1 st Problem

$$it - t^2 = (-\operatorname{Im} t - \operatorname{Re}^2 t + \operatorname{Im}^2 t) + i(\operatorname{Re} t - 2\operatorname{Re} t \operatorname{Im} t)$$

$$\Longrightarrow \operatorname{Im}(it - t^2) = \text{const} \Longleftrightarrow \operatorname{Im} t = \frac{\text{const}}{\operatorname{Re} t} + \frac{1}{2}$$

$$\operatorname{Re} e^{\lambda(it-t^2)} = \operatorname{Re}\left[e^{\lambda \operatorname{Re}(it-t^2)} e^{i\lambda \operatorname{Im}(it-t^2)}\right] =$$

$$= e^{\lambda(-\operatorname{Im} t - \operatorname{Re}^2 t - \operatorname{Im}^2 t)} \cos\left[\lambda(\operatorname{Re} t - 2\operatorname{Re} t \operatorname{Im} t)\right]$$

Let's indicate which contours correspond to the dependencies:

Red: $\operatorname{Im} t \equiv 0,\ \operatorname{Re} t \in [-3,3] \Longrightarrow \operatorname{Re} e^{\lambda(it-t^2)} =$

$$= e^{-\lambda \operatorname{Re}^2 t} \cos(\lambda \operatorname{Re} t),\ \operatorname{Re} t \in [-3,3] \Longrightarrow 1^{st} \text{ image}$$

Green:
$$\begin{cases} \operatorname{Im} t = 0, & \operatorname{Re} t \in [-3,-2] \\ \operatorname{Re} t = -2, & \operatorname{Im} t \in [0, 1/2] \\ \operatorname{Im} t = 1/2, & \operatorname{Re} t \in (-2, 2] \\ \operatorname{Re} t = 2, & \operatorname{Im} t \in (1/2, 0) \\ \operatorname{Im} t = 0, & \operatorname{Re} t \in (2, 3] \end{cases}$$

$$\operatorname{Re} e^{\lambda(it-t^2)} = \begin{cases} e^{-\lambda \operatorname{Re}^2 t} \cos(\lambda \operatorname{Re} t), & t \in [-3,-2] \\ e^{\lambda(\operatorname{Im}^2 t - \operatorname{Im} t - 4)} \cos\left[2\lambda(1 - 2\operatorname{Im} t)\right], & \operatorname{Im} t \in [0, 1/2] \\ e^{-\lambda(\operatorname{Re}^2 t + 1/4)}, & \operatorname{Re} t \in (-2, 2] \\ e^{\lambda(\operatorname{Im}^2 t - \operatorname{Im} t - 4)} \cos\left[2\lambda(1 - 2\operatorname{Im} t)\right], & \operatorname{Im} t \in (1/2, 0) \\ e^{-\lambda \operatorname{Re}^2 t} \cos(\lambda \operatorname{Re} t), & \operatorname{Re} t \in (2, 3] \end{cases} \Longrightarrow 3^{rd} \text{ image}$$

Black:
$$\operatorname{Im} t = \begin{cases} 0, & \operatorname{Re} t \in [-3,-2] \\ \frac{1}{\operatorname{Re} t} + \frac{1}{2}, & \operatorname{Re} t \in (-2,-1] \\ -1/2, & \operatorname{Re} t \in (-1,1] \\ -\frac{1}{\operatorname{Re} t} + \frac{1}{2}, & \operatorname{Re} t \in (1,2] \\ 0, & \operatorname{Re} t \in (2,3] \end{cases}$$

$$\operatorname{Re} e^{\lambda(it+t^2)} = \begin{cases} e^{-\lambda \operatorname{Re}^2 t} \cos(\lambda \operatorname{Re} t), & \operatorname{Re} t \in [-3,-2] \\ e^{-\lambda(\operatorname{Re}^2 t + \frac{1}{4} + \operatorname{Re}^{-2} t)} \cos 2\lambda, & \operatorname{Re} t \in (-2,-1] \\ e^{-\lambda(\operatorname{Re}^2 t - 3/4)} \cos(2\lambda \operatorname{Re} t), & \operatorname{Re} t \in (-1,1] \\ e^{-\lambda(\operatorname{Re}^2 t + \frac{1}{4} - \operatorname{Re}^{-2} t)} \cos 2\lambda, & \operatorname{Re} t \in (1,2] \\ e^{-\lambda \operatorname{Re}^2 t} \cos(\lambda \operatorname{Re} t), & \operatorname{Re} t \in (2,3] \end{cases}$$

