# Elements of Statistics and Probability

# STA 201

S M Rajib Hossain

MNS, BRAC University

# Lecture-13

**Exponential Distribution:** The exponential distribution is a continuous probability distribution that describes the time between independent events occurring at a constant rate. It is often used to model the time until the next event in a sequence of events, where events happen continuously and independently at a constant average rate. It is useful for modeling failure times and waiting times.

A random variable X distributed as an exponential random variable with parameter $(\boldsymbol{\lambda > 0})$ , which is written $\boldsymbol{X \sim exp(\lambda)}$.The pdf of the exponential distribution is given by the formula:

$$\boldsymbol{f(x) = \lambda e^{-\lambda x}; x \geq 0}$$

Where:

- ✓ $x$ is the random variable representing the time between events
- ✓ $\lambda$ is the rate parameter

**Mean and Variance**:

$E(X) = \frac{1}{\lambda}$ and $V(X) = \frac{1}{\lambda^2}$ [HW]

**Example-1** Suppose that you are waiting for a friend to call you and that the time you wait in minutes has an exponential distribution with parameter λ = 0.1.

i. What is the expectation of your waiting time?

ii. What is the probability that you will wait longer than 10 minutes?

iii. What is the probability that you will wait less than 5 minutes?

Solution:

X=time required

$\therefore \boldsymbol{X \sim \exp(\lambda = 0.1)}$

$\therefore \boldsymbol{f(x) = 0.1e^{-0.1x}; x \geq 0}$

i. $E(X) = 1/0.1 = 10$

ii. $P(X > 10) = \int_{10}^{\infty} f(x)dx = \int_{10}^{\infty} 0.1e^{-0.1x}dx$

$$= 0.1\left[\frac{e^{-0.1x}}{-0.1}\right]_{10}^{\infty} = -[0 - e^{-0.1*10}] = 0.3679$$

iii. $P(X < 5) = \int_{0}^{5} f(x)dx = \int_{0}^{5} 0.1e^{-0.1x}dx = 0.3935$

**Example-2** Average time required to repair a machine is 0.5 hours. What is the probability that next time required will be more than 1 hours?

Solution:

X=time required

$$\therefore \boldsymbol{X \sim \exp\left(\lambda = \frac{1}{0.5} = 2\right)}$$

$$\therefore \boldsymbol{f(x) = 2e^{-2x}; x \geq 0}$$

$$\therefore P(X > 1) = \int_1^\infty f(x)dx = \int_1^\infty 2e^{-2x}dx = 0.1353$$

Example-3 Radio of a particular brand follows exponential distribution with average lifetime 2 years. What is the probability that a randomly chosen radio will survive at least 4 years?

X = Lifetime of the radio

$$\therefore \boldsymbol{X \sim \exp\left(\lambda = \frac{1}{2} = 0.5\right)}$$

$$\therefore \boldsymbol{f(x) = 0.5e^{-0.5x}; x \geq 0}$$

$$\therefore P(X > 4) = \int_4^\infty f(x)dx = \int_4^\infty 0.5e^{-0.5x}dx = 0.1353$$

**Normal Distribution:** A normal distribution is a continuous probability distribution that is symmetric around its mean. It is fully characterized by two parameters: the mean ($\boldsymbol{\mu}$) and the variance ($\boldsymbol{\sigma^2}$). A random variable X distributed as a normal random variable with parameter $\boldsymbol{\mu}$ $(-\infty < \boldsymbol{\mu} < \infty)$ and $\boldsymbol{\sigma^2}(\boldsymbol{\sigma^2} > \mathbf{0})$ , which is written $\boldsymbol{X} \sim \boldsymbol{N}(\boldsymbol{\mu}, \boldsymbol{\sigma^2})$.The pdf of the normal distribution is given by the formula:

$$\boldsymbol{f}(\boldsymbol{x}) = \frac{\mathbf{1}}{\boldsymbol{\sigma}\sqrt{\mathbf{2}\boldsymbol{\pi}}}\boldsymbol{e}^{\frac{-(\boldsymbol{x}-\boldsymbol{\mu})^2}{2\boldsymbol{\sigma}^2}}; -\infty < \boldsymbol{x} < \infty$$

Where:

- ✓ $\mu$ is the mean and $\sigma^2$ is the variance
- ✓ $e$ is the base of the natural logarithm and $\pi = 3.1416$

**Mean and Variance:**

$E(X) = \mu$ and $V(X) = \sigma^2$

**Standard Normal Distribution:** The standard normal distribution is a specific case of the normal distribution where the mean ($\mu$) is 0 and the variance ($\sigma^2$) is 1. A random variable following a standard normal distribution is denoted as Z. The probability density function (pdf) for the standard normal distribution is given by the formula:

$$\boldsymbol{f}(\boldsymbol{z}) = \frac{\mathbf{1}}{\sqrt{\mathbf{2}\boldsymbol{\pi}}}\boldsymbol{e}^{\frac{-\boldsymbol{z}^2}{2}}; -\infty < \boldsymbol{z} < \infty$$

Values from the standard normal distribution are commonly found in statistical tables or computed using statistical software.

The relationship between a random variable X following a normal distribution with mean $\mu$ and standard deviation $\sigma$ and a corresponding standard normal variable Z is given by the formula: $Z = \frac{X-\mu}{\sigma}$.

