The Most Efficient Way to Contain a Virus with Limited Vaccinations

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***Abstract*—We have recently experienced the pandemic of COVID and know how a biological virus can spread globally in a matter of months. Initially, vaccinations were in short supply, raising the question of how to quickly contain a virus with limited vaccinations. Using the previous models of Moeller and Wang, Hartke, Messinger, and Fogarty, we will endeavor to find the most efficient use of vaccinations to contain a viral spread. We focus on a pentagon graph and a regular hexagon grid. The pentagon will have vertices with degrees of three or four. We will create different simulations where the infection starts on two vertices, either both of degree three, both of degree four or one of each. In contrast, the hexagon is regular, with all vertices having degrees of six. We will run several simulations on both graph types to see which placements of the vaccinations will be most efficient in containing the viral spread.**

***Index Terms*—COVID, vaccinations, virus**

I. Introduction

COVID-19 has shown us how quickly a virus can spread globally when there is no real containment strategy or vaccine put in place to stop the spread. New viruses have limited vaccinations due to financing shortfalls, manufacturing capacity, and logistical challenges. In this virus spread project, we observe and construct different graphs that will be used as containment strategies to stop the spread of a virus. In each graph, we begin with different scenarios and observe how the spread will react with that particular graph. For one of the triangular grids we decided to start with 1 infected and provide 3 vaccines per round, this can be seen in Figure 1.

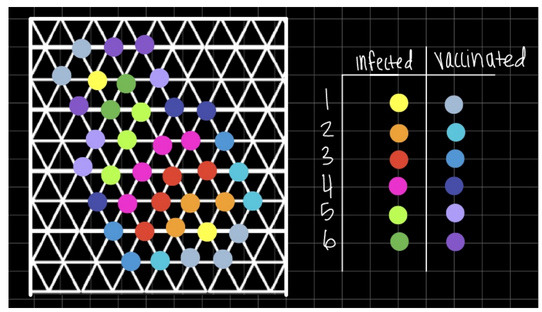

Fig. 1: This graph shows containment in 7 steps with 1 infected and 3 vaccinations

Then we used a 6- regular graph where each vertex has a degree of 6, and a pentagon graph where the vertices either have a degree of 3 or 4. For the pentagon graph, we started with 2 infections on all of them and provided 3 vaccines per round, but decided to place the infections at vertices of different degrees. We placed the infections at a degree of 4, 3, and both. Next, for the 6- regular graphs we started with 1 infection and provided 3 vaccines for both graphs, but placed the vaccines on different vertices for each graph to see how the virus spread would change. Our main goal is to find the most efficient way to contain the spread of the virus without it spreading around where we place the vaccines.

Definitions:

A 6-regular graph is a graph where each vertex has degree 6.

The triangle graph is 6-regular.

A pentagon graph is a graph where vertices have either degree 3 or 4 which can be seen in Figures 6-8.

Containment Protocol - a method determining the most effective way to deploy the vaccinations by observing how the infection spreads over two time steps and then determine what the most infectious nodes there were after the second step.

Methodology:

We start with either one or two infected vertex and calculate the number of vaccines that are used at each time step by taking the roof off half of the greatest vertex degree. We came up with a formula for calculating the number of vaccines there are at each time step by adding one to the ceiling of the highest degree divided by three. We can evaluate the effectiveness of all possible containment protocols at each time step by observing how many vertices the virus spreads to in two time steps.

II. Previous Works

In Catching the Fire on Grids [1], Fogarty uses graphs to show how to encapsulate a fire by firefighters fighting on a grid. She defines a graph as a *vertex set* $V(G) = \{v_1, v_2, ..., v_n\}$ and an *Edge Set* $E(G) = \{e_1, e_2, ..., e_m\}$, where she also states that have adjacent vertices which are adjacent to any vertex in the graph [1]. Firefighters protect vertices from catching on fire on a grid. There is a starting point of the fire which Fogarty references as the original burned vertex, and each new vertex that catches on fire is called a burned vertex. Then she uses protected vertex and saved vertex as defined by Wang and Moeller [4]. A protected vertex is where a firefighter is present and actively prevents the fire from spreading to that exact point, whereas a saved vertex is when the fire has been engulfed and that vertex has not been burned.