$$\Longrightarrow 2^{nd} \text{ image}$$

$\Rightarrow$ green contour is more suitable to evaluation of asymptotic of integral

$$I(\lambda) = \int_{-3}^{3} \frac{e^{\lambda(it-t^2)}}{1+t^2}\, dt \simeq \frac{4}{3} e^{-\lambda/4} \int_{-\infty}^{+\infty} e^{-\lambda(t-i/2)^2}\, dt$$

$$= \frac{4}{3} e^{-\lambda/4} \sqrt{\frac{\pi}{\lambda}}, \quad \lambda \to +\infty$$

2 nd Problem

$$Ai(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i(xt+\frac{t^3}{3})}\,dt$$

Let's find the directions in which the contour can go to infinity:

$$Re\left[i\left(xt+\frac{t^3}{3}\right)\right] = Re\left[x|t|e^{i(\varphi+\pi/2)} + \frac{|t|^3}{3}e^{i(3\varphi+\pi/2)}\right] =$$

$$= -x|t|\sin\varphi - \frac{|t|^3}{3}\sin 3\varphi$$

$\lim\limits_{|t|\to+\infty} Re\left[i\left(xt+\frac{t^3}{3}\right)\right]$ must be less than or equal to zero

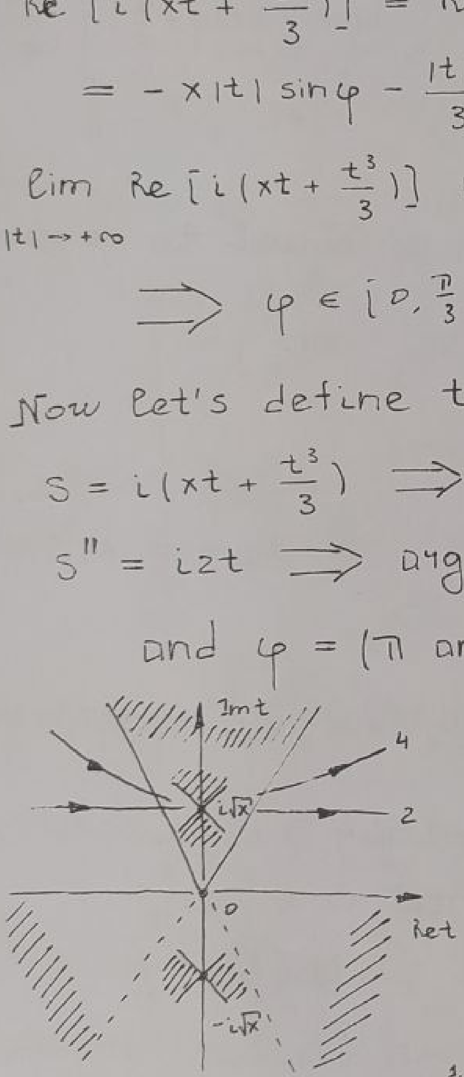
$$\Rightarrow \varphi \in \left[0,\frac{\pi}{3}\right] \cup \left[\frac{2\pi}{3},\pi\right] \cup \left(\frac{4\pi}{3},\frac{5\pi}{3}\right)$$

Now let's define the saddle points:

$$S = i\left(xt+\frac{t^3}{3}\right) \Rightarrow S' = i(x+t^2) \Rightarrow t_0 = \pm i\sqrt{x}$$

$$S'' = i2t \Rightarrow \arg S''(t_0) = \pi \text{ and } 0$$

and $\varphi = (\pi$ and $0)$ and $(-\pi/2$ and $\pi/2)$ respectively



* from here on the shaded areas are forbidden

Therefore, the suitable contours are 2 and 4.

