(19.07) 2) $\begin{cases} z_n' = n & \lim\limits_{n\to+\infty} e^{-n^2} = 0 \\ z_n'' = in & \lim\limits_{n\to+\infty} e^{n^2} = +\infty \end{cases}$

3) $\begin{cases} z_n' = \dfrac{1}{\sqrt{2n}} & \lim\limits_{n\to+\infty} \sin 2\pi n = 0 \\ z_n'' = \dfrac{1}{\sqrt{2n+\frac{1}{2}}} & \lim\limits_{n\to+\infty} \sin\left(\frac{\pi}{2} + 2\pi n\right) = 1 \end{cases}$

(19.15) 1) $\begin{cases} f(z) = \dfrac{(1+z^2)^2}{1-z^2} = \dfrac{(z-i)^2(z+i)^2}{(1-z)(1+z)} \\ f\left(\frac{1}{\xi}\right) = \dfrac{(\xi^2+1)^2}{\xi^2(\xi^2-1)} \end{cases} \Rightarrow \begin{cases} z = \pm i \text{ - нули 2-го пор.} \\ z = \pm 1 \text{ - полюса 1-го пор.} \\ z = \infty \text{ - полюс 2-го пор.} \end{cases}$

2) $\begin{cases} f(z) = \cot z \\ f'(z) = -\dfrac{1}{\sin^2 z} \\ \dfrac{1}{f(z)} = \tan z \\ \left[\dfrac{1}{f(z)}\right]' = \dfrac{1}{\cos^2 z} \end{cases} \Rightarrow \begin{cases} z = \frac{\pi}{2} + \pi k,\ k \in \mathbb{Z} \text{ - нули 1-го пор.} \\ z = \pi k,\ k \in \mathbb{Z} \text{ - полюса 1-го пор.} \end{cases}$

3) $f(z) = z\tan^2 z = \dfrac{z\sin^2 z}{\cos^2 z} \Longrightarrow \begin{cases} z = 0 \text{ - нуль 3-го пор.} \\ z = \pi k,\ k \in \mathbb{Z}\setminus\{0\} \text{ - нули 2-го пор.} \\ z = \frac{\pi}{2} + \pi k,\ k \in \mathbb{Z} \text{ - полюса 2-го пор.} \end{cases}$

20.01

1) $\begin{cases} \dfrac{1}{R} = \overline{\lim\limits_{n \to +\infty}} \sqrt[n]{2^{-n}} = \dfrac{1}{2} \implies R = 2 \\ r = \overline{\lim\limits_{n \to +\infty}} \sqrt[n]{2^{-n}} = \dfrac{1}{2} \implies r = \dfrac{1}{2} \end{cases} \implies |z| \in \left(\dfrac{1}{2}, 2\right)$

2) $\begin{cases} \dfrac{1}{R} = \overline{\lim\limits_{n \to +\infty}} \dfrac{1}{\sqrt[n]{3^{n}+1}} = \dfrac{1}{3} \implies R = 3 \\ r = \overline{\lim\limits_{n \to +\infty}} \dfrac{1}{\sqrt[n]{3^{-n}+1}} = 1 \implies r = 1 \end{cases} \implies |z| \in (1, 3)$

4) $\begin{cases} \dfrac{1}{R} = \overline{\lim\limits_{n \to +\infty}} \sqrt[n]{2^{-n^2}} = 0 \implies R = +\infty \\ r = \overline{\lim\limits_{n \to +\infty}} \sqrt[n]{2^{-n^2}} = 0 \implies r = 0 \end{cases} \implies z \in (0, +\infty)$

20.09

1) $$\frac{1}{z(z-3)^2} = \left\{ \begin{matrix} w = z-1 \\ z = w+1 \end{matrix} \right\} = \frac{1}{(w+1)(w-2)^2} = \frac{1}{9}\,\frac{1}{w+1} - \frac{1}{9}\,\frac{1}{w-2} +$$

$$+ \frac{1}{3}\,\frac{1}{(w-2)^2} = \frac{1}{9}\,\frac{1}{(-1)} \sum_{n=-\infty}^{-1} \frac{w^n}{(-1)^n} - \frac{1}{9}\,\frac{(-1)}{2} \sum_{n=0}^{+\infty} \frac{w^n}{2^n} + \frac{1}{3}\,\frac{(-1)^2}{4} \sum_{n=0}^{+\infty} (n+1)\,\frac{w^n}{2^n} =$$

$$= \sum_{n=-\infty}^{-1} \frac{(-1)^{n+1}}{9} (z-1)^n + \sum_{n=0}^{+\infty} \frac{3n+5}{9}\,\frac{(z-1)^n}{2^{n+2}} \, , \quad 1 < |z-1| < 2$$

5) $$\frac{1}{z(z-1)(z-2)} = \frac{1}{2}\,\frac{1}{z} - \frac{1}{z-1} + \frac{1}{2}\,\frac{1}{z-2} = \frac{1}{2}\,\frac{1}{z} - \frac{1}{1} \sum_{n=-\infty}^{-1} \frac{z^n}{1^n} +$$

$$+ \frac{1}{2}\,\frac{(-1)}{2} \sum_{n=0}^{+\infty} \frac{z^n}{2^n} = \sum_{n=-\infty}^{-2} (-1)\, z^n - \frac{1}{2}\,\frac{1}{z} - \sum_{n=0}^{+\infty} \frac{z^n}{2^{n+2}} \, , \quad -\frac{3}{2} \in D$$

