(1) $\Box \vec{A} = -\frac{4\pi}{c}\vec{j}$ in the Lorentz gauge.

$\vec{A} = \vec{A}(r)$ due to the symmetry of the problem.

$$\Delta \vec{A} = \begin{cases} -\frac{4\pi}{c}\rho\omega r\hat{\varphi}, & 0 \le r < a, \\ 0, & a \le r. \end{cases}$$

$$\begin{cases} \begin{cases} \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A_r}{\partial r}\right) - \frac{A_r}{r^2} = 0, \\ \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A_\varphi}{\partial r}\right) - \frac{A_\varphi}{r^2} = -\frac{4\pi}{c}\rho\omega r, \\ \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A_z}{\partial r}\right) = 0. \end{cases} \longrightarrow \begin{cases} A_r = C_1 r + \frac{C_2}{r}, \\ A_\varphi = -\frac{\pi}{2c}\rho\omega r^3 + C_3 r + \frac{C_4}{r}, \\ A_z = C_5 \ln r + C_6, \end{cases} \quad 0 \le r < a. \\ \begin{cases} \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A_r}{\partial r}\right) - \frac{A_r}{r^2} = 0, \\ \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A_\varphi}{\partial r}\right) - \frac{A_\varphi}{r^2} = 0, \\ \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A_z}{\partial r}\right) = 0. \end{cases} \longrightarrow \begin{cases} A_r = C_7 r + \frac{C_8}{r}, \\ A_\varphi = C_9 r + \frac{C_{10}}{r}, \\ A_z = C_{11} \ln r + C_{12}, \end{cases} \quad a \le r. \end{cases}$$

Due to the natural boundary condition $\lim_{r\to+\infty}\vec{A} = 0$

(except, perhaps, A_z):

$$A_r = \frac{C_8}{r}, \quad A_\varphi = \frac{C_{10}}{r}, \quad A_z = C_{12}.$$

Due to the continious differentiability of the vector potential $\vec{A}$:

$$\begin{cases} A_r = \frac{C_1}{r}, \\ A_\varphi = -\frac{\pi}{2c}\rho\omega r^3 + \frac{\pi}{c}\rho\omega a^2 r + \frac{C_2}{r}, \\ A_z = C_3, \end{cases} \quad 0 \le r < a. \qquad \begin{cases} A_r = \frac{C_1}{r}, \\ A_\varphi = \frac{C_2 + \frac{\pi}{2c}\rho\omega a^4}{r}, \\ A_z = C_3, \end{cases} \quad a \le r.$$

Since $\vec{H} = \nabla \times \vec{A}$, we get:

$$\vec{H} = \begin{cases} \frac{2\pi}{c}\rho\omega(a^2 - r^2)\hat{z}, & 0 \le r < a. \\ \vec{0}, & r \ge a. \end{cases}$$

2

$\Box \varphi = -4\pi\rho$ in the Lorentz gauge.

$\varphi = \varphi(r,\theta)$ due to the symmetry of the problem.

$$\Delta\varphi = \begin{cases} -4\pi Q\,\delta(\vec{r}), & 0 \le r < R_1, \\ 0, & R_1 \le r < R_2, \\ 0, & R_2 \le r \end{cases}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial\varphi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial\varphi}{\partial\theta}\right) = 0$$

for a homogeneous equation.

Let $\varphi(r,\theta) = R(r)\,\Theta(\theta)$, then:

$$\left(2rR' + r^2R''\right)\Theta + \left(\cot\theta\cdot\Theta' + \Theta''\right)R = 0 \implies$$

$$\implies \frac{2rR' + r^2R''}{R} = -\frac{\cot\theta\cdot\Theta' + \Theta''}{\Theta} = a \implies$$

$$\implies \begin{cases} r^2R'' + 2rR' - aR = 0 & \text{(Cauchy - Euler equation)} \\ \Theta'' + \cot\theta\cdot\Theta' + a\Theta = 0 & \text{(Legendre's equation)} \end{cases} \implies$$

$$\implies \varphi = \sum_{n=0}^{+\infty}\left(A_n r^n + \frac{B_n}{r^{n+1}}\right)P_n(\cos\theta)$$

$$\Delta\varphi = -4\pi Q\,\delta(\vec{r}),\ 0 \le r < R_1 \implies \varphi = C_1 + \frac{Q}{r}$$