In Hartke's thesis *Graph-theoretic Models of Spread and Competition*, they proved the optimal containment for square lattices set forth by Wang and Moeller, which holds under the assumption of small average vertex degree. Also by using the Markov and Semi-Markov processes they use transition probabilities and a transition matrix to show the finite state of the Markov chain. Hartke also uses linear and integer programming to find an optimal solution for encapsulating the disease.

Messinger's *Firefighting on Infinite Grids* [3] thesis examines how long it takes firefighters to contain a fire on several infinite grids, but most importantly on a triangular grid. We will be focusing on Conjecture 12 [Messinger] If a fire breaks out on the triangular grid, the minimum time at which the fire can be surrounded using three firefighters is time $t = 5$, leaving a minimum of 17 vertices burned. We will be attempting to make a formal math proof of Conjecture 12 by referencing *Fire Control on Graphs* Theorem 4.3 [Wang & Moller] [4] A fire cannot be contained in less than 8 time steps with two firefighters on a Cartesian grid as well as *Graph-Theoretic Models of Spread and Competition* Theorem 3.11 [Hartke] [2] If an outbreak of fire starts a single vertex, then when using two firefighters per time step at least 18 vertices are burned, and Theorem 3.13 which is another way to prove Wang & Moller's Theorem 4.3. We will also be attempting to find the critical density value for a triangular grid to support the Conjecture 12 proof.

III. Case Study

In order to get our study done, we used the method of containment protocol. Originally we started with one infection site, but we found that we could completely contain the virus in two steps with three vaccinations in each step. So we would place vaccines at specific spots at different locations around the infected. Then placements of the vaccine were chosen based on which limited the spread of the infection. This is shown in figure 2, as the left graph has one more possible infection site with 5 nodes versus the right graph which only has 4 nodes.

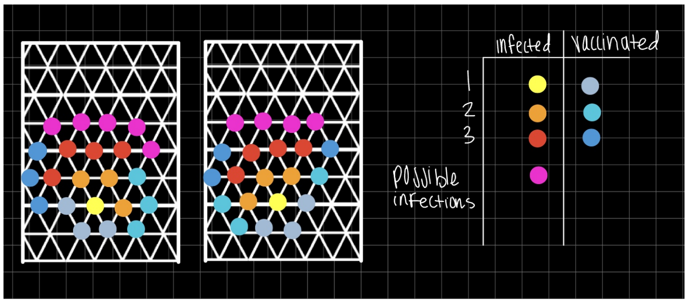

Fig. 2: In this example the left side graph shows more vertices are in risk of getting infected than the right side graph

Then we would continue with a second vaccine placement on the right graph to continue the containment protocol. There are cases where there are equal probable infection sites. Such as Figure 3 and Figure 4. This means that there is a tie-breaker to decide between the two.

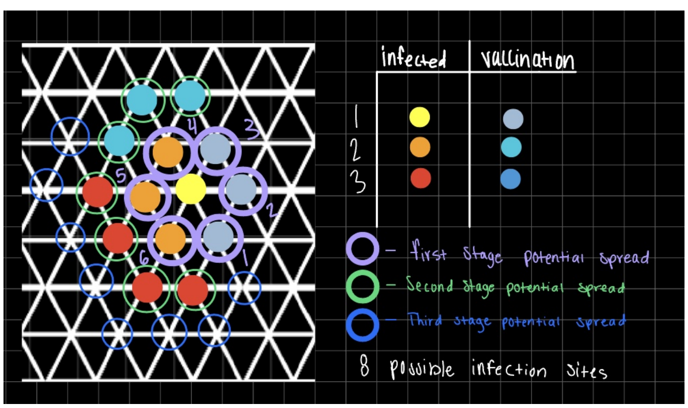

Fig. 3: This way shows that there are eight possible infection sites after two steps of the virus

If we chose poorly, then we would result in a failed containment field as shown in Figure 5.

