1. (Separable variable method)

$$\sin x\,(1+\sin y)\,dx + \cos y\,(1-\cos x)\,dy = 0$$

$$\Rightarrow \sin x\,(1+\sin y)\,dx = -\cos y\,(1-\cos x)\,dy$$

$$\Rightarrow \frac{\sin x}{1-\cos x}\,dx = -\frac{\cos y}{1+\sin y}\,dy$$

$$\Rightarrow \int \frac{\sin x}{1-\cos x}\,dx = -\int \frac{\cos y}{1+\sin y}\,dy$$

$$\Rightarrow \ln|1-\cos x| + c_1 = -\ln|1+\sin y| + c_2$$

$$\Rightarrow \ln|(1-\cos x)(1+\sin y)| + c = 0 \quad [\text{let, } c = c_1 + c_2]$$

$$\Rightarrow (1-\cos x)(1+\sin y) = e^{c}$$

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2. (Exact differential equation)

$$(e^{x}\sin y + e^{-y})\,dx + (e^{x}\cos y - xe^{-y})\,dy = 0 \quad \text{——— (1)}$$

Let, $M(x,y) = e^{x}\sin y + e^{-y}$

$N(x,y) = e^{x}\cos y - xe^{-y}$

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(e^{x}\sin y + e^{-y}) = e^{x}\cos y - e^{-y}$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(e^{x}\cos y - xe^{-y}) = e^{x}\cos y - e^{-y}$$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, so equation (1) is exact

$\int M\,dx + \int N(\text{not containing the } x \text{ having term(s)})\,dy = 0$

$\Rightarrow \int (e^{x}\sin y + e^{-y})\,dx + \int$

$\Rightarrow e^{x}\sin y + xe^{-y} + c = 0$

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3. (Integrating factor to solve)

$$\left(y+\frac{1}{3}y^3+\frac{1}{2}x^2\right)dx+\frac{1}{4}\left(x+xy^2\right)dy=0 \quad \text{--- (1)}$$

Let, $M\cancel{(x,y)} = y+\frac{1}{3}y^3+\frac{1}{2}x^2$

$N\cancel{(x,y)} = \frac{1}{4}(x+xy^2)$

$$\frac{\partial M}{\partial y} = 1+y^2$$

$$\frac{\partial N}{\partial x} = \frac{1}{4}(1+y^2)$$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, so equation (1) is non exact DE

$$\cancel{\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}} = \cancel{\frac{1+y^2-\frac{1}{4}-y^2\frac{1}{4}}{\frac{1}{4}(x+xy^2)}}$$

$$\cancel{\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{-M}} = \cancel{\frac{}{-1-y^2}}$$

$$\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x} = \frac{3}{4}(1+y^2)$$

$$\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N} = \frac{\frac{3}{4}(1+y^2)}{\frac{1}{4}(x+xy^2)} = \frac{\frac{3}{4}(1+y^2)}{\frac{1}{4}x(1+y^2)}$$

$$= \frac{3}{x}$$

$$\cancel{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}} \qquad \cancel{\frac{\frac{3}{4}(1+y^2)}{(y+\frac{1}{3}y^3+\frac{1}{2}x^2)}}$$

$$I.F = e^{\int \frac{3}{x}\, dx}$$

$$= e^{3\ln x}$$

$$= e^{\ln x^3}$$

$$= x^3$$

Now, $x^3\left(y+\frac{1}{3}y^3+\frac{1}{2}x^2\right)dx + x^3\frac{1}{4}(x+xy^2)\,dy = 0$ —— (11)

$$\frac{\partial M}{\partial y} = x^3(1+y^2)$$

$$\frac{\partial N}{\partial x} = \frac{1}{4}\cdot 4x^3(1+y^2) = x^3(1+y^2)$$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

So, equation (11) is exact

$\int M\,dx + \int N$ (not containing x having term) $dy = 0$

$$\Rightarrow \int\left(x^3y+\frac{1}{3}x^3y^3+\frac{1}{2}x^5\right)dx = 0$$

$$\Rightarrow \frac{1}{4}x^4y + \frac{1}{12}x^4y^3 + \frac{1}{12}x^6 + c = 0$$

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4. (Linear first order ODE)

