Elements of Statistics and Probability

STA 201

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Lecture-11

Joint Distribution:

In probability theory and statistics, the joint distribution of two or more random variables describes how the probabilities of these variables are related.

Joint probability mass function (joint pmf):

Let's consider two discreate random variables, X and Y. The joint probability mass function (joint pmf), denoted as $p(x, y)$, gives the probability of both $X = x$ and $Y = y$ occurring simultaneously.

Properties of $p(x, y)$

i. $p(x, y) \geq 0$

ii. $\sum_x \sum_y \; p(x, y) = 1$

Example-1

x \ y	0	1	2	3	Total
0	4/23	2/23	1/23	1/23	8/23
1	2/23	1/23	2/23	1/23	6/23
2	0	1/23	7/23	1/23	9/23
Total	6/23	4/23	10/23	3/23	

$$p(0,0) = \frac{4}{23}$$

$$p(1,0) = \frac{2}{23}$$

Marginal pmf:

$$p_x(x) = \sum_y p(x, y)$$

$$p_y(y) = \sum_x p(x, y)$$

Here, $p_x(0) = \frac{4}{23} + \frac{2}{23} + \frac{1}{23} + \frac{1}{23} = \frac{8}{23}$

$p_x(1) = \frac{2}{23} + \frac{1}{23} + \frac{2}{23} + \frac{1}{23} = \frac{6}{23}$

$p_x(2) = 0 + \frac{1}{23} + \frac{7}{23} + \frac{1}{23} = \frac{9}{23}$

∴ Marginal pmf of X is

x	0	1	2
$p_x(x)$	8/23	6/23	9/23

Now, $p_y(0) = \frac{4}{23} + \frac{2}{23} + 0 = \frac{6}{23}$

$p_y(1) = \frac{2}{23} + \frac{1}{23} + \frac{1}{23} = \frac{4}{23}$

$p_y(2) = \frac{1}{23} + \frac{2}{23} + \frac{7}{23} = \frac{10}{23}$

$p_y(3) = \frac{1}{23} + \frac{1}{23} + \frac{1}{23} = \frac{3}{23}$

∴ Marginal pmf of Y is

y	0	1	2	3
$p_y(y)$	6/23	4/23	10/23	3/23

Example-2

x \ y	1	2	5	7	Total
0	1/13	3/13	1/13	2/13	7/13
1	1/13	1/13	k	2/13	(4+13k)/13
Total	2/13	4/13	(1+13k)/13	4/13	

Determine

i. Find k
ii. Marginal pmf of Y
iii. $p(y = 5)$

Solution:

i. $\sum p_x(x) = \frac{7}{13} + \frac{4+13k}{13} = 1$

$\therefore k = \frac{2}{13}$

Similarly,

$$\sum p_y(y) = \frac{2}{13} + \frac{4}{13} + \frac{1+13k}{13} + \frac{4}{13} = 1$$

$\therefore k = \frac{2}{13}$

ii. Using k= $\frac{2}{13}$ in the joint pmf table

x \ y	1	2	5	7	Total
0	1/13	3/13	1/13	2/13	7/13
1	1/13	1/13	2/13	2/13	6/13
Total	2/13	4/13	3/13	4/13	

∴ Marginal pmf of Y is

y	1	2	5	7
$p_y(y)$	2/13	4/13	3/13	4/13

iii. $p(y = 5) = p_y(5) = \frac{1}{13} + \frac{2}{13} = \frac{3}{13}$

Conditional pmf:

$$p_{x|y}(x|y) = \frac{p(x,y)}{p_y(y)}$$

Example

x \ y	0	1	2	Total
1	1/17	3/17	3/17	7/17
2	3/17	5/17	2/17	10/17
Total	4/17	8/17	5/17	

Determine

i. $p(x = 1)$
ii. $p(y = 2)$
iii. $p_{x|y}(1|2)$
iv. Conditional pmf of y given x equal to 1

Solution:

i. $p(x = 1) = p_x(1) = \frac{1}{17} + \frac{3}{17} + \frac{3}{17} = \frac{7}{17}$
ii. $p(y = 2) = p_y(2) = \frac{3}{17} + \frac{2}{17} = \frac{5}{17}$
iii. $p_{x|y}(1|2) = \frac{p(1,2)}{p_y(2)}$

$$= \frac{\frac{3}{17}}{\frac{5}{17}} = \frac{3}{5}$$

iv. $p_{y|x}(0|1) = \frac{p(1,0)}{p_x(1)}$

$$= \frac{\frac{1}{17}}{\frac{7}{17}} = \frac{1}{7} \qquad \left[p_x(1) = \frac{7}{17}\ from\ i\right]$$

$$p_{y|x}(1|1) = \frac{p(1,1)}{p_x(1)}$$

$$= \frac{\frac{3}{17}}{\frac{7}{17}} = \frac{3}{7}$$

$$p_{y|x}(2|1) = \frac{p(1,2)}{p_x(1)}$$

$$= \frac{\frac{3}{17}}{\frac{7}{17}} = \frac{3}{7}$$

∴ The conditional pmf of y given x equal to 1

y	0	1	2
$p_{y\|x}(y\|1)$	1/7	3/7	3/7

Joint probability distribution function (joint pdf):

Let's consider two continous random variables, X and Y.

