Elements of Statistics and Probability

STA 201

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Lecture-10

Random Variable:

A random variable is a function that assigns a numerical value to each outcome of a random experiment.

Example: X = Number of heads out of two tossed of a fair coin.

So, X = (0, 1, 2)

Types of Random Variable

There are two types of random variables:

(a) Discrete random variable.

(b) Continuous random variable.

(a) Discrete random variable: If a random variable X assumes only a finite number values, then it is called a discrete random variable.

Example: Number of throws of a fair coin before the first head occurs.

(b) Continuous random variable: If a random variable is such that it assumes any value with in a given interval, then it is called as a continuous random variable.

Example: The heights of the persons collected from a crowd.

Probability mass function (Pmf): A function $p(x)$ that gives probability for different values of a discrete random variable X.

Properties of $p(x)$

i. $p(x) \geq 0$

ii. $\sum_x p(x) = 1$

Example: Two fair coins tossed once.

X = number of heads out of 2 tossed

So, X = (0, 1, 2)

Here, S = {HH, HT, TH, TT}

P(0) = 1/4

P(1) = P(HT)+P(TH)

= 1/4 +1/4

= 1/2

P(2) = 1/4

x	0	1	2
$p(x)$	1/4	1/2	1/4

Probability density function (Pdf): Let X be a random variable. A function $f(x)$ is called pdf of X if it satisfied

i. $f(x) \geq 0$

ii. $\int_x f(x)\,dx = 1$

Example: $f(x) = k(1 - x^2)$ $\quad\quad -1 < x < 1$

Determine,

i. k

ii. P$(-.5 < x < .5)$

iii. P$(x < 0)$

iv. $P(x > .5)$

v. $P(x = .7)$

Solution:

i. $\int_{-1}^{1} k(1 - x^2)\, dx = 1$

$\Rightarrow k\left(x - \frac{x^3}{3}\right)\begin{matrix} 1 \\ -1 \end{matrix} = 1$

⋮

⋮

$\therefore k = \frac{3}{4}$

$$\therefore f(x) = \frac{3}{4}(1 - x^2) \qquad\qquad -1 < x < 1$$

ii. $\int_{-.5}^{.5} \frac{3}{4}(1 - x^2)\ dx$

$= \frac{3}{4}\left(x - \frac{x^3}{3}\right)\begin{matrix} .5 \\ -.5 \end{matrix}$

⋮

⋮

$=$

iii. $\int_{-1}^{0} \frac{3}{4}(1 - x^2)\ dx$

$= \frac{3}{4}\left(x - \frac{x^3}{3}\right)\begin{matrix} 0 \\ -1 \end{matrix}$

⋮

⋮

$=$

iv. $\int_{.5}^{1} \frac{3}{4}(1-x^2)\ dx$

$= \frac{3}{4}\left(x - \frac{x^3}{3}\right)\begin{matrix}1\\.5\end{matrix}$

$\vdots$

$\vdots$

$=$

v. $P(x = .7) = 0$

As, in this case length = .7 but width = 0.

So, *area* = *length* $\times\ width = 0$ [Integration gives area]

Mathematical Expectation:

Let us consider a discrete random variable X which assumes the values x_1, x_2,..., x_n with respective probabilities p_1, p_2,..., p_n , such that $\sum_1^n p_i = 1$, then the mathematical expectation of the random variable X is given by the sum of the products of the different values of X with their

corresponding probabilities. The expectation of a random variable is generally denoted by $E(X)$.

$$E(X) = \sum_{i}^{n} x_i p_i$$

If X is a continuous random variable with probability density function $f(x) \quad -\infty < x < \infty$

Then the mathematical expectation of the random variable X is given by

$$E(X) = \int_{-\infty}^{\infty} xf(x)\,dx$$

Some properties of Mathematical Expectations:

i. If C is a constant then $E(C) = C$

ii. If C is a constant then $E(CX) = C\,E(X)$

iii. If a and c are two constants then $E(aX + c) = a\,E(X) + c$

iv. The expectation of the sum of two random variables are equal to the sum of their expectations i.e. $E\,(X + Y) = E(X) + E(Y)$

v. The expectation of the difference of two random variables are equal to the difference of their expectations i.e. $E\,(X - Y) = E(X) - E(Y)$

vi. If X and Y are two independent random variables then, $E(XY) = E(X)\,E(Y)$

Variance of a Random Variable:

Variance is an important characteristic of a random variable. It is a measure of dispersion of therandom variable. Variance of X, is denoted by $Var(X)$ or by ${\sigma_x}^2$. It is defined as the expected value of the square of the deviation of the random variable from its mean value.

$$Var(X) = E[X - E(X)]^2$$

The positive square-root of the variance is called the standard deviation and is denoted by σ_x.

