Elements of Statistics and Probability

STA 201

S M Rajib Hossain

MNS, BRAC University

Lecture-8 &9

Probability

In our daily life very often, we use the term 'Probability'. Probability or a tendency 'uncertainty' or 'chance' refers to the probable movements or to occur an event. Every day we express our thinking using the sentences like:

i. Everyone who lives will die

ii. We can live without breathing

iii. It may rain today

The idea of probability is expressed in the above three sentences. In the first sentence, we see a must, that is a certain incident or event. Every certain event is an obvious proclamation of probability. Here the value of the probability is 1. In the second sentence, the probability of living without breathing is zero. Now if we consider the third sentence, we realize that there lies a probability of rain today. Here we do not know the exact probability but the probability lies between 0 to 1.

Some important definitions related to probability:

Experiment: An experiment is an act that can be repeated under some given identical conditions.

Example: Throwing a die, tossing a coin, drawing cards from a bridge deck.

Experiments can be of two types:

1. **Deterministic or Predictable:** An experiment whose outcome is predictable in advance is called deterministic experiment. Everyone conducting that experiment will get the same outcome.

Examples:

- ✓ Measuring the aerial distance from Teknaf to Tutuila.
- ✓ Counting the number of floors in the Burj Khalifa.

2. Random or Unpredictable: A random experiment is an experiment that can be repeated any number of times under some identical conditions where the outcome of any particular trial should not be known but all possible outcomes should be known in advance.

Example: Tossing a fair coin or throwing a die and observing what the top shows.

Trial: Each of the repetitions in an experiment is called a trial. This means that the trial is a special case of the experiment. The experiment may be a trial or two or more trials.

Example: Throwing a coin once is a trial

Outcomes: The results of an experiment are known as outcomes.

Example: If we toss a coin, we get head or tail. Individually head and trial are two outcomes.

Sample space: The set of all possible outcomes of a random experiment is known as sample space.

Example: If we throw a die the outcomes are 1, 2, 3, 4, 5, and 6.

Then $S = \{1,2,3,4,5,6\}$ is a sample space.

Event: Any possible outcome or a set of possible outcomes of a random experiment is called an event.

Example: If we throw a die the outcomes are 1, 2, 3, 4, 5, and 6. Then the outcomes of even numbers are 2, 4, 6. Then A = {2,4,6} is called an event of even numbers.

Sure Event: An event is called a sure event when it always happens. The probability of a sure event is 1.

Example: Today the sun rising in the east is a sure event and its probability is 1.

Impossible Event: An event is called an impossible event when it never happens. The probability of an impossible event is 0.

Example: A river that does not contain any fish is an impossible event and its probability is 0.

Independent Events: If the occurrence of a set of events is not affected by any other set of events in any way, then the set of events is known as independent events.

Example: If we throw a coin two times, the result of the 1st and 2nd draw is independent of each other.

Dependent Events: If the occurrence or non-occurrence of an event in a trial is affected by the other subsequent trials then the events are said to be dependent events.

Example: If we consider 10 balls in a box where 5 are red and 5 are black balls, then the probability

of drawing a red/black ball in the first draw is 5/10. The ball we have drawn is red and if we do not return the ball back then the probability of drawing a red ball in the second draw is 4/9. Also, the probability of drawing a black ball in the second draw is 5/9.

Complementary Events: The complement of an event implies the non-occurrence of that event.

Example: If we throw a die and define an event A = {2,4,6} then the compliment of event A is A^c= {1,3,5}.

Equally Likely Events: Outcomes of an experiment are said to be equally likely events if we have no reason to expect anyone rather than the other.

Example: In tossing a fair coin, the outcomes 'head' and 'tail' are equally likely events.

Mutually Exclusive event: If the happening of any of the events excludes the happening of all the others, then the events would be termed as mutually exclusive events.

Example: If we toss a coin, two outcomes head (H) and tail (T) are mutually exclusive events. Because if it appears head (H) or tail (T) not both head and tail at the same time.

Exhaustive events: When the union of two or more events forms the sample space, we say that the events are exhaustive.

Example: Given the sample space, S = {1, 2, 3, 4, 5, 6}

Let, Event A = {1, 2, 3} and Event B = {4, 5, 6}. Now, A ∪ B = S.

