LECTURE 21: TWO-DIMENSIONAL WAVE EQUATIONS, DOUBLE FOURIER SERIES

In this lecture we first derive a model of two-dimensional wave equation, namely, the motion of an elastic membrane. We then solve this model for rectangular membrane by the method of separation of variables, using double Fourier series.

1. Modeling: Membrane, Two-Dimensional Wave Equation

Physical Assumptions

1. The mass of the membrane per unit area is constant ("homogeneous membrane"). The membrane is perfectly flexible and offers no resistance to bending.
2. The membrane is stretched and then fixed along its entire boundary in the xy-plane. The tension per unit length T caused by stretching the membrane is the same at all points and in all directions and does not change during the motion. There are no external forces acting on the moving membrane.
3. The deflection $u(x, y, t)$ of the membrane during the motion is small compared to the size of the membrane, and all angles of inclination are small.

Under these assumptions we regard the membrane performs small transverse motions.

Derivation of the PDE of the Model

As in lecture 17 we take an arbitrary small portion of membrane and consider the forces acting on this portion, as it is moving up and down, see Figure 1. Since the deflections of the membrane and the angles of inclination are small, the sides of the portion are approximately equal to Δx and Δy. We denote T by the force per unit length. Hence the forces acting on the sides of the portion are approximately $T\Delta x$ and $T\Delta y$. Since the membrane is perfectly flexible, these forces are tangent to the moving membrane at every instant.

Horizontal Components of the Forces. We first consider the horizontal components of the forces. These components are obtained by multiplying the tension acting on sides by the cosines of the angles of inclination. Since these angles are small, their cosines are close to 1. Hence the horizontal components of the forces at opposite sides are approximately equal. Therefore, the motion of the particles of the membrane in a horizontal direction will be negligibly small. From this we conclude that we may regard the motion of the membrane as transversal, that is, each particle moves vertically.

Vertical Components of the Forces. These components along the right side and the left side are (Figure 1) at any instant t, respectively,

$$T\Delta y \sin\beta \quad \text{and} \quad -T\Delta y \sin\alpha.$$

Here α and β are the values of the angle of inclination (which varies slightly along the edges) in the middle of the edges, and the minus sign appears because the force on the left side is directed downward. Since the angles are small, we may replace their sines by their tangents. Hence the resultant of those two vertical components is

$$\begin{aligned} T\Delta y(\sin\beta - \sin\alpha) &\approx T\Delta y(\tan\beta - \tan\alpha) \\ &= T\Delta y[u_x(x + \Delta x, y_1, t) - u_x(x, y_2, t)], \end{aligned} \tag{1.1}$$

where subscripts x denote partial derivatives and y_1 and y_2 are values between y and $y + \Delta y$. Similarly, the resultant of the vertical components of the forces acting on the other two sides of the

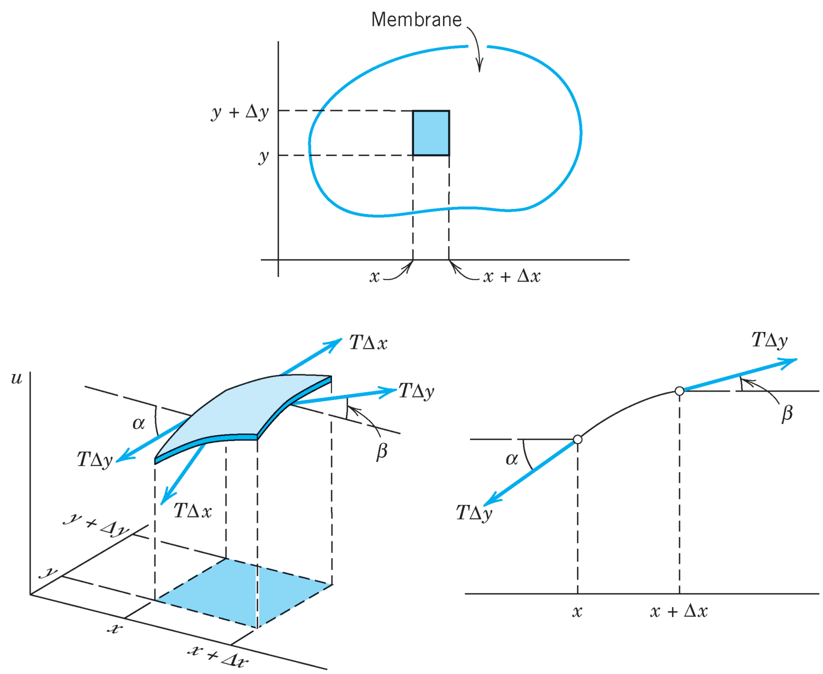

FIGURE 1. Vibrating membrane

portion at time t is

$$T\Delta x[u_y(x_1, y + \Delta y, t) - u_y(x_2, y, t)], \tag{1.2}$$

where x_1 and x_2 are values between x and $x + \Delta x$.

