1 st Problem.

1.1. $C = \int_0^{+\infty} \frac{\ln t}{t} \sin t \, dt = \left(\frac{d}{da} \int_0^{+\infty} t^{a-1} \sin t \, dt \right)\Big|_{a=0} = \left(\frac{d}{da} \operatorname{Im} \int_0^{+\infty} t^{a-1} e^{it} \, dt \right)\Big|_{a=0}$

$$\oint_{C_1} t^{a-1} e^{it} \, dt = F(a) - e^{i\pi a} \int_0^{+\infty} t^{a-1} e^{-it} \, dt$$

$\oint_{C_2} t^{a-1} e^{it} \, dt$ diverges because $\int_{C_R} t^{a-1} e^{it} \, dt \sim R^a e^R \xrightarrow[R \to +\infty]{} 0 \ \forall a \in \mathbb{R}$

$$\oint_{C_3} t^{a-1} e^{it} \, dt = F(a) - e^{i\frac{\pi}{2}a} \Gamma(a) = 0$$

Hence, $F(a) = e^{i\frac{\pi}{2}a} \Gamma(a)$ and $f(a) = \operatorname{Im} F(a) = \sin \frac{\pi}{2} a \cdot \Gamma(a)$

$$C = f'(0) = \frac{d}{da} \left[\sin \frac{\pi}{2} a \cdot \Gamma(a) \right]\Big|_{a=0} = \frac{d}{da} \left[\frac{\pi}{2} \frac{\sec \frac{\pi}{2} a}{\Gamma(1-a)} \right]\Big|_{a=0} =$$

$$= \left[\left(\frac{\pi}{2} \right)^2 \frac{\tan \frac{\pi}{2} a \cdot \sec \frac{\pi}{2} a}{\Gamma(1-a)} + \frac{\pi}{2} \psi(1-a) \sec \frac{\pi}{2} a \right]\Big|_{a=0} = -\frac{\pi \gamma}{2}$$

1.2. $\sin x \sim x, \ x \to 0 \implies I(\nu) = \int_0^{\pi/2} \sin^\nu x \, dx$ converges if $\operatorname{Re} \nu > -1$

$$I(\nu) = \int_0^{\pi/2} \sin^\nu x \, dx = (t = \sin x) = \int_0^1 \frac{t^\nu}{\sqrt{1-t^2}} \, dt = (u = t^2) =$$

$$= \frac{1}{2} \int_0^1 u^{\frac{\nu+1}{2} - 1} (1-u)^{\frac{1}{2} - 1} \, du = \frac{1}{2} B\left(\frac{\nu+1}{2}, \frac{1}{2} \right) = \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{\nu+1}{2} \right)}{\Gamma\left(\frac{\nu}{2} + 1 \right)}$$

1.3. $h(a) = \int_0^1 \frac{t^{a-1}}{1+t} \, dt = \int_0^1 t^{a-1} \sum_{n=0}^{+\infty} (-1)^n t^n \, dt = \sum_{n=0}^{+\infty} (-1)^n \int_0^1 t^{a+n-1} \, dt =$

$$= \sum_{n=0}^{+\infty} \frac{(-1)^n}{a+n} = \sum_{n=0}^{+\infty} \frac{1}{a+2n} - \sum_{n=0}^{+\infty} \frac{1}{a+2n+1} = \frac{1}{2} \left\{ \left[\sum_{n=0}^{+\infty} \left(\frac{1}{n+1} - \frac{1}{n + \frac{a+1}{2}} \right) - \gamma \right] - \right.$$

$$\left. - \left[\sum_{n=0}^{+\infty} \left(\frac{1}{n+1} - \frac{1}{n + \frac{a}{2}} \right) - \gamma \right] \right\} = \frac{1}{2} \left[\psi\left(\frac{a+1}{2} \right) - \psi\left(\frac{a}{2} \right) \right]$$

1.4. $M(a,b) = \int_0^1 \frac{t^{a-1} - t^{b-1}}{(1+t) \ln t} \, dt$ converges because $(t^{a-1} - t^{b-1})$ tends to 0 faster than $\ln t$ with $t \to 1$.