20.16

1) $$z^3 e^{1/z} = z^3 \cdot \sum_{n=0}^{+\infty} \frac{\left(\frac{1}{z}\right)^n}{n!} = z^3 + z^2 + \frac{1}{2} z + \frac{1}{6} + \sum_{n=1}^{+\infty} \frac{z^{-n}}{(n+3)!}$$

2) $$z^2 \sin \pi \frac{z+1}{z} = z^2 \sin\left(\pi + \frac{\pi}{z}\right) = - z^2 \sin \frac{\pi}{z} = - z^2 \cdot \sum_{n=1}^{+\infty} \frac{(-1)^n \left(\frac{\pi}{z}\right)^{2n+1}}{(2n+1)!} =$$

$$= - \pi z + \sum_{n=1}^{+\infty} \frac{(-1)^{n+1} \pi^{2n+1}}{(2n+1)!} z^{-2n+1}$$

3) $$z^3 \cos \frac{1}{z-2} = \begin{cases} w = z-2 \\ z = w+2 \end{cases} = (w+2)^3 \cos \frac{1}{w} = (w^3 + 6w^2 + 12w + 8) \cdot$$

$$\cdot \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n)!\, w^n} = (z-2)^3 + 6(z-2)^2 + \frac{23}{2}(z-2) + 5 + \sum_{n=1}^{+\infty} (-1)^n \frac{48n^2 + 72n + 23}{(2n+2)!} (z-2)^{-2n+1}$$

$$+ \sum_{n=1}^{+\infty} 2(-1)^n \frac{16n^2 + 24n + 5}{(2n+2)!} (z-2)^{2n}$$

20.21

1) $$\frac{z}{(z+2)^2} = \left\{ \begin{array}{l} w = z+2 \\ z = w-2 \end{array} \right\} = \frac{w-2}{w^2} = -\frac{2}{w^2} + \frac{1}{w} =$$

$$= -\frac{2}{(z+2)^2} + \frac{1}{z+2}$$

2) $$\frac{e^z+1}{e^z-1} = \left\{ \begin{array}{l} w = z - 2\pi k i,\ k \in \mathbb{Z} \\ z = w + 2\pi k i,\ k \in \mathbb{Z} \end{array} \right\} = \frac{e^w+1}{e^w-1} = \left\{ \begin{array}{l} w = 0 \text{ — полюс} \\ \text{1-го пор.} \end{array} \right\} =$$

$$= \frac{c_{-1}}{w} + O(1)$$

$$e^w + 1 = 2 + O(w) = \left[\frac{c_{-1}}{w} + O(1)\right](e^w - 1) = \left[\frac{c_{-1}}{w} + O(1)\right]\left[w + O(w^2)\right] =$$

$$= c_{-1} + O(w) \Longrightarrow c_{-1} = 2$$

$$\frac{e^z+1}{e^z-1} = \frac{2}{z - 2\pi k i} + O(1),\ z \to 2\pi k i,\ k \in \mathbb{Z}$$

3) $$\frac{z-1}{\sin^2 z} = \left\{ \begin{array}{l} z = 0 \text{ — полюс} \\ \text{2-го пор.} \end{array} \right\} = \frac{c_{-2}}{z^2} + \frac{c_{-1}}{z} + O(1)$$

$$z - 1 = \left(\frac{z-1}{\sin^2 z}\right)\sin^2 z = \left[\frac{c_{-2}}{z^2} + \frac{c_{-1}}{z} + O(1)\right]\left[z^2 + O(z^4)\right] = c_{-2} + c_{-1} z + O(z^2)$$

$$\Longrightarrow \left\{ \begin{array}{l} c_{-2} = -1 \\ c_{-1} = 1 \end{array} \right.$$

$$\frac{z-1}{\sin^2 z} = -\frac{1}{z^2} + \frac{1}{z} + O(1),\ z \to 0$$

4) $$\frac{e^{iz}}{z^2+b^2} = \left\{ \begin{array}{l} w = z - ib \\ z = w + ib \end{array} \right\} = \frac{e^{-b} e^{iw}}{w(w + i \cdot 2b)} = \left\{ \begin{array}{l} w = 0 \text{ — полюс} \\ \text{1-го пор.} \end{array} \right\} = \frac{c_{-1}}{w} + O(1)$$

$$e^{-b} e^{iw} = e^{-b} + O(w) = \left[i \cdot 2bw + O(w)\right]\left[\frac{c_{-1}}{w} + O(1)\right] = i \cdot 2b c_{-1} + O(w)$$

