04_regression_session

January 28, 2021

1 Regression Session

Last session, we looked at correlation to assess whether two variables related to each other. Unfortunately, this did not tell us anything about the direction of causality: A correlation between variable X and Y could mean that X predicts Y, but it could also mean that Y predicts X, or even that a third variable predicts both X and Y.

This session's statistical tool allows you to be a bit more explicit about your predictions. In a **regression**, you use one or more variables to predict one or more other variables. This doesn't automatically mean that regressions tell you about the direction of causality, but at least you can use it to get a bit closer.

1.1 The basics

We'll start of simple, with a regression in which one variable X predicts another variable Y. This is roughly equivalent to a correlation. In order to make this a bit more exciting, let's look at some real data!

Background to our data At the Winter Olympics, there is a sport called *speed skating.* If you're not from a select few countries, you might not have heard of it. It's essentially like running, but then you do it on ice, using special skates. In most disciplines, only two skaters compete against each other at a time. The times are compared between all athletes afterwards, and whoever had the best time wins. One of the more exciting races, is the 500 meter sprint. All competing athletes are *highly* trained, and focus on raw power and technique, as that is all that matters. Right?

Well, perhaps not. Perhaps coincidence or even bias might also play a role. You see, the starting procedure in speed skating is like most other racing sports. The referee says "Ready?", then waits for a bit, and then shoots their starting pistol. The regulations are clear on that "bit" of time: It should be 1-1.5 seconds from the moment athletes are in position, and hence it's regulated to be random.

Unlike in other racing sports, speed skaters race alone or against a single opponent, and times are compared between all skaters afterwards. (So the main competitors for the win might not directly face each other!) This means that every pair of racers starts with a different interval between their "Ready" cue and the starting shot.

Does this matter? In theory, the "Ready" signal could be considered an *alerting cue*, and we know from psychological research that the time between an alerting cue and a subsequent signal affects how quickly people respond to that signal. In practice, an interval of 500 ms results in an optimal response time, and longer intervals result in higher response times.

Now, obviously, the Winter Olympics are not some psychological experiment. These aren't a bunch of students in some dank lab room, these are highly trained athletes competing under immense pressure, looked on by thousands of spectators in the stadium, and even more watching from home through live stream or television. Surely this alerting thing will not affect their actual performance?

That sounds like an empirical question!

Loading our data The National Skating Union records not only finish times at 500 meters, but also measures athletes' time 100 meters into the race. Both of these numbers are freely available. Researchers from the Universities of Oxford (UK) and Utrecht (Netherlands) have collected data on the intervals between the onset of the "Ready" signal and the onset of the starting shot. The attached file, `speed_skating_all_races.csv`, has both the 100 and 500 meter times, and the ready-start intervals. In addition, there is a column that indicates whether an athlete fell or stumbled during a race.

You can load those data into Python using NumPy's `loadtxt` function:

[1]:
```python
import numpy
from matplotlib import pyplot

# Load the data from all individual races.
data = numpy.loadtxt("speed_skating_all_races.csv", \
    delimiter=",", dtype=float, skiprows=1, unpack=True)
```

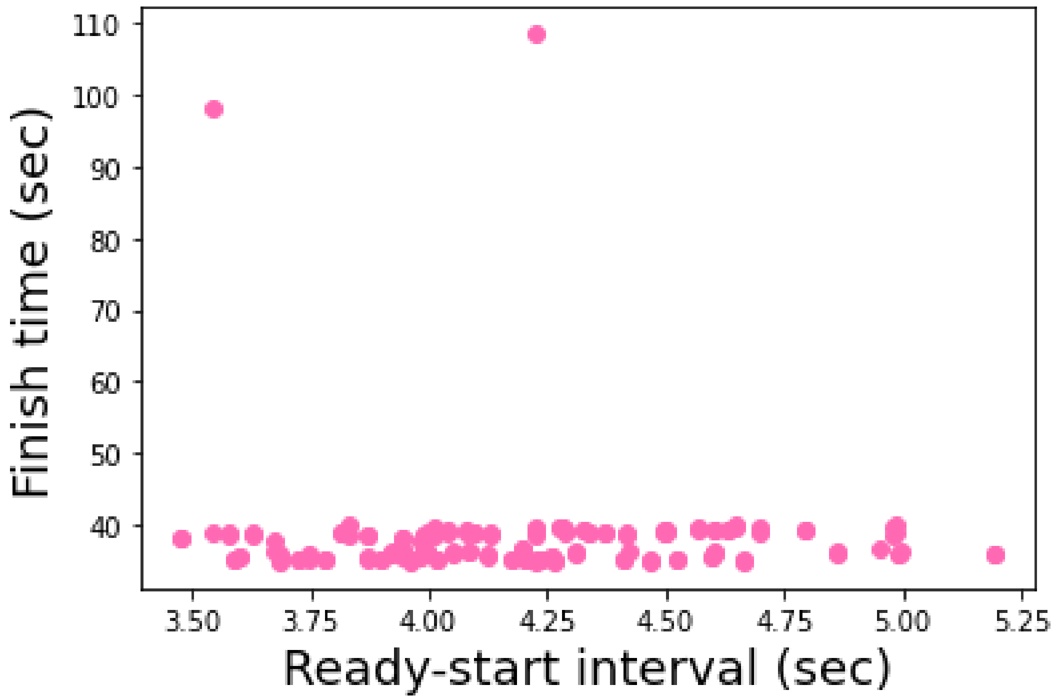
Let's do some convenience renaming on the columns in the data file:

[2]:
```python
# Get the ready-start interval data.
interval = data[0,:]
# Get the times at 100 and 500 meters.
time_100m = data[1,:]
time_500m = data[2,:]
# Get the sex data, which is formatted as a 1 for male and
# 0 for female. We can cast this into Booleans: True (1)
# or False (0) for the question "Is this athlete male?"
is_male = data[3,:].astype(bool)
# The data on falls is formatted in the same way: 1 for a
# fall/stumble, and 0 for a regular race. We can cast this
# into Booleans too: True (1) of False (0) for "Did this
# athlete fall?"
fall = data[4,:].astype(bool)
```

Getting a feel for the data Let's plot the data! Make sure to plot the (near) falls in a different colour, so we can see whether they really are different from the normal races.

```
[3]: # Plot the ready-start interval on the x-axis, and the 500
     # meter times on the y-axis.
     pyplot.plot(interval, time_500m, 'o', color="#FF69B4")

     # Add axis labels to make the plot clearer.
     pyplot.xlabel("Ready-start interval (sec)", fontsize=18)
     pyplot.ylabel("Finish time (sec)", fontsize=18)
```

```
[3]: Text(0, 0.5, 'Finish time (sec)')
```



Two points really stick out in this plot. Perhaps you've heard such points be described as "outliers" before. It is important to properly deal with outliers, as they can massively impact analyses.

You might even have heard of different ways to deal with outliers. Maybe you've been told to simply ignore points if they're above or below a certain threshold, but this is a *very* dangerous subject. Deleting or ignoring data is dodgy AF, because there is no consensus on what exactly constitutes an "outlier", and just chucking data out is considered is a highly questionable practice.