$\frac{\partial M}{\partial a} = \int_0^1 \frac{t^{a-1}}{1+t} \, dt = h(a)$ and similarly $\frac{\partial M}{\partial b} = -h(b)$

$$M(1,1) = \int_0^1 0 \, dt = 0 \implies M(a,b) = \int_1^a h(a) \, da + \int_1^b [-h(b)] \, db =$$

$$= \ln \Gamma\left(\frac{a+1}{2} \right) - \ln \Gamma\left(\frac{a}{2} \right) - \ln \Gamma\left(\frac{b+1}{2} \right) + \ln \Gamma\left(\frac{b}{2} \right) = \ln \frac{\Gamma\left(\frac{a+1}{2} \right) \Gamma\left(\frac{b}{2} \right)}{\Gamma\left(\frac{b+1}{2} \right) \Gamma\left(\frac{a}{2} \right)}$$

2 nd Problem

$$I(\alpha) = \frac{1}{\Gamma\left(\frac{\alpha+1}{2}\right)} \int_0^{+\infty} x^\alpha (1+x)^{2\alpha} (2+x)^{3\alpha} e^{-x} \, dx$$

$x^\alpha (1+x)^{2\alpha} (2+x)^{3\alpha} e^{-x} \sim 2^{3\alpha} x^\alpha$, $x \to 0$, from which it follows:

$$I(\alpha) = \frac{1}{\Gamma\left(\frac{\alpha+1}{2}\right)} \int_0^{+\infty} x^\alpha \left[(1+x)^{2\alpha} (2+x)^{3\alpha} - 2^{3\alpha} \right] e^{-x} \, dx + 2^{3\alpha} \frac{\Gamma(\alpha) \cdot \alpha}{\Gamma\left(\frac{\alpha+1}{2}\right)}$$

Further, we note the following:

$$\frac{\Gamma(\alpha)}{\Gamma\left(\frac{\alpha+1}{2}\right)} = \frac{\Gamma(\alpha)\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{\alpha+1}{2}\right)\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{1}{2}\right)} = \frac{1}{\sqrt{\pi}} \frac{\Gamma(\alpha)}{\Gamma\left(\frac{\alpha}{2}\right)} B\left(\frac{1}{2}, \frac{\alpha}{2}\right)$$

$$B\left(\frac{1}{2}, \frac{\alpha}{2}\right) = \int_0^1 x^{\frac{1}{2}-1} (1-x)^{\frac{\alpha}{2}-1} dx = (x = t^2) = 2 \int_0^1 (1-t^2)^{\frac{\alpha}{2}-1} dt =$$

$$= \int_{-1}^1 (1-t^2)^{\frac{\alpha}{2}-1} dt = \left(u = \frac{t+1}{2} \right) = 2^{\alpha-1} \int_0^1 u^{\frac{\alpha}{2}-1} (1-u)^{\frac{\alpha}{2}-1} du =$$

$$= 2^{\alpha-1} B\left(\frac{\alpha}{2}, \frac{\alpha}{2}\right) = 2^{\alpha-1} \frac{\Gamma^2\left(\frac{\alpha}{2}\right)}{\Gamma(\alpha)}$$

Hence, $2^{3\alpha} \frac{\Gamma(\alpha)\cdot\alpha}{\Gamma\left(\frac{\alpha+1}{2}\right)} = \frac{2^{4\alpha-1}}{\sqrt{\pi}} \alpha \Gamma\left(\frac{\alpha}{2}\right)$ and $I_{con}(-1) = -\frac{1}{32\sqrt{\pi}} \Gamma\left(-\frac{1}{2}\right) = +\frac{1}{16}$

Continuing to substract the counterterms, we obtain:

$$I_{con}(\alpha) = \frac{2^{4\alpha-1}}{\sqrt{\pi}} \alpha \Gamma\left(\frac{\alpha}{2}\right) \sum_{n=0}^{+\infty} \sum_{k=0}^{n} \frac{n!}{2^k} \binom{\alpha+n}{n} \binom{2\alpha}{n-k} \binom{3\alpha}{k},$$

where $\binom{\alpha}{n}$ are the generalized binomial coefficients:

$$\binom{\alpha}{n} = \begin{cases} 1, & n = 0, \\ \prod_{k=1}^{n} \frac{\alpha-k+1}{k}, & \text{otherwise}. \end{cases}$$

