1 Euclidean Vector Spaces

1.1 Linear combination [1.1]

$$\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = v_1 \begin{bmatrix} 1 \\ \vdots \\ 0 \end{bmatrix} + ... + v_n \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix} = v_1\vec{e_1} + ... + v_n\vec{e_n} \quad (1)$$

Any vector can be written as a unique linear combination of the standard basis vectors.

1.2 Vector equation of a line through point P [1.1]

$$\vec{x} = \vec{p} + t\vec{d}, t \in \mathbb{R} \quad (2)$$

1.3 Directed line segments [1.1]

Denote the directed line segment from point P to point Q by PQ.

$$\vec{PQ} = \vec{q} - \vec{p} \quad (3)$$

A directed line segment that starts at the origin is called the position vector of the point: $\vec{OP} = \vec{p}$.

1.4 Entities [1.2]

For all $\vec{w}, \vec{x}, \vec{y} \in \mathbb{R}$ and $s, t \in \mathbb{R}$ we have:

(1) $\vec{x} + \vec{y} \in \mathbb{R}$ **(closed under addition)**
(2) $\vec{x} + \vec{y} = \vec{y} + \vec{x}$ **(addition is commutative)**
(3) $(\vec{x} + \vec{y}) + \vec{w} = \vec{x} + (\vec{y} + \vec{w})$ **(addition is associative)**
(4) $\exists \vec{0} \in \mathbb{R}^n : \vec{z} + \vec{0} = \vec{z}, \forall \vec{z} \in \mathbb{R}^n$ **(zero vector)**
(5) $\forall \vec{x} \in \mathbb{R}^n : \exists -\vec{x} \in \mathbb{R}^n, \vec{x} + (-\vec{x}) = \vec{0}$ **(additive inverse)**
(6) $t\vec{x} \in \mathbb{R}$ **(closed under scalar multiplication)**
(7) $s(t\vec{x}) = (st)\vec{x}$ **(scalar multiplication is associative)**
(8) $(s + t)\vec{x} = s\vec{x} + t\vec{x}$ **(distributive la)**
(9) $t(\vec{x} + \vec{y}) = t\vec{x} + t\vec{y}$ **(distributive law)**
(10) $1\vec{x} = \vec{x}$ **(scalar multiplicative identity)**

1.5 Subspace [1.2]

A non-empty subset S of $\mathbb{R}^n$ is called a subspace of $\mathbb{R}^n$ if for all vectors $\vec{s}, \vec{y} \in S$ and $t \in \mathbb{R}$:

(1) $\vec{x} + \vec{y} \in \mathbb{R}^n$ (closed under addition)
(2) $t\vec{x} \in \mathbb{R}^n$ (closed under scalar multiplication)

The definition requires that a subspace be non-empty.

1.6 Spanning Sets and Linear Independence [1.2]

One of the main ways that subspaces arise is as the set of all linear combinations of some spanning set.

If $v_1, ..., v_k$ is a set of vectors in $\mathbb{R}^n$ and S is the set of all possible linear combinations of these vectors,

$$\begin{aligned} S &= \{t_1\vec{v_1} + ... + t_k\vec{v_k} | t_1, ..., t_k \in \mathbb{R}\} \\ &= Span\{\vec{v_1}, ..., \vec{v_k}\} = \langle S \rangle \end{aligned} \quad (4)$$

then S is a subspace of $\mathbb{R}^n$.

1.7 Linearly (in)dependent [1.2]

A set of vectors $\{\vec{v_1}, ..., \vec{v_k}\}$ is said to be **linearly dependent** if there exist coefficients $t_1, ..., t_k$ not all zero such that

$$\vec{0} = t_1\vec{v_1} + ... + t_k\vec{v_k} \quad (5)$$

A set of vectors $\{\vec{v_1}, ..., \vec{v_k}\}$ is said to be **linearly independent** if the only solution to (5) is $t_1 = t_2 = ... = t_k = 0$. This is called the **trivial solution**.

If a set of vectors $\{\vec{v_1}, ..., \vec{v_k}\}$ contains the zero vector, then it is **linearly dependent**.

1.8 Plane in $\mathbb{R}^n$ [1.2]

Let $\vec{v_1}, \vec{v_2}, \vec{p} \in \mathbb{R}^n$, with $\{\vec{v_1}, \vec{v_2}\}$ being a linearly independent set. Then the set with vector equation

$$\vec{x} = \vec{p} + t_1\vec{v_1} + t_2\vec{v_2} \quad (6)$$

with $t_1, t_2 \in \mathbb{R}$ is called a **plane** in $\mathbb{R}^n$ that passes through $\vec{p}$.

1.9 Vector length (Norm) in $\mathbb{R}^n$ [1.3]

$$||\vec{x}|| = \sqrt{{x_1}^2 + {x_2}^2 + ... + {x_n}^2} \quad (7)$$

Let $\vec{x}, \vec{y} \in \mathbb{R}^n$ and $t \in \mathbb{R}$. Then,

(1) $||\vec{x}|| \geq 0$ and $||\vec{x}|| = 0$ if and only if $\vec{x} = \vec{0}$
(2) $||t\vec{x}|| = |t|||\vec{x}||$
(3) $|\vec{x} \cdot \vec{y}| \leq ||\vec{x}||||\vec{y}||$, with equality if and only if $\{\vec{x}, \vec{y}\}$ is linearly dependent (Cauchy-Schwarz Inequality)
(4) $||\vec{x} + \vec{y}|| \leq ||\vec{x}|| + ||\vec{y}||$ (Triangle Inequality)

1.10 Angles and the Dot Product in $\mathbb{R}^2$ and $\mathbb{R}^3$ [1.3]

$$\begin{aligned} \vec{x} \cdot \vec{y} &= x_1y_1 + x_2y_2 \\ \cos\theta &= \frac{\vec{x} \cdot \vec{y}}{||\vec{x}||||\vec{y}||} \end{aligned} \quad (8)$$

where θ is always chosen to satisfy $0 < \theta < \pi$.

