610_HW2

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9/19/2022

Question 1: Hierarchical Normal Model

$$\text{Likelihood: } (y_{ij}|\mu_j, \sigma^2) \sim N(\mu_j, \sigma^2) \quad i = 1, ...n_j \quad j = 1, ..., J$$
$$\text{Prior: } (\mu_j|\tau^2) \sim N(\mu, \tau^2)$$
$$\text{Prior: } \sigma^2 \sim IG\left(\frac{\nu_0}{2}, \frac{\nu_0\sigma_0^2}{2}\right)$$
$$\text{Prior: } \mu \sim N\left(\mu_0, \frac{\tau^2}{m_0}\right)$$
$$\text{Prior: } \tau^2 \sim IG\left(\frac{m_0}{2}, \frac{m_0\tau_0^2}{2}\right)$$

a)

$$\text{Full-Conditional: } p(\mu_j|\mu_{-j}, \mu, Y, \sigma^2, \tau^2)$$

$$\begin{aligned}
p(\mu_j|-) &\propto \prod_{i=1}^{n_j} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{y_{ij}-\mu_j}{\sigma}\right)^2\right) \cdot -\frac{1}{\tau\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{\mu_j-\mu}{\tau}\right)^2\right) \\
&\propto \prod_{i=1}^{n_j} \exp\left(-\frac{1}{2}\left(\frac{y_{ij}-\mu_j}{\sigma}\right)^2\right) \cdot \exp\left(-\frac{1}{2}\left(\frac{\mu_j-\mu}{\tau}\right)^2\right) \\
&\propto \exp\left(\sum_{i=1}^{n_j} -\frac{1}{2}\left(\frac{y_{ij}-\mu_j}{\sigma}\right)^2 - \frac{1}{2}\left(\frac{\mu_j-\mu}{\tau}\right)^2\right) \\
&\propto \exp\left(\frac{1}{\sigma^2}\sum_{i=1}^{n_j}(y_{ij}-\mu_j)^2 + \frac{(\mu_j-\mu)^2}{\tau^2}\right) \\
&\propto \exp\left(\frac{1}{\sigma^2}\sum_{i=1}^{n_j}(y_{ij}^2 - 2y_{ij}\mu_j + \mu_j^2) + \frac{\mu_j^2 - 2\mu_j\mu + \mu^2}{\tau^2}\right)
\end{aligned}$$

$$
\begin{aligned}
&\propto \exp\left(\frac{1}{\sigma^2}\left(-2n_j\bar{y}_j\mu_j + n_j\mu_j^2\right) + \frac{\mu_j^2 - 2\mu_j\mu}{\tau^2}\right) \\
&\propto \exp\left(-2\left(\frac{n_j\bar{y}_j}{\sigma^2} + \frac{\mu}{\tau^2}\right)\mu_j + \left(\frac{n_j}{\sigma^2} + \frac{1}{\tau^2}\right)\mu_j^2\right) \\
&\propto \exp\left(\left(\frac{n_j}{\sigma^2} + \frac{1}{\tau^2}\right)\left(\mu_j^2 - 2\left(\frac{\frac{n_j}{\sigma^2}\bar{y}_j + \frac{1}{\tau^2}\mu}{\frac{n_j}{\sigma^2} + \frac{1}{\tau^2}}\right)\mu_j\right)\right) \\
&\text{Let } v_j = \frac{1}{\frac{n_j}{\sigma^2} + \frac{1}{\tau^2}} \quad \text{and} \quad m_j = v_j\left(\frac{n_j}{\sigma^2}\bar{y}_j + \frac{1}{\tau^2}\mu\right), \quad \text{then} \\
&\propto \exp\left(\frac{1}{v_j}(\mu_j^2 - 2m_j\mu_j)\right) \\
&\propto \exp\left(\frac{1}{v_j}(\mu_j^2 - 2m_j\mu_j + m_j^2 + m_j^2)\right) \\
&\propto \exp\left(\frac{1}{v_j}(\mu_j - m_j)^2\right) \\
p(\mu_j|-) &\sim N(m_j, v_j)
\end{aligned}
$$

b)