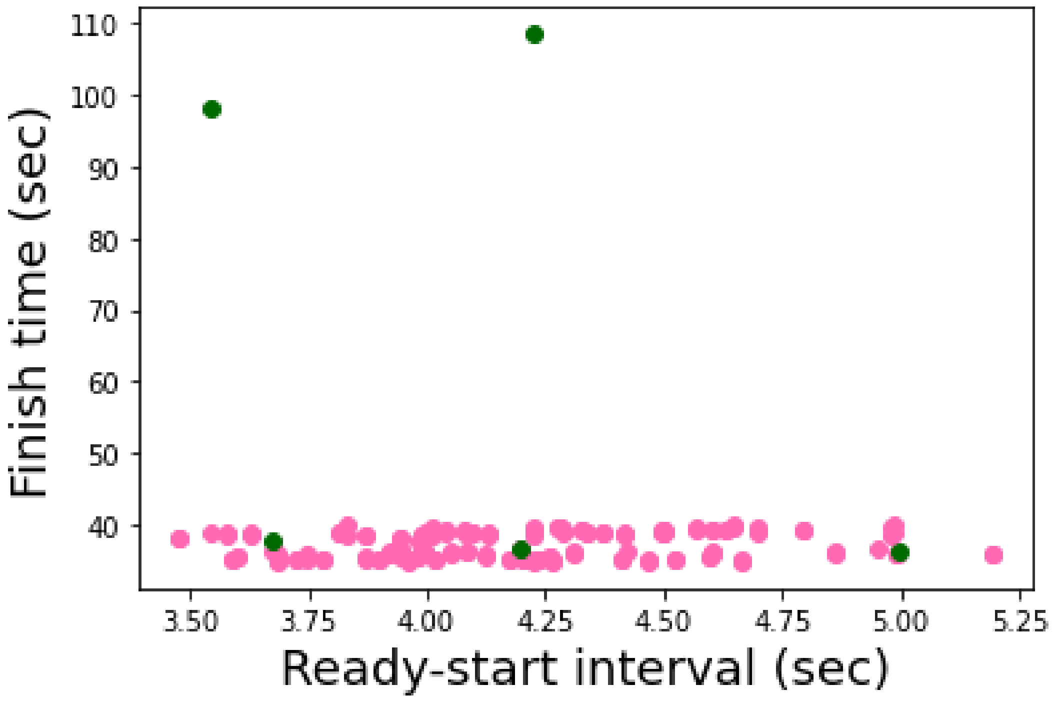
An alternative option is *Winsorisation*. (This is not to be confused with *Windsorisation*, the practice of renaming your German-heritaged royal family to avoid anti-German sympathies.) In this process, you replace all "outlier" values by the most extreme value you would not consider an outlier. For example, you would replace all values higher that 3 standard deviations from the mean, by values that are exactly three standard deviations from the mean.

Really, though, we're already well too many steps ahead here. Before even starting to think about how to deal with outliers, you re ought to find out what is causing them. In this dataset, none of the races are filtered out. This means that racers who fell or stumbled are still included. Perhaps this is the most sensible place to look for outliers first.

Assignment 1: Plotting the fallen Whether or not a race included a fall is captured in the variable `fall`. If `fall==False`, the racer did not fall. If `fall==True`, the racer did. Use this information to complete the following code:

```
[4]: # Plot the ready-start interval on the x-axis, and the 500
     # meter times on the y-axis.
     pyplot.plot(interval[fall==False], time_500m[fall==False], \
         'o', color="#FF69B4")

     # TODO: Also plot the races in which athletes fell or stumbled.
     pyplot.plot(interval[fall==True], time_500m[fall==True], \
         'o', color="#006900")

     # Add axis labels to make the plot clearer.
     pyplot.xlabel("Ready-start interval (sec)", fontsize=18)
     pyplot.ylabel("Finish time (sec)", fontsize=18)
```

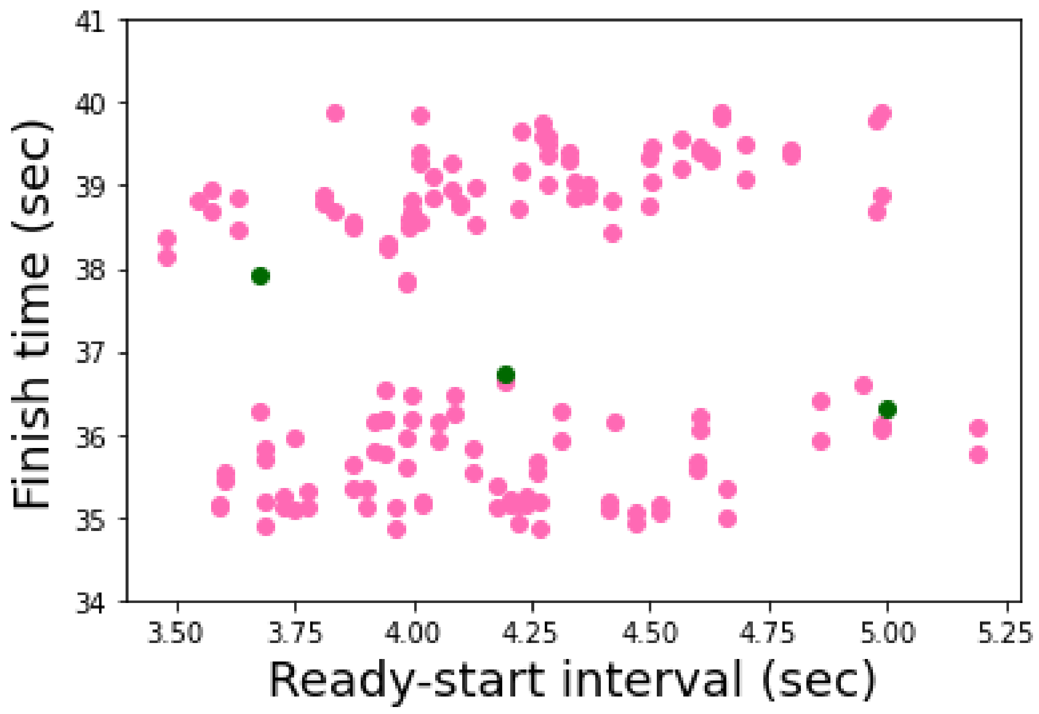
```
[4]: Text(0, 0.5, 'Finish time (sec)')
```



So the first thing you see is that there are two REALLY slow races, both due to falls. Let's set a limit on the y-axis that will allow us to actually see the data.

```
[5]: # Plot the ready-start interval on the x-axis, and the
     # 500 meter times on the y-axis.
     pyplot.plot(interval[fall==False], time_500m[fall==False], \
         'o', color="#FF69B4")
     # Also plot the races in which athletes fell or stumbled.
     pyplot.plot(interval[fall==True], time_500m[fall==True], \
         'o', color="#006900")

     # Add axis labels to make the plot clearer.
     pyplot.xlabel("Ready-start interval (sec)", fontsize=18)
     pyplot.ylabel("Finish time (sec)", fontsize=18)

     # Set the axis limit.
     pyplot.ylim([34, 41])
```

```
[5]: (34.0, 41.0)
```