Newton's Second Law Gives the PDE of the Model. By Newton's second law the sum of the forces given by (1.1) and (1.2) at vertical direction is equal to the mass $\rho\Delta A$ of that small portion times the acceleration $\dfrac{\partial^2 u}{\partial t^2}(\tilde{x}, \tilde{y}, t)$ at any instant t. Here ρ is the mass of the undeflected membrane per unit area, and $\Delta A = \Delta x\Delta y$ is the area of that portion when it is undeflected, $(\tilde{x}, \tilde{y})$ is a suitable point at the portion. Thus

$$\begin{aligned}\rho\Delta x\Delta y\frac{\partial^2 u}{\partial t^2}(\tilde{x}, \tilde{y}, t) &= T\Delta y[u_x(x + \Delta x, y_1) - u_x(x, y_2)] \\ &+ T\Delta x[u_y(x_1, y + \Delta y) - u_y(x_2, y)].\end{aligned}$$

Division by $\rho\Delta x\Delta y$ gives

$$\frac{\partial^2 u}{\partial t^2}(\tilde{x}, \tilde{y}, t) = \frac{T}{\rho}\left[\frac{u_x(x + \Delta x, y_1, t) - u_x(x, y_2, t)}{\Delta x} + \frac{u_y(x_1, y + \Delta y, t) - u_y(x_2, y, t)}{\Delta y}\right].$$

If we let Δx and Δy approach zero, we have $\tilde{x}, x_1, x_2 \to x$ and $\tilde{y}, y_1, y_2 \to y$ and we obtain the PDE of the model

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \tag{1.3}$$

where $c^2 = T/\rho$. This PDE is called the **two-dimensional wave equation**. The expression in parentheses is the Laplacian Δu of u. Hence (1.3) can be written

$$\frac{\partial^2 u}{\partial t^2} = c^2\Delta u.$$

As in lecture 17, the model of the motion of an elastic membrane consists of two-dimensional wave equation and additional conditions arising from the problem. Additional conditions contain the initial conditions and the boundary conditions. In agreement with the highest order of derivative with respect to t in equation, we need two initial conditions, that is,

$$\begin{aligned} u(x,y,0) &= f(x,y), \quad \text{the initial deflection or displacement,} \\ u_t(x,y,0) &= g(x,y), \quad \text{the initial velocity.} \end{aligned}$$

Suppose that the boundary of membrane is a curve C in xy-plane. We often consider three types of boundary conditions, similar to two dimensional heat equation in lecture 20:

(I) Dirichlet boundary condition: u is prescribed on C.

(II) Neumann boundary condition: the normal derivative $u_n = \dfrac{\partial u}{\partial n}$ is prescribed on C.

(III) Robin boundary condition $u + b\dfrac{\partial u}{\partial n}$ is prescribed on C.

For example, according to Assumption 2 (the membrane is stretched and then fixed along its entire boundary in the xy-plane), we have the boundary condition

$$u = 0 \quad \text{on } C.$$

2. Rectangular Membrane, double Fourier Series

Now we solve the model of the vibrating membrane that is a rectangular $\Omega = \{(x,y) \mid 0 < x < a,\ 0 < y < b\}$ for obtaining the displacement $u(x,y,t)$ of a point (x,y) of the membrane at time t. According to §1, this model can be characterized by a two-dimensional wave equation and some initial and boundary conditions arising from the problem. Here we assume that the membrane is fixed along the boundary in the xy-plane for all times $t \geq 0$, which implies that $u(x,y,t) = 0$ on the boundary. Also the initial displacement (initial shape) $f(x,y)$ and the initial velocity $g(x,y)$ are given. Thus we write the model

$$\text{EQ} \quad \frac{\partial^2 u}{\partial t^2} = c^2\Big(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\Big),\ 0 < x < a,\ 0 < y < b,\ t > 0, \tag{2.1}$$

$$\text{B.C.} \quad u(0,y,t) = 0,\ u(a,y,t) = 0, \quad 0 \leq y \leq b,\ t \geq 0, \tag{2.2}$$

$$\text{B.C.} \quad u(x,0,t) = 0,\ u(x,b,t) = 0, \quad 0 \leq x \leq a,\ t \geq 0, \tag{2.3}$$

$$\text{I.C.} \quad u(x,y,0) = f(x,y),\ u_t(x,y,0) = g(x,y), \quad 0 \leq x \leq a,\ 0 \leq y \leq b. \tag{2.4}$$

