}$$

Observation.

- Entropy dissipation $\mathcal{D}(f,f_\infty)$ is ONLY microscopic coercive[1], i.e.,

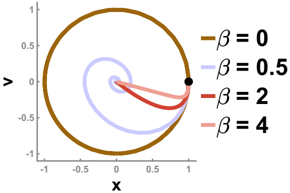
$$\|f(x,v,t) - \rho(x,t)\mathcal{M}(v)\|^2_{L^1(\mathrm{d}x\mathrm{d}v)} \lesssim \mathcal{D}(f,f_\infty).$$

- Lack of control on the distance between $\rho(x,t)$ and $\rho_\infty(x)$.

[1] Villani 2009, AMS

Hypocoercivity: a toy problem

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = +v, \\ \dfrac{\mathrm{d}v}{\mathrm{d}t} = -x - \beta v. \end{cases} \longrightarrow$$



Relative entropy. $\mathcal{H}(t) := \frac{1}{2}(|x(t)|^2 + |v(t)|^2)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(t) = -\beta\,|v(t)|^2\,.$$

Modified entropy. $\mathcal{G}(t) := \mathcal{H}(t) + \alpha x(t)v(t)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{G}(t) = -\beta\,|v(t)|^2 + \alpha(|v(t)|^2 - |x(t)|^2 - \beta x(t)v(t)).$$

By $\mathcal{G}(t) \simeq \mathcal{H}(t)$ and $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{G}(t) \leq -\kappa\mathcal{G}(t)$ so $\mathcal{H}(t) \lesssim \mathcal{H}(0)\exp(-\kappa t)$.

Outline

Weighted L^2 functional framework

Vlasov-Poisson-Fokker-Planck model.

$$\begin{cases} \partial_t f + v\partial_x f + E\partial_v f_\infty + E\partial_v (f - f_\infty) = \dfrac{1}{\tau}\partial_v (vf + T_0 \partial_v f), \\ E = -\partial_x \Phi, \quad -\partial_{xx}\Phi = \rho - \rho_0, \quad \rho = \displaystyle\int_{\mathbb{R}} f \mathrm{d}v. \end{cases}$$

Weighted L^2 free energy estimate.

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(f, f_\infty) = -\dfrac{1}{\tau}\mathcal{I}(f, f_\infty) + \mathcal{R}(f, f_\infty), \\ \mathcal{E}(f, f_\infty) := \dfrac{1}{2}\displaystyle\int_{\mathbb{T}\times\mathbb{R}} |f - f_\infty|^2 f_\infty^{-1} \mathrm{d}x\mathrm{d}v + \dfrac{1}{2T_0}\displaystyle\int_{\mathbb{T}} |E - E_\infty|^2 \mathrm{d}x, \\ \mathcal{I}(f, f_\infty) := T_0 \displaystyle\int_{\mathbb{T}\times\mathbb{R}} \left|\partial_v \left(\dfrac{f}{f_\infty}\right)\right|^2 f_\infty \mathrm{d}x\mathrm{d}v, \\ \mathcal{R}(f, f_\infty) := \dfrac{1}{2}\displaystyle\int_{\mathbb{T}\times\mathbb{R}} (E - E_\infty) |f - f_\infty|^2 \partial_v (f_\infty^{-1}) \mathrm{d}x\mathrm{d}v. \end{cases}$$

Hermite decomposition

Normalization. Let $D = \frac{f}{\sqrt{\rho_\infty}\mathcal{M}} \in L^2(\mathrm{d}x\mathcal{M}\mathrm{d}v)$ and $D_\infty = \sqrt{\rho_\infty}$,

- Spectral decomposition in Hermite basis $(H_k)_{k\in\mathbb{N}}$ of $L^2(\mathcal{M}\mathrm{d}v)$,

$$D(x,v,t) = \sum_{k\in\mathbb{N}} D_k(x,t) H_k\Big(\frac{v}{\sqrt{T_0}}\Big) \implies \|D\|^2_{L^2(\mathrm{d}x\mathcal{M}\mathrm{d}v)} = \sum_{k\in\mathbb{N}} \|D_k\|^2_{L^2(\mathrm{d}x)}.$$