We would repeat what we did in the step above to see which has the least amount of possible infection sites depending on

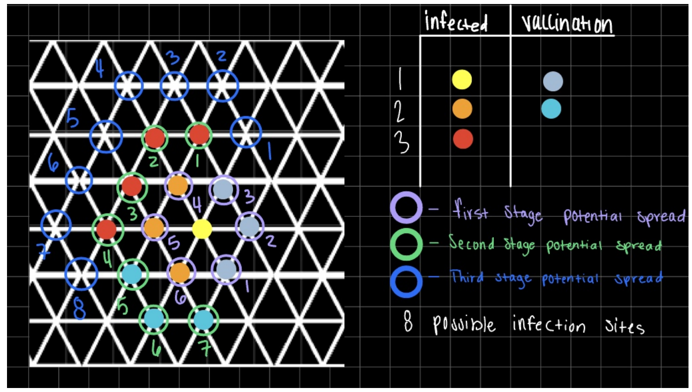

Fig. 4: Also keeping the vaccines together in a different manner also produces 8 possible infection sites as well

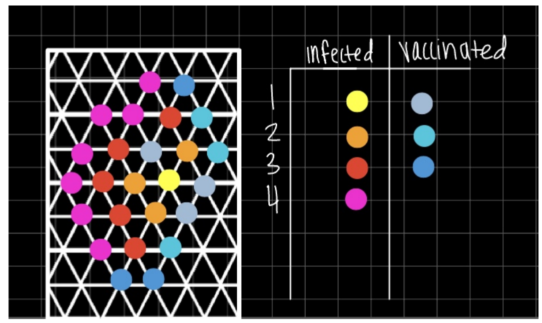

Fig. 5: Shows what the failed containment looks like

where we place our vaccines. This would continue until the virus has been contained. If there are ties there is a way to decide which way to move forward with the specific vaccine placement. We made several scenarios for the pentagon graph. In each scenario, we started at two vertices which we will define as people. Furthermore, we provided each scenario with three vaccinations per step. The first case we made started with two infections at degree four, as shown in Figure 6.

We started with applying the vaccines in unconnected nodes around the infected which was not successful in containing the virus. After using the protocol containment method, we showed that we needed to start the vaccines in adjacent vertices, then in further vaccinations to split them up.

A. Observing the Results

What we observed was that we were able to contain the infection with four steps, meaning that ten people were

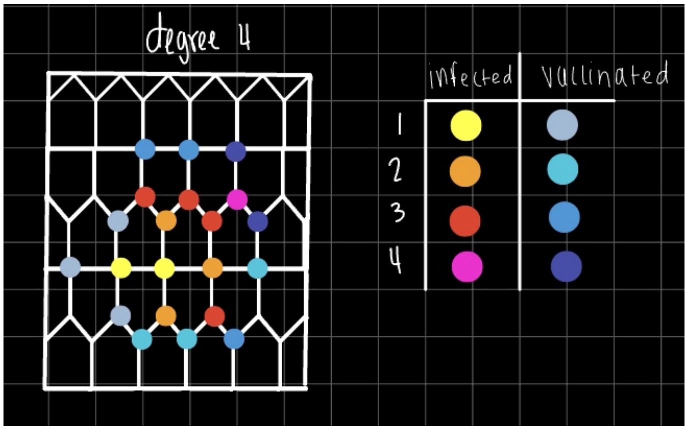

Fig. 6: Where the origin infected started on a 4 degree vertex graph

infected and eleven people were vaccinated. In the second case, we also started with two infected at degree three. We were able to contain the infection with two steps, with only four people being infected and five people being vaccinated. In the last case, we included one of each degree of three and four. In this case, we were able to contain the infection with four steps as well. Although very similar to the case of infections at degree four, only nine people were infected and ten people were vaccinated. Moving on to the triangular grid (degree 6) we started with an infection at only one vertex and also provided three vaccinations at every step. We were able to contain the virus in five steps.