$$\Longrightarrow c_{-1} = -\frac{ie^{b}}{2b}$$

$$\frac{e^{iz}}{z^2+b^2} = \frac{-\frac{ie^{b}}{2b}}{z - ib} + O(1),\ z \to ib$$

5) $$\frac{(z^2+1)^2}{z^2+b^2} = \left\{ \begin{array}{l} w = \frac{1}{z} \\ z = \frac{1}{w} \end{array} \right\} = \frac{(w^2+1)^2}{w^2(b^2w^2+1)} = \left\{ \begin{array}{l} w=0 - \text{полюс} \\ 2\text{-го пор.} \end{array} \right\} = \frac{c_{-2}}{w^2} + \frac{c_{-1}}{w} + O(1)$$

$$(w^2+1)^2 = 1 + O(w^2) = \left[\frac{c_{-2}}{w^2} + \frac{c_{-1}}{w} + O(1)\right]\left[w^2 + O(w^4)\right] = c_{-2} + c_{-1}w + O(w^2)$$

$$\Longrightarrow \left\{ \begin{array}{l} c_{-2} = 1 \\ c_{-1} = 0 \end{array} \right.$$

$$\frac{(z^2+1)^2}{z^2+b^2} = z^2 + O(1), \quad z \to \infty$$

6) $$\frac{ze^{iz}}{(z^2+b^2)^2} = \left\{ \begin{array}{l} w = z - ib \\ z = w + ib \end{array} \right\} = \frac{(w+ib)e^{-b}e^{iw}}{w^2(w+i\cdot 2b)^2} = \left\{ \begin{array}{l} w=0 - \text{полюс} \\ 2\text{-го пор.} \end{array} \right\} = \frac{c_{-2}}{w^2} + \frac{c_{-1}}{w} + O(1)$$

$$(w+ib)e^{-b}e^{iw} = ibe^{-b} + (1-b)e^{-b}w + O(w^2) = \left[\frac{c_{-2}}{w^2} + \frac{c_{-1}}{w} + O(1)\right] \cdot$$

$$\cdot \left[-4b^2w^2 + i\cdot 4bw^3 + O(w^4)\right] = -4b^2c_{-2} + (i\cdot 4bc_{-2} - 4b^2c_{-1})w + O(w^2)$$

$$\Longrightarrow \left\{ \begin{array}{l} c_{-2} = -\frac{ie^{-b}}{4b} \\ c_{-1} = \frac{e^{-b}}{4b} \end{array} \right.$$

$$\frac{ze^{iz}}{(z^2+b^2)^2} = \frac{-\frac{ie^{-b}}{4b}}{(z-ib)^2} + \frac{\frac{e^{-b}}{4b}}{z-ib} + O(1), \quad z \to ib$$

7) $$\frac{z}{(z^2+b^2)^2} = \left\{ \begin{array}{l} w = z - ib \\ z = w + ib \end{array} \right\} = \frac{w+ib}{w^2(w+i\cdot 2b)^2} = \left\{ \begin{array}{l} w=0 - \text{полюс} \\ 2\text{-го пор.} \end{array} \right\} = \frac{c_{-2}}{w^2} + \frac{c_{-1}}{w} + O(1)$$

$$ib + w = -4b^2c_{-2} + (i\cdot 4bc_{-2} - 4b^2c_{-1})w + O(w^2) \Longrightarrow \left\{ \begin{array}{l} c_{-2} = -\frac{i}{4b} \\ c_{-1} = 0 \end{array} \right.$$

$$\frac{z}{(z^2+b^2)^2} = \frac{-\frac{i}{4b}}{(z-ib)^2} + O(1), \quad z \to ib$$

8) $$\cot \pi z = \left\{ \begin{array}{l} w = z - k,\ k \in \mathbb{Z} \\ z = w + k,\ k \in \mathbb{Z} \end{array} \right\} = \frac{\cos \pi w}{\sin \pi w} = \left\{ \begin{array}{l} w=0 - \text{полюс} \\ 1\text{-го пор.} \end{array} \right\} = \frac{c_{-1}}{w} + O(1)$$

$$\cos \pi w = 1 + O(w) = \cot \pi w \cdot \sin \pi w = \left[\frac{c_{-1}}{w} + O(1)\right]\left[\pi w + O(w^2)\right] \Longrightarrow c_{-1} = \frac{1}{\pi}$$

$$\cot \pi z = \frac{\frac{1}{\pi}}{z-k} + O(1), \quad z \to k,\ k \in \mathbb{Z}$$

9) $$\frac{1}{\sin \pi z} = \left\{ \begin{array}{l} w = z - k,\ k \in \mathbb{Z} \\ z = w + k,\ k \in \mathbb{Z} \end{array} \right\} = \frac{1}{(-1)^k \sin \pi w} = \left\{ \begin{array}{l} w=0 - \text{полюс} \\ 1\text{-го пор.} \end{array} \right\} = \frac{c_{-1}}{w} + O(1)$$

$$1 = \left[\frac{c_{-1}}{w} + O(1)\right]\left[(-1)^k \pi w + O(w^2)\right] = (-1)^k \pi c_{-1} + O(w) \Longrightarrow c_{-1} = \frac{(-1)^k}{\pi}$$

$$\frac{1}{\sin \pi z} = \frac{\frac{(-1)^k}{\pi}}{z-k} + O(1), \quad z \to k,\ k \in \mathbb{Z}$$