Note of Fermi Liquid 2

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1 Some comments of previous note

Now we give the *proof* of one-dimensional Euler-Lagrange Equation here:
we first find a function f, which satisfies the boundary condition:

$$\begin{aligned} f(a) &= A \\ f(b) &= B \end{aligned} \tag{1}$$

and we construct a function(try to extrmizes the functional):

$$J = \int_a^b dx\ F(x, f(x), f'(x)) \tag{2}$$

now we give a slight perturbation of $f(x)$

$$g_\epsilon(x) = f(x) + \epsilon\eta(x) \tag{3}$$

with $\epsilon \to 0$, $\eta(a) = \eta(b) = 0$. We get:

$$J_\epsilon = \int_a^b dx\ F(x, g_\epsilon(x), g'_\epsilon(x)) = \int_a^b dx\ F_\epsilon \tag{4}$$

Then we calculate the total derivation of J_ϵ:

$$\begin{aligned} \frac{dJ_\epsilon}{d\epsilon} &= \int_a^b dx\ \frac{dF_\epsilon}{d\epsilon} \\ &= \int_a^b dx\ (\frac{dx}{d\epsilon}\frac{\partial F_\epsilon}{\partial x} + \frac{dg_\epsilon}{d\epsilon}\frac{\partial F_\epsilon}{\partial g_\epsilon} + \frac{g'_\epsilon}{d\epsilon}\frac{\partial F_\epsilon}{\partial g'_\epsilon}) \\ &= \int_a^b dx\ (\eta(x)\frac{\partial F_\epsilon}{\partial g_\epsilon} + \eta'(x)\frac{\partial F_\epsilon}{\partial g'_\epsilon}) \end{aligned} \tag{5}$$

when $\epsilon = 0, g_\epsilon = f, F_\epsilon = F(x, f(x), f'(x))$. J_ϵ should be the extreme value:

$$\begin{aligned}
\frac{dJ_\epsilon}{d\epsilon}|_{\epsilon=0} &= \int_a^b dx\ (\eta(x)\frac{\partial F}{\partial f} + \eta'(x)\frac{\partial F}{\partial f'}) \\
&= \int_a^b dx\ (\frac{\partial F}{\partial f} - \frac{d}{dx}\frac{\partial F}{\partial f'})\eta(x) + [\eta(x)\frac{\partial F}{\partial f'}]_a^b \\
&= \int_a^b dx\ (\frac{\partial F}{\partial f} - \frac{d}{dx}\frac{\partial F}{\partial f'})\eta(x) \\
&= 0
\end{aligned} \tag{6}$$

Then we get the Euler-Lagrange Equation

$$\frac{\partial F}{\partial f} - \frac{d}{dx}\frac{\partial F}{\partial f'} = 0 \tag{7}$$

2 Interaction in QFT

2.1 Some specific examples of weakly coupled theory

as we have mentioned before, with the Lagrange density:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \sum_{n\geq 3}\frac{\lambda_n}{n!}\phi^n \tag{8}$$

2.1.1 ϕ^4 Theory

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \tag{9}$$

when $\lambda \ll 1$, this part is so weak.
Then use the formula:

$$H(\phi, \phi', x, t) = \frac{\partial\phi}{\partial t}\frac{\partial\mathcal{L}}{\partial\phi} - \mathcal{L}(\phi, \nabla\phi, \frac{\partial\phi}{\partial t}, x, t) \tag{10}$$

to get the Hamiltonian and then expand ϕ^4 by creation and annihilation operators:

$$\phi^4 = a_{\vec{p}}^+ a_{\vec{p}}^+ a_{\vec{p}}^+ a_{\vec{p}}^+ + a_{\vec{p}}^+ a_{\vec{p}}^+ a_{\vec{p}}^+ a_{\vec{p}} + \cdots \tag{11}$$

and in this process we will have $[H, N] \neq 0$

2.1.2 Scalar Yukawa Theory

$$\mathcal{L} = \partial_\mu \psi^* \partial^\mu \psi + \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - M^2 \psi^* \psi - \frac{1}{2} m^2 \phi^2 - g\psi^* \psi \phi \tag{12}$$

when $g \ll M, m$ this part is weak. As we have mentioned in ϕ^4 section we will have $[H, N_\phi] \neq 0$, $[H, N_\psi] \neq 0$ but $[Q, H] = 0$(Q is the total charge)

3 Normal Fermi Liquid

3.1 The Non-Interacting Fermi Gas

recall the result from the statistic physics(research it by partition function)

when $T = 0$ we have:

density of states:

$$g(E) = \frac{1}{2\pi^2}(\frac{2m}{\hbar^2})^{\frac{3}{2}} E^{\frac{1}{2}} \tag{13}$$

Number density:

$$n = \int_0^\infty (\frac{2m}{\hbar})^{\frac{3}{2}} \frac{1}{2\pi^2} \frac{2}{3} E_F^{\frac{3}{2}} \tag{14}$$

Dispersion relation:

$$E_F = \frac{\hbar^2 k_F^2}{2m} \tag{15}$$

when $T \neq 0$ we have:

the number of states:

$$\Omega = \prod_i \frac{g_i!}{n_i!(g_i - n_i)!} \tag{16}$$

and the entropy:

$$S = k_B \sum_i g_i ln g_i - n_i ln n_i - (g_i - n_i) ln(g_i - n_i) \tag{17}$$

3.2 Weakly correlated Electronic Liquid

Before the Fermi Liquid, it's necessary to consider the Electronic Liquid. We can research the interaction of electron-electron and electron-ion.

Considering a situation: adding a defect(additional electron) in to a material, which has the ion background. What will this defect field behave?

From the analogy of the Fermi Gas we can rewrite the formula of the Number Density near(away) the defect:

$$
\begin{aligned}
n(r_{near}) &\approx \int\limits_{0}^{E_F+e\delta U(r_{near})} g(E)d(E) \\
n(r_{away}) &\approx \int\limits_{0}^{E_F} g(E)d(E)
\end{aligned}
\tag{18}
$$

Hence $\delta n(r) = n(r_{near}) - n(r_{away}) \approx e\delta U g(E_F)$

Then we use the Poisson Equation to get the change in the electrostatic potential.

$$
\begin{aligned}
&\nabla^2 \delta U = 4\pi\delta\rho = 4\pi e\delta n = 4\pi e^2 g(E_F)\delta U \\
\Rightarrow &\nabla^2 \delta U = \lambda^2 \delta U \\
\Rightarrow &\frac{1}{\lambda} = r_{T-F} = \left(4\pi e^2 g(E_F)\right)^{-\frac{1}{2}}
\end{aligned}
\tag{19}
$$

this is called Thomas–Fermi Length

3.3 Mott transition