Similar to those for the string in lecture 17. we shall solve this problem in three steps:

Step 1. By using separating variables, first setting $u(x,y,t) = F(x,y)G(t)$ and later $F(x,y) = H(x)Q(y)$ we obtain from (2.1) an ODE for G and later from a PDE for F two ODEs for H and Q.

Step 2. From the solutions of those ODEs we determine solutions of (2.1) satisfying the boundary conditions (2.2) and (2.3).

Step 3. We compose the solutions obtained in **Step 2** into a double series such that it satisfies the initial conditions (2.4).

Step 1. Three ODEs From the Wave Equation (2.1)

To obtain ODEs from (2.1), we apply two successive separations of variables. In the first separation we set $u(x,y,t) = F(x,y)G(t)$. Substitution into (2.1) gives

$$F(x,y)G''(t) = c^2(F_{xx} + F_{yy})G(t)$$

To separate the variables, we divide both sides by $c^2F(x,y)G(t)$:

$$\frac{G''(t)}{c^2G(t)} = \frac{1}{F(x,y)}(F_{xx} + F_{yy}).$$

Since the left side depends only on t, whereas the right side is independent of t, both sides must equal a constant. By a simple investigation we see that only negative values of that constant will lead to solutions that satisfy (2.2) and (2.3) without being identically zero. This is similar to lecture 18. Denoting the negative constant by $-\nu^2$, we have

$$\frac{G''(t)}{c^2G(t)} = \frac{1}{F(x,y)}(F_{xx} + F_{yy}) = -\nu^2.$$

This gives two equations: for the "time function" $G(t)$ we have the ODE

$$G''(t) + (c\nu)^2G(t) = 0, \ t > 0 \tag{2.5}$$

and for the "**amplitude function**" $F(x,y)$ a PDE, called the two-dimensional **Helmholtz equation**

$$F_{xx} + F_{yy} + \nu^2F = 0. \tag{2.6}$$

Separation of the Helmholtz equation is achieved if we set $F(x,y) = H(x)Q(y)$. By substitution of this into (2.6) we obtain

$$H''(x)Q(y) = -H(x)(Q''(y) + \nu^2Q(y)).$$

To separate the variables, dividing both sides by $H(x)Q(y)$ we have

$$\frac{H''(x)}{H(x)} = -\frac{1}{Q(y)}(Q''(y) + \nu^2Q(y)).$$

Both sides must equal a constant, by the usual argument. This constant must be negative, say,$-k^2$, because only negative values will lead to solutions that satisfy (2.2) without being identically zero. Thus

$$\frac{H''(x)}{H(x)} = -\frac{1}{Q(y)}(Q''(y) + \nu^2Q(y)) = -k^2.$$

This yields two ODEs for H and Q, namely,

$$H''(x) + k^2H(x) = 0, \ 0 < x < a \tag{2.7}$$

and

$$Q''(y) + p^2Q(y) = 0, \ 0 < y < b, \tag{2.8}$$

where $p^2 = \nu^2 - k^2$.

Step 2. Satisfying the Boundary Conditions

Boundary conditions (2.2) give $H(0)Q(y)G(t) = H(a)Q(y)G(t) = 0$, from which we have

$$H(0) = H(a) = 0.$$

Similarly, from boundary conditions (2.3) we have

$$Q(0) = Q(b) = 0.$$

We first solve the problem for $H(x)$, that is,

$$H''(x) + k^2H(x) = 0, \ 0 < x < a; \ H(0) = H(a) = 0.$$

Since the general solution is $H(x) = A\cos kx + B\sin kx$, using $H(0) = H(a) = 0$ gives

$$A = 0, \ B\sin ka = 0.$$

Here $B \neq 0$ since otherwise $H(x) \equiv 0$ and $F(x,y) \equiv 0$. Thus taking $B = 1$, we obtain

$$k = k_m = \frac{m\pi}{a}, \ H(x) = H_m(x) = \sin\frac{m\pi x}{a}, \ m = 1, 2, \cdots.$$

