SANTA MONICA COLLEGE
Department of Mathematics

Math 10– Final Exam

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June 6-8, 2021

Submit your work, typed, as a .pdf or .tex file on Canvas as usual.

This exam allows one sheet of handwritten notes. Handheld calculators are allowed. It is not open Internet. I take academic honesty during COVID even more seriously than usual, and have been known to be rather vindictive when I see solutions that were garnered online. Nobody wants that. Don't do it. I am generous with curves for honest students.

Show your work. Partial credit will be given when earned. Explanations are required unless explicitly not requested.

Distribution of Points

Question	Points	Score
1	14	
2	10	
3	4	
4	5	

Question	Points	Score
5	16	
6	14	
Total:	63	

1. For any string $s \in A^*$ where $A = \{a, b, c, d, \ldots, x, y, z\}$, say another string in A^* *avoids* s if s does not occur as a substring.

   (a) (4 points) Derive a recurrence relation for the number a_n of strings in A^* that avoid the string 'ck', including initial conditions.

   Solution: $a_1 = 26$, and $a_2 = 26 * 25$ Let c_k be the number of strings in A^* that do **not** end in *'c'* and avoid the string *'ck'*. $c_k = a_{k-1} * 25$, removing *'c'*. $a_n = 25 * c_{n-1} * 26$, avoiding *'k'* when ending with *'c'* and allowing any letter otherwise.

   $$a_n = 25 * (a_{n-2} * 25) * 26 \tag{1}$$

   (b) (3 points) By finding a closed form for a_n, how many digits does a_{50} have? You may use a programming language or WolframAlpha for this.

   (b) $\underline{1915112396073565684681314674443600704476968923258682086 5 * 10^{49}}$

   (c) (4 points) Derive a recurrence relation for the number b_n of strings that avoid the string 'ack'.

   Solution: $b_1 = 26$, $b_2 = 26 * 26$, $b_3 = 26 * 26 * 25$ Let c_k be the number of strings in A^* that do **not** end in *'ac'* and avoid the string *'ack'*. $c_k = a_{k-2} * 26 * 25$, removing *'c'*. $b_n = 25 * c_{n-1} * 26$, avoiding *'ak'* when ending with *'ac'* and allowing any letter otherwise.

   $$b_n = 25 * (b_{n-3} * 26 * 25) + 26$$

   (d) (3 points) Compute b_5 using inclusion-exclusion.

   Solution:

   Excluded Strings: $ack\boldsymbol{xx}$, $\boldsymbol{x}ack\boldsymbol{x}$, and $\boldsymbol{xx}ack$

   $$n(excluded) = 3 * 26^2$$

   $$b_5 = 26^5 - 3 * 26^2 = 11879348$$

2. (a) (4 points) Explain which axioms of an equivalence relation fail for the relation "there is a driving route of length less than 0.25 miles between x and y" on the set of all points in Santa Monica.

   Solution: The relation isn't transitive, as the route from point A to point B, and the route from point B to point C can both be less than 0.25 miles, that does not mean that the rounte $A \to B$ is less than 0.25 miles.

(b) (6 points) Give an example of an equivalence relation on the set $\mathbb{Z}$ of integers under which all primes are equivalent, and such that the number of equivalence classes equals the total number of letters in your first and last name combined.

(b) **has the same number of factors modulo 17 as**

Solution: Since all primes only have themselves and 1 as factors, they all have 2 factors.

Furthermore, since I added the modulo 17, there are only 17 equivalence classes, as it wraps back around to 0.

3. (4 points) Give a natural example of a predicate P of domain the set H of all humans such that $\forall x \in H \exists y \in H(P(x, y))$ is true but $\exists y \in H \forall x \in H(P(x, y))$ is false.

3. **P is whether x is a biological child of y**

Solution: Since all humans x have a parent (excluding the edge case of first human, as that begets the chicken-egg problem), but there is no human y that is a child to ALL humans x, as humans typically have 2 biological parents.

4. (5 points) State and prove which natural numbers can be written in the form $6x+10y$ for some $x, y \in \mathbb{N}$.

Solution:

Base case 16**:** $10(1) + 6(1) = 16$

Inductive Case: Assume $a_{x,y}$ is can be written in the form $6x + 10y$. Then $a_{x,y+1}$ can also be written as $6x + 10(y + 1)$, as $a_{x,y+1} = a_{x,y} + 10$.

Inductive Case: Assume $a_{x,y}$ is can be written in the form $6x + 10y$. Then $a_{x+1,y}$ can also be written as $6(x + 1) + 10y$, as $a_{x+1,y} = a_{x,y} + 6$.

Therefore $16, 22, 26, 28, 32, 34, 36, \ldots$ can be written in the form $6x + 10y$.

$a_1 = 16 + 6 = 22$ $a_2 = 16 + 10 = 26$ $a_3 = 16 + 6 + 6 = 28$

Since from these we can add 10 any number of times, we get that all even numbers greater than 20 can be written in the form $6x + 10y$.

5. Consider the graph $G = K_{3,4}$.

(a) (3 points) Does G have an Euler circuit? (Why not, if not?)

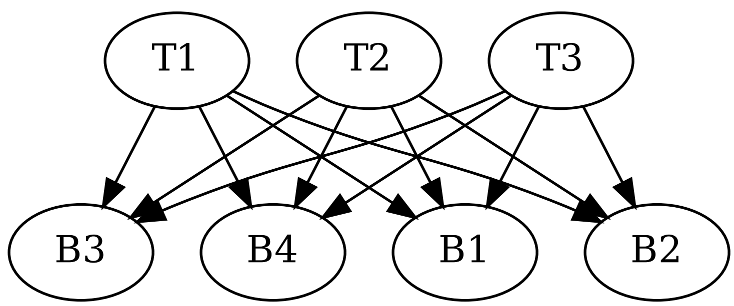
Solution: No, since there is no way to go from the last bottom node you have back to the start without going through a used top node.

(b) (3 points) Does G have a Hamilton circuit? (Why not, if not?)

Solution: No, since there is no euler circuit, it is impossible to get a Hamilton circuit.

(c) (4 points) Is G planar? If so, explain how to draw it without any edges crossing (you may upload a drawing in this case.) If not, prove it is not.

Solution:



(d) (6 points) What are all the isomorphisms $\varphi : G \to G$?

Solution: Since the top and bottom have no set order, there are $3! * 4! = 144$ ways to draw G.

6. Let's do some landscaping.

(a) (3 points) How many ways are there to choose six trees from a landscaper that sells orange, lemon, lime, and kumquat trees?

(a) $4^6 = 4096$

Solution: Since there are 6 trees to choose from 4 variants.

(b) (4 points) Supposing I bought two orange, two lemon, and two lime trees, how many ways are there to distribute my trees among three identical bins for transport home?

(b) $3 * 2 = 6$

Solution: Assuming we are distributing evenly.

Since there are 3 possible cominations to distribute the two orange trees, and another 2 possible cominations to distribute the two lime trees, and only one way to distribute the lemon trees (into the remaining spots).

(c) (7 points) Let n be the number of ways to arrange my six trees along my driveway. There are several possibilities for n depending on how my trees are distributed among the four species. What are all of these possibilities?

Solution:

$$\sum_{o=0}^{6}\sum_{l=0}^{6-o}\sum_{k=0}^{6-l-o}\frac{6!}{o!l!k!(6-o-l-k)!}$$

Counting the number of orange, lime, and kumquat (and the rest are lemon), sum all the possibilities.