CSE230: Discrete Mathematics

Practice Sheet 4: **Sequence, Summation, Recurrence Solution**

Q1	What is the term a_8 of the sequence $\{a_n\}$ if a_n equals a) 2^{n-1}? b) 7 ? c) $1 + (-1)^n$? d) $-(-2)^n$?
Q2	What are the terms a_0, a_1, a_2 and a_3 of the sequence $\{a_n\}$, where a_n equals a) $(-2)^n$? b) 3? c) $7 + 4^n$? d) $2^n + (-2)^n$?
Q3	List the first 10 terms of each of these sequences. a) The sequence that begins with 2 and in which each successive term is 3 more than the preceding term. b) The sequence that lists each positive integer three times, in increasing order. c) The sequence that lists the odd positive integers in increasing order, listing each odd integer twice. d) The sequence whose nth term is $n! - 2^n$. e) The sequence that begins with 3, where each succeeding term is twice the preceding term. f) The sequence whose first term is 2, second term is 4, and each succeeding term is the sum of the two preceding terms. g) The sequence where the nth term is the number of letters in the English word for the index n. h) The sequence obtained by starting with 10 and obtaining each term by subtracting 3 from the previous term. i) The sequence whose nth term is the sum of the first n positive integers. j) The sequence whose nth term is $3^n - 2^n$.
Q4	Find the first five terms of the sequence defined by each of the following recurrence relations and initial conditions given below: a) $a_n = 6a_{n-1}$, $a_0 = 2$. b) $a_n = a^2_{n-1}$, $a_1 = 2$. c) $a_n = a_{n-1} + 3a_{n-2}$, $a_0 = 1$, $a_1 = 2$. d) $a_n = na_{n-1} + n^2a_{n-2}$, $a_0 = 1$, $a_1 = 1$. e) $a_n = a_{n-1} + a_{n-3}$, $a_0 = 1$, $a_1 = 2$, $a_2 = 0$.

Q5	Show that the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = -3a_{n-1} + 4a_{n-2}$ if a) $a_n = 0$. b) $a_n = 1$. c) $a_n = (-4)^n$. d) $a_n = 2(-4)^n + 3$
Q6	Is the sequence $\{a_n\}$ a solution of the recurrence relation $a_n = 8a_{n-1} - 16a_{n-2}$ if a) $a_n = 0$? b) $a_n = 1$? c) $a_n = 2^n$? d) $a_n = 4^n$? e) $a_n = n4^n$? f) $a_n = 2 \cdot 4^n + 3n4^n$? g) $a_n = (-4)^n$? h) $a_n = n^2 4^n$?
Q7	Show that the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = a_{n-1} + 2a_{n-2} + 2n - 9$ if a) $a_n = -n + 2$. b) $a_n = 5(-1)^n - n + 2$. c) $a_n = 3(-1)^n + 2^n - n + 2$. d) $a_n = 7 \cdot 2^n - n + 2$.
Q8	Find the solution to each of these recurrence relations with the given initial conditions. Use an iterative approach. a) $a_n = -a_{n-1}$, $a_0 = 5$. b) $a_n = a_{n-1} + 3$, $a_0 = 1$. c) $a_n = a_{n-1} - n$, $a_0 = 4$. d) $a_n = 2a_{n-1} - 3$, $a_0 = -1$. e) $a_n = (n + 1)a_{n-1}$, $a_0 = 2$. f) $a_n = 2na_{n-1}$, $a_0 = 3$. g) $a_n = -a_{n-1} + n - 1$, $a_0 = 7$.
Q9	A person deposits \$1000 in an account that yields 9% interest compounded annually. a) Set up a recurrence relation for the amount in the account at the end of n years. b) Find an explicit formula for the amount in the account at the end of n years. c) How much money will the account contain after 100 years?