Finally:

$$Ai(x) \sim \frac{1}{2\pi}e^{-\frac{2}{3}x^{3/2}}\int_{ix^{1/2}-\infty}^{ix^{1/2}+\infty} e^{-x^{1/2}(t-ix^{1/2})^2}\,dt = \left\{u = x^{1/4}(t-ix^{1/2})\right\} =$$

$$= \frac{1}{2\pi}x^{-1/4}e^{-\frac{2}{3}x^{3/2}}\int_{-\infty}^{+\infty} e^{-u^2}\,du = \frac{1}{2\sqrt{\pi}}x^{-1/4}e^{-\frac{2}{3}x^{3/2}}, \quad x\to+\infty$$

3 $\underline{rd}$ Problem

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i(xt+t^3/3)}dt, \; x \to -\infty \iff \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i(-|x|t+t^3/3)}dt, \; |x| \to +\infty$$

Let's find the directions in which the contour can go to infinity:

$$Re\left[i\left(-|x|t+\frac{t^3}{3}\right)\right] = Re\left[|x||t|e^{i(\varphi+\frac{3\pi}{2})} + \frac{|t|^3}{3}e^{i(3\varphi+\frac{\pi}{2})}\right] =$$

$$= -\frac{|t|^3}{3}\sin 3\varphi + |x||t|\sin\varphi$$

$\lim\limits_{|t|\to+\infty} Re\left[i\left(-|x|t+\frac{t^3}{3}\right)\right]$ must be less than or equal to zero

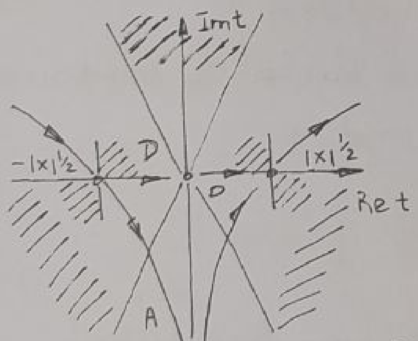
$$\Longrightarrow \varphi \in [0, \pi/3) \cup (2\pi/3, \pi] \cup [4\pi/3, 5\pi/3]$$

Now let's define the saddle points:

$$S = i(-|x|t+t^3/3) \Longrightarrow S' = -i|x| + it^2 \Longrightarrow t_0 = \pm|x|^{1/2}$$

$$S'' = i2t \Longrightarrow \arg S'' = \pi/2 \text{ and } 3\pi/2$$

and $\varphi = (-3\pi/4$ and $\pi/4)$ and $(-5\pi/4$ and $-\pi/4)$ respectively



Therefore, contour D is equivalent to the original one, and contour A is suitable for evaluation via Laplace method:

$$Ai(x) \sim \frac{1}{2\pi}\left(e^{-i\frac{2}{3}|x|^{3/2}}e^{i\pi/4}\sqrt{\pi}|x|^{-1/4} + \right.$$

$$\left. + e^{i\frac{2}{3}|x|^{3/2}}e^{-i\pi/4}\sqrt{\pi}|x|^{-1/4}\right) = \frac{|x|^{-1/4}}{\sqrt{\pi}}\cos\left(\frac{2}{3}|x|^{3/2} - \pi/4\right), \; x \to -\infty$$

4^{th} Problem

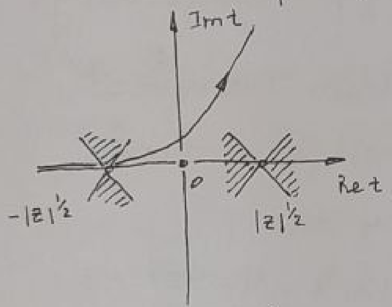
Contour can't be deformed to match the real axis since it can be shown by the method given earlier that the direction $t \to +\infty$ is forbidden.

(1) $f(z) = \int_{-\infty}^{e^{i\pi/3}\infty} e^{t^3/3 - zt}\,dt,\ z \to +\infty \iff f(z) = \int_{-\infty}^{e^{i\pi/3}\infty} e^{t^3/3 - |z|t}\,dt,\ |z| \to +\infty$

$S = t^3/3 - |z|t \Rightarrow S' = t^2 - |z| \Rightarrow t_0 = \pm |z|^{1/2}$

$S'' = 2t \Rightarrow \arg S'' = 0$ and π

and $\varphi = \left(-\frac{\pi}{2} \text{ and } \frac{\pi}{2}\right)$ and $(\pi \text{ and } 0)$ respectively



$$f(z) \simeq e^{\frac{2}{3}|z|^{3/2}} \int_{-|z|^{1/2}-\infty}^{-|z|^{1/2}+\infty} e^{-|z|^{1/2}(t+|z|^{1/2})^2}\,dt =$$