1.11 Dot Product in $\mathbb{R}^n$ [1.3]

$$\vec{x} \cdot \vec{y} = x_1y_1 + ... + x_ny_n \quad (9)$$

Let $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^n$ and $t \in \mathbb{R}$. Then,

(1) $\vec{x} \cdot \vec{x} \geq 0$ and $\vec{x} \cdot \vec{x} = 0$ if and only if $\vec{x} = \vec{0}$
(2) $\vec{x} \cdot \vec{y} = \vec{y} \cdot \vec{x}$
(3) $\vec{x} \cdot (\vec{y} + \vec{w}) = \vec{x} \cdot \vec{y} + \vec{x} \cdot \vec{w})$
(4) $(t\vec{x}) \cdot \vec{y} = t(\vec{x} \cdot \vec{y}) = \vec{x} \cdot (t\vec{y})$

1.12 Unit Vector [1.3]

$$\hat{x} = \frac{1}{||\vec{x}||}\vec{x} \quad (10)$$

1.13 Orthogonality of two vectors[1.3]

Two vectors $\vec{x}$ and $\vec{y}$ in $\mathbb{R}^n$ are orthogonal to each other if and only if $\vec{x} \cdot \vec{y} = 0$. Notice that this definition implies that $\vec{0}$ is orthogonal to every vector in $\mathbb{R}^n$.

1.14 Scalar equation of a hyperplane [1.3]

$$\begin{aligned} 0 &= \vec{n} \cdot \vec{PX} \\ \vec{n} \cdot \vec{x} &= \vec{n} \cdot \vec{p} \qquad (11) \\ n_1 x_1 + ... + n_n x_n &= \vec{n} \cdot \vec{p} = d \end{aligned}$$

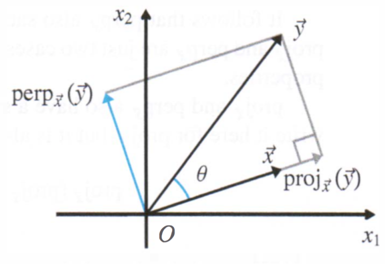
1.15 Projections [1.4]

The projection of $\vec{y}$ onto $\vec{x}$ is denoted

$$proj_{\vec{x}}\vec{y} = proj_{\hat{x}}\vec{y} = (\hat{x} \cdot \vec{y})\hat{x} = (\vec{y} \cdot \frac{\vec{x}}{\|\vec{x}\|})\hat{x} = \frac{\vec{x} \cdot \vec{y}}{\|\vec{x}\|^2}\vec{x} \qquad (12)$$

The perpendicular of a projection is very useful to calculate the minimum distance from a point to a line/plane.

$$perp_{\vec{x}}\vec{y} = \vec{y} - proj_{\vec{x}}\vec{y} \qquad (13)$$



(1) $proj_{\vec{x}}(\vec{y} + \vec{z}) = proj_{\vec{x}}\vec{y} + proj_{\vec{x}}\vec{z}$ for all $\vec{y}, \vec{z} \in \mathbb{R}^n$

(2) $proj_{\vec{x}}(t\vec{y}) = t proj_{\vec{x}}\vec{y}$ for all $\vec{y} \in \mathbb{R}^n$ and all $t \in \mathbb{R}$

1.16 Cross product [1.5]

The cross-product is a construction that is defined only in $\mathbb{R}^3$. There is a generalization to higher dimensions, but it is considerably more complicated.

$$\vec{u} \times \vec{v} = \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{bmatrix} \qquad (14)$$

(1) $\vec{x} \times \vec{y} = -\vec{y} \times \vec{x}$

(2) $\vec{x} \times \vec{x} = \vec{0}$

(3) $\vec{x} \times (\vec{y} + \vec{z}) = \vec{x} \times \vec{y} + \vec{x} \times \vec{z}$

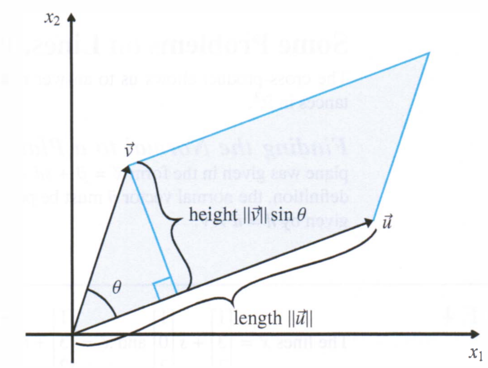
(4) $(t\vec{x}) \times \vec{y} = t(\vec{x} \times \vec{y})$

1.17 Length of the cross product [1.5]

Let $\vec{u}, \vec{v} \in \mathbb{R}^3$ and θ be the angle between $\vec{u}$ and $\vec{v}$, then

$$\|\vec{u} \times \vec{v}\| = \|\vec{u}\|\|\vec{y}\| \sin\theta \qquad (15)$$

Which is the area of the parallelogram:



2 Systems of Linear Equations

A general system of m linear equations in n variables is written in the form

$$\begin{gathered} \vdots \\ a_{i1} + a_{i2} + \cdots + a_{ij} + \cdots + a_{in} = b_i \qquad (16) \\ \vdots \end{gathered}$$

In matrix form:

$$\left[\begin{array}{cccccc|c} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & & \vdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} & b_i \\ \vdots & \vdots & & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} & b_m \end{array}\right] \qquad (17)$$

2.1 Elimination [2.1]

Types of Steps in Elimination:

(1) Multiply one equation by a non-zero constant.