$$(1+y^2)\,dx = (\tan^{-1}y - x)\,dy \quad \text{——— (i)}$$

$$\Rightarrow (1+y^2)\frac{dx}{dy} = \tan^{-1}y - x$$

$$\Rightarrow \frac{dx}{dy} = \frac{\tan^{-1}y - y}{1+y^2}$$

$$\Rightarrow \frac{dx}{dy} = \frac{\tan^{-1}y}{1+y^2} - \frac{x}{1+y^2}$$

$$\Rightarrow \frac{dx}{dy} + \frac{1}{1+y^2}x = \frac{\tan^{-1}y}{1+y^2} \quad \text{——— (ii)}$$

here, we know, general form $\frac{dx}{dy} + P(y)\,x = Q(y)$

so, $P(y) = \frac{1}{1+y^2}$

$Q(y) = \frac{\tan^{-1}(y)}{1+y^2}$

$$I.F = e^{\int \frac{1}{1+y^2}dy}$$

$$= e^{\tan^{-1}y}$$

Now, from (ii)

$$e^{\tan^{-1}y}\frac{dx}{dy} + e^{\tan^{-1}y}\frac{1}{1+y^2}x = e^{\tan^{-1}y}\frac{\tan^{-1}y}{1+y^2}$$

$$\Rightarrow \frac{d}{dy}\left(x.e^{\tan^{-1}y}\right) = e^{\tan^{-1}y}\frac{\tan^{-1}(y)}{1+y^2}$$

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$$\int \frac{d}{dy}\left(x \cdot e^{\tan^{-1} y}\right) dy = \int e^{\tan^{-1}(y)} \frac{\tan^{-1}(y)}{1+y^2}\, dy$$

$$\cancel{\Rightarrow x e^{\tan^{-1} y} = \int e^{u}\, u\, du}$$

let, $\tan^{-1}(y) = u$

$$\frac{d}{dy} \tan^{-1}(y) = \frac{du}{dy}$$

$$\Rightarrow \frac{1}{1+y^2}\, dy = du$$

Now, $\int e^{u}\, u\, du$

$$= u \cdot e^{u} - e^{u} + c$$

$$= \tan^{-1}(y) \cdot e^{\tan^{-1} y} - e^{\tan^{-1}(y)} + c$$

$$\therefore x e^{\tan^{-1} y} = \left(\tan^{-1}(y) \cdot e^{\tan^{-1} y} - e^{\tan^{-1} y} + c\right)$$

$$\Rightarrow x = \frac{1}{e^{\tan^{-1} y}} \left(\tan^{-1}(y)\, e^{\tan^{-1} y} - e^{\tan^{-1} y} + c\right)$$

$$\Rightarrow x = \tan^{-1}(y) - 1 + \frac{c}{e^{\tan^{-1} y}}$$

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5. (Homogeneous DE)

$\frac{d}{dx} e^{mx} = m e^{mx}$

$\frac{d}{dx} m e^{mx} = m^2 e^{mx}$

$\frac{d}{dx} m^2 e^{mx} = m^3 e^{mx}$

Given equation,

$$y'''' + 2y'' - 2y = 0 \text{ ——— } (1)$$

Let, $y = e^{mx}$ be a trail solution of (1)

$$(1) \Rightarrow m^4 e^{mx} + 2m^2 e^{mx} - 2e^{mx} = 0$$

$$\Rightarrow m^4 + 2m^2 - 2 = 0$$

$$\Rightarrow m^2 = -1 \pm \sqrt{3}$$

$$\Rightarrow m = \pm\sqrt{-1+\sqrt{3}}, \quad m = \pm i\sqrt{1+\sqrt{3}}$$

$$\therefore y = c_1 e^{\sqrt{\sqrt{3}-1}\,x} + c_2 e^{-\sqrt{\sqrt{3}-1}\,x} + c_3 \cos\left(\sqrt{\sqrt{3}+1}\,x\right) + c_4 \sin\left(\sqrt{\sqrt{3}+1}\,x\right)$$

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6. $y'''' + y''' + y'' = 0$ ——— ①

Let, $y = e^{mx}$ be a trail solution for ①

$① \Rightarrow m^4 e^{mx} + m^3 e^{mx} + m^2 e^{mx} = 0$

$\Rightarrow m^4 + m^3 + m^2 = 0$

$\Rightarrow m^2 (m^2 + m + 1) = 0$

$\Rightarrow m = 0, \ m = 0, \ m = \frac{-1 + i\sqrt{3}}{2}, m = \frac{-1 - \sqrt{3}i}{2}$

$\Rightarrow m = 0, \ m = 0, \ m = \frac{-1 \pm i\sqrt{3}}{2}$

So, $y = c_1 + c_2 x + e^{-\frac{x}{2}} \left(c_3 \cos\left(\frac{\sqrt{3}}{2}x\right) + c_4 \sin\left(\frac{\sqrt{3}}{2}x\right)\right)$

