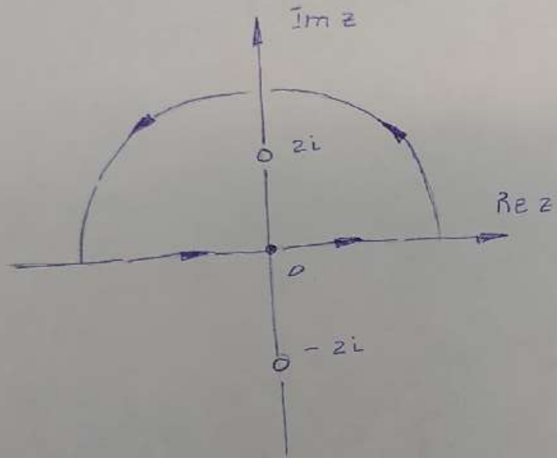
(3) $$\int_{-\infty}^{+\infty} \frac{\cos\left(x - \frac{1}{x}\right)}{1 + x^2}\, dx = \left\{ \begin{matrix} u = x - \frac{1}{x} & u = +\infty \\ du = \left(1 + \frac{1}{x^2}\right) dx & u = -\infty \end{matrix} \right\} =$$

$$= 2 \int_{-\infty}^{+\infty} \frac{\cos u}{u^2 + 4}\, du = 2\, \mathrm{Re} \int_{-\infty}^{+\infty} \frac{e^{iu}}{u^2 + 4}\, du = 2\, \mathrm{Re} \left(2\pi i \operatorname*{Res}_{u = 2i} \frac{e^{iu}}{u^2 + 4} \right) =$$

$$= 2\, \mathrm{Re} \left(2\pi i \left. \frac{e^{iu}}{2u} \right|_{u = 2i} \right) = \frac{\pi}{e^2}$$

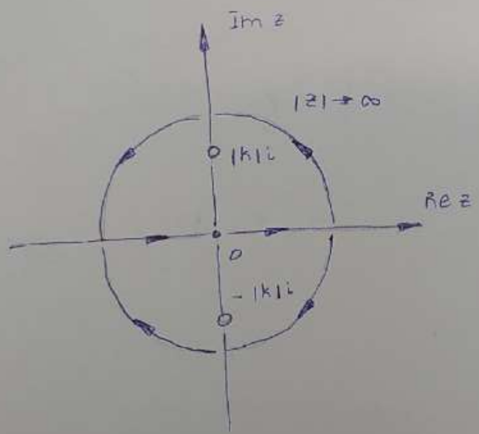


(6) $\displaystyle\int_0^{+\infty} \frac{x \sin ax}{x^2+k^2}\,dx = \frac{1}{2}\,\mathrm{Im}\int_{-\infty}^{+\infty}\frac{xe^{iax}}{x^2+k^2}\,dx =$

$$= \begin{cases} \dfrac12\, \mathrm{Im}\left(2\pi i \operatorname*{Res}_{z=|k|i} \dfrac{ze^{i|a|z}}{z^2+k^2}\right) = \dfrac12\, \mathrm{Im}\left(2\pi i\, \dfrac{ze^{i|a|z}}{z+|k|i}\Big|_{z=|k|i}\right) = \dfrac{\pi}{2}e^{-|ak|} \\ -\dfrac12\, \mathrm{Im}\left(2\pi i \operatorname*{Res}_{z=-|k|i}\dfrac{ze^{-i|a|z}}{z^2+k^2}\right) = -\dfrac{\pi}{2}e^{-|ak|}\end{cases}$$

$= \dfrac{\pi}{2}e^{-|ak|}\,\mathrm{sgn}\, a, \quad a\neq 0 \wedge k \neq 0.$

Интеграл расходится, если $a = 0 \wedge k = 0$.



(1) $f(z) = \dfrac{1}{z^3 - z^5} = \dfrac{1}{z^3(1-z^2)}$

$\operatorname*{Res}_{z=-1} f(z) = \dfrac{1}{z^3(1-z)}\Big|_{z=-1} = -\dfrac{1}{2}$

$\operatorname*{Res}_{z=1} f(z) = \dfrac{-1}{z^3(z+1)}\Big|_{z=1} = -\dfrac{1}{2}$

$\operatorname*{Res}_{z=\infty} f(z) = -\lim\limits_{|z|\to\infty} \dfrac{1}{z^2 - z^4}$ (т.к. $\lim\limits_{|z|\to\infty} f(z)) = 0$) $= 0$