(2) Interchange two equations.

(3) Add a multiple of one equation to another equation.

2.2 Row Echelon Form [2.1]

A matrix is in row echelon form **(REF)** if:

(1) When all entries in a row are zeros, this row appears below all rows that contain a non-zero entry.

(2) When two non-zero rows are compared, the first non-zero entry, called the leading entry, in the upper row is to the left of the leading entry in the lower row.

Any matrix can be row reduced to row echelon form by using the following steps:

(1) Consider the first column of the matrix; if it consists entirely of zero entries, move to the next column. If it contains some non-zero entry, interchange rows (if necessary) so that the top entry in the column is non-zero. We will call this entry a pivot.

(2) Use elementary row operations of type (3) to make all entries beneath the pivot into zeros.

(3) Next, consider the submatrix consisting of all columns to the right of the column we have just worked on and all rows below the row with the most recently obtained leading entry. Repeat the procedure described for this submatrix.

2.3 Consistent Systems and Unique Solutions [2.1]

A system that has at least one solution is called **consistent**, and a system that does not have any solutions is called **inconsistent**.

Suppose that the augmented matrix $[A|\vec{b}]$ of a system of linear equations is row equivalent to $[S|\vec{c}]$, which is in row echelon form.

(1) The given system is inconsistent if and only if some row of $[S|\vec{c}]$ is of the form $[0\,0\,\cdots\,0 \mid c]$, with $c \neq 0$.

(2) If the system is consistent, there are two possibilities. Either the number of pivots in S is equal to the number of variables in the system and the system has a unique solution, or the number of pivots is less than the number of variables and the system has infinitely many solutions.

2.4 Reduced Row Echelon Form [2.2]

A matrix R is said to be in **reduced row echelon form (RREF)** if

(1) It is in row echelon form.
(2) All leading entries are 1, called a **leading 1**.
(3) In a column with a leading 1, all the other entries are zeros.

For any given matrix A there is a unique matrix in reduced row echelon form that is row equivalent to A.

2.5 Rank of a Matrix [2.2]

The rank of a matrix A is the number of leading 1s in its reduced row echelon form and is denoted by $rank(A)$.

Let $[A|\vec{b}]$ be a system of m linear equations in n variables.

(1) The system is consistent if and only if the rank of the coefficient matrix A is equal to the rank of the augmented matrix $[A|\vec{b}]$.
(2) If the system is consistent, then the number of parameters in the general solution is the number of variables minus the rank of the matrix: #of parameters $= n - rank(A)$

Let $[A|\vec{b}]$ be a system of m linear equations in n variables. Then $[A|\vec{b}]$ is consistent for all $\vec{b}$ if and only if $rank(A) = m$.

2.6 Homogeneous Linear Equations [2.2]

A linear equation is **homogeneous** if the right-hand side is zero. A system of linear equations is **homogeneous** if all of the equations of the system are homogeneous.

Observe that every homogeneous system is consistent as the zero vector $\vec{0}$ (the trivial solution) will certainly be a solution.

2.7 Spanning Problems [2.3]

A set of k vectors $\{\vec{v_1}, ..., \vec{v_k}\}$ in $\mathbb{R}^n$ spans $\mathbb{R}^n$ if and only if the rank of the coefficient matrix of the system $t_1\vec{v_1} + ... + t_k\vec{v_k} = \vec{v}$ is n.

Let $\{\vec{v_1}, ..., \vec{v_k}\}$ be a set of k vectors in $\mathbb{R}^n$. If $Span\{\vec{v_1}, ..., \vec{v_k}\} = \mathbb{R}^n$, then $k \leq n$.

2.8 Linear Independence Problems [2.3]

A set of k vectors $\{\vec{v_1}, ..., \vec{v_k}\}$ in $\mathbb{R}^n$ is linearly independent if and only if the rank of the coefficient matrix of the system $t_1\vec{v_1} + ... + t_k\vec{v_k} = \vec{0}$ is k.

If $\{\vec{v_1}, ..., \vec{v_k}\}$ is a linearly independent set of vectors in $\mathbb{R}^n$, then $k \leq n$.

2.9 Bases of Subspaces [2.3]

A set of k vectors $\{\vec{v_1}, ..., \vec{v_n}\}$ is a basis for $\mathbb{R}^n$ if and only if the rank of the coefficient matrix of $t_1\vec{v_1} + ... + t_n\vec{v_n} = \vec{v}$ is n.

Suppose that S is a non-trivial subspace of $\mathbb{R}^n$ and $Span\{\vec{v_1}, ..., \vec{v_l}\} = S$. If $\{\vec{u_1}, ..., \vec{u_k}\}$ is a linearly independent set of vectors in S, then $k \leq l$.

