Math 741 Midterm 1

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Math 741 - Abstract Algebra

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Author's Note: Through the process of slaving away many hours over a hot notebook, I believe I have answered every question to at least some degree of correctness. Numerous times throughout this exam I reference code and programs to compute certain results. In order to reduce paper waste, rather than including all of the code in this printout, all of the code can be found at `https://github.com/eschlossberg/algebra741-midterm1`. This repository will be private in the interest of academic integrity up until Tuesday, October 9th 2018 at around 1:00PM when the exam is due. Afterwards, the repository will go public at the soonest possible opportunity so the code and computations can be examined and critiqued for correctness.

Exercise 1. Let X be a set, and suppose $R \subseteq X \times X$ is a relation. Define

$$\overline{R} = \bigcap_E E,$$

where the intersection ranges over all equivalence relations $E \subseteq X \times X$ with $R \subseteq E$. Prove that $R \subseteq \overline{R}$ and that $\overline{R}$ is an equivalence relation. If $X = S_5$ and $R = \{(x, y) \in S_5 \times S_5 : xy = yx\}$, find the cardinality of $X/\overline{R}$.

Proof. First, to see that $R \subseteq \overline{R}$, let $(x, y) \in R$. Then for each equivalence relation E containing R, we have $(x, y) \in E$. Hence $(x, y) \in \bigcap E = \overline{R}$.

Now if $(x, y) \in \overline{R}$, then for every equivalence relation E containing R, we have $(x, y) \in E$. Since E is an equivalence relation, we have $(y, x) \in E$ as well, so $(y, x) \in \overline{R}$ and hence $\overline{R}$ is symmetric. Similarly, for each $x \in X$, we have $(x, x) \in E$ for all E, so $(x, x) \in \overline{R}$, and hence $\overline{R}$ is reflexive. Finally, if $(x, y), (y, z) \in \overline{R}$ then $(x, y), (y, z) \in E$ for each E, and thus $(x, z) \in E$, so $(x, z) \in \overline{R}$, concluding the proof that $\overline{R}$ is an equivalence relation.

Now in the program `Exercise1.py`, we define a graph `G` where the nodes are permutations in S_5 and the edges are elements of the relation R. Finding the cardinality of $S_5/\overline{R}$ is equivalent to finding the number of connected components in this graph, since each equivalence class can be considered as a connected component in the graph. Running the computation yields an answer of 1 connected component, hence $|S_5/\overline{R}| = 1$. This result can be explained since for each $\sigma \in S_5$ we have $(1)\sigma = \sigma(1)$. Since $\overline{R}$ is an equivalence relation we thus have $\sigma \sim \tau$ for each $\tau \in S_5$.

Natural Number: 1 ∎

Exercise 2. Let G be the subgroup of S_4 generated by the permutations (123) and (234). Compute a list of right G-sets $(X_1)_G, (X_2)_G, \dots, (X_k)_G$ so that any transitive right G-set Y_G has $Y_S G \cong (X_i)_G$ for exactly one $i \in \{1, \dots, k\}$. Up to isomorphism, how many right G-sets are there with exactly 12 elements?

Proof. In the program `Exercise2.py` it is computed that the group $G = \langle(123), (234) >\subseteq S_4$ is precisely the alternating group A_4. In lecture it was shown that any transitive right G-set X_G is isomorphic to $\mathrm{Stab}(x)\backslash G$ for some $x \in X$. Now for any subgroup $H \leq G$, we will take for granted that we can define an action on some set X such that $H = \mathrm{Stab}(x)$ for some $x \in X$. Thus it suffices to consider $(X_G)_i H\backslash A_4$ for some $H \leq A_4$. The following list of subgroups of A_4, grouped by equivalence up to inner automorphism, was taken from [1]:

1) $H_1 = \{(1)\}$
2) $H_2 = A_4$
3) $H_3\langle(123)\rangle$, $\langle(234)\rangle$, $\langle(134)\rangle$, $\langle(124)\rangle$
4) $H_4 = \langle(12)(34)\rangle$, $\langle(13)(24)\rangle$, $\langle(14)(23)\rangle$
5) $H_5 = \langle(12)(34), (13)(24), (14)(23)\rangle$.