We now solve the problem for $Q(y)$, namely,

$$Q''(y) + p^2 Q(y) = 0,\ 0 < y < b;\ Q(0) = Q(b) = 0.$$

In precisely the same fashion we conclude that

$$p = p_n = \frac{n\pi}{b},\quad Q_n(y) = \sin\frac{n\pi y}{b},\ n = 1, 2, \cdots.$$

Moveover, we obtain

$$\nu_{mn}^2 = k_m^2 + p_n^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2,\ m, n = 1, 2, \cdots. \tag{2.9}$$

As in the case of the vibrating string, it is not necessary to consider $m, n = -1, -2, \cdots$ since the corresponding solutions are essentially the same as for positive m and n, expect for a factor -1. Hence the functions

$$F_{mn}(x, y) = H_m(x)Q_n(y) = \sin\frac{m\pi x}{a}\sin\frac{m\pi y}{b},\ m, n = 1, 2, \cdots \tag{2.10}$$

are solutions of the Helmholtz equation (2.6) that are zero on the boundary of membrane.

We finally solve equation (2.11). By (2.9), it becomes

$$G''(t) + \lambda_{mn}^2 G(t) = 0,\ t > 0,$$

where

$$\lambda = \lambda_{mn} = c\pi\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}},\ m, n = 1, 2, \cdots. \tag{2.11}$$

It has general solution

$$G_{mn}(t) = A_{mn}\cos\lambda_{mn}t + B_{mn}\sin\lambda_{mn}t.$$

It follows that the functions

$$u_{mn}(x, y, t) = (A_{mn}\cos\lambda_{mn}t + B_{mn}\sin\lambda_{mn}t)\sin\frac{m\pi x}{a}\sin\frac{m\pi y}{b},\ m, n = 1, 2, \cdots \tag{2.12}$$

satisfy the wave equation (2.1) and the boundary conditions (2.2), (2.3). These functions are called the **eigenfunctions** or characteristic functions, and the numbers λ_{mn} are called the **eigenvalues** or characteristic values of the vibrating membrane. The frequency of u_{mn} is $\lambda_{mn}/(2\pi)$.

Discussion of Eigenfunctions. It is very interesting that, depending on a and b, several functions F_{mn} may correspond to the same eigenvalue. Physically this means that there may exists vibrations having the same frequency but entirely different **nodal lines** (curves of points on the membrane that do not move). Let us illustrate this with the following example.

Example 1. (Eigenvalues and Eigenfunctions of the Square Membrane) Consider the square membrane with $a = b = 1$. From (2.11) we obtain its eigenvalues

$$\lambda_{mn} = c\pi\sqrt{m^2 + n^2}.$$

Hence $\lambda_{mn} = \lambda_{nm}$, but for $m \neq n$ the corresponding functions

$$F_{mn} = \sin m\pi x \sin n\pi y \text{ and } F_{nm} = \sin n\pi x \sin m\pi y$$

are certainly different. For example, to $\lambda_{12} = \lambda_{21} = c\pi\sqrt{5}$ there correspond the two functions

$$F_{12} = \sin\pi x\sin 2\pi y \quad\text{and}\quad F_{21} = \sin 2\pi x\sin\pi y.$$

Hence the corresponding solutions

$$u_{12} = (A_{12}\cos c\pi\sqrt{5}t + B_{12}\sin c\pi\sqrt{5}t)F_{12} \quad\text{and}\quad u_{21} = (A_{21}\cos c\pi\sqrt{5}t + B_{21}\sin c\pi\sqrt{5}t)F_{21}$$

have the nodal lines $y = 1/2$ and $x = 1/2$ respectively (see Figure 2). Taking $A_{12} = 1$ and $B_{12} = B_{21} = 0$ we obtain

$$u_{12} + u_{21} = \cos c\pi\sqrt{5}t(F_{12} + A_{21}F_{21}) \tag{2.13}$$

which represents another vibration corresponding to the eigenvalue $c\pi\sqrt{5}t$. The nodal line of this function is the solution of the equation

$$F_{12} + B_{21}F_{21} = \sin \pi x \sin 2\pi y + A_{21} \sin 2\pi x \sin \pi y = 0.$$

Since $\sin 2\alpha = 2 \sin \alpha \cos \alpha$,

$$\sin \pi x \sin \pi y(\cos \pi y + B_{21} \cos \pi x) = 0.$$

This solution depends on the value of B_{21} (see Figure 2).