$f(x, y)$ is called joint pdf of X and Y if

i. $f(x, y) \geq 0$

ii. $\int_x \int_y f(x, y) dy\, dx = 1$

Example-1

$$f(x,y) = 6e^{-2x}e^{-3y} \qquad\qquad x > 0,\ y > 0$$

i. Check whether $f(x,y)$ is a joint pdf.

ii. Calculate $p(x < 1, y < 2)$

Solution:

i. $$\int_0^\infty \int_0^\infty 6e^{-2x}e^{-3y}dy\,dx$$

$$= \int_0^\infty 6e^{-2x}\left[\frac{e^{-3y}}{-3}\right]_0^\infty dx$$

$$= \int_0^\infty -2e^{-2x}[0-1]dx$$

$$= \int_0^\infty 2e^{-2x}dx$$

$$= 2\left[\frac{e^{-2x}}{-2}\right]_0^\infty$$

$$= -[0-1]$$

$$= 1$$

ii. $$\int_0^1 \int_0^2 6e^{-2x}e^{-3y}dy\,dx$$

$$= \int_0^1 6e^{-2x}\left[\frac{e^{-3y}}{-3}\right]_0^2 dx$$

$$= \int_0^1 -2e^{-2x}[e^{-6}-1]dx$$

$$= -2[e^{-6}-1]\int_0^1 e^{-2x}dx$$

$$= -2[e^{-6}-1]\left[\frac{e^{-2x}}{-2}\right]_0^1$$

$= [e^{-6} - 1][e^{-2} - 1]$

=0.86

Marginal pdf:

Marginal pdf of X

$$f_x(x) = \int_y f\,(x, y)dy$$

Marginal pdf of Y

$$f_y(y) = \int_x f\,(x, y)dx$$

iii. $f_x(x) = \int_0^\infty 6e^{-2x}e^{-3y}dy$

$= 6e^{-2x}\left[\frac{e^{-3y}}{-3}\right]_0^\infty$

$= -2e^{-2x}[0 - 1]$

$= 2e^{-2x}$

$\therefore f_x(x) = 2e^{-2x} \qquad\qquad x > 0$

$$\therefore p(x < 3) = \int_0^3 f_x(x)dx$$

$$= \int_0^3 2e^{-2x}\,dx$$

$= 2\left[\frac{e^{-2y}}{-2}\right]_0^3$

$= -[e^{-6} - 1]$

$= [1 - e^{-6}]$

$= 0.99$

Example-2

$$f(x, y) = x + y \qquad 0 < x < 1, \qquad 0 < y < 1$$

i. Find the marginal pdf of X and Y

ii. Calculate $p(.3 < x < .7)$

iii. Calculate $p(.5 < y < .9)$

Solution:

i. $f_X(x) = \int_0^1 (x + y) dy$

$= \int_0^1 x\, dy + \int_0^1 y\, dy$

$= x[y]_0^1 + \left(\frac{y^2}{2}\right)_0^1$

$= x + \frac{1}{2}$

$\therefore f_X(x) = x + \frac{1}{2} \quad 0 < x < 1$

$f_Y(y) = \int_0^1 (x + y) dx$

$= \int_0^1 x\, dx + \int_0^1 y\, dx$

$= \left(\frac{x^2}{2}\right)_0^1 + y[x]_0^1$

$= y + \frac{1}{2}$

$$\therefore f_y(y) = y + \frac{1}{2} \quad 0 < y < 1$$

ii. $p(.3 < x < .7) = \int_{.3}^{.7} f_x(x)dx$

$$= \int_{.3}^{.7} \left(x + \tfrac{1}{2}\right) dx$$

iii. $p(.5 < y < .9) = \int_{.5}^{.9} f_y(y)dy$

$$= \int_{.5}^{.9} \left(y + \tfrac{1}{2}\right) dy$$

Conditional pdf:

$$f_{y|x}(y|x) = \frac{f(x,y)}{f_x(x)}$$

$$f_{x|y}(x|y) = \frac{f(x,y)}{f_y(y)}$$

From example 2

i. $f_{y|x}(y|0.7)$

ii. $p(.3 < y < .6|x = 0.7)$

iii. $p(.2 < x < .4|y = 0.5)$

Solution:

i. $f_{y|x}(y|0.7) = \frac{f(0.7,y)}{f_x(0.7)}$

$$f(x,y) = x + y \qquad 0 < x < 1,\ 0 < y < 1$$

$$\therefore\ f(0.7, y) = 0.7 + y$$

$$f_x(x) = x + \tfrac{1}{2} \qquad 0 < x < 1$$

$$\therefore f_x(0.7) = 0.7 + \tfrac{1}{2}$$

$$= 1.2$$

$$\therefore\ f_{y|x}(y|0.7) = \frac{0.7 + y}{1.2}$$

$$= \tfrac{1}{1.2}(0.7 + y) \qquad 0 < y < 1$$

ii. $p(.3 < y < .6|x = 0.7) = \int_{.3}^{.6} \frac{1}{1.2}(0.7 + y)dy$

iii. HW

Check of Independence

For joint pmf X and Y are independent if

$p(x, y) = p_x(x) \times p_y(y)$ for all x, y

For joint pdf X and Y are independent if

$f_x(x) * f_y(y) = f(x, y)$ for all x, y

Examples will be practice in your class