IV. Conclusion

By putting the original infected with the virus on the 3 degree vertices in the pentagon graph, we were able to contain the virus in two steps, see Figure 7.

Then we tried putting the original infected on one 3 degree vertex and one 4 degree vertex, we were able to contain that spread in three steps, please see Figure 8.

In the triangle graph we discovered that our best solution to containing the virus can be done in 7 steps. Though when we started we kept all the vaccines together to create a line,

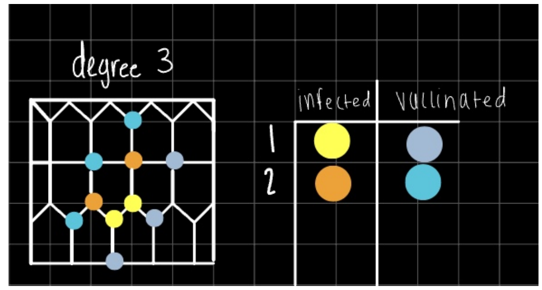

Fig. 7: Where the origin infected started on a 3 degree vertex graph

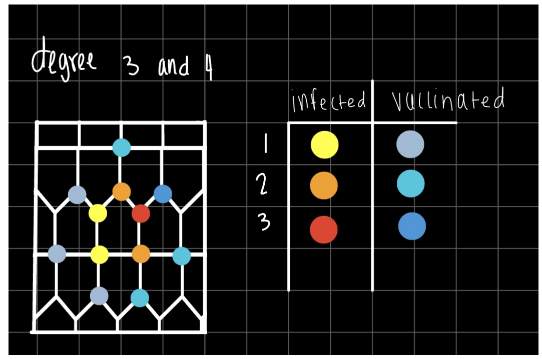

Fig. 8: It took only three steps to contain the virus with an infected on each degree vertices

this was ineffective as it allowed the virus to go around the 'wall' that we created. Please see Figure 9.

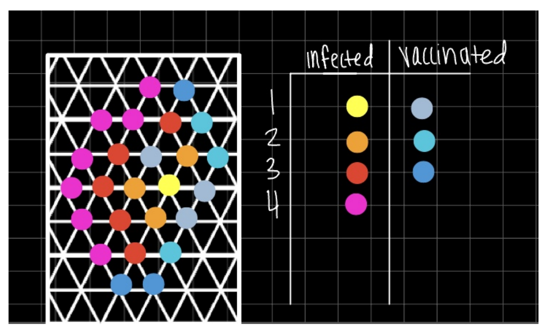

Fig. 9: Shows a failed containment where the vaccines go around the wall

However, as Conjecture 12 of Messinger's thesis states, it is possible to contain the spread in as little as 5 steps as long as we begin and add 3 vaccines per step. As we have worked on it, we were unable to do so in 5 steps. The first one we did had a starting infection of two people and was contained in 13 steps as shown in Figure 10.

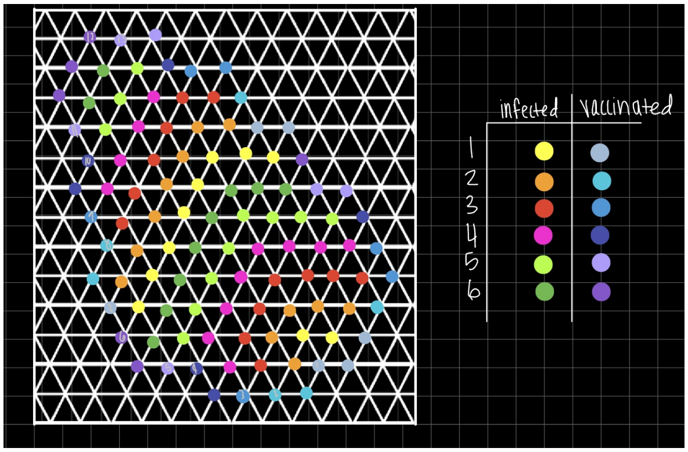

Fig. 10: Containment in 13 steps

Then we did some more testing and was able to improve from 13 to 12 as shown in Figure 11 Then we tried doing