Full-Conditional: $p(\sigma^2|\mu_1, .., \mu_J, \mu, Y, \tau^2)$

$$
\begin{aligned}
p(\sigma^2|-) &\propto \prod_{j=1}^{J}\prod_{i=1}^{n_j}\frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{1}{2}\left(\frac{y_{ij}-\mu_j}{\sigma}\right)^2\right)\cdot\frac{(\frac{v_0\sigma_0^2}{2})^{(\frac{v_0}{2})}}{\Gamma\left(\frac{v_0}{2}\right)}(\sigma^2)^{(-\frac{v_0}{2}-1)}\cdot\exp\left(-\left(\frac{\frac{v_0\sigma_0^2}{2}}{\sigma^2}\right)\right) \\
&\propto \prod_{j=1}^{J}\prod_{i=1}^{n_j}(\sigma)^{-\frac{1}{2}}\exp\left(\frac{-\frac{1}{2}(y_{ij}-\mu_j)^2}{\sigma^2}\right)\cdot(\sigma^2)^{(-\frac{v_0}{2}-1)}\cdot\exp\left(-\frac{1}{\sigma^2}\left(\frac{v_0\sigma_0^2}{2}\right)\right) \\
&\propto \left(\sigma^2\right)^{\left(-\frac{1}{2}\sum_{j=1}^{J}n_j\right)}\cdot\exp\left(-\frac{1}{\sigma^2}\sum_{j=1}^{J}\sum_{i=1}^{n_j}\frac{(y_{ij}-\mu_j)^2}{2}\right)\cdot(\sigma^2)^{(-\frac{v_0}{2}-1)}\cdot\exp\left(-\frac{1}{\sigma^2}\left(\frac{v_0\sigma_0^2}{2}\right)\right) \\
&\propto (\sigma^2)^{\left(-\frac{1}{2}\sum_{j=1}^{J}n_j-\frac{v_0}{2}-1\right)}\cdot\exp\left(-\frac{1}{\sigma^2}\sum_{j=1}^{J}\sum_{i=1}^{n_j}\frac{(y_{ij}-\mu_j)^2}{2}+\frac{v_0\sigma_0^2}{2}\right) \\
p(\sigma^2|-) &\sim IG\left(\frac{1}{2}\sum_{j=1}^{J}n_j + \frac{v_0}{2},\ \frac{\sum_{j=1}^{J}\sum_{i=1}^{n_j}(y_{ij}-\mu_j)^2 + v_0\sigma_0^2}{2}\right)
\end{aligned}
$$

Question 2: Unequal Variances by Group

$$\text{Likelihood: } (y_{ij}|\mu_j, \sigma_j^2) \sim N(\mu_j, \sigma_j^2) \quad i = 1, ...n_j \quad j = 1, ..., J$$

$$\text{Prior: } (\mu_j|\tau^2) \sim N(\mu, \tau^2)$$

$$\text{Prior: } \sigma_j^2 \sim IG\left(\frac{\nu_0}{2}, \frac{\nu_0\sigma_0^2}{2}\right)$$

$$\text{Prior: } \mu \sim N\left(\mu_0, \frac{\tau^2}{m_0}\right)$$

$$\text{Prior: } \tau^2 \sim IG\left(\frac{m_0}{2}, \frac{m_0\tau_0^2}{2}\right)$$

$$\text{Prior: } \sigma_0^2 \sim Gamma(a, b)$$

There is no conjugate form available for v_0 the Hoff textbook recomends using:

$$p(v_0) \ \propto \ e^{-\rho v_0} \ for \ some \ \rho > 0 \ on \ (1, 2, ...)$$

i)

$$\text{Full-Conditional: } p(\mu_j|\mu_{-j}, \mu, Y, \sigma_1^2, ...\sigma_J^2, \tau^2, \sigma_0^2, v_0)$$

