Formula Sheet

1. Bisection Method

Applicability Criteria:

- The function $f(x)$ is continuous on the interval $[a, b]$.
- $f(a)\,f(b) < 0$, i.e., the signs of $f(a)$ and $f(b)$ are opposite.

Algorithm:

$$\text{Given } a, b \text{ with } f(a)f(b) < 0:$$
$$x_{\text{mid}} = \frac{a+b}{2},$$
$$\text{Check the sign of } f(x_{\text{mid}}):$$
$$\begin{cases} a \leftarrow x_{\text{mid}}, & \text{if } f(a)\,f(x_{\text{mid}}) > 0, \\ b \leftarrow x_{\text{mid}}, & \text{otherwise.} \end{cases}$$

Repeat until convergence or maximum iterations.

2. Newton-Raphson Method

Formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Failure Criteria:

- If $f'(x_n) = 0$ at any iteration, the method fails or stalls.
- If iterations diverge or oscillate indefinitely.
- Poor initial guess or a function whose derivative is too small near the root can cause divergence.

3. Fixed-Point Iteration

General Form:

$$x_{n+1} = g(x_n).$$

Convergence Criterion (Sufficient Condition):

$$|g'(x)| < 1 \quad \text{in a neighborhood of the root.}$$

If $|g'(x)| \geq 1$, the iteration may fail to converge.

4. Numerical Differentiation

First Derivative Approximations

(i) First-Order Forward Difference (Order h):

$$f'(x_0) \approx \frac{f(x_0 + h) - f(x_0)}{h}.$$

(ii) First-Order Backward Difference (Order h):

$$f'(x_0) \approx \frac{f(x_0) - f(x_0 - h)}{h}.$$

(iii) First-Order Central Difference (Order h^2):

$$f'(x_0) \approx \frac{f(x_0 + h) - f(x_0 - h)}{2h}.$$

Second-Order Approximations for the First Derivative

(i) Second-Order Forward Difference (Order h or h^2 depending on exact formula): One common second-order accurate formula (for the first derivative) is:

$$f'(x_0) \approx \frac{-3f(x_0) + 4f(x_0 + h) - f(x_0 + 2h)}{2h} \quad \text{(often } O(h^2)\text{)}.$$

(ii) Second-Order Backward Difference (similarly $O(h^2)$):

$$f'(x_0) \approx \frac{3f(x_0) - 4f(x_0 - h) + f(x_0 - 2h)}{2h}.$$

(iii) Second-Order Central Difference (Order h^2): A commonly used second-order central approximation for the first derivative is

$$f'(x_0) \approx \frac{-f(x_0 + 2h) + 8f(x_0 + h) - 8f(x_0 - h) + f(x_0 - 2h)}{12h},$$

which has $O(h^4)$ error. However, a simpler form for central difference that achieves $O(h^2)$ is:

$$f'(x_0) \approx \frac{f(x_0 + h) - f(x_0 - h)}{2h}$$

but it is typically referred to as the *first-order central difference with a second-order error term.*

5. Numerical Integration

Trapezoidal Rule (Composite)

For n sub-intervals, with $x_i = a + i\,h$ and $h = (b - a)/n$:

$$\int_a^b f(x)\,dx \approx \frac{h}{2}\Big[f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)\Big].$$

Simpson's $\frac{1}{3}$ Rule

$$\int_a^b f(x)\,dx \approx \frac{h}{3}\Big[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\Big],$$

where n (number of sub-intervals) is **even**.

Applicability Criterion: The number of intervals n (sub-intervals) must be even (so there are $n+1$ data points).

Simpson's $\frac{3}{8}$ Rule

$$\int_a^b f(x)\,dx \approx \frac{3h}{8}\Big[f(x_0)+3f(x_1)+3f(x_2)+2f(x_3)+3f(x_4)+3f(x_5)+2f(x_6)+\cdots+f(x_n)\Big],$$

where n is a multiple of 3.

Applicability Criterion: The number of intervals n must be a multiple of 3.

6. Euler Method for ODEs

For the initial value problem

$$\frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0,$$

with step size h, Euler's method is:

$$y_{n+1} = y_n + h\,f(x_n, y_n).$$

Here $x_{n+1} = x_n + h$.