Therefore:

$$\operatorname*{Res}_{z=-2} I_{con}(z) = -\frac{1}{32\sqrt{\pi}} \operatorname*{Res}_{z=-2} \Gamma\left(\frac{z}{2}\right) = -\frac{1}{16\sqrt{\pi}} \operatorname*{Res}_{z=-1} \Gamma(z) = \frac{1}{16\sqrt{\pi}}$$

3 th Problem

$$G(in) = \sum_{k=1}^{+\infty} \left(\frac{1}{-a+ik+in} - \frac{1}{-a-ik+in} + \frac{2i}{k} \right) = i \left\{ \sum_{k=1}^{+\infty} \left[\frac{1}{k} - \frac{1}{k+(n+ia)} \right] + \right.$$

$$\left. + \sum_{k=1}^{+\infty} \left[\frac{1}{k} - \frac{1}{k-(n+ia)} \right] - 2\gamma + 2\gamma \right\} = i \left[\psi(n+ia+1) + \psi(-n-ia+1) + 2\gamma \right]$$

$$in \to z \implies G(in) \to G(z) = i \left\{ \psi[1+i(z-a)] + \psi[1-i(z-a)] + 2\gamma \right\}$$

Since the poles of $\psi[1+i(z-a)]$ are $z_n = a + in$, $n \in \mathbb{N}$, then this is the source of undesired poles.

$$\psi(1-x) = \psi(x) + \pi \cot \pi x \implies \psi[1-(n+ia)] = \psi(n+ia) + \pi \cot \pi(n+ia) =$$

$$= \psi(n+ia) - i\pi \coth \pi a$$

$$in \to z \implies G(in) \to G(z) = i \left\{ 2\gamma + \psi[1+i(a-z)] + \psi[i(a-z)] - \pi i \coth \pi a \right\}$$

4^{th} Problem. Dirichlet Beta function

$$L(1) = \int_0^{+\infty} \frac{e^{-x}}{1+e^{-2x}}dx = (t = e^x) = \int_1^{+\infty} \frac{dt}{t^2+1} = \operatorname{atan} t \Big|_1^{+\infty} = \frac{\pi}{4}$$

$$L(0) = \lim_{z\to 0+0} \frac{1}{\Gamma(z)} \int_0^{+\infty} \frac{x^{z-1}e^{-x}}{1+e^{-2x}}dx = \lim_{z\to 0+0} \frac{\frac{1}{z}\frac{x^z e^{-x}}{1+e^{-2x}}\Big|_0^{+\infty} - \frac{1}{z}\int_0^{+\infty} x^z\left(\frac{e^{-x}}{1+e^{-2x}}\right)' dx}{\Gamma(z)} =$$

$$= \lim_{z\to 0+0} \frac{-\frac{1}{z}\int_0^{+\infty} x^z\left(\frac{e^{-x}}{1+e^{-2x}}\right)' dx}{\frac{1}{z}+o(1)} = -\frac{e^{-x}}{1+e^{-2x}}\Big|_0^{+\infty} = \frac{1}{2}$$

$$L(z) = \frac{1}{\Gamma(z)}\int_0^{+\infty} x^{z-1}e^{-x}\sum_{n=0}^{+\infty}(-1)^n e^{-2nx}dx = \frac{1}{\Gamma(z)}\sum_{n=0}^{+\infty}(-1)^n\int_0^{+\infty} x^{z-1}e^{-(2n+1)x}dx =$$

$$= (t = (2n+1)x) = \frac{1}{\Gamma(z)}\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n+1)^z}\int_0^{+\infty} t^{z-1}e^{-t}dt = \sum_{n=0}^{+\infty}(-1)^n(2n+1)^{-z}$$

$$S = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \sum_{n=0}^{+\infty}(-1)^n(2n+1)^{-1} = L(1) = \frac{\pi}{4}$$

$$L'(z) = \sum_{n=0}^{+\infty}(-1)^{n+1}(2n+1)^{-z}\ln(2n+1) = \ln\prod_{n=0}^{+\infty} e^{(-1)^{n+1}(2n+1)^{-z}\ln(2n+1)}$$