Since the subgroups in a line are isomorphic, we may consider a single representative of each line. This yields the following list, calculated in `Exercise2.py`, that describes every transitive right A_4-set up to isomorphism:

1) $X_1 = H_1\backslash A_4 = A_4$
2) $X_2 = H_2\backslash A_4 = (1)$
3) $X_3 = H_3\backslash A_4 = \{\overline{(1)}, \overline{(234)}, \overline{(143)}, \overline{(243)}\}$
4) $X_4 = H_4\backslash A_4 = \{\overline{(1)}, \overline{(123)}, \overline{(132)}, \overline{(134)}, \overline{(234)}, \overline{(14)(23)}\}$
5) $X_5 = H_5\backslash A_4 = \{\overline{(1)}, \overline{(123)}, \overline{(132)}\}$.

From the classification of right G-sets proved in lecture, every right G-set is of the form $\coprod H_i\backslash G$, where each H_i is a subgroup of G. Using this and the classification of transitive right A_4-sets just computed, we thus enumerate all right A_4-sets of order 12 up to isomorphism:

1) X_1
2) $\coprod_{i\leq 12} x_2$
3) $\coprod_{i\leq 3} X_3$
4) $X_4 \coprod X_4$
5) $\coprod_{i\leq 4} X_5$
6) $X_3 \coprod X_3 \coprod_{i\leq 4} X_2$
7) $X_3 \coprod X_3 \coprod X_5 \coprod X_2$
8) $X_3 \coprod_{i\leq 8} X_2$
9) $X_3 \coprod X_5 \coprod_{i\leq 5} X_2$
10) $X_3 \coprod_{i\leq 2} X_2 \coprod_{i\leq 2} X_5$
11) $X_4 \coprod X_3 \coprod_{i\leq 2} X_2$
12) $X_4 \coprod_{i\leq 6} X_2$
13) $X_4 \coprod X_5 \coprod X_5$

14) $X_4 \coprod X_5 \coprod_{i\leq 3} X_2$

15) $\coprod_{i\leq 4} X_5$

16) $\coprod_{i\leq 3} X_5 \coprod_{i\leq 3} X_2$

17) $\coprod_{i\leq 2} X_5 \coprod_{i\leq 6} X_2$

18) $X_5 \coprod_{i\leq 9} X_2$

Natural Number: 18 ☐

Exercise 3. Let G be the finitely presented group $\langle x, y | yxy = x\rangle$. Prove that there are infinitely many group homomorphisms $G \to GL_2(\mathbb{Z})$. How many group homomorphisms are there $G \to GL_2(\mathbb{Z}/3)$?

Proof. Consider the family of maps $G \xrightarrow{\phi_a} \mathrm{GL}_2(\mathbb{Z})$ indexed by $a \in \mathbb{Z}$ such that $x \mapsto \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix}$ and $y \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. We claim that each ϕ_a is a group homomorphism. Indeed, we have

$$\phi_a(yxy) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix} = \phi_a(x)$$

as required. Now suppose $g, h \in G$ such that gh contains n multiples of x. Since $y \mapsto I$, we have $\phi_a(gh) = \phi_a(x)^n$, and using basic properties of the identity we may insert enough copies of I into $\phi_a(x)^n$ such that $\phi_a(x)^n = \phi_a(g)\phi_a(h)$, thus proving that ϕ_a is an infinite family of group homomorphisms $G \to \mathrm{GL}_2(\mathbb{Z})$.

Since any homomorphism can be uniquely determined by defining the maps of the generators, in order to determine $|\mathrm{Hom}(G, \mathrm{GL}_2(\mathbb{Z}/3)|$ it suffices to find all pairs of matrices $(A, B) \in \mathrm{GL}_2(\mathbb{Z}/3)$ such that $BAB = A$. Such a computation was performed in `Exercise3.py`, in which it was determined that there are precisely 288 pairs of matrices satisfying that condition, and thus 288 homomorphisms from G to $\mathrm{GL}_2(\mathbb{Z}/3)$.

