Typing the Y Combinator

Esther Wang

Software Engineer - Airbnb

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Why have recursive types?

The Y combinator

$$\lambda\ \mathtt{f\ .\ (}\lambda\ \mathtt{x\ .\ f\ (x\ x))(}\lambda\ \mathtt{x\ .\ f\ (x\ x))}$$

The Y combinator

`λ f . (λ x . f (x x))(λ x . f (x x))`

What if we want static types?

Omega: a smaller example

λ `x . x x`

Omega: a smaller example

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1. `x` must be a function
2. `x :: a -> b`
3. `a` must be the type of `x`, so `a = a -> b`
4. `a -> b = (a -> b) -> b = ((a -> b) -> b) -> b = ...`

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```

Solution: recursive types

Examples of recursive types

Recursive types are ubiquitous

```
data Nat = Zero | Succ Nat
data IntList = Nil | Cons Int IntList
data StringTree = Leaf String | Node StringTree StringTree
```

Recursive types in type theory

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- Recursive types have the form $\mu a.F\ a$, where F is a type expression
- μ is the type-level **fixpoint operator**
- Adding μ to our type system allows us to type any term from the untyped lambda calculus

Example: Nat

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```
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1. `sumAllInts :: Int -> ???`
2. `sumAllInts :: Int -> (Int + ???)`
3. `sumAllInts ::` μ `a . Int -> (Int + a)`

Fixpoints

Definitions

Fixed point x

$$x = f\,x = f\,(f\,x) = \cdots$$

Fixpoint combinator **fix**

$$\mathrm{fix}\,f = x$$

Combined definitions

$$\mathrm{fix}\,f = f\,(\mathrm{fix}\,f)$$

Fixpoint combinators

- The Y Combinator is **fix** on the term level
- μ is **fix** on the type level

Defining μ

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Defining μ

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- By the definition of a fixpoint combinator, $\mu\ F = x$, where x is a fixed point
- By the definition of fixed points, $x = F\ x$
- Substituting for x,

$$\mu\ F = F\ (\mu\ F)$$

Defining μ

μ is defined as

$$\mu\ F = F\ (\mu\ F)$$

You can substitute the recursive type into itself

```
Nat = μ a . 1 + a = 1 + (μ a . 1 + a) = ...
```

Typing the Y combinator

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Typing the Y combinator

λ `f . (`λ` x . f (x x))(`λ` x . f (x x))`

1. `f :: a -> b`
2. `x :: c -> a`
3. `c = c -> a`, so `c = `μ` c . c -> a`
4. The Y combinator has type `(a -> b) -> b`, for fixed `a` and `b`

Equi vs. iso-recursive types

Equi-recurisive types

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- In our examples so far, we've implicitly used

$$\mu\, F = F\,(\mu\, F)$$

- equi-recursive approach, where a recursive type is interchangeable with its expansion
- Easy to add to a type system
- Difficult to implement in a typechecker

Iso-recursive types

- In the iso-recursive approach,

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- A recursive type and its expansion are *isomorphic*
- The functions `roll` and `unroll` witness the isomorphism

Statics: iso-recursive types

Let `S = `μ` a . T`, where `T = F a`

$$\texttt{unroll[S] :: S -> [a} \mapsto \texttt{S] T}$$

Statics: iso-recursive types

Let `S = ` μ ` a . T`, where `T = F a`

$$\texttt{unroll[S] :: S -> [a} \mapsto \texttt{S] T}$$
$$\texttt{roll[S] :: [a} \mapsto \texttt{S] T -> S}$$

Dynamics: Iso-recursive types

Roll and unroll are inverses

$$\texttt{unroll[S] (roll[T] (e))} \rightarrow \texttt{e}$$

Use in production languages

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Recursive types in Haskell

Fix in Haskell

Haskell has recursion, so no need to use the Y combinator

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If this feature didn't exist, would Haskell still be Turing-complete?

Naive implementation of the Y combinator

```
Prelude> y = \f -> (\x -> f (x x)) (\x -> f (x x))

<interactive>:7:23: error:
  • Occurs check: cannot construct the infinite type:
      t0 ~ t0 -> t
    Expected type: t0 -> t
      Actual type: (t0 -> t) -> t
  • In the first argument of ‘x’, namely ‘x’
    In the first argument of ‘f’, namely ‘(x x)’
    In the expression: f (x x)
  • Relevant bindings include
      x :: (t0 -> t) -> t (bound at <interactive>:7:13)
      f :: t -> t (bound at <interactive>:7:6)
      y :: (t -> t) -> t (bound at <interactive>:7:1)
```

Implementing the Y combinator

Define μF

```
newtype Mu f = Mu (f (Mu f))
unroll (Mu f) = f
roll = Mu
```

Remember the types

$$\lambda \texttt{ f . (}\lambda\texttt{ x . f (x x))(}\lambda\texttt{ x . f (x x))}$$

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1. `f :: a -> b`
2. `x ::` μ `c . c -> a`
3. We need to be able to write the type of `x` in terms of $\mu a.F\ a$
4. `x ::` μ `c . F(c)`, where `F(c) = c -> a` for some fixed `a`

Implementing the Y combinator

Define `F(c) = c -> a`

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newtype Mu f = Mu (f (Mu f)
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newtype F' c a = F' (c -> a)
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type F c = Mu (F' c)
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y f = (\x -> f (unroll' x x))
  $ roll' (\x -> f (unroll' x x))
```

Recursion schemes

`Mu` is the same as `Fix` from the *recursion-schemes* library!

```
newtype Fix f = Fix (f (Fix f))

unfix (Fix f) = f
fix = Fix
```

Some additional theory

Data and codata

- Data consists of indefinitely large, but finite structures

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Why it matters

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- For category theorists, data is an initial F-algebra, codata is a terminal F-coalgebra

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- μ is defined as a type-level fixpoint combinator: $\mu a.F\ a$
- The equi-recursive approach treats a recursive type as *equal* to its expansion
- The iso-recursive approach treats a recursive type as *isomorphic* to its expansion

Further reading

1. *Types and Programming Languages* by Benjamin Pierce
2. “Recursive types for free!” by Philip Wadler
3. Recursion schemes
4. Fixpoints and iso-recursive types
5. Data and codata

Questions?