Since the term $c_0 = 0$, adding the subsequent adjacent terms in pairs and renumbering them we get:

$$c_n = \frac{\ln(4n+3)}{(4n+3)^z} - \frac{\ln(4n+5)}{(4n+5)^z}$$

$$F_1 = \ln\prod_{n=0}^{N-1}\frac{4n+3}{4n+5} = \ln\prod_{n=0}^{N-1}\frac{n+3/4}{n+5/4} = \ln\left[N^{-1/2}\frac{(N-1)!\,N^{5/4}}{\prod_{n=0}^{N-1}(n+5/4)} \Big/ \frac{(N-1)!\,N^{3/4}}{\prod_{n=0}^{N-1}(n+3/4)}\right] \simeq$$

$$\simeq \ln\left[\frac{\Gamma(5/4)}{\Gamma(3/4)}N^{-1/2}\right], \ N \gg 1$$

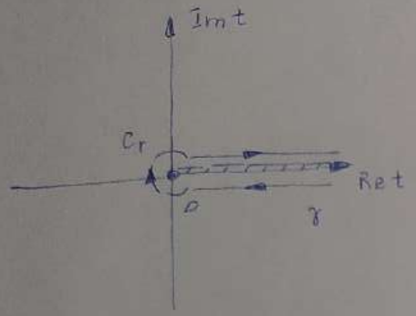
$$F_2 = \int_N^{+\infty}\left[\frac{\ln(4n+3)}{(4n+3)^z} - \frac{\ln(4n+5)}{(4n+5)^z}\right]dn = \frac{1}{4}\int_{4N+3}^{4N+5}\frac{\ln x}{x^z}dx = \frac{1}{4}\frac{(z-1)\ln x+1}{(z-1)^2 t^{z-1}}\Bigg|_{4N+5}^{4N+3} \xrightarrow[z\to 0+0]{}$$

$$\xrightarrow[z\to 0+0]{} \frac{1}{4}\left[(4N+5)\ln(4N+5) - (4N+3)\ln(4N+3) - 2\right] = \frac{1}{2}\ln N + \ln 2 +$$

$$+ \sum_{n=1}^{+\infty}(-1)^{n+1}\frac{5^{n+1}-3^{n+1}}{n(n+1)4^{n+1}}N^{-n}, \ N \gg 1 \simeq \ln 2 - \frac{1}{2}\ln\frac{1}{N}, \ N \gg 1$$

$$L'(0) = F_1 + F_2 = \ln\frac{\Gamma(5/4)}{\Gamma(3/4)} + \frac{1}{2}\ln\frac{1}{N} + \ln 2 - \frac{1}{2}\ln\frac{1}{N} = \ln\frac{2\Gamma(5/4)}{\Gamma(3/4)}$$

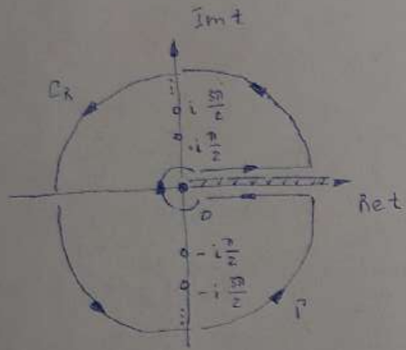
5 th Problem. Analytic continuation: Dirichlet beta function



$$\frac{1}{\Gamma(z)} \oint_{\gamma} \frac{e^{-t}\, t^{z-1}}{1+e^{-2t}}\, dt = \left(1-e^{i2\pi z}\right) L(z),$$

because $\displaystyle\int_{C_r} \frac{e^{-t}\, t^{z-1}}{1+e^{-2t}}\, dt \sim r^{z} \xrightarrow[r\to 0]{} 0,\ \operatorname{Re} z > 0.$

Hence: $$L(z) = \frac{1}{1-e^{i2\pi z}} \frac{1}{\Gamma(z)} \oint_{\gamma} \frac{e^{-t}\, t^{z-1}}{1+e^{-2t}}\, dt$$



$$\frac{1}{\Gamma(z)} \oint_{\Gamma} \frac{e^{-t}\, t^{z-1}}{1+e^{-2t}}\, dt = \left(1-e^{i2\pi z}\right) L(z),$$

because $\displaystyle\int_{C_R} \frac{e^{-t}\, t^{z-1}}{1+e^{-2t}}\, dt \sim R^{z} e^{-R} \xrightarrow[R\to +\infty]{} 0 \ \forall\, z \in \mathbb{C}.$