Q10	Suppose that the number of bacteria in a colony triples every hour. a) Set up a recurrence relation for the number of bacteria after n hours have elapsed. b) If 100 bacteria are used to begin a new colony, how many bacteria will be in the colony in 10 hours?
Q11	Assume that the population of the world in 2017 was 7.6 billion and is growing at the rate of 1.12% a year. a) Set up a recurrence relation for the population of the world n years after 2017. b) Find an explicit formula for the population of the world n years after 2017. c) What will the population of the world be in 2050?
Q12	A factory makes custom sports cars at an increasing rate. In the first month only one car is made, in the second month two cars are made, and so on, with n cars made in the nth month. a) Set up a recurrence relation for the number of cars produced in the first n months by this factory. b) How many cars are produced in the first year? c) Find an explicit formula for the number of cars produced in the first n months by this factory.
Q13	An employee joined a company in 2017 with a starting salary of $50,000 per year. Every year this employee receives a raise of $1000 plus 5% of the salary of the previous year. a) Set up a recurrence relation for the salary of this employee n years after 2017. b) What will the salary of this employee be in 2025? c) Find an explicit formula for the salary of this employee n years after 2017.
Q14	For each of these lists of integers, provide a simple formula or rule that generates the terms of an integer sequence that begins with the first term in each given list. Assuming that your formula or rule is correct, determine the next three terms of the sequence. a) 1, 0, 1, 1, 0, 0, 1, 1, 1, 0, 0, 0, 1, ... b) 1, 2, 2, 3, 4, 4, 5, 6, 6, 7, 8, 8, ... c) 1, 0, 2, 0, 4, 0, 8, 0, 16, 0, ... d) 3, 6, 12, 24, 48, 96, 192, ... e) 15, 8, 1, −6, −13, −20, −27, ... f) 3, 5, 8, 12, 17, 23, 30, 38, 47, ... g) 2, 16, 54, 128, 250, 432, 686, ... h) 2, 3, 7, 25, 121, 721, 5041, 40321, ...

Q15	For each of these lists of integers, provide a simple formula or rule that generates the terms of an integer sequence that begins with the first term in each given list. Assuming that your formula or rule is correct, determine the next three terms of the sequence. a) 3, 6, 11, 18, 27, 38, 51, 66, 83, 102, ... b) 7, 11, 15, 19, 23, 27, 31, 35, 39, 43, ... c) 1, 10, 11, 100, 101, 110, 111, 1000, 1001, 1010, 1011, ... d) 1, 2, 2, 2, 3, 3, 3, 3, 3, 5, 5, 5, 5, 5, 5, 5, ... e) 0, 2, 8, 26, 80, 242, 728, 2186, 6560, 19682, ... f) 1, 3, 15, 105, 945, 10395, 135135, 2027025, 34459425, ... g) 1, 0, 0, 1, 1, 1, 0, 0, 0, 0, 1, 1, 1, 1, 1, ... h) 2, 4, 16, 256, 65536, 4294967296, ...
Q16	Find the closed formula for each of the following sequences by relating them to a well known sequence. Assume the first term given is a_1. (a) 2, 5, 10, 17, 26, . . . (b) 0, 2, 5, 9, 14, 20, . . . (c) 8, 12, 17, 23, 30, . . . (d) 1, 5, 23, 119, 719, . . .
Q17	Write down the first 5 terms (starting with a_0) for each of the sequences described below. Then give either a closed formula or a recursive definition for the sequence (whichever is NOT given in the problem). (a) $a_n = ½ (n^2 + n)$. (b) $a_n = 2a_{n-1} - a_{n-2}$ with $a_0 = 0$ and $a_1 = 1$. (c) $a_n = na_{n-1}$ with $a_0 = 1$.
Q18	Consider the three sequences below. For each sequence, find a recursive definition. How are these sequences related? (a) 2, 4, 6, 10, 16, 26, 42, ... (b) 5, 6, 11, 17, 28, 45, 73, ... (c) 0, 0, 0, 0, 0, 0, 0, ...
Q19	Write down the first few terms of the sequence given by $a_1 = 3$; $a_n = 2a_{n-1} + 4$. Then find a recursive definition for the sequence 10, 24, 52, 108, ...
Q20	Show that $a_n = 3 \cdot 2^n + 7 \cdot 5^n$ is a solution to the recurrence relation an $7a_{n-1} - 10a_{n-2}$. What would the initial conditions needed for this to be the closed formula of the sequence?