Therefore, events A and B are exhaustive.

Definition of Probability: There are mainly three definitions of probability, namely

1. Mathematical or classical or a priori definition of probability.
2. Statistical or empirical or frequency or a posteriori definition of probability.
3. Subjective Approach.

Classical or a priori probability: If a random experiment can result in n(S) mutually exclusive, exhaustive, and equally likely outcomes and if n(A) of these outcomes are favorable to an event A, then the probability of A is the ratio of n(A) to n(S). In symbol, $P(A) = \frac{n(A)}{n(S)}$

$$P(A) = \frac{Number\ of\ Favourable\ Outcomes}{Number\ of\ All\ Possible\ Outcomes}$$

Statistical or empirical probability: If an experiment is repeated n times under similar conditions as a result of which an event A occurs m times, then the limit of the ratio m/n tends to an idealized value as n becomes infinitely large. This idealized value is called the probability of the event A. Symbolically, $P(A) = \lim_{n\to\infty} (\frac{m}{n})$

Example: Consider the experiment of throwing a 6-sided die.

S = {1, 2, 3, 4, 5, 6}

Let the favourable event be E={3}

After repeating the experiment 1200 times, 3 occurred a total of 198 times

Therefore, $P(A) = \lim_{n\to\infty} (\frac{198}{1200}) \approx \frac{1}{6}$

Subject approach: Subjective probability is the probability that an individual assign to an event on the basis of his/her own experience, judgment, guesses, intuition, prior information, or beliefs.

For example: You go out with a friend to a park, and seeing the clouds in the sky, your friend says there is a 60% chance of it raining today.

Axioms of Probability:

The axioms of probability are a set of postulates that define probability as a set function:

Axiom 1: The probability of any event is non-negative: $P(A) \geq 0$.

Axiom 2: The probability of the sample space is equal to 1: $P(S) = 1$

Axiom 3: The probability of the union of any collection of mutually exclusive events is the sum of the individual probabilities:

$P(A_1 \cup A_2 \cup \dots) = \sum_{i=1}^{\infty} P(A_i)$

Laws of Probability: There are two types of laws of probability

- Additive law of probability (For mutually exclusive and non-mutually exclusive events)
- Multiplicative law of probability (For independent and dependent events)

Additive law of probability (For two mutually exclusive events): The probability of occurrence of any one event of two mutually exclusive events is equal to the sum of their individual probabilities.

If A and B are two mutually exclusive events, then $P(A \cup B) = P(A) + P(B)$

Additive law of probability (For two non-mutually exclusive events): The probability of occurrence of any one event of two non-mutually exclusive events is equal to the sum of their individual probabilities minus the probability that both events occur. If A and B are two non-mutually exclusive events, then $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Example:

Experiment: Drawing a single card from a standard deck of 52 cards.

Mutually Exclusive Events:

Events:

A: Drawing a heart.

B: Drawing a spade.

So, $P(A \cup B) = P(A) + P(B)$

$$= \frac{13}{52} + \frac{13}{52}$$

$$= \frac{1}{2}$$

In this case, drawing a heart and drawing a spade are mutually exclusive because a single card cannot be both a heart and a spade. The sum of their probabilities equals $\frac{1}{2}$

Non-Mutually Exclusive Events:

Events:

C: Drawing a face card (king, queen, or jack).

D: Drawing a red card.

In this case,

$P(C \cup D) = P(C) + P(D) - P(C \cap D)$

$$= \frac{12}{52} + \frac{26}{52} - \frac{6}{52}$$

$$= \frac{8}{13}$$

In this case, drawing a face card and drawing a red card are not mutually exclusive because a single card can be both a face card and red (e.g., a red queen). The subtraction term P(C∩D) accounts for the double counting of red face cards.

Multiplicative law of probability (For two independent events): The probability of joint occurrence of the two independent events is equal to the product of their individual probabilities.

If A and B are two independent events, then $P(A \cap B) = P(A) . P(B)$

Multiplicative law of probability (For two dependent events): The probability of joint occurrence of the two dependent events is equal to the product of the unconditional probability of it's anyone event and the conditional probability of another event. If A and B are two dependent events, then

$P(A \cap B) = P(B) . P(A|B)$

Or, $P(A \cap B) = P(A) . P(B|A)$

Example:

Drawing two balls successively from a bag containing red and white balls **without replacement**. The bag contains 4 red balls and 3 white balls.