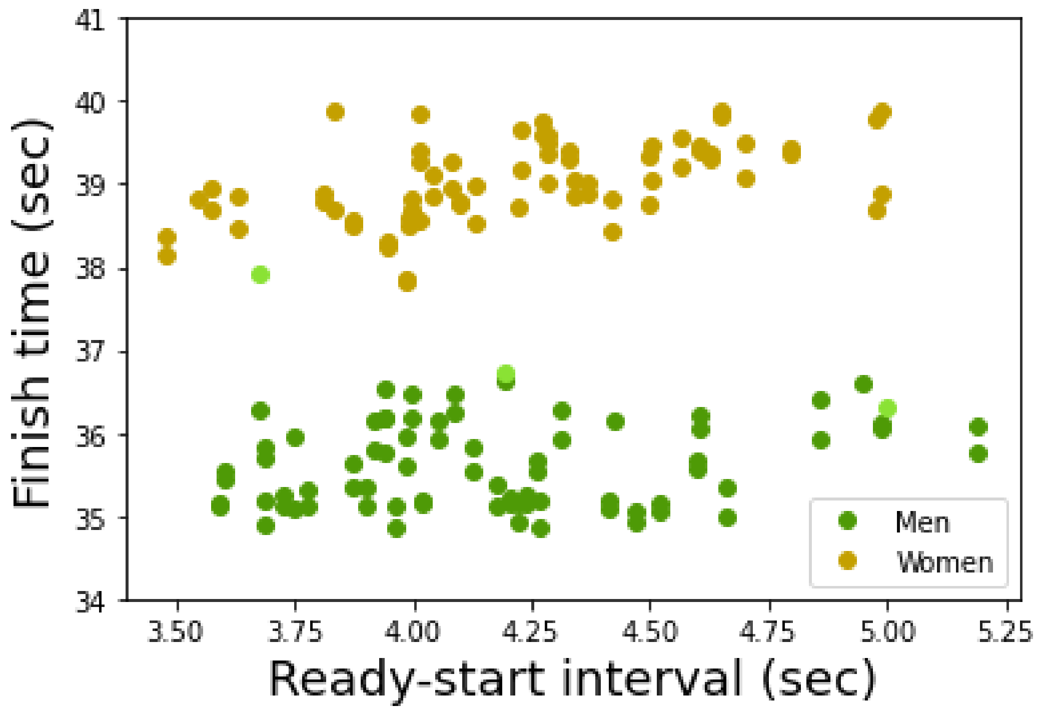
Assignment 2: Let's plot about sex, baby It's quite clear there are two separate sub-groups in the data here... One reason for this could be that the men's and women's competitions are both included in the same dataset. Information on this is captured in the `is_male` variable, which is `True` for races in the men's competition, and `False` for the women's.

Complete the code below to plot the men's and the women's races separately within the same figure. Use the following colour scheme:

- Men, regular race: "#4e9a06"
- Men, fell in race: "#8ae234"
- Women, regular race: "#c4a000"
- Women, fell in race: "#fce94f"

```
[6]: # MEN'S RACES
     # Plot the ready-start interval on the x-axis, and the 500
     # meter times on the y-axis.
     pyplot.plot(interval[(is_male == True) & (fall==False)], \
         time_500m[(is_male == True) & (fall==False)], 'o', \
         color="#4e9a06", label="Men")
     # TODO: Also plot the races in which athletes fell or stumbled.
     pyplot.plot(interval[(is_male == True) & (fall==True)], \
         time_500m[(is_male == True) & (fall==True)], 'o', \
         color="#8ae234")

     # WOMEN'S RACES
     # TODO: Plot the ready-start interval on the x-axis, and the 500
     # meter times on the y-axis.
     pyplot.plot(interval[(is_male == False) & (fall==False)], \
         time_500m[(is_male == False) & (fall==False)], 'o', \
         color="#c4a000", label="Women")
     # TODO: Also plot the races in which athletes fell or stumbled.
     pyplot.plot(interval[(is_male == False) & (fall==True)], \
         time_500m[(is_male == False) & (fall==True)], 'o', \
         color="#fce94f")

     # Add axis labels to make the plot clearer.
     pyplot.xlabel("Ready-start interval (sec)", fontsize=18)
     pyplot.ylabel("Finish time (sec)", fontsize=18)

     # Set the axis limit.
     pyplot.ylim([34, 41])

     # Add a legend.
     pyplot.legend(loc="lower right")
```

```
[6]: <matplotlib.legend.Legend at 0x7fc1fbba9850>
```



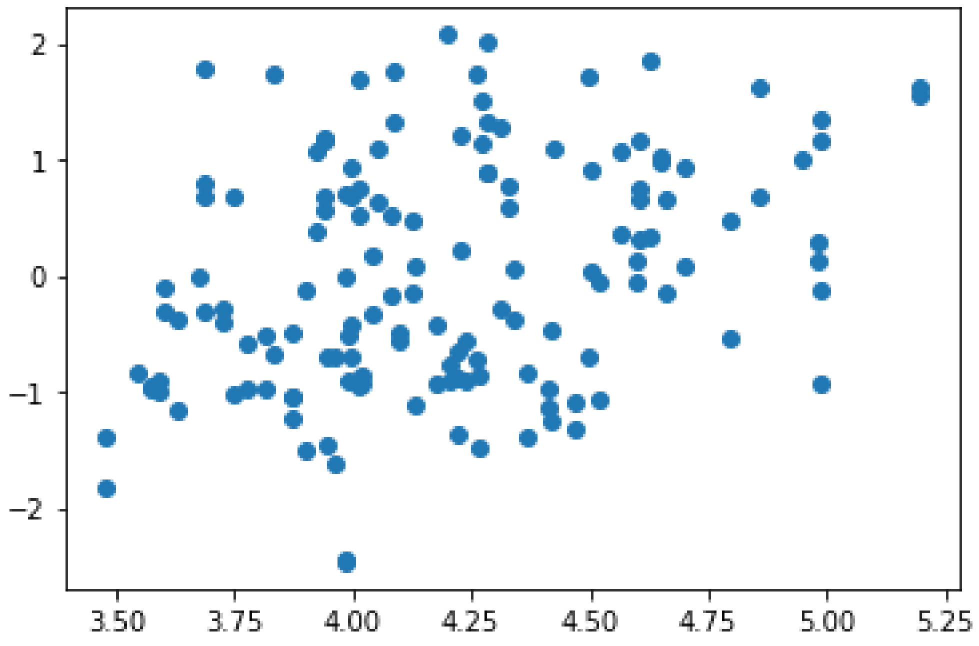
Z-scoring within sub-groups It seems that on average, men were about 4 seconds quicker than women. That means we can't just compute a Pearson correlation on the entire dataset as one group: Clearly there are two separate underlying distributions. However, we don't really care about the difference between men and women here. Instead, we're only interested in whether or not there is an effect of ready-start interval on finish times.