Natural Number: 288 ☐

Exercise 4. Let $\mathrm{Hom}([6], [2])$ be the set of functions from $[6] = \{1, 2, 3, 4, 5, 6\}$ to $[2] = \{1, 2\}$, considered as an (S_6, S_2)-biset. Let $G \leq S_6$ be the subgroup generated by the permutations (123456) and $(12)(36)(45)$. Define a set

$$N = * \times_G S_6 \times_{S_6} \mathrm{Hom}([6], [2]) \times_{S_2} *.$$

What is the cardinality of N? You may assume that the compatible products associate. Hint: the letter N stands for "necklace."

Proof. Since the compatible product is assumed to be associative, we begin by breaking down individual components of the product to determine the structure of N. By exercise 8 in homework 3, we have

$$\mathrm{Hom}([6], [2]) \times_{S_2} * \cong \mathrm{Hom}([6], [2])/S_2. \tag{1}$$

Similarly, by exercise 6 in homework 3 we have

$${}_G S_6 \times_{S_6} \mathrm{Hom}([6], [2]) \cong {}_G\mathrm{Hom}([6], [2]),$$

and finally by exercise 4 in the same homework we conclude

$$* \times_G \mathrm{Hom}([6],[2]) \cong G\backslash \mathrm{Hom}([6],[2]). \tag{2}$$

Now by combining (1) and (2), we get the apparent result that

$$N \cong G\backslash \mathrm{Hom}([6],[2])/S_2.$$

Now it may be vain to attempt to discuss the specific structure of N, but we may discuss some of the substructure imposed on it by each quotient. First, the right quotient by S_2 imposes the condition that any function f may be identified with its "mirror" $\bar{f}$, that is to say if $f(x) = 1$ then $\bar{f}(x) = 2$ and vice versa. Hence the quotient by S_2 halves the size of $\mathrm{Hom}([6],[2])$, so $|N| \leq 32$.

The quotient by G is significantly more complicated to describe, but can be simplified when considering each generator of G. The generator $(12)(36)(45)$ generates a subgroup of order 2 and permutes every element in $[6]$, so taking the quotient by $\langle(12)(36)(45)\rangle$ identifies all maps f, g such that $f(1) = g(2)$, $f(3) = g(6)$ and $f(4) = g(5)$, effectively halving the size of $\mathrm{Hom}([6],[2])$. The quotient by the generator (123456) is somewhat more subtle. This identifies all functions in the orbit of a function f. At first glance one would assume that this divides the size of $\mathrm{Hom}([6],[2])$ by 6, but this is not the case. Consider for instance the function f such that

$$\begin{array}{ccc} 1 \mapsto 1 & 2 \mapsto 2 & 3 \mapsto 1 \\ 4 \mapsto 2 & 5 \mapsto 1 & 6 \mapsto 2. \end{array}$$

The orbit of this function when acted upon by (123456) is 2, and they are mirror images of each other. Now note that the quotient by the other generator, $(12)(36)(45)$, as well as the quotient by S_2 identifies f and its mirror as well, so when considering the whole set N it is important to consider the overlap in equivalence relations. It was at this discovery that the intrepid author was severely deterred from continuing to manually compute this structure. The result of this deterrence is thus the existence of the program `Exercise4.py`, in which a final calculation revealed that $|N| = 10$.

