1 Introduction

1.1 Events [2.2]

(1) The result u_1 of a random trial is called **outcome**.
(2) The set of all possible outcomes is called **sample space** Ω.
(3) A collection of outcomes A is called **event**.

Thus an event is a subset of the sample space (or possibly the whole sample space).

The events $A_1, ..., A_n$ are said to be **mutually exclusive** (or **mutually disjoint**) if all pairs A_i, A_j are exclusive; that is, if it is impossible that two or more of these events take place at the same time.

1.2 Sample spaces [2.2]

If the number of outcomes is finite or denumerably infinite, Ω is said to be a **discrete sample space**. More particularly, if the number is finite, Ω is said to be a **finite sample space**.

If the number of outcomes is neither finite nor denumerably infinite, Ω is said to be a **continuous sample space**.

1.3 Kolmogorov's System of Axioms [2.3]

The following axioms for the probabilities $P(\cdot)$ should be fulfilled:

A1. If A is any event, then $0 \leq P(A) \leq 1$.
A2. If Ω is the entire sample space, then $P(\Omega) = 1$.
A3. ("Addition Formula"). If A and B are mutually exclusive events, then $P(A \cup B) = P(A) + P(B)$.

The sample space Ω and the probabilities $P(\cdot)$ together constitute a **probability space**.

1.4 Results from Kolmogorov's system [2.3]

Complement Theorem

$$P(A*) = 1 - P(A) \tag{1}$$

$$P(\varnothing) = 0 \tag{2}$$

Addition Theorem

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) + P(A \cap B) \\ P(A \cup B \cup C) &= P(A) + P(B) + P(C) - P(A \cap B) \\ &\quad - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C) \end{aligned} \tag{3}$$

Boole's Inequality

$$\begin{aligned} P(A \cup B) &\leq P(A) + P(B) \\ P(A \cup B \cup C) &\leq P(A) + P(B \cup C) \\ &\leq P(A) + P(B) + P(C) \end{aligned} \tag{4}$$

1.5 Probabilities in Discrete Sample Spaces [2.4]

$$P(u_i) = 1/m \quad (i = 1, 2, \ldots, m) \tag{5}$$

If (5), we have a **uniform probability distribution**.

An event A with g outcomes, has g **favourable cases**. That means there are m **possible cases** in Ω. We then determine $P(A)$ from the addition formula by adding the probabilities m of the g outcomes. Hence,

$$P(A) = g/m \tag{6}$$

1.6 Urn model [2.4]

Let's examine the probability of drawing exactly k white balls from an urn containing a white and b black balls, n balls are drawn.

If one ball at a time is drawn at random without returning it to the urn, the drawings are said to be made **without replacement**. The probability is then:

$$\frac{\binom{a}{k}\binom{b}{n-k}}{\binom{a+b}{n}} \tag{7}$$

If one ball at a time is drawn at random and is returned before the next ball is drawn, the drawings are said to be made **with replacement**. The probability is then:

$$\binom{n}{k} p^k q^{n-k} \tag{8}$$

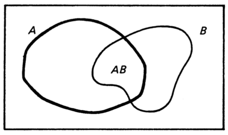
where

$$p = \frac{a}{a+b}, \quad q = 1 - p = \frac{b}{a+b} \tag{9}$$

1.7 Conditional Probability [2.5]

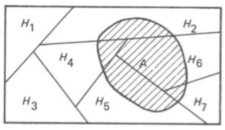
$$\begin{aligned} P(B|A) &= \frac{P(A \cap B)}{P(A)} \\ P(A \cap B) &= P(A)P(B|A) = P(B)P(A|B) \end{aligned} \tag{10}$$

is called the **conditional probability of B given A**.



From this we can derive the **Total Probability Theorem**. If the events $H_1, H_2, \ldots, H_n$ are mutually exclusive, have positive probabilities, and together fill Ω completely, any event A satisfies the formula:

$$P(A) = \sum_{i=1}^{n} P(H_i)P(A|H_i) \tag{11}$$



And **Bayes' Theorem**:

$$P(H_i|A) = \frac{P(H_i)P(A|H_i)}{\sum_{j=1}^{n} P(H_j)P(A|H_j)} \tag{12}$$

1.8 Independent Events [2.6]

$$P(A \cap B) = P(A)P(B), \tag{13}$$

If (13), then A and B are said to be **independent events**. This can be extended to three events. Consequence:

If the events $A_1, A_2, \ldots, A_n$ are independent and $P(A_i) = p_i$, then the probability that at least one of them occurs is equal to

$$1 - (1 - p_1)(1 - p_2) \cdots (1 - p_n) \tag{14}$$

If the events A_i are independent and each one of them occurs with probability p, then the probability that at least one of them occurs is equal to $1-(1-p)^n$.

Two trials as **independent**, if the result of one does not affect, or is not affected by, the result of the other. An important special case is provided by **repeated trials**.

1.9 Theorems in Combinatorics [2.7]

Drawing n elements from N:

	with replacement	without replacement
order	N^n	$N(N-1)\cdots(N-n+1)$
no order	$\binom{N+n-1}{n}$	$\binom{N}{n}$

The **multiplication principle** states: If choice 1 can be performed in a_1 ways and choice 2 in a_2 ways, then there are a_1a_2 ways to perform both choices. In the case of three choices, the number of ways is $a_1a_2a_3$, and so on.