① $1^3+2^3+3^3+\dots+n^3 = \frac{n^2(n+1)^2}{4}$

base case (1):

LHS $= 1^3$ RHS $= \frac{(1)^2(1+1)^2}{4} = \frac{4}{4} = 1$

base case is true.

inductive step,

$P(k)$, we assume $P(k)$ is true

$P(k) \Rightarrow 1^3+2^3+3^3+\dots+k^3 = \frac{k^2(k+1)^2}{4}$ —①

$P(k+1)$ $1^3+2^3+3^3+\dots+k^3+(k+1)^3 = \frac{(k+1)(k+2)^2}{4}$ —⑪

adding $(k+1)^3$ on both sides of ①

$1^3+2^3+3^3+\dots+k^3+(k+1)^3 = \cancel{\frac{(k+1)^2(k+2)^2}{4}}\ \frac{k^2(k+1)^2}{4}+(k+1)^3$

simplifying RHS

$\frac{k^2(k+1)^2}{4}+(k+1)^3$

$\Rightarrow \frac{k^2(k+1)^2+4(k+1)^3}{4}$

$\Rightarrow \frac{(k+1)^2(k^2+4k+4)}{4}$

$\Rightarrow \frac{(k+1)^2(k+2)^2}{4} = RHS$ $\therefore LHS = RHS$

$\therefore$ $p(k+1)$ is true (proved)

② $3^n > n^2$ for n, a positive integer

base case,

$p(1) \Rightarrow 3^1 > 1^2$

$\Rightarrow 3 > 1$ $\therefore$ base case is true.

inductive case,

$\cancel{(P)}$ $P(k) \Rightarrow 3^k > k^2$ —(i) $\because$ we assume $P(k)$ is true.

$P(k+1) \Rightarrow 3^{k+1} > (k+1)^2$ —(ii)

Multiplying 3 to both sides of (i):

$3^k \times 3 > k^2 \times 3$

$\Rightarrow 3^{k+1} > 3k^2$ —(iii)

we need to prove $3k^2 > (k+1)^2$

expanding the equation,

$(3k^2 + 2k^2) > 2k + k^2 + 1$

$\Rightarrow 2k^2 > 2k + 1$ [removing k^2 from both sides]

for $k > 1$, $2k^2 > 2k+1$

$\therefore 3k^2 > (k+1)^2$

$\therefore 3^{k+1} > 3k^2 > (k+1)^2$

$\therefore 3^{k+1} > (k+1)^2$

$\therefore$ $P(k+1)$ is true for a positive integer > 1 and k is true for 1. Hence, it covers all positive integers. (proved)

③ $111\ldots1 = \frac{(10^n - 1)}{9}$

base case,

$$1 = \frac{10^1 - 1}{9}$$

$$= 1$$

base case is true.

inductive case,

$$P(k) \Rightarrow 111\ldots1 = \frac{10^k - 1}{9} \text{ — (i)}$$ we assume $p(k)$ is true.

$$P(k+1) \Rightarrow 111\ldots11 = \frac{10^{k+1} - 1}{9} \text{ — (ii)}$$

~~adding~~

Multiplying to 10 and adding 1 to both sides of (i):

$$111\ldots1 \times 10 + 1 = \frac{10^k - 1}{9} \times 10 + 1$$

~~simplifying~~

simplifying RHS:

$$\frac{10^k - 1}{9} \times 10 + 1$$

$$\Rightarrow \frac{10^{k+1} - 10}{9} + 1 \Rightarrow \frac{10^{k+1} - 10 + 9}{9} \Rightarrow \frac{10^{k+1} - 1}{9} \Rightarrow RHS$$

$\therefore p(k+1)$ is true (proved)

(4) $n! < n^n$ for $n \geq 2$

base case,

$P(2) = 2! < 2^2$

$= 2 < 4$

base case is true

inductive case

$P(k) \Rightarrow k! < k^k$ — (1) we assume $P(k)$ is true

$P(k+1) \Rightarrow (k+1)! < (k+1)^{(k+1)}$

Multiplying $(k+1)$ to both sides of (1):

$(k+1)\, k! < (k+1)\, k^k$

since,

$(k+1)\, k^k < (k+1)(k+1)^k$,

$(k+1)\, k! < (k+1)(k+1)^k$

$(k+1)! < (k+1)^{k+1}$ $\therefore$ LHS = RHS

$\therefore P(k+1)$ is true (proved)

⑤ any integer ≥ 60, ~~eq~~ can be changed by 6-cent and 11-cent coins.