$$\begin{aligned}
p(\mu_j|-) \quad &\propto \quad \prod_{i=1}^{n_j} \frac{1}{\sigma_j\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{y_{ij}-\mu_j}{\sigma_j}\right)^2\right) \cdot \ -\frac{1}{\tau\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{\mu_j-\mu}{\tau}\right)^2\right) \\
&\propto \quad \prod_{i=1}^{n_j} \exp\left(-\frac{1}{2}\left(\frac{y_{ij}-\mu_j}{\sigma_j}\right)^2\right) \cdot \ \exp\left(-\frac{1}{2}\left(\frac{\mu_j-\mu}{\tau}\right)^2\right) \\
&\propto \quad \exp\left(\sum_{i=1}^{n_j} -\frac{1}{2}\left(\frac{y_{ij}-\mu_j}{\sigma_j}\right)^2 - \frac{1}{2}\left(\frac{\mu_j-\mu}{\tau}\right)^2\right) \\
&\propto \quad \exp\left(\frac{1}{\sigma_j^2}\sum_{i=1}^{n_j}(y_{ij}-\mu_j)^2 + \frac{(\mu_j-\mu)^2}{\tau^2}\right) \\
&\propto \quad \exp\left(\frac{1}{\sigma_j^2}\sum_{i=1}^{n_j}(y_{ij}^2 - 2y_{ij}\mu_j + \mu_j^2) + \frac{\mu_j^2 - 2\mu_j\mu + \mu^2}{\tau^2}\right) \\
&\propto \quad \exp\left(\frac{1}{\sigma_j^2}\left(-2n_j\bar{y}_j\mu_j + n_j\mu_j^2\right) + \frac{\mu_j^2 - 2\mu_j\mu}{\tau^2}\right) \\
&\propto \quad \exp\left(-2\left(\frac{n_j\bar{y}_j}{\sigma_j^2} + \frac{\mu}{\tau^2}\right)\mu_j + \left(\frac{n_j}{\sigma_j^2} + \frac{1}{\tau^2}\right)\mu_j^2\right) \\
&\propto \quad \exp\left(\left(\frac{n_j}{\sigma_j^2} + \frac{1}{\tau^2}\right)\left(\mu_j^2 - 2\left(\frac{\frac{n_j}{\sigma_j^2}\bar{y}_j + \frac{1}{\tau^2}\mu}{\frac{n_j}{\sigma_j^2} + \frac{1}{\tau^2}}\right)\mu_j\right)\right)
\end{aligned}$$

$$\text{Let } v_j = \frac{1}{\frac{n_j}{\sigma_j^2} + \frac{1}{\tau^2}} \quad \text{and} \quad m_j = v_j\left(\frac{n_j}{\sigma_j^2}\bar{y}_j + \frac{1}{\tau^2}\mu\right), \quad \text{then}$$

$$
\begin{aligned}
&\propto \exp\left(\frac{1}{v_j}(\mu_j^2 - 2m_j\mu_j)\right) \\
&\propto \exp\left(\frac{1}{v_j}(\mu_j^2 - 2m_j\mu_j + m_j^2 + m_j^2)\right) \\
&\propto \exp\left(\frac{1}{v_j}(\mu_j - m_j)^2\right) \\
p(\mu_j|-) &\sim N(m_j, v_j)
\end{aligned}
$$

ii)

$$\text{Full-Conditional: } p(\sigma_j^2|\sigma_{-j}^2, \mu_1, ..., \mu_J, \mu, Y, \tau^2, \sigma_0^2, v_0)$$