In order to look at this, we could z-score the data within each group. This subtracts the group mean from every sample, and then divides it by the group standard deviation. The z-scored finish times for men and women should both have a mean of 0 and a standard deviation of 1, making the two groups directly comparable. (And, more importantly, combinable!)

[7]:
```
from scipy.stats import zscore

# Compute the z-scored finish times.
z_time_500m_male = \
    zscore(time_500m[(is_male==True) & (fall==False)])
z_time_500m_female = \
    zscore(time_500m[(is_male==False) & (fall==False)])

# Combine the two vectors into one.
z_time_500m = \
    numpy.hstack([z_time_500m_male, z_time_500m_female])
```

```
[8]: # Import the Pearson correlation function.
     from scipy.stats import pearsonr

     # Compute the correlation between interval and finish time.
     r, p = pearsonr(interval[fall==False], z_time_500m)

     # Produce a quick plot of the z-scored times.
     pyplot.plot(interval[fall==False], z_time_500m, 'o')

     # Report the R and p value.
     print("R={}, p={}".format(round(r, ndigits=2), \
         round(p, ndigits=3)))
```

```
R=0.34, p=0.0
```



OK, so we now know that there is a *statistically* significant correlation between ready-start interval and finish time. However, we don't know whether this correlation is *practically* significant. Ideally, we would like to know exactly how a longer ready-start interval affects the finish time. In other words: If the referee waits 1 second longer to shoot the starting pistol, how much slower does an athlete become?

To answer this question, we can use *regression*.

1.2 Linear regression

One thing that wasn't mentioned about the 500 meter sprint in speed skating, is that athletes race twice. After the first round of races, pairs are mixed up, and all athletes race again. Their times of both races are combined, and whoever has the lowest summed time wins a gold medal.

This is not unlike an experimental manipulation: It sounds a bit like a researcher used two trials per participant to estimate the effect of ready-start interval on finish time. This is exactly how you can use the data!

For our each athlete, you can compute the difference in ready-start interval between both races. This will be your *predictor* variable `x`.

You can also compute the difference in finish times between both races. This will be your *outcome* variable `y`.

In a regression, you try to predict the outcome with one or more predictors. Or, in an equation:

$y = \beta_0 + x_1\beta_1 + \epsilon$

Where y is the outcome variable, β_0 is the intercept (what is y when all x values are zero?), x_1 is the first predictor variable, β_1 is a free variable that determines how much x_1 affects y, and ϵ is the *error term* that determines how much was unaccounted for. Sometimes this is called *noise*, because it refers to all unpredicted things.

Or, in code:

```
from scipy.stats import linregress
slope, intercept, r_value, p_value, std_err = linregress(x, y)
```

First, we need to compute our predictor and outcome: The differences between the two races of each athlete in ready-start interval and finish time. We can load the data for individual races from the file `speed_skating_paired_races.csv`.

[9]:
```
import numpy

# Load the data from all individual races.
data = numpy.loadtxt("speed_skating_paired_races.csv", \
    delimiter=",", dtype=float, skiprows=1, \
    usecols=range(1,9), unpack=True)

# Rename the variables for our convenience.
interval_1 = data[0,:]
time_100m_1 = data[1,:]
time_500m_1 = data[2,:]
interval_2 = data[3,:]
time_100m_2 = data[4,:]
time_500m_2 = data[5,:]
is_male = data[6,:].astype(bool)
exclude = data[7,:].astype(bool)
```

Now we can compute the differences in ready-start interval and finish time between the two races.

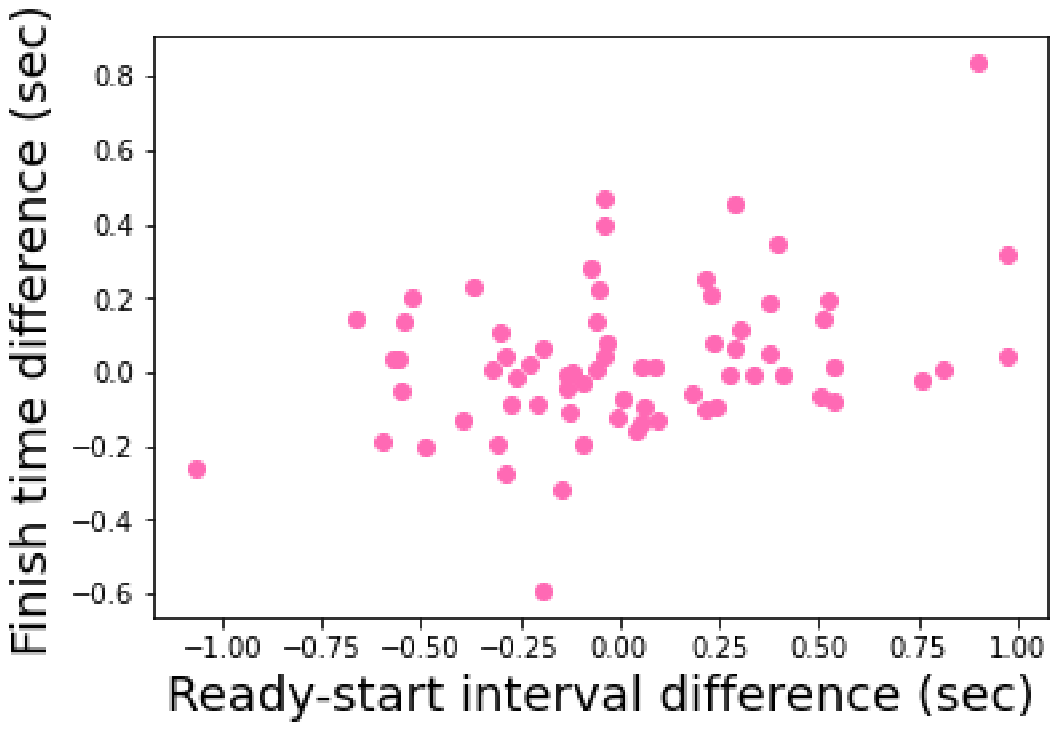
We'll exclude all the athletes who (nearly) fell, but we won't separate men and women again. This is because we have no reason to assume that alerting effects are any different between men and women, and thus our hypothesis should be that *all* athletes are affected, regardless of sex.

[10]:
```python
# Compute the interval difference between race 1 and 2.
interval_d = \
    interval_1[exclude==False] - interval_2[exclude==False]

# Compute the finish time difference between race 1 and 2.
time_500m_d = \
    time_500m_1[exclude==False] - time_500m_2[exclude==False]
```

Let's do a quick check to see what our data looks like:

[11]:
```python
# Plot the values.
pyplot.plot(interval_d, time_500m_d, 'o', color="#FF69B4")

# Add axis labels.
pyplot.xlabel("Ready-start interval difference (sec)", fontsize=18)
pyplot.ylabel("Finish time difference (sec)", fontsize=18)
```

[11]:
```
Text(0, 0.5, 'Finish time difference (sec)')
```



In the graph, we can see that ready-start interval differences lie between -1 and 1 seconds. We can also see that finish times are quite stable within each individual athlete: Most differences are between -0.4 and 0.4 seconds!