base case,

$P(60) \Rightarrow$ 6-cent coins $\times 10$, 11-cent coins $\times 0$

base case is true.

inductive case,

$P(k) \rightarrow 6a + 11b$ [Here a and b are multiples of 6 and 11]

if p(k) has no 11,

$p(k+1) \Rightarrow$ replace 9 6-cent coins with 5 11-cent coins.

if p(k) has at least one 11,

$p(k+1) \Rightarrow$ replace one 11-cent coin with 2 6-cent coins.

assuming P(k) is true, P(k+1) is also true (proved

⑥ $1\times 2 + 2\times 2^2 + 3\times 2^3 + \dots + n\times 2^n = (n-1)\,2^{n+1} + 2$

base case,

$P(1) \Rightarrow 1\times 2 = (1-1)\,2^{1+1} + 2$

$\Rightarrow 2 = 2$

base case is true.

inductive step,

$P(k) \Rightarrow 1\times 2 + 2\times 2^2 + \dots + k\times 2^k = (k-1)\,2^{k+1} + 2$ — (i)

$P(k+1) \Rightarrow 1\times 2 + 2\times 2^2 + \dots + k\times 2^k + (k+1)\,2^{(k+1)} = k\,2^{k+2} + 2$ — (ii)

we assume $P(k)$ is true.

adding $(k+1)\,2^{(k+1)}$ to both sides of (i),

$\cancel{1\times 2 + 2\times 2^2 + \dots + k\times 2^k = (k-1)\,2^{k+1} + 2 + (k-1)}$

$1\times 2 + 2\times 2^2 + \dots + k\times 2^k + (k+1)\,2^{(k+1)} = (k-1)\,2^{(k+1)} + 2 + (k+1)\,2^{(k+1)}$

simplifying RHS

$\Rightarrow (k-1)\,2^{k+1} + 2 + (k+1)\,2^{k+1}$

$\Rightarrow 2^{k+1}(k-1+k+1) + 2 \Rightarrow k\,2^{k+2} + 2 \Rightarrow \text{RHS}$

$\therefore \text{LHS} = \text{RHS}$

$\therefore P(k+1)$ is true (proved)

⑦ $1\times3+2\times3^2+3\times3^3+\dots+n\times3^n = \frac{3}{4}[(2n-1)3^n+1]$

base case,

$P(1) \Rightarrow 1\times3 = \frac{3}{4}[(2.1-1)3^1+1]$

$\Rightarrow 3 = 3$

base case is true.

inductive case,

$P(k) \Rightarrow 1\times3+2\times3^2+\dots+k\times3^k = \frac{3}{4}[(2k-1)3^k+1]$ —— ①

we assume $p(k)$ is true.

$P(k+1) \Rightarrow 1\times3+2\times3^2+\dots+k3^k+(k+1)3^{k+1}$

$= \frac{3}{4}[(2(k+1)-1)3^{k+1}+1]$ —— ⑪

adding $(k+1)3^{k+1}$ to both sides of ①

$1\times3+2\times3^2+\dots+k3^k+(k+1)3^{k+1} = \frac{3}{4}[(2k-1)3^k+1]+(k+1)3^{k+1}$

Simplifying RHS

$\frac{3}{4}[(2k-1)3^k+1]+(k+1)3^{k+1}$

$= \frac{3[(2k-1)3^k+1]+4(k+1)3^{k+1}}{4}$

$= \frac{3[(2k-1)3^k+1+(4k+4)3^k]}{4} = \frac{3[3^k(2k-1+4k+4)+1]}{4}$

$= \frac{3[3^k(6k+3)+1]}{4} = \frac{3[3^{k+1}(2k+1)+1]}{4}$

$= \frac{3}{4}[(2k+1)3^{k+1}+1] = RHS$

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$\therefore$ LHS = RHS

$\therefore$ p(k+1) is true (proved)

(8) $\frac{3}{1\times2\times2}+\frac{4}{2\times3\times2^2}+\frac{5}{3\times4\times2^3}+\cdots+\frac{n+2}{n\times(n+1)\times2^n}=1-\frac{1}{(n+1)\times2^n}$

base case,

$P(1)\Rightarrow\frac{3}{1\times2\times2}=1-\frac{1}{(1+1)\times2^1}$

$\Rightarrow\frac{3}{4}=1-\frac{1}{4}\Rightarrow\frac{3}{4}=\frac{3}{4}$

base case is true.

inductive case,

$P(k)\Rightarrow\frac{3}{1\times2\times2}+\frac{4}{2\times3\times2^2}+\cdots+\frac{k+2}{k(k+1)2^k}=1-\frac{1}{(k+1)2^k}$ — (i)

we assume p(k) is true.

