Elements of Statistics and Probability

STA 201

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Lecture-12

Bernoulli Trials:

A Bernoulli trial is a random experiment with only two possible outcomes: success and failure.

The key characteristics of a Bernoulli trial are:

i. Two Outcomes: The experiment results in one of two possible outcomes success and failure.
ii. Constant Probability: The probability of success, denoted as p, remains constant for each trial.
iii. Independent Trials: The outcome of one trial does not affect the outcome of another.

Binomial Distribution:

The Binomial distribution arises when we conduct a fixed number

n of independent Bernoulli trials, each with the same probability of success p. The random variable X represents the number of successes in these trials.

Then $X\sim$ binomial (n, p) [X follows binomial distribution]

Here, n and p are two parameters of binomial distribution.

The probability mass function (pmf) of the Binomial distribution is given by:

$$p(X = x) = \binom{n}{x} p^x (1-p)^{n-x} \qquad x = 0, 1, 2, \dots, n$$

Where:

- ✓ n is the number of trials.
- ✓ x is the number of successes.
- ✓ p is the probability of success in a single trial.

- ✓ $(1-p)$ is the probability of failure in a single trial.
- ✓ $\binom{n}{x}$ is the binomial coefficient.

Mean and Variance:

$E(X) = np$

$V(X) = np(1-p)$

Problem:

An MCQ test has 10 questions each with 5 possible choices. A student did not study at all and chose all the answer randomly.

i. Probability that 2 answers will be correct.
ii. Probability that at least 1 answer will be correct.
iii. Probability that at most 2 answers will be correct.
iv. Probability that less than 1 answer will be correct.
v. Average number of correct answers.
vi. Variance of this distribution.

Solution:

Here, X = number of successes

$\therefore X \sim$ binomial (n = 10, p = $\frac{1}{5}$ = 0.2)

$\therefore p(X = x) = \binom{10}{x} 0.2^x (1-0.2)^{10-x} \qquad x = 0, 1, 2, \ldots, 10$

i. $p(X = 2) = \binom{10}{2} 0.2^2 (1-0.2)^{10-2}$

ii. $p(X \geq 1) = 1 - p(X = 0)$

$$= 1 - \binom{10}{0} 0.2^0 (1-0.2)^{10-0}$$

iii. $p(X \leq 2) = p(X = 0) + p(X = 1) + p(X = 2)$

iv. $p(X < 1) = p(X = 0)$

v. $E(X) = np = 10 * 0.2 = 2$

vi. $V(X) = np(1-p) = 10 * 0.2 * (1 - 0.2) = 1.6$

Practice problems:

i. A factory produces light bulbs, and 90% of them pass quality control. If a sample of 8 bulbs is randomly selected, what is the probability that exactly 7 pass quality control?

Solution:

$X =$ number of successes.

$\therefore X \sim$ binomial (n = 8, p = 0.90)

$\therefore p(X = x) = \binom{8}{x} 0.9^x (1 - 0.9)^{8-x} \qquad x = 0, 1, 2, \ldots, 8$

So, $p(X = 7) = \binom{8}{7} 0.9^7 (1 - 0.9)^{8-7}$

ii. A basketball player has a free throw success rate of 80%. If the player takes 7 free throws, what is the probability of making at least 5 of them?

Solution:

$X =$ number of successes

$\therefore X \sim$ binomial (n = 7, p = 0.80)

$\therefore p(X = x) = \binom{7}{x} 0.8^x (1 - 0.8)^{7-x} \qquad x = 0, 1, 2, \ldots, 7$

So, $p(X \geq 5) = p(X = 5) + p(X = 6) + p(X = 7)$

Geometric Distribution:

The Geometric distribution models the number of Bernoulli trials needed for the first success. It is derived from a sequence of independent Bernoulli trials, each with the same probability of success p.

The random variable X represents the number of trials required until the first success.

Then $X\sim$ geometric (p) [X follows geometric distribution]

Here, p is the parameter of geometric distribution.

The probability mass function (pmf) of the Geometric distribution is given by: $p(X = x) = p(1 - p)^{x-1}$ $x = 1, 2, 3, \dots\dots\dots\dots$

Where,

- ✓ x is the number of trials required until the first success.
- ✓ p is the probability of success in a single trial.