since $\Delta\frac{1}{r} = -4\pi\,\delta(\vec{r})$

$$\begin{cases} \Delta\varphi \equiv 0, \\ \vec{E} = -\nabla\varphi \equiv 0, \end{cases} \quad R_1 \le r < R_2 \implies \varphi = C_2$$

since the induced surface charge eliminates the field inside the ball

$$\begin{cases} \Delta\varphi \equiv 0, \\ \lim\limits_{r\to+\infty}\vec{E} = E\cos\theta\cdot\hat{r} - E\sin\theta\cdot\hat{\theta} \end{cases}, \quad R_2 \le r \implies$$

$$\implies \varphi = \left(C_3 + \frac{C_4}{r}\right) + \left(-Er + \frac{C_5}{r^2}\right)\cos\theta$$

Due to the continuity of the electrostatic potential φ:

$$\begin{cases} C_1 + \dfrac{Q}{R_1} = C_2 \\ C_2 = \left(C_3 + \dfrac{C_4}{R_2} \right) + \left(-ER_2 + \dfrac{C_5}{R_2^2} \right) \cos\theta \end{cases} \Longrightarrow$$

$$\Longrightarrow \varphi = \begin{cases} C_1 + \dfrac{Q}{r}, & 0 \le r < R_1 \\ C_2, & R_1 \le r < R_2 \\ \left[C_3 + \dfrac{(C_1 + C_3) R_2 + Q^{R_2/R_1}}{r} \right] + \left(-Er + \dfrac{ER_2^3}{r^2} \right) \cos\theta, & r \ge R_2. \end{cases}$$

According to Gauss' law in integral form:

$$\oint_{\Sigma} \vec{E} \cdot d\vec{S} = 4\pi Q \Longrightarrow C_1 = Q \left(\frac{1}{R_1} - \frac{1}{R_2} \right) + C_3$$

Choosing the gauge of the electrostatic potential φ so that $\lim\limits_{r \to +\infty} \varphi = -Er\cos\theta$, we get:

$$\varphi = \begin{cases} \dfrac{Q}{r} - Q \left(\dfrac{1}{R_1} - \dfrac{1}{R_2} \right), & 0 \le r < R_1, \\ \dfrac{Q}{R_2}, & R_1 \le r < R_2, \\ \dfrac{Q}{r} - E \left(r - \dfrac{R_2^3}{r^2} \right) \cos\theta, & r \ge R_2. \end{cases}$$

Since $\vec{E} = -\nabla\varphi$, we get:

$$\vec{E} = \begin{cases} \dfrac{Q}{r^2} \hat{r}, & 0 \le r < R_1, \\ 0, & R_1 \le r < R_2, \\ \left[\dfrac{Q}{r^2} + E \left(1 + \dfrac{2R_2^3}{r^3} \right) \cos\theta \right] \hat{r} - E \left(1 - \dfrac{R_2^3}{r^3} \right) \sin\theta \cdot \hat{\theta}, & R_2 \le r. \end{cases}$$

3. According to Cauchy's integral formula:

$$P_\ell(x) = \frac{1}{2^\ell \ell!}\frac{d^\ell}{dx^\ell}(x^2-1)^\ell = \frac{1}{2\pi i}\frac{1}{2^\ell}\oint_{\partial D}\frac{(t^2-1)^\ell}{(t-x)^{\ell+1}}\,dt,$$

where ∂D is the rectifiable (i.e. having a finite length) boundary of the simply connected open subset of the complex plane D containing a point x, oriented counterclockwise.

Suppose that $(1-x^2)P_\ell''(x) - 2xP_\ell'(x) + \ell(\ell+1)P_\ell(x) = 0$, then:

$$\frac{1}{2\pi i}\frac{\ell+1}{2^\ell}\oint_{\partial D}\frac{(t^2-1)^\ell[\ell t^2 - 2(\ell+1)xt + (\ell+2)]}{(t-x)^{\ell+3}}\,dt = 0.$$

(therefore, $F(t,x) = \ell t^2 - 2(\ell+1)xt + (\ell+2)$ and $a = \ell+3$).

The integrand is $\dfrac{(t^2-1)^\ell[\ell t^2 - 2(\ell+1)xt + (\ell+2)]}{(t-x)^{\ell+3}} = \dfrac{d}{dt}\dfrac{(t^2-1)^{\ell+1}}{(t-x)^{\ell+2}}$,

what completes the proof.