Hence: $$L(z) = \frac{1}{1-e^{i2\pi z}} \frac{1}{\Gamma(z)}\, 2\pi i \sum_{n=-\infty}^{+\infty} \operatorname*{Res}_{t=i\frac{\pi}{2}(2n+1)} \frac{e^{-t}\, t^{z-1}}{1+e^{-2t}}$$

$$\operatorname*{Res}_{\substack{t=i\frac{\pi}{2}(2n+1)\\ n\in\mathbb{N}}} \frac{e^{-t}\, t^{z-1}}{1+e^{-2t}} = \left(-\frac{e^{-t}\, t^{z-1}}{2e^{-2t}}\right)\Bigg|_{\substack{t=\frac{\pi}{2}(2n+1)e^{i\frac{\pi}{2}}\\ n\in\mathbb{N}}} = \frac{e^{-i\frac{\pi}{2}}}{2i} \left(\frac{\pi}{2}\right)^{z-1} (-1)^n (2n+1)^{z-1} e^{i\frac{\pi}{2}z}$$

$$\operatorname*{Res}_{\substack{t=-i\frac{\pi}{2}(2n+1)\\ n\in\mathbb{N}}} \frac{e^{-t}\, t^{z-1}}{1+e^{-2t}} = \left(-\frac{e^{-t}\, t^{z-1}}{2e^{-2t}}\right)\Bigg|_{\substack{t=\frac{\pi}{2}(2n+1)e^{i\frac{3\pi}{2}}\\ n\in\mathbb{N}}} = \frac{e^{-i\frac{\pi}{2}}}{2i} \left(\frac{\pi}{2}\right)^{z-1} (-1)^n (2n+1)^{z-1} e^{i\frac{3\pi}{2}z}$$

$$L(z) = \frac{1}{1-e^{i2\pi z}} \frac{1}{\Gamma(z)}\, 2\pi i\, \frac{e^{-i\frac{\pi}{2}}}{2i} \left(\frac{\pi}{2}\right)^{z-1} \left(e^{i\frac{\pi}{2}z} + e^{i\frac{3\pi}{2}z}\right) \sum_{n=0}^{+\infty} (-1)^n (2n+1)^{-(1-z)} =$$

$$= \left(\frac{\pi}{2}\right)^{z} \csc\frac{\pi}{2}z \cdot \frac{L(1-z)}{\Gamma(z)} \implies L(1-z) = \left(\frac{\pi}{2}\right)^{-z} \sin\frac{\pi}{2}z \cdot \Gamma(z)\, L(z)$$

$$L(-2k-1) = L[1-2(k+1)] = \left(\frac{\pi}{2}\right)^{-2(k+1)} \sin\pi(k+1)\, \Gamma[2(k+1)]\, L[2(k+1)] = 0,\ k\in\mathbb{N}.$$

6 $\underline{th}$ Problem.

$$L(z) = \frac{1}{2}\frac{1}{\Gamma(z)}\int_0^{+\infty}\frac{x^{z-1}}{\cosh x}\,dx \implies 2\frac{d}{dz}\Gamma(z)L(z) = \int_0^{+\infty}\frac{x^{z-1}\ln x}{\cosh x}\,dx$$

Therefore, $C = \int_0^{+\infty}\frac{\ln x\,dx}{\cosh x} = 2\frac{d}{dz}\left[\Gamma(z)L(z)\right]\Big|_{z=1}$

$$C = 2\frac{d}{dz}\left[\left(\frac{\pi}{2}\right)^z \csc\frac{\pi z}{2}\, L(1-z)\right]\Big|_{z=1} =$$

$$= 2\left\{\left[\left(\frac{\pi}{2}\right)^z \ln\frac{\pi}{2}\csc\frac{\pi z}{2} - \left(\frac{\pi}{2}\right)^z \cot\frac{\pi z}{2}\csc\frac{\pi z}{2}\cdot\frac{\pi}{2}\right]L(1-z) - \left(\frac{\pi}{2}\right)^z\csc\frac{\pi z}{2}\,L'(1-z)\right\}\Big|_{z=1} =$$

$$= \pi\ln\frac{\pi}{2}\,L(0) - \pi L'(0) = \pi\ln\frac{\Gamma^2\left(\frac{3}{4}\right)}{\sqrt{\pi}}$$