$$
\begin{aligned}
p(\sigma_j^2|-) &\propto \prod_{i=1}^{n_j} \frac{1}{\sigma_j\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{y_{ij}-\mu_j}{\sigma_j}\right)^2\right) \cdot \frac{(\frac{v_0\sigma_0^2}{2})^{(\frac{v_0}{2})}}{\Gamma\left(\frac{v_0}{2}\right)}(\sigma_j^2)^{(-\frac{v_0}{2}-1)} \cdot \exp\left(-\left(\frac{\frac{v_0\sigma_0^2}{2}}{\sigma_j^2}\right)\right) \\
&\propto \prod_{i=1}^{n_j}(\sigma_j)^{-\frac{1}{2}} \exp\left(\frac{-\frac{1}{2}(y_{ij}-\mu_j)^2}{\sigma_j^2}\right) \cdot (\sigma_j^2)^{(-\frac{v_0}{2}-1)} \cdot \exp\left(-\frac{1}{\sigma_j^2}\left(\frac{v_0\sigma_0^2}{2}\right)\right) \\
&\propto \left(\sigma_j^2\right)^{\left(-\frac{n_j}{2}\right)} \cdot \exp\left(-\frac{1}{\sigma_j^2}\sum_{i=1}^{n_j}\frac{(y_{ij}-\mu_j)^2}{2}\right) \cdot (\sigma_j^2)^{(-\frac{v_0}{2}-1)} \cdot \exp\left(-\frac{1}{\sigma_j^2}\left(\frac{v_0\sigma_0^2}{2}\right)\right) \\
&\propto (\sigma_j^2)^{\left(-\frac{n_j}{2}-\frac{v_0}{2}-1\right)} \cdot \exp\left(-\frac{1}{\sigma_j^2}\sum_{i=1}^{n_j}\frac{(y_{ij}-\mu_j)^2}{2} + \frac{v_0\sigma_0^2}{2}\right) \\
p(\sigma_j^2|-) &\sim IG\left(\frac{n_j+v_0}{2},\ \frac{\sum_{i=1}^{n_j}(y_{ij}-\mu_j)^2 + v_0\sigma_0^2}{2}\right)
\end{aligned}
$$

iii)

$$\text{Full-Conditional: } p(\mu|\mu_1, ..., \mu_J, \sigma_1^2, ...\sigma_J^2, Y, \tau^2, \sigma_0^2, v_0)$$

$$
\begin{aligned}
p(\mu|-) &\propto \prod_{j=1}^{J}\frac{1}{\tau\sqrt{2\pi}}\exp\left(-\frac{1}{2}\frac{(\mu_j-\mu)^2}{\tau^2}\right) \cdot \frac{1}{\sqrt{\frac{2\pi\tau^2}{m_0}}}\exp\left(-\frac{1}{2}\frac{(\mu-\mu_0)^2}{\frac{\tau^2}{m_0}}\right) \\
&\propto \prod_{j=1}^{J}\exp\left(-\frac{(\mu_j-\mu)^2}{2\tau^2}\right) \cdot \exp\left(-\frac{m_0(\mu-\mu_0)^2}{2\tau^2}\right) \\
&\propto \exp\left(-\frac{\sum_{j=1}^{J}(\mu_j-\mu)^2}{2\tau^2}\right) \cdot \exp\left(-\frac{m_0(\mu-\mu_0)^2}{2\tau^2}\right)
\end{aligned}
$$

$$
\begin{aligned}
&\propto \quad \exp\left(-\frac{\sum_{j=1}^{J}(\mu_j-\mu)^2}{2\tau^2}-\frac{m_0(\mu-\mu_0)^2}{2\tau^2}\right)\\
&\propto \quad \exp\left(\frac{-\sum_{j=1}^{J}\mu_j^2+2\mu\sum_{j=1}^{J}\mu_j-J\mu^2-m_0\mu^2+2m_0\mu\mu_0-m_0\mu_0^2}{2\tau^2}\right)\\
&\propto \quad \exp\left(\frac{2\mu\sum_{j=1}^{J}\mu_j-J\mu^2-m_0\mu^2+2m_0\mu\mu_0}{2\tau^2}\right)\\
&\propto \quad \exp\left(\frac{(-J-m_0)\mu^2+(2\sum_{j=1}^{J}\mu_j+2m_0\mu_0)\mu}{2\tau^2}\right)\\
&\propto \quad \exp\left(\left(\frac{-J-m_0}{2\tau^2}\right)\left(\mu^2+\mu\left(\frac{2\sum_{j=1}^{J}\mu_j+2m_0\mu_0}{-J-m_0}\right)\right)\right)\\
&\propto \quad \exp\left(\left(\frac{-(J+m_0)}{2\tau^2}\right)\left(\mu^2-\mu\left(\frac{2\sum_{j=1}^{J}\mu_j+2m_0\mu_0}{J+m_0}\right)+\left(\frac{\sum_{j=1}^{J}\mu_j+m_0\mu_0}{J+m_0}\right)^2-\left(\frac{\sum_{j=1}^{J}\mu_j+m_0\mu_0}{J+m_0}\right)^2\right)\right)\\
&\propto \quad \exp\left(\left(\frac{-(J+m_0)}{2\tau^2}\right)\left(\mu-\frac{\sum_{j=1}^{J}\mu_j+m_0\mu_0}{J+m_0}\right)^2\right)\\
p(\mu|-) &\sim \quad N\left(\frac{\sum_{j=1}^{J}\mu_j+m_0\mu_0}{J+m_0},\ \frac{\tau^2}{J+m_0}\right)
\end{aligned}
$$