What is the probability of drawing a red ball on the first draw and a white ball on the second draw?

Solution:

Let's denote the events as follows:

R_1: Drawing a red ball on the first draw.

W_2 : Drawing a white ball on the second draw.

The probability of drawing a red ball on the first draw P(R_1) is the ratio of the number of red balls to the total number of balls before the first draw:

$$\mathrm{P}(R_1) = \frac{4}{7}$$

Now, after drawing one ball, there are 6 balls left in the bag (3 red and 3 white). The probability of drawing a white ball on the second draw given that a red ball was drawn on the first draw $\mathrm{P}(W_2|R_1) = \frac{3}{6}$

The probability of both events happening (drawing a red ball on the first draw and a white ball on the second draw) is the product of their individual probabilities:

$$\mathrm{P}\,(R_1 \cap W_2) = \mathrm{P}(R_1)\ .\ \mathrm{P}(W_2|R_1)$$

$$= \frac{4}{7} \times \frac{3}{6}$$

Now, if drawing two balls successively from a bag containing red and white balls **with replacement**.

As the ball is replaced after the first draw, the probability of drawing a white ball on the second draw P(W_2) is also the ratio of the number of white balls to the total number of balls:

So, P(W_2) $=\frac{3}{7}$

The probability of both independent events happening (drawing a red ball on the first draw and a white ball on the second draw) is the product of their individual probabilities:

P ($R_1 \cap W_2$) = P(R_1) . P(W_2)

$$= \frac{4}{7} \times \frac{3}{7}$$

Conditional probability:

Conditional probability is the probability of an event occurring given that another event has already occurred. It is denoted by P(A|B), read as "the probability of event A given event B."

Formula

P(A|B) $= \frac{\text{P (A} \cap \text{B)}}{\text{P (B)}}$

P(A∩B): Probability of both events A and B occurring.

P(B): Probability of event B occurring.

Similarly, P(B|A) $= \frac{\text{P (A} \cap \text{B)}}{\text{P (A)}}$

Example-1

In a school 30% students are female. 10% of female students are color blind. If a student is selected at random. What is the probability that the student will be color blind and female?

Solution:

Here, $P(F) = 0.30$ F for female

$P(C|F) = 0.10$ C for color blind

$P(C\cap F) = ?$

We know, $P(C|F) = \frac{P\,(C \cap F)}{P\,(F)}$

$\therefore P(C\cap F) = P(C|F) \,.\, P(F)$

$= 0.10 \times 0.30$

$= 0.03$

Example-2

The probability that a married man watches a certain TV show is 0.4 and that a married woman watches the show is 0.5. The probability that a man watches the show, given that his wife does, is 0.7.

a) Find the probability that a married couple watches the show.
b) Find the probability that a wife watches the show given that her husband does.
c) Find the probability that at least one person of a married couple will watch the show.

Solution:

Let us define two events H and W as follows:

H: Husband watches the show

W: Wife watches the show

It is given that, $P(H) = 0.4, P(W) = 0.5$ and $P(H|W) = 0.7$

a) $P(H \cap W) = P(H|W) \times P(W)$

$= 0.7 \times 0.5$

$= 0.35$

b) $P(W|H) = \frac{P(H \cap W)}{P(H)} = \frac{0.35}{0.4} = 0.875$

c) $P(H \cup W) = P(H) + P(W) - P(H \cap W)$

$= 0.4 + 0.5 - 0.35 = 0.55$

Independence of events:

Two events A and B are independent if occurrence of B does not change probability A. That means, P(A|B) = P(A)

Example-1

In a commodity 20% families have cars, 10% families having washing machines and 7% families have both. Does having washing machine depend on having a car?

Solution:

W for washing machine

C for cars

$P(C) = 0.20$

$P(W) = 0.10$

$P(C \cap W) = 0.07$

$$P(W|C) = \frac{P\,(C \cap W)}{P\,(C)}$$

$$= \frac{0.07}{0.20}$$

$$= 0.35 \neq P(W)$$

So, two events are not independent. That's means, having washing machine depend on having a car.