If $\{\vec{v_1}, ..., \vec{v_l}\}$ and $\{\vec{u_1}, ..., \vec{v_k}\}$ are both bases of a non-trivial subspace S of $\mathbb{R}^n$, then $k = l$.

If S is a non-trivial subspace of $\mathbb{R}^n$ with a basis containing k vectors, then we say that the **dimension** of S is k and write:

$$dim(S) = k$$

3 Matrices, Linear Mappings, and Inverses

We denote the ij-th entry of a matrix A by $(A)_{ij}$.

3.1 Triangular Matrix [3.1]

A **square matrix** U is **upper triangular** if the entries beneath the main diagonal are all zero. The **square matrix** L is **lower triangular** if the entries above the main diagonal are all zero.

A matrix D that is **both upper and lower triangular** is called a **diagonal matrix**. We denote an $n \times n$ diagonal matrix by $D = diag(d_{11}, d_{22}, \cdots, d_{nn})$.

3.2 Addition and Scalar Multiplication [3.1]

$$\begin{gathered}(A+B)_{ij} = A_{ij} + B_{ij} \\ (tA_{ij}) = t(A_{ij})\end{gathered} \tag{18}$$

Let A, B, C be $m \times n$ matrices and let $s, t \in \mathbb{R}$. Then

(1) $A + B$ is an $m \times n$ matrix **(closed under addition)**
(2) $A + B = B + A$ **(addition is commutative)**
(3) $(A + B) + C = A + (B + C)$ **(addition is associative)**
(4) $\exists\ Om, n\ :\ A + Om, n = A$; It is the $m \times n$ matrix with all entries as zero, called the zero matrix **(zero vector)**
(5) $\forall A : \exists - (A), A + (-A) = Om, n$ **(additive inverse)**
(6) sA is an $m \times n$ matrix **(closed under scalar multiplication)**
(7) $s(tA) = (st)A$ **(scalar multiplication is associative)**
(8) $(s + t)A = sA + tA$ **(distributive la)**
(9) $s(A + B) = sA + sB$ **(distributive law)**
(10) $1A = A$ **(scalar multiplicative identity)**

3.3 Span and linear dependence [3.1]

Let $\mathcal{B} = \{A_1, \cdots, A_k\}$ be a set of $m \times n$ matrices. Then the span of $\mathcal{B}$ is defined as:

$$Span\mathcal{B} = \{t_1A_1, \cdots, t_kA_k \mid t_1, \cdots, t_k \in \mathbb{R}\} \tag{19}$$

$\mathcal{B}$ is **linearly independent** if the only solution to the equation $t_1A_1 + \cdots + t_kA_k = O_{mn}$ is $t_1 = \cdots = t_k = 0$; otherwise $\mathcal{B}$ is **linearly dependent**.

3.4 Transpose [3.1]

$$(A^T)_{ij} = (A)_{ji} \tag{20}$$

For any matrices A and B and scalar $s \in \mathbb{R}$, we have

(1) $(A^T)^T = A$
(2) $(A + B)^T = A^T + B^T$
(3) $(sA)^T = sA^T$

3.5 Matrix Multiplication [3.1]

$$A\vec{x} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_{11}x_1 & a_{12}x_2 \\ a_{21}x_1 & a_{22}x_2 \end{bmatrix} \tag{21}$$

The matrix multiplication BA of a $m \times n$ matrix B and $n \times p$ matrix A returns the $m \times p$ matrix whose ij-th entry is

$$(BA)_{ij} = \vec{b_i} \cdot \vec{a_j} \tag{22}$$

If A, B, and C are matrices of the correct size so that the required products are defined, and $t \in \mathbb{R}$, then

(1) $A(B+C) = AB + AC$
(2) $(A+B)C = AC + BC$
(3) $t(AB) = (tA)B = A(tB)$
(4) $A(BC) = (AB)C$
(5) $(AB)^T = B^T A^T$

There is **no division for matrices** because of the missing cancellation law for matrix multiplication.

The **identity matrix** with any $m \times n$ matrix:

$$I_m A = A = AI_n \; ; \; I_n = \begin{bmatrix} \vec{e_1} & \cdots & \vec{e_n} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{23}$$

3.6 Matrix Mappings [3.2]

For any $m \times n$ matrix A, we define a function $f_A : \mathbb{R}^n \to \mathbb{R}^m$ called the **matrix mapping**, corresponding to A by

$$f_A(\vec{x}) : A\vec{x} \tag{24}$$

for any $\vec{x} \in \mathbb{R}^n$.

Then, for any $\vec{x}, \vec{y} \in \mathbb{R}^n$ and any $t \in \mathbb{R}$

(L1) $f_A(\vec{x} + \vec{y}) = f_A(\vec{x}) + f_A(\vec{y})$
(L2) $f_A(t\vec{x}) = tf_A(\vec{x})$

3.7 Linear Mappings [3.2]

A function $L : \mathbb{R}^n \to \mathbb{R}^m$ is called a **linear mapping (or linear transformation)** if for every $\vec{x}, \vec{y} \in \mathbb{R}^n$ and any $t \in \mathbb{R}$ it satisfies the following properties:

(L1) $L(\vec{x} + \vec{y}) = L(\vec{x}) + L(\vec{y})$
(L2) $L(t\vec{x}) = tL(\vec{x})$

A **linear operator** L **on** $\mathbb{R}^n$ is a linear mapping whose domain and codomain are the same.