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7. $16y'''' + 24y'' + 9y = 0$ ——— ①

Let, $y = e^{mx}$ be a trail solution of ①

① $\Rightarrow 16m^4 e^{mx} + 24m^2 e^{mx} + 9e^{mx} = 0$

$\Rightarrow 16m^4 + 24m^2 + 9 = 0$

$\Rightarrow m^2 = -\frac{3}{4}$

$\Rightarrow m = \pm i\frac{\sqrt{3}}{2}$

So, $y = c_1 \cos\left(\frac{\sqrt{3}}{2}x\right) + c_2 \sin\left(\frac{\sqrt{3}}{2}x\right)$

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8.

$2xy\,y' = 1+y^2$

$\Rightarrow 2xy \dfrac{dy}{dx} = 1+y^2$

$\Rightarrow \int \dfrac{2y}{1+y^2}\, dy = \int \dfrac{1}{x}\, dx$

$\Rightarrow \ln(1+y^2) = \ln|x| + \ln c$

$\Rightarrow 1+y^2 = xc$

$\Rightarrow y^2 = xc - 1$

$\Rightarrow y = \pm\sqrt{xc-1}$ ——— ①

as, $y(2)=3$, $x=2$, $y=3$

$3 = \pm\sqrt{2c-1}$

$\therefore c = 5$

So, $y = \sqrt{5x-1}$ [from ①, as $y(2)=3$ positive]

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So, $y = \sqrt{5x-1}$ [From (1), as $y(2)=3$ positive]

Q. 16.

$(e^{2y} - y\cos xy)\,dx + (2xe^{2y} - x(\cos xy + 2y))\,dy = 0$ ——— ①

Let, $M = e^{2y} - y\cos(xy)$

$\frac{\partial M}{\partial y} = 2e^{2y} + yx\sin(xy) - \cos(xy)$

$N = 2xe^{2y} - x(\cos xy) + 2y$

$\frac{\partial N}{\partial x} = 2e^{2y} + xy\sin(xy) - \cos(xy)$

as $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, so, equation ① is exact

$\int M\,dx + \int N$ (not containing x having term) dy

$\Rightarrow e^{2y}x + y\sin(xy) + \int 2y\,dy = 0$

$= e^{2y}x + y\sin(xy) + \cancel{2}y^2 + C = 0$

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(Non homogeneous)

9. $y'' - y' + y = 2\sin 3x$ —— (1)

Let, $y'' - y' + y = 0$ —— (11)

Let, $y = e^{mx}$ be a trail solution for (11)

(11) $\Rightarrow m^2 e^{mx} - me^{mx} + e^{mx} = 0$

$\Rightarrow m^2 - m + 1 = 0$

$\Rightarrow m = \frac{1}{2} \pm \frac{i\sqrt{3}}{2}$

$\therefore y_c = e^{\frac{x}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}}{2}x\right) + c_2 \sin\left(\frac{\sqrt{3}}{2}x\right)\right)$

Now, Let, $y_p = A \cos(3x) + B\sin(3x)$ be a trail solution of (1)

$y_p' = -3A\sin(3x) + 3B\cos(3x$

$y_p'' = -9A\cos(3x) - 9B\sin(3x)$

(1) $\Rightarrow$

$(-9A\cos(3x - 9B\sin(3x)) - (-3A\sin(3x)$

$+ 3B\cos(3x)) + (A\cos(3x) + B\sin(3x)) = 2\sin(3x$

$\Rightarrow (-9A + 3B + A)\cos(3x) + (-9B + 3A + B)\sin(3x)$
$= 2\sin(3x)$

$\Rightarrow (-8A + 3B)\cos(3x) + (-10B + 3A)\sin(3x)$
$= 2\sin(3x)$

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equating the coefficient of $\cos(3x)$ and $\sin(3x)$

$$-8A + 3B = 0$$

$$-10B + 3A = 2$$

So, $B = \frac{8A}{3}$

$$A = -\frac{6}{71}$$

$$B = -\frac{16}{71} \cancel{\sin(3x)}$$

$$\therefore y_p = -\frac{6}{71}\cos(3x) - \frac{16}{71}\sin(3x)$$

We know, General solution

$$y = y_c + y_p$$

$$= e^{\frac{x}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}}{2}x\right) + c_2 \sin\left(\frac{\sqrt{3}}{2}x\right)\right) - \frac{6}{71}\cos(3x) - \frac{16}{71}\sin(3x)$$

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