Assignment 3: To Z or not to Z? Should the finish time and ready-start interval differences (the values along the y- and x-axes in the above plot) be z-scored first? Why?

DOUBLE-CLICK HERE TO WRITE YOUR ANSWER

ANSWER: Running a regression on the non-standardised values allows one to estimate the real-life effect. Saying "there is a statistically significant correlation between ready-start interval and finish time" does not tell you how meaningful this correlation is. In order to tell whether this correlation has a noticable effect, you should quantify it in real numbers. For example, here we would like to know how many seconds a racer would finish slower if the referee waits for half a second longer between saying "Ready?" and firing the starting pistol.

Regression of ready-start interval and finish time Just by eyeballing the above graph, it looks like there might be a positive correlation between the ready-start interval difference and the finish time difference. Let's quantify this relation by using a regression:

[12]:
```python
from scipy.stats import linregress

# Run the regression analysis.
slope, intercept, r, p, std_err = \
    linregress(interval_d, time_500m_d)

# Report the regression's results.
print("R={}, p={}".format(round(r, ndigits=2), \
    round(p, ndigits=3)))
print("slope={}, intercept={}".format(
    round(slope, ndigits=2), round(intercept, ndigits=2)))
```

```
R=0.35, p=0.003
slope=0.17, intercept=0.03
```

From the output, we can learn that there is a statistically significant correlation with a Pearson R of 0.35. In addition, we now know how to quantify the effect:

$y = 0.03 + 0.17x_1$

or:

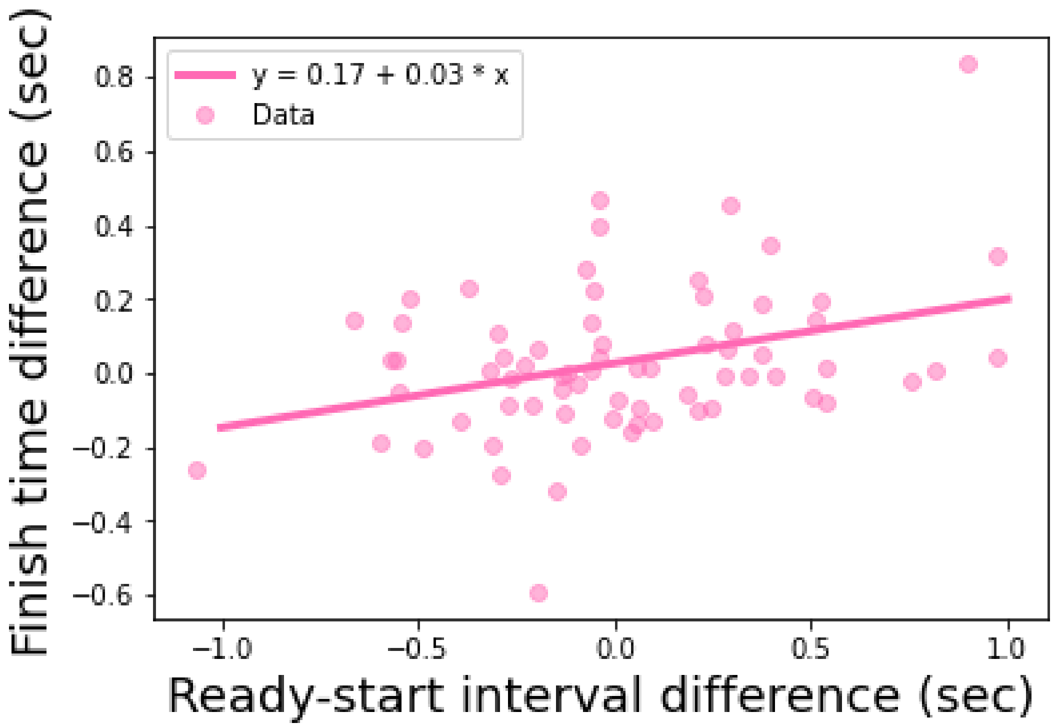
$\Delta_{finish} = 0.03 + 0.17\Delta_{interval}$

In practice, this means that for every second the referee waits between "Ready" and the starting shot, they add (on average) 0.17 seconds to an atheletes finish time. You can draw this line into your graph:

[13]:
```python
# Generate x values that cover the range of the real
# x-axis data (ready-start interval differences range
# from about -1 to 1 seconds).
```

```
x = numpy.arange(-1, 1.1, 0.1)
# Compute what the predicted y values would be for each
# of these x values.
y = intercept + slope * x

# Plot the line into a graph.
pyplot.plot(x, y, '-', linewidth=3, color="#FF69B4", \
    label="y = {} + {} * x".format( \
    round(slope, ndigits=2), round(intercept, ndigits=2)))

# Plot the measured values in the same graph.
pyplot.plot(interval_d, time_500m_d, 'o', color="#FF69B4", \
    alpha=0.5, label="Data")

# Add axis labels.
pyplot.xlabel("Ready-start interval difference (sec)", \
    fontsize=18)
pyplot.ylabel("Finish time difference (sec)", fontsize=18)

# Add a legend.
pyplot.legend(loc="upper left")
```

```
[13]: <matplotlib.legend.Legend at 0x7fc1f20ac290>
```



Is this a problem? Well, at Vancouver in 2010, the total difference between gold and silver was 0.16 seconds. It is not uncommen at all for speed skaters to get even closer: In 2014, the difference between gold and silver was 0.01 seconds, and between gold and bronze it was 0.15. In 2018, the difference between gold and silver was 0.01 seconds again.

Clearly, the margins are small, and thus this ready-start interval effect might have real-life consequences.

References If you're interested in the background to the data, you can read the following two articles. They're very short, and not very technical:

- Dalmaijer, E.S., Nijenhuis, B.G., & Van der Stigchel, S. (2015). Life is unfair, and so are racing sports: Some athletes can randomly benefit from alerting effects due to inconsistent starting procedures. Frontiers in Psychology, 6(1618). doi:10.3389/fpsyg.2015.01618
- Dalmaijer, E.S., Nijenhuis, B.G., & Van der Stigchel, S. (2016). Commentary: Life is unfair, and so are racing sports: Some athletes can randomly benefit from alerting effects due to inconsistent starting procedures. Frontiers in Psychology, 7(119). doi:10.3389/fpsyg.2016.00119

1.3 How does regression work?

In regression, you have an explicit *model* for your data. Specifically, it says that there is a linear relationship between variables y and x:

$y = \beta_0 + x_1\beta_1 + \epsilon$

You know what the values for y are, because you measured those. You also know what the values for x are, because you manipulated (or measured) those. But how do you know what the β values are? And what that ϵ is?

Let's talk ϵ first, because it is surprisingly simple: You simply forget about it. This is the "error", and thus by definition the extent to which the data does not abide by your model. This can be because of measurement error, but also because it is influenced by variables that were not included in your model.