- Translation:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(D,D_\infty) = -\dfrac{1}{\tau}\mathcal{I}(D,D_\infty) + \mathcal{R}(D,D_\infty), \\ \mathcal{E}(D,D_\infty) := \dfrac{1}{2}\displaystyle\int_{\mathbb{T}\times\mathbb{R}} |D-D_\infty|^2 \,\mathrm{d}x\mathcal{M}\mathrm{d}v + \dfrac{1}{2T_0}\int_{\mathbb{T}} |E-E_\infty|^2\,\mathrm{d}x, \\ \mathcal{I}(D,D_\infty) := T_0 \displaystyle\int_{\mathbb{T}\times\mathbb{R}} |\partial_v D|^2 \,\mathrm{d}x\mathcal{M}\mathrm{d}v, \\ \mathcal{R}(D,D_\infty) := \dfrac{1}{T_0}\displaystyle\int_{\mathbb{T}\times\mathbb{R}} v(E-E_\infty)\,|D-D_\infty|^2\,\mathrm{d}x\mathcal{M}\mathrm{d}v. \end{cases}$$

Hermite formulation

Vlasov-Poisson-Fokker-Planck model on $D = (D_k)_{k\in\mathbb{N}}$.

$$\begin{cases} \underbrace{\partial_t D_k + \sqrt{k}\mathcal{A}D_{k-1} - \sqrt{k+1}\mathcal{A}^* D_{k+1}}_{\text{linearized transport}} - \underbrace{\sqrt{k\rho_\infty}QD_{k-1}}_{\text{nonlinear interaction}} = \underbrace{-\frac{k}{\tau}D_k}_{\text{collision}}, \quad \forall k \in \mathbb{N}, \\ \rho_\infty Q = -\mathcal{A}\Psi, \quad -\mathcal{A}^*\rho_\infty^{-1}\mathcal{A}\Psi = D_0 - D_{\infty,0}, \quad D_{-1} = 0. \end{cases}$$

- Modified electric potential Ψ and field Q,

$$\sqrt{\rho_\infty}Q = \frac{E - E_\infty}{\sqrt{T_0}}, \quad \frac{\Psi}{\sqrt{\rho_\infty}} = \frac{\Phi - \Phi_\infty}{T_0}.$$

- Linear operator $\mathcal{A}$ and its adjoint $\mathcal{A}^*$

$$\begin{cases} \mathcal{A}u = +\sqrt{T_0}\partial_x u - \frac{E_\infty}{2\sqrt{T_0}}u = +\sqrt{T_0\rho_\infty}\partial_x(\frac{u}{\sqrt{\rho_\infty}}), \\ \mathcal{A}^*u = -\sqrt{T_0}\partial_x u - \frac{E_\infty}{2\sqrt{T_0}}u = -\sqrt{\frac{T_0}{\rho_\infty}}\partial_x(\sqrt{\rho_\infty}u). \end{cases}$$

Discontinuous Galerkin scheme

Finite element space. Quasi uniform mesh $\mathbb{T}_h$, polynomial space $\mathbb{P}^m$,

$$V_h = \left\{ u\,;\, u\big|_K \in \mathbb{P}^m(K), \quad \forall K \in \mathbb{T}_h \right\}.$$

DG scheme. Find $D_{h,k} \in V_h$ for all $k \in \mathbb{N}$, $Q_h \in W_h$ and $\Psi_h \in V_h$,

$$\begin{cases} \partial_t D_{h,0} - \mathcal{A}_h^* D_{h,0} = 0, \\ \partial_t D_{h,1} + \mathcal{A}_h D_{h,0} - \sqrt{2}\mathcal{A}_h^* D_{h,2} + \mathcal{A}_h \Psi_h - \sqrt{\rho_{\infty,h}} Q_h (D_{h,0} - D_{\infty,h,0}) = -\frac{1}{\tau} D_{h,1}, \\ \partial_t D_{h,k} + \sqrt{k}\mathcal{A}_h D_{h,k-1} - \sqrt{k+1}\mathcal{A}_h^* D_{h,k+1} - \sqrt{k\rho_{\infty,h}} Q_h D_{h,k-1} = -\frac{k}{\tau} D_{h,k}, \, \forall k \geq 2, \\ \rho_{\infty,h} Q_h = -\mathcal{B}_h \Psi_h, \quad -\mathcal{B}_h^* Q_h = D_{h,0} - D_{\infty,h,0}. \end{cases}$$