If $L : \mathbb{R}^n \to \mathbb{R}^m$ is a linear mapping, then L can be represented as a matrix mapping, with corresponding $m \times n$ matrix $[L]$ given by

$$[L] = \begin{bmatrix} L(\vec{e_1}) & L(\vec{e_2}) & \cdots & L(\vec{e_n}) \end{bmatrix} \tag{25}$$

3.8 Compositions and Linear Combinations of Linear Mappings [3.2]

If L and M are linear mappings from $\mathbb{R}^n \to \mathbb{R}^m$ and $t \in \mathbb{R}$,then $(L+M)$ and (tL) are linear mappings.

Let $L : \mathbb{R}^n \to \mathbb{R}^m$ and $M : \mathbb{R}^m \to \mathbb{R}^p$ be linear mappings. The **composition** $M \circ L : \mathbb{R}^n \to \mathbb{R}^p$ is defined by

$$(M \circ L)(\vec{x}) = M(L(\vec{x})) \tag{26}$$

for all $\vec{x} \in \mathbb{R}^n$.

Since compositions and linear combinations of linear mappings are linear mappings, it is natural to ask about the standard matrix of these new linear mappings.

Let $L : \mathbb{R}^n \to \mathbb{R}^m$, $M : \mathbb{R}^n \to \mathbb{R}^m$ and $N : \mathbb{R}^m \to \mathbb{R}^p$ be linear mappings and $t \in \mathbb{R}$. Then

$$[L+M] = [L] + [M], \; [tL] = t[L], \; [N \circ L] = [N][L] \tag{27}$$

3.9 Identity Mapping [3.2]

The linear mapping a $Id : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$Id(\vec{x}) = \vec{x} \tag{28}$$

is called the **identity mapping**.

3.10 Geometrical Transformations [3.3]

3.10.1 Rotations in the plane

$R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ is defined to be the transformation that rotates $\vec{x}$ counterclockwise through angle θ to the image $R_\theta(\vec{x})$.

$$R_\theta(\vec{x}) = \begin{bmatrix} x_1 \cos\theta - x_2 \sin\theta \\ x_1 \sin\theta + x_2 \cos\theta \end{bmatrix} \tag{29}$$

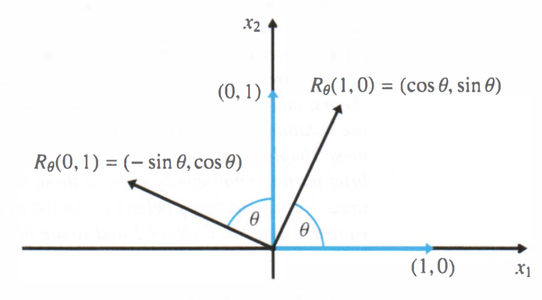


Figure 3.3.2 Image of the standard basis vectors under R_θ.

3.10.2 Rotations in the plane

$R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ is defined to be the transformation that rotates $\vec{x}$ counterclockwise through angle θ to the image $R_\theta(\vec{x})$.

$$R_\theta(\vec{x}) = \begin{bmatrix} x_1 \cos\theta - x_2 \sin\theta \\ x_1 \sin\theta + x_2 \cos\theta \end{bmatrix} \tag{30}$$

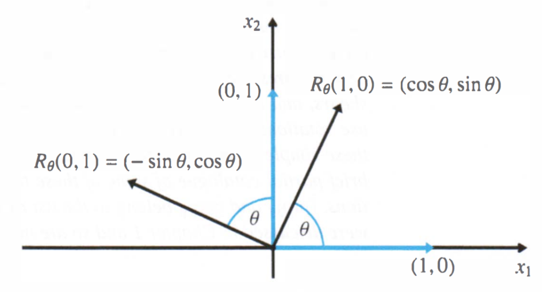


Figure 3.3.2 Image of the standard basis vectors under R_θ.

3.10.3 Rotation Through Angle θ about the x_3-axis in $\mathbb{R}^3$

$$[R] = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{31}$$

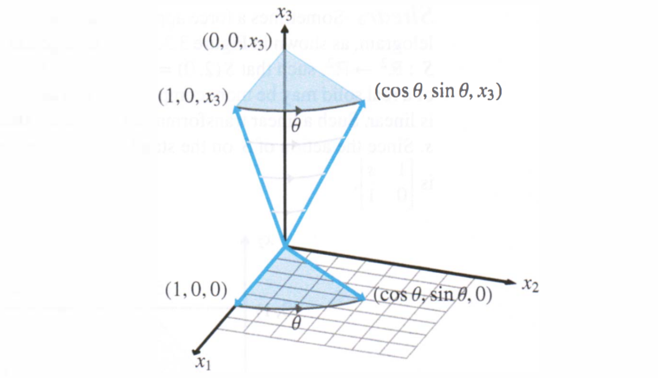


Figure 3.3.3 A right-handed counterclockwise rotation about the x_3-axis in $\mathbb{R}^3$.

3.10.4 Stretch/Shrink

Example of a stretch (shrink for $t < 1$) in x_1-direction in $\mathbb{R}^2$:

$$[R] = \begin{bmatrix} t & 0 \\ 0 & 1 \end{bmatrix} \tag{32}$$

Stretching / shrinking in all directions is called **dilation** / **contraction**.