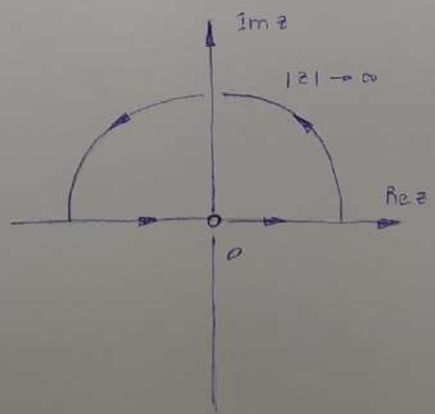
$\operatorname*{Res}_{z=0} f(z) = 0 - \left(-\dfrac{1}{2} - \dfrac{1}{2} + 0\right) = 1$

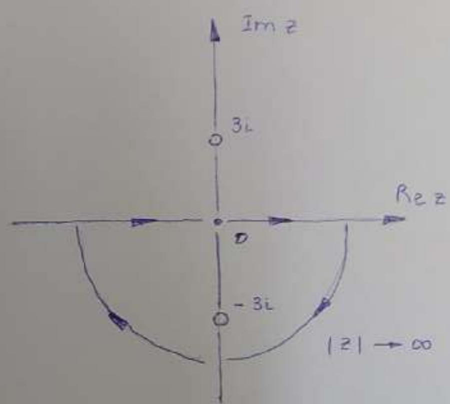
(2) $\displaystyle\int_0^{+\infty} \frac{x - \sin x}{x^3}\,dx = \frac{1}{2}\,\mathrm{Im}\int_{-\infty}^{+\infty} \frac{z - e^{iz}}{z^3}\,dz = \frac{1}{2}\,\mathrm{Im}\left\{\pi i \operatorname*{Res}_{z=0} \frac{z - \left[1 + iz - \frac{z^2}{2!} + O(z^3)\right]}{z^3}\right\}$

(т.к. полюс $z=0$ лежит на контуре) $= \dfrac{1}{2}\,\mathrm{Im}\left[\pi i \operatorname*{Res}_{z=0}\left(\ldots + \dfrac{1}{2z} + \ldots\right)\right] =$

$= \dfrac{\pi}{4}$

$\displaystyle\int_{-\infty}^{+\infty} \frac{e^{iz}}{z^2+9}\,dz = -2\pi i \operatorname*{Res}_{z=-3i} \frac{e^{-iz}}{z^2+9} = -2\pi i \,\frac{e^{-iz}}{2z}\Big|_{z=-3i} = \frac{\pi}{3e^3}$





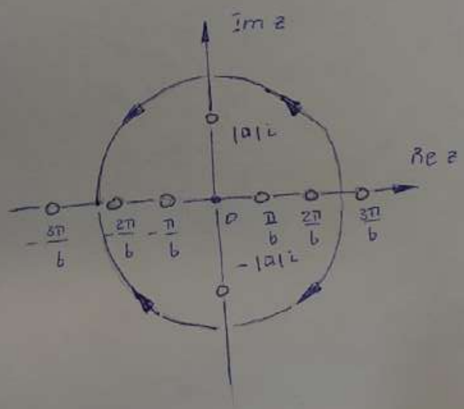
(5) $$\lim_{|z|\to\infty} \int_{\arg z \in [0,\pi]} e^{iz}\,dz = \lim_{R\to\infty} \int_{R} e^{iz}\,dz = \lim_{R\to\infty} \frac{1}{i} e^{iz} \Big|_{R}^{-R} =$$

$$= \lim_{R\to\infty} (-2\sin R) \implies \nexists \lim_{R\to\infty} \int_{C_R} e^{iz}\,dz$$

(7) $$p.v. \int_0^{+\infty} \frac{x\,dx}{(x^2+a^2)\sin bx} = p.v.\ \frac{1}{2} \int_{-\infty}^{+\infty} \frac{x\,dx}{(x^2+a^2)\sin bx} =$$

$$= \frac{1}{2} \cdot 2\pi i \operatorname*{Res}_{z=|a|i} \frac{z}{(z^2+a^2)\sin bz} = \pi i \cdot \frac{z}{(z+|a|i)\sin bz}\Bigg|_{z=|a|i} =$$

$$= \frac{\pi}{2\sinh(|a|b)} \operatorname{sgn} a = \frac{\pi}{2\sinh ab}$$



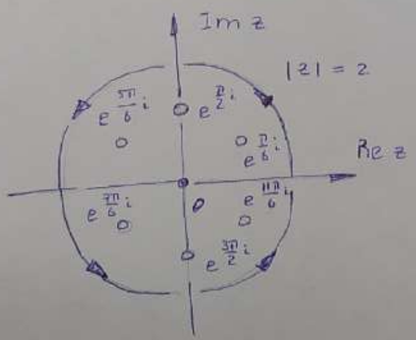
(2.) $\operatorname*{Res}_{z=\infty} z^3 \cos\frac{1}{z-2} = -\operatorname*{Res}_{z=2} z^3 \cos\frac{1}{z-2} = \left\{ \begin{matrix} w = z-2 \\ z = w+2 \end{matrix} \right\} =$