**Example1**: Suppose that $Z \sim N(0,1)$. Find: $(a)$ $P(Z < 1.34)$ $(b)$ $P(Z > -0.22)$ (c) $P(-2.19 < Z < 0.43)$ $(d)$ $P(0.09 < Z < 1.76)$ $(e)$ $P(|Z| < 0.38)$ (f) $P(|Z| > 0.38)$

Solution:

(a) $\boldsymbol{P(Z < 1.34) = \Phi(1.34) = 0.9099}$

[where $\mathbf{\Phi(Z)}$ is the cdf(cumulative distribution function) of Z, $\mathbf{\Phi(Z)}$=P(Z< $\boldsymbol{z}$)]

(b) $\boldsymbol{P(Z > -0.22) = 1 - P(Z < -0.22)}$

$$\boldsymbol{= 1 - \Phi(-0.22) = 1 - 0.41294 = 0.58706}$$

(c) $\boldsymbol{P(-2.19 < Z < 0.43) = \Phi(0.43) - \Phi(-2.19)}$

$$\boldsymbol{= 0.6664 - 0.0143 = 0.6521}$$

(d) $P(0.09 < Z < 1.76) = \Phi(1.76) - \Phi(0.09)$

$$= 0.9608 - 0.53586 = 0.42494$$

(e) $P(|Z| < 0.38) = P(-0.38 < Z < 0.38)$

$$= \Phi(0.38) - \Phi(-0.38) = 0.6480 - 0.352 = 0.296$$

(f) $P(|Z| > 0.38) = 1 - P(|Z| < 0.38)$

$$1 - 0.296 = 0.704$$

**Example2:** If X is a normal random variable with parameters $\mu = \mathbf{3}$ and $\boldsymbol{\sigma^2}$ = 9, find $(\boldsymbol{a})$ $\boldsymbol{P\{2 < X < 5\}}$; $(\boldsymbol{b})$ $\boldsymbol{P\{X > 0\}}$; $(\boldsymbol{c})$ $\boldsymbol{P\{|X - 3| > 6\}}$.

Solution:

(a) $\mathbf{P\{2 < X < 5\} = P\left\{\frac{2-3}{3} < \frac{x-3}{3} < \frac{5-3}{3}\right\} = P\left\{\frac{-1}{3} < Z < \frac{2}{3}\right\} = \Phi\left(\frac{2}{3}\right) - \Phi\left(\frac{-1}{3}\right)}$

$$\mathbf{= 0.74751 - 0.36944 = 0.378}$$

(b) $\mathbf{P\{X > 0\} = P\left\{\frac{x-3}{3} > \frac{0-3}{3}\right\} = P\{Z > -1\} = 1 - \Phi(-1) = 1 - 0.15866 = 0.84135}$

(c) $\mathbf{P\{|X - 3| > 6\} = 1 - P\{|X - 3| < 6\}}$

$$= 1 - \mathrm{P}\{-6 < X - 3 < 6\}$$

$$= 1 - P\left\{\frac{-6}{3} < \frac{\mathrm{X}-3}{3} < \frac{6}{3}\right\}$$

$$= 1 - \mathrm{P}\{-2 < Z < 2\}$$

$$= 1 - [\,\Phi(2) - \Phi(-2)]$$

$$= 1 - [0.9772 - 0.0228]$$

$$= 0.0456$$

**Example** 3: The time taken by employees to complete a certain task follows a normal distribution with a mean of 2.1 minutes and a standard deviation of 0.6 minutes.

i. Find the probability that a randomly selected employee will take at most 2 minutes to complete the task?

ii. Find the probability that a randomly selected employee completes the task in between 2.5 minutes and 3.5 minutes?

iii. Find the probability that a randomly selected employee will take at least 3 minutes to complete the task?

**Solution:**

Here, $X \sim N(2.1, 0.6^2)$

i. The probability that a randomly selected employee will take at most 2 minutes to complete the task is:

$$P(X \leq 2) = P\left(\frac{X-2.1}{0.6} \leq \frac{2-2.1}{0.6}\right) = P(Z \leq -0.17) = \Phi(-0.17) = 0.433$$

ii. The probability that a randomly selected employee completes the task in between 2.5 minutes and 3.5 minutes is:

$$P(2.5 \leq X \leq 3.5) = P\left(\frac{2.5-2.1}{0.6} \leq \frac{X-2.1}{0.6} \leq \frac{3.5-2.1}{0.6}\right)$$
$$= P(0.67 \leq Z \leq 2.33) = \Phi(2.33) - \Phi(0.67) = 0.990 - 0.748 = 0.242$$

iii. The probability that a randomly selected employee will take at least 3 minutes to complete the task is:

$$P(X \geq 3) = P\left(\frac{X-2.1}{0.6} \geq \frac{3-2.1}{0.6}\right) = P(Z \geq 1.5) = 1 - P(Z < 1.5)$$
$$= 1 - \Phi(1.5) = 1 - 0.933 = 0.067$$

**Example** 4: The weights of adult men in a city are normally distributed with a mean of 75 kg and a standard deviation of 8 kg.

i. Find the probability that a randomly selected man weighs more than 82 kg. **[Ans: 0.191]**
ii. Find the probability that a randomly selected man weighs less than 68 kg. **[Ans: 0.191]**
iii. Find the probability that a randomly selected man weighs in between 70 kg and 85 kg? **[Ans: 0.628]**