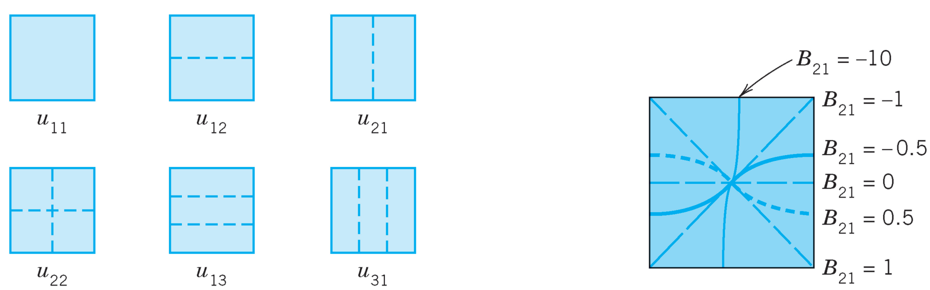

FIGURE 2. Left: Nodal of the solutions u_{11}, u_{12}, u_{21}, u_{22}, u_{13}, u_{31} in the case of the square membrane Right: Nodal lines of the solution (2.12) for some values of B_{21}

Furthermore, more than two functions may correspond to the same λ_{mn}. For example, the four functions F_{18}, F_{81}, F_{47} and F_{74} correspond to the value

$$\lambda_{18} = \lambda_{81} = \lambda_{47} = \lambda_{74} = c\pi\sqrt{65},$$

because $1^2 + 8^2 = 4^2 + 7^2 = 65$. This happens because 65 can be expressed as the sum of two squares of positive integers in several ways. According to a theorem by Gauss, this is the case for every sum of two squares among whose prime factors there are at least two different ones of the form $4n + 1$ where n is a positive integer. In our case we have $65 = 5 \cdot 13 = (4 + 1)(12 + 1)$.

Step 3. Solution of the Model (2.1)-(2.4), Double Fourier Series

In general, the solutions (2.12) only satisfy (2.1)-(2.3). To obtain the solution that also satisfies the initial conditions (2.4), we consider the double series as in lecture 18

$$\begin{aligned} u(x,y,t) &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} u_{mn}(x,y,t) \\ &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} (A_{mn} \cos \lambda_{mn} t + B_{mn} \sin \lambda_{mn} t) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}. \end{aligned} \tag{2.14}$$

Setting $t = 0$, we have

$$u(x,y,0) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} = f(x,y). \tag{2.15}$$

Suppose that $f(x,y)$ can be represented by (2.15). (Sufficient condition for this is the continuity of f, $\partial f/\partial x$, $\partial f/\partial y$, $\partial^2 f/\partial x\partial y$ in a rectangular domain Ω.) Then (2.15) is called the double Fourier series of $f(x,y)$. Its coefficients can be determined as follows. Setting

$$K_m(y) = \sum_{n=1}^{\infty} A_{mn} \sin \frac{n\pi y}{b}, \ m = 1, 2, \cdots, \tag{2.16}$$

then we can write (2.15) in the form

$$f(x,y) = \sum_{n=1}^{\infty} K_m(y) \sin\frac{m\pi x}{a}.$$

For fixed y this is the Fourier sine series of $f(x,y)$, considered as a function of x. Thus the coefficients of this expansion are

$$K_m(y) = \frac{2}{a}\int_0^a f(x,y)\sin\frac{m\pi x}{a}\mathrm{d}x, \;\; m = 1, 2, \cdots.$$

Furthermore, since (2.16) is the Fourier sine series of $K_m(y)$ for every fixed m, we obtain **generalized Euler formula**

$$\begin{aligned} A_{mn} &= \frac{2}{b}\int_0^b K_m(y)\sin\frac{n\pi y}{b}\mathrm{d}y \\ &= \frac{4}{ab}\int_0^b\int_0^a f(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\mathrm{d}x\mathrm{d}y, \quad m,n = 1,2,\cdots \end{aligned} \tag{2.17}$$

for the **Fourier coefficients** of $f(x,y)$ in the double Fourier series (2.15).