$= -\operatorname*{Res}_{w=0} \left\{ (w^3 + 6w^2 + 12w + 8)\left[1 - \frac{1}{2!\,w^2} + \frac{1}{4!\,w^4} + O\left(\frac{1}{w^6}\right) \right] \right\} =$

$= -\operatorname*{Res}_{w=0} \left[\ldots + \left(-\frac{12}{2!} + \frac{1}{4!} \right)\frac{1}{w} + \ldots \right] = \frac{143}{24}$

(3.) $\oint\limits_{|z|=2} \frac{z^5}{1+z^6}\,dz = -2\pi i \operatorname*{Res}_{z=\infty} f(z) = 2\pi i \lim\limits_{|z|\to\infty} \frac{z^6}{1+z^6}$

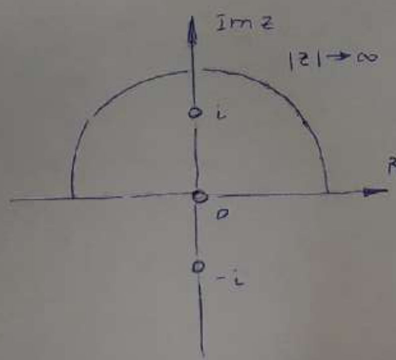
$\left(\text{т.к. } \lim\limits_{|z|\to\infty} f(z) = 0 \right) = 2\pi i$



(4.) $\int\limits_{-\infty}^{+\infty} \frac{\sin^2 x\,dx}{x^2(x^2+1)} = \operatorname{Im} \int\limits_{-\infty}^{+\infty} \frac{e^{ix}\sin x}{x^2(x^2+1)}\,dx = \operatorname{Im}\left[2\pi i \operatorname*{Res}_{z=i} f(z) + \pi i \operatorname*{Res}_{z=0} f(z) \right]$

(т.к. полюс $z=0$ лежит на контуре) $= \operatorname{Im}\left[2\pi i \left. \frac{e^{iz}\sin z}{z^2(z+i)} \right|_{z=i} + \right.$

$\left. + \pi i \left. \frac{e^{iz}\sin z}{z(z^2+1)} \right|_{z=0} \right] = \frac{\pi}{2}\left(1 + \frac{1}{e^2} \right)$



1.

$$\int_{-\infty}^{+\infty} \frac{x^4}{1+x^6}\,dx = 2\pi i \left[\operatorname*{Res}_{z=\frac{\pi}{6}i} f(z) + \operatorname*{Res}_{z=\frac{\pi}{2}i} f(z) + \operatorname*{Res}_{z=\frac{5\pi}{6}i} f(z) \right] =$$

$$= 2\pi i \cdot \frac{1}{6}\left(e^{-\frac{\pi}{6}i} + e^{-\frac{\pi}{2}i} + e^{-\frac{5\pi}{6}i} \right) = \frac{\pi i}{3}\cdot(-2i) = \frac{2\pi}{3}$$

$$\int_0^{2\pi} \frac{\cos 2\theta}{2+\cos\theta}\,d\theta = \oint_{|z|=1} \frac{\frac{1}{4}\left(z+\frac{1}{z}\right)^2 + \frac{1}{4}\left(z-\frac{1}{z}\right)^2}{2+\frac{1}{2}\left(z+\frac{1}{z}\right)}\frac{dz}{iz} =$$

$$= \oint_{|z|=1} \frac{z^4+1}{iz^2(z^2+4z+1)}\,dz = 2\pi i \left[\operatorname*{Res}_{z=0} f(z) + \operatorname*{Res}_{z=\sqrt{3}-2} f(z) \right] =$$

$$= 2\pi i \left[\frac{d}{dz}\frac{z^4+1}{i(z^2+4z+1)}\bigg|_{z=0} + \frac{z^4+1}{iz^2(z+2+\sqrt{3})}\bigg|_{z=\sqrt{3}-2} \right] = 2\pi\left(\frac{7}{\sqrt{3}} - 4\right)$$

$$\int_{-\infty}^{+\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)^2} = 2\pi i \left[\operatorname*{Res}_{z=|a|i} f(z) + \operatorname*{Res}_{z=|b|i} f(z) \right] =$$

$$= 2\pi i \left[\frac{1}{(z+|a|i)(z^2+b^2)^2}\bigg|_{z=|a|i} + \frac{d}{dz}\frac{1}{(z^2+a^2)(z+|b|i)^2}\bigg|_{z=|b|i} \right] =$$

$$= \frac{\pi}{2}\,\frac{|a|+2|b|}{|ab^3|(|a|+|b|)^2}, \quad a \neq 0 \wedge b \neq 0.$$

Интеграл расходится, если $a = 0 \vee b = 0$.