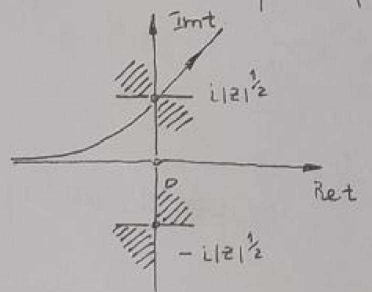
$$= \sqrt{\pi}\,|z|^{-1/4} e^{\frac{2}{3}|z|^{3/2}},\quad z \to +\infty$$

(2) $f(z) = \int_{-\infty}^{e^{i\pi/3}\infty} e^{t^3/3 - zt}\,dt,\ z \to -\infty \iff f(z) = \int_{-\infty}^{e^{i\pi/3}\infty} e^{t^3/3 + |z|t}\,dt,\ |z| \to +\infty$

$S = t^3/3 + |z|t \Rightarrow S' = t^2 + |z| \Rightarrow t_0 = \pm i|z|^{1/2}$

$S'' = 2t \Rightarrow \arg S'' = \frac{\pi}{2}$ and $-\frac{\pi}{2}$

and $\varphi = \left(-\frac{3\pi}{4} \text{ and } \frac{\pi}{4}\right)$ and $\left(-\frac{\pi}{4} \text{ and } \frac{3\pi}{4}\right)$ respectively



$$f(z) \sim e^{i\frac{2}{3}|z|^{3/2}} \int_{i|z|^{1/2}+e^{-i3\pi/4}\infty}^{i|z|^{1/2}+e^{i\pi/4}\infty} e^{i|z|^{1/2}(t - i|z|^{1/2})^2}\,dt =$$

$$= e^{i\pi/4}\sqrt{\pi}\,|z|^{1/4} e^{i\frac{2}{3}|z|^{3/2}},\quad z \to -\infty$$

5^{th} Problem

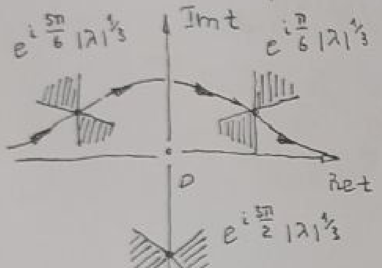
$I(\lambda) = \int\limits_{-\infty}^{+\infty} e^{-\frac{x^4}{4} + i\lambda x}\, dx$ defines the real function of parameter λ because $\operatorname{Im} I(\lambda) = \int\limits_{-\infty}^{+\infty} e^{-x^4/4} \sin \lambda x \, dx \equiv 0$ due to the fact that the integrand is an odd function.

(1.) $\int\limits_{-\infty}^{+\infty} e^{-x^4/4 + i\lambda x} dx, \ \lambda \to +\infty \iff \int\limits_{-\infty}^{+\infty} e^{-x^4/4 + i|\lambda| x} dx, \ |\lambda| \to +\infty$

$S = -\frac{x^4}{4} + i|\lambda| x \implies S' = -x^3 + i|\lambda| \implies x_0 \in \{ e^{i\frac{\pi}{6}}|\lambda|^{\frac{1}{3}}, e^{i\frac{5\pi}{6}}|\lambda|^{\frac{1}{3}}, e^{i\frac{3\pi}{2}}|\lambda|^{\frac{1}{3}} \}$

$S'' = -3x^2 \implies \arg S'' \in \{ 4\pi/3, 2\pi/3, 0 \}$

and $\varphi = (-\frac{\pi}{6}, \frac{5\pi}{6}), (-\frac{5\pi}{6}, \frac{\pi}{6})$ and $(-\frac{\pi}{2}, \frac{\pi}{2})$ respectively



The direction of the contour traversal is inverse to the defined directions of steepest descent for the saddle point $x_0 = e^{i\pi/6}|\lambda|^{1/3}$, therefore:

$\implies A + 4$

$$I(\lambda) \sim \sqrt{\frac{2\pi}{3}} |\lambda|^{-1/3} e^{\frac{3}{4} e^{i 4\pi/3} |\lambda|^{4/3} + i\frac{\pi}{6}} -$$

$$- \sqrt{\frac{2\pi}{3}} |\lambda|^{-1/3} e^{\frac{3}{4} e^{i 2\pi/3} |\lambda|^{4/3} + i 5\pi/6} =$$