Next up are the β values. One way would to identify the β values is to simply try all possible values of each β, and see when the resulting line fits best. How do you know what set of β values fits best? Simple: It's when the difference between your predicted values of y and your measured values of y is the smallest.

Let's give this approach a go with the skating dataset:

[14]:
```python
# First, define the ranges along which we need to
# search for the best fitting betas.
b0_range = numpy.arange(0, 1, 0.01)
b1_range = numpy.arange(-10, 10, 0.01)

# Count the number of values we will try for each beta.
n_b0 = len(b0_range)
n_b1 = len(b1_range)
```

```
# Second, define some starting values. The first are the
# betas, which will be None to start with.
beta = (None, None)
# We also need to start with a difference between the y
# values and the predicted y values. This
# will start at infinitely high:
min_s = numpy.inf

# Finally, we loop through every possible combination of
# b0 and b1.
for i, b0 in enumerate(b0_range):
    for j, b1 in enumerate(b1_range):
        # Predict y using the current betas.
        y_pred = b0 + b1 * interval_d
        # Compute the difference between the predicted y
        # and the measured y for each observation.
        d = time_500m_d - y_pred
        # Compute the sum of squares of the differences
        # (residuals).
        s = numpy.sum(d**2)
        # Remember the current betas if the sum of squares
        # is lower than the previously lowest.
        if s < min_s:
            betas = (b0, b1)
            min_s = s

print("Best fit: b0={}, b1={}".format(round(betas[0], ndigits=2), \
    round(betas[1], ndigits=2)))
```

```
Best fit: b0=0.03, b1=0.17
```

As you might recall, these are the same values that you obtained through using the `linregress` function earlier!

Residuals and parameter space The reason that the above works, is because you travelled through *parameter space*, and at each point computed the *sum of squares* of the difference between the actual and your predicted values. This difference is called the *residuals*. You kept track of which point in parameter space was associated with the lowest *squared residuals*. This is called *least-squares regression*.

You can actually plot parameter space and the associated residual squares:

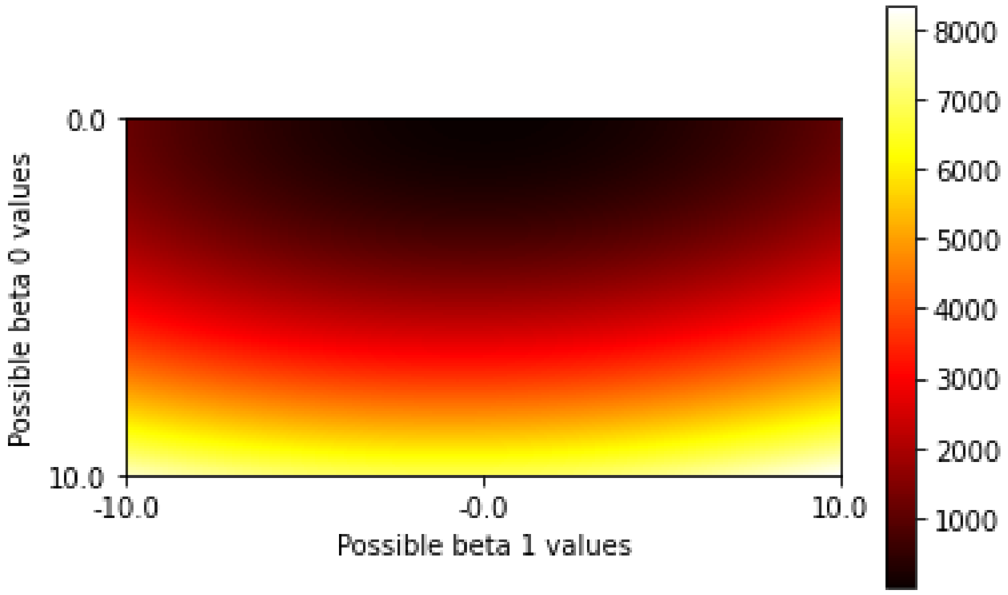
[15]:
```
# First, define the ranges along which we need to search
# for the best fitting betas.
b0_range = numpy.arange(0, 10, 0.01)
b1_range = numpy.arange(-10, 10, 0.01)

# Count the number of values we will try for each beta.
```

```python
n_b0 = len(b0_range)
n_b1 = len(b1_range)

# Second, define some starting values. The first are the
# betas, which will be None to start with.
beta = (None, None)
# We also need to start with a difference between the y
# values and the predicted y values. This
# will start at infinitely high:
min_s = numpy.inf
# Keep track of the residuals at every point in parameter
# space. This starts as a matrix filled with NaN (not a
# number), and one value will be added on every iteration.
s = numpy.zeros((n_b0,n_b1), dtype=float) * numpy.NaN

# Finally, we loop through every possible combination of
# b0 and b1.
for i, b0 in enumerate(b0_range):
    for j, b1 in enumerate(b1_range):
        # Predict y using the current betas.
        y_pred = b0 + b1 * interval_d
        # Compute the difference between the predicted y
        # and the measured y for each observation.
        d = time_500m_d - y_pred
        # Compute the sum of squares of the differences
        # (residuals).
        s[i,j] = numpy.sum(d**2)
        # Remember the current betas if the sum of squares
        # is lower than the previously lowest.
        if s[i,j] < min_s:
            betas = (b0, b1)
            min_s = s[i,j]

print("Best fit: b0={}, b1={}".format(round(betas[0], ndigits=2), \
    round(betas[1], ndigits=2)))

# Now plot the residual squares in parameter space:
pyplot.imshow(s, cmap="hot")
# Set the tick labels on the x and y axes.
x_ticks = [0, n_b1//2, n_b1-1]
x_tick_labels = numpy.round(b1_range[x_ticks])
pyplot.xticks(x_ticks, x_tick_labels)
pyplot.xlabel("Possible beta 1 values")
y_ticks = [0, n_b0-1]
y_tick_labels = numpy.round(b0_range[y_ticks])
pyplot.yticks(y_ticks, y_tick_labels)
pyplot.ylabel("Possible beta 0 values")
```

```
# Draw an colour bar to show the resulting sums of residual
# squares.
pyplot.colorbar()
```

```
Best fit: b0=0.03, b1=0.17
```