iv)

Full-Conditional: $p(\tau^2|\mu_1,...,\mu_J,\sigma_1^2,...\sigma_J^2,Y,\mu,\sigma_0^2,v_0)$

$$
\begin{aligned}
p(\tau^2|-) &\propto \quad \prod_{j=1}^{J}\frac{1}{\tau\sqrt{2\pi}}\exp\left(-\frac{(\mu_j-\mu)^2}{2\tau^2}\right)\cdot\frac{1}{\sqrt{\frac{\tau^2 2\pi}{m_0}}}\exp\left(-\frac{m_0(\mu-\mu_0)^2}{2\tau^2}\right)\cdot\frac{\frac{m_0\tau_0^2}{2}}{\Gamma\frac{m_0}{2}}(\tau^2)^{-\frac{m_0}{2}-1}\exp\left(-\frac{\frac{m_0\tau_0^2}{2}}{\tau^2}\right)\\
&\propto \quad \prod_{j=1}^{J}(\tau^2)^{-\frac{1}{2}}\exp\left(-\frac{(\mu_j-\mu)^2}{2\tau^2}\right)\cdot(\tau^2)^{-\frac{1}{2}}\exp\left(-\frac{m_0(\mu-\mu_0)^2}{2\tau^2}\right)\cdot(\tau^2)^{-\frac{m_0}{2}-1}\exp\left(-\frac{m_0\tau_0^2}{2\tau^2}\right)\\
&\propto \quad (\tau^2)^{-\frac{J}{2}}\exp\left(-\frac{\sum_{j=1}^{J}(\mu_j-\mu)^2}{2\tau^2}\right)\cdot(\tau^2)^{-\frac{1}{2}}\exp\left(-\frac{m_0(\mu-\mu_0)^2}{2\tau^2}\right)\cdot(\tau^2)^{-\frac{m_0}{2}-1}\exp\left(-\frac{m_0\tau_0^2}{2\tau^2}\right)\\
&\propto \quad (\tau^2)^{-\frac{J+m_0+1}{2}-1}\exp\left(-\frac{\sum_{j=1}^{J}(\mu_j-\mu)^2+m_0(\mu-\mu_0)^2+m_0\tau_0^2}{2\tau^2}\right)\\
p(\tau^2|-) &\sim \quad IG\left(\frac{m_0+J+1}{2},\frac{\sum_{j=1}^{J}(\mu_j-\mu)^2+m_0(\mu-\mu_0)^2+m_0\tau_0^2}{2}\right)
\end{aligned}
$$

v)

$$\text{Full-Conditional: } p(\sigma_0^2|\mu_1,...,\mu_J,\sigma_1^2,...\sigma_J^2,Y,\mu,\tau^2,v_0)$$