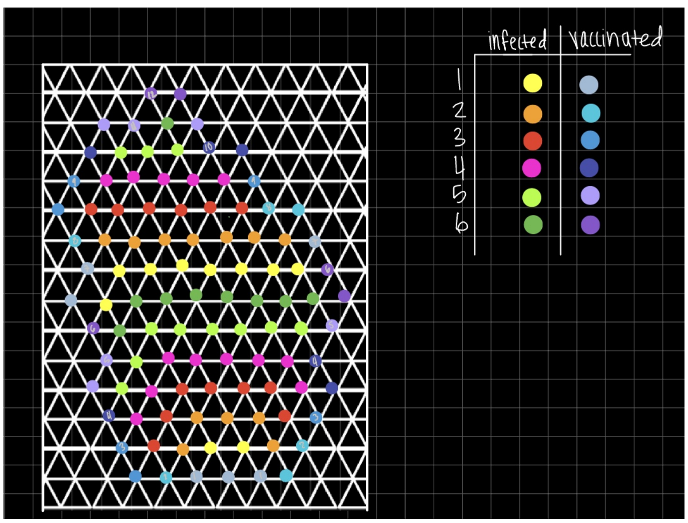

Fig. 11: Containment in 12 steps

with a starting three person infection. It can still be contained, however the steps to containment jumped from 12 to 23, which is a far cry from Messinger's Conjecture. Please see Figure 12 to show the steps

We are unable to get this down any further at this point, we will continue our work to see if we can prove the Conjecture. For an added bonus please scan this QR code that will allow you to see the virus spread and the containment protocol work in moving time.

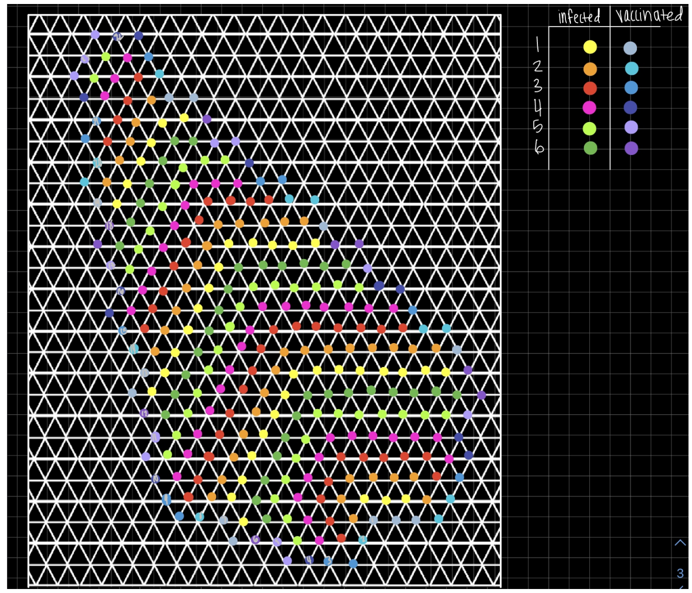

Fig. 12: This graph shows containment in 23 steps far from Messinger's thesis

Fig. 13: Please use this QR code to see an animated version of the virus containment protocol

V. Acknowledgment

We would like to thank Dr. Thomas Mattman for his contribution and dedication in helping us complete this paper. Without his knowledge and compassion, this paper would not have been completed.

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