The A_{mn} in (2.14) are now determined in terms of $f(x,y)$. To determine the B_{mn}, we differentiate (2.14) termwise with respect to t. Using the second initial condition, we obtain

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn}\lambda_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} = g(x,y).$$

Suppose that $g(x,y)$ can be developed in this double Fourier series. Then, proceeding as before, we find that the coefficients are

$$B_{mn} = \frac{4}{ab\lambda_{mn}}\int_0^b\int_0^a g(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\mathrm{d}x\mathrm{d}y, \;\; m,n = 1,2,\cdots. \tag{2.18}$$

Result. If f and g in (2.4) are such that u can be represented by (2.14), then (2.14) with coefficients (2.17) and (2.18) is the solution of the model (2.1)-(2.4).

Example 2. *Find the vibrations of a rectangular membrane of sides* $a = 4$ ft *and* $b = 2$ ft. *Assume that the tension is* 12.5 lb/ft, *the density is* 2.5 slugs/ft^2, *the initial velocity is* 0, *and the initial displacement is*

$$f(x,y) = 0.1(4x - x^2)(2x - y^2) \text{ ft}.$$

Solution. $c^2 = T/\rho = 12.5/2.5 = 5$ [ft^2/sec^2]. Also $B_{mn} = 0$ from (2.18). From (2.17) we have

$$\begin{aligned} A_{mn} &= \frac{4}{4\cdot 2}\int_0^2\int_0^4 0.1(4x - x^2)(2y - y^2)\sin\frac{m\pi x}{4}\sin\frac{n\pi y}{2}\mathrm{d}x\mathrm{d}y \\ &= \frac{1}{20}\int_0^4 (4x - x^2)\sin\frac{m\pi x}{4}\mathrm{d}x\int_0^2 (2y - y^2)\sin\frac{n\pi y}{2}\mathrm{d}y. \end{aligned}$$

Using integrations by parts we have for the first integral on the right

$$\int_0^4 (4x - x^2)\sin\frac{m\pi x}{4}\mathrm{d}x = \frac{128}{m^3\pi^3}[1 - (-1)^m]$$

and for the second integral

$$\int_0^2 (2y - y^2)\sin\frac{n\pi y}{2}\mathrm{d}y = \frac{16}{n^3\pi^3}[1 - (-1)^n].$$

Hence we have

$$A_{mn} = \begin{cases} \dfrac{256 \cdot 32}{20m^3n^3\pi^6} \approx \dfrac{0.426050}{m^3n^3}, & m \text{ and } n \text{ both odd,} \\ 0, & \text{otherwise.} \end{cases}$$

From this we obtain the solution

$$\begin{aligned} u(x,y,t) &= 0.426050 \sum_{m,n} \sum_{\text{odd}} \frac{1}{m^3n^3} \cos\left(\frac{\sqrt{5}\pi}{4}\sqrt{m^2+4n^2}t\right) \sin\frac{m\pi x}{4} \sin\frac{n\pi y}{2} \\ &= 0.426050 \Big[\cos\left(\frac{\sqrt{5}\pi\sqrt{5}}{4}t\right) \sin\frac{\pi x}{4} \sin\frac{\pi y}{2} + \frac{1}{27}\cos\left(\frac{\sqrt{5}\pi\sqrt{37}}{4}t\right) \sin\frac{\pi x}{4} \sin\frac{3\pi y}{2} \\ &\quad + \frac{1}{27}\cos\left(\frac{\sqrt{5}\pi\sqrt{13}}{4}t\right) \sin\frac{3\pi x}{4} \sin\frac{\pi y}{2} + \frac{1}{729}\cos\left(\frac{\sqrt{5}\pi\sqrt{45}}{4}t\right) \sin\frac{3\pi x}{4} \sin\frac{3\pi y}{2} + \cdots \Big]. \end{aligned}$$

To discuss this solution, we note that the first term is very similar to the initial shape of the membrane, has no nodal lines, and is by far the dominating term because the coefficients of the next terms are much smaller. The second term has two horizontal nodal lines ($y = \frac{2}{3}, \frac{4}{3}$), the third term two vertical ones ($x = \frac{4}{3}, \frac{8}{3}$), the fourth term two horizontal and two vertical ones, and so on.