$$\begin{aligned}
p(\sigma_0^2|-) &\propto \prod_{j=1}^{J}\left(\frac{v_0\sigma_0^2}{2}\right)^{\frac{v_0}{2}}\exp\left(-\frac{v_0\sigma_0^2}{2\sigma_j^2}\right)\cdot(\sigma_0^2)^{a-1}\exp(-b\sigma_0^2) \\
&\propto (\sigma_0^2)^{\frac{Jv_0}{2}+a-1}\exp\left(-\left(b+\frac{v_0}{2}\sum_{j=1}^{J}\frac{1}{\sigma_j^2}\right)\sigma_0^2\right) \\
p(\sigma_0^2|-) \sim &\quad Gamma\left(a+\frac{Jv_0}{2},\ b+\frac{v_0}{2}\sum_{j=1}^{J}\frac{1}{\sigma_j^2}\right)
\end{aligned}$$

vi)

$$\text{Full-Conditional: } p(v_0|\mu_1,...,\mu_J,\sigma_1^2,...\sigma_J^2,Y,\mu,\tau^2,\sigma_0^2)$$

$$\begin{aligned}
p(v_0|-) &\propto \left(\prod_{j=1}^{J}\left(\frac{\frac{v_0\sigma_0^2}{2}}{\Gamma\frac{v_0}{2}}\right)^{\frac{v_0}{2}}(\sigma_j^2)^{-\frac{v_0}{2}-1}\exp\left(-\frac{v_0\sigma_0^2}{2\sigma_j^2}\right)\right)\cdot\exp(-\rho v_o) \\
&\propto \left(\frac{v_0\sigma_0^2}{2}\right)^{\frac{Jv_0}{2}}\Gamma\left(\frac{v_0}{2}\right)^{J}\left(\prod_{j=1}^{J}\sigma_j^2\right)^{-\frac{v_0}{2}}\exp\left(-v_0\left(\rho+\frac{\sigma_0^2}{2}\sum_{j=1}^{J}\frac{1}{\sigma_j^2}\right)\right)
\end{aligned}$$

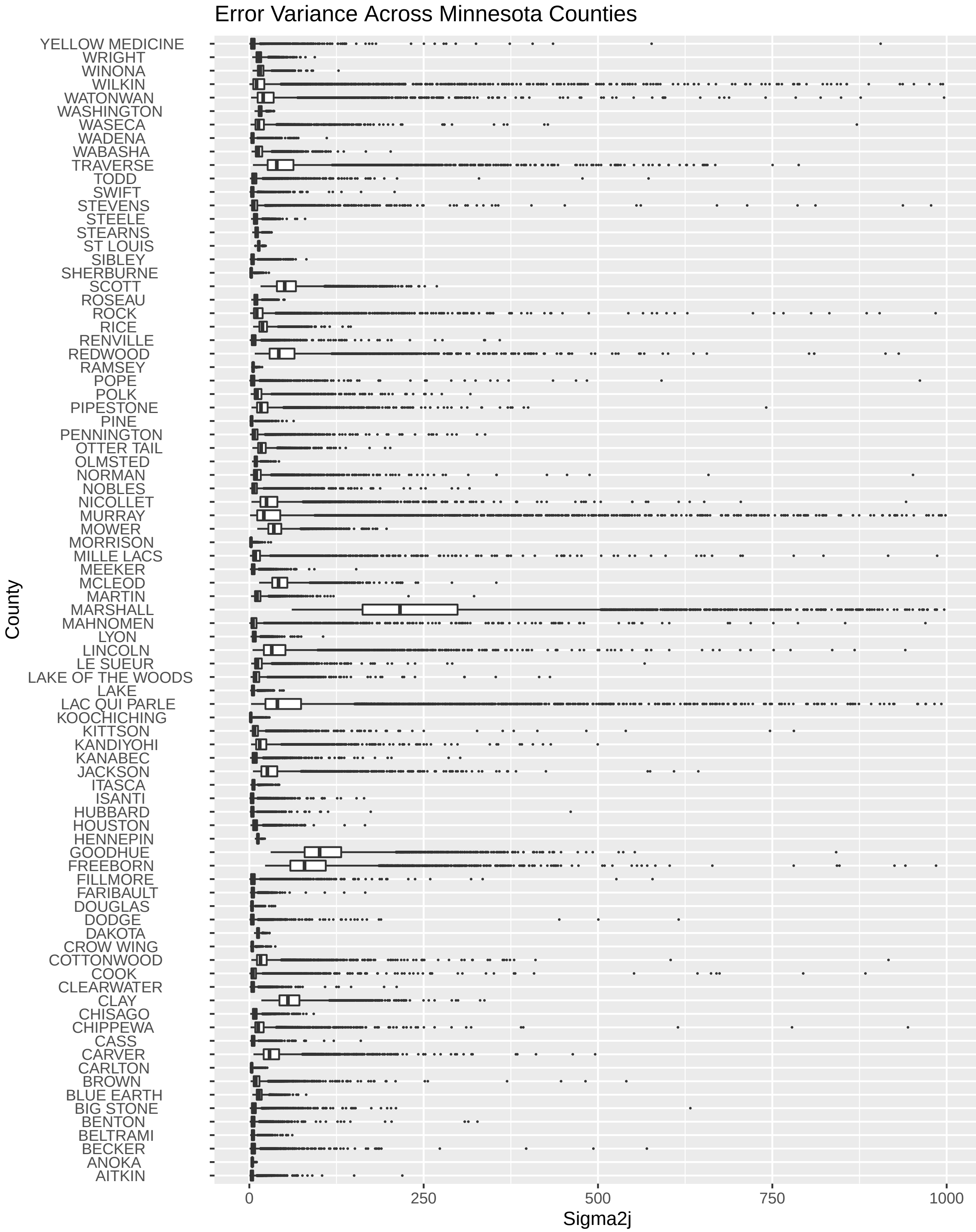
This is not a known distribution, but it can be discretized. We can do this by taking the density of a gamma and multiplying by the exponential portion (the first line of the full conditional). From there we can divide by the summation and sample from a grid of values with these probabilities.