Natural Number: 10 □

Exercise 5. Define a graph with vertex set $V = \{0,1\}^3$, and with undirected edges

$$E = \{\{(x_1,x_2,x_3),(y_1,y_2,y_3)\} : \sum_{i=1}^{3} |x_i - y_i| = 1\}.$$

Write $\mathrm{Sym}(V)$ for the symmetric group on the vertices, so that $\mathrm{Sym}(V) \cong S_8$ and V is a left $\mathrm{Sym}(V)$-set. Let $G \leq \mathrm{Sym}(V)$ be the subgroup of consisting of those permutations that send edges to edges. Let $\binom{E}{2}$ be the set of unordered pairs of distinct edges. Prove that there are no G-equivariant maps $V \to \binom{E}{2}$. What is the cardinality of ${}_G\mathrm{Hom}(\binom{E}{2}, V)$?

Proof. The group G, among other components of this problem, are calculated in the program `Exercise5.py`. From this the result is obtained that $|G| = 48$, so we will not attempt to enumerate all of G. We do however name one specific element of G for the sake of our argument: $\sigma = (235)(476)$. The other elements of G can be seen by running the specified program. Now for each vertex $v = (x_1, x_2, x_3) \in V$, we identify it with it's image under the mapping $v \mapsto 1 + x_1 + 2x_2 + 4x_3$ so as to easier describe the action of the permutations using conventional notation.

Suppose briefly that $\{\{v_1, v_2\}, \{v_3, v_4\}\} \in \binom{E}{2}$. Now $v_1 \neq v_2$ and $v_3 \neq v_4$, so there are at least 2 distinct vertices. Suppose then without loss of generality that $v_1 = v_3$. If it was the case that $v_2 = v_4$ then we would have $\{v_1, v_2\} = \{v_3, v_4\}$, but this is impossible by the definition of $\binom{E}{2}$. Thus in every element of $\binom{E}{2}$ there are at least 3 distinct vertices.

Now σ as defined before permutes all but two elements of V, so consider one of the fixed points, 1 (considering the earlier identification). Suppose $f \in_G \mathrm{Hom}(V, \binom{E}{2})$ such that $f(1) = \{\{v_1, v_2\}, \{v_3, v_4\}\}$. By our earlier argument, there are at least three different vertices in $f(1)$, but there are only two points fixed by σ. This plus the fact that $|\sigma| = 3$ implies $\sigma f(1) \neq f(1)$, but $f(\sigma 1) = f(1)$. Since it was assumed that f was a map of G-sets, we have arrived at a contradiction. Therefore there are no G-equivariant maps $V \to \binom{E}{2}$.

In order to compute the number of G-equivariant maps $\binom{E}{2} \to V$, we use the basic notion from group theory that any map must preserve orbits. Note that the orbit of any element of V under the action of G is the entire set V. Since it suffices to examine orbits, we have thus stumbled upon a simple algorithm for computing the number of possible mappings:

```
product = 1     // the total number of viable mappings, initialized to 1
foreach orbit do
    rep = a representative of the orbit
    valid_mappings = 0  // number of valid maps for the orbit
    foreach vertex v in V do
        // the mapping is represented as a hashmap
        mapping = {rep: v}
        valid_map = true
        foreach g in G do
            p_rep = g.rep
            p_vert = g.v
            if p_rep in mapping.keys and mapping[p_rep] is not p_vert:
                // the mapping is invalid as a $G$–map
                try a mapping with a different start vertex
        if the mapping was valid
            increment valid_mappings
    product *= valid_mappings
return product
```

This pseudocode was implemented in the program `Exercise5.py`. Running that code yields the fact that there are no G-equivariant mappings $\binom{E}{2} \to V$. Despite many hours of effort, a reasonable explanation for this phenomenon has yet to reveal itself. It can be conjectured that this relates specifically to the structure of the orbits. Consider a fixed permutation σ, and let $(e_1, e_2) \in \binom{E}{2}$ and $v \in V$. If the orbits of (e_1, e_2) and v under the action of the subgroup generated by σ have coprime orders, then we would find ourselves in a similar situation to the present. We may surmise that this is the case, however future investigations into this subject must be left for a later date.

Natural Number: 0 $\square$

References

[1] Vipul Naik. *Subgroup structure of alternating group:A4*. 2013. URL: https://groupprops.subwiki.org/wiki/Subgroup_structure_of_alternating_group:A4.