Example 3. *Find the solution of the problem*

$$\begin{cases} u_{tt} = c^2(u_{xx} + u_{yy}), & 0 < x < a,\ 0 < y < b,\ t > 0, \\ u(0,y,t) = 0,\ u(a,y,t) = 0, & 0 \le y \le b,\ t \ge 0, \\ u_y(x,0,t) = 0,\ u_y(x,b,t) = 0, & 0 \le x \le a,\ t \ge 0, \\ u(x,y,0) = x(a-x)y,\ u_t(x,y,0) = x, & 0 \le x \le a,\ 0 \le y \le b. \end{cases}$$

Solution. Let $u(x,y,t) = H(x)Q(y)G(t)$ and substitute it into the wave equation as before, we get three ODEs

$$\begin{aligned} &G''(t) + (c\nu)^2 G(t) = 0,\ t > 0, \\ &H''(x) + k^2 H(x) = 0,\ 0 < x < a, \\ &Q''(y) + p^2 Q(y) = 0,\ 0 < y < b, \end{aligned}$$

where $\nu^2 = k^2 + p^2$. From boundary conditions we have $H(0)Q(y)G(t) = H(a)Q(y)G(t) = 0,\ H(x)Q'(0)G(t) = H(x)Q'(b)G(t) = 0$, which implies

$$H(0) = H(a) = 0,\ Q'(0) = Q'(b) = 0.$$

We first solve the problem for $H(x)$, that is,

$$H''(x) + k^2 H(x) = 0,\ 0 < x < a;\ H(0) = H(a) = 0.$$

As before, we obtain

$$k = k_m = \frac{m\pi}{a},\ H(x) = H_m(x) = \sin\frac{m\pi x}{a},\ m = 1, 2, \cdots.$$

We then solve the problem for $Q(y)$, namely,

$$Q''(y) + p^2 Q(y) = 0,\ 0 < y < b;\ Q'(0) = Q'(b) = 0.$$

Similarly, it has solutions

$$p = p_n = \frac{n\pi}{b},\quad Q_n(y) = \cos\frac{n\pi y}{b},\ n = 0, 1, 2, \cdots.$$

Moveover, we obtain

$$\nu_{mn}^2 = k_m^2 + p_n^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2,\ m = 1, 2, \cdots; n = 0, 1, 2, \cdots.$$

We finally solve equation for $G(t)$ with $\nu = \nu_{mn}$. It has general solution

$$G_{mn}(t) = A_{mn}\cos\lambda_{mn}t + B_{mn}\sin\lambda_{mn}t,\ \ \lambda_{mn} = c\pi\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}.$$

Hence we obtain the functions

$$u_{mn}(x,y,t) = (A_{mn}\cos\lambda_{mn}t + B_{mn}\sin\lambda_{mn}t)\sin\frac{m\pi x}{a}\cos\frac{m\pi y}{b},\ \ m = 1,2,\cdots; n = 0,1,\cdots,$$

which satisfy wave equation and the boundary conditions, does not satisfy the initial conditions.

To obtain the solution that also satisfies the initial conditions, we consider the double series

$$\begin{aligned} u(x,y,t) &= \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} u_{mn}(x,y,t) \\ &= \sum_{m=1}^{\infty}\sum_{n=0}^{\infty}(A_{mn}\cos\lambda_{mn}t + B_{mn}\sin\lambda_{mn}t)\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}. \end{aligned}$$

Setting $t = 0$, we have

$$u(x,y,0) = \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} A_{mn}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b} = x(a-x)y.$$

This gives

$$\begin{aligned} A_{m0} &= \frac{2}{ab}\int_0^b\int_0^a x(a-x)y\sin\frac{m\pi x}{a}\mathrm{d}x\mathrm{d}y = \frac{2ba^2}{(m\pi)^3}[1-(-1)^m],\quad m = 1,2,\cdots, \\ A_{mn} &= \frac{4}{ab}\int_0^b\int_0^a x(a-x)y\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\mathrm{d}x\mathrm{d}y \\ &= -\frac{8ba^2}{n^2m^3\pi^5}[1-(-1)^m][1-(-1)^m],\quad m,n = 1,2,\cdots. \end{aligned}$$

Differentiating termwise with respect to t at $t = 0$ gives

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} B_{mn}\lambda_{mn}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b} = x.$$

