The major findings of your study

The meaning of those findings

How these findings relate to what others have done

Limitations of your findings

An explanation for any surprising, unexpected, or inconclusive results

Suggestions for further research

[**Summary**](https://www.scribbr.com/working-with-sources/how-to-summarize/)**:** A brief recap of your key results

The objective of this thesis was to Elaborate a method that adapt the finding of \desai2022oneshot to stiff problem. Namely, the contributors in the original work were able to solve a variety of differential equations and subsequently transfer learn the solutions. In this work we adapt these methodologies to stiffness by introducing the stiffness parameter $\alpha$ in the equation in infer this parameter in the transfer learning. The figure \ref{mae\_comparative\_general} summarize the result in term of mean absolute error showing that our method outperform the vanilla PINNS or other optimizing way of transfer learning. A major difficulty in stiffnes as explain in Section \ref{{sec:stiffness} is to be able to encode the solution in the vicinity of the transient phase. The Figure \ref{maxAEy2\_comparative\_general} show that our method outperform other PINNs based method by being able to encode it further in the stiff regime. When comparing with numerical methods Figures \ref{transfer\_stiff\_increase, exemple1\_transfer\_stiff\_increase, duffing\_transfer\_stiff\_increase} show that our solution computational time is independent of the stiffness and can compete (but not beat) the Radau method wich is well design to enhance stiffness. Figures \ref{ multi\_inference\_1000\_IC, multi\_inference\_1000\_force, duffing\_inference\_1000\_IC} show that in term of computational time our methodologie outperforms numerical methods when infering of the initial condition or the force function inside a stiff domain.

The aim of this thesis was to develop a methodology that adapts the approaches outlined in \cite{desai2022oneshot} to address stiff problems. The original work demonstrated the capability to solve various differential equations and subsequently transfer learned solutions. In our research, we extend these methodologies to address stiffness by incorporating the stiffness parameter $\alpha$ into the equation and inferring this parameter during the transfer learning process. The results, presented in Figure \ref{mae\_comparative\_general}, are summarized in terms of mean absolute error, indicating that our approach surpasses both vanilla Physics-Informed Neural Networks (PINNS) and other optimization-based transfer learning methods. One significant challenge in addressing stiffness, as discussed in Section \ref{sec:stiffness}, is the ability to encode the solution in the vicinity of the transient phase. Figure \ref{maxAEy2\_comparative\_general} illustrates that Our method outperforms other PINNS-based approaches for encoding this behavior with a maximum absolute error that is lower in stiffness regimes.. A comparative analysis with numerical methods, as depicted in Figures \ref{transfer\_stiff\_increase}, \ref{exemple1\_transfer\_stiff\_increase}, and \ref{duffing\_transfer\_stiff\_increase}, reveals that our solution's computational time remains independent of stiffness, offering competitive performance, albeit falling slightly short of the Radau method, a well-designed approach tailored for handling stiffness. Furthermore, Figures \ref{multi\_inference\_1000\_IC}, \ref{multi\_inference\_1000\_force}, and \ref{duffing\_inference\_1000\_IC} highlight the superior computational efficiency of our methodology when inferring initial conditions or force functions within a stiff domain. In comparison to numerical methods, our approach demonstrates a notable advantage in terms of computational time, showcasing its efficacy in practical applications.

**Interpretations:** What do your results mean?

**Implications:** Why do your results matter?

Train in a non stiff regime -> transfer to stiff

In comparison to numerical methods, our approach demonstrates a notable advantage in terms of computational time, showcasing its efficacy in practical applications

The results suggest that we can calculate a solution of a stiff problem by training the model in a less stiff or even a non-stiff regime. This result matter when we know the difficulty to train a PINNs in a stiff regime as present in \\cite{wang2020understanding}, which investigated  
a fundamental mode of failure of PINN that is related to stiffness. While previous research \cite{ baty2023solving, Ji\_2021} has focused on *solving stiff problem by training a PINNs evry time the regime of the stiffness, initales condition, force function change* , these results demonstrate we can compute the solution in “one shot” without retraining the model. This is particulary valuable in term of computational time and make our methods competitive in a computational time point of view with numerical methods. Even more for inference of initials condition and force function the computation of the solution can be faster than numerical methods. For example (give an applied example of where it can be used like an inference on the initiale condition of a stiff problem, an example in the industry or the reacherche).

The findings of this study indicate that the solution to a stiff problem can be computed by training the model in a less stiff or even a non-stiff regime. This observation holds significant implications, especially considering the challenges highlighted in prior research such as Wang et al. (2020) \cite{wang2020understanding}, about training in a stiff domain.

While previous investigations (\cite{baty2023solving, Ji\_2021}) have primarily focused on addressing stiff problems by retraining PINNs each time the stiffness regime, initial conditions, or force function changes. Our results demonstrate the feasibility of obtaining the solution without retraining a model. Such an approach holds considerable value in terms of computational efficiency and it’s independent of the stiffness of the equation. Rendering it competitive from a computational time perspective when compared to traditional numerical techniques.