Bayes' Theorem:

Bayes' Theorem is a fundamental concept in probability theory named after Reverend Thomas Bayes. It provides a systematic way to update the probability of a hypothesis in light of new evidence.

If A and B are two events, then the formula for the Bayes theorem is given by:

$$P(A|B) = \frac{P\,(A \cap B)}{P\,(B)}$$

$$= \frac{P(A).P(B|A)}{P\,(B)}$$

Priori Probability: The probability P(A) is considered as the priori probability.

Posteriori Probability: The probability P(A|B) is considered as the posteriori probability.

Example-1

A blood test gives 97% positive (+) result for diseased people and 2% positive result for healthy people. In a community 1% people have diseases. A person is randomly selected from the community and gives the test.

i. What is the probability that the test will be positive?

ii. If the test is positive, what is the probability that the person has the disease?

iii. If the test is positive, what is the probability that the person has not the disease?

iv. If the test is negative, what is the probability that the person has not the disease?

Solution:

D for disease

H for healthy

+ for positive test result

$P(+|D) = 0.97$

$P(+|H) = 0.02$

$P(D) = 0.01$

$P(H) = 0.99$

i. $P(+) = P(D \cap +) + P(H \cap +)$

$= P(D).\ P(+|D) + P(H).P(+|H)$

$$= 0.01\times 0.97 \ + 0.99 \times 0.02$$

$$= 0.0295$$

ii. $P(D|+) = \frac{P(D\cap +)}{P(+)}$

$$= \frac{P(D).P(+|D)}{P(+)}$$

$$= \frac{0.01\times 0.97}{0.0295}$$

$$= 0.329$$

iii. HW

iv. HW

Example-2

3 Machines produce disks.

First machine produces 30%, of which 5% are defective.

Second machine produces 50%, of which 3% are defective.

Third machine produces 20%, of which 10% are defective.

A disk is randomly selected from the mixed lot.

i. What is the probability that the disk is defective?
ii. If the disk is defective, what is the probability that it was produced by second machine?
iii. If the disk is defective, what is the probability that it was not produced by second machine?
iv. If the disk is not defective, what is the probability that it was produced by second machine?

Solution:

M_1 for first machine

M_2 for second machine

M_3 for third machine

D for defective disk

$P(M_1) = 0.30$

$P(M_2) = 0.50$

$P(M_3) = 0.20$

$P(D|M_1) = 0.05$

$P(D|M_2) = 0.03$

$P(D|M_3) = 0.10$

i. $P(D) = P(D\cap M_1) + P(D\cap M_2) + P(D \cap M_3)$

$= P(M_1) \times P(D|M_1) + P(M_2) \times P(D|M_2) + P(M_3) \times P(D|M_3)$

$= 0.30\times 0.05 + 0.50\times 0.03 + 0.20 \times 0.10$

$= 0.05$

ii. $P(M_2|D) = \frac{P(D\cap M_2)}{P(D)}$

$= \frac{P(M_2).P(D|M_2)}{P(D)}$

$= \frac{0.50\times 0.03}{0.05}$

$= 0.30$

iii. HW

iv. HW

Example-3

Bag I contains 4 white and 6 black balls while Bag II contains 4 white and 3 black balls. One ball is drawn at random from one of the bags, and it is found to be black. Find the probability that it was drawn from Bag I.

Solution:

B_1 for choosing bag I

B_2 for choosing bag II

B for choosing black ball

$P(B_1) = 0.50$

$P(B_2) = 0.50$

$P(B|B_1) = \frac{6}{10} = 0.60$

$P(B|B_2) = \frac{3}{7} = 0.43$

$$P(B_1|B) = \frac{P(B_1 \cap B)}{P(B)}$$

$$= \frac{P(B_1).P(B|B_1)}{P(B_1 \cap B) + P(B_2 \cap B)}$$

$$= \frac{P(B_1).P(B|B_1)}{P(B_1).P(B|B_1) + P(B_2).P(B|B_2)}$$

$$= \frac{0.50 \times 0.60}{0.50 \times 0.60 + 0.50 \times 0.43}$$

$$= 0.583$$