Requirements for discrete operators

Table: Requirements for $\mathcal{A}_h$ and $\mathcal{A}_h^*$

Property	Preservation
$\langle \mathcal{A}_h u, \phi \rangle_{L^2} = \langle u, \mathcal{A}_h^* \phi \rangle_{L^2}$	Duality structure
$\sqrt{\rho_{\infty,h}} \in \ker \mathcal{A}_h$	Kernel structure
$\|u\|_{L^2} \lesssim \|\mathcal{A}_h u\|_{L^2}$	Macroscopic coercivity
$\|\mathcal{A}_h(\sqrt{\rho_{\infty,h}} u)\|_{L^2} \simeq \|\mathcal{A}_h^*(\frac{u}{\sqrt{\rho_{\infty,h}}})\|_{L^2}$	Primal-dual balance

Table: Requirements for $\mathcal{B}_h$ and $\mathcal{B}_h^*$

Property	Preservation
$\langle \mathcal{B}_h u, \phi \rangle_{L^2} = \langle u, \mathcal{B}_h^* \phi \rangle_{L^2}$	Duality structure
$\|\frac{\mathcal{B}_h \Psi_h}{\sqrt{\rho_{\infty,h}}}\|_{L^\infty} \lesssim \|D_{h,0} - D_{\infty,h,0}\|_{L^2}$	L^∞ bound of electric field

Main results

Proposition (Discrete L^2 free energy estimate)

The DG scheme preserves the L^2 free energy estimate:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_h(D_h, D_{\infty,h}) = -\dfrac{1}{\tau}\mathcal{I}_h(D_h, D_{\infty,h}) + \mathcal{R}_h(D_h, D_{\infty,h}), \\ \mathcal{E}_h(D_h, D_{\infty,h}) = \dfrac{1}{2}\displaystyle\sum_{k\in\mathbb{N}} \|D_{h,k} - D_{\infty,h,k}\|^2_{L^2(\mathbb{T}_h)} + \dfrac{1}{2}\|\sqrt{\rho_{\infty,h}}Q_h\|^2_{L^2(\mathbb{T}_h)}, \\ \mathcal{I}_h(D_h, D_{\infty,h}) = \displaystyle\sum_{k\in\mathbb{N}^+} k\|D_{h,k} - D_{\infty,h,k}\|^2_{L^2(\mathbb{T}_h)}, \\ \mathcal{R}_h(D_h, D_{\infty,h}) = \displaystyle\sum_{k\in\mathbb{N}} \sqrt{k+1}\left\langle \sqrt{\rho_{\infty,h}}Q_h(D_{h,k} - D_{\infty,h,k}), D_{h,k+1}\right\rangle_{L^2(\mathbb{T}_h)}. \end{cases}$$

Observation. Lack of dissipation in the first Hermite mode $D_{h,0}$.

Main results

Theorem (Discrete hypocoercive estimate)

If $\mathcal{E}_h(D_h(0), D_{\infty,h})$ is small enough, there exists positive constants $C > 0$ and $\kappa > 0$ independent of ε and h such that

$$\mathcal{E}_h(D_h(t), D_{\infty,h}) \leq C\mathcal{E}_h(D_h(0), D_{\infty,h}) \exp\left(-\kappa \min\left(\tau, \tau^{-1}\right) t\right).$$

Main idea. Define a modified discrete free energy by

$$\mathcal{G}_h(D_h, D_{\infty,h}) = \mathcal{E}_h(D_h, D_{\infty,h}) + \lambda_0 \left\langle D_{h,1}, q_h \right\rangle_{L^2},$$

where $\lambda_0 > 0$ is small enough and (ψ_h, q_h) satisfies

$$\rho_{\infty,h} q_h = \mathcal{A}_h \psi_h, \quad \mathcal{A}_h^* q_h = D_{h,0} - D_{\infty,h,0}.$$

such that

$$\|q_h\|_{L^2} + \|\mathcal{A}_h q_h\|_{L^2} \lesssim \|D_{h,0} - D_{\infty,h,0}\|_{L^2}.$$

Perspectives

- couple the DG scheme with high-order time discretization;
- numerical verification for the nonlinear model in a perturbative setting;
- quantitative estimate in diffusive regime;
- extends the method/analysis to other kinetic models (Boltzmann equation, radiative transfer equation, etc.);
- discrete framework for logarithmic relative entropy?

Thank you for your attention!