Question 3: R Function for Heterogeneous Variances

```r
# Function for sampling v0
sample_v0 <- function(v0_grid, s02, sigma2j, rho) {
  values <- rep(0, length(v0_grid))
  for(k in 1:length(v0_grid)) {
    v0 <- v0_grid[k]
    values[k] <- sum(dinvgamma(sigma2j, v0/2, (v0*s02)/2), log=TRUE) - (rho*v0)
  }
  values <- exp(values-max(values))+max(values)
  values <- values/(sum(values))

  v0 <- sample(v0_grid, size=1, prob=values)

  return(v0)
}

gibbs_norm <- function(Y, groups, iter= 10000, burn=1000, t02=1, mu0=0,
                       m0=1, n0=1, a=1, b=1, rho=1) {
  # Number of groups
  J <- length(unique(groups))
  # Store MUj, SIGMA2j, TAU2, MU, SIGMA20, V0
  MUj <- matrix(NA, iter, J)
  SIGMA2j <- matrix(NA, iter, J)
  TAU2 <- rep(NA, iter)
  MU <- rep(NA, iter)
  SIGMA20 <- rep(NA, iter)
  V0 <- rep(NA, iter)

  ybarj <- aggregate(Y, list(groups), mean)$x
  nj <- table(groups)
  group_names <- names(nj)

  # Initialize parameters
  sigma2j <- rep(1,J)
  tau2 <- 1
  mu <- 0
  muj <- rep(0, J)
  s02 <- 1
  v0 <- 1
  v0_grid <- seq(0.01, 5, .01)

  for(r in 1:(iter+burn)) {
    if(r%%1000 == 0) {
      print(r)
    }
    # Sample MUj
```

```r
    for(j in 1:J) {
      nuj <- 1/((nj[j]/sigma2j[j])+(1/tau2))
      muj[j] <- rnorm(1, nuj*((nj[j]/sigma2j[j])*ybarj[j]+(1/tau2)*mu), sqrt(nuj))
    }
    # Sample SIGMA2j
    for(j in 1:J) {
      sigma2j[j] <- 1/rgamma(1, (v0+nj[j])/2, (v0*s02+sum((Y[groups == group_names[j]]-muj[j])
    }


    # Sample TAU2
    tau2 <- 1/rgamma(1, (m0+J+1)/2, (m0*t02 + m0*(mu-mu0)^2+sum((muj-mu)^2))/2)

    # Sample MU
    mu <- rnorm(1, (m0*mu0+sum(muj))/(mu0+J), sqrt(tau2/(m0+J)))

    # SIGMA20
    s02 <- rgamma(1, a + (J*v0)/2, b + (v0/2)*sum((1/sigma2j)))

    #V0
    v0 <-sample_v0(v0_grid, s02, sigma2j, rho)

    # Store values
    if(r > burn) {
      MUj[r-burn, ] <- muj
      SIGMA2j[r-burn, ] <- sigma2j
      TAU2[r-burn] <- tau2
      MU[r-burn] <- mu
      SIGMA20[r-burn] <- s02
      V0[r-burn] <- v0

    }
  }
  return(list(MUj = MUj, SIGMA2j = SIGMA2j, TAU2 = TAU2, MU =MU, SIGMA20=SIGMA20,
              V0=V0))
}
```