Q21	Find a closed formula of the sequence with recursive definition $a_n = 2a_{n-1} - a_{n-2}$ and with $a_1 = 1$ and $a_2 = 2$.
Q22	Suppose you draw n lines in the plane so that every pair of lines cross (no lines are parallel) and no three lines cross at the same point. This will create some number of regions in the plane, including some unbounded regions. Call the number of regions R_n. Find a recursive formula for the number of regions created by n lines, and justify why your recursion is correct.
Q23	Consider the sequence 5, 9, 13, 17, 21, . . . with $a_1 = 5$. (a) Give a recursive definition for the sequence. (b) Give a closed formula for the nth term of the sequence. (c) Is 2013 a term in the sequence? Explain. (d) How many terms does the sequence 5, 9, 13, 17, 21, . . . , 533 have? (e) Find the sum: $5 + 9 + 13 + 17 + 21 + \cdots + 533$.
Q24	Consider the sum $4 + 11 + 18 + 25 + \cdots + 249$. How many terms (summands) are in the sum?
Q25	Consider the sequence 1, 7, 13, 19, . . . , $6n + 7$. (a) How many terms are there in the sequence? Your answer will be in terms of n. (b) What is the second-to-last term? (c) Find the sum of all the terms in the sequence, in terms of n.
Q26	Find $5 + 15 + 45 + \cdots + 5 \cdot 3^{20}$.
Q27	Solve the recurrence relation an $a_{n-1} + 2^n$ with $a_0 = 5$.
Q28	Show that 4^n is a solution to the recurrence relation $a_n = 3a_{n-1} + 4a_{n-2}$.
Q29	Find the solution to the recurrence relation $a_n = 3a_{n-1} + 4a_{n-2}$ with initial terms $a_0 = 2$ and $a_1 = 3$.
Q30	Think back to the magical candy machine at your neighborhood grocery store. Suppose that the first time a quarter is put into the machine 1 Skittle comes out. The second time, 4 Skittles, the third time 16 Skittles, the fourth time 64 Skittles, etc. Find a recursive formula for how many Skittles the nth customer gets.

Q31	Let a_n be the number of $1 \times n$ tile designs you can make using 1×1 squares available in 4 colors and 1×2 dominoes available in 5 colors. (a) First, find a recurrence relation to describe the problem. Explain why the recurrence relation is correct (in the context of the problem). (b) Write out the first 6 terms of the sequence $a_1, a_2, \ldots.$ (c) Solve the recurrence relation.
Q32	Solve the recurrence relation $a_n = 2a_{n-1} - a_{n-2}$. (a) What is the solution if the initial terms are $a_0 = 1$ and $a_1 = 2$? (b) What do the initial terms need to be in order for $a_9 = 30$?
Q33	The population of Utopia increases 5 percent each year. In 2000 the population was 10,000. What was the population in 1970?
Q34	Assume that a Database table has 0 records at time n=0. Suppose that at time n, 100n new records are populated into that table. The number of records increases 20 percent per minute. Write a recurrence relation and an initial condition that defines the number of records in the database table.
Q35	A wildlife reserve starts with 200 deers. The population increases by 15% annually due to reproduction and natural growth. Write a recurrence relation for the deer population at the end of n years.
Q36	A cup of coffee is initially at 90°C and is left to cool down in a room at a constant temperature of 25°C. The temperature of the coffee decreases by 10% of the difference between its current temperature and the room temperature every minute. Write a recurrence relation for the temperature of the coffee at the end of n minutes.
Q37	A new machine has been purchased for $50,000. Its value depreciates by 12% of its current value each year. Generate a recurrence relation for the value of the machine at the end of n years.
Q38	A television channel holds a contest everyday. On day 1, the prize money is $1000. For each day, the prize money increases by 10%, plus an additional $5 from the CEO. Model recurrence relations for the contest. (Mention the base case). Solve the recurrence relation and find what will be the prize money on the 11th day according to the Contest.
Q39	A person drops a tennis ball from the top of a 100 meter tall building. After each drop on the ground, the ball jumps up to the two-third of its previous height. Find a recurrence relation expressing the total distance covered by the ball before its nth drop on the ground.
Q40	Solve the following recurrence relation: $2a_{n+2} = 4a_{n+1} + 126a_n + 2.5^n$. Here $a_0 = 0,\ a_1 = 5$.

Q41	Expand the following sums and products. That is, write them out the long way. (a) $\sum_{k=1}^{100}(3+4k)$. (b) $\sum_{k=0}^{n}2^k$. (c) $\sum_{k=2}^{50}\frac{1}{(k^2-1)}$. (d) $\prod_{k=2}^{100}\frac{k^2}{(k^2-1)}$. (e) $\prod_{k=0}^{n}(2+3k)$.