[15]:
```
<matplotlib.colorbar.Colorbar at 0x7fc1fbc7ffd0>
```

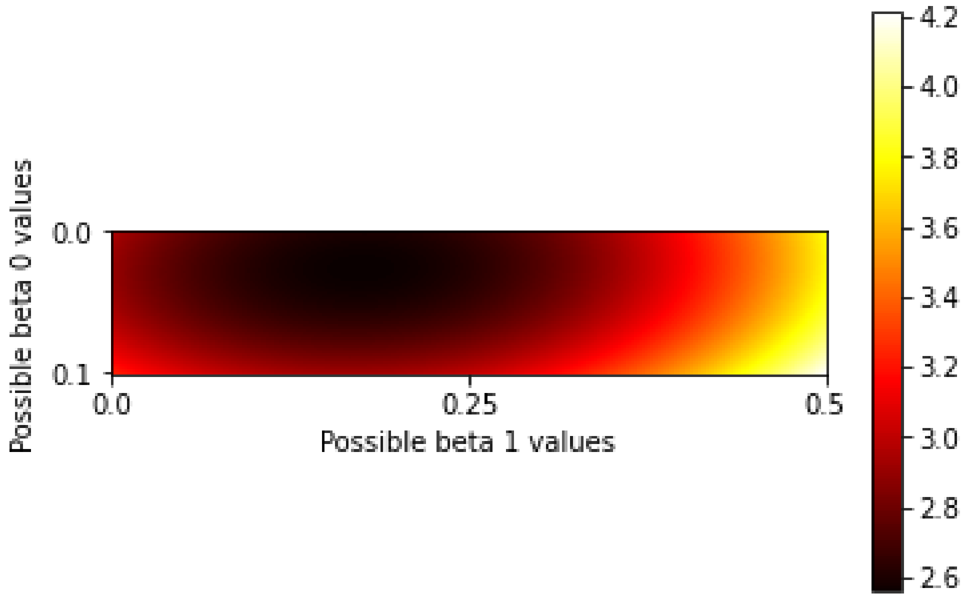


From this plot, you can see that the optimal combination of betas is close to point (0,0). However, you can also see that we cast a *very* wide net. Perhaps it would have been better to choose a smaller search space. For example, instead of using ranges $[0, 10]$ for β_0 and $[-10, 10]$ for β_1, we could have used $[0, 0.1]$ and $[0, 0.5]$

[16]:
```
# First, define the ranges along which we need to
# search for the best fitting betas.
b0_range = numpy.arange(0, 0.1, 0.001)
b1_range = numpy.arange(0, 0.5, 0.001)

# Count the number of values we will try for each beta.
n_b0 = len(b0_range)
n_b1 = len(b1_range)

# Second, define some starting values. The first are the
# betas, which will be None to start with.
beta = (None, None)
# We also need to start with a difference between the y
# values and the predicted y values. This
# will start at infinitely high:
min_s = numpy.inf
# Keep track of the residuals at every point in parameter
# space. This starts as a matrix filled with NaN (not a
# number), and one value will be added on every iteration.
s = numpy.zeros((n_b0,n_b1), dtype=float) * numpy.NaN

# Finally, we loop through every possible combination of
# b0 and b1.
for i, b0 in enumerate(b0_range):
    for j, b1 in enumerate(b1_range):
        # Predict y using the current betas.
        y_pred = b0 + b1 * interval_d
        # Compute the difference between the predicted y
        # and the measured y for each observation.
        d = time_500m_d - y_pred
        # Compute the sum of squares of the differences
        # (residuals).
        s[i,j] = numpy.sum(d**2)
        # Remember the current betas if the sum of squares
        # is lower than the previously lowest.
        if s[i,j] < min_s:
            betas = (b0, b1)
            min_s = s[i,j]

print("Best fit: b0={}, b1={}".format(round(betas[0], ndigits=2), \
    round(betas[1], ndigits=2)))

# Now plot the residual squares in parameter space:
pyplot.imshow(s, cmap="hot")
# Set the tick labels on the x and y axes.
x_ticks = [0, n_b1//2, n_b1-1]
x_tick_labels = numpy.round(b1_range[x_ticks], 2)
pyplot.xticks(x_ticks, x_tick_labels)
pyplot.xlabel("Possible beta 1 values")
y_ticks = [0, n_b0-1]
y_tick_labels = numpy.round(b0_range[y_ticks], 2)
pyplot.yticks(y_ticks, y_tick_labels)
pyplot.ylabel("Possible beta 0 values")

# Draw an colour bar to show the resulting sums of residual
# squares.
pyplot.colorbar()
```

```
Best fit: b0=0.03, b1=0.17
```

```
[16]: <matplotlib.colorbar.Colorbar at 0x7fc1f2047d10>
```



Here, it's a lot clearer that the best fitting combination of predictor values is around (0.17, 0.03).

Using minimisation in least-squares regression What you just did is a *full space estimate*. The advantage of such an approach is that you tried every possible point within a pre-defined grid. A significant downside, however, is that it takes ages to complete. Especially if you have no clue where your possible β values are going to be, and/or if you want a high resolution estimate (smaller step sizes between your points), you would have to try a very large number of combinations. In addition, if you want to add additional predictors, your grid will grow exponentially.

Fortunately, there are *minimisation algorithms*. These will walk through parameter space in a clever way. Most work by randomly starting at one particular point, computing the residual squares for that particular set of β values, and then they try a nearby point to compute the residual squares again. By using the slope between these points, the algorithm knows where to go: Because the best fitting solution is at the point with the lowest residual square, the algorithm simply has to follow the slope downwards until it reaches a point where it can no longer go down any further. This is the best fit!

Let's try one of these minimisation algorithms. It needs a function to minimise the value for (i.e. a function that computes the residual squares):

```
[17]: # Import the minimize function from SciPy.
      from scipy.optimize import minimize

      # Define a function to compute the sum of residual
      # squares based on our model.
```

```python
def residuals(betas, x, y):
    # Compute the predicted value of y.
    y_pred = betas[0] + betas[1] * x
    # Compute the residuals.
    res = y - y_pred
    # Compute the sum of squared residuals.
    s = numpy.sum(res**2)
    # Return the squared residuals.
    return s

# Choose values that the algorithm uses as an initial guess.
initial_guess = (0.0, 0.0)
# Use the mimize function to compute the best fit.
model = minimize(residuals, initial_guess, \
    args=(interval_d, time_500m_d), method="L-BFGS-B")

# Report the betas.
betas = model.x
print("Best fit: b0={}, b1={}".format(round(betas[0], ndigits=2), \
    round(betas[1], ndigits=2)))
```

```
Best fit: b0=0.03, b1=0.17
```

This is the same result as the full space estimate provided earlier. However, this method is **much** faster.

You could ask what the benefit of using `minimize` is over simply using `linregress`. Both functions allowed us to fit our data, both gave us the same answer, and both were very quick about it too. The neat thing about using the `minimize` approach is that it is very flexible. You could have used *any* model, regardless of how many predictors you would have liked. You could even have used a non-linear model. Or a different way of computing the ”best fit”, for example one that doesn't rely on residuals.