It gives the coefficients

$$\begin{aligned} B_{m0} &= \frac{2}{ab\lambda_{m0}}\int_0^b\int_0^a x\sin\frac{m\pi x}{a}\mathrm{d}x\mathrm{d}y = \frac{2a(-1)^{m-1}}{m\lambda_{m0}\pi} = \frac{2a^2(-1)^{m-1}}{m^2c\pi^2},\ \ m = 1,2,\cdots, \\ B_{mn} &= \frac{4}{ab\lambda_{mn}}\int_0^b\int_0^a x\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\mathrm{d}x\mathrm{d}y = 0,\ \ m,n = 1,2,\cdots. \end{aligned}$$

Therefore, the solution is

$$\begin{aligned} u(x,y,t) &= \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} u_{mn}(x,y,t) \\ &= \sum_{m=1}^{\infty}\left(\frac{2ba^2}{(m\pi)^3}[1-(-1)^m]\cos\frac{cm\pi t}{a} + \frac{2a^2(-1)^{m-1}}{m^2c\pi^2}\sin\frac{cm\pi t}{a}\right)\sin\frac{m\pi x}{a} \\ &\quad - \sum_{m,n=1}^{\infty}\sum_{\text{odd}}\frac{32ba^2}{n^2m^3\pi^5}\cos\lambda_{mn}t\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}. \end{aligned}$$

3. Problems

1. How does the frequency of the eigenfunctions of the rectangular membrane change (a) If we double the tension? (b) If we take a membrane of half the density of the original one? (c) If we double the sides of the membrane? Give reasons.
2. Find eigenvalues of the rectangular membrane of sides $a = 2$ and $b = 1$ to which there correspond two or more different (independent) eigenfunctions.
3. Show that among all rectangular membranes of the same area $A = ab$ and the same c the square membrane is that for which u_{11} has the lowest frequency.
4. Solve the solutions of following problems

$$(1)\begin{cases} u_{tt} - c^2(u_{xx} + u_{yy}) = 0, & 0 < x < 1, 0 < y < 1,\ t > 0, \\ u(0, y, t) = 0,\ u(1, y, t) = 0, & 0 < y < 1,\ t > 0, \\ u(x, 0, t) = 0,\ u(x, 1, t) = 0, & 0 < x < 1,\ t > 0, \\ u(x, y, 0) = 0, u_t(x, y, 0) = xy, & 0 \le x \le 1,\ 0 \le y \le 1. \end{cases}$$

$$(2)\begin{cases} u_{tt} - c^2(u_{xx} + u_{yy}) = 0, & 0 < x < 1, 0 < y < 1,\ t > 0, \\ u(0, y, t) = 0,\ u(1, y, t) = 0, & 0 < y < 1,\ t > 0, \\ u(x, 0, t) = 0,\ u(x, 1, t) = 0, & 0 < x < 1,\ t > 0, \\ u(x, y, 0) = \sin \pi x \sin 2\pi y, u_t(x, y, 0) = y, & 0 \le x \le 1,\ 0 \le y \le 1. \end{cases}$$

$$(3)\begin{cases} u_{tt} - c^2(u_{xx} + u_{yy}) = 0, & 0 < x < 1, 0 < y < \pi,\ t > 0, \\ u_x(0, y, t) = 0,\ u(1, y, t) = 0, & 0 < y < \pi,\ t > 0, \\ u(x, 0, t) = 0,\ u(x, \pi, t) = 0, & 0 < x < 1,\ t > 0, \\ u(x, y, 0) = 0, u_t(x, y, 0) = x, & 0 \le x \le 1,\ 0 \le y \le 1. \end{cases}$$

$$(4)\begin{cases} u_{tt} - (u_{xx} + u_{yy}) = 0, & 0 < x < \pi, 0 < y < \pi,\ t > 0, \\ u_x(0, y, t) = 0,\ u_x(\pi, y, t) = 0, & 0 < y < \pi,\ t > 0, \\ u(x, 0, t) = 0,\ u(x, \pi, t) = 0, & 0 < x < \pi,\ t > 0, \\ u(x, y, 0) = xy(\pi - x)(\pi - y), u_t(x, y, 0) = 1, & 0 \le x \le 1,\ 0 \le y \le 1. \end{cases}$$

5. Show that forced vibrations of a membrane are modeled by the PDE $u_{tt} = c^2\nabla^2 u + \frac{P}{\rho}$, where $P(x, y, t)$ is the external force per unit area acting perpendicular to the xy-plane.