$P(k+1)\Rightarrow\frac{3}{1\times2\times2}+\frac{4}{2\times3\times2^2}+\cdots+\frac{k+2}{k(k+1)2^k}+\frac{(k+1)+2}{(k+1)(k+2)2^{k+1}}$

$=\cancel{1-\frac{1}{(k+1)2^k}+\frac{(k+1)+2}{(k+1)(k+2)2^{k+1}}}$

$=1-\frac{1}{(k+2)2^{k+1}}$ — (ii)

adding $\frac{(k+1)+2}{(k+1)(k+2)2^{k+1}}$ to both sides of (i)

$\frac{3}{1\times2\times2}+\frac{4}{2\times3\times2^2}+\cdots+\frac{k+2}{k(k+1)2^k}+\frac{(k+1)+2}{(k+1)(k+2)2^{k+1}}=1-\frac{1}{(k+1)2^k}+\frac{(k+1)+2}{(k+1)(k+2)2^{k+1}}$

Simplifying RHS

$$1 - \frac{1}{(k+1)2^k} + \frac{k+3}{(k+1)(k+2)2^{k+1}}$$

$$\Rightarrow 1 + \frac{-1}{(k+1)2^k} + \frac{k+3}{(k+1)(k+2)2^{k+1}}$$

$$\Rightarrow 1 + \frac{-2(k+2) + k+3}{(k+1)(k+2)2^{k+1}}$$

$$\Rightarrow 1 + \frac{-2k - 4 + k + 3}{(k+1)(k+2)2^{k+1}} \Rightarrow 1 + \frac{-k-1}{(k+1)(k+2)2^{k+1}}$$

$$\Rightarrow 1 - \frac{1}{(k+2)2^{k+1}} \Rightarrow \text{RHS}$$

$\therefore$ LHS = RHS

$\therefore$ P(k+1) is true. (proved)

9 $5^{2n+1} - 12 \times 16^{2n}$ is divisible by 7

base case,

$$P(1) \rightarrow \frac{5^{2.1+1} - 12 \times 16^{2.1}}{7} = -421$$

base case is true.

inductive case,

$P(k) \Rightarrow 5^{2k+1} - 12 \times 16^{2k} = 7p$ — ① [d is multiple of 7]

we assume $P(k)$ is true.

$P(k+1) \Rightarrow 5^{2(k+1)+1} - 12 \times 16^{2(k+1)}$ — ⑪

extracting ① from ⑪:

$5^{2k+3} - 12 \times 16^{2k+2}$

$\Rightarrow 5^{2k+1} \times 5^2 - 12 \times 16^{2k} \times 16^2$

$\Rightarrow 5^2 \times 5^{2k+1} - 5^2 \times 12 \times 16^{2k} - 231 \times 12 \times 16^{2k}$

$\Rightarrow 5^2 (5^{2k+1} - 12 \times 16^{2k}) - 231 (12 \times 16^{2k})$

$\Rightarrow 5^2 (5^{2k+1} - 12 \times 16^{2k}) - 7 (33 \times 12 \times 16^{2k})$

$\Rightarrow 5^2 \times 7p - 7 (33 \times 12 \times 16^{2k})$

which is divisible by 7

$\therefore P(k+1)$ is true (Proved)

(10) $4^{2n+1} + 10 \times 11^{2n}$ divisible by 7

base case,

$$P(1) = \frac{4^{2.1+1} + 10 \times 11^{2.1}}{7} = 182$$

$\therefore$ base case is true

inductive case,

$P(k) \Rightarrow 4^{2k+1} + 10 \times 11^{2k} = 7x$ — (i) [x is multiple of 7]

we assume $p(k)$ is true.

$P(k+1) \Rightarrow 4^{2(k+1)+1} + 10 \times 11^{2(k+1)}$ — (ii)

extracting (i) from (ii):

$4^{2k+3} + 10 \times 11^{2k+2}$

$\Rightarrow 4^{2k+1} \times 4^2 + 10 \times 11^{2k} \times 11^2$

$\Rightarrow 4^2 \times 4^{2k+1} + 4^2 \times 10 \times 11^{2k} + 105\,(11^{2k} \times 10)$

$\Rightarrow 4^2(4^{2k+1} + 10 \times 11^{2k}) + 7\,(15 \times 10 \times 11^{2k})$

$\Rightarrow 4^2(7x) + 7\,(15 \times 10 \times 11^{2k})$

which is divisible by 7

$\therefore$ $P(k+1)$ is true. (proved)

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