Mean and Variance:

$$E(X) = \frac{1}{p}$$

$$V(X) = \frac{1-p}{p^2}$$

Problem:

A die is thrown until 6 occurs for the first time. What is the probability that

i. Exactly 3 tosses will be required.

ii. More than two tosses will be required.

iii. Find the average number of tosses required.

iv. Find the variance of this distribution.

Solution:

$X =$ number of trials required until 6 occurs for the first time.

$\therefore X \sim$ geometric ($p = \frac{1}{6}$)

$$\therefore p(X = x) = \frac{1}{6}\left(1 - \frac{1}{6}\right)^{x-1} \qquad x = 1, 2, 3, \dots \dots \dots \dots$$

i. $p(X = 3) = \frac{1}{6}\left(1 - \frac{1}{6}\right)^{3-1}$

$= \frac{1}{6}\left(\frac{5}{6}\right)^{2}$

$= \frac{25}{36}$

ii. $p(X > 2) = 1 - p(X = 1) - p(X = 2)$

iii. $E(X) = \frac{1}{p} = \frac{1}{\frac{1}{6}} = 6$

iv. $V(X) = \frac{1-p}{p^2} = \frac{1-\frac{1}{6}}{\left(\frac{1}{6}\right)^2}$

Practice problems:

i. A biased coin has a probability of landing heads equal to 0.2. What is the probability that the first head occurs on the 4th toss?

Solution:

$X =$ number of trials required until head occurs for the first time.

$\therefore X \sim$ geometric (p =0.2)

$\therefore p(X = x) = 0.2(1 - 0.2)^{x-1}$ $\qquad x = 1, 2, 3, \dots \dots \dots \dots$

So, $p(X = 4) = 0.2(1 - 0.2)^{4-1}$

ii. A student is preparing for a multiple-choice exam. The student knows that the probability of guessing the correct answer to a question is 0.25. What is the probability that the student guesses the correct answer on the 5th attempt?

Solution:

$X =$ number of attempts required until guessing the correct answer to a question.

$\therefore X \sim$ geometric (p =0.25)

$\therefore p(X = x) = 0.25(1 - 0.25)^{x-1}$ $\qquad x = 1, 2, 3, \dots \dots \dots \dots$

So, $p(X = 5) = 0.25(1 - 0.25)^{5-1}$

Poisson distribution:

A probability distribution that models the number of events that occur in a fixed interval of time or space under the assumption that these events occur with a known constant mean rate and are independent of the time since the last event. Usually rare events.

The random variable X represents the number of events in an interval of time or in a region.

Then $X \sim$ poisson (λ) [X follows poisson distribution]

Here, rate of event λ is the parameter of geometric distribution.

The probability mass function (pmf) of the poisson distribution is given by:

$p(X = x) = \frac{e^{-\lambda}\lambda^x}{x!} \qquad x = 0, 1, 2, \dots \dots \dots \dots$

Where,

- ✓ x is the number of events in an interval of time or in a region.
- ✓ λ is the rate of event.

Mean and Variance:

$E(X) = \lambda$

$V(X) = \lambda$

Problem-1

A call center receives an average of 5 customer service calls per hour. What is the probability that the call center receives exactly 3 calls in the next hour?

Solution:

$X =$ number of calls receives by a call center in an hour.

$\therefore X \sim$ poisson (λ =5)

$\therefore p(X = x) = \frac{e^{-5}5^x}{x!} \qquad x = 0, 1, 2, \dots \dots \dots \dots$

So, $p(X = 3) = \frac{e^{-5}5^3}{3!} = 0.1404$

Problem-2

On average, a factory experiences 2 machine failures per day. What is the probability that the factory will have no machine failures on a given day?

Solution:

$X =$ number of machine failures on a given day.

$\therefore X \sim$ poisson (λ =2)

$$\therefore p(X = x) = \frac{e^{-2}2^x}{x!} \qquad x = 0, 1, 2, \dots\dots\dots\dots$$

So, $p(X = 0) = \frac{e^{-2}2^0}{0!} = 0.1353$

Problem-3

A website receives an average of 8 new user registrations per hour. What is the probability that the website receives at least 1 new registrations in the next hour?

Solution:

$X =$ number of new user registrations per hour.

$\therefore X \sim$ poisson (λ =8)

$$\therefore p(X = x) = \frac{e^{-8} 8^x}{x!} \qquad x = 0, 1, 2, \ldots \ldots \ldots \ldots$$

So, $p(X \geq 1) = 1 - p(X = 0) = 1 - \frac{e^{-8} 8^0}{0!} = 0.999$