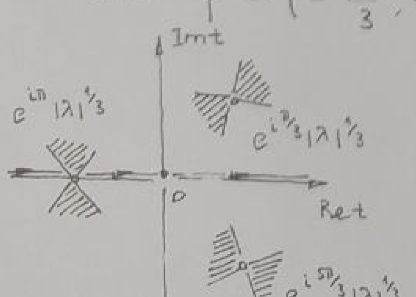
$$= \sqrt{\frac{2\pi}{3}} |\lambda|^{-1/3} e^{-\frac{3}{8}|\lambda|^{4/3}} \cos\left(\frac{3\sqrt{3}}{8} |\lambda|^{4/3} - \frac{\pi}{6} \right), \ \lambda \to +\infty$$

(2) $\int\limits_{-\infty}^{+\infty} e^{-x^4/4 + i\lambda x} dx, \ \lambda \to +i\infty \iff \int\limits_{-\infty}^{+\infty} e^{-x^4/4 - |\lambda| x} dx, \ |\lambda| \to +\infty$

$S = -\frac{x^4}{4} - |\lambda| x \implies S' = -x^3 - |\lambda| \implies x_0 \in \{ e^{i\pi/3}|\lambda|^{\frac{1}{3}}, e^{i\pi}|\lambda|^{\frac{1}{3}}, e^{i 5\pi/3}|\lambda|^{\frac{1}{3}} \}$

$S'' = -3x^2 \implies \arg S'' \in \{ \frac{5\pi}{3}, \pi, \frac{\pi}{3} \}$

and $\varphi = (-\frac{4\pi}{3}, -\frac{\pi}{3}), (\pi, 0)$ and $(-\frac{2\pi}{3}, \frac{\pi}{3})$ respectively



$$I(\lambda) \sim \sqrt{\frac{2\pi}{3}} |\lambda|^{-1/3} e^{\frac{3}{4}|\lambda|^{4/3}}, \ \lambda \to +i\infty$$

$\implies B + 7$

6 th Problem

$$I(\lambda) = \int_{-\infty}^{+\infty} e^{-\lambda(x^2-3ix)} F(x)\,dx$$

$$S = -x^2 + 3ix \implies S' = -2x + 3i \implies x_0 = i\,3/2$$

$$F(x) = \int_0^{+\infty} \frac{(1+ix)\,y^{ix}}{(1+y)^{2+2ix}} e^{-y}\,dy$$

$$\implies F(i\,3/2) = -\frac{1}{2}\int_0^{+\infty} \frac{1+y}{y^{3/2}} e^{-y}\,dy = -\frac{1}{2}\left(\int_0^{+\infty} y^{-\frac{1}{2}-1} e^{-y}\,dy + \int_0^{+\infty} y^{\frac{1}{2}-1} e^{-y}\,dy\right) =$$

$$= -\frac{\Gamma(1/2)+\Gamma(-1/2)}{2} = \sqrt{\pi}/2$$

Finally, $I(\lambda) \sim \frac{\pi}{2}\frac{1}{\sqrt{\lambda}} e^{-9/4\,\lambda}$, $\lambda \to +\infty$

The contour deformation we had in mind was in fact possible, since finding the analytical continuation of $F(x)$ we get:

$$F(x) = \int_0^{+\infty} (1+ix)\left[\frac{1}{(1+y)^{2+2ix}} - 1\right] y^{ix} e^{-y}\,dy + (1+ix)\int_0^{+\infty} y^{(1+ix)-1} e^{-y}\,dy,$$

where the 1st integral converges at $\operatorname{Im} x < 2$, and the 2nd is equal to $\Gamma(2+ix)$ that has simple poles at the points $x = i(2+n)$, $n \in \mathbb{N}$.

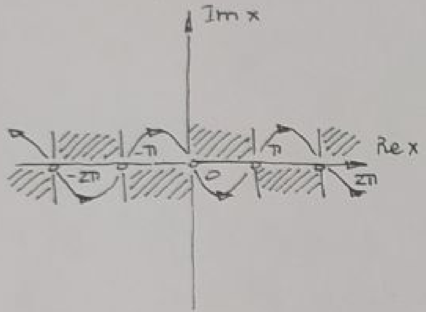
7 th Problem

$$I(\lambda) = \int_{-\infty}^{+\infty} \cos(\lambda \cos x) \frac{\sin x}{x} dx = \operatorname{Re} \int_{-\infty}^{+\infty} e^{\lambda i \cos x} \frac{\sin x}{x} dx, \ \lambda \to +\infty$$