Furthermore, our results highlight the potential for expedited computations, in the inference of initial conditions and force functions. The efficiency gained through our approach surpasses that of conventional numerical methods. To illustrate this, consider a practical scenario, with a stiff equation in chemical kinetics due to the rate of change of the component being of different magnitude like the Roberston equation. One can transfer to various initales condiction of the chemical composent within the stiff domain very fast.

To illustrate this concept, let's consider a practical scenario, such as chemical kinetics, involving a rigid equation, such as the Robertson equation. The stiffness arises because of an importante variation in the rate of change among the chemical components .Our approach can facilitate a rapid computation of the solution through different initial conditions in the stiff domain.

**Limitations:** What can’t your results tell us?

A first limitation of this work is that we can’t transfer to an infinite stiffness regime without losing to much accuracy on the solution. In deed even if the jump in the stiffness can be significatively importante it depand on the stiffness regime of the training process. But the more far in the stiffness we can train the more far we can transfer afterward.

An other limitation considering these methods is for the purpose of non linear equations. In deed the perturbation expansion transfer learning method invaloves that it exist a perturaiton solution of the system. In a lot of stiff problem like the Van der Pols, Roberston equation, It appears that there is no finite perturbation solution of these equation and it require an infinite degree of the expansion ($p=\inifite$)

On top of this , it is worth noting that this a relatively new area of research that does not have the well established theory and vast number of contributions which numerical methods provide such as error bound or robustness. Namely, it is known that neural networks do not provide a guarantee in terms of training stability or convergence. Hence, for most real-world, critical applications where accuracy of solutions is paramount, our network-based solution should be used in conjunction with numerical approaches.

Can’t transfer to inifite stiffness (it depend on the stiffness training regime)

Cannot be apply to all non linear methods (Van der pols…) because there is no perturbation solution

Limitation when comparing to numerical methods (robustess, accuracy, error bound…)

A primary constraint of this work is that we cannot transfer to an infinite stiffness regime without sacrificing accuracy in the solution. Even if the jump in stiffness can be significant, it depends on the stiffness regime of the training process. However, the more we train in a stiff regime, the more stiff we'll be able to transfer later.

in this study is the inability to seamlessly transition to an infinite stiffness regime without incurring a substantial loss in solution accuracy. The extent of this loss depends on the stiffness regime during the training process, with notable limitations encountered when attempting to navigate towards infinitely stiff scenarios. While substantial progress can be achieved in training for higher levels of stiffness, the challenge remains in maintaining solution accuracy during subsequent transfers.

Another notable limitation pertains to the applicability of these methods in the context of nonlinear equations. Specifically, the perturbation expansion transfer learning method assumes the existence of a perturbation solution for the system. However, in numerous stiff problems such as the Van der Pol and Robertson equations, it becomes evident after some experiement that no finite perturbation solution exists, necessitating an infinite degree of expansion (i.e., \(p=\infty\)).

Additionally, it is crucial to acknowledge that this field of research is relatively nascent, lacking a well-established theoretical framework and the extensive contributions seen in numerical methods, such as error bounds and robustness assessments. Notably, the inherent nature of neural networks introduces uncertainties related to training stability and convergence. Consequently, in critical real-world applications where solution accuracy is paramount, it is advisable to complement our network-based approach with traditional numerical methods, given their proven reliability and established theoretical underpinnings.

**Recommendations:** Avenues for further studies or analyses

Try on pdes

Be able to train in a stiffer domain to then transfer further

Quantify mathematically the range of stiffness until when we can transfer.

Avenues for future research include the adaptation of this methods for PDEs as its notice in \cite {wang2020understanding} it could be extend to for linear PDEs or non linear pde that have a perturbation solution. It could also include to try to be able to train in a stiffer domain to then transfer further with more computational recources during training or bigger model.

Future research endeavors could explore the adaptation of these methods to Partial Differential Equations (PDEs), as suggested by Wang et al. (2020) \cite{wang2020understanding}. This extension might encompass linear PDEs or non-linear PDEs possessing a perturbation solution.

Furthermore, there is potential for advancing the methodology by enhancing its capacity to train in stiffer domains. This could be achieved through the allocation of additional computational resources during the training process or by employing larger models. By pushing the boundaries of stiffness that the model can handle during training, may subsequent in transferring to even more stiff regime. Exploring these avenues would provide valuable insights into the scalability of the proposed approach, in the realm of solving stiff problems.

Parler de VDP, Roberston

How these methodologies can be adapted to effectively tackle stiffness, aiming to rival numerical methods in both computational efficiency and accuracy for some linear and nonlinear stiff ordinary differential equations?