3.10.5 Shear

$$[R] = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} \tag{33}$$

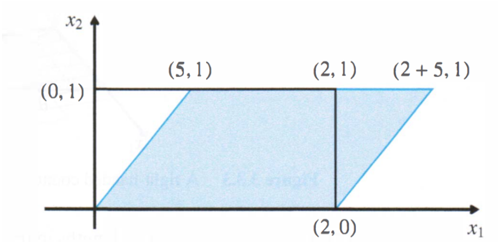


Figure 3.3.5 A shear in the direction of x_1 by amount s.

3.10.6 Reflections

Examples: $a)$ reflection in x_1-axis, $b)$ reflection in the x_1x_2-plane.

$$a) \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad b) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{34}$$

3.10.7 General reflections

We consider only reflections in (or "across") lines in $\mathbb{R}^2$ or planes in $\mathbb{R}^3$ that pass through the origin. Reflections in lines or planes not containing the origin involve translations (which are not linear) as well as linear mappings.

Consider the plane in $\mathbb{R}^3$ with equation $\vec{n} \cdot \vec{x} = 0$. Since a reflection is related to $proj_{\vec{n}}$, a **reflection in the plane with normal vector** $\vec{n}$ will be denoted $refl_{\vec{n}}$.

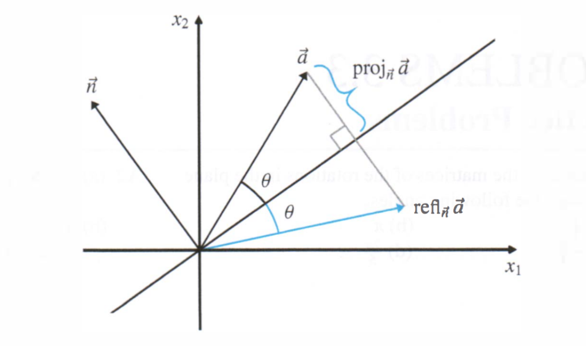


Figure 3.3.7 A reflection in $\mathbb{R}^2$ over the line with normal vector $\vec{n}$.

3.11 Solution Space [3.4]

The set $S = \{\vec{x} \in \mathbb{R}^n | A\vec{x} = \vec{0}\}$ of all solutions to a homogeneous system $A\vec{x} = \vec{0}$ is called the **solution space** of the system $A\vec{x} = \vec{0}$.

3.12 Null Space / Kernel [3.4]

The **nullspace/kernel** of a linear mapping L is the set of all vectors whose image under L is the zero vector $\vec{0}$. This is also valid for matrices, where A is an $m \times n$ matrix.

$$\begin{aligned} Null(L) = ker(L) = \{\vec{x} \in \mathbb{R}^n | L(\vec{x}) = \vec{0}\} \\ Null(A) = ker(A) = \{\vec{x} \in \mathbb{R}^n | A\vec{x} = \vec{0}\} \end{aligned} \tag{35}$$

3.13 Solution Set of $A\vec{x} = \vec{b}$ [3.4]

Let $\vec{p}$ be a solution of the system of linear equations $A\vec{x} = \vec{b}, \vec{b} \neq \vec{0}$.

(1) If $\vec{v}$ is any other solution of the same system, then $A(\vec{p} - \vec{v}) = \vec{0}$, so that $\vec{p} - \vec{v}$ is a solution of the corresponding homogeneous system $A\vec{x} = \vec{0}$.

(2) If $\vec{h}$ is any solution of the corresponding system $A\vec{x} = \vec{0}$, then $\vec{p} + \vec{h}$ is a solution of the system $A\vec{x} = \vec{b}$.

3.14 Range of L and Columnspace of A [3.4]

Range of linear mapping $L : \mathbb{R}^n \to \mathbb{R}^m$ is defined to be the set

$$Range(L) = \{L(\vec{v}) \in \mathbb{R}^m | \vec{x} \in \mathbb{R}^n\} \tag{36}$$

Columnspace of $m \times n$ matrix A is the set $Col(A)$ defined by

$$Col(A) = \{A\vec{v} \in \mathbb{R}^m | \vec{x} \in \mathbb{R}^n\} \tag{37}$$

The system of equations $A\vec{x} = \vec{b}$ is consistent if and only if b is in the range of the linear mapping L with standard matrix A (or, equivalently, if and only if b is in the columnspace of A).

3.15 Rowspace of A [3.4]

Given an $m \times n$ matrix A, the **rowspace** of A is the subspace spanned by the rows of A (regarded as vectors) and is denoted $Row(A)$. Example:

$$Row(\begin{bmatrix} 1 & 3 \\ 2 & -1 \\ 0 & 1 \end{bmatrix}) = Span\{\begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\} \tag{38}$$

3.16 Bases for $Row(A)$, $Col(A)$, and $Null(A)$ [3.4]

Rowspace: Let B be the reduced row echelon form of an $m \times n$ matrix A.Then the non-zero rows of B form a basis for $Row(A)$, and hence the dimension of $Row(A)$ equals the rank of A.

Columnspace: Suppose that B is the reduced row echelon form of A. Then the columns of A that correspond to the columns of B with leading 1s form a basis of the columnspace of A. Hence, the dimension of the columnspace equals the rank of A.

Let A be an $m \times n$ matrix. We call the dimension of the nullspace of A the **nullity** of A and denote it by $nullity(A)$.

Nullspace/Kernel: Let A be an $m \times n$ matrix with $rank(A) = r$. Then the spanning set for the general solution of the homogeneous system $A\vec{x} = \vec{0}$ is a basis for $Null(A)$ and the nullity of A is $n - r$.