$$S = i\cos x \implies S' = -i \sin x \implies x_0 = \pi k, \ k \in \mathbb{Z}$$

$$S'' = -i\cos x \implies \arg S'' = \begin{cases} 3\pi/2, & k = 2n, \ n \in \mathbb{Z} \\ \pi/2, & k = 2n+1, \ n \in \mathbb{Z} \end{cases}$$

and $\varphi = (-5\pi/4$ and $-\pi/4)$ and $(-3\pi/4$ and $\pi/4)$ respectively



$$I(\lambda) \simeq \operatorname{Re}\left[e^{i\lambda} e^{-i\pi/4} \sqrt{\frac{2\pi}{\lambda}}\right] =$$

$$= \sqrt{\frac{2\pi}{\lambda}} \cos\left(\lambda - \frac{\pi}{4}\right), \ \lambda \to +\infty$$

Let's find the subleading term of the asymptotics:

$\implies$ contour B

$$\operatorname{Re} \int_{-\infty}^{+\infty} e^{\lambda i \cos x} \frac{\sin x}{x} dx = \operatorname{Re} \sum_{k=-\infty}^{+\infty}{}' \int_{\pi k + e^{-i(-1)^k \pi/4 - \pi}\infty}^{\pi k + e^{-i(-1)^k \pi/4}\infty} e^{\lambda i \cos x} \frac{\sin x}{x} dx =$$

$$= \{\varepsilon = x - \pi k\} = \operatorname{Re} \sum_{k=-\infty}^{+\infty}{}' \int_{e^{-i(-1)^k \pi/4 - \pi}\infty}^{e^{-i(-1)^k \pi/4}\infty} e^{\lambda i (-1)^k \cos \varepsilon} \frac{(-1)^k \sin \varepsilon}{\varepsilon + \pi k} d\varepsilon =$$

$$= \{\varepsilon = e^{-i(-1)^k \pi/4} \rho\} = \operatorname{Re} \sum_{k=-\infty}^{+\infty}{}' e^{i(-1)^k \lambda} e^{-i(-1)^k \pi/4} \int_{-\infty}^{+\infty} e^{\lambda [i(-1)^k \cos(e^{-i(-1)^k \pi/4}\rho) - i(-1)^k]} \cdot$$

$$\cdot \frac{(-1)^k \sin(e^{-i(-1)^k \pi/4}\rho)}{e^{-i(-1)^k \pi/4}\rho + \pi k} d\rho = \begin{Bmatrix} -u^2 = i(-1)^k [\cos(e^{-i(-1)^k \pi/4}\rho) - 1] \\ \rho = \sqrt{2} u - \frac{\sqrt{2}}{12} i(-1)^k u^3 + O(u^3), \ u \to 0 \end{Bmatrix}$$

$$= \operatorname{Re}\left\{e^{i(\lambda - \pi/4)} \int_{-\infty}^{+\infty} e^{-\lambda u^2} \left\{1 - \frac{1/3}{2!} e^{-i\pi/2} \left[\sqrt{2} u - \frac{\sqrt{2}}{12} i u^3 + O(u^3)\right]^2 + O(u^2)\right\} \cdot \right.$$

$$\left. \cdot \left[\sqrt{2} - \frac{\sqrt{2}}{4} i u^2 + O(u^2)\right] du \right\} + \operatorname{Re} \sum_{k=-\infty}^{+\infty}{}' \left\{ e^{i(-1)^k (\lambda - \pi/4)} \int_{-\infty}^{+\infty} e^{-\lambda u^2} \cdot \right.$$

$$\cdot \left\{ \frac{(-1)^k}{\pi k} e^{-i(-1)^k \pi/4} \left[\sqrt{2} u - \frac{\sqrt{2}}{12} i (-1)^k u^3 + O(u^3)\right] - \frac{2(-1)^k}{(\pi k)^2} e^{-i(-1)^k \pi/2} \cdot \right.$$

$$\left. \cdot \left[\sqrt{2} u - \frac{\sqrt{2}}{12} i(-1)^k u^3 + O(u^3)\right]^2 + O(u^2) \right\} \left[\sqrt{2} - \frac{\sqrt{2}}{4} i (-1)^k u^2 + O(u^2)\right] du \Big\}$$

$$\sim \frac{1}{8} \sqrt{\frac{2\pi}{\lambda^3}} \sin(\lambda - \pi/4) + I_{lead}(\lambda), \ \lambda \to +\infty$$