A simplified expression for the variance can be derived as:

$$Var(X) = E(X^2) - \{E(X)\}^2$$

Some properties of variance of a Random Variable:

i. If C is a constant then $Var(C) = 0$

ii. If C is a constant then we have $Var(CX) = C^2\, Var(X)$

iii. If A and C are two constants then $Var(AX + C) = A^2\, Var(X)$

iv. If X and Y are two independent random variables then, $Var(X + Y) = Var(X) + Var(Y)$

$Var(X - Y) = Var(X) + Var(Y)$

v. If X and Y are dependent random variables then,

$Var(X + Y) = Var(X) + Var(Y) + 2Cov(X, Y)$

$Var(X - Y) = Var(X) + Var(Y) - 2Cov(X, Y)$

Example-1

Let X be a random variable with probability distribution

x	0	1	2	3
$p(x)$	1/3	1/2	0	1/6

Determine,

i. $E(X)$

ii. $E(X^2)$

iii. $E[(X-1)^2]$

iv. $P(X > 1)$

v. $P(X \leq 2)$

vi. $Var(X)$

Solution:

i. $E(X) = \sum_i^n x_i p_i$

$$= 0 \times \frac{1}{3} + 1 \times \frac{1}{2} + 2 \times 0 + 3 \times \frac{1}{6}$$

$$= 1$$

ii. $E(X^2) = \sum_i^n x_i{}^2 p_i$

$$= 0^2 \times \frac{1}{3} + 1^2 \times \frac{1}{2} + 2^2 \times 0 + 3^2 \times \frac{1}{6}$$

$$= 2$$

iii. $E[(X-1)^2] = \sum_i^n (x_i - 1)^2 p_i$

$$= (0-1)^2 \times \frac{1}{3} + (1-1)^2 \times \frac{1}{2} + (2-1)^2 \times 0 + (3-1)^2 \times \frac{1}{6}$$

$$= 1$$

iv. $P(X > 1) = P(X = 2) + P(X = 3)$

$$= 0 + \frac{1}{6}$$

$$= \frac{1}{6}$$

v. $P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2)$

$$= \frac{1}{3} + \frac{1}{2} + 0$$

$$= \frac{5}{6}$$

vi. $Var(X) = E(X^2) - \{E(X)\}^2$

$$= 2 - 1^2$$

$$= 1$$

Example-2

$f(x) = 2x, 0 \leq x \leq 1$

$E(X) = ?$

$V(X) = ?$

Solution:

$$E(X) = \int_0^1 x\ f(x)dx$$

$$= \int_0^1 x.2x\ dx$$

$$= 2\int_0^1 x^2\ dx$$

$$= 2\left[\frac{x^3}{3}\right]\begin{matrix}1\\0\end{matrix}$$

$$= 2\left[\frac{1}{3} - 0\right]$$

$$= \frac{2}{3}$$

$$E(X^2) = \int_0^1 x^2\ f(x)dx$$

Then,

$$Var(X) = E(X^2) - \{E(X)\}^2$$

Example-3

$$f(x) = \begin{cases} x, & 0 \le x < 1 \\ 2 - x, & 1 \le x < 2 \\ 0 \; otherweise \end{cases}$$

$E(X) = ?$

$V(X) = ?$

Solution:

$$E(X) = \int_0^2 x \; f(x) dx$$

$$= \int_0^1 x.x \; dx + \int_1^2 x.(2 - x) \; dx$$

$$= \int_0^1 x^2 \; dx + \int_1^2 (2x - x^2) \; dx$$

$$\vdots$$

$$\vdots$$

$$= \frac{1}{3} + \frac{2}{3}$$

$$= 1$$

$$E(X^2) = \int_0^2 x^2 \; f(x) dx$$

$$= \int_0^1 x^2 . x \; dx + \int_1^2 x^2 . (2 - x) \; dx$$

$$= \int_0^1 x^3 \, dx + \int_1^2 (2x^2 - x^3) \, dx$$

Then,

$$Var(X) = E(X^2) - \{E(X)\}^2$$