3.17 Rank Theorem A [3.4]

If A is any $m \times n$ matrix, then

$$rank(A) + nullity(A) = n \tag{39}$$

Facts about Rank:

$$\begin{aligned} rank(A) &= \text{the number of leading 1s in the RREF of} A \\ &= \text{the number of non-zero rows in any REF of} A \\ &= dimRow(A) \\ &= dimCol(A) \\ &= n - dimNull(A) \end{aligned} \tag{40}$$

3.18 Inverse [3.5]

Let A be an $n \times n$ matrix. If there exists an $n \times n$ matrix B such that $AB = I = BA$, then A is said to be invertible, and B is called the **inverse** of A (and A is the inverse of B). The inverse of A is denoted A^{-1}.

Suppose that A and B are $n \times n$ matrices:

(1) $BA = AB = I$ and $CA = AC = I$. Then $B = C$.

(2) $AB = I$. Then $BA = I$, so that $B = A^{-1}$. Moreover, B and A have rank n.

Suppose that A and B are invertible matrices and that t is a non-zero real number:

(1) $(tA)^{-1} = \frac{1}{t}A^{-1}$

(2) $(AB)^{-1} = B^{-1}A^{-1}$

(3) $(A^T)^{-1} = (A^{-1})^T$

3.19 Finding A^{-1}[3.5]

To find the inverse of a square matrix A,

(1) Row reduce the multi-augmented matrix $[\,A \mid I\,]$ so that the left block is in RREF.

(2) If the RREF is $[\,I \mid B\,]$ then $A^{-1} = B$.

(3) Else A is not invertible.

3.20 Invertible Matrix Theorem [3.5]

Suppose that A is an $n \times n$ matrix. Then the following statements are equivalent (that is, one is true if and only if each of the others is true).

(1) A is invertible.

(2) $rank(A) = n$

(3) The reduced row echelon form of A is I.

(4) For all $b \in \mathbb{R}^n$, the system $A\vec{x} = \vec{b}$ is consistent and has a unique solution.

(5) The columns of A are linearly independent.

(6) The columnspace of A is $\mathbb{R}^n$.

Suppose that $L : \mathbb{R}^n \to \mathbb{R}^n$ is a linear mapping with standard matrix $A = [l]$. Then, the following statements are equivalent to each other and to the statements above.

(7) L is invertible.

(8) $range(L) = \mathbb{R}^n$

(9) $Null(L) = \{\vec{0}\}$

3.21 Elementary Matrices [3.6]

Not part of this course.

3.22 LU-Decomposition [3.7]

Not part of this course.

4 Vector Spaces

4.1 Spaces of polynomials [4.1]

4.1.1 Addition and Scalar Multiplication of Polynomials

Let $p(x)$, $q(x)$ and $r(x)$ be polynomials of degree at most n and let $s, t \in \mathbb{R}$. Then:

(1) $p(x) + q(x)$ is a polynomial of degree at most n

(2) $p(x) + q(x) = q(x) + p(x)$

(3) $(p(x) + q(x)) + r(x) = p(x) + (q(x) + r(x))$

(4) The polynomial $0 = 0 + 0x + \cdots + 0x^n$, called the **zero polynomial**, satisfies $p(x) + 0 = p(x) = 0 + p(x)$ for any polynomial $p(x)$

(5) For each polynomial $p(x)$, there exists an additive inverse, denoted $(-p)(x)$, with the property that $p(x) + (-p)(x) = 0$; in particular, $(-p)(x) = -p(x)$

(6) $tp(x)$ is a polynomial of degree at most n

(7) $s(tp(x)) = (st)p(x)$

(8) $(s + t)p(x) = sp(x) + tp(x)$

(9) $t(p(x) + q(x)) = tp(x) + tq(x)$

(10) $1p(x) = p(x)$

4.1.2 Span

Let $\mathcal{B} = \{p_1(x), ..., p_k(x)\}$ be a set of polynomials of degree at most n. Then the **span** of $\mathcal{B}$ is defined as

$$Span\mathcal{B} = \{t_1p_1(x) + \cdots + t_kp_k(x) | t_1, ..., t_k \in \mathbb{R}\} \tag{41}$$

4.1.3 Linear Independent

The set $\mathcal{B} = \{p_1(x), ..., p_k(x)\}$ is said to **linearly independent** if the only solution to the equation

$$t_1p_1(x) + \cdots + t_kp_k(x) = 0 \tag{42}$$

is $t_1 = \cdots = t_k = 0$; otherwise, $\mathcal{B}$ is **linearly dependent**.

4.2 Vector Spaces [4.2]

A **vector space over** $\mathbb{R}$ is a set $\mathbb{V}$ together with an operation of **addition** ($\oplus$), usually denoted $x+y$ for any $x, y \in \mathbb{V}$, and an operation of **scalar multiplication** ($\odot$), usually denoted sx for any $x \in \mathbb{V}$ and $s \in \mathbb{R}$, such that for any $x, y, z \in \mathbb{V}$ and $s, t \in \mathbb{R}$ we have all of the following properties:

V(1) $x + y \in \mathbb{V}$ **(closed under addition)**
V(2) $x + y = y + x$ **(addition is commutative)**
V(3) $(x + y) + z = x + (y + z)$ **(addition is associative)**
V(4) $\exists$ **0** $\in \mathbb{V} : x + 0 = x = 0 + x$ **(zero vector)**
V(5) $\forall x \in \mathbb{V} : \exists - x \in \mathbb{V}, x + (-x) = 0$ **(additive inverse)**
V(6) $tx \in \mathbb{V}$ **(closed under scalar multiplication)**
V(7) $s(tx) = (st)x$ **(scalar multiplication is associative)**
V(8) $(s + t)x = sx + tx$ **(distributive la)**
V(9) $t(x + y) = tx + ty$ **(distributive law)**
V(10) $1x = x$ **(scalar multiplicative identity)**

Let $\mathbb{V}$ be a vector space. Then

(1) $0x = 0 \forall x \in \mathbb{V}$
(2) $(-1)x = -x \forall x \in \mathbb{V}$
(3) $t0 = 0 \forall t \in \mathbb{R}$

4.2.1 Subspaces

Suppose that $\mathbb{V}$ is a vector space. A non-empty subset $\mathbb{U}$ of $\mathbb{V}$ is a **subspace** $\mathbb{V}$ if it satisfies the following two properties:

S(1) $x + y \in \mathbb{U},\ \forall x, y \in \mathbb{V}$ ($\mathbb{U}$ is closed under addition)
S(2) $tx \in \mathbb{U}, \forall x \in \mathbb{U}$ and $t \in \mathbb{R}$ ($\mathbb{U}$ is closed under scalar multiplication)

Equivalent definition: If $\mathbb{U}$ is a subset of a vector space $\mathbb{V}$ and $\mathbb{U}$ is also a vector space using the same operations as $\mathbb{V}$, then $\mathbb{U}$ is a **subspace** of $\mathbb{V}$.

4.2.2 Set of all possible linear combinations

If $\{v_1(x), ..., v_k(x)\}$ is a set of vectors in a vector space $\mathbb{V}$ and $\mathbb{S}$ is the set of all possible linear combinations of these vectors,

$$\mathbb{S} = \{t_1v_1 + \cdots + t_kv_k \,|t_1, ..., t_k \in \mathbb{R}\} \tag{43}$$

then $\mathbb{S}$ is a subspace of $\mathbb{V}$.

4.2.3 Span, Spanning Set

If $\mathbb{S}$ is the subspace of the vector space $\mathbb{V}$ consisting of all possible linear combinations of the vectors $v_1, ..., v_k \in \mathbb{V}$, then $\mathbb{S}$ is called the subspace spanned by $\mathcal{B} = \{v_1, ..., v_k\}$, and we say that set $\mathcal{B}$ **spans** $\mathbb{S}$. The set $\mathcal{B}$ is called a **spanning set** for the subspace $\mathbb{S}$. We denote $\mathbb{S}$ by

$$\mathbb{S} = Span\{t_1p_1(x) + \cdots + t_kp_k(x)\} = Span\mathcal{B} \tag{44}$$

4.2.4 Linear Independence

If $\mathcal{B} = \{p_1(x), ..., p_k(x)\}$ is a set of vectors in vector space $\mathbb{V}$, then $\mathcal{B}$ is said to **linearly independent** if the only solution to the equation

$$t_1p_1(x) + \cdots + t_kp_k(x) = 0 \tag{45}$$

is $t_1 = \cdots = t_k = 0$; otherwise, $\mathcal{B}$ is **linearly dependent**.

4.3 Bases [4.3]

Let $\mathcal{B} = \{v_1, ..., v_n\}$ be a spanning set for a vector space $\mathbb{V}$. Then every vector in $\mathbb{V}$ can be expressed in a *unique way* as a linear combination of the vectors of $\mathcal{B}$ if and only if the set $\mathcal{B}$ is linearly independent.

A set $\mathbb{B}$ of vectors in a vector space $\mathcal{V}$ is a **basis** if it is a linearly independent spanning set for $\mathbb{V}$.

4.3.1 Dimension

If a vector space $\mathbb{V}$ has a basis of n vectors, then the **dimension** of $\mathbb{V}$ is n.

$$dim(\mathbb{V}) = n \tag{46}$$

4.3.2 Coordinate Vector

The **coordinate vector** of x with respect to the basis $\mathcal{B}$ is denoted by

$$[x]_{\mathcal{B}} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \tag{47}$$

4.3.3 Change of Coordinates Matrix

Let $\mathcal{B}$ and $\mathcal{C} = \{w_1, ..., w_n\}$ bot be bases for a vector space $\mathbb{V}$. The matrix $\mathcal{P} = \left[[w_1]_{\mathcal{B}} \cdots [w_n]_{\mathcal{B}}\right]$ is called the **change of coordinates matrix** from $\mathcal{C}$-coordinates to $\mathcal{B}$-coordinates and satisfies

$$[x]_{\mathcal{B}} = \mathcal{P}[x]_{\mathcal{C}} \tag{48}$$

then $\mathcal{P}$ is invertible and $\mathcal{P}^{-1}$ is the change of coordinates matrix from $\mathcal{B}$-coordinates to $\mathcal{C}$-coordinates.

4.4 Linear Mappings [4.5]

If $\mathbb{V}$ and $\mathbb{W}$ are vector spaces over $\mathbb{R}$, a function $L : \mathbb{V} \to \mathbb{W}$ is a linear mapping if it satisfies the linearity properties

L1 L(x + y) = L(x) = L(y)
L2 L(t x) = t L(x)

for all $x, y \in \mathbb{V}$ and $t \in \mathbb{R}$. If $\mathbb{W} = \mathbb{V}$, then L may be called a **linear operator**.

5 Appendix

//todo