1.4 How good is a regression's fit?

The one thing we didn't do yet, is computing how good a fit really is. The usual measure for this is the *coefficient of determination*, or R^2. This is computed by dividing the sum of squares of the residuals by the total sum of squares:

$R^2 = 1 - \frac{SS_{res}}{SS_{total}}$

Or, more scary-looking, but also more helpful:

$R^2 = 1 - \frac{\Sigma_{i=1}^{n}(y_i - f_i)^2}{\Sigma_{i=1}^{n}(y_i - \bar{y})^2}$

Where n is the number of observations, $\bar{y}$ is the mean of y, and f_i is the predicted value of y_i given the model. For example:

$f_i = \beta_0 + \beta_1 x_i$

Or, in code:

[18]:
```
# Compute the predicted y values based on the fitted betas.
y_pred = betas[0] + betas[1] * interval_d

# Compute the residual sum of squares.
ss_res = numpy.sum((time_500m_d - y_pred)**2)

# Compute the total sum of squares.
ss_tot = numpy.sum((time_500m_d - numpy.mean(time_500m_d))**2)

# Compute R square.
r_sq = 1.0 - (ss_res / ss_tot)

print("R squared = {}".format(round(r_sq, ndigits=2)))
```

```
R squared = 0.12
```

If R^2 is 1, all variance in outcome y is predicted by predictor x. If R^2 is 0, none of the variance in outcome y is explained by predictor x. Here, the ready-start interval differences can explain 12% of the variance in finish time difference.

In the case of high-level sporting events, the amount of variance explained by anything other than athlete's ability should ideally be 0%. Here, a random variation in pre-start time that is introduced by the person holding the starting pistol was 12%. That's probably not ideal.

Assignment 4: Linear regression in JASP Run the regression of ready-start time difference on race difference in JASP. You will have to run through the following steps:

1. Open the file `speed_skating_paired_races.csv` in JASP.
2. Exclude the samples that should not be included using JAPS's filter functions.
3. Compute the difference between the ready-start intervals between both races, using JASP's new variable calculation functions.
4. Compute the difference between the finish times between both races, using JASP's new variable calculation functions.
5. Compute a linear regression of ready-start time difference and finish time difference, using JASP's "classic" (frequentist) linear regression function.

1.5 Multi-variable regression

Now that you know how to do a regression with a single predictor, we can turn to regressions with multiple predictors. The general format of multi-variable regression looks very similar to the single-variable version:

$y = \beta_0 + x_1\beta_1 + ... + x_n\beta_n + \epsilon$

Where n is the number of variables you might have. For example, the equation for three predictors would look like this:

$y = \beta_0 + x_1\beta_1 + x_2\beta_2 + x_3\beta_3 + \epsilon$

Here, x_1, x_2, and x_3 are three different predictors. β_1, β_2, and β_3 indicate the magnitude and direction of the effect of each predictor on y. As before, ϵ captures the "noise": all variance in y that we could not explain using the predictors.

Another example dataset Last session, we looked at a dataset that contained the number of minutes each participant listened to Taylor Swift, and their happiness ratings. We collected similar data, but now included a measure of IQ too. These data can be found in the attached file `taytay_revisited.csv`.

We can load the dataset using NumPy's `loadtxt` function:

[19]:
```python
import numpy
from matplotlib import pyplot

# Load the data.
data = numpy.loadtxt("taytay_revisited.csv", dtype=float, \
    delimiter=",", skiprows=1, unpack=True)

# Create some easy variable names to point to the data.
tay_minutes = data[0,:]
happy = data[1,:]
iq = data[2,:]

# Use the two predictors together into a single variable.
predictors = numpy.vstack([iq, happy])
```

Our objective is to predict the number of minutes someone listens to Taylor Swift by using their IQ score and happiness rating. Or, in an equation:

$Swifting = \beta_0 + happiness * \beta_1 + IQ * \beta_2 + \epsilon$

Let's write a function to model this:

[20]:
```python
def residuals(betas, x, y):
    # Compute the predicted y values.
    y_pred = betas[0] + betas[1] * x[0,:] + \
        betas[2] * x[1,:]
    # Compute the residuals.
    res = y - y_pred
    # Compute the sum of squared residuals.
    s = numpy.sum(res**2)
    # Return the SSres
    return s
```

Now we can use SciPy's `minimize` function to fit out model to the data:

[21]:
```python
from scipy.optimize import minimize

# Set an initial guess for the betas.
```

```
initial_guess = [0.0, 0.0, 0.0]

# Fit the model.
model = minimize(residuals, initial_guess, \
    args=(predictors, tay_minutes), method="L-BFGS-B")

# Report the betas.
betas = model.x
print("Best fit: b0={}, b1={}, b2={}".format( \
    round(betas[0], ndigits=2), round(betas[1], ndigits=2), \
    round(betas[2], ndigits=2)))
```

```
Best fit: b0=34.19, b1=0.16, b2=1.88
```

OK! So the best fit is the following:

$Swifting = 34.19 + happiness * 1.88 + IQ * 0.16 + \epsilon$

Our β values are 1.88 for happiness and 0.16 for IQ. Does that mean IQ is less important than happiness for determining Taylor Swift listening? It doesn't necessarily, because we're currently looking at *unstandardised coefficients*. The values for IQ are larger than the values for happiness: IQ, per definition, has a mean of 100 and a standard deviation of 15, whereas happiness was rated on a 0-10 scale. This difference in range alters the magnitudes of β values.

In order to directly compare the parameters, we'll need the *standardised coefficients*. You can compute those by simply z-scoring predictors (and outcomes!) **before** running your regression:

[22]:
```
from scipy.stats import zscore
from scipy.optimize import minimize

# Use the two predictors together into a single variable.
z_predictors = numpy.vstack([zscore(iq), zscore(happy)])
z_minutes = zscore(tay_minutes)

# Set an initial guess for the betas.
initial_guess = [1.0, 1.0, 1.0]

# Fit the model.
stand_model = minimize(residuals, initial_guess, \
    args=(z_predictors, z_minutes), method="L-BFGS-B")

# Report the betas.
stand_betas = stand_model.x
print("Best fit: b0={}, b1={}, b2={}".format( \
    round(stand_betas[0], ndigits=2), \
    round(stand_betas[1], ndigits=2), \
    round(stand_betas[2], ndigits=2)))
```

```
Best fit: b0=0.0, b1=0.13, b2=0.3
```

From this, we can really tell that happiness has a bigger effect on Taylor Swift listening than IQ does.

Let's compute how much of the variance we can explain with the current fit:

```
[23]: # Compute the predicted y values based on the fitted betas.
      y_pred = betas[0] + betas[1] * iq + betas[2] * happy

      # Compute the residual sum of squares.
      ss_res = numpy.sum((tay_minutes - y_pred)**2)

      # Compute the total sum of squares.
      ss_tot = numpy.sum((tay_minutes - numpy.mean(tay_minutes))**2)

      # Compute R square.
      r_sq = 1.0 - (ss_res / ss_tot)

      print("R squared = {}".format(round(r_sq, ndigits=2)))
```

```
R squared = 0.11
```

An R^2 of 0.11 is pretty decent!

Or is it? We don't really know what it means in context, or whether it's statistically significant. Let's shelf that thought for a second, as we will return to it later.

Assignment 5: Multi-variable linear regression in JASP The above code is not always the easiest way to do regressions. In fact, it requires a few further computations to assess whether a multi-variable regression produced any statistically significant results.

To simplify the process, you could turn to JASP. For this assignment, you should follow these steps:

1. Open the `taytay_revisited.csv` file in JASP.
2. Run a ("classic") linear regression with outcome variable `minutes`, and predictors `happy` and `IQ`.