After running the Gibbs sampler, a plot of the error variances across Minnesota counties indicates that heterogeneous error variances are present in the radon data. This evident by the clear difference in boxplots from county to county, as seen in the plot below.


Error Variance Across Minnesota Counties
County
Sigma2j
0
250
500
750
1000
YELLOW MEDICINE
WRIGHT
WINONA
WILKIN
WATONWAN
WASHINGTON
WASECA
WADENA
WABASHA
TRAVERSE
TODD
SWIFT
STEVENS
STEELE
STEARNS
ST LOUIS
SIBLEY
SHERBURNE
SCOTT
ROSEAU
ROCK
RICE
RENVILLE
REDWOOD
RAMSEY
POPE
POLK
PIPESTONE
PINE
PENNINGTON
OTTER TAIL
OLMSTED
NORMAN
NOBLES
NICOLLET
MURRAY
MOWER
MORRISON
MILLE LACS
MEEKER
MCLEOD
MARTIN
MARSHALL
MAHNOMEN
LYON
LINCOLN
LE SUEUR
LAKE OF THE WOODS
LAKE
LAC QUI PARLE
KOOCHICHING
KITTSON
KANDIYOHI
KANABEC
JACKSON
ITASCA
ISANTI
HUBBARD
HOUSTON
HENNEPIN
GOODHUE
FREEBORN
FILLMORE
FARIBAULT
DOUGLAS
DODGE
DAKOTA
CROW WING
COTTONWOOD
COOK
CLEARWATER
CLAY
CHISAGO
CHIPPEWA
CASS
CARVER
CARLTON
BROWN
BLUE EARTH
BIG STONE
BENTON
BELTRAMI
BECKER
ANOKA
AITKIN

Question 4: Correlation One

I would propose a random effects model with the random effect being on school.

$$\text{Model: } y_{ij} = \mu_j(school) + \epsilon_{ij}$$

$$\text{Assumptions:}(y_{ij}|\mu_j, \sigma_j^2) \sim N(\mu_j, \sigma_j^2) \quad i = 1, ...n_j \quad j = 1, ..., J$$

$$\mu_j \sim N(\mu,\ \tau^2)$$

$$\sigma_j^2 \sim IG\left(\frac{v_0}{2},\ \frac{v_0\sigma_0^2}{2}\right)$$

$$\mu \sim N\left(\mu_0,\ \frac{\tau^2}{m_0}\right)$$

$$\tau^2 \sim IG\left(\frac{m_0}{2},\ \frac{m_0\tau_0^2}{2}\right)$$

The ranking on my model would be different than that of one just observing the mean from each school because I have accounted for different variances across schools. As mentioned in the problem, some of the schools have a large number of participants, while others have a much smaller pool. The schools that have larger sample sizes will have smaller variances than schools with smaller sample sizes. My model takes these differences into consideration. To employ my ranking system I would rank the schools by their posterior mean. To achieve an uncertainty quantification I would compute the probability that each school is among the best 10 in the country. This can be accomplished by ranking the schools in each iteration of the Gibbs sampler and calculating the average number of times each school falls in the top 10. I would expect that for most schools it would be 0.